Exploring the abelian 4D, $\mathcal{N} = 4$ vector-tensor supermultiplet and its off-shell central charge structure

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ABSTRACT: An abelian 4D, $\mathcal{N} = 4$ vector supermultiplet allows for a duality transformation to be applied to one of its spin-0 states. The resulting theory can be described as an abelian 4D, $\mathcal{N} = 4$ vector-tensor supermultiplet. It is seen to decompose into a direct sum of an off-shell 4D, $\mathcal{N} = 2$ vector supermultiplet and an off-shell 4D, $\mathcal{N} = 2$ tensor supermultiplet. The commutator algebra of the other two supersymmetries are still found to be on-shell. However, the central charge structure in the resulting 4D, $\mathcal{N} = 4$ vector-tensor supermultiplet is considerably simpler than that of the parent abelian 4D, $\mathcal{N} = 4$ vector supermultiplet. This appears to be due to the replacement of the usual $SO(4)$ symmetry associated with the abelian 4D, $\mathcal{N} = 4$ vector supermultiplet being replaced by a $GL(2, \mathbb{R}) \otimes GL(2, \mathbb{R})$ symmetry in the 4D, $\mathcal{N} = 4$ vector-tensor supermultiplet. The *Mathematica* code detailing the calculations is available open-source at the [HEPTools Data Repository](#) on GitHub.

KEYWORDS: Duality in Gauge Field Theories, Extended Supersymmetry, Supersymmetric Gauge Theory, Supersymmetry and Duality

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1 Introduction

There currently exists in the physics literature very few examples of four dimensional relativistic quantum field theories that realize 4D, $\mathcal{N} = 4$ supersymmetry. In fact, to our knowledge the only known examples currently in the literature are: (a.) 4D, $\mathcal{N} = 4$ supergravity theories [1–3], and (b.) 4D, $\mathcal{N} = 4$ super Yang-Mills theories [4, 5]. With such a paucity of these types of supermultiplets, we believe it might prove useful to cast the dual version of the 4D, $\mathcal{N} = 4$ super Yang-Mills theories in a new light as pertains to the internal isospin structure. Specifically, in this paper we will focus on the 4D, $\mathcal{N}=4$ vector-tensor multiplet originally described in [6]. As is well-known the spectrum of the 4D, $\mathcal{N} = 4$ abelian vector supermultiplet contains one spin-1 boson and six spin-0 bosons. One can perform a duality transformation on one of the spin-0 bosons to replace it by a second rank anti-symmetric tensor. In [6], a 4D, $\mathcal{N} = 4$ vector-tensor multiplet was presented with an $SP(4)$ internal symmetry of the scalars and fermions. Effectively, this multiplet was an $SP(4)$ extension of the 4D, $\mathcal{N}=2$ vector-tensor multiplet, which has central charges. Dimensionally reducing to the 4D, $\mathcal{N} = 4$ vector-tensor multiplet was discussed in [7]. More recently, other facets of vector-tensor multiplets have been developed such as couplings to supergravity, couplings to Chern-Simons, and self-interactions [8–13].

In this note, we express the 4D, $\mathcal{N} = 4$ vector-tensor multiplet in an explicit $GL(2, \mathbb{R}) \otimes GL(2, \mathbb{R})$ isospin form, where one $GL(2, \mathbb{R})$ corresponds to a 4D, $\mathcal{N} = 2$ vector supermultiplet and the other $GL(2, \mathbb{R})$ corresponds to a 4D, $\mathcal{N} = 2$ tensor supermultiplet, neither of which have central charges. The 4D, $\mathcal{N}=2$ vector supermultiplet contains a 4D,

$\mathcal{N} = 1$ vector supermultiplet and a 4D, $\mathcal{N} = 1$ chiral supermultiplet. In a similar manner, the 4D, $\mathcal{N} = 2$ tensor supermultiplet contains a 4D, $\mathcal{N} = 1$ tensor supermultiplet and a 4D, $\mathcal{N} = 1$ chiral supermultiplet.

Our examination begins with the 4D, $\mathcal{N} = 2$ vector supermultiplet and the 4D, $\mathcal{N} = 2$ tensor supermultiplet, with their manifest off-shell 4D, $\mathcal{N} = 2$ supersymmetry, and constructs from them a 4D, $\mathcal{N} = 4$ vector-tensor supermultiplet. We will examine the resulting supermultiplet in a Majorana notation as in [14], but cast into a $\text{GL}(2, \mathbb{R}) \otimes \text{GL}(2, \mathbb{R})$ isospin form where the underlying 4D, $\mathcal{N} = 2$ supersymmetric tensor and vector supermultiplets take a similar form to those investigated in [15].

This paper is organized as follows. The main result of the 4D, $\mathcal{N} = 4$ vector-tensor supermultiplet with $\text{GL}(2, \mathbb{R}) \otimes \text{GL}(2, \mathbb{R})$ isospin structure is given in section 2 and the appendices referenced therein. In contrast, section 3 and the appendices referenced therein display the usual 4D, $\mathcal{N} = 4$ vector multiplet, expressed in terms of a $\text{GL}(2, \mathbb{R}) \otimes \text{GL}(2, \mathbb{R})$ isospin structure. The version of the 4D, $\mathcal{N} = 4$ vector multiplet investigated in [15] is reviewed in appendix B. A Majorana notation is used throughout the paper.

2 Moving from 4D, $\mathcal{N} = 2$ to 4D, $\mathcal{N} = 4$ SUSY

The supersymmetry transformation properties and Lagrangians for both the 4D, $\mathcal{N} = 2$ vector supermultiplet $\{A, B, F, G, A_\mu, d, \Psi_c^k\}$ and the 4D, $\mathcal{N} = 2$ tensor supermultiplet $\{\tilde{A}, \tilde{B}, \tilde{F}, \tilde{G}, \tilde{\varphi}, B_{\mu\nu}, \tilde{\Psi}_c^k\}$ are given in the appendix. The expectation is that if we add their two respective Lagrangians $\mathcal{L}_{(2VS)}$ and $\mathcal{L}_{(2TS)}$ together, the sum of these should be able to realize two additional supersymmetries. Therefore, we introduce a “second doublet” of supersymmetrical covariant derivative operators denoted by $\tilde{D}_a^i$ where we use the “tilde” in the notation $\tilde{D}_a^i$ to distinguish these from the covariant derivatives D_a^i associated with the two manifest supersymmetries. The first set of transformations involving D_a^i is given in appendix A. The second supersymmetry covariant derivative $\tilde{D}_a^i$ is represented by a transformation with the property that under its action, each field in the 4D, $\mathcal{N} = 2$ vector supermultiplet is transformed into a field in the 4D, $\mathcal{N} = 2$ tensor supermultiplet and vice versa. We also require these transformations to act linearly on the field variables. We are thus motivated to make an ansatz that requires the introduction of two sets of twelve matrices in $\text{GL}(2, \mathbb{R})$ $\{(\mathcal{V}_1)^{ij}, \dots, (\mathcal{V}_{12})^{ij}\}$ and $\{(\mathcal{U}_1)^{ij}, \dots, (\mathcal{U}_{12})^{ij}\}$ that are used to write a realization of the action of $\tilde{D}_a^i$ according to

$$\begin{aligned}
 \tilde{D}_a^i A &= (\mathcal{V}_1)^{ij} \tilde{\Psi}_a^j, \\
 \tilde{D}_a^i B &= i(\mathcal{V}_2)^{ij} (\gamma^5)_a^b \tilde{\Psi}_b^j, \\
 \tilde{D}_a^i F &= (\mathcal{V}_3)^{ij} (\gamma^\mu)_a^b \partial_\mu \tilde{\Psi}_b^j, \\
 \tilde{D}_a^i G &= i(\mathcal{V}_4)^{ij} (\gamma^5 \gamma^\mu)_a^b \partial_\mu \tilde{\Psi}_b^j, \\
 \tilde{D}_a^i A_\mu &= i(\mathcal{V}_5)^{ij} (\gamma_\mu)_a^b \tilde{\Psi}_b^j, \\
 \tilde{D}_a^i d &= i(\mathcal{V}_6)^{ij} (\gamma^5 \gamma^\mu)_a^b \partial_\mu \tilde{\Psi}_b^j, \\
 \tilde{D}_a^i \Psi_b^j &= [-(\mathcal{V}_7)^{ij} (\gamma^5 \gamma^\mu)_{ab} \partial_\mu \tilde{B} - i(\mathcal{V}_8)^{ij} C_{ab} \tilde{F} + (\mathcal{V}_9)^{ij} (\gamma^5)_{ab} \tilde{G}] + i(\mathcal{V}_{10})^{ij} (\gamma^\mu)_{ab} \partial_\mu \tilde{A} \\
 &\quad + i(\mathcal{V}_{11})^{ij} (\gamma^\mu)_{ab} \partial_\mu \tilde{\varphi} - i(\mathcal{V}_{12})^{ij} \epsilon_\mu^{\nu\alpha\beta} (\gamma^5 \gamma^\mu)_{ab} \partial_\nu \tilde{B}_{\alpha\beta}, \quad (2.1)
 \end{aligned}$$

on the fields in the 4D, $\mathcal{N} = 2$ vector supermultiplet and as

$$\begin{aligned}
 \tilde{D}_a^i \tilde{A} &= (\mathcal{U}_1)^{ij} \Psi_a^j \quad , \\
 \tilde{D}_a^i \tilde{B} &= i(\mathcal{U}_2)^{ij} (\gamma^5)_a{}^b \Psi_b^j \quad , \\
 \tilde{D}_a^i \tilde{F} &= (\mathcal{U}_3)^{ij} (\gamma^\mu)_a{}^b \partial_\mu \Psi_b^j \quad , \\
 \tilde{D}_a^i \tilde{G} &= i(\mathcal{U}_4)^{ij} (\gamma^5 \gamma^\mu)_a{}^b \partial_\mu \Psi_b^j \quad , \\
 \tilde{D}_a^i \tilde{\varphi} &= (\mathcal{U}_5)^{ij} \Psi_a^j \quad , \\
 \tilde{D}_a^i \tilde{B}_{\mu\nu} &= -i \frac{1}{4} (\mathcal{U}_6)^{ij} ([\gamma_\mu, \gamma_\nu])_a{}^b \Psi_b^j \quad , \\
 \tilde{D}_a^i \tilde{\Psi}_b^j &= [i(\mathcal{U}_7)^{ij} (\gamma^\mu)_{ab} \partial_\mu A - (\mathcal{U}_8)^{ij} (\gamma^5 \gamma^\mu)_{ab} \partial_\mu B - i(\mathcal{U}_9)^{ij} C_{ab} F] + (\mathcal{U}_{10})^{ij} (\gamma^5)_{ab} G \\
 &\quad + (\mathcal{U}_{11})^{ij} (\gamma^5)_{ab} d + \frac{1}{4} (\mathcal{U}_{12})^{ij} ([\gamma^\mu, \gamma^\nu])_{ab} (\partial_\mu A_\nu - \partial_\nu A_\mu) \quad ,
 \end{aligned} \tag{2.2}$$

on the fields in the 4D, $\mathcal{N} = 2$ tensor supermultiplet.

We next seek solutions for the $(\mathcal{U}_n)^{ij}$ and $(\mathcal{V}_n)^{ij}$ matrices that lead to invariance of the Lagrangian $\mathcal{L}_{(4TV)} = \mathcal{L}_{(2VS)} + \mathcal{L}_{(2TS)}$:

$$\tilde{D}_a^i \mathcal{L}_{(4TV)} = 0 + \text{total derivative} \quad . \tag{2.3}$$

The solution to this condition is given by

$$\begin{aligned}
 (\mathcal{U}_1)^{ij} &= (\mathcal{V}_{10})^{ij} \quad , & (\mathcal{U}_2)^{ij} &= (\mathcal{V}_7)^{ij} \quad , & (\mathcal{U}_3)^{ij} &= (\mathcal{V}_8)^{ij} \quad , & (\mathcal{U}_4)^{ij} &= (\mathcal{V}_9)^{ij} \quad , \\
 (\mathcal{U}_5)^{ij} &= (\mathcal{V}_{11})^{ij} \quad , & (\mathcal{U}_6)^{ij} &= (\mathcal{V}_{12})^{ij} \quad , & (\mathcal{U}_7)^{ij} &= (\mathcal{V}_1)^{ij} \quad , & (\mathcal{U}_8)^{ij} &= (\mathcal{V}_2)^{ij} \quad , \\
 (\mathcal{U}_9)^{ij} &= (\mathcal{V}_3)^{ij} \quad , & (\mathcal{U}_{10})^{ij} &= (\mathcal{V}_4)^{ij} \quad , & (\mathcal{U}_{11})^{ij} &= (\mathcal{V}_6)^{ij} \quad , & (\mathcal{U}_{12})^{ij} &= (\mathcal{V}_5)^{ij} \quad .
 \end{aligned} \tag{2.4}$$

For no choice of the $(\mathcal{V}_n)^{ij}$ matrices does the algebra close for $\{\tilde{D}_a^i, \tilde{D}_b^j\}$ without central charges. This has been verified via *Mathematica* code that we have made available open-source at the [HEPThools Data Repository](#). In contrast, there are 327,680 ways to close the $\{\tilde{D}_a^i, \tilde{D}_b^j\}$ portion of the algebra, up to gauge transformations. One of these choices is as follows (the ij indices are suppressed below, and also below I refers to the 2×2 identity matrix):

$$\mathcal{V}_n = \{i\sigma_2, i\sigma_2, i\sigma_2, \sigma_1, -iI, -\sigma_3, i\sigma_2, i\sigma_2, i\sigma_2, -\sigma_1, \sigma_3, -iI\} \quad . \tag{2.5}$$

For this choice, the $\{\tilde{D}_a^i, \tilde{D}_b^j\}$ portion of the algebra takes the form

$$\{\tilde{D}_a^i, \tilde{D}_b^j\} \chi = i2\delta^{ij} (\gamma^\mu)_{ab} \partial_\mu \chi \quad , \tag{2.6}$$

$$\{\tilde{D}_a^i, \tilde{D}_b^j\} A_\mu = i2\delta^{ij} (\gamma^\nu)_{ab} F_{\nu\mu} - i2(\sigma^2)^{ij} \partial_\mu [iC_{ab} A - (\gamma^5)_{ab} B] \quad , \tag{2.7}$$

$$\{\tilde{D}_a^i, \tilde{D}_b^j\} \tilde{B}_{\mu\nu} = i2\delta^{ij} (\gamma^\alpha)_{ab} \partial_\alpha \tilde{B}_{\mu\nu} - \partial_{[\mu} \tilde{\Lambda}_{\nu]ab}^{ij} \quad , \tag{2.8}$$

where $\chi = \{A, B, F, G, d, \Psi_c^k, \tilde{A}, \tilde{B}, \tilde{F}, \tilde{G}, \tilde{\varphi}, \tilde{\Psi}_c^k\}$ and

$$\tilde{\Lambda}_{\nu ab}^{ij} = 2i\delta^{ij} (\gamma^\alpha)_{ab} B_{\alpha\nu} + i(\sigma^1)^{ij} (\gamma_\nu)_{ab} \tilde{A} + i(\sigma^2)^{ij} (\gamma^5 \gamma_\nu)_{ab} \tilde{B} - i(\sigma^3)^{ij} (\gamma_\nu)_{ab} \tilde{\varphi} \quad . \tag{2.9}$$

The cross terms $\{D_a^i, \tilde{D}_b^j\}$ take the form

$$\{D_a^i, \tilde{D}_b^j\}A = -2i(\sigma^1)^{ij}(\gamma^\mu)_{ab}\partial_\mu\tilde{A} + 2i(\sigma^3)^{ij}(\gamma^\mu)_{ab}\partial_\mu\tilde{\varphi} \quad , \quad (2.10)$$

$$\{D_a^i, \tilde{D}_b^j\}B = 2(\sigma^2)^{ij}(\gamma^\mu)_{ab}\partial_\mu\tilde{B} + 2i\delta^{ij}\epsilon^{\mu\nu\alpha\beta}(\gamma_\mu)_{ab}\partial_\nu\tilde{B}_{\alpha\beta} \quad , \quad (2.11)$$

$$\begin{aligned} \{D_a^i, \tilde{D}_b^j\}F &= 2(\sigma^2)^{ij}(\gamma^\mu)_{ab}\partial_\mu\tilde{F} + i\delta^{ij}([\gamma^\alpha, \gamma^\beta])_{ab}\square\tilde{B}_{\alpha\beta} \\ &\quad - 2i\delta^{ij}([\gamma^\alpha, \gamma^\mu])_{ab}\partial_\mu\partial^\beta\tilde{B}_{\alpha\beta} \quad , \end{aligned} \quad (2.12)$$

$$\begin{aligned} \{D_a^i, \tilde{D}_b^j\}G &= 2i(\sigma^2)^{ij}(\gamma^5)_{ab}\square\tilde{A} + 2i(\sigma^1)^{ij}C_{ab}\square\tilde{B} + 2(\sigma^1)^{ij}(\gamma^5\gamma^\mu)_{ab}\partial_\mu\tilde{F} \\ &\quad - (\sigma^3)^{ij}(\gamma^5[\gamma^\alpha, \gamma^\beta])_{ab}\square\tilde{B}_{\alpha\beta} + 2(\sigma^3)^{ij}(\gamma^5[\gamma^\alpha, \gamma^\mu])_{ab}\partial_\mu\partial^\beta\tilde{B}_{\alpha\beta} \quad , \end{aligned} \quad (2.13)$$

$$\begin{aligned} \{D_a^i, \tilde{D}_b^j\}d &= -2i(\sigma^3)^{ij}C_{ab}\square\tilde{B} - 2(\sigma^3)^{ij}(\gamma^5\gamma^\mu)_{ab}\partial_\mu\tilde{F} + 2i(\sigma^2)^{ij}(\gamma^5)_{ab}\square\tilde{\varphi} \\ &\quad - (\sigma^1)^{ij}(\gamma^5[\gamma^\alpha, \gamma^\beta])_{ab}\square\tilde{B}_{\alpha\beta} + 2(\sigma^1)^{ij}(\gamma^5[\gamma^\alpha, \gamma^\mu])_{ab}\partial_\mu\partial^\beta\tilde{B}_{\alpha\beta} \quad , \end{aligned} \quad (2.14)$$

$$\{D_a^i, \tilde{D}_b^j\}\tilde{A} = -2i(\sigma^1)^{ij}(\gamma^\mu)_{ab}\partial_\mu A - 2i(\sigma^2)^{ij}(\gamma^5)_{ab}G \quad , \quad (2.15)$$

$$\begin{aligned} \{D_a^i, \tilde{D}_b^j\}\tilde{B} &= 2(\sigma^2)^{ij}(\gamma^\mu)_{ab}\partial_\mu B - 2i(\sigma^1)^{ij}C_{ab}G + 2i(\sigma^3)^{ij}C_{ab}d \\ &\quad + \delta^{ij}(\gamma^5[\gamma^\mu, \gamma^\nu])_{ab}\partial_\mu A_\nu \quad , \end{aligned} \quad (2.16)$$

$$\begin{aligned} \{D_a^i, \tilde{D}_b^j\}\tilde{F} &= 2(\sigma^2)^{ij}(\gamma^\mu)_{ab}\partial_\mu F + 2(\sigma^1)^{ij}(\gamma^5\gamma^\mu)_{ab}\partial_\mu G - 2(\sigma^3)^{ij}(\gamma^5\gamma^\mu)_{ab}\partial_\mu d \\ &\quad - 2i\delta^{ij}(\gamma^\mu)_{ab}\square A_\mu + 2i\delta^{ij}(\gamma^\mu)_{ab}\partial_\mu\partial^\nu A_\nu \quad , \end{aligned} \quad (2.17)$$

$$\{D_a^i, \tilde{D}_b^j\}\tilde{G} = 0 \quad , \quad (2.18)$$

$$\{D_a^i, \tilde{D}_b^j\}\tilde{\varphi} = 2i(\sigma^3)^{ij}(\gamma^\mu)_{ab}\partial_\mu A - 2i(\sigma^2)^{ij}(\gamma^5)_{ab}d \quad , \quad (2.19)$$

$$\begin{aligned} \{D_a^i, \tilde{D}_b^j\}A_\mu &= -\delta^{ij}(\gamma^5[\gamma_\mu, \gamma_\nu])_{ab}\partial^\nu\tilde{B} + 2i\delta^{ij}(\gamma_\mu)_{ab}\tilde{F} \\ &\quad - 2(\sigma^2)^{ij}([\gamma^\alpha, \gamma^\beta])_{ab}\partial_\alpha\tilde{B}_{\beta\mu} - \partial_\mu\tilde{\lambda}_{ab}^{ij} \quad , \end{aligned} \quad (2.20)$$

$$\begin{aligned} \{D_a^i, \tilde{D}_b^j\}\tilde{B}_{\mu\nu} &= -\frac{1}{2}\delta^{ij}(\gamma^5\gamma^\alpha[\gamma_\mu, \gamma_\nu])_{ab}\partial_\alpha B - i\frac{1}{2}\delta^{ij}([\gamma_\mu, \gamma_\nu])_{ab}F \\ &\quad + \frac{1}{2}(\sigma^3)^{ij}(\gamma^5[\gamma_\mu, \gamma_\nu])_{ab}G + \frac{1}{2}(\sigma^1)^{ij}(\gamma^5[\gamma_\mu, \gamma_\nu])_{ab}d \\ &\quad - \frac{1}{2}(\sigma^2)^{ij}([\gamma_\alpha, \gamma_{[\mu}])_{|ab|}\partial^\alpha A_{\nu]} - \partial_{[\mu}\tilde{\Lambda}_{\nu]ab}^{ij} \quad , \end{aligned} \quad (2.21)$$

$$\begin{aligned} \{D_a^i, \tilde{D}_b^j\}\Psi_c^k &= 2iZ_1^{ijkl}(\gamma^\mu)_{ab}\partial_\mu\tilde{\Psi}_c^l + 2iZ_2^{ijkl}([\gamma^\mu, \gamma^\nu])_{ab}(\gamma_\nu)_c{}^d\partial_\mu\tilde{\Psi}_d^l \\ &\quad + 2iZ_3^{ijkl}(\gamma_\nu)_{ab}([\gamma^\nu, \gamma^\mu])_c{}^d\partial_\mu\tilde{\Psi}_d^l + 2iZ_4^{ijkl}(\gamma^5[\gamma^\mu, \gamma^\nu])_{ab}(\gamma^5\gamma_\nu)_c{}^d\partial_\mu\tilde{\Psi}_d^l \\ &\quad + 2iZ_5^{ijkl}(\gamma^5\gamma^\mu)_{ab}(\gamma^5)_c{}^d\partial_\mu\tilde{\Psi}_d^l + 2iZ_6^{ijkl}(\gamma^5\gamma_\nu)_{ab}(\gamma^5[\gamma^\nu, \gamma^\mu])_c{}^d\partial_\mu\tilde{\Psi}_d^l \\ &\quad + 2iZ_7^{ijkl}(\gamma^5)_{ab}(\gamma^5\gamma^\mu)_c{}^d\partial_\mu\tilde{\Psi}_d^l + 2iZ_8^{ijkl}C_{ab}(\gamma^\mu)_c{}^d\partial_\mu\tilde{\Psi}_d^l \quad , \end{aligned} \quad (2.22)$$

$$\begin{aligned} \{D_a^i, \tilde{D}_b^j\}\tilde{\Psi}_c^k &= 2i\tilde{Z}_1^{ijkl}(\gamma^\mu)_{ab}\partial_\mu\Psi_c^l + 2i\tilde{Z}_2^{ijkl}([\gamma^\mu, \gamma^\nu])_{ab}(\gamma_\nu)_c{}^d\partial_\mu\Psi_d^l \\ &\quad + 2i\tilde{Z}_3^{ijkl}(\gamma_\nu)_{ab}([\gamma^\nu, \gamma^\mu])_c{}^d\partial_\mu\Psi_d^l + 2i\tilde{Z}_4^{ijkl}(\gamma^5[\gamma^\mu, \gamma^\nu])_{ab}(\gamma^5\gamma_\nu)_c{}^d\partial_\mu\Psi_d^l \\ &\quad + 2i\tilde{Z}_5^{ijkl}(\gamma^5\gamma^\mu)_{ab}(\gamma^5)_c{}^d\partial_\mu\Psi_d^l + 2i\tilde{Z}_6^{ijkl}(\gamma^5\gamma_\nu)_{ab}(\gamma^5[\gamma^\nu, \gamma^\mu])_c{}^d\partial_\mu\Psi_d^l \\ &\quad + 2i\tilde{Z}_7^{ijkl}(\gamma^5)_{ab}(\gamma^5\gamma^\mu)_c{}^d\partial_\mu\Psi_d^l + 2i\tilde{Z}_8^{ijkl}C_{ab}(\gamma^\mu)_c{}^d\partial_\mu\Psi_d^l \quad , \end{aligned} \quad (2.23)$$

where the parameters that are associated with the commutators of the gauge fields above

are given by

$$\tilde{\lambda}_{ab}^{ij} = (\sigma^2)^{ij}([\gamma^\alpha, \gamma^\beta])_{ab} \tilde{B}_{\alpha\beta} + 2i(\sigma^3)^{ij} C_{ab} \tilde{A} + 2i(\sigma^1)^{ij} C_{ab} \tilde{\varphi} \quad , \quad (2.24)$$

$$\tilde{\Lambda}_{\nu ab}^{ij} = -\frac{1}{2}(\sigma^2)^{ij}[\gamma_\nu, \gamma_\alpha]_{ab} A^\alpha - i\delta^{ij}(\gamma_\nu)_{ab} A - \delta^{ij}(\gamma^5 \gamma_\nu)_{ab} B \quad . \quad (2.25)$$

In a similar manner, the Z -factors and $\tilde{Z}$ -factors that appear in the commutators associated with the spinors are defined by

$$\begin{aligned} Z_1^{ijkl} &= -\frac{3}{2}(\sigma^1)^{ij}(\sigma^3)^{kl} - \frac{1}{4}(\sigma^1)^{ik}(\sigma^3)^{jl} + \frac{3}{4}(\sigma^3)^{il}(\sigma^1)^{jk} + (\sigma^1)^{il}(\sigma^3)^{jk} \quad , \\ Z_2^{ijkl} &= -\frac{1}{4}(\sigma^1)^{ij}(\sigma^3)^{kl} + \frac{1}{8}(\sigma^1)^{ik}(\sigma^3)^{jl} + \frac{1}{8}(\sigma^3)^{il}(\sigma^1)^{jk} \quad , \\ Z_3^{ijkl} &= \frac{1}{8}(\sigma^3)^{il}(\sigma^1)^{jk} - \frac{1}{8}(\sigma^1)^{ik}(\sigma^3)^{jl} \quad , \\ Z_4^{ijkl} &= \frac{1}{8}(\sigma^1)^{ik}(\sigma^3)^{jl} - \frac{1}{8}(\sigma^3)^{il}(\sigma^1)^{jk} \quad , \\ Z_5^{ijkl} &= -\frac{1}{2}(\sigma^1)^{ij}(\sigma^3)^{kl} - \frac{3}{4}(\sigma^1)^{ik}(\sigma^3)^{jl} + \frac{3}{4}(\sigma^3)^{il}(\sigma^1)^{jk} + \frac{1}{2}(\sigma^1)^{il}(\sigma^3)^{jk} \quad , \\ Z_6^{ijkl} &= -\frac{3}{8}(\sigma^1)^{ik}(\sigma^3)^{jl} + \frac{1}{8}(\sigma^3)^{il}(\sigma^1)^{jk} + \frac{1}{4}(\sigma^1)^{il}(\sigma^3)^{jk} \quad , \\ Z_7^{ijkl} &= \frac{1}{4}(\sigma^1)^{ik}(\sigma^3)^{jl} + \frac{1}{4}(\sigma^3)^{il}(\sigma^1)^{jk} - \frac{1}{2}(\sigma^1)^{il}(\sigma^3)^{jk} \quad , \\ Z_8^{ijkl} &= -\frac{1}{2}(\sigma^1)^{ij}(\sigma^3)^{kl} - \frac{3}{4}(\sigma^1)^{ik}(\sigma^3)^{jl} + \frac{3}{4}(\sigma^3)^{il}(\sigma^1)^{jk} + \frac{1}{2}(\sigma^1)^{il}(\sigma^3)^{jk} \quad , \\ \tilde{Z}_1^{ijkl} &= -\frac{3}{2}(\sigma^1)^{ij}(\sigma^3)^{kl} + \frac{1}{4}(\sigma^1)^{ik}(\sigma^3)^{jl} + \frac{3}{4}(\sigma^3)^{il}(\sigma^1)^{jk} + \frac{1}{2}(\sigma^1)^{il}(\sigma^3)^{jk} \quad , \\ \tilde{Z}_2^{ijkl} &= -\frac{1}{4}(\sigma^1)^{ij}(\sigma^3)^{kl} - \frac{1}{8}(\sigma^1)^{ik}(\sigma^3)^{jl} + \frac{1}{8}(\sigma^3)^{il}(\sigma^1)^{jk} + \frac{1}{4}(\sigma^1)^{il}(\sigma^3)^{jk} \quad , \\ \tilde{Z}_3^{ijkl} &= \frac{1}{8}(\sigma^3)^{il}(\sigma^1)^{jk} - \frac{1}{8}(\sigma^1)^{ik}(\sigma^3)^{jl} \quad , \\ \tilde{Z}_4^{ijkl} &= \frac{1}{8}(\sigma^1)^{ik}(\sigma^3)^{jl} - \frac{1}{8}(\sigma^3)^{il}(\sigma^1)^{jk} \quad , \\ \tilde{Z}_5^{ijkl} &= \frac{1}{2}(\sigma^1)^{ij}(\sigma^3)^{kl} + \frac{1}{4}(\sigma^1)^{ik}(\sigma^3)^{jl} - \frac{3}{4}(\sigma^3)^{il}(\sigma^1)^{jk} \quad , \\ \tilde{Z}_6^{ijkl} &= -\frac{1}{8}(\sigma^1)^{ik}(\sigma^3)^{jl} - \frac{1}{8}(\sigma^3)^{il}(\sigma^1)^{jk} + \frac{1}{4}(\sigma^1)^{il}(\sigma^3)^{jk} \quad , \\ \tilde{Z}_7^{ijkl} &= \frac{3}{4}(\sigma^1)^{ik}(\sigma^3)^{jl} - \frac{1}{4}(\sigma^3)^{il}(\sigma^1)^{jk} - \frac{1}{2}(\sigma^1)^{il}(\sigma^3)^{jk} \quad , \\ \tilde{Z}_8^{ijkl} &= \frac{1}{2}(\sigma^1)^{ij}(\sigma^3)^{kl} + \frac{1}{4}(\sigma^1)^{ik}(\sigma^3)^{jl} - \frac{3}{4}(\sigma^3)^{il}(\sigma^1)^{jk} \quad . \end{aligned} \quad (2.26)$$

To summarize the results seen here, the 4D, $\mathcal{N} = 4$ vector-tensor supermultiplet can be realized in terms of one 4D, $\mathcal{N} = 2$ vector-supermultiplet and one 4D, $\mathcal{N} = 2$ tensor-supermultiplet. The form of the super algebra of its four supercharges D_a^i and $\tilde{D}_a^i$ is

$$\begin{aligned} \{D_a^i, D_b^j\} &= \{\tilde{D}_a^i, \tilde{D}_b^j\} = i2\delta^{ij}(\gamma^\mu)_{ab}\partial_\mu \quad , \\ \{D_a^i, \tilde{D}_b^j\} &= \text{central charge} \quad , \end{aligned} \quad (2.27)$$

uniformly on all of the component fields, up to gauge transformations.

3 Hat derivative transformation laws

As the spins of the states in the 2D, $\mathcal{N} = 2$ vector-tensor supermultiplet are the same as in the Wess-Fayet 4D, $\mathcal{N} = 2$ supermultiplet [16, 17], there must be a formulation of the latter that is similar to the construction in section two. We thus introduce an ansatz of the form

$$\begin{aligned}
 \hat{D}_a^i A &= (\mathcal{W}_1)^{ij} \hat{\Psi}_a^j, \\
 \hat{D}_a^i B &= i(\mathcal{W}_2)^{ij} (\gamma^5)_a{}^b \hat{\Psi}_b^j, \\
 \hat{D}_a^i F &= (\mathcal{W}_3)^{ij} (\gamma^\mu)_a{}^b \partial_\mu \hat{\Psi}_b^i, \\
 \hat{D}_a^i G &= i(\mathcal{W}_4)^{ij} (\gamma^5 \gamma^\mu)_a{}^b \partial_\mu \hat{\Psi}_b^j, \\
 \hat{D}_a^i A_\mu &= (\mathcal{W}_5)^{ij} (\gamma_\mu)_a{}^b \hat{\Psi}_b^j, \\
 \hat{D}_a^i d &= i(\mathcal{W}_6)^{ij} (\gamma^5 \gamma^\mu)_a{}^b \partial_\mu \hat{\Psi}_b^j, \\
 \hat{D}_a^i \Psi_b^j &= i(\mathcal{W}_7)^{ij} (\gamma^\mu)_{ab} \partial_\mu \hat{A} - i(\mathcal{W}_8)^{ij} C_{ab} \hat{F} - (\mathcal{W}_9)^{ij} (\gamma^5 \gamma^\mu)_{ab} \partial_\mu \hat{B} + (\mathcal{W}_{10})^{ij} (\gamma_{ab}^5) \hat{G} \\
 &\quad + i(\mathcal{W}_{11})^{ij} (\gamma^\mu)_{ab} \partial_\mu \check{A} - i(\mathcal{W}_{12})^{ij} C_{ab} \check{F} - (\mathcal{W}_{13})^{ij} (\gamma^5 \gamma^\mu)_{ab} \partial_\mu \check{B} + (\mathcal{W}_{14})^{ij} (\gamma_{ab}^5) \check{G}
 \end{aligned} \tag{3.1}$$

$$\begin{aligned}
 \hat{D}_a^i \hat{A} &= (\mathcal{X}_1)^{ij} \Psi_a^j, \\
 \hat{D}_a^i \hat{B} &= i(\mathcal{X}_2)^{ij} (\gamma^5)_a{}^b \Psi_b^j, \\
 \hat{D}_a^i \hat{F} &= (\mathcal{X}_3)^{ij} (\gamma^\mu)_a{}^b \partial_\mu \Psi_b^j, \\
 \hat{D}_a^i \hat{G} &= i(\mathcal{X}_4)^{ij} (\gamma^5 \gamma^\mu)_a{}^b \partial_\mu \Psi_b^j, \\
 \hat{D}_a^i \check{A} &= (\mathcal{X}_5)^{ij} \Psi_a^j, \\
 \hat{D}_a^i \check{B} &= i(\mathcal{X}_6)^{ij} (\gamma^5)_a{}^b \Psi_b^j, \\
 \hat{D}_a^i \check{F} &= (\mathcal{X}_7)^{ij} (\gamma^\mu)_a{}^b \partial_\mu \Psi_b^j, \\
 \hat{D}_a^i \check{G} &= i(\mathcal{X}_8)^{ij} (\gamma^5 \gamma^\mu)_a{}^b \partial_\mu \Psi_b^j, \\
 \hat{D}_a^i \hat{\Psi}_b^j &= i(\mathcal{X}_9)^{ij} (\gamma^\mu)_{ab} \partial_\mu A - (\mathcal{X}_{10})^{ij} (\gamma^5 \gamma^\mu)_{ab} \partial_\mu B - i(\mathcal{X}_{11})^{ij} C_{ab} F + (\mathcal{X}_{12})^{ij} (\gamma_{ab}^5) G \\
 &\quad + (\mathcal{X}_{13})^{ij} (\gamma_{ab}^5) d - \frac{1}{4} i(\mathcal{X}_{14})^{ij} ([\gamma^\mu, \gamma^\nu])_{ab} (\partial_\mu A_\nu - \partial_\nu A_\mu)
 \end{aligned} \tag{3.2}$$

for the fields of the 4D, $\mathcal{N} = 2$ vector supermultiplet combined with the fields of the 4D, $\mathcal{N} = 2$ W-F supermultiplet.

We next seek solutions for the $(\mathcal{W}_n)^{ij}$ and $(\mathcal{X}_n)^{ij}$ that lead to invariance of the Lagrangian $\mathcal{L}_{(4CV)} = \mathcal{L}_{(2VS)} + \mathcal{L}_{(2CC)}$:

$$\hat{D}_a^i \mathcal{L}_{(4CV)} = 0 + \text{total derivative}. \tag{3.3}$$

This solution is

$$\begin{aligned}
 (\mathcal{X}_1)^{ij} &= (\mathcal{W}_7)^{ij}, & (\mathcal{X}_2)^{ij} &= (\mathcal{W}_9)^{ij}, & (\mathcal{X}_3)^{ij} &= (\mathcal{W}_8)^{ij}, & (\mathcal{X}_4)^{ij} &= (\mathcal{W}_{10})^{ij}, \\
 (\mathcal{X}_5)^{ij} &= (\mathcal{W}_{11})^{ij}, & (\mathcal{X}_6)^{ij} &= (\mathcal{W}_{13})^{ij}, & (\mathcal{X}_7)^{ij} &= (\mathcal{W}_{12})^{ij}, & (\mathcal{X}_8)^{ij} &= (\mathcal{W}_{14})^{ij}, \\
 (\mathcal{X}_9)^{ij} &= (\mathcal{W}_1)^{ij}, & (\mathcal{X}_{10})^{ij} &= (\mathcal{W}_2)^{ij}, & (\mathcal{X}_{11})^{ij} &= (\mathcal{W}_3)^{ij}, & (\mathcal{X}_{12})^{ij} &= (\mathcal{W}_4)^{ij}, \\
 (\mathcal{X}_{13})^{ij} &= (\mathcal{W}_6)^{ij}, & (\mathcal{X}_{14})^{ij} &= (\mathcal{W}_5)^{ij}.
 \end{aligned} \tag{3.4}$$

With the use of open-source *Mathematica* code that can be found at the [HEPTools Data Repository](#), we have found that even without imposing the above Lagrangian constraints, for no choice of $(\mathcal{W}_n)^{ij}$ does the algebra $\{\hat{D}_a^i, \hat{D}_b^j\}$ or $\{D_a^i, \hat{D}_b^j\}$ close on the fields of the chiral-chiral W-F hypermultiplet. That is for any possible choice of $(\mathcal{W}_n)^{ij}$, we necessarily have

$$\{\hat{D}_a^i, \hat{D}_b^j\}X \neq \delta^{ij}(\gamma^\mu)_{ab}\partial_\mu X \quad , \quad \{D_a^i, \hat{D}_b^j\}X \neq 0 \quad , \quad (3.5)$$

where X is at least one of the fields in the list

$$X \subset \hat{A}, \check{A}, \hat{B}, \check{B}, \hat{F}, \check{F}, \hat{G}, \check{G}, \hat{\Psi}_a^i \quad . \quad (3.6)$$

As such, we choose the $(\mathcal{W}_n)^{ij}$ such that the $\hat{D}_a^i$ transformation laws parallel those of the D_a^i transformation laws, i.e., $D_a^i \rightarrow \hat{D}_a^i$ and $\Psi_a^i \rightarrow \hat{\Psi}_a^i$:

$$\mathcal{W}_n = \{ I, I, I, \sigma^3, i\sigma^2, \sigma^1, \sigma^3, \sigma^3, I, I, \sigma^1, \sigma^1, i\sigma^2, i\sigma^2 \} \quad , \quad (3.7)$$

which upon enforcing eq. (3.4) demands

$$\mathcal{X}_n = \{ \sigma^3, I, \sigma^3, I, \sigma^1, i\sigma^2, \sigma^1, i\sigma^2, I, I, I, \sigma^3, \sigma^1, i\sigma^2 \} \quad . \quad (3.8)$$

With the above choices for $(\mathcal{W}_n)^{ij}$ and $(\mathcal{X}_n)^{ij}$, the transformation laws satisfy the following $\{\hat{D}_a^i, \hat{D}_b^j\}$ algebra on the fields of the $\mathcal{N} = 2$ vector multiplet

$$\{\hat{D}_a^i, \hat{D}_b^j\}\chi = i2\delta^{ij}(\gamma^\mu)_{ab}\partial_\mu\chi \quad , \quad (3.9)$$

$$\{\hat{D}_a^i, \hat{D}_b^j\}A_\mu = i2\delta^{ij}(\gamma^\nu)_{ab}F_{\nu\mu} + i2(\sigma^2)^{ij}\partial_\mu[iC_{ab}A - (\gamma^5)_{ab}B] \quad , \quad (3.10)$$

where

$$\chi \in \{A, B, F, G, d, \Psi_c^k\} \quad . \quad (3.11)$$

For the fields of the $\mathcal{N} = 2$ W-F hypermultiplet, the $\{\hat{D}_a^i, \hat{D}_b^j\}$ algebra is

$$\begin{aligned} \{\hat{D}_a^i, \hat{D}_b^j\}\hat{A} &= \delta^{ij}2i(\gamma^\mu)_{ab}\partial_\mu\hat{A} + i(\sigma^2)^{ij}2iC_{ab}\check{F} \quad , \\ \{\hat{D}_a^i, \hat{D}_b^j\}\check{A} &= \delta^{ij}2i(\gamma^\mu)_{ab}\partial_\mu\check{A} - i(\sigma^2)^{ij}2iC_{ab}\hat{F} \quad , \\ \{\hat{D}_a^i, \hat{D}_b^j\}\hat{B} &= \delta^{ij}2i(\gamma^\mu)_{ab}\partial_\mu\hat{B} + i(\sigma^2)^{ij}2iC_{ab}\check{G} \quad , \\ \{\hat{D}_a^i, \hat{D}_b^j\}\check{B} &= \delta^{ij}2i(\gamma^\mu)_{ab}\partial_\mu\check{B} - i(\sigma^2)^{ij}2iC_{ab}\hat{G} \quad , \\ \{\hat{D}_a^i, \hat{D}_b^j\}\hat{F} &= \delta^{ij}2i(\gamma^\mu)_{ab}\partial_\mu\hat{F} + i(\sigma^2)^{ij}2iC_{ab}\square\check{A} \quad , \\ \{\hat{D}_a^i, \hat{D}_b^j\}\check{F} &= \delta^{ij}2i(\gamma^\mu)_{ab}\partial_\mu\check{F} - i(\sigma^2)^{ij}2iC_{ab}\square\hat{A} \quad , \\ \{\hat{D}_a^i, \hat{D}_b^j\}\hat{G} &= \delta^{ij}2i(\gamma^\mu)_{ab}\partial_\mu\hat{G} + i(\sigma^2)^{ij}2iC_{ab}\square\check{B} \quad , \\ \{\hat{D}_a^i, \hat{D}_b^j\}\check{G} &= \delta^{ij}2i(\gamma^\mu)_{ab}\partial_\mu\check{G} - i(\sigma^2)^{ij}2iC_{ab}\square\hat{B} \quad , \\ \{\hat{D}_a^i, \hat{D}_b^j\}\hat{\Psi}_c^k &= \delta^{ij}2i(\gamma^\mu)_{ab}\partial_\mu\hat{\Psi}_c^k - (\sigma^2)^{ij}(\sigma^2)^{kr}2iC_{ab}(\gamma^\mu)_c^d\partial_\mu\hat{\Psi}_d^r \quad . \end{aligned} \quad (3.12)$$

For the cross terms $\{D_a^i, \hat{D}_b^j\}$, we have the following algebra for the bosons

$$\left\{D_a^i, \hat{D}_b^j\right\} A = 2i(\sigma^3)^{ij}(\gamma^\mu)_{ab}\partial_\mu \hat{A} + 2i(\sigma^1)^{ij}(\gamma^\mu)_{ab}\partial_\mu \check{A} - 2i(\sigma^2)^{ij}(\gamma^5\gamma^\mu)_{ab}\partial_\mu \check{B} \\ + 2i(\sigma^2)^{ij}(\gamma^5)_{ab}\check{G} \ , \quad (3.13)$$

$$\left\{D_a^i, \hat{D}_b^j\right\} B = 2i\delta^{ij}(\gamma^\mu)_{ab}\partial_\mu \hat{B} - 2(\sigma^2)^{ij}C_{ab}\check{G} \ , \quad (3.14)$$

$$\left\{D_a^i, \hat{D}_b^j\right\} F = 2i(\sigma^3)^{ij}(\gamma^\mu)_{ab}\partial_\mu \hat{F} + 2i(\sigma^2)^{ij}(\gamma^5)_{ab}\square\check{B} \\ + 2i(\sigma^1)^{ij}(\gamma^\mu)_{ab}\partial_\mu \check{F} - 2i(\sigma^2)^{ij}(\gamma^5\gamma^\mu)_{ab}\partial_\mu \check{G} \ , \quad (3.15)$$

$$\left\{D_a^i, \hat{D}_b^j\right\} G = 2i(\sigma^3)^{ij}(\gamma^\mu)_{ab}\partial_\mu \hat{G} - 2i(\sigma^2)^{ij}(\gamma^5)_{ab}\square\check{A} \\ - 2i(\sigma^2)^{ij}(\gamma^5\gamma^\mu)_{ab}\partial_\mu \check{F} - 2i(\sigma^1)^{ij}(\gamma^\mu)_{ab}\partial_\mu \check{G} \ , \quad (3.16)$$

$$\left\{D_a^i, \hat{D}_b^j\right\} d = 2i(\sigma^2)^{ij}(\gamma^5)_{ab}\square\hat{A} + 2i(\sigma^2)^{ij}(\gamma^5\gamma^\mu)_{ab}\partial_\mu \hat{F} \\ + 2i(\sigma^1)^{ij}(\gamma^\mu)_{ab}\partial_\mu \hat{G} + 2i(\sigma^3)^{ij}(\gamma^\mu)_{ab}\partial_\mu \check{G} \ , \quad (3.17)$$

$$\left\{D_a^i, \hat{D}_b^j\right\} A_\mu = -i(\sigma^1)^{ij}[\gamma_\mu, \gamma_\nu]_{ab}\partial^\nu \hat{A} - 2i(\sigma^1)^{ij}(\gamma_\mu)_{ab}\hat{F} + 2i(\sigma^2)^{ij}(\gamma^5\gamma_\mu)_{ab}\hat{G} \\ + i(\sigma^3)^{ij}[\gamma_\mu, \gamma_\nu]_{ab}\partial^\nu \check{A} - \delta^{ij}(\gamma^5[\gamma_\mu, \gamma_\nu])_{ab}\partial^\nu \check{B} \\ + 2i(\sigma^3)^{ij}(\gamma_\mu)_{ab}\check{F} - \partial_\mu \hat{\lambda}_{ab}^{ij} \ , \quad (3.18)$$

$$\left\{D_a^i, \hat{D}_b^j\right\} \hat{A} = 2i(\sigma^3)^{ij}(\gamma^\mu)_{ab}\partial_\mu A - 2i(\sigma^2)^{ij}(\gamma^5)_{ab}d + i\frac{1}{2}(\sigma^1)^{ij}[\gamma^\mu, \gamma^\nu]_{ab}F_{\mu\nu} \ , \quad (3.19)$$

$$\left\{D_a^i, \hat{D}_b^j\right\} \check{A} = 2i(\sigma^1)^{ij}(\gamma^\mu)_{ab}\partial_\mu A + 2i(\sigma^2)^{ij}(\gamma^5)_{ab}G - i\frac{1}{2}(\sigma^3)^{ij}[\gamma^\mu, \gamma^\nu]_{ab}F_{\mu\nu} \ , \quad (3.20)$$

$$\left\{D_a^i, \hat{D}_b^j\right\} \hat{B} = 2i\delta^{ij}(\gamma^\mu)_{ab}\partial_\mu B \ , \quad (3.21)$$

$$\left\{D_a^i, \hat{D}_b^j\right\} \check{B} = -2i(\sigma^2)^{ij}(\gamma^5\gamma^\mu)_{ab}\partial_\mu A - 2i(\sigma^2)^{ij}(\gamma^5)_{ab}F \\ + \frac{1}{2}\delta^{ij}(\gamma^5[\gamma^\mu, \gamma^\nu])_{ab}F_{\mu\nu} \ , \quad (3.22)$$

$$\left\{D_a^i, \hat{D}_b^j\right\} \hat{F} = 2i(\sigma^3)^{ij}(\gamma^\mu)_{ab}\partial_\mu F + 2i(\sigma^2)^{ij}(\gamma^5\gamma^\mu)_{ab}\partial_\mu d - 2i(\sigma^1)^{ij}(\gamma^\mu)_{ab}\partial^\nu F_{\mu\nu} \ , \quad (3.23)$$

$$\left\{D_a^i, \hat{D}_b^j\right\} \check{F} = 2i(\sigma^1)^{ij}(\gamma^\mu)_{ab}\partial_\mu F - 2i(\sigma^2)^{ij}(\gamma^5\gamma^\mu)_{ab}\partial_\mu G + 2i(\sigma^3)^{ij}(\gamma^\mu)_{ab}\partial^\nu F_{\mu\nu} \ , \quad (3.24)$$

$$\left\{D_a^i, \hat{D}_b^j\right\} \hat{G} = 2i(\sigma^3)^{ij}(\gamma^\mu)_{ab}\partial_\mu G + 2i(\sigma^1)^{ij}(\gamma^\mu)_{ab}\partial_\mu d + 2i(\sigma^2)^{ij}(\gamma^5\gamma^\mu)_{ab}\partial^\nu F_{\mu\nu} \ , \quad (3.25)$$

$$\left\{D_a^i, \hat{D}_b^j\right\} \check{G} = -2i(\sigma^2)^{ij}(\gamma^5)_{ab}\square A + 2(\sigma^2)^{ij}C_{ab}\square B - 2i(\sigma^2)^{ij}(\gamma^5\gamma^\mu)_{ab}\partial_\mu F \\ - 2i(\sigma^1)^{ij}(\gamma^\mu)_{ab}\partial_\mu G + 2i(\sigma^3)^{ij}(\gamma^\mu)_{ab}\partial_\mu d \ . \quad (3.26)$$

The gauge term in the above for the vector field A_μ is

$$\hat{\lambda}_{ab}^{ij} = 2i(\sigma^2)^{ij}(\gamma^5)_{ab}\hat{B} \ . \quad (3.27)$$

For the fermions we have

$$\left\{D_a^i, \hat{D}_b^j\right\} \Psi_c^k = 2i\hat{Z}_1^{ijkl}(\gamma^\mu)_{ab}\partial_\mu \hat{\Psi}_c^l + 2i\hat{Z}_2^{ijkl}([\gamma^\mu, \gamma^\nu])_{ab}(\gamma_\nu)_c^d\partial_\mu \hat{\Psi}_d^l \\ + 2i\hat{Z}_3^{ijkl}(\gamma_\nu)_{ab}([\gamma^\nu, \gamma^\mu])_c^d\partial_\mu \hat{\Psi}_d^l + 2i\hat{Z}_4^{ijkl}(\gamma^5[\gamma^\mu, \gamma^\nu])_{ab}(\gamma^5\gamma_\nu)_c^d\partial_\mu \hat{\Psi}_d^l \\ + 2i\hat{Z}_5^{ijkl}(\gamma^5\gamma^\mu)_{ab}(\gamma^5)_c^d\partial_\mu \hat{\Psi}_d^l + 2i\hat{Z}_6^{ijkl}(\gamma^5\gamma_\nu)_{ab}(\gamma^5[\gamma^\nu, \gamma^\mu])_c^d\partial_\mu \hat{\Psi}_d^l \\ + 2i\hat{Z}_7^{ijkl}(\gamma^5)_{ab}(\gamma^5\gamma^\mu)_c^d\partial_\mu \hat{\Psi}_d^l + 2i\hat{Z}_8^{ijkl}C_{ab}(\gamma^\mu)_c^d\partial_\mu \hat{\Psi}_d^l \ , \quad (3.28)$$

$$\begin{aligned}
 \{D_a^i, \hat{D}_b^j\} \hat{\Psi}_c^k &= 2i \hat{\hat{Z}}_1^{ijkl} (\gamma^\mu)_{ab} \partial_\mu \Psi_c^l + 2i \hat{\hat{Z}}_2^{ijkl} ([\gamma^\mu, \gamma^\nu])_{ab} (\gamma_\nu)_c^d \partial_\mu \Psi_d^l \\
 &+ 2i \hat{\hat{Z}}_3^{ijkl} (\gamma_\nu)_{ab} ([\gamma^\nu, \gamma^\mu])_c^d \partial_\mu \Psi_d^l + 2i \hat{\hat{Z}}_4^{ijkl} (\gamma^5 [\gamma^\mu, \gamma^\nu])_{ab} (\gamma^5 \gamma_\nu)_c^d \partial_\mu \Psi_d^l \\
 &+ 2i \hat{\hat{Z}}_5^{ijkl} (\gamma^5 \gamma^\mu)_{ab} (\gamma^5)_c^d \partial_\mu \Psi_d^l + 2i \hat{\hat{Z}}_6^{ijkl} (\gamma^5 \gamma_\nu)_{ab} (\gamma^5 [\gamma^\nu, \gamma^\mu])_c^d \partial_\mu \Psi_d^l \\
 &+ 2i \hat{\hat{Z}}_7^{ijkl} (\gamma^5)_{ab} (\gamma^5 \gamma^\mu)_c^d \partial_\mu \Psi_d^l + 2i \hat{\hat{Z}}_8^{ijkl} C_{ab} (\gamma^\mu)_c^d \partial_\mu \Psi_d^l \quad , \quad (3.29)
 \end{aligned}$$

where the “ Z -factors” take the forms

$$\begin{aligned}
 \hat{\hat{Z}}_1^{ijkl} &= \frac{3}{4} (\sigma^1)^{ik} (\sigma^1)^{jl} + \frac{1}{4} (\sigma^1)^{il} (\sigma^1)^{jk} + (\sigma^3)^{il} (\sigma^3)^{jk} \quad , \\
 \hat{\hat{Z}}_2^{ijkl} &= \frac{1}{8} (\sigma^1)^{il} (\sigma^1)^{jk} - \frac{1}{8} (\sigma^1)^{ik} (\sigma^1)^{jl} \quad , \\
 \hat{\hat{Z}}_3^{ijkl} &= \frac{1}{8} (\sigma^1)^{ik} (\sigma^1)^{jl} - \frac{1}{8} (\sigma^1)^{il} (\sigma^1)^{jk} \quad , \\
 \hat{\hat{Z}}_4^{ijkl} &= \frac{3}{8} (\sigma^1)^{il} (\sigma^1)^{jk} - \frac{3}{8} (\sigma^1)^{ik} (\sigma^1)^{jl} \quad , \\
 \hat{\hat{Z}}_5^{ijkl} &= -\frac{1}{2} (\sigma^1)^{ij} (\sigma^1)^{kl} + \frac{1}{4} (\sigma^1)^{ik} (\sigma^1)^{jl} + \frac{1}{4} (\sigma^1)^{il} (\sigma^1)^{jk} \quad , \\
 \hat{\hat{Z}}_6^{ijkl} &= \frac{1}{4} (\sigma^1)^{ij} (\sigma^1)^{kl} - \frac{1}{8} (\sigma^1)^{ik} (\sigma^1)^{jl} - \frac{1}{8} (\sigma^1)^{il} (\sigma^1)^{jk} \quad , \\
 \hat{\hat{Z}}_7^{ijkl} &= -\frac{1}{2} (\sigma^1)^{ij} (\sigma^1)^{kl} + \frac{1}{4} (\sigma^1)^{ik} (\sigma^1)^{jl} + \frac{1}{4} (\sigma^1)^{il} (\sigma^1)^{jk} \quad , \\
 \hat{\hat{Z}}_8^{ijkl} &= -\frac{1}{2} (\sigma^1)^{ij} (\sigma^1)^{kl} - \frac{1}{4} (\sigma^1)^{ik} (\sigma^1)^{jl} + \frac{3}{4} (\sigma^1)^{il} (\sigma^1)^{jk} \quad (3.30)
 \end{aligned}$$

$$\begin{aligned}
 \hat{\hat{Z}}_1^{ijkl} &= \frac{5}{4} (\sigma^1)^{ik} (\sigma^1)^{jl} - \frac{1}{4} (\sigma^1)^{il} (\sigma^1)^{jk} + (\sigma^3)^{il} (\sigma^3)^{jk} \quad , \\
 \hat{\hat{Z}}_2^{ijkl} &= \frac{1}{8} (\sigma^1)^{ik} (\sigma^1)^{jl} - \frac{1}{8} (\sigma^1)^{il} (\sigma^1)^{jk} \quad , \\
 \hat{\hat{Z}}_3^{ijkl} &= \frac{1}{8} (\sigma^1)^{il} (\sigma^1)^{jk} - \frac{1}{8} (\sigma^1)^{ik} (\sigma^1)^{jl} \quad , \\
 \hat{\hat{Z}}_4^{ijkl} &= \frac{3}{8} (\sigma^1)^{ik} (\sigma^1)^{jl} - \frac{3}{8} (\sigma^1)^{il} (\sigma^1)^{jk} \quad , \\
 \hat{\hat{Z}}_5^{ijkl} &= \frac{1}{2} (\sigma^1)^{ij} (\sigma^1)^{kl} - \frac{1}{4} (\sigma^1)^{ik} (\sigma^1)^{jl} - \frac{1}{4} (\sigma^1)^{il} (\sigma^1)^{jk} \quad , \\
 \hat{\hat{Z}}_6^{ijkl} &= -\frac{1}{4} (\sigma^1)^{ij} (\sigma^1)^{kl} + \frac{1}{8} (\sigma^1)^{ik} (\sigma^1)^{jl} + \frac{1}{8} (\sigma^1)^{il} (\sigma^1)^{jk} \quad , \\
 \hat{\hat{Z}}_7^{ijkl} &= \frac{1}{2} (\sigma^1)^{ij} (\sigma^1)^{kl} - \frac{1}{4} (\sigma^1)^{ik} (\sigma^1)^{jl} - \frac{1}{4} (\sigma^1)^{il} (\sigma^1)^{jk} \quad , \\
 \hat{\hat{Z}}_8^{ijkl} &= \frac{1}{2} (\sigma^1)^{ij} (\sigma^1)^{kl} - \frac{3}{4} (\sigma^1)^{ik} (\sigma^1)^{jl} + \frac{1}{4} (\sigma^1)^{il} (\sigma^1)^{jk} \quad . \quad (3.31)
 \end{aligned}$$

To compare the results seen here against the ones found in section two for the 4D, $\mathcal{N} = 4$ vector-tensor supermultiplet, the form of the super algebra of its four supercharges D_a^i and $\hat{D}_a^i$ in this section is

$$\begin{aligned}
 \{D_a^i, D_b^j\} &= i2\delta^{ij} (\gamma^\mu)_{ab} \partial_\mu + \text{central charge} \quad , \\
 \{\hat{D}_a^i, \hat{D}_b^j\} &= i2\delta^{ij} (\gamma^\mu)_{ab} \partial_\mu + \text{central charge} \quad , \\
 \{D_a^i, \hat{D}_b^j\} &= \text{central charge} \quad , \quad (3.32)
 \end{aligned}$$

on the fields in the W-F 4D, $\mathcal{N} = 2$ submultiplet. An examination of the results in (3.12) and in (A.12) reveals that the central charges that appear in the first two equations above are identical by our choice of the ansatz in (3.1) and (3.2). The four supercharges close on the fields of the 4D, $\mathcal{N} = 2$ vector submultiplet without central charges.

4 Conclusion

This short note expressed the 4D, $\mathcal{N}=4$ vector-tensor supermultiplet in terms of a $\text{GL}(2, \mathbb{R}) \otimes \text{GL}(2, \mathbb{R})$ isospin structure, rather than the original $\text{SP}(4)$ isospin structure presented in [6]. The $\text{GL}(2, \mathbb{R}) \otimes \text{GL}(2, \mathbb{R})$ structure maintains the manifest off-shell closure of the 4D, $\mathcal{N} = 2$ vector and tensor submultiplets. Using modern computing techniques, we have exhaustively analyzed all possible ways of marrying the two 4D, $\mathcal{N} = 2$ off-shell supermultiplet discussed in this paper into a 4D, $\mathcal{N}=4$ off-shell multiplets, and found no possible way of closing the resulting $\mathcal{N}=4$ algebra without central charges. The associated *Mathematica* code is available open-source at the [HEPThools Data Repository](#).

Thus, our work has extended a familiar construction that has long been used in supersymmetrical field theories. One can begin with two 4D, $\mathcal{N} = 1$ chiral supermultiplets in the context of a non-linear σ -model and extend this to an $\mathcal{N} = 2$ non-linear σ -model by introducing a complex structure f_j^j . In the current discussion, the analogs of such a complex structure consist of the set of twelve matrices shown in (2.5) for combining the 4D, $\mathcal{N} = 2$ vector and tensor supermultiplets to form a 4D, $\mathcal{N} = 4$ representation or the set of twelve matrices shown in (3.8) for combining the 4D, $\mathcal{N} = 2$ vector and W-F supermultiplets to form a 4D, $\mathcal{N} = 4$ representation. So this immediately raises the question of precisely what mathematical structure is being described by these sets of matrices?

There are several possible future avenues that suggest themselves for further study. One obvious one is the mathematical structure of non-linear σ -models related to these 4D, $\mathcal{N} = 4$ supermultiplets. This is a question to be answered both in four dimensions and in the dimensional reduction of such models. Another distinct direction is to use these sorts of discussions to drive exploration of the representation theory of supersymmetrical models via the approach of adinkras [18–24] and corresponding methods in four dimensional theories [25–29].

It should be clear that the doublets of supercovariant derivatives either given by $(D_a^i, \tilde{D}_b^j)$ or $(D_a^i, \hat{D}_b^j)$ can be regarded as the components of a single supercovariant derivative $D_a^{i\hat{A}}$ that possesses a pair of “isospin” indices that each take on two values. Given that the 4D, $\mathcal{N} = 4$ vector-tensor supermultiplet possesses the simplest central charge structure, it might be profitable to study the supervector fields associated with the supermanifold whose coordinates are dual to $D_a^{i\hat{A}}$ in the context of the 4D, $\mathcal{N} = 4$ superconformal symmetry.

“If you obey all the rules, you miss all the fun.”

— Katharine Hepburn

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A Reviewing 4D, $\mathcal{N} = 2$ supersymmetry results

In this appendix, for the convenience of the reader we simply present the form of the 4D, $\mathcal{N} = 2$ supermultiplets used in our text. We use the index convention $i = 1, 2$ to label the two supersymmetries. Furthermore our definitions are such that

$$\begin{aligned}
 (\sigma^0)^{ij} &= \delta^{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & (\sigma^1)^{ij} &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
 (\sigma^2)^{ij} &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, & (\sigma^3)^{ij} &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\
 \eta_{\mu\nu} &= \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, & \epsilon_{0123} &= 1.
 \end{aligned} \tag{A.1}$$

A.1 4D, $\mathcal{N}=2$ vector supermultiplet

The corresponding transformation laws are

$$\begin{aligned}
 D_a^i A &= \Psi_a^i, \\
 D_a^i B &= i(\gamma^5)_a{}^b \Psi_b^i, \\
 D_a^i F &= (\gamma^\mu)_a{}^b \partial_\mu \Psi_b^i, \\
 D_a^i G &= i(\sigma^3)^{ij} (\gamma^5 \gamma^\mu)_a{}^b \partial_\mu \Psi_b^j, \\
 D_a^i A_\mu &= i(\sigma^2)^{ij} (\gamma_\mu)_a{}^b \Psi_b^j, \\
 D_a^i d &= i(\sigma^1)^{ij} (\gamma^5 \gamma^\mu)_a{}^b \partial_\mu \Psi_b^j, \\
 D_a^i \Psi_b^j &= \delta^{ij} [i(\gamma^\mu)_{ab} \partial_\mu A - (\gamma^5 \gamma^\mu)_{ab} \partial_\mu B - iC_{ab} F] + (\sigma^3)^{ij} (\gamma^5)_{ab} G \\
 &\quad + (\sigma^1)^{ij} (\gamma^5)_{ab} d + \frac{1}{4} (\sigma^2)^{ij} ([\gamma^\mu, \gamma^\nu])_{ab} (\partial_\mu A_\nu - \partial_\nu A_\mu).
 \end{aligned} \tag{A.2}$$

The transformation laws satisfy the algebra

$$\{D_a^i, D_b^j\} \chi = i2\delta^{ij} (\gamma^\mu)_{ab} \partial_\mu \chi, \tag{A.3}$$

$$\{D_a^i, D_b^j\} A_\mu = i2\delta^{ij} (\gamma^\nu)_{ab} F_{\nu\mu} + i(\sigma^2)^{ij} [i2C_{ab} \partial_\mu A - 2(\gamma^5)_{ab} \partial_\mu B], \tag{A.4}$$

where $\chi \in \{A, B, F, G, d, \Psi_c^k\}$ and the Lagrangian is

$$\begin{aligned}
 \mathcal{L}_{(2VS)} &= -\frac{1}{2} \partial_\mu A \partial^\mu A - \frac{1}{2} \partial_\mu B \partial^\mu B + \frac{1}{2} F^2 + \frac{1}{2} G^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} d^2 \\
 &\quad + i\frac{1}{2} (\gamma^\mu)^{bc} \Psi_b^i \partial_\mu \Psi_c^i,
 \end{aligned} \tag{A.5}$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

A.2 4D, $\mathcal{N}=2$ tensor supermultiplet

The corresponding transformation laws are

$$\begin{aligned}
 D_a^i \tilde{A} &= (\sigma^3)^{ij} \tilde{\Psi}_a^j, \\
 D_a^i \tilde{B} &= i(\gamma^5)_a{}^b \tilde{\Psi}_b^i, \\
 D_a^i \tilde{F} &= (\gamma^\mu)_a{}^b \partial_\mu \tilde{\Psi}_b^i, \\
 D_a^i \tilde{G} &= i(\gamma^5 \gamma^\mu)_a{}^b \partial_\mu \tilde{\Psi}_b^i, \\
 D_a^i \tilde{\varphi} &= (\sigma^1)^{ij} \tilde{\Psi}_a^j, \\
 D_a^i \tilde{B}_{\mu\nu} &= -i \frac{1}{4} (\sigma^2)^{ij} ([\gamma_\mu, \gamma_\nu])_a{}^b \tilde{\Psi}_b^j, \\
 D_a^i \tilde{\Psi}_b^j &= \delta^{ij} [-(\gamma^5 \gamma^\mu)_{ab} \partial_\mu \tilde{B} - i C_{ab} \tilde{F} + (\gamma^5)_{ab} \tilde{G}] + i(\sigma^3)^{ij} (\gamma^\mu)_{ab} \partial_\mu \tilde{A} \\
 &\quad + i(\sigma^1)^{ij} (\gamma^\mu)_{ab} \partial_\mu \tilde{\varphi} - i(\sigma^2)^{ij} \epsilon_\mu{}^{\nu\alpha\beta} (\gamma^5 \gamma^\mu)_{ab} \partial_\nu \tilde{B}_{\alpha\beta},
 \end{aligned} \tag{A.6}$$

The transformation laws satisfy the algebra

$$\{D_a^i, D_b^j\} \chi = i 2 \delta^{ij} (\gamma^\mu)_{ab} \partial_\mu \chi, \tag{A.7}$$

$$\begin{aligned}
 \{D_a^i, D_b^j\} \tilde{B}_{\mu\nu} &= i 2 \delta^{ij} (\gamma^\alpha)_{ab} H_{\alpha\mu\nu} \\
 &\quad + i(\gamma_{[\mu} \partial_{\nu]})_{ac} \partial_\nu \left[(\sigma^1)^{ij} \delta_b{}^c \tilde{A} + (\sigma^2)^{ij} (\gamma^5)_b{}^c \tilde{B} - (\sigma^3)^{ij} \delta_b{}^c \tilde{\varphi} \right],
 \end{aligned} \tag{A.8}$$

where $\chi \in \{\tilde{A}, \tilde{B}, \tilde{F}, \tilde{G}, \tilde{\varphi}, \tilde{\Psi}_c^k\}$ and the Lagrangian is

$$\begin{aligned}
 \mathcal{L}_{(2TS)} &= -\frac{1}{2} \partial_\mu \tilde{A} \partial^\mu \tilde{A} - \frac{1}{2} \partial_\mu \tilde{B} \partial^\mu \tilde{B} + \frac{1}{2} \tilde{F}^2 + \frac{1}{2} \tilde{G}^2 - \frac{1}{3} H_{\mu\nu\alpha} H^{\mu\nu\alpha} - \frac{1}{2} \partial_\mu \tilde{\varphi} \partial^\mu \tilde{\varphi} \\
 &\quad + i \frac{1}{2} (\gamma^\mu)^{bc} \tilde{\Psi}_b^i \partial_\mu \tilde{\Psi}_c^i,
 \end{aligned} \tag{A.9}$$

with $H_{\mu\nu\alpha} = \partial_\mu \tilde{B}_{\nu\alpha} + \partial_\nu \tilde{B}_{\alpha\mu} + \partial_\alpha \tilde{B}_{\mu\nu}$.

A.3 4D, $\mathcal{N}=2$ W-F hypermultiplet

The transformation laws for the Wess-Fayet (W-F) hypermultiplet containing the Chiral-Chiral multiplet combination are

$$\begin{aligned}
 D_a^i \hat{A} &= (\sigma^3)^{ij} \hat{\Psi}_a^j, \\
 D_a^i \hat{B} &= i(\gamma^5)_a{}^b \hat{\Psi}_b^i, \\
 D_a^i \hat{F} &= (\sigma^3)^{ij} (\gamma^\mu)_a{}^b \partial_\mu \hat{\Psi}_b^j, \\
 D_a^i \hat{G} &= i(\gamma^5 \gamma^\mu)_a{}^b \partial_\mu \hat{\Psi}_b^i, \\
 D_a^i \check{A} &= (\sigma^1)^{ij} \hat{\Psi}_a^j, \\
 D_a^i \check{B} &= -(\sigma^2)^{ij} (\gamma^5)_a{}^b \hat{\Psi}_b^j, \\
 D_a^i \check{F} &= (\sigma^1)^{ij} (\gamma^\mu)_a{}^b \partial_\mu \hat{\Psi}_b^j, \\
 D_a^i \check{G} &= -(\sigma^2)^{ij} (\gamma^5 \gamma^\mu)_a{}^b \partial_\mu \hat{\Psi}_b^j, \\
 D_a^i \hat{\Psi}_b^j &= i(\sigma^3)^{ij} \left((\gamma^\mu)_{ab} \partial_\mu \hat{A} - C_{ab} \hat{F} \right) + \delta^{ij} \left(-(\gamma^5 \gamma^\mu)_{ab} \partial_\mu \hat{B} + (\gamma_{ab}^5) \hat{G} \right) \\
 &\quad + i(\sigma^1)^{ij} \left((\gamma^\mu)_{ab} \partial_\mu \check{A} - C_{ab} \check{F} \right) + i(\sigma^2)^{ij} \left(-(\gamma^5 \gamma^\mu)_{ab} \partial_\mu \check{B} + (\gamma_{ab}^5) \check{G} \right),
 \end{aligned} \tag{A.10}$$

The following Lagrangian is invariant with respect to these transformations:

$$\begin{aligned} \mathcal{L}_{(2CC)} = & -\frac{1}{2}\partial_\mu \hat{A} \partial^\mu \hat{A} - \frac{1}{2}\partial_\mu \check{A} \partial^\mu \check{A} - \frac{1}{2}\partial_\mu \hat{B} \partial^\mu \hat{B} - \frac{1}{2}\partial_\mu \check{B} \partial^\mu \check{B} \\ & + \frac{1}{2}\hat{F}^2 + \frac{1}{2}\check{F}^2 + \frac{1}{2}\hat{G}^2 + \frac{1}{2}\check{G}^2 + \frac{1}{2}i(\gamma^\mu)^{cd}\hat{\Psi}_c^i \partial_\mu \hat{\Psi}_d^i \quad , \end{aligned} \quad (\text{A.11})$$

which easily seen to be the direct sum of the $\mathcal{N} = 1$ invariant Lagrangians for the separate $(\hat{A}, \hat{B}, \hat{\Psi}_c^1, \hat{F}, \hat{G})$ chiral supermultiplet and the $(\check{A}, \check{B}, \check{\Psi}_c^2, \check{F}, \check{G})$ chiral supermultiplet. Direct calculation yields the following algebra:

$$\begin{aligned} \{D_a^i, D_b^j\} \hat{A} &= \delta^{ij} 2i(\gamma^\mu)_{ab} \partial_\mu \hat{A} + i(\sigma^2)^{ij} 2iC_{ab} \check{F} \quad , \\ \{D_a^i, D_b^j\} \check{A} &= \delta^{ij} 2i(\gamma^\mu)_{ab} \partial_\mu \check{A} - i(\sigma^2)^{ij} 2iC_{ab} \hat{F} \quad , \\ \{D_a^i, D_b^j\} \hat{B} &= \delta^{ij} 2i(\gamma^\mu)_{ab} \partial_\mu \hat{B} + i(\sigma^2)^{ij} 2iC_{ab} \check{G} \quad , \\ \{D_a^i, D_b^j\} \check{B} &= \delta^{ij} 2i(\gamma^\mu)_{ab} \partial_\mu \check{B} - i(\sigma^2)^{ij} 2iC_{ab} \hat{G} \quad , \\ \{D_a^i, D_b^j\} \hat{F} &= \delta^{ij} 2i(\gamma^\mu)_{ab} \partial_\mu \hat{F} + i(\sigma^2)^{ij} 2iC_{ab} \square \check{A} \quad , \\ \{D_a^i, D_b^j\} \check{F} &= \delta^{ij} 2i(\gamma^\mu)_{ab} \partial_\mu \check{F} - i(\sigma^2)^{ij} 2iC_{ab} \square \hat{A} \quad , \\ \{D_a^i, D_b^j\} \hat{G} &= \delta^{ij} 2i(\gamma^\mu)_{ab} \partial_\mu \hat{G} + i(\sigma^2)^{ij} 2iC_{ab} \square \check{B} \quad , \\ \{D_a^i, D_b^j\} \check{G} &= \delta^{ij} 2i(\gamma^\mu)_{ab} \partial_\mu \check{G} - i(\sigma^2)^{ij} 2iC_{ab} \square \hat{B} \quad , \\ \{D_a^i, D_b^j\} \hat{\Psi}_c^k &= \delta^{ij} 2i(\gamma^\mu)_{ab} \partial_\mu \hat{\Psi}_c^k - (\sigma^2)^{ij} (\sigma^2)^{kr} 2iC_{ab} (\gamma^\mu)_c^d \partial_\mu \hat{\Psi}_d^r \quad . \end{aligned} \quad (\text{A.12})$$

As can easily be seen (and is well-known) the pair-results in (A.3)–(A.4) and as well as the pair-results (A.7)–(A.8) describe closure of the SUSY algebra *without* central charges nor use of equations of motion for fermion fields. This is very different than the results in (A.12) where the closure of the algebra requires both the presence of central charges realized on the bosons and the enforcement of equations of motion on the fermions.

B Reviewing 4D, $\mathcal{N} = 4$ supersymmetry results

In this section, we review the Abelian 4D, $\mathcal{N} = 4$ SUSY-YM system in a Majorana representation as presented in [14].

B.1 $\mathcal{N} = 4$ transformation laws

The Lagrangian for the Abelian $d = 4$, $\mathcal{N} = 4$ SUSY-YM system

$$\begin{aligned} \mathcal{L} = & -\frac{1}{2}(\partial_\mu A^J)(\partial^\mu A^J) - \frac{1}{2}(\partial_\mu B^J)(\partial^\mu B^J) + i\frac{1}{2}(\gamma^\mu)^{ab}\psi_a^J \partial_\mu \psi_b^J + \frac{1}{2}(F^J)^2 + \frac{1}{2}(G^J)^2 \\ & - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}i(\gamma^\mu)^{cd}\lambda_c \partial_\mu \lambda_d + \frac{1}{2}d^2 \end{aligned} \quad (\text{B.1})$$

is invariant with respect to the global supersymmetric transformations

$$\begin{aligned} D_a A^J &= \psi_a^J \quad , \\ D_a B^J &= i(\gamma^5)_a^b \psi_b^J \quad , \end{aligned}$$

$$\begin{aligned}
 D_a \psi_b^J &= i(\gamma^\mu)_{ab} \partial_\mu A^J - (\gamma^5 \gamma^\mu)_{ab} \partial_\mu B^J - i C_{ab} F^J + (\gamma^5)_{ab} G^J, \\
 D_a F^J &= (\gamma^\mu)_a{}^b \partial_\mu \psi_b^J, \\
 D_a G^J &= i(\gamma^5 \gamma^\mu)_a{}^b \partial_\mu \psi_b^J.
 \end{aligned} \tag{B.2}$$

$$\begin{aligned}
 D_a A_\mu &= (\gamma_\mu)_a{}^b \lambda_b, \\
 D_a \lambda_b &= -\frac{1}{2}(\sigma^{\mu\nu})_{ab} F_{\mu\nu} + (\gamma^5)_{ab} d, \\
 D_a d &= i(\gamma^5 \gamma^\mu)_a{}^b \partial_\mu \lambda_b.
 \end{aligned} \tag{B.3}$$

$$\begin{aligned}
 D_a^I A^J &= \delta^{IJ} \lambda_a - \epsilon^{IJ}_K \psi_a^K, \\
 D_a^I B^J &= i(\gamma^5)_a{}^b [\delta^{IJ} \lambda_b + \epsilon^{IJ}_K \psi_b^K], \\
 D_a^I \psi_b^J &= \delta^{IJ} \left[\frac{1}{2}(\sigma^{\mu\nu})_{ab} F_{\mu\nu} + (\gamma^5)_{ab} d \right] \\
 &\quad - \epsilon^{IJ}_K [-i(\gamma^\mu)_{ab} \partial_\mu A^K - (\gamma^5 \gamma^\mu)_{ab} \partial_\mu B^K + i C_{ab} F^K + (\gamma^5)_{ab} G^K],
 \end{aligned} \tag{B.4}$$

$$\begin{aligned}
 D_a^I F^J &= (\gamma^\mu)_a{}^b \partial_\mu [\delta^{IJ} \lambda_b - \epsilon^{IJ}_K \psi_b^K], \\
 D_a^I G^J &= i(\gamma^5 \gamma^\mu)_a{}^b \partial_\mu [-\delta^{IJ} \lambda_b + \epsilon^{IJ}_K \psi_b^K], \\
 D_a^I A_\mu &= -(\gamma_\mu)_a{}^b \psi_b^I, \\
 D_a^I \lambda_b &= i(\gamma^\mu)_{ab} \partial_\mu A^I - (\gamma^5 \gamma^\mu)_{ab} \partial_\mu B^I - i C_{ab} F^I - (\gamma^5)_{ab} G^I, \\
 D_a^I d &= i(\gamma^5 \gamma^\mu)_a{}^b \partial_\mu \psi_b^I.
 \end{aligned} \tag{B.5}$$

where

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu], \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{B.6}$$

and our conventions for the gamma matrices are as in appendix A of [21].

B.2 Algebra

Using the shorthand

$$\chi = (A^I, B^I, F^I, G^I, d, \psi_c^J, \lambda_c), \tag{B.7}$$

the algebra can be written

$$\{D_a, D_b\} \chi = 2i(\gamma^\mu)_{ab} \partial_\mu \chi, \quad \{D_a, D_b\} A_\nu = 2i(\gamma^\mu)_{ab} F_{\mu\nu} \tag{B.8}$$

and

$$\begin{aligned}
 \{D_a^I, D_b^J\} A^K &= 2i\delta^{IJ}(\gamma^\mu)_{ab} \partial_\mu A^K - 2\epsilon^{IJK}(\gamma^5)_{ab} d - 2Z^{IJKM} [iC_{ab} F^M + (\gamma^5)_{ab} G^M], \\
 \{D_a^I, D_b^J\} B^K &= 2i\delta^{IJ}(\gamma^\mu)_{ab} \partial_\mu B^K + 2i\epsilon^{IJK} C_{ab} d, \\
 \{D_a^I, D_b^J\} F^K &= 2i\delta^{IJ}(\gamma^\mu)_{ab} \partial_\mu F^K + 2\epsilon^{IJK}(\gamma^5 \gamma^\mu)_{ab} \partial_\mu d \\
 &\quad + 2Z^{IJKM} [-iC_{ab} \square A^M + (\gamma^5 \gamma^\mu)_{ab} \partial_\mu G^M] \\
 \{D_a^I, D_b^J\} G^K &= 2i\delta^{IJ}(\gamma^\mu)_{ab} \partial_\mu G^K - 2\epsilon^{IJK}(\gamma^5 \gamma^\mu)_{ab} \partial^\nu F_{\mu\nu} \\
 &\quad - 2Z^{IJKM} [(\gamma^5)_{ab} \square A^M + (\gamma^5 \gamma^\mu)_{ab} \partial_\mu F^M] \\
 \{D_a^I, D_b^J\} d &= 2i\delta^{IJ}(\gamma^\mu)_{ab} \partial_\mu d + 2\epsilon^{IJK} ((\gamma^5)_{ab} \square A^K - iC_{ab} \square B^K + (\gamma^5 \gamma^\mu)_{ab} \partial_\mu F^K)
 \end{aligned} \tag{B.9}$$

$$\begin{aligned}
 \{D_a^I, D_b^J\}A_\nu &= 2i\delta^{IJ}(\gamma^\mu)_{ab}F_{\mu\nu} + 2\epsilon^{IJK}(iC_{ab}\partial_\nu A^K + (\gamma^5)_{ab}\partial_\nu B^K - (\gamma^5\gamma_\nu)_{ab}G^K) \\
 \{D_a^I, D_b^J\}\lambda_c &= 2i\delta^{IJ}(\gamma^\mu)_{ab}\partial_\mu\lambda_c + i\epsilon^{IJK}[-C_{ab}(\gamma^\mu)_c^d + (\gamma^5)_{ab}(\gamma^5\gamma^\mu)_c^d + \\
 &\quad + (\gamma^5\gamma^\nu)_{ab}(\gamma^5\gamma_\nu\gamma^\mu)_c^d]\partial_\mu\psi_c^K \\
 \{D_a^I, D_b^J\}\psi_c^K &= 2i\delta^{IJ}(\gamma^\mu)_{ab}\partial_\mu\psi_c^K \\
 &\quad - i\epsilon^{IJK}[-C_{ab}(\gamma^\mu)_c^d + (\gamma^5)_{ab}(\gamma^5\gamma^\mu)_c^d + (\gamma^5\gamma^\nu)_{ab}(\gamma^5\gamma_\nu\gamma^\mu)_c^d]\partial_\mu\lambda_d \\
 &\quad - iZ^{IJKM}[C_{ab}(\gamma^\mu)_c^d + (\gamma^5)_{ab}(\gamma^5\gamma^\mu)_c^d + (\gamma^5\gamma^\nu)_{ab}(\gamma^5\gamma_\nu\gamma^\mu)_c^d]\partial_\mu\psi_d^M
 \end{aligned} \tag{B.10}$$

and for the cross terms

$$\begin{aligned}
 \{D_a, D_b^I\}A^J &= 2i\epsilon^{IJK}C_{ab}F^K \\
 \{D_a, D_b^I\}B^J &= 2i\epsilon^{IJK}C_{ab}G^K \\
 \{D_a, D_b^I\}F^J &= 2i\epsilon^{IJK}C_{ab}\square A^K \\
 \{D_a, D_b^I\}G^J &= 2i\epsilon^{IJK}C_{ab}\square B^K \\
 \{D_a, D_b^I\}\lambda_c &= 0 \\
 \{D_a, D_b^I\}d &= 0
 \end{aligned} \tag{B.11}$$

$$\begin{aligned}
 \{D_a, D_b^I\}A_\nu &= 2iC_{ab}\partial_\nu A^I - 2(\gamma^5)_{ab}\partial_\nu B^I \\
 \{D_a, D_b^I\}\psi_c^J &= 2i\epsilon^{IJK}C_{ab}(\gamma^\mu)_c^d\partial_\mu\psi_d^K
 \end{aligned} \tag{B.12}$$

where

$$Z^{IJKM} \equiv \delta^{IM}\delta^{JK} - \delta^{IK}\delta^{JM} \tag{B.13}$$

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