

Soft corrections to inclusive deep-inelastic scattering at four loops and beyond

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ABSTRACT: We study the threshold corrections for inclusive deep-inelastic scattering (DIS) and their all-order resummation. Using recent results for the QCD form factor, related anomalous dimensions and Mellin moments of DIS structure functions at four loops we derive the complete soft and collinear contributions to the DIS Wilson coefficients at four loops. For a general $SU(n_c)$ gauge group the results are exact in the large- n_c approximation and for QCD with $n_c = 3$ we present precise approximations. We extend the threshold resummation exponent G^N in Mellin- N space to the fifth logarithmic (N^4LL) order collecting the terms $\alpha_s^3(\alpha_s \ln N)^n$ to all orders in the strong coupling constant α_s . We study the numerical effect of the N^4LL corrections using both the fully exponentiated form and the expansion of the coefficient function in towers of logarithms. As a byproduct, we derive a numerical result for the complete pole structure of the QCD form factor in the parameter of dimensional regularization ε at four loops.

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1 Introduction

The cross section for inclusive deep-inelastic scattering (DIS) of charged leptons or neutrinos off a nucleon target can be expressed in terms of structure functions, which factorize into the product of operator matrix elements for the nucleon under consideration and the DIS Wilson coefficients. The latter parametrize the hard partonic scattering cross section driven by short-distance physics and are calculable in perturbative Quantum Chromodynamics (QCD) order by order in the strong coupling constant α_s . However, only a few terms in this expansion can be calculated completely. Currently, the QCD corrections at next-to-next-to-next-to-leading order (N³LO) for the neutral- and charged-current DIS structure functions F_1 , F_2 and F_3 are known [1, 2]. In the exceptional region of phase space, on the other hand, when the Bjorken scaling variable x is close to unity, $x \rightarrow 1$, the DIS Wilson coefficients develop large Sudakov logarithms at all orders in the form $\alpha_s^n \ln^l(1-x)/(1-x)_+$, with $l = 2n - 1, \dots, 0$. These are a consequence of the constrained phase space available for the real emission of soft and collinear gluons.

The DIS Wilson coefficients are subject to an additional factorization for $x \rightarrow 1$ (or $N \rightarrow \infty$ in Mellin- N space) which allows for resummation of those Sudakov double logarithms [3–7]. Based on the fixed-order QCD results up to N³LO for the DIS structure

functions [1, 2] this resummation has been performed to next-to-next-to-next-to-leading logarithmic ($N^3\text{LL}$) accuracy in Mellin- N space [8], where the resummation takes the form of an exponentiation which organizes the respective logarithms $\alpha_s^n \ln^l N$, with $l = 2n, \dots, 1$ (see also [9–12] for related work in the Soft-Collinear Effective Theory).

From a phenomenological point of view, resummation in DIS is essential to obtain reliable QCD predictions in the (small) region of phase space dominated by such large threshold logarithms. This is of particular relevance since DIS structure functions also receive power corrections beyond the QCD factorization at leading twist and such higher-twist contributions are supposed to be significant at large x . Therefore, good control of the perturbative series at leading twist is important to distinguish those effects in the analyses of DIS data, see for instance [13].

From a field-theoretic point of view, on the other hand, the threshold limit in DIS is of interest, since the factorization of the DIS Wilson coefficients for $x \rightarrow 1$ links a number of fundamental quantities in QCD with each other due to the universality of the soft and collinear limit. In particular, this comprises the QCD form factor which features an all-order exponentiation in dimensional regularization with $d = 4 - 2\epsilon$ dimensions [14–19] and the QCD splitting functions, currently fully known to next-to-next-to-leading order (NNLO) [20, 21]. The necessary cancellation of soft and collinear singularities in ϵ in inclusive observables due to the Kinoshita-Lee-Nauenberg theorem [22, 23] offers a constructive approach to the DIS Wilson coefficients near threshold [24], see also [25–27].

Recent progress in the computation of QCD corrections has been obtained in particular for the quark form factor [28] and the non-singlet quark splitting functions [29], which have been calculated at four-loop order in the limit of large- n_c for a general $SU(n_c)$ gauge group. In addition, a low number of fixed Mellin moments for the DIS structure functions are available at four loops [30], as well as information on the n_f -dependence of the QCD form factor [30–34] and on new colour factors (quartic Casimirs) [33, 35–39]. Taken together [40], these results allow us to extend the threshold resummation exponent G^N

for DIS Wilson coefficients in Mellin- N space to the fifth logarithmic ($N^4\text{LL}$) order. At this accuracy we collect and resum in G^N the terms $\alpha_s^3 (\alpha_s \ln N)^n$ to all orders in α_s . To that end, we extract the resummation exponent B^{DIS} of the quark jet function collecting final-state collinear emissions at the fourth order in α_s , and we address the term $\alpha_s^4/(1-x)_+$ in the fixed-order expansion of the DIS Wilson coefficients at four loops. With the help of the available four-loop QCD results, we can provide exact expressions in the large- n_c limit and adequate numerical approximations for full QCD with $n_c = 3$. As a byproduct, we also derive a numerical result including full colour dependence for the single pole $1/\epsilon$ of the dimensional regulated QCD form factor at four loops.

This article is organized as follows. In section 2 we recall the general structure of the threshold resummation for DIS and provide the required formulae to perform resummation to $N^4\text{LL}$ accuracy. In section 3 we describe the extraction of the resummation coefficients and the calculation of the DIS Wilson coefficient in the soft and collinear limit. We present a numerical study for the exponentiated resummed DIS Wilson coefficients and their tower expansion in section 4 and summarize in section 5. Exact expressions for the DIS Wilson coefficients are given in appendix A.

2 Theoretical framework

The general structure of the resummed DIS Wilson coefficients takes the following form in the Mellin N -space [3, 4],

$$C_{a,q}^N(Q^2) = g_0^a(Q^2) \cdot \exp[G^N(Q^2)] + \mathcal{O}(N^{-1} \ln^n N), \quad (2.1)$$

for the neutral- or charged-current DIS structure functions F_a , where $a = 1, 2, 3$. In Mellin space they factorize in a sum over all partons i as $F_a^N(Q^2) = C_{a,i}^N(Q^2) f_i^N$ with f_i^N denoting the Mellin transformed parton distributions $f_i(x)$ for the parton i , and the Born contributions to the are normalized as $C_{a,q}^N|_{\text{LO}} = 1$. Here the resummation exponent G^N contains all terms of the form $\ln^k N$ to all orders in α_s , and the pre-factor g_0 encompasses the N - independent terms. G^N and g_0 depend on the physical hard scale Q^2 ($= -q^2$ in DIS, with q the four-momentum of the exchanged gauge boson), and on the renormalization and factorization scales μ and μ_f , suppressed for brevity. For inclusive DIS G^N reads in the notation of [8, 41]

$$G^N = \ln \Delta_q + \ln J_q + \ln \Delta_{\text{DIS}}^{\text{int}}, \quad (2.2)$$

where the radiation factors ($\Delta_q, J_q, \Delta_{\text{DIS}}^{\text{int}}$) are given by well-known integrals over functions of the running coupling. The collinear soft-gluon radiation off an initial-state quark is collected by

$$\ln \Delta_q(Q^2, \mu_f^2) = \int_0^1 dz \frac{z^{N-1} - 1}{1-z} \int_{\mu_f^2}^{(1-z)^2 Q^2} \frac{dq^2}{q^2} A^q(\alpha_s(q^2)), \quad (2.3)$$

with the light-like cusp anomalous dimension $A^q(\alpha_s)$ addressed below. The collinear emissions from an ‘unobserved’ final-state quark are summarized in the jet function,

$$\ln J_q(Q^2) = \int_0^1 dz \frac{z^{N-1} - 1}{1-z} \left[\int_{(1-z)^2 Q^2}^{(1-z)Q^2} \frac{dq^2}{q^2} A^q(\alpha_s(q^2)) + B^{\text{DIS}}(\alpha_s([1-z]Q^2)) \right] \quad (2.4)$$

including the additional function $B^{\text{DIS}}(\alpha_s)$. All expansions in terms of the strong coupling α_s are normalized as

$$A^q(\alpha_s) = \sum_{n=1}^{\infty} \left(\frac{\alpha_s}{4\pi} \right)^n A_n \equiv \sum_{n=1}^{\infty} a_s^n A_n \quad (2.5)$$

etc. Any process-dependent contributions from large-angle soft gluons emissions are contained in the function $\Delta_{\text{DIS}}^{\text{int}}$ which, however, evaluates to unity, i.e., $\Delta_{\text{DIS}}^{\text{int}} = 1$, since the corresponding evolution kernels vanish at all order in α_s for inclusive DIS [42, 43]. Thus, the last term in eq. (2.2) is absent for inclusive DIS. The running of the strong coupling is governed by the QCD beta function

$$\beta(a_s) = - \sum_{n=0}^{\infty} a_s^{n+2} \beta_n, \quad (2.6)$$

with $\beta_0 = (11/3)C_A - (2/3)n_f$ and so on.

The evaluation of the integrals in eqs. (2.3) and (2.4) proceeds in a standard manner, see for instance [8, 44], leading to the following expression of the resummation exponent G^N ,

$$G^N = \ln \tilde{N} g_1^{\text{DIS}}(\lambda) + g_2^{\text{DIS}}(\lambda) + a_s g_3^{\text{DIS}}(\lambda) + a_s^2 g_4^{\text{DIS}}(\lambda) + a_s^3 g_5^{\text{DIS}}(\lambda), \quad (2.7)$$

where $\lambda = \beta_0 a_s \ln \tilde{N}$ with $\tilde{N} = N \exp(\gamma_E)$, where $\gamma_E \simeq 0.57721566$ is the Euler-Mascheroni constant. The number of terms on the right hand side of G^N provides the resummation accuracy, namely LL, NLL ..., respectively. For notational brevity we have organized the logarithms in Mellin space as $\ln \tilde{N}$. Eq. (2.7) is accurate to N⁴LL order and the coefficient $g_5^{\text{DIS}}(\lambda)$ collects all large logarithms up to $\alpha_s^3 (\alpha_s \ln \tilde{N})^n$ in G^N . The complete results for g_i^{DIS} with $i \leq 5$, including for consistency the well-known lower order results [3, 4, 8, 44, 45], read

$$\begin{aligned} g_1^{\text{DIS}}(\lambda) &= (1 + \lambda^{-1} L_{\lambda_1} - L_{\lambda_1}) A_1, \\ g_2^{\text{DIS}}(\lambda) &= -(L_{\lambda_1} + \lambda) A_2 + L_{\lambda_1} B_1^{\text{DIS}} + (L_{\lambda_1} + L_{\lambda_1}^2/2 + \lambda) A_1 \beta_1 + L_{\text{qr}} L_{\lambda_1} A_1 + L_{\text{fr}} \lambda A_1, \\ g_3^{\text{DIS}}(\lambda) &= -(1 - \lambda_1^{-1} + \lambda) A_3/2 + (1 - \lambda_1^{-1})(B_2^{\text{DIS}} - A_1 \zeta_2/2) \\ &\quad - (1 - \lambda_1^{-1} - 2\lambda_1^{-1} L_{\lambda_1} - \lambda_1^{-1} L_{\lambda_1}^2 + 2L_{\lambda_1} + \lambda) A_1 \beta_1^2/2 \\ &\quad - (1 - \lambda_1^{-1} - \lambda_1^{-1} L_{\lambda_1}) B_1^{\text{DIS}} \beta_1 + (3 - 3\lambda_1^{-1} - 2\lambda_1^{-1} L_{\lambda_1} + \lambda) A_2 \beta_1/2 \\ &\quad - (1 - \lambda_1^{-1} - 2L_{\lambda_1} - \lambda) A_1 \beta_2/2 \\ &\quad + L_{\text{qr}} \{ (1 - \lambda_1^{-1})(A_2 - B_1^{\text{DIS}}) - (1 - \lambda_1^{-1} - \lambda_1^{-1} L_{\lambda_1}) A_1 \beta_1 \} \\ &\quad - L_{\text{qr}}^2 (1 - \lambda_1^{-1}) A_1/2 + L_{\text{fr}} \lambda A_2 - L_{\text{fr}}^2 \lambda A_1/2, \\ g_4^{\text{DIS}}(\lambda) &= -(1 - \lambda_1^{-2} + 2\lambda) A_4/6 + (1 - \lambda_1^{-2})(B_3^{\text{DIS}}/2 + B_1^{\text{DIS}} \zeta_2/2 - A_2 \zeta_2/2 - A_1 \zeta_3/3) \\ &\quad - (1 - \lambda_1^{-2} - 2\lambda_1^{-2} L_{\lambda_1}) B_2^{\text{DIS}} \beta_1/2 + (5 - 5\lambda_1^{-2} - 6\lambda_1^{-2} L_{\lambda_1} + 4\lambda) A_3 \beta_1/12 \\ &\quad - (11 + \lambda_1^{-2} - 6\lambda_1^{-2} L_{\lambda_1} - 6\lambda_1^{-2} L_{\lambda_1}^2 - 12\lambda_1^{-1} + 4\lambda) A_2 \beta_1^2/12 \\ &\quad - \lambda_1^{-2} L_{\lambda_1} A_1 \beta_1 \zeta_2/2 - (1 - \lambda_1^{-2} - 6L_{\lambda_1} - 4\lambda) A_1 \beta_3/12 \\ &\quad + (2 + \lambda_1^{-2} - 3\lambda_1^{-1} + \lambda) A_2 \beta_2/3 + (2 + \lambda_1^{-2} + 3/2 \lambda_1^{-2} L_{\lambda_1} \\ &\quad - 1/2 \lambda_1^{-2} L_{\lambda_1}^3 - 3\lambda_1^{-1} - 3\lambda_1^{-1} L_{\lambda_1} + 3/2 L_{\lambda_1} + \lambda) A_1 \beta_1^3/3 \\ &\quad - (7 + 5\lambda_1^{-2} + 6\lambda_1^{-2} L_{\lambda_1} - 12\lambda_1^{-1} - 12\lambda_1^{-1} L_{\lambda_1} + 12L_{\lambda_1} + 8\lambda) A_1 \beta_1 \beta_2/12 \\ &\quad + (1 + \lambda_1^{-2} - \lambda_1^{-2} L_{\lambda_1}^2 - 2\lambda_1^{-1}) B_1^{\text{DIS}} \beta_1^2/2 - (1 + \lambda_1^{-2} - 2\lambda_1^{-1}) B_1^{\text{DIS}} \beta_2/2 \\ &\quad + L_{\text{qr}} \{ (1 - \lambda_1^{-2})(A_3/2 - B_2^{\text{DIS}} + A_1 \zeta_2/2) - (1 - \lambda_1^{-2} - 2\lambda_1^{-2} L_{\lambda_1}) A_2 \beta_1/2 \\ &\quad + (1 + \lambda_1^{-2} - \lambda_1^{-2} L_{\lambda_1}^2 - 2\lambda_1^{-1}) A_1 \beta_1^2/2 - (1 + \lambda_1^{-2} - 2\lambda_1^{-1}) A_1 \beta_2/2 \\ &\quad - \lambda_1^{-2} L_{\lambda_1} B_1^{\text{DIS}} \beta_1 \} - L_{\text{qr}}^2 \{ (1 - \lambda_1^{-2})(A_2/2 - B_1^{\text{DIS}}/2) + (\lambda_1^{-2} L_{\lambda_1}) A_1 \beta_1/2 \} \\ &\quad + L_{\text{qr}}^3 (1 - \lambda_1^{-2}) A_1/6 + L_{\text{fr}} \lambda A_3 - L_{\text{fr}}^2 \lambda (A_2 + A_1 \beta_1/2) + L_{\text{fr}}^3 \lambda A_1/3. \end{aligned} \quad (2.8)$$

For brevity, we use here the short-hand notations $\lambda_1 = 1 - \lambda$, $L_{\lambda_1} = \ln(1 - \lambda)$, $L_{\text{qr}} = \ln(Q^2/\mu^2)$ and $L_{\text{fr}} = \ln(\mu_f^2/\mu^2)$. Also we suppress factors of β_0 which can be restored easily by the substitutions $A_k \rightarrow A_k/\beta_0^k$, $B_k^{\text{DIS}} \rightarrow B_k^{\text{DIS}}/\beta_0^k$ and $\beta_k \rightarrow \beta_k/\beta_0^{k+1}$. In addition, g_3 needs to be multiplied by β_0 and g_4 by β_0^2 .

The new function g_5 (computed with methods of [8] and to be multiplied by β_0^3) is given by

$$\begin{aligned}
g_5^{\text{DIS}}(\lambda) = & \\
= & (1 - \lambda_1^{-3})(B_4^{\text{DIS}}/3 + B_2^{\text{DIS}}\zeta_2 + 2B_1^{\text{DIS}}\zeta_3/3 - A_3\zeta_2/2 - 2A_2\zeta_3/3 - 9A_1\zeta_4/8) \\
& - (1 - \lambda_1^{-3} + 3\lambda)A_5/12 + (1 - \lambda_1^{-3} + 2\lambda_1^{-3}L_{\lambda_1})(B_1^{\text{DIS}}\beta_1\zeta_2/2 - A_1\beta_1\zeta_3/3) \\
& - (1 - \lambda_1^{-3} - 3\lambda_1^{-3}L_{\lambda_1})B_3^{\text{DIS}}\beta_1/3 + (7 - 7\lambda_1^{-3} - 12\lambda_1^{-3}L_{\lambda_1} + 9\lambda)A_4\beta_1/36 \\
& - (1 - \lambda_1^{-3} - 12L_{\lambda_1} - 9\lambda)A_1\beta_4/36 + (5 + \lambda_1^{-3} - 6\lambda_1^{-1} + 3\lambda)A_2\beta_3/12 \\
& - (5 - 5\lambda_1^{-3} + 18\lambda_1^{-2} - 18\lambda_1^{-1} + 12L_{\lambda_1} + 9\lambda)A_1\beta_2^2/36 \\
& - (7 - \lambda_1^{-3} + 2\lambda_1^{-3}L_{\lambda_1} + 6\lambda_1^{-3}L_{\lambda_1}^2 + 2\lambda_1^{-3}L_{\lambda_1}^3 - \lambda_1^{-3}L_{\lambda_1}^4 + 6\lambda_1^{-2} - 6\lambda_1^{-2}L_{\lambda_1}^2 - 12\lambda_1^{-1} \\
& - 6\lambda_1^{-1}L_{\lambda_1} + 4L_{\lambda_1} + 3\lambda)A_1\beta_1^4/12 + (41 - 5\lambda_1^{-3} + 12\lambda_1^{-3}L_{\lambda_1} + 18\lambda_1^{-3}L_{\lambda_1}^2 + 36\lambda_1^{-2} \\
& - 18\lambda_1^{-2}L_{\lambda_1}^2 - 72\lambda_1^{-1} - 36\lambda_1^{-1}L_{\lambda_1} + 36L_{\lambda_1} + 27\lambda)A_1\beta_1^2\beta_2/36 \\
& - (2 + \lambda_1^{-3} - 3\lambda_1^{-1})B_1^{\text{DIS}}\beta_3/6 + (1 + \lambda_1^{-3} - 2\lambda_1^{-2} + \lambda)A_3\beta_2/4 \\
& - (7 + 2\lambda_1^{-3} + 3\lambda_1^{-3}L_{\lambda_1} - 9\lambda_1^{-1} - 9\lambda_1^{-1}L_{\lambda_1} + 12L_{\lambda_1} + 9\lambda)A_1\beta_1\beta_3/18 \\
& - (20 + 7\lambda_1^{-3} + 12\lambda_1^{-3}L_{\lambda_1} - 9\lambda_1^{-2} - 18\lambda_1^{-2}L_{\lambda_1} - 18\lambda_1^{-1} + 9\lambda)A_2\beta_1\beta_2/18 \\
& - (13 + 5\lambda_1^{-3} - 12\lambda_1^{-3}L_{\lambda_1} - 18\lambda_1^{-3}L_{\lambda_1}^2 - 18\lambda_1^{-2} + 9\lambda)A_3\beta_1^2/36 \\
& + (25 + 11\lambda_1^{-3} + 24\lambda_1^{-3}L_{\lambda_1} - 12\lambda_1^{-3}L_{\lambda_1}^3 - 18\lambda_1^{-2} - 36\lambda_1^{-2}L_{\lambda_1} - 18\lambda_1^{-1} + 9\lambda)A_2\beta_1^3/36 \\
& + (2 + \lambda_1^{-3} + 3\lambda_1^{-3}L_{\lambda_1} - 3\lambda_1^{-2}L_{\lambda_1} - 3\lambda_1^{-1})B_1^{\text{DIS}}\beta_1\beta_2/3 - \lambda_1^{-3}L_{\lambda_1}A_2\beta_1\zeta_2 \\
& - (2 + \lambda_1^{-3} + 6\lambda_1^{-3}L_{\lambda_1} + 3\lambda_1^{-3}L_{\lambda_1}^2 - 2\lambda_1^{-3}L_{\lambda_1}^3 - 6\lambda_1^{-2}L_{\lambda_1} - 3\lambda_1^{-1})B_1^{\text{DIS}}\beta_1^3/6 \\
& + (1 + 2\lambda_1^{-3} - 3\lambda_1^{-3}L_{\lambda_1}^2 - 3\lambda_1^{-2})B_2^{\text{DIS}}\beta_1^2/3 - (1 + 2\lambda_1^{-3} - 3\lambda_1^{-2})B_2^{\text{DIS}}\beta_2/3 \\
& - (\lambda_1^{-3} + \lambda_1^{-3}L_{\lambda_1} - \lambda_1^{-3}L_{\lambda_1}^2 - \lambda_1^{-2})A_1\beta_1^2\zeta_2/2 + (\lambda_1^{-3} - \lambda_1^{-2})A_1\beta_2\zeta_2/2 \\
& + L_{\text{qr}}\{(1 - \lambda_1^{-3})(A_4/3 - B_3^{\text{DIS}} - B_1^{\text{DIS}}\zeta_2 + A_2\zeta_2 + 2A_1\zeta_3/3) \\
& - (2 + \lambda_1^{-3} - 3\lambda_1^{-1})A_1\beta_3/6 + (2 + \lambda_1^{-3} + 3\lambda_1^{-3}L_{\lambda_1} - 3\lambda_1^{-2}L_{\lambda_1} - 3\lambda_1^{-1})A_1\beta_1\beta_2/3 \\
& + (1 - \lambda_1^{-3} + 2\lambda_1^{-3}L_{\lambda_1})A_1\beta_1\zeta_2/2 - (1 - \lambda_1^{-3} - 3\lambda_1^{-3}L_{\lambda_1})A_3\beta_1/3 \\
& - (2 + \lambda_1^{-3} + 6\lambda_1^{-3}L_{\lambda_1} + 3\lambda_1^{-3}L_{\lambda_1}^2 - 2\lambda_1^{-3}L_{\lambda_1}^3 - 6\lambda_1^{-2}L_{\lambda_1} - 3\lambda_1^{-1})A_1\beta_1^3/6 \\
& + (1 + 2\lambda_1^{-3} - 3\lambda_1^{-3}L_{\lambda_1}^2 - 3\lambda_1^{-2})A_2\beta_1^2/3 + (\lambda_1^{-3} - \lambda_1^{-2})B_1^{\text{DIS}}\beta_2 - 2(\lambda_1^{-3}L_{\lambda_1})B_2^{\text{DIS}}\beta_1 \\
& - (1 + 2\lambda_1^{-3} - 3\lambda_1^{-2})A_2\beta_2/3 - (\lambda_1^{-3} + \lambda_1^{-3}L_{\lambda_1} - \lambda_1^{-3}L_{\lambda_1}^2 - \lambda_1^{-2})B_1^{\text{DIS}}\beta_1^2\} \\
& + L_{\text{qr}}^2\{(1 - \lambda_1^{-3})(B_2^{\text{DIS}} - A_3/2 - A_1\zeta_2/2) + (1 - \lambda_1^{-3} + 2\lambda_1^{-3}L_{\lambda_1})B_1^{\text{DIS}}\beta_1/2 \\
& - (\lambda_1^{-3} + \lambda_1^{-3}L_{\lambda_1} - \lambda_1^{-3}L_{\lambda_1}^2 - \lambda_1^{-2})A_1\beta_1^2/2 + (\lambda_1^{-3} - \lambda_1^{-2})A_1\beta_2/2 - (\lambda_1^{-3}L_{\lambda_1})A_2\beta_1\} \\
& + L_{\text{qr}}^3\{(1 - \lambda_1^{-3})(A_2 - B_1^{\text{DIS}})/3 + (1 - \lambda_1^{-3} + 2\lambda_1^{-3}L_{\lambda_1})A_1\beta_1/6\} \\
& - L_{\text{qr}}^4(1 - \lambda_1^{-3})A_1/12 + L_{\text{fr}}\lambda A_4 - L_{\text{fr}}^2\lambda(3A_3/2 + A_2\beta_1 + A_1\beta_2/2) \\
& + L_{\text{fr}}^3\lambda(A_2 + 5A_1\beta_1/6) - L_{\text{fr}}^4\lambda A_1/4. \tag{2.9}
\end{aligned}$$

Note that for resummation to N⁴LL accuracy the function $g_5^{\text{DIS}}(\lambda)$ needs the five-loop coefficient β_4 of the QCD beta function in eq. (2.6) which is available due to [46–49], as well as the cusp anomalous dimension up to five loops, i.e., the coefficient A_5 which has recently been estimated [50] (see below). In addition, one also needs the evolution kernel $B^{\text{DIS}}(\alpha_s)$ of the jet function in eq. (2.4) to four-loop order, i.e., the term B_4^{DIS} , which will be addressed below. We will collect and discuss all necessary resummation coefficients in the next section.

3 Resummation coefficients

We present results for a general $SU(n_c)$ gauge group with n_c colours and n_f massless fermions; the QCD expressions can always be recovered by setting $n_c = 3$. In this way, the resummation coefficients are expressible in terms of the usual $SU(n_c)$ quadratic Casimir factors $C_A = n_c$ and $C_F = (n_c^2 - 1)/(2n_c)$, i.e., $C_A = 3$, $C_F = 4/3$ in QCD. Starting from three-loop order in perturbation theory, higher group invariants enter, such as the square of the symmetric part of the trace of three $SU(n_c)$ generators T_F^a in the fundamental representation,

$$\frac{d^{abc}d_{abc}}{n_c} = \frac{1}{16 n_c^2} (n_c^2 - 1)(n_c^2 - 4), \tag{3.1}$$

that is $d^{abc}d_{abc}/n_c = 5/18$ in QCD, cf. [51] for conventions on the normalization. At four loops we also have for the first time contributions with quartic colour factors $d_{xy}^{(4)} \equiv d_x^{abcd}d_y^{abcd}$, where x, y labels the representations with generators T_r^a and

$$d_r^{abcd} = \frac{1}{6} \text{Tr} (T_r^a T_r^b T_r^c T_r^d + \text{five } bcd \text{ permutations}), \tag{3.2}$$

which leads to

$$\frac{d_{FA}^{(4)}}{n_c} = \frac{1}{48} (n_c^2 - 1)(n_c^2 + 6), \quad \frac{d_{FF}^{(4)}}{n_c} = \frac{1}{96 n_c^3} (n_c^2 - 1)(n_c^4 - 6 n_c^2 + 18). \tag{3.3}$$

In QCD these factors evaluate to $d_{FA}^{(4)}/n_c = 5/2$ and $d_{FF}^{(4)}/n_c = 5/36$.

3.1 Splitting functions at large x

The coefficients of the cusp anomalous dimension A_n appear in the large- x expansion of the diagonal parts of the splitting functions. For a quark field, we consider the large- x behaviour of the non-singlet splitting functions $P_{\text{ns}}^{(n-1)}(x)$ in the $\overline{\text{MS}}$ scheme at n -loops, recall eq. (2.5) for the normalization of the expansion in powers of α_s . Disregarding terms that vanish for $x \rightarrow 0$, these can be written [52, 53]

$$P_{\text{ns}}^{(n-1)}(x) = \frac{A_n}{(1-x)_+} + B_n^q \delta(1-x), \tag{3.4}$$

where $1/(1-x)_+$ on the right hand side represents the usual plus-distribution; B_n^q is sometimes referred to as virtual anomalous dimension.

The cusp anomalous dimension of a quark field A^q is known up to third order for quite a long time [20] and the perturbative expansion reads according to eq. (2.5),

$$\begin{aligned} A_1 &= 4C_F, \\ A_2 &= 8C_F \left\{ \left(\frac{67}{18} - \zeta_2 \right) C_A - \frac{5}{9} n_f \right\}, \\ A_3 &= 16C_F \left\{ C_A^2 \left(\frac{245}{24} - \frac{67}{9} \zeta_2 + \frac{11}{6} \zeta_3 + \frac{11}{2} \zeta_4 \right) + C_F n_f \left(-\frac{55}{24} + 2\zeta_3 \right) \right. \\ &\quad \left. + C_A n_f \left(-\frac{209}{108} + \frac{10}{9} \zeta_2 - \frac{7}{3} \zeta_3 \right) + n_f^2 \left(-\frac{1}{27} \right) \right\}. \end{aligned} \tag{3.5}$$

The four-loop contribution A_4 has been of subject to intensive recent studies. Combining all available results it reads for quark fields,

$$\begin{aligned}
 A_4 = & C_F C_A^3 \left(\frac{84278}{81} - \frac{88400}{81} \zeta_2 + \frac{20944}{27} \zeta_3 + 1804 \zeta_4 - \frac{352}{3} \zeta_2 \zeta_3 - \frac{3608}{9} \zeta_5 \right. \\
 & \left. - 16 \zeta_3^2 - \frac{2504}{3} \zeta_6 \right) + \frac{d_{FA}^{(4)}}{n_c} \left(-128 \zeta_2 + \frac{128}{3} \zeta_3 + \frac{3520}{3} \zeta_5 - 384 \zeta_3^2 - 992 \zeta_6 \right) \\
 & + C_F^3 n_f \left(\frac{572}{9} + \frac{592}{3} \zeta_3 - 320 \zeta_5 \right) \\
 & + C_F^2 C_A n_f \left(-\frac{34066}{81} + \frac{440}{3} \zeta_2 + \frac{3712}{9} \zeta_3 - 176 \zeta_4 - 128 \zeta_2 \zeta_3 + 160 \zeta_5 \right) \\
 & + C_F C_A^2 n_f \left(-\frac{24137}{81} + \frac{20320}{81} \zeta_2 - \frac{23104}{27} \zeta_3 - \frac{176}{3} \zeta_4 + \frac{448}{3} \zeta_2 \zeta_3 + \frac{2096}{9} \zeta_5 \right) \\
 & + n_f \frac{d_{FF}^{(4)}}{n_c} \left(256 \zeta_2 - \frac{256}{3} \zeta_3 - \frac{1280}{3} \zeta_5 \right) + C_F^2 n_f^2 \left(\frac{2392}{81} - \frac{640}{9} \zeta_3 + 32 \zeta_4 \right) \\
 & + C_F C_A n_f^2 \left(\frac{923}{81} - \frac{608}{81} \zeta_2 + \frac{2240}{27} \zeta_3 - \frac{112}{3} \zeta_4 \right) - C_F n_f^3 \left(\frac{32}{81} - \frac{64}{27} \zeta_3 \right). \quad (3.6)
 \end{aligned}$$

This expression combines available exact results in the large- n_c limit of QCD [28, 29], as well as for the terms proportional to n_f [32, 33], to n_f^2 [31, 54] and to n_f^3 [55, 56], the latter have been known for a long time. The quartic colour factors have been completed in exact form recently [33, 36, 38], and found to be in agreement with numerical estimates [29, 35].

Finally, the five-loop quark cusp anomalous dimension in QCD has been estimated in [50] as

$$A_5 = (1.7 \pm 0.5, 1.1 \pm 0.5, 0.7 \pm 0.5) \cdot 10^5 \quad \text{for } n_f = 3, 4, 5. \quad (3.7)$$

Inserting the numerical values in eqs. (3.5)–(3.7) one obtains the perturbative expansion for $A^q(\alpha_s)$ according to eq. (2.5) through five loops in powers of α_s for the physically relevant values of n_f as [50]

$$\begin{aligned}
 A^q(n_f=3) &= 0.42441 \alpha_s (1 + 0.7266 \alpha_s + 0.7341 \alpha_s^2 + 0.665 \alpha_s^3 + (1.3 \pm 0.4) \alpha_s^4 + \dots), \\
 A^q(n_f=4) &= 0.42441 \alpha_s (1 + 0.6382 \alpha_s + 0.5100 \alpha_s^2 + 0.317 \alpha_s^3 + (0.8 \pm 0.4) \alpha_s^4 + \dots), \\
 A^q(n_f=5) &= 0.42441 \alpha_s (1 + 0.5497 \alpha_s + 0.2840 \alpha_s^2 + 0.013 \alpha_s^3 + (0.5 \pm 0.4) \alpha_s^4 + \dots). \quad (3.8)
 \end{aligned}$$

The $\delta(1-x)$ parts of the n -loop non-singlet splitting function $P_{\text{ns}}^{(n-1)}(x)$, i.e. the coefficients B_n^q in eq. (3.4) are also known exactly to three loops [20],

$$\begin{aligned}
 B_1^q &= 3C_F, \\
 B_2^q &= 4C_F \left\{ C_F \left(\frac{3}{8} - 3\zeta_2 + 6\zeta_3 \right) + C_A \left(\frac{17}{24} + \frac{11}{3}\zeta_2 - 3\zeta_3 \right) - n_f \left(\frac{1}{12} + \frac{2}{3}\zeta_2 \right) \right\}, \\
 B_3^q &= 16C_F \left\{ C_F^2 \left(\frac{29}{32} + \frac{9}{8}\zeta_2 + \frac{17}{4}\zeta_3 + 9\zeta_4 - 2\zeta_2\zeta_3 - 15\zeta_5 \right) \right. \\
 &\quad + C_F C_A \left(\frac{151}{64} - \frac{205}{24}\zeta_2 + \frac{211}{12}\zeta_3 - \frac{247}{24}\zeta_4 + \zeta_2\zeta_3 + \frac{15}{2}\zeta_5 \right) \\
 &\quad - C_F n_f \left(\frac{23}{16} - \frac{5}{12}\zeta_2 + \frac{17}{6}\zeta_3 - \frac{29}{12}\zeta_4 \right) + C_A n_f \left(\frac{5}{4} - \frac{167}{54}\zeta_2 + \frac{25}{18}\zeta_3 + \frac{1}{8}\zeta_4 \right) \\
 &\quad \left. - C_A^2 \left(\frac{1657}{576} - \frac{281}{27}\zeta_2 + \frac{97}{9}\zeta_3 + \frac{5}{16}\zeta_4 - \frac{5}{2}\zeta_5 \right) - n_f^2 \left(\frac{17}{144} - \frac{5}{27}\zeta_2 + \frac{1}{9}\zeta_3 \right) \right\}. \tag{3.9}
 \end{aligned}$$

At four loops the exact result in the large- n_c limit of QCD with an overall factor of C_F reads [29]

$$\begin{aligned}
 B_4^q \Big|_{L n_c} &= C_F n_c^3 \left(-\frac{1379569}{5184} + \frac{24211}{27}\zeta_2 - \frac{9803}{162}\zeta_3 - \frac{9382}{9}\zeta_4 + \frac{838}{9}\zeta_2\zeta_3 + 1002\zeta_5 \right. \\
 &\quad \left. + \frac{16}{3}\zeta_3^2 + 135\zeta_6 - 80\zeta_2\zeta_5 + 32\zeta_3\zeta_4 - 560\zeta_7 \right) \\
 &\quad + C_F n_c^2 n_f \left(\frac{353}{3} - \frac{85175}{162}\zeta_2 - \frac{137}{9}\zeta_3 + \frac{16186}{27}\zeta_4 - \frac{584}{9}\zeta_2\zeta_3 - \frac{248}{3}\zeta_5 \right. \\
 &\quad \left. - \frac{16}{3}\zeta_3^2 - 144\zeta_6 \right) \\
 &\quad + C_F n_c n_f^2 \left(-\frac{127}{18} + \frac{5036}{81}\zeta_2 - \frac{932}{27}\zeta_3 - \frac{1292}{27}\zeta_4 + \frac{160}{9}\zeta_2\zeta_3 + \frac{32}{3}\zeta_5 \right) \\
 &\quad + C_F n_f^3 \left(-\frac{131}{81} + \frac{32}{81}\zeta_2 + \frac{304}{81}\zeta_3 - \frac{32}{27}\zeta_4 \right). \tag{3.10}
 \end{aligned}$$

The full colour dependence can be parametrized with coefficients $b_{4,C_F^4}^q$, $b_{4,C_F^3 C_A}^q$, $b_{4,C_F^2 C_A^2}^q$, $b_{4,n_f C_F^3}^q$, $b_{4,n_f C_F^2 C_A}^q$, $b_{4,d_{FA}^{(4)}}^q$, and $b_{4,d_{FF}^{(4)}}^q$, which are given in numerical form in table 1. The numerical values in table 1 have been improved considerably compared to [29], thanks to the computation of more Mellin moments of the corresponding anomalous dimension and the exact result for A_4 in eq. (3.6). All parts proportional to n_f^2 and n_f^3 are known in analytic form [54, 55]. The result for B_4^q can then be written as

$$\begin{aligned}
 B_4^q &= C_F^4 b_{4,C_F^4}^q + C_F^3 C_A b_{4,C_F^3 C_A}^q + C_F^2 C_A^2 b_{4,C_F^2 C_A^2}^q + C_F C_A^3 \left(-\frac{1379569}{5184} + \frac{24211}{27}\zeta_2 \right. \\
 &\quad - \frac{9803}{162}\zeta_3 - \frac{9382}{9}\zeta_4 + \frac{838}{9}\zeta_2\zeta_3 + 1002\zeta_5 + \frac{16}{3}\zeta_3^2 + 135\zeta_6 - 80\zeta_2\zeta_5 + 32\zeta_3\zeta_4 \\
 &\quad \left. - 560\zeta_7 - \frac{1}{2} b_{4,C_F^2 C_A^2}^q - \frac{1}{4} b_{4,C_F^3 C_A}^q - \frac{1}{8} b_{4,C_F^4}^q - \frac{1}{24} b_{4,d_{FA}^{(4)}}^q \right) + \frac{d_{FA}^{(4)}}{n_c} b_{4,d_{FA}^{(4)}}^q
 \end{aligned}$$

C_F^4	$C_F^3 C_A$	$C_F^2 C_A^2$	$C_F C_A^3$	$d_{FA}^{(4)}/n_F$
$196.5 \pm 1.$	-687.5 ± 1.5	$1219.5 \pm 2.$	295.7 ± 0.5	-998.0 ± 0.2
$n_f C_F^3$	$n_f C_F^2 C_A$	$n_f C_F C_A^2$	$n_f d_{FF}^{(4)}/n_F$	
80.780 ± 0.005	-455.247 ± 0.005	-274.466 ± 0.01	-143.6 ± 0.2	
$C_F n_c^3$		$n_f C_F n_c^2$		
716.5577		-484.8864		
$n_f^2 C_F^2$	$n_f^2 C_F C_A$		$n_f^3 C_F$	
-5.775288	51.03056		2.261237	

Table 1. Numerical values for the colour coefficients of the $\delta(1-x)$ part B_4^q in eq. (3.4) at fourth order. All exact values for the coefficients have been rounded to seven digits and the errors are correlated due to the known exact results at n_f^0 and n_f^1 in the large- n_c limit given in the third row.

$$\begin{aligned}
 & + C_F^3 n_f b_{4,n_f C_F^3}^q + C_F^2 C_A n_f b_{4,n_f C_F^2 C_A}^q + C_F C_A^2 n_f \left(\frac{353}{3} - \frac{85175}{162} \zeta_2 - \frac{137}{9} \zeta_3 \right. \\
 & + \frac{16186}{27} \zeta_4 - \frac{584}{9} \zeta_2 \zeta_3 - \frac{248}{3} \zeta_5 - \frac{16}{3} \zeta_3^2 - 144 \zeta_6 - \frac{1}{2} b_{4,n_f C_F^2 C_A}^q - \frac{1}{4} b_{4,n_f C_F^3}^q \\
 & \left. - \frac{1}{48} b_{4,d_{FF}^{(4)}}^q \right) + n_f \frac{d_{FF}^{(4)}}{n_c} b_{4,d_{FF}^{(4)}}^q + C_F^2 n_f^2 \left(-\frac{188}{27} + \frac{1244}{27} \zeta_2 + \frac{56}{27} \zeta_3 \right. \\
 & \left. - \frac{2104}{27} \zeta_4 - \frac{160}{9} \zeta_2 \zeta_3 + \frac{368}{9} \zeta_5 \right) + C_F C_A n_f^2 \left(-\frac{193}{54} + \frac{3170}{81} \zeta_2 - \frac{320}{9} \zeta_3 - \frac{80}{9} \zeta_4 \right. \\
 & \left. + \frac{80}{3} \zeta_2 \zeta_3 - \frac{88}{9} \zeta_5 \right) + C_F n_f^3 \left(-\frac{131}{81} + \frac{32}{81} \zeta_2 + \frac{304}{81} \zeta_3 - \frac{32}{27} \zeta_4 \right). \quad (3.11)
 \end{aligned}$$

Taken together, eqs. (3.9) and (3.11) lead to the following perturbative expansion for $B^q(\alpha_s)$ through four loops,

$$\begin{aligned}
 B^q(n_f=3) &= 0.31831 \alpha_s (1 + 0.9971 \alpha_s + 1.2412 \alpha_s^2 + 1.080 \alpha_s^3 + \dots), \\
 B^q(n_f=4) &= 0.31831 \alpha_s (1 + 0.8719 \alpha_s + 0.9783 \alpha_s^2 + 0.566 \alpha_s^3 + \dots), \\
 B^q(n_f=5) &= 0.31831 \alpha_s (1 + 0.7467 \alpha_s + 0.7191 \alpha_s^2 + 0.109 \alpha_s^3 + \dots), \quad (3.12)
 \end{aligned}$$

where the error on the four-loop result from the numerical uncertainty in respective coefficients in table 1 is of $\mathcal{O}(10^{-4})$, i.e., beyond the accuracy quoted here.

In order to obtain the resummation coefficient B_4^{DIS} in eq. (2.9) relevant at N⁴LL accuracy for the jet function in eq. (2.4) we have to combine the results summarized above with those for the QCD form factor and the DIS cross sections in the soft and collinear limit at four-loop order; this will be done next.

3.2 Quark form factor

The quark form factor summarizes the QCD corrections to the vertex of a photon with virtuality Q^2 and a massless external quark / anti-quark, the relevant amplitude being

$$\Gamma_\mu = ie_q (\bar{\psi} \gamma_\mu \psi) \mathcal{F}(\alpha_s, Q^2), \quad (3.13)$$

where e_q denotes the quark's electric charge and the scalar function \mathcal{F} the quark form factor. As we are interested in neutral- and charged-current DIS, we note that analogous expressions hold for general vector or axial-vector currents with the appropriate replacements of the couplings $e_q \rightarrow v_q$ and $e_q \rightarrow a_q$, i.e., for DIS including Z -boson or W^\pm -boson exchange. The latter case is distinguished by certain diagram classes involving higher group theory invariants which will be addressed below.

\mathcal{F} is gauge invariant, but divergent. It has been computed in dimensional regularization with $d = 4 - 2\varepsilon$ at four-loop order in the large- n_c limit of QCD [28]. In addition, all terms proportional to n_f^2 [31] and the quartic colour factor $d_{FF}^{(4)}/n_c$ at four loops are also known [36, 37]. For lower order results to sufficient depth in ε , see, e.g., [19, 57–59]. The exponentiation of \mathcal{F} is achieved by solving the well-known evolution equations [14–18] in $d = 4 - 2\varepsilon$ dimensions,

$$Q^2 \frac{\partial}{\partial Q^2} \ln \mathcal{F} \left(\alpha_s, \frac{Q^2}{\mu^2}, \varepsilon \right) = \frac{1}{2} \left[K(\alpha_s, \varepsilon) + G \left(\frac{Q^2}{\mu^2}, \alpha_s, \varepsilon \right) \right], \quad (3.14)$$

where μ again represents the renormalization scale. Here K is a counter term containing all the poles in ε whereas the function G is finite in the limit $\varepsilon \rightarrow 0$. The functions G and K follow the renormalization group equations [14],

$$\begin{aligned} \left(\mu^2 \frac{\partial}{\partial \mu^2} + \beta(\alpha_s, \varepsilon) \frac{\partial}{\partial \alpha_s} \right) G \left(\frac{Q^2}{\mu^2}, \alpha_s, \varepsilon \right) &= A^q(\alpha_s), \\ \left(\mu^2 \frac{\partial}{\partial \mu^2} + \beta(\alpha_s, \varepsilon) \frac{\partial}{\partial \alpha_s} \right) K(\alpha_s, \varepsilon) &= -A^q(\alpha_s), \end{aligned} \quad (3.15)$$

with the standard cusp anomalous dimension A^q discussed above.

The solution of eqs. (3.14) and (3.15) order by order in perturbation theory is straightforward, see, e.g., [18, 19, 26, 27], and provides a perturbative expansion of the bare (unrenormalized) quark form factor in terms of the bare strong coupling a_s^b (recall the normalization $a_s = \alpha_s/(4\pi)$ in eq. (2.5)), as

$$\mathcal{F}^b(a_s^b, Q^2) = 1 + \sum_{n=1}^{\infty} (a_s^b)^n \left(\frac{Q^2}{\mu^2} \right)^{-n\varepsilon} \mathcal{F}_n. \quad (3.16)$$

In terms of the n -th order coefficients A_n in eqs. (3.5) and (3.6) and the coefficients $G_n(\varepsilon)$ of the function G in eq. (3.15), which are still functions of the parameter ε , the expansion of eq. (3.16) up to four loops leads to [19, 26],

$$\mathcal{F}_1 = -\frac{1}{2} \frac{1}{\varepsilon^2} A_1 - \frac{1}{2} \frac{1}{\varepsilon} G_1, \quad (3.17)$$

$$\mathcal{F}_2 = \frac{1}{8} \frac{1}{\varepsilon^4} (A_1)^2 + \frac{1}{8} \frac{1}{\varepsilon^3} A_1 (2G_1 - \beta_0) + \frac{1}{8} \frac{1}{\varepsilon^2} \left((G_1)^2 - A_2 - 2\beta_0 G_1 \right) - \frac{1}{4} \frac{1}{\varepsilon} G_2, \quad (3.18)$$

$$\begin{aligned} \mathcal{F}_3 = & -\frac{1}{48} \frac{1}{\varepsilon^6} (A_1)^3 - \frac{1}{16} \frac{1}{\varepsilon^5} (A_1)^2 (G_1 - \beta_0) - \frac{1}{144} \frac{1}{\varepsilon^4} A_1 \left(9(G_1)^2 - 9A_2 - 27\beta_0 G_1 \right. \\ & \left. + 8\beta_0^2 \right) - \frac{1}{144} \frac{1}{\varepsilon^3} \left(3(G_1)^3 - 9A_2 G_1 - 18A_1 G_2 + 4\beta_1 A_1 - 18\beta_0 (G_1)^2 + 16\beta_0 A_2 \right. \\ & \left. + 24\beta_0^2 G_1 \right) + \frac{1}{72} \frac{1}{\varepsilon^2} \left(9G_1 G_2 - 4A_3 - 6\beta_1 G_1 - 24\beta_0 G_2 \right) - \frac{1}{6} \frac{1}{\varepsilon} G_3, \end{aligned} \quad (3.19)$$

$$\begin{aligned}
 \mathcal{F}_4 = & \frac{1}{384} \frac{1}{\varepsilon^8} (A_1)^4 + \frac{1}{192} \frac{1}{\varepsilon^7} (A_1)^3 (2G_1 - 3\beta_0) + \frac{1}{1152} \frac{1}{\varepsilon^6} (A_1)^2 \left(18(G_1)^2 - 18A_2 \right. \\
 & \left. - 72\beta_0 G_1 + 41\beta_0^2 \right) + \frac{1}{576} \frac{1}{\varepsilon^5} A_1 \left(6(G_1)^3 - 18A_2 G_1 - 18A_1 G_2 + 8\beta_1 A_1 \right. \\
 & \left. - 45\beta_0 (G_1)^2 + 41\beta_0 A_2 + 82\beta_0^2 G_1 - 18\beta_0^3 \right) + \frac{1}{1152} \frac{1}{\varepsilon^4} \left(3(G_1)^4 - 18A_2 (G_1)^2 \right. \\
 & \left. + 9(A_2)^2 - 72A_1 G_1 G_2 + 32A_1 A_3 + 64\beta_1 A_1 G_1 - 36\beta_0 (G_1)^3 + 100\beta_0 A_2 G_1 \right. \\
 & \left. + 228\beta_0 A_1 G_2 - 48\beta_0 \beta_1 A_1 + 132\beta_0^2 (G_1)^2 - 108\beta_0^2 A_2 - 144\beta_0^3 G_1 \right) \\
 & + \frac{1}{288} \frac{1}{\varepsilon^3} \left(-9(G_1)^2 G_2 + 8A_3 G_1 + 9A_2 G_2 + 24A_1 G_3 - 3\beta_2 A_1 + 12\beta_1 (G_1)^2 \right. \\
 & \left. - 9\beta_1 A_2 + 66\beta_0 G_1 G_2 - 27\beta_0 A_3 - 48\beta_0 \beta_1 G_1 - 108\beta_0^2 G_2 \right) + \frac{1}{96} \frac{1}{\varepsilon^2} \left(3(G_2)^2 \right. \\
 & \left. + 8G_1 G_3 - 3A_4 - 4\beta_2 G_1 - 12\beta_1 G_2 - 36\beta_0 G_3 \right) - \frac{1}{8} \frac{1}{\varepsilon} G_4. \tag{3.20}
 \end{aligned}$$

As observed at lower fixed orders [60, 61] and generalized in [62] the function G in eq. (3.15) generates the subleading poles in ε at each order. G is the sum of three terms: twice the coefficient B^q in eq. (3.4) of the $\delta(1-x)$ part in the relevant parton splitting function, the single-logarithmic anomalous dimension of the eikonal form factor, and a term associated with the QCD beta function. Thus, the perturbative coefficients $G_n(\varepsilon)$ satisfy the following relations

$$\begin{aligned}
 G_1 &= 2B_1^q + f_1^q + \varepsilon f_{01}^q, \\
 G_2 &= 2B_2^q + (f_2^q + \beta_0 f_{01}^q) + \varepsilon f_{02}^q, \\
 G_3 &= 2B_3^q + (f_3^q + \beta_1 f_{01}^q + \beta_0 f_{02}^q) + \varepsilon f_{03}^q, \\
 G_4 &= 2B_4^q + (f_4^q + \beta_2 f_{01}^q + \beta_1 f_{02}^q + \beta_0 f_{03}^q) + \varepsilon f_{04}^q, \tag{3.21}
 \end{aligned}$$

where the functions $f_{0n}^q(\varepsilon)$ at n loops are polynomials in ε . For consistency, their lower order expressions are needed to sufficient depth in ε in eqs. (3.17)–(3.20).

The finite coefficients f_n^q in the eq. (3.21) are related to the anomalous dimension of the eikonal form factor [62], see also the recent work [63]. In analogy to the cusp anomalous dimensions A^q , they exhibit a maximally non-Abelian colour structure. This implies that the n_f -independent colour factor for quarks in f_n^q is proportional to $C_F C_A^{(n-1)}$ up to three loops, where the explicit results read [61]

$$\begin{aligned}
 f_1^q &= 0, \\
 f_2^q &= C_F \left\{ C_A \left(\frac{808}{27} - \frac{22}{3} \zeta_2 - 28\zeta_3 \right) + n_f \left(-\frac{112}{27} + \frac{4}{3} \zeta_2 \right) \right\}, \\
 f_3^q &= C_F \left\{ C_A^2 \left(\frac{136781}{729} - \frac{12650}{81} \zeta_2 - \frac{1316}{3} \zeta_3 + 176\zeta_4 + \frac{176}{3} \zeta_2 \zeta_3 + 192\zeta_5 \right) \right. \\
 & \quad \left. + C_F n_f \left(-\frac{1711}{27} + 4\zeta_2 + \frac{304}{9} \zeta_3 + 16\zeta_4 \right) + C_A n_f \left(-\frac{11842}{729} + \frac{2828}{81} \zeta_2 \right. \right. \\
 & \quad \left. \left. + \frac{728}{27} \zeta_3 - 48\zeta_4 \right) + n_f^2 \left(-\frac{2080}{729} - \frac{40}{27} \zeta_2 + \frac{112}{27} \zeta_3 \right) \right\}. \tag{3.22}
 \end{aligned}$$

With the currently available results for B_4^q in eq. (3.11) and G_4 in eq. (3.21) in the planar limit we can determine f_4^q for quarks completely in the large- n_c limit as

$$\begin{aligned}
 f_4^q \Big|_{Ln_c} &= C_F n_c^3 \left(\frac{9364079}{6561} - \frac{1186735}{729} \zeta_2 - \frac{837988}{243} \zeta_3 + \frac{115801}{27} \zeta_4 + \frac{11896}{9} \zeta_2 \zeta_3 \right. \\
 &\quad \left. + 3952 \zeta_5 - \frac{4796}{9} \zeta_3^2 - \frac{129547}{54} \zeta_6 - 416 \zeta_2 \zeta_5 - 720 \zeta_3 \zeta_4 - 1700 \zeta_7 \right) \\
 &\quad + C_F n_c^2 n_f \left(-\frac{247315}{432} + \frac{412232}{729} \zeta_2 + \frac{102205}{243} \zeta_3 - \frac{7589}{6} \zeta_4 - \frac{824}{9} \zeta_2 \zeta_3 \right. \\
 &\quad \left. - \frac{740}{9} \zeta_5 + \frac{2816}{9} \zeta_3^2 + \frac{15611}{27} \zeta_6 \right) \\
 &\quad + C_F n_c n_f^2 \left(\frac{329069}{17496} - \frac{22447}{729} \zeta_2 + \frac{25300}{243} \zeta_3 + \frac{140}{3} \zeta_4 - \frac{176}{9} \zeta_2 \zeta_3 - \frac{856}{9} \zeta_5 \right) \\
 &\quad + C_F n_f^3 \left(-\frac{16160}{6561} - \frac{16}{81} \zeta_2 - \frac{400}{243} \zeta_3 + \frac{128}{27} \zeta_4 \right). \tag{3.23}
 \end{aligned}$$

With these results the perturbative expansion of f^q through four loops reads

$$\begin{aligned}
 f^q(n_f=3) &= -0.44960 \alpha_s^2 (1 + 0.9300 \alpha_s + (1.086) \Big|_{Ln_c} \alpha_s^2 + \dots), \\
 f^q(n_f=4) &= -0.46610 \alpha_s^2 (1 + 0.8043 \alpha_s + (0.753) \Big|_{Ln_c} \alpha_s^2 + \dots), \\
 f^q(n_f=5) &= -0.48261 \alpha_s^2 (1 + 0.6881 \alpha_s + (0.438) \Big|_{Ln_c} \alpha_s^2 + \dots), \tag{3.24}
 \end{aligned}$$

where we have used the full QCD coefficients f_n^q in eq. (3.22) and, as indicated, the large- n_c result of eq. (3.23). Based on experience with other anomalous dimensions and lower orders, the latter is expected to provide a very good approximation of full QCD for the coefficients of individual powers of n_f , as well as the complete result for the four loop term, putting $n_f = 3, 4$ or 5 .

With the results of section 3.1 and using all available results for the quark form factor we can, in addition, also determine the complete colour decomposition of the four-loop coefficient f_4^q . Given its relation to the anomalous dimension of the eikonal form factor, we assume here that f_4^q exhibits the same generalized maximal non-Abelian property in the presence of quartic colour factors as the cusp anomalous dimension A_4 , see [35]. This assumption, supported also by recent studies of amplitude factorization in [63], implies the absence of the colour coefficients C_F^4 , $C_F^3 C_A$ and $C_F^2 C_A^2$ in f_4^q , which leads to the result

$$\begin{aligned}
 f_4^q &= C_F C_A^3 \left(\frac{9364079}{6561} - \frac{1186735}{729} \zeta_2 - \frac{837988}{243} \zeta_3 + \frac{115801}{27} \zeta_4 + \frac{11896}{9} \zeta_2 \zeta_3 \right. \\
 &\quad \left. + 3952 \zeta_5 - \frac{4796}{9} \zeta_3^2 - \frac{129547}{54} \zeta_6 - 416 \zeta_2 \zeta_5 - 720 \zeta_3 \zeta_4 - 1700 \zeta_7 - \frac{1}{24} f_{4, d_{FA}^{(4)}}^q \right) \\
 &\quad + \frac{d_{FA}^{(4)}}{n_c} f_{4, d_{FA}^{(4)}}^q + C_F^3 n_f f_{4, n_f C_F^3}^q + C_F^2 C_A n_f f_{4, n_f C_F^2 C_A}^q + C_F C_A^2 n_f \left(-\frac{243859}{432} \right. \\
 &\quad \left. + \frac{389228}{729} \zeta_2 + \frac{105193}{243} \zeta_3 - \frac{22667}{18} \zeta_4 - \frac{848}{9} \zeta_2 \zeta_3 - \frac{860}{27} \zeta_5 + \frac{2740}{9} \zeta_3^2 + \frac{5179}{9} \zeta_6 \right. \\
 &\quad \left. + \frac{1}{24} b_{4, d_{FF}^{(4)}}^q - \frac{1}{2} f_{4, n_f C_F^2 C_A}^q - \frac{1}{4} f_{4, n_f C_F^3}^q \right) + n_f \frac{d_{FF}^{(4)}}{n_c} \left(-384 + \frac{4544}{3} \zeta_2 \right)
 \end{aligned}$$

$$\begin{aligned}
& -\frac{5312}{9}\zeta_3 - \frac{800}{3}\zeta_4 + 128\zeta_2\zeta_3 - \frac{21760}{9}\zeta_5 + \frac{1216}{3}\zeta_3^2 + \frac{1184}{9}\zeta_6 - 2b_{4,d_{FF}^{(4)}}^q \Big) \\
& + C_F^2 n_f^2 \left(\frac{16733}{486} - \frac{172}{9}\zeta_2 - \frac{4568}{81}\zeta_3 + \frac{64}{9}\zeta_4 + \frac{32}{3}\zeta_2\zeta_3 + \frac{304}{9}\zeta_5 \right) \\
& + C_F C_A n_f^2 \left(\frac{27875}{17496} - \frac{15481}{729}\zeta_2 + \frac{32152}{243}\zeta_3 + \frac{388}{9}\zeta_4 - \frac{224}{9}\zeta_2\zeta_3 - 112\zeta_5 \right) \\
& + C_F n_f^3 \left(-\frac{16160}{6561} - \frac{16}{81}\zeta_2 - \frac{400}{243}\zeta_3 + \frac{128}{27}\zeta_4 \right), \tag{3.25}
\end{aligned}$$

with three still unknown coefficients $f_{4,n_f C_F^3}^q$, $f_{4,n_f C_F^2 C_A}^q$ and $f_{4,d_{FA}^{(4)}}^q$.

We do have, however, a low number of fixed Mellin moments for the DIS structure functions at four loops at our disposal [30]. As will be explained in the section 3.3 below, this information can be used to constrain the DIS Wilson coefficients at large x , specifically the term proportional to the plus-distribution $[1/(1-x)]_+$, so that we can extract numerical values for the unknown colour coefficients of f_4^q in eq. (3.25). We obtain

$$\begin{aligned}
f_{4,n_f C_F^3}^q &= (-0.4 \pm 6.) \cdot 10^3 \\
f_{4,n_f C_F^2 C_A}^q &= (2. \pm 6.) \cdot 10^3 \\
f_{4,d_{FA}^{(4)}}^q &= (-1. \pm 1.) \cdot 10^2, \tag{3.26}
\end{aligned}$$

where the errors are correlated due to the known exact results in the large- n_c limit. These are the best estimations for these coefficients given our current knowledge.

The four-loop contributions in eq. (3.26) display still a significant uncertainty, but in the bare form factor \mathcal{F}_4 at four loops in eq. (3.20) they are numerically not dominant. This leads to the expression for the full colour dependence of the single pole in ε at four loops

$$\begin{aligned}
\mathcal{F}_4 \Big|_{1/\varepsilon} &= C_F^4 (-2212.8 \pm 0.3) + C_F^3 C_A (-1601.9 \pm 0.5) + C_F^2 C_A^2 (19661.7 \pm 0.5) \\
&+ C_F C_A^3 (-13274.1 \pm 1.0) + \frac{d_{FA}^{(4)}}{n_c} (262.3 \pm 12.5) + C_F^3 n_f (2140. \pm 750.) \\
&+ C_F^2 C_A n_f (-12800. \pm 750.) + C_F C_A^2 n_f (10320. \mp 560.) + n_f \frac{d_{FF}^{(4)}}{n_c} (53.12744) \\
&+ C_F^2 n_f^2 (1604.851) + C_F C_A n_f^2 (-2304.682) + C_F n_f^3 (158.0655), \tag{3.27}
\end{aligned}$$

where all exact values have been rounded to seven digits and the errors are inherited from the numerical results in table 1 and eq. (3.26) and therefore are correlated.

Note that the form factor \mathcal{F} in eq. (3.13) receives additional corrections due to a new flavour structure starting from three loops, where the photon couples to a closed quark loop [1]. This requires the summation over the charges of all quark flavours in the loop, leading to additional terms with a relative factor $(\sum_{q'} e_{q'})/e_q = n_f \langle e \rangle / e_q$ and proportional to the group invariant $(d^{abc} d_{abc})/n_c$ in eq. (3.1). For the analogous expression of a charged-current, i.e., the form factor with the coupling of a W^\pm -boson, such contributions are absent. For the photon form factor the respective terms proportional to $(d^{abc} d_{abc})/n_c$ are

finite at three loops, thus appearing in f_{03}^q in G_3 in eq. (3.21). At four loops, those terms enter in G_4 through $\beta_0 f_{03}^q$ in eq. (3.21) and generate a single pole in ε in the bare form factor, as confirmed by the exact result for the coefficient of $n_f(d^{abc}d_{abc})/n_c$ in [37]. This colour factor has been omitted in eq. (3.27).

Eqs. (3.23), (3.25) and (3.27) are new results of the present paper, with the large- n_c results in eq. (3.23) being exact. Eq. (3.25) for f_4^q and, as a consequence, eq. (3.27) for the single pole in ε of \mathcal{F}_4 are based on the (by now well-supported) conjecture that f_4^q has the same generalized maximal non-Abelian colour structure as the cusp anomalous dimension.

3.3 DIS Wilson coefficients at large x

The direct link between the QCD form factor and the DIS Wilson coefficients near threshold for $x \rightarrow 1$ through factorization in the soft and collinear limit allows to relate the knowledge on either quantity at a given order in perturbation theory. To that end, we write the partonic coefficient function W^b as a series in bare coupling a_s^b , see [19, 24]. With the convention in eq. (3.16) and choosing the scale $\mu = Q$, the expansion coefficients W_n^b have the following structure up to four loops,

$$\begin{aligned}
 W_0^b &= \delta(1-x), \\
 W_1^b &= \mathcal{S}_1 + 2\mathcal{F}_1 \delta(1-x), \\
 W_2^b &= \mathcal{S}_2 + 2\mathcal{F}_1 \mathcal{S}_1 + (2\mathcal{F}_2 + (\mathcal{F}_1)^2) \delta(1-x), \\
 W_3^b &= \mathcal{S}_3 + 2\mathcal{F}_1 \mathcal{S}_2 + (2\mathcal{F}_2 + (\mathcal{F}_1)^2) \mathcal{S}_1 + (2\mathcal{F}_3 + 2\mathcal{F}_1 \mathcal{F}_2) \delta(1-x), \\
 W_4^b &= \mathcal{S}_4 + 2\mathcal{F}_1 \mathcal{S}_3 + (2\mathcal{F}_2 + (\mathcal{F}_1)^2) \mathcal{S}_2 + (2\mathcal{F}_3 + 2\mathcal{F}_1 \mathcal{F}_2) \mathcal{S}_1 \\
 &\quad + (2\mathcal{F}_4 + (\mathcal{F}_2)^2 + 2\mathcal{F}_1 \mathcal{F}_3) \delta(1-x).
 \end{aligned} \tag{3.28}$$

Here \mathcal{F}_n are the space-like form factors discussed in the previous section 3.2, whereas \mathcal{S}_n denote the real emission contributions. In the limit Bjorken $x \rightarrow 1$, the singular terms are proportional to $\delta(1-x)$ and to the usual plus-distributions,

$$\mathcal{D}_k = \left[\frac{\ln^k(1-x)}{1-x} \right]_+, \quad k = 1, \dots, 2n-1. \tag{3.29}$$

The dependence of the pure real-emission contributions \mathcal{S}_n on the scaling variable x is given by the d -dimensional plus-distributions up to $f_{n,\varepsilon}$ defined by

$$f_{k,\varepsilon}(x) = \varepsilon[(1-x)^{-1-k\varepsilon}]_+ = -\frac{1}{k} \delta(1-x) + \sum_{i=0} \frac{(-k\varepsilon)^i}{i!} \varepsilon \mathcal{D}_i. \tag{3.30}$$

The bare strong coupling in the coefficients W_n^b has to be renormalized (in the $\overline{\text{MS}}$ scheme) according to

$$a_s^b = a_s \left\{ 1 - \frac{\beta_0}{\varepsilon} a_s + \left(\frac{\beta_0^2}{\varepsilon^2} - \frac{\beta_1}{2\varepsilon} \right) a_s^2 - \left(\frac{\beta_0^3}{\varepsilon^3} - \frac{7\beta_1\beta_0}{6\varepsilon^2} + \frac{\beta_2}{3\varepsilon} \right) a_s^3 \right\}. \tag{3.31}$$

With the ingredients of sections 3.1 and 3.2, using eq. (3.28), the renormalized coupling in eq. (3.31), and the known results for the DIS Wilson coefficients up to third

order [1] we can then derive the coefficient functions $C_{a,q}$ for the DIS structure functions F_a for $a = 1, 2, 3$ in the soft and collinear limit up to four-loop order. In x -space, with distributions f_i for the parton i , the DIS structure functions F_a factorize as $F_a(x, Q^2) = [C_{a,i}(\alpha_s, \mu^2/Q^2) \otimes f_i(\mu^2)](x)$, cf. eq. (2.1) for the corresponding Mellin space expressions. With the standard normalization

$$C_{a,q} = \delta(1-x) + a_s c_{i,q}^{(1)} + a_s^2 c_{i,q}^{(2)} + a_s^3 c_{i,q}^{(3)} + a_s^4 c_{2,q}^{(4)}, \quad (3.32)$$

for $a = 1, 2, 3$, we determine the coefficient proportional to \mathcal{D}_0 in $c_{a,q}^{(4)}$. For the structure function F_2 it takes the following form,

$$\begin{aligned} c_{2,q}^{(4)} \Big|_{\mathcal{D}_0} = & -f_4^q - B_4^q - (B_3^q + f_3^q) \left(g_{01}^{\text{DIS}} - \frac{3}{2} \zeta_2 A_1 \right) + \frac{1}{3} \zeta_3 (B_1^q + f_1^q)^4 \\ & - (B_2^q + f_2^q) \left(g_{02}^{\text{DIS}} - \frac{3}{2} \zeta_2 A_2 - \frac{3}{2} \zeta_2 A_1 g_{01}^{\text{DIS}} + \frac{15}{16} \zeta_4 A_1^2 \right) \\ & + (B_1^q + f_1^q)^3 \left(\frac{1}{2} \zeta_2 g_{01}^{\text{DIS}} - \frac{5}{8} \zeta_4 A_1 \right) + \frac{3}{2} \zeta_2 (B_1^q + f_1^q)^2 (B_2^q + f_2^q) \\ & + 4 \zeta_3 A_1 (B_1^q + f_1^q) (B_2^q + f_2^q) + 2 \zeta_3 A_1 A_3 + \zeta_3 A_1^2 g_{02}^{\text{DIS}} \\ & + (B_1^q + f_1^q)^2 \left(2 \zeta_3 A_2 + 2 \zeta_3 A_1 g_{01}^{\text{DIS}} - \frac{15}{2} \zeta_2 \zeta_3 A_1^2 + 9 \zeta_5 A_1^2 \right) \\ & - (B_1^q + f_1^q) \left(g_{03}^{\text{DIS}} - \frac{3}{2} \zeta_2 A_3 - \frac{3}{2} \zeta_2 A_2 g_{01}^{\text{DIS}} - \frac{3}{2} \zeta_2 A_1 g_{02}^{\text{DIS}} \right. \\ & \quad \left. + \frac{15}{16} \zeta_4 A_1^2 g_{01}^{\text{DIS}} + \frac{15}{8} \zeta_4 A_1 A_2 + \frac{35}{6} \zeta_3^2 A_1^3 - \frac{525}{128} \zeta_6 A_1^3 \right) \\ & + A_2 \left(\zeta_3 A_2 + 9 \zeta_5 A_1^2 - \frac{15}{2} \zeta_2 \zeta_3 A_1^2 + 2 \zeta_3 A_1 g_{01}^{\text{DIS}} \right) \\ & - A_1^3 g_{01}^{\text{DIS}} \left(\frac{5}{2} \zeta_2 \zeta_3 - 3 \zeta_5 \right) + A_1^4 \left(\frac{35}{16} \zeta_3 \zeta_4 - \frac{21}{2} \zeta_2 \zeta_5 + 15 \zeta_7 \right) \\ & - \beta_2 \left(f_{01}^q + g_{01}^{\text{DIS}} - \frac{1}{2} \zeta_2 A_1 \right) + \beta_0 \beta_1 \left(\frac{5}{2} \zeta_2 f_1^q + \frac{5}{2} \zeta_2 B_1^q + \frac{5}{3} \zeta_3 A_1 \right) \\ & + \beta_0^3 \left(3 \zeta_2 f_{01}^q + 3 \zeta_2 g_{01}^{\text{DIS}} + 2 \zeta_3 f_1^q + 2 \zeta_3 B_1^q - \frac{3}{8} \zeta_4 A_1 \right) \\ & + \beta_0^2 \left(\frac{11}{3} \zeta_3 (B_1^q + f_1^q)^2 + (B_1^q + f_1^q) \left(\frac{9}{2} \zeta_2 f_{01}^q + \frac{11}{2} \zeta_2 g_{01}^{\text{DIS}} - \frac{5}{2} \zeta_4 A_1 \right) \right. \\ & \quad \left. + 3 \zeta_2 (B_2^q + f_2^q) + 2 \zeta_3 A_2 + \zeta_3 A_1 \left(\frac{16}{3} f_{01}^q + 6 g_{01}^{\text{DIS}} \right) - A_1^2 \left(\frac{20}{3} \zeta_2 \zeta_3 - 8 \zeta_5 \right) \right) \\ & + \beta_1 \left(\frac{3}{2} \zeta_2 (B_1^q + f_1^q)^2 + \frac{10}{3} \zeta_3 (B_1^q + f_1^q) A_1 - f_{02}^q \right. \\ & \quad \left. - 2 g_{02}^{\text{DIS}} + \zeta_2 A_2 - g_{01}^{\text{DIS}} f_{01}^q + A_1 \left(2 \zeta_2 g_{01}^{\text{DIS}} + \frac{3}{2} \zeta_2 f_{01}^q - \frac{5}{8} \zeta_4 A_1 \right) \right) \\ & + \beta_0 \left(\frac{3}{2} \zeta_2 A_3 - f_{03}^q - 3 g_{03}^{\text{DIS}} + A_2 \left(\frac{3}{2} \zeta_2 f_{01}^q - \frac{15}{8} \zeta_4 A_1 \right) \right. \\ & \quad \left. + \frac{3}{2} \zeta_2 A_1 f_{02}^q - g_{02}^{\text{DIS}} \left(f_{01}^q - \frac{7}{2} \zeta_2 A_1 \right) - A_1^3 \left(\frac{35}{6} \zeta_3^2 - \frac{525}{128} \zeta_6 \right) \right) \end{aligned}$$

$$\begin{aligned}
& -\frac{15}{16}\zeta_4 A_1^2 f_{01}^q - g_{01}^{\text{DIS}} \left(f_{02}^q - \frac{5}{2}\zeta_2 A_2 - \frac{3}{2}\zeta_2 A_1 f_{01}^q + \frac{25}{16}\zeta_4 A_1^2 \right) \\
& + (B_2^q + f_2^q) \left(\frac{9}{2}\zeta_2 B_1^q + \frac{9}{2}\zeta_2 f_1^q + \frac{16}{3}\zeta_3 A_1 \right) + 2\zeta_3 (B_1^q + f_1^q)^3 \\
& + (B_1^q + f_1^q)^2 \left(\frac{3}{2}\zeta_2 f_{01}^q + 3\zeta_2 g_{01}^{\text{DIS}} - \frac{5}{2}\zeta_4 A_1 \right) + (B_1^q + f_1^q) \left(\frac{14}{3}\zeta_3 A_2 \right. \\
& \quad \left. - \frac{35}{2}\zeta_2 \zeta_3 A_1^2 + 21\zeta_5 A_1^2 + 4\zeta_3 A_1 f_{01}^q + \frac{22}{3}\zeta_3 A_1 g_{01}^{\text{DIS}} \right) \Big), \quad (3.33)
\end{aligned}$$

where g_{0i}^{DIS} denote the coefficients of the perturbative expansion of the function $g_0(Q^2)$ in eq. (2.1), which collects the constant terms in N . Explicit expressions for g_{0i}^{DIS} can be obtained from eqs. (4.6)–(4.8) of [8] by omitting all terms proportional to γ_E and are collected in eqs. (B.5)–(B.7). The other functions have been defined in eqs. (3.5), (3.6), (3.9), (3.10), (3.11), (3.22), (3.23) and (3.25) respectively.

The full colour decomposition of $c_{2,q}^{(4)}$ is given in eq. (A.4) in appendix A, where also the lower order soft-virtual expressions for $C_{2,q}$ from [8] have been collected for convenience. The expressions for the DIS Wilson coefficients in eq. (3.32) for neutral- or charged-current DIS are identical up to two loops. They start to differ from three-loop order onwards, because in the case of neutral-current DIS additional corrections need to be considered, where the exchanged virtual photon couples to a closed quark loop [1]. These contributions give rise to a new flavour structure, depending on the quark flavour composition of the nucleon target through their charges e_q . In the normalization of [1] this leads to a relative factor $fl_{11} = 3\langle e \rangle = (3/n_f) \sum_q e_q$ in the non-singlet and $fl_{11} = (1/n_f) \langle e \rangle^2 / \langle e^2 \rangle = (1/n_f) (\sum_{q'} e_{q'}^2) / (\sum_q e_q^2)$ in the singlet case, respectively. These fl_{11} terms are implicitly contained in g_{03}^{DIS} in eq. (3.33), which is given in eq. (B.7), and spelled out explicitly in eqs. (A.3) and (A.4).

As mentioned already above, the threshold expansion in eq. (A.4) with complete dependence on all colour factors can then be used as part of an ansatz for the full four-loop Wilson coefficient $c_{2,q}^{(4)}$. Following a well-established procedure (see, e.g., [29]) the unknown coefficients of a given functional form for $c_{2,q}^{(4)}$ can be approximately determined using the available Mellin moments for $N \leq 9$ of the DIS Wilson coefficients at four loops [30]. In this way, fixing the individual colour coefficients in terms proportional to \mathcal{D}_0 , the numerical constraints for the remaining unknown coefficients, i.e., $f_{4,n_f C_F^3}^q$, $f_{4,n_f C_F^2 C_A}^q$ and $f_{4,d_{FA}}^q$ have been derived. The terms presented in eq. (3.26) display still a significant uncertainty, whereas the results in large- n_c limit are exact.

Therefore, we determine the best estimate for the term proportional to \mathcal{D}_0 in $c_{2,q}^{(4)}$ in eq. (3.33) by using the large- n_c limit of eq. (3.23) for the terms proportional to n_f^1 in f_4^q and keeping the full colour dependence, i.e. putting $n_c = 3$, for the n_f^0 , n_f^2 and n_f^3 terms of B_4^q and f_4^q given in eqs. (3.11) and (3.25) and in all lower order terms. Using this information then leads to the following numerical result

$$\begin{aligned}
c_{2,q}^{(4)} \Big|_{\mathcal{D}_0, \text{best}} & = (3.874 \pm 0.010) \cdot 10^4 + (-3.494 \pm 0.032) \cdot 10^4 n_f + 2062.715 n_f^2 \\
& \quad - 12.08488 n_f^3 + 47.55183 n_f fl_{11}, \quad (3.34)
\end{aligned}$$

where all exact values have been rounded to seven digits. The error on the term proportional to n_f^0 is dominated by the uncertainty for $f_{4,d_{FA}}^q$ in eq. (3.26), while for the term proportional to n_f^1 it indicates the effect of varying the large- n_c limit for f_4^q of eq. (3.23) by 10%.

Note that the $\delta(1-x)$ coefficient in $c_{2,q}^{(4)}$ is still unknown and requires a complete four-loop computation of DIS structure functions including all virtual corrections. Numerical estimates based on the currently available number of Mellin moments are provided in eq. (A.5) in appendix A.

3.4 Determination of B^{DIS} coefficient

The resummation coefficient B_n^{DIS} which enters the jet function in eq. (2.4) at n loops can finally be derived from the single logarithms $\ln N$ of the Mellin transforms of the DIS coefficient functions $c_{2,q}^{(n)}(x)$. Expanding eqs. (2.8)–(2.9) leads to the relations

$$B_1^{\text{DIS}} = -f_1^q - B_1^q, \quad (3.35)$$

$$B_2^{\text{DIS}} = -f_2^q - B_2^q - \beta_0 \left(f_{01}^q + g_{01}^{\text{DIS}} - \frac{1}{2} \zeta_2 A_1 \right), \quad (3.36)$$

$$B_3^{\text{DIS}} = -f_3^q - B_3^q - \beta_1 \left(f_{01}^q + g_{01}^{\text{DIS}} - \frac{1}{2} \zeta_2 A_1 \right) + \beta_0^2 \left(\zeta_2 f_1^q + \zeta_2 B_1^q + \frac{2}{3} \zeta_3 A_1 \right) - \beta_0 \left(f_{02}^q + 2g_{02}^{\text{DIS}} - (g_{01}^{\text{DIS}})^2 - \zeta_2 A_2 \right), \quad (3.37)$$

$$B_4^{\text{DIS}} = -f_4^q - B_4^q - \beta_2 \left(f_{01}^q + g_{01}^{\text{DIS}} - \frac{1}{2} \zeta_2 A_1 \right) + \beta_0^3 \left(3\zeta_2 f_{01}^q + 3\zeta_2 g_{01}^{\text{DIS}} + 2\zeta_3 f_1^q + 2\zeta_3 B_1^q - \frac{3}{8} \zeta_4 A_1 \right) + \beta_0 \beta_1 \left(\frac{5}{2} \zeta_2 f_1^q + \frac{5}{2} \zeta_2 B_1^q + \frac{5}{3} \zeta_3 A_1 \right) + \beta_0^2 (3\zeta_2 f_2^q + 3\zeta_2 B_2^q + 2\zeta_3 A_2) - \beta_1 \left(f_{02}^q + 2g_{02}^{\text{DIS}} - (g_{01}^{\text{DIS}})^2 - \zeta_2 A_2 \right) - \beta_0 \left(f_{03}^q + 3g_{03}^{\text{DIS}} - 3g_{02}^{\text{DIS}} g_{01}^{\text{DIS}} + (g_{01}^{\text{DIS}})^3 - \frac{3}{2} \zeta_2 A_3 \right), \quad (3.38)$$

where references for all quantities have been already given below eq. (3.33). This leads to

$$B_1^{\text{DIS}} = -3C_F, \quad (3.39)$$

$$B_2^{\text{DIS}} = C_F^2 \left(-\frac{3}{2} + 12 \zeta_2 - 24 \zeta_3 \right) + C_F C_A \left(-\frac{3155}{54} + \frac{44}{3} \zeta_2 + 40 \zeta_3 \right) + C_F n_f \left(\frac{247}{27} - \frac{8}{3} \zeta_2 \right), \quad (3.40)$$

$$B_3^{\text{DIS}} = C_F^3 \left(-\frac{29}{2} - 18 \zeta_2 - 68 \zeta_3 - 144 \zeta_4 + 32 \zeta_2 \zeta_3 + 240 \zeta_5 \right) + C_A C_F^2 \left(-46 + 287 \zeta_2 - \frac{712}{3} \zeta_3 - 136 \zeta_4 - 16 \zeta_2 \zeta_3 - 120 \zeta_5 \right) + C_A^2 C_F \left(-\frac{599375}{729} + \frac{32126}{81} \zeta_2 + \frac{21032}{27} \zeta_3 - \frac{326}{3} \zeta_4 - \frac{176}{3} \zeta_2 \zeta_3 - 232 \zeta_5 \right)$$

$$\begin{aligned}
& +C_F^2 n_f \left(\frac{5501}{54} - 50 \zeta_2 + \frac{32}{9} \zeta_3 \right) + C_F n_f^2 \left(-\frac{8714}{729} + \frac{232}{27} \zeta_2 - \frac{32}{27} \zeta_3 \right) \\
& + C_A C_F n_f \left(\frac{160906}{729} - \frac{9920}{81} \zeta_2 - \frac{776}{9} \zeta_3 + \frac{104}{3} \zeta_4 \right). \tag{3.41}
\end{aligned}$$

Eqs. (3.39)–(3.41) are, of course, well-known results [8]. In the large- n_c limit we can determine B_4^{DIS} with eqs. (3.10) and (3.23) as

$$\begin{aligned}
B_4^{\text{DIS}} \Big|_{\text{Lnc}} &= C_F n_c^3 \left(-\frac{2040092429}{139968} + \frac{23011973}{1944} \zeta_2 + \frac{517537}{36} \zeta_3 - \frac{312481}{36} \zeta_4 \right. \\
&\quad \left. - \frac{39838}{9} \zeta_2 \zeta_3 - \frac{50680}{9} \zeta_5 - 988 \zeta_3^2 + \frac{12467}{6} \zeta_6 + 496 \zeta_2 \zeta_5 + 688 \zeta_3 \zeta_4 + 2260 \zeta_7 \right) \\
&+ C_F n_c^2 n_f \left(\frac{83655179}{11664} - \frac{5160215}{972} \zeta_2 - \frac{639191}{162} \zeta_3 + \frac{24856}{9} \zeta_4 + \frac{8624}{9} \zeta_2 \zeta_3 \right. \\
&\quad \left. + 200 \zeta_5 - 32 \zeta_3^2 - \frac{1201}{3} \zeta_6 \right) + C_F n_c n_f^2 \left(-\frac{5070943}{5832} + \frac{160903}{243} \zeta_2 \right. \\
&\quad \left. + \frac{14618}{81} \zeta_3 - \frac{2110}{9} \zeta_4 - \frac{400}{9} \zeta_2 \zeta_3 + \frac{904}{9} \zeta_5 \right) \\
&+ C_F n_f^3 \left(\frac{50558}{2187} + \frac{80}{81} \zeta_3 - \frac{1880}{81} \zeta_2 + \frac{40}{9} \zeta_4 \right). \tag{3.42}
\end{aligned}$$

Using instead eqs. (3.11) and (3.25) with the full colour dependence in B_4^{DIS} gives the following expression

$$\begin{aligned}
B_4^{\text{DIS}} &= -C_F^4 b_{4,C_F^4}^q + C_F^3 C_A \left(-\frac{10769}{12} - \frac{2717}{6} \zeta_2 - 4565 \zeta_3 - 2200 \zeta_4 + 1496 \zeta_2 \zeta_3 - 616 \zeta_5 \right. \\
&\quad \left. + \frac{2992}{3} \zeta_3^2 + \frac{48620}{9} \zeta_6 - b_{4,C_F^3 C_A}^q \right) + C_F^2 C_A^2 \left(-\frac{51325}{72} + \frac{307033}{54} \zeta_2 + \frac{160994}{27} \zeta_3 \right. \\
&\quad \left. - \frac{481637}{54} \zeta_4 - \frac{8932}{9} \zeta_2 \zeta_3 - \frac{17776}{9} \zeta_5 + \frac{616}{3} \zeta_3^2 - \frac{19558}{9} \zeta_6 - b_{4,C_F^2 C_A^2}^q \right) \\
&+ C_F C_A^3 \left(-\frac{1958802125}{139968} + \frac{2213182}{243} \zeta_2 + \frac{676939}{54} \zeta_3 - \frac{198203}{54} \zeta_4 - \frac{38738}{9} \zeta_2 \zeta_3 \right. \\
&\quad \left. - \frac{40406}{9} \zeta_5 - 1340 \zeta_3^2 + \frac{10883}{6} \zeta_6 + 496 \zeta_2 \zeta_5 + 688 \zeta_3 \zeta_4 + 2260 \zeta_7 + \frac{1}{2} b_{4,C_F C_A^3}^q \right. \\
&\quad \left. + \frac{1}{4} b_{4,C_F^3 C_A}^q + \frac{1}{8} b_{4,C_F^4}^q + \frac{1}{24} f_{4,d_{FA}^{(4)}}^q + \frac{1}{24} b_{4,d_{FA}^{(4)}}^q \right) \\
&- \frac{d_{FA}^{(4)}}{n_c} \left(f_{4,d_{FA}^{(4)}}^q + b_{4,d_{FA}^{(4)}}^q \right) + C_F^3 n_f \left(\frac{482}{3} + \frac{19}{3} \zeta_2 + 806 \zeta_3 + 564 \zeta_4 \right. \\
&\quad \left. - 272 \zeta_2 \zeta_3 + 112 \zeta_5 - \frac{544}{3} \zeta_3^2 - \frac{8840}{9} \zeta_6 - f_{4,n_f C_F^3}^q - b_{4,n_f C_F^3}^q \right) \\
&+ C_F^2 C_A n_f \left(\frac{2465317}{972} - \frac{70060}{27} \zeta_2 - \frac{164470}{81} \zeta_3 + \frac{75011}{27} \zeta_4 + \frac{1976}{9} \zeta_2 \zeta_3 + \frac{2704}{9} \zeta_5 \right. \\
&\quad \left. - \frac{112}{3} \zeta_3^2 + \frac{3556}{9} \zeta_6 - f_{4,n_f C_F^2 C_A}^q - b_{4,n_f C_F^2 C_A}^q \right) + C_F C_A^2 n_f \left(\frac{68301461}{11664} \right. \\
&\quad \left. - \frac{1935001}{486} \zeta_2 - \frac{254678}{81} \zeta_3 + \frac{66211}{54} \zeta_4 + \frac{8272}{9} \zeta_2 \zeta_3 - \frac{772}{27} \zeta_5 + \frac{364}{9} \zeta_3^2 - \frac{9439}{27} \zeta_6 \right)
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2} f_{4,n_f}^q C_F^2 C_A + \frac{1}{2} b_{4,n_f}^q C_F^2 C_A + \frac{1}{4} f_{4,n_f}^q C_F^3 + \frac{1}{4} b_{4,n_f}^q C_F^3 - \frac{1}{48} b_{4,d_{FF}^{(4)}}^q \\
& + n_f \frac{d_{FF}^{(4)}}{n_c} \left(384 - \frac{4544}{3} \zeta_2 + \frac{5312}{9} \zeta_3 + \frac{800}{3} \zeta_4 - 128 \zeta_2 \zeta_3 + \frac{21760}{9} \zeta_5 - \frac{1216}{3} \zeta_3^2 \right. \\
& \left. - \frac{1184}{9} \zeta_6 + b_{4,d_{FF}^{(4)}}^q \right) + C_F^2 n_f^2 \left(-\frac{218239}{486} + \frac{2150}{9} \zeta_2 + \frac{5684}{27} \zeta_3 - \frac{380}{3} \zeta_4 \right. \\
& \left. - 64 \zeta_5 \right) + C_F C_A n_f^2 \left(-\frac{3761509}{5832} + \frac{131878}{243} \zeta_2 + \frac{6092}{81} \zeta_3 - \frac{1540}{9} \zeta_4 - \frac{400}{9} \zeta_2 \zeta_3 \right. \\
& \left. + \frac{1192}{9} \zeta_5 \right) + C_F n_f^3 \left(\frac{50558}{2187} + \frac{80}{81} \zeta_3 - \frac{1880}{81} \zeta_2 + \frac{40}{9} \zeta_4 \right) \tag{3.43}
\end{aligned}$$

which can be evaluated numerically using the results in table 1 and eq. (3.26).

As discussed above, the four-loop terms in eq. (3.26) are not very well constrained and therefore we follow the same approach as in the derivation of eq. (3.34) to determine the present best estimate for B_4^{DIS} in eq. (3.38). We use the large- n_c limit of eq. (3.23) for terms proportional to n_f^1 in f_4^q and keep full colour dependence in all others terms, i.e., all those proportional to the QCD beta-function coefficients in eq. (3.38) and in those proportional to n_f^0 , n_f^2 and n_f^3 in B_4^q and f_4^q . This leads to

$$\begin{aligned}
B_4^{\text{DIS}} \Big|_{\text{appr}} &= (10.68 \pm 0.01) \cdot 10^4 + (-2.025 \pm 0.032) \cdot 10^4 n_f + 798.0698 n_f^2 \\
&\quad - 12.08488 n_f^3, \tag{3.44}
\end{aligned}$$

and amounts to a perturbative expansion through four loops

$$\begin{aligned}
B_4^{\text{DIS}}(n_f=3) &= -0.31831 \alpha_s (1 - 1.1004 \alpha_s - 3.623 \alpha_s^2 - (6.67 \pm 0.13) \alpha_s^3 + \dots), \\
B_4^{\text{DIS}}(n_f=4) &= -0.31831 \alpha_s (1 - 1.2267 \alpha_s - 3.405 \alpha_s^2 - (4.77 \pm 0.17) \alpha_s^3 + \dots), \\
B_4^{\text{DIS}}(n_f=5) &= -0.31831 \alpha_s (1 - 1.3530 \alpha_s - 3.190 \alpha_s^2 - (3.03 \pm 0.21) \alpha_s^3 + \dots), \tag{3.45}
\end{aligned}$$

where the numerical uncertainties at four loops follow from varying the large- n_c limit for the terms proportional to n_f^1 in f_4^q of eq. (3.23) by 10%, as done also in eq. (3.34). With the knowledge of all resummation coefficients we are now ready to perform the numerical analysis up to N⁴LL accuracy.

4 Numerical results

4.1 Resummation at N⁴LL

In the following we present numerical results for the resummed series up to N⁴LL accuracy. We use the approximate value for B_4^{DIS} as provided in eq. (3.44) and study the convergence of the resummed series and also the soft-virtual (SV) results at the fourth order in strong coupling and beyond. We set the renormalization and factorization scales to be equal to the momentum transfer squared ($Q^2 = -q^2$). Since we are interested in the region of (very) large x at moderate Q^2 , the charm and bottom quarks cannot be treated as effectively massless. Hence we will focus on the results for $n_f = 3$ light quark flavours, which implies

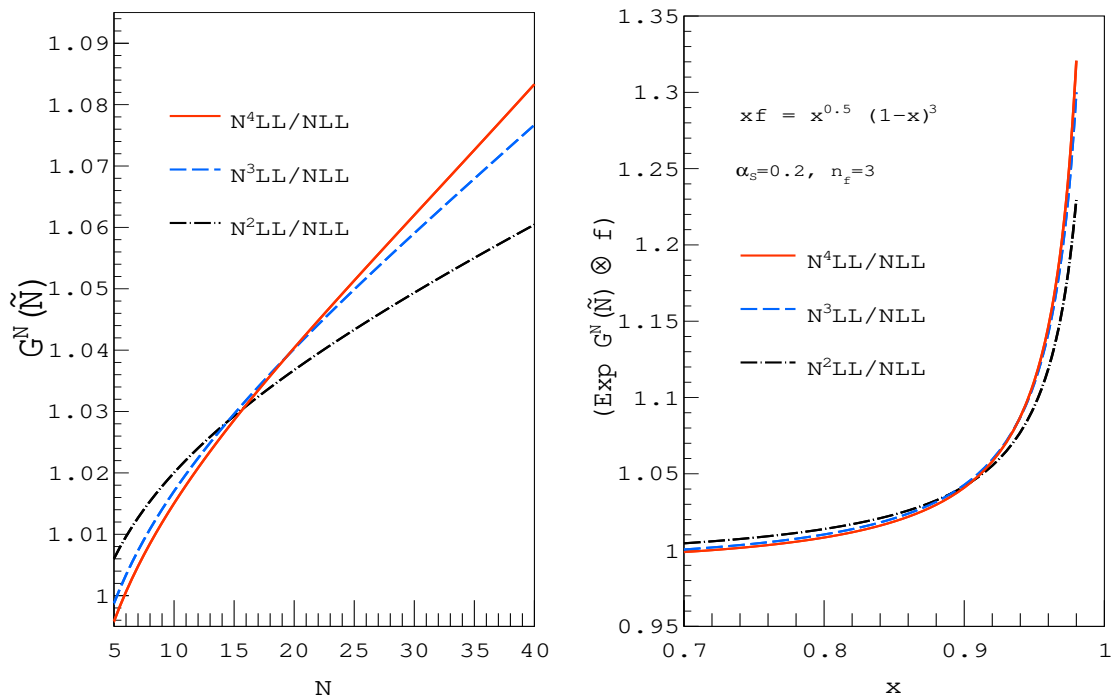


Figure 1. Left: the resummation exponent G^N , normalized to the NLL results, for DIS in eq. (2.7) plotted successively up to N⁴LL for $\alpha_s = 0.2$ and $n_f = 3$. Right: the resummed series is convoluted with a typical shape for a quark distribution up to N⁴LL.

$fl_{11} = 0$ for the new flavour structure due to a vanishing combination of the quark charges. For $n_f \neq 3$ this flavour structure contributes to neutral-current DIS. We will come back to the resulting numerical differences between neutral- and charged-current DIS at the end.

In the left panel of figure 1, we present the resummation exponent up to N⁴LL, normalized to the NLL results for a better visibility of the small higher-order effects. Recall that the resummation accuracy is defined by truncating the resummation exponent in eq. (2.7) to a particular order; i.e., the first term $\ln \tilde{N}g^{(1)}$ defines the LL approximation, the first two terms $\ln \tilde{N}g^{(1)} + g^{(2)}$ define the NLL accuracy and so on. Using the order-independent value $\alpha_s = 0.2$ for the strong coupling, we observe a good perturbative convergence up to N⁴LL level. At the largest N -value shown, $N = 40$, the exponent increases from LL to NLL by around 66.7%, from NLL to NNLL by around 6.0%, from NNLL to N³LL by around 1.5% and from N³LL to N⁴LL by around 0.6%. The resummed series thus shows a good stabilization at this order. The uncertainty in B_4^{DIS} quoted in eq. (3.44) as a result of changing the respective large- n_c terms by $\pm 10\%$ does not affect the cross section much. It amounts at most to 0.1%, which is small compared to the size of the N⁴LL correction.

In the right panel of figure 1 we plot $\exp(G^N) \otimes f$, the resummation exponential convoluted with a schematic but sufficiently typical shape for a quark distribution given by $xf = x^{0.5}(1-x)^3$. For the Mellin inversion we have used the minimal prescription [6] with the standard contour choice as in QCD-PEGASUS [64]. Also here we find a good perturbative convergence up to N⁴LL. At $x = 0.9$ the cross section changes from LL to

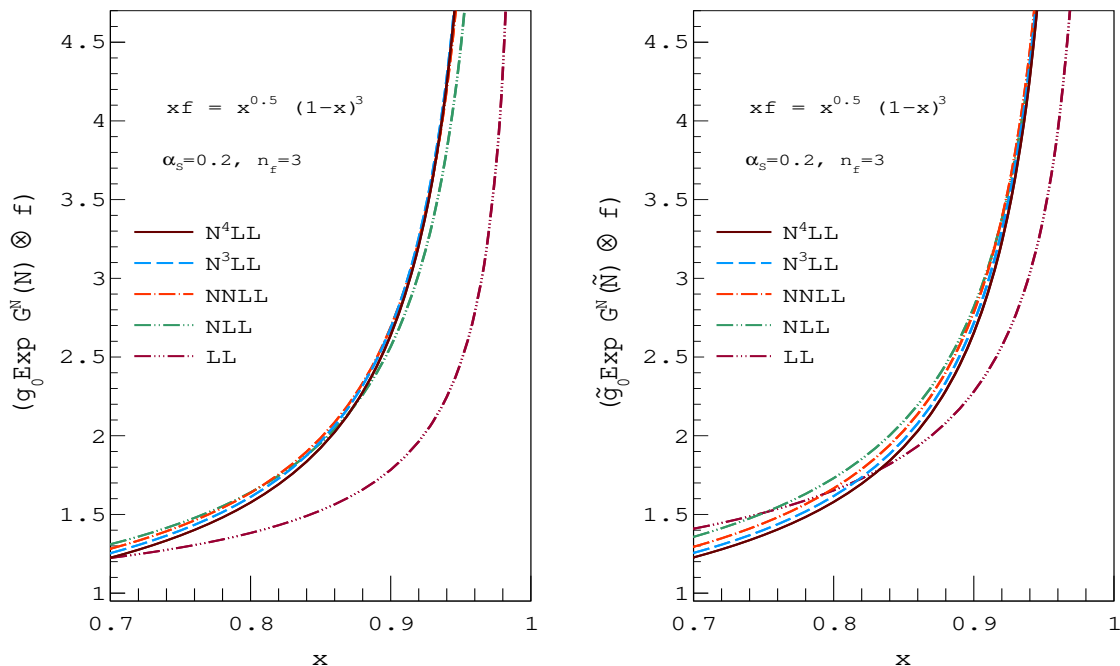


Figure 2. Comparison between the N -exponentiation (left) and \tilde{N} -exponentiation (right) as discussed in the text.

NLL by 67.1%, from NLL to NNLL by 4.2%, from NNLL to N^3 LL by 0.09% and from N^3 LL to N^4 LL by -0.17% . At $x = 0.95$ the corresponding changes are much larger, yet the N^4 LL effect stays well within 0.5%.

For $n_f = 4$ light quark flavours on the other hand, we see an even faster perturbative convergence for the resummation exponent as well as for the convolution of exponential with the same input shape. For example, at $N = 40$ we observe a 61.4% increase from LL to NLL, another increase of 1.8% from NLL to NNLL, and changes of -1.1% from NNLL to N^3 LL and of -0.9% from N^3 LL to N^4 LL accuracy. For the case of the convolution of the exponential with the above input shape, the corresponding values read 61%, 0.3%, -1.7% and -0.8% , respectively, at $x = 0.9$.

The threshold limit in the Mellin- N space is based on the large- N limit, which we have discussed so far in terms of the variable $\tilde{N} = N \exp(\gamma_E)$, including the exponentiation of all $\ln \tilde{N}$ terms. One can, in principle, define the large Mellin variable as N only and collect only those terms in the resummed exponent which are enhanced as $N \rightarrow \infty$. As far as the threshold limit is concerned, both, $\tilde{N} \rightarrow \infty$ and $N \rightarrow \infty$ are physically equivalent. However, numerically the two options differ since in the former case all γ_E terms associated with N are also exponentiated to all orders. Due to this feature the results for both cases are different starting already at LL accuracy. The resummed exponent shows a faster convergence for the N exponentiation. However, note that now also the function g_0 in eq. (2.1) differs between the two cases. Whereas in the \tilde{N} exponentiation, the γ_E terms are collected by definition in \tilde{N} as the large- N variable, for the N -exponentiation they appear in the finite function g_0 and in the resummed exponent G . As mentioned already, we collect the explicit results for the perturbative expansion of g_0 in the appendix B.

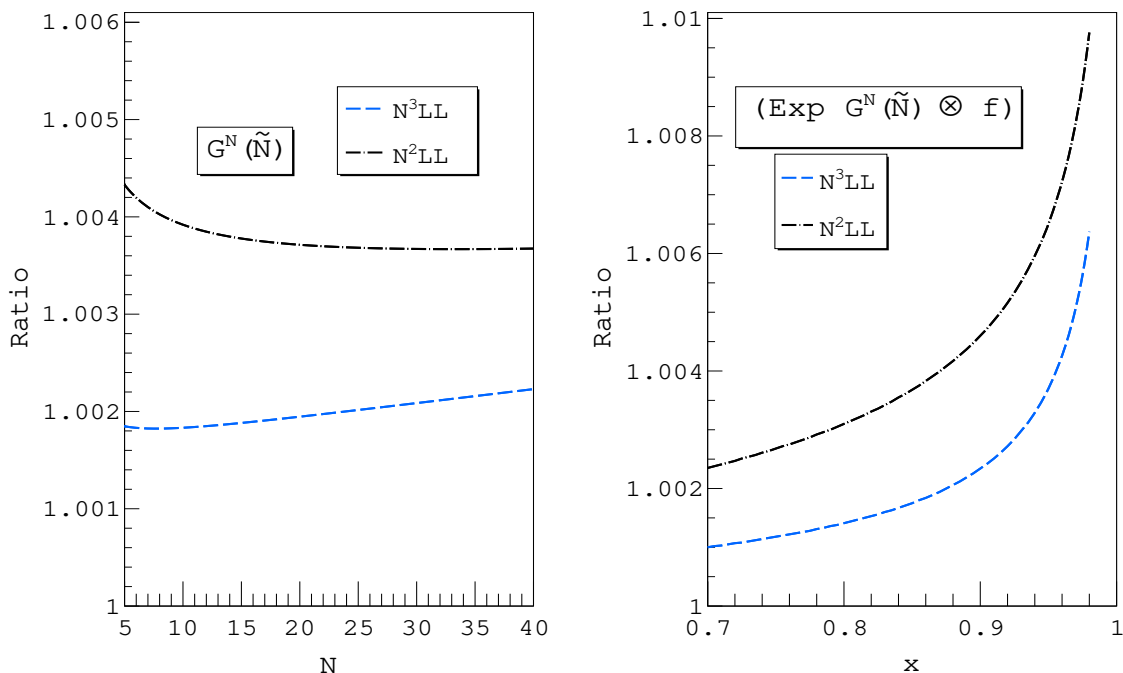


Figure 3. The ratio of large- n_c approximation and the exact results at NNLL and N^3LL for $\alpha_S = 0.2$ and $n_f = 3$ light flavours. Left: the ratio for DIS resummed exponent G^N as a function of Mellin- N . Right: the ratio for the resummed series convoluted with the typical input shape used in figure 1 plotted against x .

A useful comparison for these two approaches is therefore to check the convergence at the cross-section level itself. We provide this comparison in figure 2. For the N -exponentiation satisfactory convergence occurs in the threshold region only beyond NLL accuracy whereas the \tilde{N} -exponentiation shows a systematic behaviour for perturbative convergence for the successive orders of the resummation and good convergence in the threshold region is achieved already at NLL accuracy. At sufficiently high logarithmic order both approaches converge. However they start to differ away from the threshold. At $x = 0.7$, the LL result in \tilde{N} -exponentiation differs by as large as 15% compared to the N -exponentiation, whereas at the N^4LL level this difference decreases to 0.1% showing that at higher logarithmic accuracy they tend to converge. We see similar observation in the higher x region as well. In the rest of the article, we apply the standard \tilde{N} exponentiation throughout.

It is instructive to study the quality of the specific large- n_c approximation used to derive the approximate value for B_4^{DIS} in eq. (3.44) at each lower order. To that end, we have compared the large- n_c behaviour of results at lower loops with the exact full-colour expressions available up to three loops. Note that in order to have a meaningful estimate of the resummed coefficient function, we take the large- n_c approximation only for the process-dependent pieces appearing at that order, whereas all the lower order terms are kept exact. For example, at the third order, we take the large- n_c approximation only for the combination $-(f_3^q + B_3^q)$ appearing in eq. (3.37) whereas all the other terms proportional to coefficients of the QCD beta function are kept exact. The LL result is always exact since

it only depends on the universal cusp anomalous dimension A_1 while the first process-dependent coefficients enter as $-(f_1^q + B_1^q)$ at NLL order. Note that up to the third order the large- n_c limit for the terms proportional to n_f^1 and n_f^0 in f_i^q and B_i^q coincides with the exact large- n_c expressions of these coefficients when restoring the overall factor C_F . However this is not true anymore at the N⁴LO level.

In figure 3, we compare the large- n_c and the exact results for the resummation exponent (left) and the resummed series convoluted with the same quark distribution xf used above (right). In particular we have taken the ratio between the large- n_c and exact results at lower orders. We have taken $n_f = 3$ light quark flavours also for this study. Up to NLL the large- n_c result is exact. The difference in the cross section starts from NNLL and is about 0.4% in the large- N region ($N = 40$), whereas at N³LL it is about 0.2%; the large- n_c approximation always overestimates the exact result in the region of interest. These differences are to be compared with the absolute size of the exact corrections. These are much larger with about 6.0% at $N = 40$, when increasing the logarithmic accuracy from NLL to NNLL, and 1.5% for NNLL to N³LL. Similar observations also hold at lower N values.

Using the approximate value for B_4^{DIS} in eq. (3.44) with the specific large- n_c limit, the step from N³LL to N⁴LL accuracy amounts to a correction of about 0.6% at $N = 40$. Based on the lower order studies, we thus expect that the exact result at fourth order for the complete N⁴LL calculation should not differ by more than $\pm 0.2\%$ from our prediction for the resummed exponent at large- n_c . In summary, the above procedure of taking the specific large- n_c approximation provides robust estimates for the cross sections of interest.

4.2 Soft-virtual cross section at the fourth order

The convergence of the threshold expansion in Bjorken- x or Mellin- N space can be illustrated by means of the successive addition of the plus-distributions \mathcal{D}_k , $k = 2n - 1, \dots, 0$ in the SV cross section in x -space compared to the successive addition of the logarithms $\ln^i N$, $i = 2n, \dots, 1$ in N -space.¹ The former procedure shows poor convergence, a fact that has been already observed in [6] for two ($n = 2$) and in [8] for three ($n = 3$) loops. In figure 4 we compare the convergence of the DIS Wilson coefficient at four loops $c_{2,q}^{(4)}$ constructed in this manner either in x - or in N -space and convoluted with the above input shape xf . As expected we see a very similar behaviour compared to the known lower orders. The x -space result converges poorly and only stabilizes after the addition of the six highest plus-distributions \mathcal{D}_k at this order. On the other hand in N -space, the same convergence is achieved much faster after adding only the four highest logarithms $\ln^i N$ in the series. This confirms that N -space is better suited for studies of the threshold limit also at the fourth order.

The exact expression of \mathcal{D}_0 at fourth order still contains the poorly constrained coefficients $f_{4,d_{FA}}^q$, $f_{4,n_f C_F^3}^q$ and $f_{4,n_f C_F^2 C_A}^q$ in eq. (3.26) which drop out in the large- n_c limit. We have improved the \mathcal{D}_0 coefficient in eq. (3.34) through the best estimate for B_4^{DIS} in eq. (3.44). Using instead the central value for the above f_4 coefficients (see eq. (3.26)) the \mathcal{D}_0 coefficient changes by around 6.5% compared to the value in eq. (3.34). This is a

¹Note that we use powers of $\ln N$, and not $\ln \tilde{N}$, from now on.

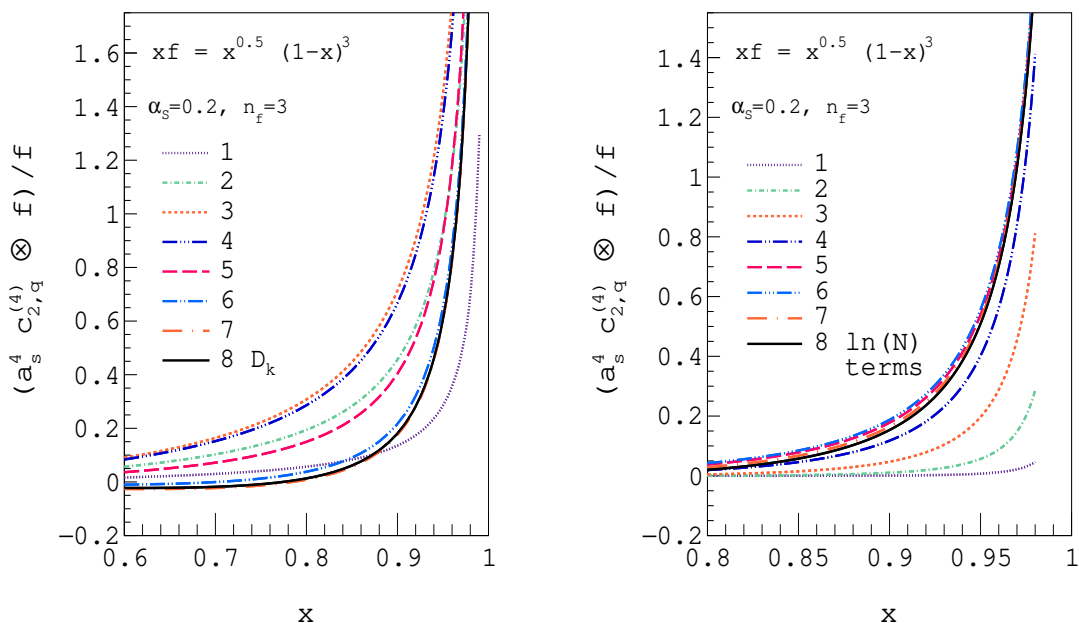


Figure 4. Left: the DIS Wilson coefficient $c_{2,q}^{(4)}$ in eq. (3.33) convoluted with the input shape xf of figure 1 with the successive addition of the plus-distributions \mathcal{D}_k starting from the highest term. Right: the same with the successive addition of the N -space logarithms.

rather small change considering the errors in these coefficients in eq. (3.26). On the other hand setting the f_4 coefficients in eq. (3.26) to zero, which is within the uncertainty range quoted, yields a value for the \mathcal{D}_0 coefficient which differs only by 0.5% from its best prediction in eq. (3.34). These are small effects given that the best estimate for \mathcal{D}_0 in eq. (3.34) changes the whole N⁴LO SV cross-section (up to the contribution proportional to $\delta(1-x)$ in eq. (A.5)) by only 0.9% even at $x = 0.95$. Altogether, this demonstrates a small dependence of the total cross section on those, as of yet, still poorly known coefficients. At the same time, it corroborates the assumptions made in the derivation of the best estimate for \mathcal{D}_0 in eq. (3.34).

4.3 Tower expansion vs exponentiation

The resummed cross section in Mellin N -space can also be reorganized in a different manner [8, 65] by re-expanding the exponential in eq. (2.1) as follows:

$$C_{a,q}^{N}(Q^2) = 1 + \sum_{k=1}^{\infty} a_s^k \sum_{l=1}^{2k} c_{kl} \ln^{2k-l+1} N, \tag{4.1}$$

where the coefficients c_{kl} for DIS are given in table 2 for all known terms. By successive addition of terms in this expansion, one can also predict the resummed cross section up to a certain accuracy. The truncation of this series at any particular order will, of course, recover the fixed-order SV result in N -space. The coefficients c_{kl} of the logarithms in the above

series vanish factorially at sufficiently higher orders in α_s for a fixed logarithmic structure (i.e., fixed by l). The coefficients c_{kl} are determined up to $l = 1, 2, 3$ by the complete one loop calculation with NLL resummation. The complete two-loop calculation along with the NNLL resummation fixes the next two towers ($l = 4, 5$), whereas the complete three-loop calculation along with the N³LL resummation determines again the next two towers, i.e., up to $l = 7$. The N⁴LL resummation derived here allows to completely fix the eighth tower, which corresponds to $l = 8$. Note that the quartic colour factors, i.e., $d_{FA}^{(4)}$ and $d_{FF}^{(4)}$ in eq. (3.3), appear in this tower for the first time. To obtain all terms up to the ninth tower ($l = 9$), the complete four-loop calculation is needed, which is currently not available. However from the knowledge of higher moments for DIS, we can estimate approximate values for the unknown $\delta(1-x)$ coefficient (see appendix A) at the four loops which in turn can be used to extract the coefficient g_{04}^{DIS} . For the standard \tilde{N} exponentiation, then one can have at the fourth order,

$$\begin{aligned}
 g_{04}^{\text{DIS}} &= (-22.51 \pm 3.) \cdot 10^4, \\
 &(-13.75 \pm 3.) \cdot 10^4, \\
 &(-6.88 \pm 3.) \cdot 10^4 \quad \text{for } n_f = 3, 4, 5, \text{ respectively.}
 \end{aligned}
 \tag{4.2}$$

With the lower orders coefficient the series for g_0 looks as follows

$$\begin{aligned}
 g_0(n_f = 3) &= 1 - 16.3865 a_s - 147.955 a_s^2 - 4528.85 a_s^3 - (22.51 \pm 3) \cdot 10^4 a_s^4 \\
 g_0(n_f = 4) &= 1 - 16.3865 a_s - 109.603 a_s^2 - 2918.08 a_s^3 - (13.75 \pm 3) \cdot 10^4 a_s^4 \\
 g_0(n_f = 5) &= 1 - 16.3865 a_s - 71.2505 a_s^2 - 1477.81 a_s^3 - (6.88 \pm 3) \cdot 10^4 a_s^4.
 \end{aligned}
 \tag{4.3}$$

The error in the g_{04}^{DIS} coefficient introduces an uncertainty of around 6% in the $\ln^2 N$ term of the a_s^5 coefficient. The uncertainty in column c_{k8} is due to only the uncertainty coming from the estimate for B_4^{DIS} . On the other hand the uncertainty due to both B_4^{DIS} and g_{04}^{DIS} are added in quadrature to get the final uncertainty in column c_{k9} . Although we note that this uncertainty is mostly dominated by the uncertainty in g_{04}^{DIS} estimate.

In figure 5 on the left we illustrate the expansion in decreasing powers of $\ln N$ for the five-loop coefficient function $c_{2,q}^{(5)}$ in the same manner as for the four-loop coefficient function $c_{2,q}^{(4)}$ in the right part of figure 4 above. We confirm the general pattern, that the first l logarithms provide a good estimate up to the l -th order in α_s , i.e., $l = 5$ for the case at hand. In the right panel of figure 5 we compare the resummed result (exponentiated) for the Wilson coefficient with the tower expansion given in table 2. Although they both converge in the region of $x \simeq 0.5$, they start to diverge at large x where exponentiation shows a better stability compared to the tower expansion. In fact the expansion with a fixed number of towers severely underestimates the effect of the coefficient functions of much higher orders, which become more important at the region of very large- x around $x \simeq 0.9$.

So far we have considered $fl_{11} = 0$ for the flavour structure, which holds for $n_f = 3$, due to the vanishing combination of the quark charges. Starting from three loops, the neutral-current DIS coefficient functions differ due to $fl_{11} \neq 0$, which captures the effect of incoming and outgoing photons coupling to quark lines of different flavour, see e.g., [1].

k	c_{k1}	c_{k2}	c_{k3}	c_{k4}	c_{k5}	c_{k6}	c_{k7}	$c_{k8} (\times 10^{-3})$	$c_{k9} (\times 10^{-4})$
1	2.66667	7.0785	—	—	—	—	—	—	—
2	3.55556	26.8760	46.082	-52.31	—	—	—	—	—
3	3.16049	46.5013	262.409	606.02	-379.3	-1607	—	—	—
4	2.10700	50.8159	526.224	2901.29	7563.9	3839	-30240	-50.45 ± 1.1	—
5	1.12373	40.1983	643.688	5931.23	32776.2	102186	111002	-223.83 ± 2.8	-135.79 ± 8.5
6	0.49944	24.8103	562.907	7611.78	66550.1	380864	1341323	2326.76 ± 3.8	-136.09 ± 13.4
7	0.19026	12.5251	381.022	7043.11	87178.2	749259	4455641	17525.89 ± 3.4	3930.14 ± 15.2
8	0.06342	5.3423	209.584	5057.26	83343.1	983401	8443647	52347.85 ± 2.2	22576.03 ± 13.2
9	0.01879	1.9710	96.844	2951.41	62206.6	956176	10993901	95211.30 ± 1.2	61509.98 ± 9.0
10	0.00501	0.6404	38.511	1445.50	37852.3	731699	10762856	122209.70 ± 0.5	107329.05 ± 5.0
11	0.00121	0.1858	13.424	608.30	19356.2	458590	8364106	119639.50 ± 0.2	135303.85 ± 2.4
12	0.00027	0.0487	4.161	223.97	8508.0	242196	5352115	93780.42 ± 0.1	131835.19 ± 1.0
13	0.00006	0.0116	1.161	73.19	3271.3	110118	2895802	60869.59 ± 0	103720.30 ± 0.3
14	0.00001	0.0026	0.294	21.48	1115.7	43830	1351835	33532.90 ± 0	67950.87 ± 0.1
15	0.00000	0.0005	0.068	5.72	341.4	15479	553238	15980.28 ± 0	37933.17 ± 0

Table 2. The coefficients of the tower expansion for $C^N(Q^2)$ in eq. (4.1) for $n_f = 3$ light quark flavours, where they are the same for F_1, F_2 and F_3 since $fl_{11} = 0$. The first seven columns are exact up to the numerical truncation. The eighth column uses the estimates for B_4^{DIS} and corresponding errors and the ninth column in addition requires the estimate for g_{04}^{DIS} in eq. (4.2).

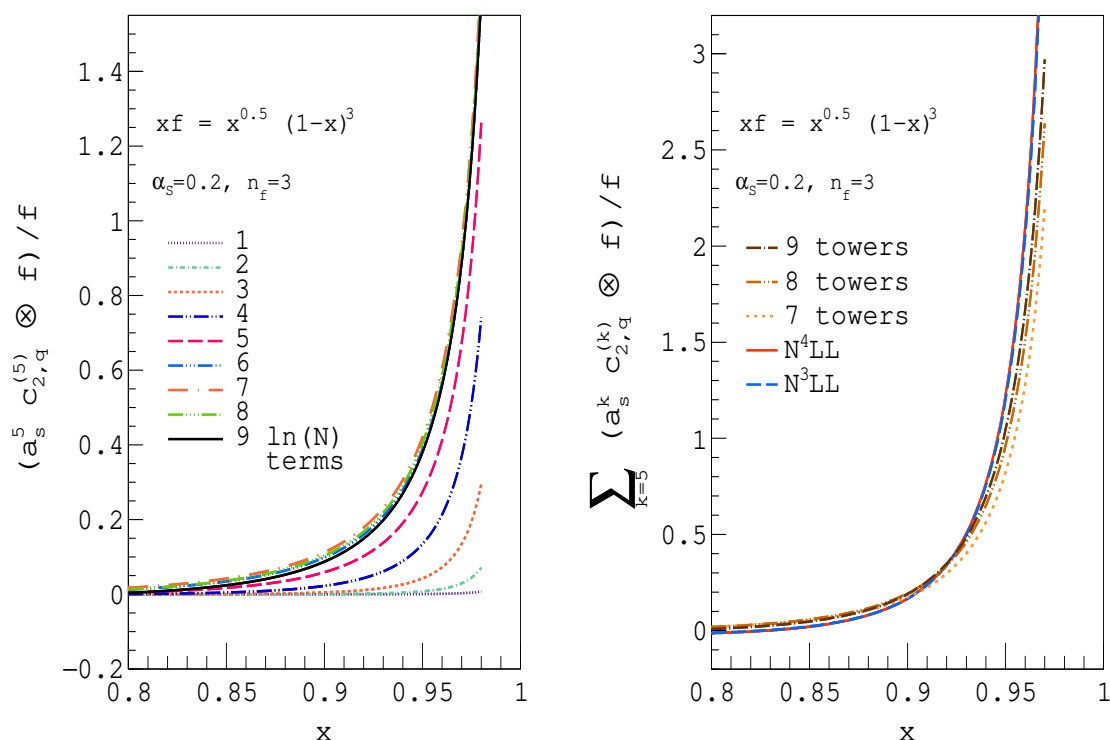


Figure 5. Left: the successive approximations of the five-loop coefficient function $c_{2,q}^{(5)}$ by the large- N terms specified in table 2, illustrated by the convolution with the input shape xf of figure 1. Right: the corresponding results for the effect of higher terms beyond α_s^4 as obtained from the tower expansion up to nine towers and from the exponentiation up to $N^4\text{LL}$ accuracy on the Wilson coefficient.

The effect of this contribution is numerically insignificant as already pointed out in [8]. To estimate its numerical impact on the SV cross section as well as in the tower expansion, we use $n_f = 4$ and keep the exact value $fl_{11} = 1/10$. Note that a non-zero fl_{11} will change the tower expansion starting from seventh tower, i.e., the coefficients c_{k7} , and from the lowest order for this towers ($k = 3$). In the seventh tower, the contribution proportional to fl_{11} changes the finite piece of third order SV result, whereas at the fourth order in addition it also changes the eight tower. Keeping exact value for fl_{11} we get a difference of 0.1% in the coefficient of the constant term at third order and a change of 0.09% in the last logarithmic term at fourth order.

The additional new term in the charged-current DIS coefficient function $C_{3,q}$ proportional to the group invariant $(d^{abc}d_{abc})/n_c$ in eq. (3.1) which corresponds to incoming and outgoing W^\pm -bosons coupling to a closed internal quark loop [2]. This term is power suppressed for $x \rightarrow 1$, hence does not contribute to the threshold approximation considered here.

5 Summary

We have applied recent progress in the calculation of the cusp anomalous dimension, the quark (non-singlet) splitting functions and the quark form factor to obtain predictions for the DIS cross sections with charged- and neutral-current interactions at four loops in perturbation theory. We have performed the threshold resummation in the large- n_c limit up to N⁴LL collecting the terms $\alpha_s^3(\alpha_s \ln N)^k$ to all orders in α_s in the threshold resummation exponent G^N in Mellin- N space. To that end, the relevant quantity B^{DIS} controlling the resummation of the quark jet function has been extracted at four-loop order.

We have also derived the full colour dependence of the DIS Wilson coefficients at four loops down to the term proportional to the plus-distribution $[1/(1-x)]_+$ using a conjecture about the colour structure of the anomalous dimension of the eikonal form factor, namely that it exhibits the same generalized maximal non-Abelian property regarding quadratic and quartic Casimir coefficients as the cusp anomalous dimension. As a by-product, also the single pole of QCD form factor in the parameter of dimensional regularization ε has been determined numerically to good accuracy.

Using the analytical expressions for the resummed DIS Wilson coefficient at the N⁴LL level, we have shown that the lower order corrections are sizable and that the threshold logarithms at N⁴LL resummed order amount to corrections well within 1% and improve the perturbative convergence. We have compared the effect of successive plus-distributions in x -space as well as the logarithms in N -space for the cross section up to the fourth order and we confirm known observations that the convergence of the logarithmic series in N -space is much smoother compared to the one in x -space. We have also reorganized the resummed series as a tower of logarithms up to the ninth tower. While the tower expansion is useful to study the effect of higher logarithms, it lacks contributions at very large x compared to the exponentiated result. The latter exhibits an even better stability even in the very large- x region. Near threshold the difference between DIS Wilson coefficients for charged- and

neutral-current exchange, which manifests itself in the flavour structure fl_{11} at three loops and beyond, has only a minor numerical effect (for $n_f = 4$) on the respective cross sections.

The current accuracy can be significantly improved with the knowledge of the complete QCD form factor at four loops or an exact computation of the corresponding eikonal anomalous dimension at this order. In particular analytic results for all n_f dependent terms in those quantities would greatly reduce the present residual numerical uncertainties. At the accuracy reached with N⁴LL resummation also power corrections proportional to $1/N$ in Mellin- N space or logarithmic enhancement $\ln^k(1-x)$ in x -space become numerically relevant, see [66, 67]. Finally, the effects from the massive quarks are important in the threshold region (see for example [68, 69]) and a proper treatment of heavy quark mass effects becomes essential for phenomenological applications.

Note added in proof. During review of the present paper, the complete analytic four-loop anomalous dimensions in massless QCD from form factors have been obtained in [72]. This provides information on the missing four-loop contributions $f_{4,n_f C_F^3}^q$ and $f_{4,n_f C_F^2 C_A}^q$ in eq. (3.26), which we can determine as

$$\begin{aligned}
 f_{4,n_f C_F^3}^q &= \frac{27949}{108} + 322\zeta_2 - \frac{1120}{9}\zeta_3 - 334\zeta_4 - \frac{512}{3}\zeta_2\zeta_3 + \frac{3872}{3}\zeta_5 + 368\zeta_3^2 \\
 &\quad - \frac{14668}{9}\zeta_6 - 2b_{4,n_f C_F^3}^q \\
 &= -9.626 \pm 0.010, \\
 f_{4,n_f C_F^2 C_A}^q &= -\frac{1092511}{972} + \frac{673}{27}\zeta_2 - \frac{23518}{81}\zeta_3 + \frac{52744}{27}\zeta_4 + \frac{3904}{9}\zeta_2\zeta_3 - \frac{4472}{3}\zeta_5 \\
 &\quad - \frac{3400}{3}\zeta_3^2 + 718\zeta_6 - 2b_{4,n_f C_F^2 C_A}^q \\
 &= -2.351 \pm 0.010.
 \end{aligned}
 \tag{5.1}$$

Their numerical values are in agreement with those quoted in eq. (3.26). The remaining uncertainties in eq. (5.1) are due to the numerical values in table 1. We have verified that the large- n_c results for $n_f = 3, 4$ or 5 at four loops in eq. (3.24) reproduce full QCD within $\mathcal{O}(1\%)$.

This new information in eq. (5.1) leads to the following numerical result for the coefficient proportional to \mathcal{D}_0 in the DIS Wilson coefficients $c_{a,q}^{(4)}$ in eq. (3.34),

$$\begin{aligned}
 c_{2,q}^{(4)} \Big|_{\mathcal{D}_0} &= (3.874 \pm 0.010) \cdot 10^4 + (-3.4965 \pm 0.0287) \cdot 10^4 n_f + 2062.715 n_f^2 \\
 &\quad - 12.08488 n_f^3 + 47.55183 n_f fl_{11},
 \end{aligned}
 \tag{5.2}$$

where all exact values have been rounded to seven digits. The remaining errors come either from the uncertainty for $f_{4,d_{FA}^{(4)}}^q$ in eq. (3.26) (term proportional to n_f^0) or from the numerical values in table 1 (term proportional to n_f^1). The new numerical value in eq. (5.2) agrees very well with the best estimate in eq. (3.34). This also corroborates the arguments based on the large- n_c limit.

With the full colour dependence in eq. (5.2) at hand, the resummation coefficient B_4^{DIS} in eq. (3.43) then becomes

$$B_4^{\text{DIS}} = (10.68 \pm 0.01) \cdot 10^4 + (-2.0275549 \pm 0.0000029) \cdot 10^4 n_f + 798.0698 n_f^2 - 12.08488 n_f^3, \quad (5.3)$$

where the uncertainty in the term proportional to n_f^1 stems from the numerical values in table 1. Again, this agrees very well with the previous best estimate in eq. (3.44) based on the large- n_c limit. The perturbative expansion of B^{DIS} through four loops (replacing eq. (3.45)) reads then

$$\begin{aligned} B^{\text{DIS}}(n_f=3) &= -0.31831 \alpha_s (1 - 1.1004 \alpha_s - 3.623 \alpha_s^2 - (6.659 \pm 0.013) \alpha_s^3 + \dots), \\ B^{\text{DIS}}(n_f=4) &= -0.31831 \alpha_s (1 - 1.2267 \alpha_s - 3.405 \alpha_s^2 - (4.753 \pm 0.013) \alpha_s^3 + \dots), \\ B^{\text{DIS}}(n_f=5) &= -0.31831 \alpha_s (1 - 1.3530 \alpha_s - 3.190 \alpha_s^2 - (3.010 \pm 0.013) \alpha_s^3 + \dots), \end{aligned} \quad (5.4)$$

which eliminates any residual uncertainty in the resummation at N⁴LL accuracy.

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A DIS Wilson coefficients

For convenience of the reader we have collected all available information on the soft-virtual expressions for the DIS coefficient functions $c_{2,q}^{(n)}$ through four loops ($n \leq 4$). The leading order is $c_{2,q}^{(0)} = \delta(1-x)$ and the one-loop result $c_{2,q}^{(1)}$ in eq. (A.1) has been obtained in [73]. The two- and three-loop expressions $c_{2,q}^{(2)}$ and $c_{2,q}^{(3)}$ in eqs. (A.2), (A.3) are due to [74, 75] and [1], respectively. At four loops all plus-distributions \mathcal{D}_k with $k \geq 2$ in $c_{2,q}^{(4)}$ have been given in [8]. The terms \mathcal{D}_1 with the complete colour dependence of A_4 inserted and \mathcal{D}_0 in $c_{2,q}^{(4)}$ are new results. Analytical results for the coefficients $f_{4,n_f C_F^3}^q$, $f_{4,n_f C_F^2 C_A}^q$ and $f_{4,d_{FA}^{(4)}}^q$ can be expected from the computation of the form factor at four loops soon. The remaining coefficients $b_{4,C_F^4}^q$, $b_{4,C_F^3 C_A}^q$, $b_{4,C_F^2 C_A^2}^q$, $b_{4,n_f C_F^3}^q$, $b_{4,n_f C_F^2 C_A}^q$, $b_{4,d_{FA}^{(4)}}^q$, and $b_{4,d_{FF}^{(4)}}^q$ of the virtual anomalous dimension B_4^q require an analytical result for the four-loop splitting function, which is not easily available. The factors fl_{11} in eqs. (A.3), (A.4) do not appear in charged-current DIS.

$$c_{2,q}^{(1)} = 4C_F \mathcal{D}_1 - 3C_F \mathcal{D}_0 - C_F (9 + 4\zeta_2) \delta(1-x), \quad (A.1)$$

$$\begin{aligned}
 c_{2,q}^{(2)} = & 8C_F^2\mathcal{D}_3 - \mathcal{D}_2 \left\{ 18C_F^2 + \frac{22}{3}C_FC_A - \frac{4}{3}C_Fn_f \right\} + \mathcal{D}_1 \left\{ -C_F^2(27 + 32\zeta_2) \right. \\
 & + C_FC_A \left(\frac{367}{9} - 8\zeta_2 \right) - \frac{58}{9}C_Fn_f \left. \right\} + \mathcal{D}_0 \left\{ C_F^2 \left(\frac{51}{2} + 36\zeta_2 - 8\zeta_3 \right) \right. \\
 & - C_FC_A \left(\frac{3155}{54} - \frac{44}{3}\zeta_2 - 40\zeta_3 \right) + C_Fn_f \left(\frac{247}{27} - \frac{8}{3}\zeta_2 \right) \left. \right\} + \delta(1-x) \left\{ C_F^2 \left(\frac{331}{8} \right. \right. \\
 & + 69\zeta_2 - 78\zeta_3 + 6\zeta_2^2 \left. \right) - C_FC_A \left(\frac{5465}{72} + \frac{251}{3}\zeta_2 - \frac{140}{3}\zeta_3 - \frac{71}{5}\zeta_2^2 \right) \\
 & \left. \left. + C_Fn_f \left(\frac{457}{36} + \frac{38}{3}\zeta_2 + \frac{4}{3}\zeta_3 \right) \right\}, \tag{A.2}
 \end{aligned}$$

$$\begin{aligned}
 c_{2,q}^{(3)} = & 8C_F^3\mathcal{D}_5 + \mathcal{D}_4 \left\{ -30C_F^3 - \frac{220}{9}C_F^2C_A + \frac{40}{9}C_F^2n_f \right\} + \mathcal{D}_3 \left\{ -C_F^3(36 + 96\zeta_2) \right. \\
 & + C_F^2C_A \left(\frac{1732}{9} - 32\zeta_2 \right) + \frac{484}{27}C_FC_A^2 - \frac{280}{9}C_F^2n_f - \frac{176}{27}C_FC_An_f + \frac{16}{27}C_Fn_f^2 \left. \right\} \\
 & + \mathcal{D}_2 \left\{ C_F^3 \left(\frac{279}{2} + 288\zeta_2 + 16\zeta_3 \right) - C_F^2C_A \left(\frac{8425}{18} - \frac{724}{3}\zeta_2 - 240\zeta_3 \right) \right. \\
 & - C_FC_A^2 \left(\frac{4649}{27} - \frac{88}{3}\zeta_2 \right) + C_F^2n_f \left(\frac{683}{9} - \frac{112}{3}\zeta_2 \right) + C_FC_An_f \left(\frac{1552}{27} - \frac{16}{3}\zeta_2 \right) \\
 & - \frac{116}{27}C_Fn_f^2 \left. \right\} + \mathcal{D}_1 \left\{ C_F^3 \left(\frac{187}{2} + 240\zeta_2 - 360\zeta_3 + \frac{376}{5}\zeta_2^2 \right) - C_F^2C_A \left(\frac{5563}{18} \right. \right. \\
 & + 972\zeta_2 + \frac{160}{3}\zeta_3 - \frac{764}{5}\zeta_2^2 \left. \right) + C_FC_A^2 \left(\frac{50689}{81} - \frac{680}{3}\zeta_2 - 264\zeta_3 + \frac{176}{5}\zeta_2^2 \right) \\
 & + C_F^2n_f \left(\frac{83}{9} + 168\zeta_2 + \frac{112}{3}\zeta_3 \right) - C_FC_An_f \left(\frac{15062}{81} - \frac{512}{9}\zeta_2 - 16\zeta_3 \right) \\
 & + C_Fn_f^2 \left(\frac{940}{81} - \frac{32}{9}\zeta_2 \right) \left. \right\} + \mathcal{D}_0 \left\{ -C_F^3 \left(\frac{1001}{8} + 429\zeta_2 - 274\zeta_3 + 210\zeta_2^2 - 32\zeta_2\zeta_3 \right. \right. \\
 & - 432\zeta_5 \left. \right) + C_F^2C_A \left(\frac{16981}{24} + \frac{26885}{27}\zeta_2 - \frac{3304}{9}\zeta_3 - 209\zeta_2^2 - 400\zeta_2\zeta_3 - 120\zeta_5 \right) \\
 & - C_FC_A^2 \left(\frac{599375}{729} - \frac{32126}{81}\zeta_2 - \frac{21032}{27}\zeta_3 + \frac{652}{15}\zeta_2^2 + \frac{176}{3}\zeta_2\zeta_3 + 232\zeta_5 \right) \\
 & - C_F^2n_f \left(\frac{2003}{108} + \frac{4226}{27}\zeta_2 + 60\zeta_3 - 16\zeta_2^2 \right) + C_FC_An_f \left(\frac{160906}{729} - \frac{776}{9}\zeta_3 - \frac{9920}{81}\zeta_2 \right. \\
 & + \frac{208}{15}\zeta_2^2 \left. \right) - C_Fn_f^2 \left(\frac{8714}{729} - \frac{232}{27}\zeta_2 + \frac{32}{27}\zeta_3 \right) \left. \right\} + \delta(1-x) \left\{ -C_F^3 \left(\frac{7255}{24} + \frac{3379}{6}\zeta_2 \right. \right. \\
 & + 318\zeta_3 + \frac{2148}{5}\zeta_2^2 - 808\zeta_2\zeta_3 - 1240\zeta_5 + \frac{304}{3}\zeta_3^2 - \frac{4184}{315}\zeta_2^3 \left. \right) + C_F^2C_A \left(\frac{9161}{12} \right. \\
 & + \frac{104117}{54}\zeta_2 - \frac{6419}{3}\zeta_3 + \frac{87632}{135}\zeta_2^2 - \frac{6644}{9}\zeta_2\zeta_3 - \frac{4952}{9}\zeta_5 + \frac{1016}{3}\zeta_3^2 - \frac{33556}{315}\zeta_2^3 \left. \right) \\
 & - C_FC_A^2 \left(\frac{1909753}{1944} + \frac{143255}{81}\zeta_2 - \frac{105712}{81}\zeta_3 - \frac{25184}{135}\zeta_2^2 - 540\zeta_2\zeta_3 + \frac{416}{3}\zeta_5 \right. \\
 & \left. \left. + \frac{248}{3}\zeta_3^2 + \frac{3512}{63}\zeta_2^3 \right) - C_F^2n_f \left(\frac{341}{36} + \frac{5491}{27}\zeta_2 - \frac{1348}{3}\zeta_3 + \frac{16472}{135}\zeta_2^2 + \frac{352}{9}\zeta_2\zeta_3 \right)
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{592}{9} \zeta_5) + C_F C_A n_f \left(\frac{142883}{486} + \frac{40862}{81} \zeta_2 - \frac{18314}{81} \zeta_3 - \frac{2488}{135} \zeta_2^2 - \frac{56}{3} \zeta_2 \zeta_3 + \frac{8}{3} \zeta_5 \right) \\
 & - C_F n_f^2 \left(\frac{9517}{486} + \frac{860}{27} \zeta_2 + \frac{152}{81} \zeta_3 + \frac{32}{27} \zeta_2^2 \right) + fl_{11} n_f \frac{d^{abc} d_{abc}}{n_c} \left(64 + 160 \zeta_2 + \frac{224}{3} \zeta_3 \right. \\
 & \left. - \frac{32}{5} \zeta_2^2 - \frac{1280}{3} \zeta_5 \right) \Big\}, \tag{A.3}
 \end{aligned}$$

$$\begin{aligned}
 c_{2,q}^{(4)} = & \frac{16}{3} C_F^4 \mathcal{D}_7 + \mathcal{D}_6 \left\{ -28 C_F^4 - \frac{308}{9} C_F^3 C_A + \frac{56}{9} C_F^3 n_f \right\} + \mathcal{D}_5 \left\{ -C_F^4 (18 + 128 \zeta_2) \right. \\
 & + C_F^3 C_A \left(\frac{998}{3} - 48 \zeta_2 \right) + \frac{1936}{27} C_F^2 C_A^2 - \frac{164}{3} C_F^3 n_f - \frac{704}{27} C_F^2 C_A n_f + \frac{64}{27} C_F^2 n_f^2 \Big\} \\
 & + \mathcal{D}_4 \left\{ C_F^4 \left(210 + 600 \zeta_2 + \frac{400}{3} \zeta_3 \right) - C_F^3 C_A \left(\frac{27835}{27} - \frac{6800}{9} \zeta_2 - 400 \zeta_3 \right) \right. \\
 & - C_F^2 C_A^2 \left(\frac{24040}{27} - \frac{440}{3} \zeta_2 \right) - \frac{1331}{27} C_F C_A^3 + C_F^3 n_f \left(\frac{4630}{27} - \frac{1040}{9} \zeta_2 \right) \\
 & \left. + C_F^2 C_A n_f \left(\frac{8120}{27} - \frac{80}{3} \zeta_2 \right) + \frac{242}{9} C_F C_A^2 n_f - \frac{640}{27} C_F^2 n_f^2 - \frac{44}{9} C_F C_A n_f^2 + \frac{8}{27} C_F n_f^3 \right\} \\
 & + \mathcal{D}_3 \left\{ C_F^4 \left(\frac{113}{2} + 264 \zeta_2 - 1072 \zeta_3 + \frac{1392}{5} \zeta_2^2 \right) - C_F^3 C_A \left(\frac{1534}{3} + \frac{41824}{9} \zeta_2 \right. \right. \\
 & \left. \left. + \frac{8800}{9} \zeta_3 - \frac{3128}{5} \zeta_2^2 \right) + C_F^2 C_A^2 \left(\frac{2154563}{486} - \frac{52912}{27} \zeta_2 - \frac{13024}{9} \zeta_3 + \frac{864}{5} \zeta_2^2 \right) \right. \\
 & + C_F C_A^3 \left(\frac{55627}{81} - \frac{968}{9} \zeta_2 \right) - C_F^2 C_A n_f \left(\frac{339134}{243} - \frac{14096}{27} \zeta_2 - \frac{1216}{9} \zeta_3 \right) \\
 & - C_F^3 n_f \left(\frac{280}{3} - \frac{7216}{9} \zeta_2 - \frac{1888}{9} \zeta_3 \right) - C_F C_A^2 n_f \left(\frac{9502}{27} - \frac{352}{9} \zeta_2 \right) \\
 & \left. + C_F^2 n_f^2 \left(\frac{24238}{243} - \frac{928}{27} \zeta_2 \right) + C_F C_A n_f^2 \left(\frac{1540}{27} - \frac{32}{9} \zeta_2 \right) - \frac{232}{81} C_F n_f^3 \right\} \\
 & + \mathcal{D}_2 \left\{ -C_F^4 \left(\frac{1299}{2} + 2808 \zeta_2 - 1392 \zeta_3 + 1836 \zeta_2^2 + 640 \zeta_2 \zeta_3 - 4128 \zeta_5 \right) \right. \\
 & + C_F^3 C_A \left(\frac{13990}{3} + \frac{30704}{3} \zeta_2 + \frac{2716}{3} \zeta_3 - \frac{12906}{5} \zeta_2^2 - 3648 \zeta_2 \zeta_3 - 720 \zeta_5 \right) \\
 & - C_F^2 C_A^2 \left(\frac{2254339}{243} - \frac{86804}{9} \zeta_2 - \frac{24544}{3} \zeta_3 + \frac{4034}{3} \zeta_2^2 + 832 \zeta_2 \zeta_3 + 1392 \zeta_5 \right) \\
 & - C_F C_A^3 \left(\frac{649589}{162} - \frac{4012}{3} \zeta_2 - 1452 \zeta_3 + \frac{968}{5} \zeta_2^2 \right) - C_F^3 n_f \left(\frac{145}{9} + \frac{5132}{3} \zeta_2 + 936 \zeta_3 \right. \\
 & \left. - \frac{1032}{5} \zeta_2^2 \right) + C_F^2 C_A n_f \left(\frac{713162}{243} - \frac{10600}{9} \zeta_3 - \frac{82004}{27} \zeta_2 + \frac{3772}{15} \zeta_2^2 \right) \\
 & + C_F C_A^2 n_f \left(\frac{17189}{9} - \frac{5096}{9} \zeta_2 - 352 \zeta_3 + \frac{176}{5} \zeta_2^2 \right) - C_F^2 n_f^2 \left(\frac{52678}{243} - \frac{6104}{27} \zeta_2 \right. \\
 & \left. - \frac{304}{9} \zeta_3 \right) - C_F C_A n_f^2 \left(\frac{7403}{27} - \frac{688}{9} \zeta_2 - 16 \zeta_3 \right) + C_F n_f^3 \left(\frac{940}{81} - \frac{32}{9} \zeta_2 \right) \Big\} \\
 & + \mathcal{D}_1 \left\{ -C_F^4 \left(\frac{19885}{24} + \frac{3101}{3} \zeta_2 + 1374 \zeta_3 + 1098 \zeta_2^2 - 5824 \zeta_2 \zeta_3 - 64 \zeta_5 + \frac{256}{3} \zeta_3^2 \right) \right.
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{64192}{315} \zeta_2^3 \Big) + C_F^3 C_A \left(\frac{2829}{2} + \frac{202705}{27} \zeta_2 - \frac{333526}{27} \zeta_3 + \frac{1043747}{135} \zeta_2^2 + \frac{23104}{9} \zeta_2 \zeta_3 \right. \\
 & - \frac{77216}{9} \zeta_5 + \frac{5984}{3} \zeta_3^2 - \frac{55472}{63} \zeta_2^3 \Big) - C_F^2 C_A^2 \left(\frac{21987073}{5832} + \frac{713440}{27} \zeta_2 + \frac{238400}{81} \zeta_3 \right. \\
 & - \frac{812819}{135} \zeta_2^2 - \frac{24896}{3} \zeta_2 \zeta_3 - \frac{6472}{3} \zeta_5 - \frac{3808}{3} \zeta_3^2 + \frac{51848}{63} \zeta_2^3 \Big) + C_F C_A^3 \left(\frac{16865531}{1458} \right. \\
 & - \frac{156238}{27} \zeta_2 - \frac{78296}{9} \zeta_3 + \frac{17996}{15} \zeta_2^2 + 528 \zeta_2 \zeta_3 + \frac{19360}{9} \zeta_5 - 16 \zeta_3^2 - \frac{20032}{105} \zeta_2^3 \Big) \\
 & + \frac{d_F^{abcd} d_A^{abcd}}{n_F} \left(-128 \zeta_2 + \frac{128}{3} \zeta_3 + \frac{3520}{3} \zeta_5 - 384 \zeta_3^2 - \frac{7936}{35} \zeta_2^3 \right) \\
 & + C_F^3 n_f \left(\frac{1775}{18} - \frac{7912}{27} \zeta_2 + \frac{78328}{27} \zeta_3 - \frac{195716}{135} \zeta_2^2 + \frac{6368}{9} \zeta_5 - \frac{6592}{9} \zeta_2 \zeta_3 \right) \\
 & - C_F C_A^2 n_f \left(\frac{2452247}{486} - \frac{68836}{27} \zeta_2 - \frac{5752}{3} \zeta_3 + \frac{3944}{15} \zeta_2^2 - 32 \zeta_2 \zeta_3 + \frac{2080}{9} \zeta_5 \right) \\
 & - C_F^2 C_A n_f \left(\frac{326570}{729} - \frac{694654}{81} \zeta_2 - \frac{158740}{81} \zeta_3 + \frac{178642}{135} \zeta_2^2 + \frac{2848}{3} \zeta_2 \zeta_3 + \frac{208}{3} \zeta_5 \right) \\
 & + n_f \frac{d_F^{abcd} d_A^{abcd}}{n_F} \left(256 \zeta_2 - \frac{256}{3} \zeta_3 - \frac{1280}{3} \zeta_5 \right) + C_F^2 n_f^2 \left(\frac{239633}{1458} - \frac{50140}{81} \zeta_2 - \frac{19304}{81} \zeta_3 \right. \\
 & + \frac{1312}{27} \zeta_2^2 \Big) + C_F C_A n_f^2 \left(\frac{315755}{486} - \frac{9848}{27} \zeta_2 - \frac{688}{9} \zeta_3 + \frac{64}{5} \zeta_2^2 \right) - C_F n_f^3 \left(\frac{17716}{729} \right. \\
 & - \frac{464}{27} \zeta_2 \Big) + fl_{11} C_F n_f \frac{d^{abc} d_{abc}}{n_c} \left(256 + 640 \zeta_2 + \frac{896}{3} \zeta_3 - \frac{128}{5} \zeta_2^2 - \frac{5120}{3} \zeta_5 \right) \Big\} \\
 & + \mathcal{D}_0 \left\{ C_F^4 \left(\frac{15605}{16} + 2803 \zeta_2 + 1064 \zeta_3 + \frac{19839}{5} \zeta_2^2 - 5168 \zeta_2 \zeta_3 - 6744 \zeta_5 + 736 \zeta_3^2 \right. \right. \\
 & + \frac{17624}{21} \zeta_2^3 - 4992 \zeta_2 \zeta_5 - \frac{752}{5} \zeta_2^2 \zeta_3 + 3840 \zeta_7 - b_{4, C_F^4}^q \Big) - C_F^3 C_A \left(\frac{137084}{27} \right. \\
 & + 16683 \zeta_2 - \frac{27860}{3} \zeta_3 + \frac{1975109}{270} \zeta_2^2 - \frac{44332}{9} \zeta_2 \zeta_3 - \frac{29240}{3} \zeta_5 + \frac{12008}{3} \zeta_3^2 - \frac{906268}{315} \zeta_2^3 \\
 & + 192 \zeta_2 \zeta_5 - \frac{7992}{5} \zeta_2^2 \zeta_3 + b_{4, C_F^3 C_A}^q \Big) + C_F^2 C_A^2 \left(\frac{54699043}{3888} + \frac{16063075}{729} \zeta_2 \right. \\
 & - \frac{6500}{3} \zeta_3 - \frac{7771513}{810} \zeta_2^2 - \frac{456400}{27} \zeta_2 \zeta_3 + \frac{20248}{9} \zeta_5 - \frac{1136}{3} \zeta_3^2 + \frac{17228}{63} \zeta_2^3 \\
 & + 1856 \zeta_2 \zeta_5 + \frac{25544}{15} \zeta_2^2 \zeta_3 - b_{4, C_F^2 C_A^2}^q \Big) - C_F C_A^3 \left(\frac{1958802125}{139968} - \frac{2213182}{243} \zeta_2 \right. \\
 & - \frac{676939}{54} \zeta_3 + \frac{198203}{135} \zeta_2^2 + \frac{38738}{9} \zeta_2 \zeta_3 + \frac{40406}{9} \zeta_5 + 1340 \zeta_3^2 - \frac{43532}{105} \zeta_2^3 - 496 \zeta_2 \zeta_5 \\
 & - \frac{1376}{5} \zeta_2^2 \zeta_3 - 2260 \zeta_7 - \frac{1}{2} b_{4, C_F^2 C_A^2}^q - \frac{1}{4} b_{4, C_F^3 C_A}^q - \frac{1}{8} b_{4, C_F^4}^q - \frac{1}{24} f_{4, d_{FA}^{(4)}}^q \\
 & - \frac{1}{24} b_{4, d_{FA}^{(4)}}^q \Big) - \frac{d_F^{abcd} d_A^{abcd}}{n_F} \left(f_{4, d_{FA}^{(4)}}^q + b_{4, d_{FA}^{(4)}}^q \right) - C_F^3 n_f \left(\frac{39775}{108} \right. \\
 & - \frac{38278}{27} \zeta_2 + 1518 \zeta_3 - \frac{37502}{27} \zeta_2^2 - \frac{9664}{9} \zeta_2 \zeta_3 + \frac{2816}{3} \zeta_5 + \frac{64}{3} \zeta_3^2 + \frac{93232}{315} \zeta_2^3
 \end{aligned}$$

$$\begin{aligned}
& + f_{4,n_f C_F^3}^q + b_{4,n_f C_F^3}^q \Big) - C_F^2 C_A n_f \left(\frac{859349}{486} + \frac{5245922}{729} \zeta_2 + \frac{191764}{81} \zeta_3 \right. \\
& - \frac{1064717}{405} \zeta_2^2 - 2576 \zeta_2 \zeta_3 + \frac{1000}{3} \zeta_5 - \frac{752}{3} \zeta_3^2 + \frac{2152}{45} \zeta_2^3 + f_{4,n_f C_F^2 C_A}^q + b_{4,n_f C_F^2 C_A}^q \Big) \\
& + C_F C_A^2 n_f \left(\frac{68301461}{11664} - \frac{1935001}{486} \zeta_2 - \frac{254678}{81} \zeta_3 + \frac{66211}{135} \zeta_2^2 + \frac{8272}{9} \zeta_2 \zeta_3 - \frac{772}{27} \zeta_5 \right. \\
& + \frac{364}{9} \zeta_3^2 - \frac{75512}{945} \zeta_2^3 + \frac{1}{2} f_{4,n_f C_F^2 C_A}^q + \frac{1}{2} b_{4,n_f C_F^2 C_A}^q + \frac{1}{4} f_{4,n_f C_F^3}^q + \frac{1}{4} b_{4,n_f C_F^3}^q \\
& - \frac{1}{48} b_{4,d_{FF}^{(4)}}^q \Big) + n_f \frac{d^{abcd} d^{abcd}}{n_F} \left(384 - \frac{4544}{3} \zeta_2 + \frac{5312}{9} \zeta_3 + \frac{320}{3} \zeta_2^2 - 128 \zeta_2 \zeta_3 \right. \\
& + \frac{21760}{9} \zeta_5 - \frac{9472}{315} \zeta_2^3 - \frac{1216}{3} \zeta_3^2 + b_{4,d_{FF}^{(4)}}^q \Big) - C_F^2 n_f^2 \left(\frac{161929}{972} - \frac{385300}{729} \zeta_2 \right. \\
& - \frac{3812}{9} \zeta_3 + \frac{19904}{135} \zeta_2^2 + \frac{1376}{27} \zeta_2 \zeta_3 + \frac{64}{9} \zeta_5 \Big) - C_F C_A n_f^2 \left(\frac{3761509}{5832} - \frac{131878}{243} \zeta_2 \right. \\
& - \frac{6092}{81} \zeta_3 + \frac{616}{9} \zeta_2^2 + \frac{400}{9} \zeta_2 \zeta_3 - \frac{1192}{9} \zeta_5 \Big) + C_F n_f^3 \left(\frac{50558}{2187} - \frac{1880}{81} \zeta_2 + \frac{80}{81} \zeta_3 \right. \\
& + \frac{16}{9} \zeta_2^2 \Big) - fl_{11} C_F n_f \frac{d^{abc} d_{abc}}{n_c} \left(192 + 480 \zeta_2 + 224 \zeta_3 - \frac{96}{5} \zeta_2^2 - 1280 \zeta_5 \right) \Big\} \\
& + \delta(1-x) c_{4,\delta,\text{DIS}}.
\end{aligned} \tag{A.4}$$

The coefficient $c_{4,\delta,\text{DIS}}$ of $\delta(1-x)$ in eq. (A.4) is currently unknown. However, we estimate the size of these coefficients for fixed values of n_f from the knowledge of moments of the DIS structure functions [30],

$$\begin{aligned}
c_{4,\delta,\text{DIS}}(n_f = 3) &= (-13.5 \pm 3) \cdot 10^4, \\
c_{4,\delta,\text{DIS}}(n_f = 4) &= (-6.0 \pm 3) \cdot 10^4, \\
c_{4,\delta,\text{DIS}}(n_f = 5) &= (-0.5 \pm 3) \cdot 10^4.
\end{aligned} \tag{A.5}$$

These predictions have been used to extract the g_{04}^{DIS} coefficients quoted in eq. (4.2).

B g_0 coefficients

Here we collect the g_0 coefficients that appear in eq. (2.1) for both N -exponentiation and \tilde{N} -exponentiation. Note that the g_0 coefficients have the following perturbative expansion:

$$g_0 = 1 + \sum_{n=1}^{\infty} a_s^n g_{0n}. \tag{B.1}$$

For \tilde{N} -exponentiation the g_0 coefficients (denoted as \tilde{g}_0) read as

$$\tilde{g}_{01} = C_F \left\{ -2\zeta_2 - 9 \right\}, \quad (\text{B.2})$$

$$\begin{aligned} \tilde{g}_{02} = & C_A C_F \left\{ \frac{51}{5} \zeta_2^2 - \frac{1139}{18} \zeta_2 + \frac{464}{9} \zeta_3 - \frac{5465}{72} \right\} + C_F^2 \left\{ \frac{4}{5} \zeta_2^2 + \frac{111}{2} \zeta_2 - 66 \zeta_3 + \frac{331}{8} \right\} \\ & + C_F n_f \left\{ \frac{85}{9} \zeta_2 + \frac{4}{9} \zeta_3 + \frac{457}{36} \right\}, \end{aligned} \quad (\text{B.3})$$

$$\begin{aligned} \tilde{g}_{03} = & C_A^2 C_F \left\{ -\frac{12016}{315} \zeta_2^3 + \frac{13151}{135} \zeta_2^2 + \frac{3496}{9} \zeta_2 \zeta_3 - \frac{78607}{54} \zeta_2 - \frac{248}{3} \zeta_3^2 + \frac{115010}{81} \zeta_3 \right. \\ & \left. - \frac{416}{3} \zeta_5 - \frac{1909753}{1944} \right\} + C_A C_F^2 \left\{ -\frac{23098}{315} \zeta_2^3 + \frac{11419}{27} \zeta_2^2 - 828 \zeta_2 \zeta_3 + \frac{191545}{108} \zeta_2 \right. \\ & \left. + \frac{536}{3} \zeta_3^2 - \frac{49346}{27} \zeta_3 - \frac{3896}{9} \zeta_5 + \frac{9161}{12} \right\} + C_A C_F n_f \left\{ \frac{164}{135} \zeta_2^2 - \frac{64}{9} \zeta_2 \zeta_3 + \frac{33331}{81} \zeta_2 \right. \\ & \left. - \frac{21418}{81} \zeta_3 + \frac{8}{3} \zeta_5 + \frac{142883}{486} \right\} + C_F^3 \left\{ \frac{8144}{315} \zeta_2^3 - \frac{1791}{5} \zeta_2^2 + 556 \zeta_2 \zeta_3 - \frac{6197}{12} \zeta_2 \right. \\ & \left. - \frac{176}{3} \zeta_3^2 - 411 \zeta_3 + 1384 \zeta_5 - \frac{7255}{24} \right\} + C_F^2 n_f \left\{ -\frac{10802}{135} \zeta_2^2 - \frac{40}{3} \zeta_2 \zeta_3 - \frac{10733}{54} \zeta_2 \right. \\ & \left. + \frac{10766}{27} \zeta_3 - \frac{784}{9} \zeta_5 - \frac{341}{36} \right\} + C_F n_f^2 \left\{ -\frac{292}{135} \zeta_2^2 - \frac{2110}{81} \zeta_2 + \frac{80}{81} \zeta_3 - \frac{9517}{486} \right\} \\ & + n_f f l_{11} \frac{d_{abc} d^{abc}}{n_c} \left\{ -\frac{32}{5} \zeta_2^2 + 160 \zeta_2 + \frac{224}{3} \zeta_3 - \frac{1280}{3} \zeta_5 + 64 \right\}. \end{aligned} \quad (\text{B.4})$$

For N -exponentiation, the g_0 coefficients are given as below,

$$g_{01} = C_F \left\{ -2\zeta_2 + 2\gamma_E^2 + 3\gamma_E - 9 \right\}, \quad (\text{B.5})$$

$$\begin{aligned} g_{02} = & C_A C_F \left\{ \frac{51}{5} \zeta_2^2 - 4\zeta_2 \gamma_E^2 - \frac{22}{3} \zeta_2 \gamma_E - \frac{1139}{18} \zeta_2 - 40 \zeta_3 \gamma_E + \frac{464}{9} \zeta_3 + \frac{22}{9} \gamma_E^3 + \frac{367}{18} \gamma_E^2 \right. \\ & \left. + \frac{3155}{54} \gamma_E - \frac{5465}{72} \right\} + C_F^2 \left\{ \frac{4}{5} \zeta_2^2 - 4\zeta_2 \gamma_E^2 - 18 \zeta_2 \gamma_E + \frac{111}{2} \zeta_2 + 24 \zeta_3 \gamma_E - 66 \zeta_3 \right. \\ & \left. + 2\gamma_E^4 + 6\gamma_E^3 - \frac{27}{2} \gamma_E^2 - \frac{51}{2} \gamma_E + \frac{331}{8} \right\} + C_F n_f \left\{ \frac{4}{3} \zeta_2 \gamma_E + \frac{85}{9} \zeta_2 + \frac{4}{9} \zeta_3 - \frac{4}{9} \gamma_E^3 \right. \\ & \left. - \frac{29}{9} \gamma_E^2 - \frac{247}{27} \gamma_E + \frac{457}{36} \right\}, \end{aligned} \quad (\text{B.6})$$

$$\begin{aligned} g_{03} = & C_A^2 C_F \left\{ -\frac{12016}{315} \zeta_2^3 + \frac{88}{5} \zeta_2^2 \gamma_E^2 + \frac{212}{15} \zeta_2^2 \gamma_E + \frac{13151}{135} \zeta_2^2 + \frac{176}{3} \zeta_2 \zeta_3 \gamma_E + \frac{3496}{9} \zeta_2 \zeta_3 \right. \\ & \left. - \frac{88}{9} \zeta_2 \gamma_E^3 - \frac{778}{9} \zeta_2 \gamma_E^2 - \frac{18179}{81} \zeta_2 \gamma_E - \frac{78607}{54} \zeta_2 - \frac{248}{3} \zeta_3^2 - 132 \zeta_3 \gamma_E^2 - \frac{6688}{9} \zeta_3 \gamma_E \right. \\ & \left. + \frac{115010}{81} \zeta_3 + 232 \zeta_5 \gamma_E - \frac{416}{3} \zeta_5 + \frac{121}{27} \gamma_E^4 + \frac{4649}{81} \gamma_E^3 + \frac{50689}{162} \gamma_E^2 + \frac{599375}{729} \gamma_E \right. \\ & \left. - \frac{1909753}{1944} \right\} + C_A C_F^2 \left\{ -\frac{23098}{315} \zeta_2^3 + \frac{142}{5} \zeta_2^2 \gamma_E^2 + \frac{299}{3} \zeta_2^2 \gamma_E + \frac{11419}{27} \zeta_2^2 \right. \\ & \left. + 96 \zeta_2 \zeta_3 \gamma_E - 828 \zeta_2 \zeta_3 - 8 \zeta_2 \gamma_E^4 - \frac{284}{9} \zeta_2 \gamma_E^3 - \frac{592}{3} \zeta_2 \gamma_E^2 - \frac{28495}{54} \zeta_2 \gamma_E + \frac{191545}{108} \zeta_2 \right. \end{aligned}$$

$$\begin{aligned}
& + \frac{536}{3}\zeta_3^2 - 80\zeta_3\gamma_E^3 + \frac{640}{9}\zeta_3\gamma_E^2 + 752\zeta_3\gamma_E - \frac{49346}{27}\zeta_3 + 120\zeta_5\gamma_E - \frac{3896}{9}\zeta_5 \\
& + \frac{44}{9}\gamma_E^5 + \frac{433}{9}\gamma_E^4 + \frac{8425}{54}\gamma_E^3 - \frac{5563}{36}\gamma_E^2 - \frac{16981}{24}\gamma_E + \frac{9161}{12} \Big\} + C_A C_F n_f \Big\{ \\
& - \frac{128}{15}\zeta_2^2\gamma_E + \frac{164}{135}\zeta_2^2 - \frac{64}{9}\zeta_2\zeta_3 + \frac{16}{9}\zeta_2\gamma_E^3 + \frac{56}{3}\zeta_2\gamma_E^2 + \frac{5264}{81}\zeta_2\gamma_E + \frac{33331}{81}\zeta_2 \\
& + 8\zeta_3\gamma_E^2 + \frac{1976}{27}\zeta_3\gamma_E - \frac{21418}{81}\zeta_3 + \frac{8}{3}\zeta_5 - \frac{44}{27}\gamma_E^4 - \frac{1552}{81}\gamma_E^3 - \frac{7531}{81}\gamma_E^2 \\
& - \frac{160906}{729}\gamma_E + \frac{142883}{486} \Big\} + C_F^3 \Big\{ \frac{8144}{315}\zeta_2^3 + \frac{8}{5}\zeta_2^2\gamma_E^2 + 84\zeta_2^2\gamma_E - \frac{1791}{5}\zeta_2^2 \\
& - 80\zeta_2\zeta_3\gamma_E + 556\zeta_2\zeta_3 - 4\zeta_2\gamma_E^4 - 36\zeta_2\gamma_E^3 + 66\zeta_2\gamma_E^2 + \frac{579}{2}\zeta_2\gamma_E - \frac{6197}{12}\zeta_2 \\
& - \frac{176}{3}\zeta_3^2 + 48\zeta_3\gamma_E^3 - 60\zeta_3\gamma_E^2 - 346\zeta_3\gamma_E - 411\zeta_3 - 240\zeta_5\gamma_E + 1384\zeta_5 + \frac{4}{3}\gamma_E^6 \\
& + 6\gamma_E^5 - 9\gamma_E^4 - \frac{93}{2}\gamma_E^3 + \frac{187}{4}\gamma_E^2 + \frac{1001}{8}\gamma_E - \frac{7255}{24} \Big\} + C_F^2 n_f \Big\{ -\frac{8}{3}\zeta_2^2\gamma_E \\
& - \frac{10802}{135}\zeta_2^2 - \frac{40}{3}\zeta_2\zeta_3 + \frac{32}{9}\zeta_2\gamma_E^3 + \frac{112}{3}\zeta_2\gamma_E^2 + \frac{2177}{27}\zeta_2\gamma_E - \frac{10733}{54}\zeta_2 + \frac{8}{9}\zeta_3\gamma_E^2 \\
& - \frac{20}{9}\zeta_3\gamma_E + \frac{10766}{27}\zeta_3 - \frac{784}{9}\zeta_5 - \frac{8}{9}\gamma_E^5 - \frac{70}{9}\gamma_E^4 - \frac{683}{27}\gamma_E^3 + \frac{83}{18}\gamma_E^2 + \frac{2003}{108}\gamma_E \\
& - \frac{341}{36} \Big\} + C_F n_f^2 \Big\{ -\frac{292}{135}\zeta_2^2 - \frac{8}{9}\zeta_2\gamma_E^2 - \frac{116}{27}\zeta_2\gamma_E - \frac{2110}{81}\zeta_2 + \frac{64}{27}\zeta_3\gamma_E + \frac{80}{81}\zeta_3 \\
& + \frac{4}{27}\gamma_E^4 + \frac{116}{81}\gamma_E^3 + \frac{470}{81}\gamma_E^2 + \frac{8714}{729}\gamma_E - \frac{9517}{486} \Big\} + n_f f l_{11} \frac{d_{abc} d^{abc}}{n_c} \Big\{ -\frac{32}{5}\zeta_2^2 \\
& + 160\zeta_2 + \frac{224}{3}\zeta_3 - \frac{1280}{3}\zeta_5 + 64 \Big\}. \tag{B.7}
\end{aligned}$$

The resummed exponent for the N -exponentiation can be found for example in [8].

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