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T, Q and periods in SU(3) $\mathcal{N}=2$ SYM

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Abstract: We consider the third order differential equation derived from the deformed Seiberg-Witten differential for pure $\mathcal{N}=2$ SYM with gauge group SU(3) in Nekrasov-Shatashvili limit of Ω-background. We show that this is the same differential equation that emerges in the context of Ordinary Differential Equation/Integrable Models (ODE/IM) correspondence for $2d$ A_2 Toda CFT with central charge $c=98$. We derive the corresponding QQ and related TQ functional relations and establish the asymptotic behaviour of Q and T functions at small instanton parameter $q \to 0$. Moreover, numerical integration of the Floquet monodromy matrix of the differential equation leads to evaluation of the A-cycles $a_{1,2,3}$ at any point of the moduli space of vacua parametrized by the vector multiplet scalar VEVs $\langle \mathbf{tr}\,\phi^2 \rangle$ and $\langle \mathbf{tr}\,\phi^3 \rangle$ even for large values of q which are well beyond the reach of instanton calculus. The numerical results at small q are in excellent agreement with instanton calculation. We conjecture a very simple relation between Baxter's T-function and A-cycle periods $a_{1,2,3}$, which is an extension of Alexei Zamolodchikov's conjecture about Mathieu equation.

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Contents

Introduction. Ever since 1994 when Seiberg and Witten derived exact low-energy Wilsonian effective action of (pure) SU(2) $\mathcal{N} = 2$ SYM [1], the interest in this kind of theories has been remaining extremely high. The reason is their remarkably rich physical and mathematical content. In fact, these theories provide a framework to address in a precise manner such problems as strong coupling, non-perturbative effects and confinement in non-Abelian gauge theory (so relevant, for instance, in the Standard Model). The impact of Seiberg-Witten theory in pure mathematics is also very substantial. Likely, the most famous applications are in algebraic geometry and topology of four-dimensional differentiable manifolds where e.g. the notion of Seiberg-Witten [2] invariants is of primary importance.

The effective action of $\mathcal{N} = 2$ SYM is given in terms of prepotential: a holomorphic function of the vacuum expectation values (VEV) of the vector multiplet scalar field. Large VEV expansion of the prepotential reveals its structure as sum of classical, one-loop and instanton contributions. Many researchers tried to restore the instanton contributions directly from the microscopic theory, but they succeeded only in the case of the first few instantons [3]. Actual progress has been achieved with the idea of using equivariant localization techniques in the moduli space of instantons [4, 5], especially in combination with the introduction of the so-called Ω background (see [6] and further developments [7–9]). Considering theory in Ω-background effectively embeds the system in a finite volume $\sim \frac{1}{\epsilon_1\epsilon_2}$, where the parameters ϵ_1, ϵ_2 are sort of angular velocities on orthogonal planes of (Euclidean) 4d space-time. This makes the partition function a finite, well defined quantity (commonly referred as Nekrasov partition function). Then the corresponding free energy coincides with (generalized) prepotential. The usual SW prepotential is recovered simply by sending the parameters $\epsilon_{1,2} \to 0$. Another, crucial consequence of introducing this background is the fact that instanton moduli integrals are localized at finitely many points. This property eventually leads to an elegant combinatorial formula for instanton contributions [7].

Later developments are even more surprising. It appears that introduction of Ω background is not merely a regularization trick. Thus, keeping $\epsilon_{1,2}$ finite a deep relation between conformal blocks of 2d CFT and Nekrasov partition function [10] (see also [11] for the higher rank case) emerges, so that the Virasoro central charge is related to these parameters, the masses of hypermultiplets specify inserted primary fields, while VEVs identify the states of the intermediate channel.

The special case $\epsilon_1 = -\epsilon_2$ bridges the theory with topological string, ϵ-expansion of Nekrasov partition function coinciding with topological (string) genus expansion [6].

Another special case of great interest is the Nekrasov-Shatashvili (NS) limit [12] when one of parameters, say ϵ_1 is kept finite while $\epsilon_2 = 0$. From AGT point of view this case corresponds to semiclassical CFT when the central charge $c \to \infty$. Besides this, another interesting link to quantum integrable system emerges, now the remaining nonzero parameter ϵ_1 being related to Plank constant. In NS limit many quantities familiar from original Seiberg-Witten theory become deformed or quantized in rather simple manner.

In particular the algebraic equation defining Seiberg-Witten curve, becomes a finite difference equation [13], which in terms of related integrable system is nothing but Baxter's TQ equation (for an earlier approach see [14] and [15] for a later development). Through discrete Fourier transform one gets a linear differential equation [16], which from 2d CFT perspective is the null vector decoupling equation [17] in the semiclassical limit. This relation was an object of intensive investigations in the last decade (see e.g. [18–28]).

More recently, moving from Gaiotto's idea of looking at these equations as quantum versions of the (suitable power of the) SW differential [29],[1] it has been proposed to investigate their monodromies (quantum periods over cycles) through the connection (Stokes) multipliers appearing in the ODE/IM correspondence [30, 31]: [32] describes the general idea by exemplifying it in the simple case of pure SU(2) gauge theory[2] and in particular the link between the a-period and the Baxter transfer matrix T function. In this perspective,

[1] In other words, the SW differential gives way to the oper upon quantisation.

[2] Very few details are given for the cases with matter in the fundamental.

Thermodynamic Bethe Ansatz (TBA)-like considerations about pure SU(2) gauge theory were initiated in [33] at zero modulus (of the Coulomb branch) for the dual period a_D, and then more recently pursued by [34].

In fact, in this paper we show how to compute the gauge A-periods of the pure SU(3) theory (without any matter hypermultiplet: cf. [35] and [36] for what concerns the generalization of SW theory to higher rank gauge groups) as Floquet monodromy coefficients of the aforementioned differential equation (in the complex domain). Then, we propose a connexion between them and the integrable Baxter's T function which extends non trivially what happens in the SU(2) case and shows that the latter is not an accident. More in details, we obtain a third order linear differential equation with some similarities (and differences[3]) with the third order oper of ODE/IM correspondence in [37, 38]. As the latter correspond to some 'minimal' case $M > 1/2$, we may conjecture, along the lines of [39] and [40], that we are describing A_2-Toda CFT with central charge $c = 98$.

In a very interesting unfinished paper [40] Alexei Zamolodchikov has proposed ODE/IM for the Liouville CFT TBA. Special attention has been payed to the self-dual case $c = 25$, when the related ODE becomes the *modified* Mathieu equation and a elegant relationship between Floquet exponent and Baxter's T function has been suggested. As written, the implication of this conjecture for the period on the A-cycle of (effective) SU(2) gauge theory has been highlighted and used by [32]. But it was not clear from there if and how it is possible to generalize this beautiful connection between transfer matrix and periods for higher rank groups.

In the details of this paper we derive QQ and TQ functional relations (see eqs. (4.6), (4.8)) and extend Zamolodchikov's conjecture for the case of gauge group SU(3) (see eq. (5.14)). We show that numerical integration of the differential equation leads to evaluation of the 'quantum' A-cycle periods $a_{1,2,3}$ at any point of the moduli space of vacua parametrized by the vector multiplet scalar VEV's $u_2 = \langle \mathbf{tr}\,\phi^2 \rangle$ and $u_3 = \langle \mathbf{tr}\,\phi^3 \rangle$ even for large values of q at which the instanton series diverges. We have checked that the numerical results at small q are in excellent agreement with instanton calculation. Thus the main message of this paper is that the differential equation provides an excellent tool for investigation of deformed SW theory in its entire range from weak to strong coupling.

The paper is organized as follows: section 1 is a short review on instanton calculus for SU(N) SW theory without hypers in Ω-background. Here one can find explicit expressions as a sum over (multiple) Young diagrams for Nekrasov partition function and VEV's $\langle \mathbf{tr}\,\phi^J \rangle$.

Section 2 is a brief introduction to deformed SW theory. We present the main results of [13] in a form convenient for our present purposes. Starting from section 3 we consider the case of SU(3) theory. The main tool of our investigation, a third order linear ODE is derived and its asymptotic solutions are found.

In section 4 we identify a unique solution $\chi(x)$ which rapidly vanishes for large negative values of the argument $x \to -\infty$. The three quantities $Q_{1,2,2}$ are defined as coefficients of

[3] The main relevant difference is, as in simpler SU(2) case [40] and [32], the exit of the oper parameter ($M > 1/2$ in [30]) from the range of validity with the appearance of a extra irregular singularity in zero (besides that at ∞).

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expansion of $\chi(x)$ in terms of three independent solutions $U_{1,2,3}(x)$ defined in asymptotic region $x \gg 0$. Investigating symmetries of the differential equation we find a system of difference equations for Q_k and their analogs $\bar{Q}_k$ obtained by flipping the sign of parameter $u_3 \to -u_3$. Based on this QQ system we introduce Baxter's T function and write down corresponding TQ relations.

In section 5 we show how numerical integration of the differential equation along imaginary direction with standard boundary conditions allows one to find the monodromy matrix and corresponding Floquet exponents, which in the context of gauge theory, coincide with the A-cycle periods $a_{1,2,3}$. We have convincingly demonstrated the correctness of this identities trough comparison with instanton computation. But the main value of this method is that it makes accessible also the region of large coupling constants, which is beyond the reach of instanton calculus. Eventually, we close this section by suggesting a simple relation between Baxter's T-function and A-cycle periods $a_{1,2,3}$ of SU(3) theory, which can be thought of as a natural extension of Alexei Zamolodchikov's conjecture relating Floquet exponent of Mathieu equation to Baxter's T function in $c = 25$ Liouville CFT.

Finally appendix A contains few technical details for derivation of the TQ relation.

1 Nekrasov partition function and the VEVs $\langle \mathrm{tr}\, \phi^J \rangle$

Consider pure SU(N) theory without hypers in Ω-background. The instanton part of partition function is given by [6]

$$Z_{\text{inst}}(\mathbf{a}, \epsilon_1, \epsilon_2, q) = \sum_{\vec{Y}} Z_{\vec{Y}} \left((-)^N q\right)^{|\vec{Y}|} , \tag{1.1}$$

where sum runs over all N-tuples of Young diagrams $\vec{Y} = (Y_1, \cdots, Y_N)$, $|\vec{Y}|$ is the total number all boxes, $\mathbf{a} = (a_1, a_2, \cdots, a_N)$ are VEV's of adjoint scalar from $\mathcal{N} = 2$ vector multiplet, ϵ_1, ϵ_2, as already mentioned, parametrize the Ω-background and the instanton counting parameter $q = \exp 2\pi i\tau$, with $\tau = \frac{i}{g^2} + \frac{\theta}{2\pi}$ being the (complexified) coupling constant. The coefficients $Z_{\vec{Y}}$ are factorized as

$$Z_{\vec{Y}} = \prod_{u,v=1}^{N} \frac{1}{P(Y_u, a_u|Y_v, a_v)} , \tag{1.2}$$

where the factors $P(\lambda, a|\mu, b)$ for arbitrary pair of Young diagrams λ, μ and associated VEV parameters a, b, are given explicitly by the formula [7]

$$P(\lambda, a|\mu, b) = \prod_{s\in\lambda}(a-b+\epsilon_1(1+L_\mu(s))-\epsilon_2 A_\lambda(s)) \prod_{s\in\mu}(a-b-\epsilon_1 L_\lambda(s)+(1+\epsilon_2 A_\lambda(s))) \tag{1.3}$$

If one specifies location of a box s by its horizontal and vertical coordinates (i, j), so that $(1, 1)$ corresponds to the corner box, its leg length $L_\lambda(s)$ and arm length $A_\lambda(s)$ with respect to the diagram λ (s does not necessarily belong to λ) are defined as

$$A_\lambda(s) = \lambda_i - j; \qquad L_\lambda(s) = \lambda'_j - i , \tag{1.4}$$

where λ_i (λ'_j) is i-th column (j-th row) of diagram λ with convention that when i exceeds the number of columns (j exceeds the number of rows) of λ, one simply sets $\lambda_i = 0$ ($\lambda'_j = 0$). The instanton part of (deformed) prepotential is given by [6]

$$F_{\text{inst}}(\mathbf{a}, q) = -\epsilon_1\epsilon_2 \log Z_{\text{inst}} \,. \tag{1.5}$$

Instanton calculus allows one to obtain also the VEV's $\langle \mathbf{tr}\,\phi^J \rangle$, ϕ being the adjoint scalar of vector multiplet:

$$\langle \mathbf{tr}\,\phi^J \rangle = \sum_{i=1}^{N} a_u^J + Z_{\text{inst}}^{-1} \sum_{\vec{Y}} Z_{\vec{Y}} \mathcal{O}_{\vec{Y}}^J q^{|\vec{Y}|} \,, \tag{1.6}$$

where $Z_{\vec{Y}}$ is already defined by (1.2), (1.3), and [41, 42]

$$\begin{aligned} \mathcal{O}_{\vec{Y}}^J = \sum_{u=1}^{N} \sum_{(i,j)\in Y_u} \Big(& \left(a_u + \epsilon_1 i + \epsilon_2(j-1)\right)^J + \left(a_u + \epsilon_1(i-1) + \epsilon_2 j\right)^J \\ & - \left(a_u + \epsilon_1(i-1) + \epsilon_2(j-1)\right)^J - \left(a_u + \epsilon_1 i + \epsilon_2 j\right)^J \Big) \,. \end{aligned} \tag{1.7}$$

2 A Baxter difference equation

2.1 Bethe ansatz equation for NS limit

It was shown in [13] that in NS limit $\epsilon_2 \to 0$, the sum (1.1) is dominated by a single term corresponding to a unique array of Young diagrams $\vec{Y}^{(\text{cr})}$ specified by properties (the i-th column length of a diagram Y_u will be denoted as $Y_{u,i}$):

- Though the total number of boxes $\to \infty$ in $\epsilon_2 \to 0$ limit the rescaled column lengths $\epsilon_2 Y_{u,i}^{(\text{cr})}$, converge to finite values

$$\xi_{u,i} = \lim_{\epsilon_2 \to 0} \epsilon_2 Y_{u,i}^{(\text{cr})} \,.$$

- The rescaled column lengths at small q behave as $\xi_{u,i} \sim O(q^i)$. This means in particular, that in order to achieve accuracy up to q^L, it is consistent to consider restricted Young diagrams with number of columns $\leq L$.

- Up to arbitrary order q^L the quantities

$$x_{u,i} = a_u + \epsilon_1(i-1) + \xi_{u,i}$$

  satisfy the Bethe-ansatz equations (for each $u = 1, 2, \cdots N$)

$$-q \prod_{v,j}^{N,L} \frac{(x_{u,i} - x_{v,j} - \epsilon_1)(x_{u,i} - x_{v,j}^0 + \epsilon_1)}{(x_{u,i} - x_{v,j} + \epsilon_1)(x_{u,i} - x_{v,j}^0 - \epsilon_1)} = \prod_{v=1}^{N} (x_{u,i} - a_v + \epsilon_1)(a_v - x_{u,i}) \,, \tag{2.1}$$

  where, by definition

$$x_{u,i}^0 = a_u + \epsilon_1(i-1) \,.$$

The system of equations (2.1) together with the property $\xi_{u,i} \sim O(q^i)$ uniquely fixes the quantities $x_{u,i}$ up to order q^L. Of course, calculations become more cumbersome if one increases L. Examples of explicit computations for first few values of L can be found in [13].

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2.2 Baxter's difference equation and deformed Seiberg-Witten 'curve'

The BA equations can be transformed into a difference equation [13]

$$Y(z+\epsilon_1)+\frac{q}{\epsilon_1^{2N}}Y(z-\epsilon_1)=\epsilon_1^{-N}P_N(z+\epsilon_1)\,Y(z)\,, \tag{2.2}$$

where $Y(z)$ is an entire function with zeros located at $z=x_{u,i}$:

$$Y(z)=\prod_{u=1}^{N}e^{\frac{z}{\epsilon_1}\psi(\frac{a_u}{\epsilon_1})}\prod_{i=1}^{\infty}\left(1-\frac{z}{x_{u,i}}\right)e^{z/x^0_{u,i}}\,, \tag{2.3}$$

and

$$\psi(x)=\frac{d}{dx}\log\Gamma(x)$$

is the logarithmic derivative of Gauss' gamma-function. Finally $P_N(z)$ is an N-th order polynomial which parametrises the Coulomb branch of the theory. Explicit expressions of coefficients of this polynomial in terms of VEVs

$$u_J\equiv\langle\mathbf{tr}\phi^J\rangle \tag{2.4}$$

will be presented later for the case of our current interest $N=3$. For more general cases one can refer to [13]. Let us note also that (2.2) coincides with Baxter's equation for the N-body quantum Toda chain, and in relation to gauge theory has been considered earlier in [12, 14] from a different perspective.

Now, let us briefly recall how the difference equation (2.2) is related to the Seiberg-Witten curve. Introducing the function

$$y(z)=\epsilon_1^N\;\frac{Y(z)}{Y(z-\epsilon_1)}$$

one can rewrite (2.2) as

$$y(z)+\frac{q}{y(z-\epsilon_1)}=P_N(z)\,. \tag{2.5}$$

At large z the function $y(z)$ behaves as

$$y(z)=z^N(1+O(1/z))\,.$$

Notice that setting $\epsilon_1=0$ in (2.5) one obtains an equation of hyperelliptic curve, which is just the Seiberg-Witten curve. When $\epsilon_1\neq 0$, everything goes surprisingly similar to the original Seiberg-Witten theory. For example the rôle of Seiberg-Witten differential is played anew by the quantity

$$\lambda_{SW}=z\frac{d}{dz}\log y(z)\,,$$

and, as in the undeformed theory, the expectation values (2.4) are given by the contour integrals

$$\langle\mathbf{tr}\,\phi^J\rangle=\oint_{\mathcal{C}}\frac{dz}{2\pi i}z^J\partial_z\log y(z)\,, \tag{2.6}$$

where $\mathcal{C}$ is a large contour, enclosing all zeros and poles of $y(z)$.

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2.3 Details on SU(3) theory

Without any essential loss of generality, from now on we will assume that

$$u_1 \equiv \langle \mathbf{tr}\, \phi \rangle = a_1 + a_2 + a_3 = 0\,. \tag{2.7}$$

Representing $y(z)$ as a power series in $1/z$

$$y(z) = z^3(1 + c_1 z^{-1} + c_2 z^{-2} + c_3 z^{-3} + \cdots) \tag{2.8}$$

and inserting in eq. (2.6) one easily finds the relations

$$c_1 = 0; \qquad c_2 = -\frac{u_2}{2}; \qquad c_3 = -\frac{u_3}{3}\,. \tag{2.9}$$

Now, consistency of (2.8), (2.9) and (2.5) immediately specifies the polynomial $P_3(z)$ (we omit the subscript 3, since only the case $N = 3$ will be considered later on)

$$P(z) = z^3 - \frac{u_2}{2}\, z - \frac{u_3}{3}\,. \tag{2.10}$$

3 The differential equation and its asymptotic solutions

3.1 Derivation of the differential equation

To keep expressions simple, from now on we will set $\epsilon_1 = 1$. In fact, at any stage the ϵ_1 dependence can be easily restored on dimensional grounds. Taking the results of previous subsection, the difference equation for $N = 3$ case (2.2) can be rewritten as

$$Y(z) - \left(z^3 - \frac{u_2}{2} z - \frac{u_3}{3}\right) Y(z-1) + q\, Y(z-2) = 0\,, \tag{3.1}$$

By means of inverse Fourier transform, following [16, 22, 25], from (3.1) we can derive a third order linear differential equation for the function

$$f(x) = \sum_{z \in \mathbb{Z}+a} e^{x(z+1)} Y(z)\,. \tag{3.2}$$

At least when $|q|$ is sufficiently small, it is expected that the series is convergent for finite x, provided a takes one of the three possible values a_1, a_2 or a_3. Taking into account the difference relation (3.1), one can easily check that the function (3.2) solves the differential equation

$$-f'''(x) + \frac{u_2}{2}\, f'(x) + \left(e^{-x} + q\, e^{x} + \frac{u_3}{3}\right) f(x) = 0\,. \tag{3.3}$$

Denoting

$$q = \Lambda^6$$

and shifting the variable

$$x \to x - \log \Lambda^3$$

the differential equation (3.3) may be cast into a more symmetric form

$$-f'''(x) + \frac{u_2}{2}\, f'(x) + \left(\Lambda^3 (e^{x} + e^{-x}) + \frac{u_3}{3}\right) f(x) = 0\,. \tag{3.4}$$

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3.2 Solutions at $x \to \pm\infty$

Physics leads us to introduce parameters p_1, p_2, p_3 satisfying $p_1 + p_2 + p_3 = 0$ such that

$$u_2 = p_1^2 + p_2^2 + p_3^2 = 2(p_1^2 + p_2^2 + p_1 p_2); \quad u_3 = p_1^3 + p_2^3 + p_3^3 = -3p_1 p_2 (p_1 + p_2)\,, \tag{3.5}$$

as in the weak coupling limit $\Lambda \to 0$ the parameters p_i and a_i, respectively, coincide.

At large positive values $x \gg 3\ln\Lambda$ the term $\Lambda^3 e^{-x}$ in (3.4) can be neglected. In this region the differential equation can be solved in terms of hypergeometric function ${}_0F_2(a, b; z)$ defined by the power series

$${}_0F_2(a, b; z) = \sum_{k=0}^{\infty} \frac{z^k}{k!(a)_k (b)_k}\,, \tag{3.6}$$

where

$$(x)_k = x(x + 1) \cdots (x + k - 1) \tag{3.7}$$

is the Pochhammer symbol. Three linearly independent solutions can be chosen as

$$U_i(x) \approx e^{(x+3\theta)p_i}\, {}_0F_2(1 + p_i - p_j, 1 + p_i - p_k; e^{x+3\theta})\,, \tag{3.8}$$

where by definition

$$\Lambda \equiv \exp\theta \tag{3.9}$$

and the indices (i, j, k) are cyclic permutations of $(1, 2, 3)$. We used the symbol $\approx$ in (3.8) to mean that the approximations of the solutions hold, striktly speaking, only for $x \gg 3\theta$ (at leading order). In the end, we must verify that the Wronskian of the three solutions (3.8) (below and later on, for brevity, we use the notation $p_{ij} \equiv p_i - p_j$)

$$Wr[U_1(x), U_2(x), U_3(x)] \equiv \det \begin{pmatrix} U_1(x) & U_2(x) & U_3(x) \\ U_1'(x) & U_2'(x) & U_3'(x) \\ U_1''(x) & U_2''(x) & U_3''(x) \end{pmatrix} = p_{12} p_{23} p_{31} \tag{3.10}$$

is not zero provided the parameters p_i are pairwise different. Thus, (3.10) confirms that generically the $U_i(x)$ are linearly independent and constitute a basis in the space of all solutions.

Similarly in region $x \ll -3\theta$ the term $\Lambda^3 e^x$ of (3.4) becomes negligible and one can write down the three linear independent solutions

$$V_i(x) \approx e^{(x-3\theta)p_i}\, {}_0F_2(1 - p_i + p_j, 1 - p_i + p_k; -e^{-x+3\theta})\,. \tag{3.11}$$

In fact, we obtain the same result for the Wronskian

$$Wr[V_1(x), V_2(x), V_3(x)] = p_{12} p_{23} p_{31}\,. \tag{3.12}$$

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4 The functional relations

4.1 The QQ relations

All three solutions $V_i(x)$ grow very fast at $x \to -\infty$, but there is a special linear combination (unique, up to a common constant factor) which vanishes in this limit. If it is the fastest one (as we suspect), this solution is usually referred to as subdominant. Using formulae for asymptotics of ${}_0F_2$, which can be found e.g. in [43], we are able to establish that the correct combination is

$$\chi(x) = \frac{\Gamma(p_{12})\Gamma(p_{13})}{4\pi^2} V_1(x) + \frac{\Gamma(p_{23})\Gamma(p_{21})}{4\pi^2} V_2(x) + \frac{\Gamma(p_{31})\Gamma(p_{32})}{4\pi^2} V_3(x) . \tag{4.1}$$

Its asymptotic expansion at $x \to -\infty$ is given by

$$\begin{aligned}\chi(x) \, = \, & \frac{v^{-\frac{1}{3}} e^{-3v^{1/3}}}{2\pi\sqrt{3}} \left(1 - \left(\frac{1}{9} - \frac{u_2}{2}\right) v^{-\frac{1}{3}} + \left(\frac{u_2^2}{8} - \frac{5u_2}{36} + \frac{u_3}{6} + \frac{2}{81}\right) v^{-\frac{2}{3}} \right. \\ & \left. - \left(-\frac{u_2^3}{48} + \frac{u_2^2}{18} - \frac{u_3 u_2}{12} - \frac{13u_2}{324} + \frac{7u_3}{54} + \frac{14}{2187}\right) v^{-1} + O\left(v^{-\frac{4}{3}}\right)\right) , \end{aligned} \tag{4.2}$$

where we denoted

$$v = \exp(3\theta - x)$$

and u_2, u_3 are defined in terms of p_i in (3.5).

Since $U_i(x)$ constitute a complete set of solutions one can represent $\chi(x)$ as a linear combination

$$\chi(x,\theta) = \sum_{n=1}^{3} Q_n(\theta)\Gamma(p_{nj})\Gamma(p_{nk}) e^{-3p_n\theta} U_n(x,\theta), \tag{4.3}$$

where the important quantities $Q_n(\theta)$, are expected to be entire functions of θ (and also of parameters $\mathbf{p}$, whose dependence will be displayed explicitly only if necessary) thanks to a rather general reasoning in the ODE/IM correspondence (cf. for instance the book [44] and the research papers [37–39] and [45] along with their references) which relies on the simple dependence in the basic differential equation, (3.4). Of course, this property is crucial for all the integrability aspects and hence in our further developments. Moreover, we would expect also a crucial rôle for the Qs in gauge theory as connected to the exponentials of the dual periods, in analogy with the SU(2) case [32].

For the time being, we wish to prove some fundamental functional relations, typical of an integrable theory. To this aim, the following easy to check property plays an essential role: namely the Wronskian of any two solutions $f(x)$, $g(x)$ of the differential equation (3.4)

$$W[f(x), g(x)] \equiv f(x)g'(x) - g(x)f'(x)$$

satisfies the *adjoint* equation, i.e. the one obtained by reversing the signs $\mathbf{p} \to -\mathbf{p}$ and $\Lambda^3 \to -\Lambda^3$. Taking inspiration from this property, it is then possible to show exactly that

$$Wr\left[\chi\left(x, \theta + \frac{i\pi}{3}\right), \chi\left(x, \theta - \frac{i\pi}{3}\right)\right] = -\frac{i}{2\pi}\bar{\chi}(x,\theta) \, , \tag{4.4}$$

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where $\bar{\chi}(\theta) = \chi(\theta, -\mathbf{p})$. In fact, the property entails that the l.h.s. of (4.4) satisfies the differential equation (3.4) with substitution $\mathbf{p} \to -\mathbf{p}$. Besides, by using the identity[4]

$$\mathrm{Wr}\left[e^{\frac{2-a-b}{3}x}\,{}_0F_2(a,b,-e^{-x})\,,e^{\frac{2a-b-1}{3}x}\,{}_0F_2(2-a,1-a+b,-e^{-x})\right]$$
$$= (a-1)e^{\frac{1+a-2b}{3}x}\,{}_0F_2(b,1-a+b,e^{-x})\,, \tag{4.5}$$

it is not difficult to show the match of the $x \to -\infty$ asymptotics of both sides. Of course, the combination of these two statements implies the equality (4.4) everywhere.

Let us investigate the $x \to \infty$ limit of (4.4). Taking into account (4.3) and using the identity (4.5) (with x substituted by $-x$), we obtain the functional relations

$$\frac{\sin(\pi p_{jk})}{2i\pi^2}\,\bar{Q}_n(\theta) = Q_j\left(\theta + \frac{i\pi}{3}\right)Q_k\left(\theta - \frac{i\pi}{3}\right) - Q_j\left(\theta - \frac{i\pi}{3}\right)Q_k\left(\theta + \frac{i\pi}{3}\right), \tag{4.6}$$

where again, the bar on Q_n indicates the sign change $\mathbf{p} \to -\mathbf{p}$

$$\bar{Q}_n(\theta, \mathbf{p}) \equiv Q_n(\theta, -\mathbf{p})$$

and (n, j, k) is a permutations of $(1, 2, 3)$.

At the end of this section let us establish the $\theta \to -\infty$ asymptotics of $Q_k(\theta)$ and $\bar{Q}_k(\theta)$. Obviously, in this case both (3.8) and (3.11) are approximate solutions of (3.4) at $x \sim 0$. Thus, comparison of (4.1) with (4.3) ensures that for $\theta \ll 0$

$$Q_k(\theta) \sim \frac{\exp(-3\theta p_k)}{4\pi^2}\,; \qquad \bar{Q}_k(\theta) \sim \frac{\exp(3\theta p_k)}{4\pi^2}\,. \tag{4.7}$$

It is easy to see that above asymptotic behavior is fully consistent with functional relations (4.6).

4.2 SU(3) version of Baxter's TQ relation

The functional relations (4.6) suggest the following SU(3) analog of Baxter's TQ equations:

$$T(\theta)Q_j\left(\theta - \frac{\pi i}{6}\right)\bar{Q}_k\left(\theta + \frac{\pi i}{6}\right) = Q_j\left(\theta - \frac{5\pi i}{6}\right)\bar{Q}_k\left(\theta + \frac{\pi i}{6}\right) + Q_j\left(\theta + \frac{\pi i}{2}\right)\bar{Q}_k\left(\theta - \frac{\pi i}{2}\right)$$
$$+ Q_j\left(\theta - \frac{\pi i}{6}\right)\bar{Q}_k\left(\theta + \frac{5\pi i}{6}\right) \tag{4.8}$$

for $j, k \in \{1, 2, 3\}$ with $j \neq k$. To uncover the essence of this construction, notice that for a fixed pair of indices (i, j) (4.8) can be thought as definition of function $T(\theta)$ in terms of Q's. Then the nontrivial question is "do other choices of (j, k) lead to the same T?" Fortunately, elementary algebraic manipulations with the help of (4.6) ensure that the answer is positive. As mentioned earlier, $Q_i(\theta)$ are entire functions. A thorough analysis shows that due to (4.6) all potential poles of $T(\theta)$ have zero residue. Thus $T(\theta)$ is an entire function too. Details on proofs of above two statements can be found in appendix A. The Bethe ansatz equations can be represented as (see equality (A.4))

$$\frac{Q_j(\theta_\ell - \frac{2\pi i}{3})\bar{Q}_k\left(\theta_\ell + \frac{\pi i}{3}\right)}{Q_j\left(\theta_\ell + \frac{2\pi i}{3}\right)\bar{Q}_k\left(\theta_\ell - \frac{\pi i}{3}\right)} = -1\,, \tag{4.9}$$

[4] It can be proven, for instance, by expanding both sides in powers of e^{-x}.

where θ_ℓ are the zeros of $Q_j(\theta)$. In appendix B we will show that there are 6 independent BA equations.

Functional relations similar to (4.6) and (4.8) emerge also in the context of ODE/IM for 'minimal' 2d CFT with extra spin 3 current (W_3 symmetry) [37, 38]. From there we can extrapolate that our case might correspond to the special choice of Virasoro central charge $c = 98$ for Toda CFT. In fact, this value of the central charge lies outside the region discussed in above references. Nevertheless, it should be possible to derive the corresponding TBA equations: we leave this task for future publication.

5 Quantum periods and prepotential from Floquet monodromies and extension of Zamolodchikov's conjecture

5.1 The Floquet-Bloch monodromy matrix

Consider the basis of solutions $f_1(x)$, $f_2(x)$, $f_3(x)$ of (3.4) with standard initial conditions $(n, k \in \{1, 2, 3\})$

$$f_n^{(k-1)}(x)\Big|_{x=0} = \delta_{k,n}\,. \tag{5.1}$$

Since the functions $f_n(x + 2\pi i)$ are solutions too, we can define the monodromy matrix $M_{k,n}$ as

$$f_n(x + 2\pi i) = \sum_{k=1}^{3} f_k(x) M_{k,n} \tag{5.2}$$

Clearly

$$M_{k,n} = f_n^{(k-1)}(2\pi i)\,. \tag{5.3}$$

The solutions (3.2) with $a \in \{a_1, a_2, a_3\}$ have diagonal monodromies and can be represented as certain linear combinations of $f_n(x)$. In other words the eigenvalues of the monodromy matrix $M_{k,n}$ must be identified with $\exp(2\pi i a_k)$, with $k = 1, 2, 3$ (below the set of eigenvalues of M is denoted as $Spec(M)$):

$$Spec(M) = \{\exp(2\pi i a_1), \exp(2\pi i a_2), \exp(2\pi i a_3)\}\,. \tag{5.4}$$

For any fixed values of parameters Λ, $\mathbf{p}$, it is easy to integrate numerically the differential equation (3.4) with boundary conditions (5.1), find the matrix $M_{k,n}$ and then its eigenvalues $\exp(2\pi i a_n)$. Taking into account Matone relation [46], valid also in the presence of Ω-background [42],

$$u_2 \equiv \langle \mathbf{tr}\,\phi^2 \rangle = \sum_{n=1}^{3} a_n^2 + 2q\partial_q F_{\text{inst}}(q, \mathbf{a})\,, \tag{5.5}$$

we can access the deformed prepotential for any value of the coupling constant.

5.2 Comparison of the instanton counting against numerical results

Using formula of section 1 it is straightforward to calculate $\langle \mathbf{tr}\,\phi^2\rangle$ or $\langle \mathbf{tr}\,\phi^3\rangle$ as a power series in q. Here are the 3-instanton results (it is assumed that $a_1+a_2+a_3=0$ and by definition $a_{jk}\equiv a_j-a_k$)

$$\langle \mathbf{tr}\,\phi^2\rangle = \sum_{k=1}^{3} a_k^2 - \frac{12(1-h_2)q}{\prod_{j<k}(a_{jk}^2-1)} + \frac{P_{2,2}q^2}{\prod_{j<k}(a_{jk}^2-1)^3(a_{jk}^2-4)} + O(q)^4 \tag{5.6}$$

$$\begin{aligned}\langle \mathbf{tr}\,\phi^3\rangle &= \sum_{k=1}^{3} a_k^3 + \frac{54h_3q}{\prod_{j<k}(a_{jk}^2-1)} + \frac{P_{3,2}q^2}{\prod_{j<k}(a_{jk}^2-1)^3(a_{jk}^2-4)} \\ &\quad - \frac{P_{3,3}q^3}{\prod_{j<k}(a_{jk}^2-1)^5(a_{jk}^2-4)(a_{jk}^2-9)} + O(q)^4\,,\end{aligned} \tag{5.7}$$

where

$$h_2 = \frac{a_1^2+a_2^2+a_3^2}{2}\,; \qquad h_3 = -a_1a_2a_3\,, \tag{5.8}$$

and

$$\begin{aligned}P_{2,2} &= 36(220-1027h_2+1659h_2^2-698h_2^3-958h_2^4+1257h_2^5-521h_2^6 \\ &\quad +68h_2^7-13959h_3^2+33804h_2h_3^2-25434h_2^2h_3^2+5292h_2^3h_3^2 \\ &\quad +297h_2^4h_3^2+13851h_3^4-5103h_2h_3^4)\end{aligned} \tag{5.9}$$

$$\begin{aligned}P_{3,2} &= -162h_3(455-2487h_2+4602h_2^2-3286h_2^3+291h_2^4+573h_2^5-148h_2^6 \\ &\quad -8073h_3^2+14985h_2h_3^2-7695h_2^2h_3^2+783h_2^3h_3^2+1458h_3^4)\end{aligned} \tag{5.10}$$

$$\begin{aligned}P_{3,3} &= -108h_3(12078563-109310145h_2+400164948h_2^2-722480972h_2^3 \\ &\quad +538752402h_2^4+275687658h_2^5-946955868h_2^6+865056708h_2^7 \\ &\quad -391259133h_2^8+81882223h_2^9-2063856h_2^{10}-1715472h_2^{11}+162944h_2^{12} \\ &\quad -984855213h_3^2+6130798389h_2h_3^2-14569978437h_2^2h_3^2+16850898261h_2^3h_3^2 \\ &\quad -9439886367h_2^4h_3^2+1593033399h_2^5h_3^2+730653777h_2^6h_3^2-352792017h_2^7h_3^2 \\ &\quad +42690240h_2^8h_3^2-562032h_2^9h_3^2+7812512937h_3^4-22941081063h_2h_3^4 \\ &\quad +24720233994h_2^2h_3^4-11808597150h_2^3h_3^4+2295385533h_2^4h_3^4-64422459h_2^5h_3^4 \\ &\quad -14031792h_2^6h_3^4-3311723799h_3^6+3321565299h_2h_3^6-982634409h_2^2h_3^6 \\ &\quad +65800269h_2^3h_3^6+29760696h_3^8)\end{aligned} \tag{5.11}$$

We have calculated also 4 and 5 instanton corrections, but the formulae are too lengthy to be presented here.

By means of numerical integration of the differential equation (3.4) along the line indicated in section 5.1 we have computed the eigenvalues of monodromy matrix (5.3) for several values of the instanton parameter $q=\Lambda^6$, namely for the values

$$\Lambda = \exp\left(\frac{k-1}{20}-5\right)\,, \qquad k=1,2,\cdots,120\,, \tag{5.12}$$

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Λ	a_1	a_2
0.00822974704902	0.1200000000131	0.169999999982
0.0223707718562	0.1200000053049	0.169999992932
0.0608100626252	0.1200021402877	0.169997148430
0.165298888222	0.1208841761521	0.168828966405
0.246596963942	0.1349151981823	0.151933010167
0.272531793034	0.142136769453 − 0.019455438633 i	0.142136769453 + 0.019455438633 i
0.449328964117	0.092117229441 − 0.135924390553 i	0.092117229441 + 0.135924390553 i
0.740818220682	0.003727137475 − 0.568756791077 i	0.003727137475 + 0.568756791077 i
1.22140275816	0.000899023180 − 1.071594057757 i	0.000899023180 + 1.071594057757 i
2.01375270747	0.00036203460 − 1.78605985179 i	0.00036203460 + 1.78605985179 i
3.32011692274	0.00013130957 − 2.96965962318 i	0.00013132399 + 2.96965962932 i

Table 1. The values a_1, a_2 obtained through numerical integration of the differential equation (3.4) with initial conditions (5.1) for $p_1 = 0.12$, $p_2 = 0.28$.

Λ	$\langle \mathbf{tr}\, \phi^2 \rangle$	$\langle \mathbf{tr}\, \phi^3 \rangle$
0.00822974704902	0.1274000000000	−0.0177480000000
0.0223707718562	0.1274000000000	−0.0177480000000
0.0608100626252	0.1274000000000	−0.0177480000000
0.165298888222	0.1274000000000	−0.0177480000000
0.246596963942	0.1273999999998	−0.0177480000000
0.272531793034	0.1273999999922	−0.0177479999994
0.449328964117	0.1273774046391	−0.0177462190257
0.740818220682	0.1313057536866	−0.0178774876030

Table 2. The values $\langle \mathbf{tr}\, \phi^2 \rangle$, $\langle \mathbf{tr}\, \phi^3 \rangle$ obtained by inserting the values of a_1, a_2 from table 1 into (5.6), (5.7) supplemented by q^4 and q^5 corrections. To be compared with (by definition) $\langle \mathbf{tr}\, \phi^2 \rangle = p_1^2 + p_2^2 + p_3^2 = 0.1274$ and $\langle \mathbf{tr}\, \phi^3 \rangle = p_1^3 + p_2^3 + p_3^3 = -0.017748$.

and fixed values of parameters

$$p_1 = 0.12\,; \qquad p_2 = 0.17\,; \qquad p_3 = -0.29\,.$$

Due to identification (5.4) this allows to find the corresponding A-cycle periods a_1, a_2, a_3. In table 1 we present some characteristic excerpt from the resulting data. In table 1 as well as in table 2 below we display stabilized digits, which do not change further upon increasing the accuracy of calculations.

Inserting the values of a_k, Λ in (5.6), (5.7) supplemented by q^4 and q^5 corrections we have calculated $\langle \mathbf{tr}\, \phi^2 \rangle$ and $\langle \mathbf{tr}\, \phi^3 \rangle$. The consistency requires that at small values of q for at which instanton expansion is valid one should always obtain the same expectation values $\langle \mathbf{tr}\, \phi^2 \rangle = p_1^2 + p_2^2 + p_3^2 = 0.1274$ and $\langle \mathbf{tr}\, \phi^3 \rangle = p_1^3 + p_2^3 + p_3^3 = -0.017748$. Table 2 displays the results of actual computations. One expects an essential deviation from the instanton series starting from the value of Λ at which the polynomial

$$\left(z^3 - \frac{u_2}{2}\, z - \frac{u_3}{3} \right)^2 - 4\Lambda^6$$

acquires coinciding zeros, i.e. at the point where its discriminant

$$1024\Lambda^{18}\left(216\Lambda^6-72\Lambda^3u_3-u_2^3+6u_3^2\right)\left(216\Lambda^6+72\Lambda^3u_3-u_2^3+6u_3^2\right)$$

vanishes. Such points correspond to massless dyons or monopoles. It is easy to check that within the range of Λ (5.12) the only zero is at $\Lambda = 0.1822359934629\cdots$ for which the last factor of discriminant vanishes. And in fact, inspecting table 2 one sees that for the greater values of Λ's, the mismatch becomes significant while for smaller values the agreement is quite impressive. Notice also from table 1 that for $\Lambda > 0.24659696394$ we encountered complex values for a_1 and a_2. Emergence of this imaginary parts indicate presence of a branch point non-analyticity, so that the convergence radius of instanton series can be estimated to be $\Lambda \sim 0.2466$.

We have calculated the contribution of the highest term $\sim q^6$ in $\langle \mathbf{tr}\,\phi^2\rangle$, $\langle \mathbf{tr}\,\phi^3\rangle$ and found that for values $\Lambda \leq 0.27$ the relative error of truncating the series at the order $\sim q^5$ instead of $\sim q^6$ is less than 10^{-9}, for $\Lambda \sim 0.74$ it is of order 10^{-2} and for $\Lambda \sim 1$ the relative error becomes of order 1, so the instanton series can not be trusted any more. This is the reason why in table 2 we have presented results up to $\Lambda \sim 0.74$.

5.3 Extension of Zamolodcikov's conjecture to SU(3)

The simpler case of the gauge group SU(2) has been analyzed recently in [32]. In this case one has to deal with the Mathieu equation. Corresponding TQ relation was investigated in [40], where Al. Zamolodchikov conjectured (and demonstrated numerically) an elegant relationship between T-function and Floquet exponent ν of Mathieu equation:

$$T = \cosh(2\pi\nu)\,. \tag{5.13}$$

Here we suggest a natural extension of Zamolodchikov's conjecture for SU(3) case:

$$T(\theta) = \sum_{n=1}^{3} e^{2\pi i a_n}\,. \tag{5.14}$$

Notice, that at $\theta \ll 0$ the asymptotic (4.7) leads to

$$T(\theta) \sim \sum_{n=1}^{3} e^{2\pi i p_n}\;, \tag{5.15}$$

which is consistent with (5.14), since for $\theta \ll 0$ instanton corrections disappear and a_k coincides with p_k.

6 Few perspectives

It would be very interesting to have a TBA for our case and check our conjecture (5.14) as it was done by Al. Zamolodchikov in [40]. Actually, even relevant would be a gauge TBA that may shed light on the dual B-cycle periods $\mathbf{a_D}$ along the route presented in [32] for the SU(2) case (see also the presentation of [33] and [34]).

It is well known that pure $\mathcal{N}=2$ SU(N) theories with $N>2$ are endowed with special points in there moduli spaces of vacua at which mutually nonlocal dyons become massless (Argyres-Douglas points) [47]. It would be interesting to investigate this, as well as the closely related wall crossing phenomena within our approach (for the NS regime of the Ω background) with the help of our numerical method.

Furthermore, for generic groups of gauge theories (starting with SU(2) and general Liouville ODE/IM correspondence) it is very intriguing to investigate the form of 'potentials' of the ODE describing excited states of the IM (cf. [45, 48] for what we know about 'ordinary' ODE/IM). In fact, the latter should be obtainable also via analytic continuation in the parameters/moduli, and thus, these non-trivial monodromies would be of great interest in gauge theories.

Of course, it is very plausible that the imaginable generalizations of our results, and in particular of (5.14), might hold for arbitrary SU(N) gauge groups. In fact, for those higher order differential equations we shall have also the enlightening treatment of a mathematically similar problem with only one irregular singularity (at ∞), the case of gluon scattering amplitudes/Wilson loops at strong coupling in planar $\mathcal{N}=4$ SYM [49, 50]. Because of its different physical nature, this problem allow for a beautiful and all-coupling exact Operator Product Expansion [51, 52], whose strong coupling limit reproduces interestingly the integrable TBA [53] of [49, 50]: the similar mathematical structures and ideas of these two different fields should bear fruit, in future, for a deeper understanding.

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A Proving the TQ relations

In this appendix we prove that different choices of indices $j\neq k$ in TQ relations (4.8) are consistent with QQ relations (4.6).

For example let us choose $j=1$, $k=2$

$$T(\theta)Q_1\left(\theta-\frac{\pi i}{6}\right)\bar{Q}_2\left(\theta+\frac{\pi i}{6}\right)=Q_1\left(\theta-\frac{5\pi i}{6}\right)\bar{Q}_2\left(\theta+\frac{\pi i}{6}\right)+Q_1\left(\theta+\frac{\pi i}{2}\right)\bar{Q}_2\left(\theta-\frac{\pi i}{2}\right)$$
$$+Q_1\left(\theta-\frac{\pi i}{6}\right)\bar{Q}_2\left(\theta+\frac{5\pi i}{6}\right),\tag{A.1}$$

and $j=1$, $k=3$

$$T(\theta)Q_1\left(\theta-\frac{\pi i}{6}\right)\bar{Q}_3\left(\theta+\frac{\pi i}{6}\right)=Q_1\left(\theta-\frac{5\pi i}{6}\right)\bar{Q}_3\left(\theta+\frac{\pi i}{6}\right)+Q_1\left(\theta+\frac{\pi i}{2}\right)\bar{Q}_3\left(\theta-\frac{\pi i}{2}\right)$$
$$+Q_1\left(\theta-\frac{\pi i}{6}\right)\bar{Q}_3\left(\theta+\frac{5\pi i}{6}\right),\tag{A.2}$$

Multiplying (A.1) by $\bar{Q}_3\left(\theta+\frac{\pi i}{6}\right)$, (A.2) by $\bar{Q}_2\left(\theta+\frac{\pi i}{6}\right)$ and taking difference, the right hand side becomes

$$\begin{aligned}
&\bar{Q}_3\left(\theta+\frac{\pi i}{6}\right)\left(Q_1\left(\theta+\frac{\pi i}{2}\right)\bar{Q}_2\left(\theta-\frac{\pi i}{2}\right)+Q_1\left(\theta-\frac{\pi i}{6}\right)\bar{Q}_2\left(\theta+\frac{5\pi i}{6}\right)\right)\\
&\qquad -\bar{Q}_2\left(\theta+\frac{\pi i}{6}\right)\left(Q_1\left(\theta+\frac{\pi i}{2}\right)\bar{Q}_3\left(\theta-\frac{\pi i}{2}\right)+Q_1\left(\theta-\frac{\pi i}{6}\right)\bar{Q}_3\left(\theta+\frac{5\pi i}{6}\right)\right)\\
&=Q_1\left(\theta+\frac{\pi i}{2}\right)\left(\bar{Q}_3\left(\theta+\frac{\pi i}{6}\right)\bar{Q}_2\left(\theta-\frac{\pi i}{2}\right)-\bar{Q}_2\left(\theta+\frac{\pi i}{6}\right)\bar{Q}_3\left(\theta-\frac{\pi i}{2}\right)\right)\\
&\qquad +Q_1\left(\theta-\frac{\pi i}{6}\right)\left(\bar{Q}_3\left(\theta+\frac{\pi i}{6}\right)\bar{Q}_2\left(\theta+\frac{5\pi i}{6}\right)-\bar{Q}_2\left(\theta+\frac{\pi i}{6}\right)\bar{Q}_3\left(\theta+\frac{5\pi i}{6}\right)\right).
\end{aligned} \tag{A.3}$$

Obviously the last expression vanishes due to (4.6) as consistency requires.

Now let us show that $T(\theta)$ does not have any pole. It follows from (A.1), that a potential pole of $T(\theta)$ can be found either among the zeros of $Q_1\left(\theta-\frac{\pi i}{6}\right)$ or $\bar{Q}_2\left(\theta+\frac{\pi i}{6}\right)$. For definiteness let us assume that it belongs to zero set of $Q_1\left(\theta-\frac{\pi i}{6}\right)$ (the other option can be considered in completely analogues manner). Let $Q_1(\theta_\ell)=0$. Then for $\theta=\theta_\ell+\frac{i\pi}{6}$ the r.h.s. of (A.1) is equal to

$$\begin{aligned}
&Q_1\left(\theta_\ell-\frac{2\pi i}{3}\right)\bar{Q}_2\left(\theta_\ell+\frac{\pi i}{3}\right)+Q_1\left(\theta_\ell+\frac{2\pi i}{3}\right)\bar{Q}_2\left(\theta_\ell-\frac{\pi i}{3}\right)\\
&\qquad=\frac{2i\pi^2}{\sin(\pi p_{31})}\left(Q_1\left(\theta_\ell-\frac{2\pi i}{3}\right)Q_1\left(\theta_\ell+\frac{2\pi i}{3}\right)Q_3\left(\theta_\ell\right)\right.\\
&\qquad\qquad\qquad\left.-Q_1\left(\theta_\ell+\frac{2\pi i}{3}\right)Q_1\left(\theta_\ell-\frac{2\pi i}{3}\right)Q_3\left(\theta_\ell\right)\right)\\
&\qquad=0,
\end{aligned} \tag{A.4}$$

where the relations (4.6) for both $\bar{Q}_2\left(\theta_\ell\pm\frac{\pi i}{3}\right)$ has been used. So $T(\theta_\ell)$ is finite, thus proving that $T(\theta)$ is entire.

B Various forms of Bethe-ansatz equations

Here using the (4.6) relation we recast the Bethe-ansatz equations (4.9) in various useful forms. In particular, this analysis shows that there are 6 independent equations which matches the number of functions Q_i, $\bar{Q}_i$. Namely, using the QQ relations

$$\frac{\sin(\pi p_{jk})}{2i\pi^2}\bar{Q}_n(\theta)=Q_j\left(\theta+\frac{i\pi}{3}\right)Q_k\left(\theta-\frac{i\pi}{3}\right)-Q_j\left(\theta-\frac{i\pi}{3}\right)Q_k\left(\theta+\frac{i\pi}{3}\right), \tag{B.1}$$

$$\frac{\sin(\pi p_{jk})}{2i\pi^2}Q_n(\theta)=-\bar{Q}_j\left(\theta+\frac{i\pi}{3}\right)\bar{Q}_k\left(\theta-\frac{i\pi}{3}\right)+\bar{Q}_j\left(\theta-\frac{i\pi}{3}\right)\bar{Q}_k\left(\theta+\frac{i\pi}{3}\right), \tag{B.2}$$

we will show that there are only 6 intendant BA equations. Remind that (see eq. (4.9)) we have

$$\frac{Q_j(\theta_\ell^{(j)}-\frac{2\pi i}{3})\bar{Q}_k\left(\theta_\ell^{(j)}+\frac{\pi i}{3}\right)}{Q_j\left(\theta_\ell^{(j)}+\frac{2\pi i}{3}\right)\bar{Q}_k\left(\theta_\ell^{(j)}-\frac{\pi i}{3}\right)}=-1\,, \tag{B.3}$$

where $\theta_\ell^{(j)}$ are the zeros of $Q_j(\theta)$ and also

$$\frac{\bar{Q}_j(\bar{\theta}_\ell^{(j)} - \frac{2\pi i}{3})Q_k\left(\bar{\theta}_\ell^{(j)} + \frac{\pi i}{3}\right)}{\bar{Q}_j\left(\bar{\theta}_\ell^{(j)} + \frac{2\pi i}{3}\right)Q_k\left(\bar{\theta}_\ell^{(j)} - \frac{\pi i}{3}\right)} = -1\,, \tag{B.4}$$

where $\bar{\theta}_\ell^{(j)}$ are the zeros of $\bar{Q}_j(\theta)$.

From (B.1), (B.2) we get

$$\frac{\sin(\pi p_{jk})}{2i\pi^2}\bar{Q}_n\left(\theta_\ell^{(j)} - \frac{\pi i}{3}\right) = -Q_j\left(\theta_\ell^{(j)} - \frac{2\pi i}{3}\right)Q_k\left(\theta_\ell^{(j)}\right)\,; \tag{B.5}$$

$$\frac{\sin(\pi p_{jk})}{2i\pi^2}\bar{Q}_n\left(\theta_\ell^{(j)} + \frac{\pi i}{3}\right) = Q_j\left(\theta_\ell^{(j)} + \frac{2\pi i}{3}\right)Q_k\left(\theta_\ell^{(j)}\right)\,. \tag{B.6}$$

Now we immediately see that

$$\frac{\bar{Q}_n\left(\theta_\ell^{(j)} - \frac{\pi i}{3}\right)}{\bar{Q}_n\left(\theta_\ell^{(j)} + \frac{\pi i}{3}\right)} = -\frac{Q_j\left(\theta_\ell^{(j)} - \frac{2\pi i}{3}\right)}{Q_j\left(\theta_\ell^{(j)} + \frac{2\pi i}{3}\right)}\,; \tag{B.7}$$

$$\frac{Q_n\left(\bar{\theta}_\ell^{(j)} - \frac{\pi i}{3}\right)}{Q_n\left(\bar{\theta}_\ell^{(j)} + \frac{\pi i}{3}\right)} = -\frac{\bar{Q}_j\left(\bar{\theta}_\ell^{(j)} - \frac{2\pi i}{3}\right)}{\bar{Q}_j\left(\bar{\theta}_\ell^{(j)} + \frac{2\pi i}{3}\right)}\,. \tag{B.8}$$

Inserting (B.7) in (B.3) we obtain

$$\frac{\bar{Q}_n\left(\theta_\ell^{(j)} - \frac{\pi i}{3}\right)\bar{Q}_k\left(\theta_\ell^{(j)} + \frac{\pi i}{3}\right)}{\bar{Q}_n\left(\theta_\ell^{(j)} + \frac{\pi i}{3}\right)\bar{Q}_k\left(\theta_\ell^{(j)} - \frac{\pi i}{3}\right)} = 1\,. \tag{B.9}$$

Obviously this equation is invariant with respect to the replacement $n \leftrightarrow k$. Insertion (B.8) in (B.4) ensures that also the equation

$$\frac{Q_n\left(\bar{\theta}_\ell^{(j)} - \frac{\pi i}{3}\right)Q_k\left(\bar{\theta}_\ell^{(j)} + \frac{\pi i}{3}\right)}{Q_n\left(\bar{\theta}_\ell^{(j)} + \frac{\pi i}{3}\right)Q_k\left(\bar{\theta}_\ell^{(j)} - \frac{\pi i}{3}\right)} = 1 \tag{B.10}$$

is valid. With the help of these equations it not difficult to get convinced that in (B.3), (B.4) one can restrict the choices of pairs (j,k) e.g. by $(1,2)$, $(1,3)$ or $(2,3)$.

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