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Twin conformal field theories

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Abstract: Supersymmetric theories with the same bosonic content but different fermions, aka *twins*, were thought to exist only for supergravity. Here we show that pairs of super conformal field theories, for example exotic $\mathcal{N} = 3$ and $\mathcal{N} = 1$ theories in $D = 4$ spacetime dimensions, can also be twin. We provide evidence from three different perspectives: (i) a twin S-fold construction, (ii) a double-copy argument and (iii) by identifying candidate twin holographically dual gauged supergravity theories. Furthermore, twin W-supergravity theories then follow by applying the double-copy prescription to exotic super conformal field theories.

Keywords: Conformal Field Theory, Supersymmetric Gauge Theory, Conformal Field Models in String Theory, Supergravity Models

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Contents

1 Introduction

We argue that there exist *twin* super conformal field theories (SCFTs) having the same bosonic sectors but distinct supersymmetric completions. In particular, the exotic S-fold $\mathcal{N} = 3$ SCFTs [1–4] have exotic $\mathcal{N} = 1$ twins based on a closely analogous construction.

In the context of supergravity, it has been long-known [5–10] that there exist twins with identical bosonic sectors, both in terms of content and couplings, but distinct degrees of supersymmetry $\mathcal{N}_b > \mathcal{N}_l$. We denote such pairs by $\{\mathcal{N}_b, \mathcal{N}_l\}$, where b and l refer to the 'big' and 'little' twin, respectively. This is made possible by the presence of $\mathcal{N}_b$ spin-3/2 gravitini in the $\mathcal{N}_b$-extended gravity multiplet, some of which can be traded-in for spin-1/2 fields living in $\mathcal{N}_l$-extended matter multiplets. This observation would seem to rule out the possibility of twin field theories with rigid supersymmetry. However, this naïve obstruction is circumvented through S-foldings that completely remove the massless states. As described in [11], the lowest order operator, aside from the $12 + 12$ super and superconformal charges, preserved by the S-fold projecting onto the exotic $\mathcal{N} = 3$ SCFT is the supercurrent multiplet. It corresponds to the $\mathcal{N}=3$ super-Weyl multiplet, which was fully constructed in [12], consisting of the massive Spin(3) $\times$ Sp(3) states,

$$[3,2] = (\mathbf{5},\mathbf{1}) \oplus (\mathbf{4},\mathbf{6}) \oplus (\mathbf{3},\mathbf{14}+\mathbf{1}) \oplus (\mathbf{2},\mathbf{14}'+\mathbf{6}) \oplus (\mathbf{1},\mathbf{14}), \tag{1.1}$$

where we denote by $[\mathcal{N}, j_{\max}]$ the massive $\mathcal{N}$-extended long supermultiplet with top spin $j_{\max}$, as constructed in [13]. We will refer to the exotic non-perturbative SCFTs of this type, obtained through an S-folding, as W-SCFTs.[1] The S-foldings preserving $\mathcal{N}=3$ supersymmetry are by now reasonably well characterised and possess a number of intriguing features [2–4, 16–20]. The focus on the $\mathcal{N}=3$ case is motivated, in part, by the fact that it was previously thought that for rigid supersymmetry in $D=4$ spacetime dimensions $\mathcal{N}=3$ necessarily implies $\mathcal{N}=4$. However, the logic underlying this conclusion relies on the existence of a perturbative limit, which fails for the intrinsically non-perturbative S-foldings. This in itself does not rule out an enhancement to $\mathcal{N}=4$, but the S-fold invariant operators do not fall into $\mathrm{SU}(4)_R$ representations, excluding this possibility [3]. However, there is no reason to think S-foldings are necessarily $\mathcal{N}=3$ and in the context of twin theories the absence of a massless sector and the presence of both spin-3/2 and spin-1/2 states in the set of lowest dimension operators suggests the possibility that the same bosonic content can admit different fermionic completions. Indeed, consulting table 2, a straightforward comparison reveals that the bosonic content of (1.1) is uniquely matched by

$$[1,2] \oplus 14[1,1], \tag{1.2}$$

which provides the lowest order spectrum of our candidate little $\mathcal{N}_l=1$ twin W-SCFT, as obtained via a twin S-folding in section 3.1. Of course, this is not enough to declare them to be twin theories as, without a Lagrangian description, we have no immediate handle on the interactions. However, there are twin $\mathcal{N}_b=6$ and $\mathcal{N}_l=2$ supergravities in $D=5$, with identical bosonic sectors determined by the common scalar coset $\mathrm{SU}^\star(6)/\mathrm{Sp}(3)$, that can be gauged with respect to the same subgroup $\mathrm{SU}(3)\times \mathrm{U}(1) \subset \mathrm{Sp}(3)$. The gauged $\mathcal{N}_b=6$ supergravity (or more precisely, an S-duality fibration thereof) provides the bulk holographic dual of the exotic $\mathcal{N}_b=3$ SCFT [1, 3], while its $\mathcal{N}_l=2$ twin provides the candidate bulk holographic dual of the proposed exotic $\mathcal{N}_l=1$ twin SCFT. Note, all twin Poincaré supergravity theories can be obtained through the "square" or "double-copy" of conventional super Yang-Mills theories [10], as summarised in table 1. As one might anticipate, for each twin supergravity pair in $D=5$ there is a candidate twin pair of dual $D=4$ W-SCFTs, that admit a twin S-fold construction and may also be deduced through the double-copy of massive spin-1 multiplets following the procedure of [10], as we describe in section 3.1 and section 3.2. Since the W-SCFTs are intrinsically non-perturbative the use of "double-copy" here is meant rather heuristically; it is essentially an exercise in representation theory.

Remarkably, just as the double-copy of conventional super Yang-Mills theories gives conventional supergravity theories, it has been argued that the "double-copy" of W-SCFTs

[1] The nomenclature reflects: (i) the role of Weyl multiplets [12, 14, 15] in characterising the W-SCFT spectra [1, 11]; (ii) that the product of two W-SCFTs yield a W-supergravity [11], in analogy to the double-copy of conventional super Yang-Mills theories, which yields conventional supergravity theories.

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yields exotic massive higher spin W-supergravity theories [11]. The chief example is the $\mathcal{N}=7$ W-supergravity, which follows from the product of $\mathcal{N}=4$ super Yang-Mills with the $\mathcal{N}=3$ W-SCFT [11] and contains a single spin-4 and 1000 spin-2 states. Note, the existence of an $\mathcal{N}=7$ W-supergravity theory is the direct analog of the existence of an $\mathcal{N}=3$ W-SCFT, in the sense that for locally supersymmetric theories in $D=4$ with a perturbative limit, $\mathcal{N}=7$ implies $\mathcal{N}=8$. Again, the loop-hole is the intrinsically non-perturbative nature of the $\mathcal{N}=7$ W-supergravity, which in this case can be traced back to its $\mathcal{N}=3$ W-SCFT factor in the double-copy. The $\mathcal{N}=7$ W-supergravity has been proposed to be the effective field theory limit of a type II W-superstring theory [11]. From this perspective, the $[\mathcal{N}=4]\times[\mathcal{N}=3]$ product corresponds to the product of $\mathcal{N}=4$ left-moving and $\mathcal{N}=3$ right-moving fermionic strings, which follows from a non-perturbative string S-folding involving a T-fold and S-duality twist that acts only on the right-movers of the conventional type II string [11]. Now, given a double-copy construction of W-supergravities and an array of W-SCFTs with $\mathcal{N}=1,2,3$ one can follow [10] to generate candidate twin W-supergravities, as done in section 4. In this case we cannot directly appeal to AdS/CFT, so for the time-being their twinness is confined to spectra and symmetries alone.

2 $D=4$ massive multiplets with spin ≤ 4

We shall need in the following all long massive supermultiplets with spins ranging from 0 to 4. For $\mathcal{N}$-extended supersymmetry, the long massive spin-$\left(\frac{\mathcal{N}}{2}+j\right)$ multiplet is obtained by tensoring the smallest long massive spin-$(\frac{\mathcal{N}}{2})$ multiplet by a spin-j state [13]. This yields the list of multiplets given table 2. The unitary R-symmetry representations may be collected into representations of $\mathrm{Sp}(\mathcal{N})$, the automorphism algebra of the massive $\mathcal{N}$-extended supersymmetry algebra, and the states are accordingly labelled by $\mathrm{Spin}(3)\times\mathrm{Sp}(\mathcal{N})$ representations. Note the coincidences in [11] and [21], which both make use of table 2, suggesting a possible relation to bound p-branes. Specifically, the N, L, q multiplets of [21] are related to the $\mathcal{N}, j_{\max}$ multiplets of [11] and table 2 by

$$[\mathcal{N}, j_{\max}] = [N-q, (N-q+2L)/2]. \tag{2.1}$$

3 Twin superconformal field theories

In this section we shall construct the twin W-SCFTs in $D=4$. The $\{\mathcal{N}_b, \mathcal{N}_l\}$ twin pair can be obtained by twin S-fold operators, denoted $S_b^{\{\mathcal{N}_b,\mathcal{N}_l\}}$ and $S_l^{\{\mathcal{N}_b,\mathcal{N}_l\}}$, for the big and little twin, respectively. They each have holographic duals given by gauged twin supergravities obtained via non-perturbative projections of type IIB on $\mathrm{AdS}_5\times S^5$. We will also show how they may deduced from using the *gauge*$\times$*gauge* construction of [10], using a Cartan involution and $(-1)^F$. We will review the known $\mathcal{N}=3$ theories, before describing the $\{3,1\}$ and $\{2,1\}$ twins. The $\mathcal{N}=2,1$ W-SCFTs are new, to the best of our knowledge.

3.1 The $\{3,1\}$ twins

Before giving the S-fold construction of the twin pair, let us summarise their spectra. Let $[\mathcal{N}, j_{\max}]_{B(F)}$ denote the bosonic (fermionic) sector of $[\mathcal{N}, j_{\max}]$.

Consider the bosonic sector of $\mathcal{N}=3$ spin-2 multiplet,

$$[3,2]_B = (\mathbf{5},\mathbf{1}) \oplus (\mathbf{3},\mathbf{14}+\mathbf{1}) \oplus (\mathbf{1},\mathbf{14}), \tag{3.1}$$

and the $\mathcal{N}=1$ spin-2 multiplet,

$$[1,2]_B = (\mathbf{5},\mathbf{1}) \oplus (\mathbf{3},\mathbf{1}). \tag{3.2}$$

Evidently, to match the bosonic content of the $\mathcal{N}=3$ theory we must add 14 $\mathcal{N}=1$ spin-1 multiplets transforming in the **14** of Sp(3), giving

$$[1,2] \oplus 14[1,1] = (\mathbf{5},\mathbf{1},\mathbf{1}) \oplus (\mathbf{4},\mathbf{2},\mathbf{1}) \oplus (\mathbf{3},\mathbf{1},\mathbf{14}+\mathbf{1}) \oplus (\mathbf{2},\mathbf{2},\mathbf{14}) \oplus (\mathbf{1},\mathbf{1},\mathbf{14}), \tag{3.3}$$

such that

$$([1,2] \oplus 14[1,1])_B = [3,2]_B. \tag{3.4}$$

The $\{3,1\}$ twin multiplets, $[3,2]$ and $[1,2] \oplus 14[1,1]$, can be obtained via complementary truncations of $[4,2]$. First, decompose $[4,2]$ with respect to $\mathrm{Sp}(1) \times \mathrm{Sp}(3) \subset \mathrm{Sp}(4)$

$$\begin{aligned}
(\mathbf{5},\mathbf{1}) &\to (\mathbf{5},\mathbf{1},\mathbf{1}) \\
(\mathbf{4},\mathbf{8}) &\to (\mathbf{4},\mathbf{2},\mathbf{1}) + (\mathbf{4},\mathbf{1},\mathbf{6}) \\
(\mathbf{3},\mathbf{27}) &\to (\mathbf{3},\mathbf{1},\mathbf{14}) + (\mathbf{3},\mathbf{2},\mathbf{6}) + (\mathbf{3},\mathbf{1},\mathbf{1}) \\
(\mathbf{2},\mathbf{48}) &\to (\mathbf{2},\mathbf{1},\mathbf{14}') + (\mathbf{2},\mathbf{2},\mathbf{14}) + (\mathbf{2},\mathbf{1},\mathbf{6}) \\
(\mathbf{1},\mathbf{42}) &\to (\mathbf{1},\mathbf{1},\mathbf{14}) + (\mathbf{1},\mathbf{2},\mathbf{14}')
\end{aligned} \tag{3.5}$$

The $[3,2]$ multiplet is then given by truncating to the Sp(1) invariant subsector:

$$\begin{aligned}
(\mathbf{5},\mathbf{1}) &\to (\mathbf{5},\mathbf{1},\mathbf{1}) \\
(\mathbf{4},\mathbf{8}) &\to (\mathbf{4},\mathbf{1},\mathbf{6}) \\
(\mathbf{3},\mathbf{27}) &\to (\mathbf{3},\mathbf{1},\mathbf{14}+\mathbf{1}) \\
(\mathbf{2},\mathbf{48}) &\to (\mathbf{2},\mathbf{1},\mathbf{14}'+\mathbf{6}) \\
(\mathbf{1},\mathbf{42}) &\to (\mathbf{1},\mathbf{1},\mathbf{14})
\end{aligned} \tag{3.6}$$

Its twin $[1,2] \oplus 14[1,1]$ multiplet is given by retaining the same bosonic subector, but the complementary fermionic subsector, that is all fermions transforming as the **2** of Sp(1):

$$\begin{aligned}
(\mathbf{5},\mathbf{1}) &\to (\mathbf{5},\mathbf{1},\mathbf{1}) \\
(\mathbf{4},\mathbf{8}) &\to (\mathbf{4},\mathbf{2},\mathbf{1}) \\
(\mathbf{3},\mathbf{27}) &\to (\mathbf{3},\mathbf{1},\mathbf{1}) + (\mathbf{3},\mathbf{1},\mathbf{14}) \\
(\mathbf{2},\mathbf{48}) &\to \phantom{(\mathbf{3},\mathbf{1},\mathbf{1})} + (\mathbf{2},\mathbf{2},\mathbf{14}) \\
(\mathbf{1},\mathbf{42}) &\to \phantom{(\mathbf{3},\mathbf{1},\mathbf{1})} + (\mathbf{1},\mathbf{1},\mathbf{14})
\end{aligned} \tag{3.7}$$

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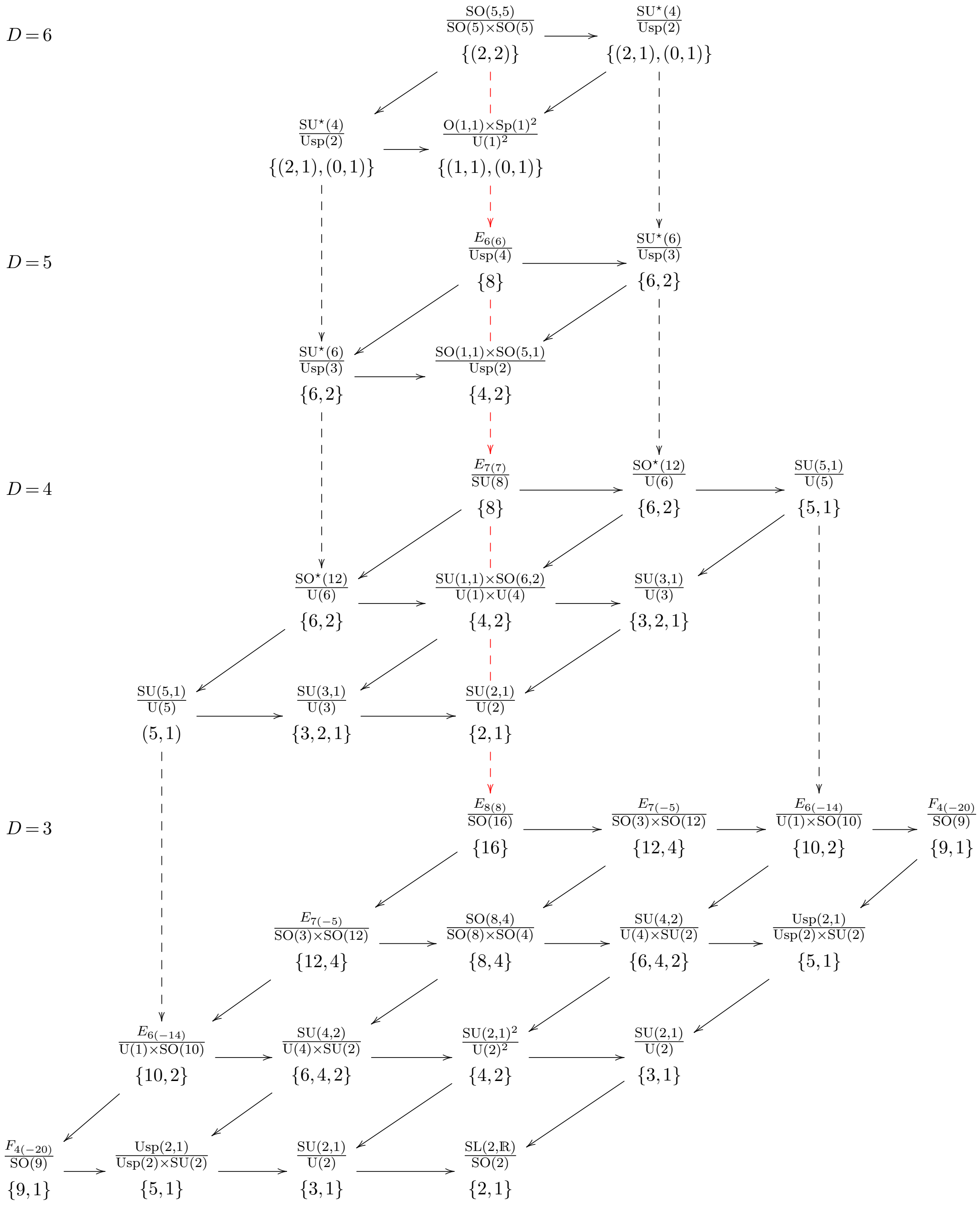

Table 1. Pyramid of twin supergravities generated by the product of left and right super Yang-Mills theories in $D = 3,4,5,6$. Each level is related by dimensional reduction as indicated by the vertical arrows. The horizontal arrows indicate consistent truncations effected by truncating the left or right Yang-Mills multiplets. All such supergravity theories have a twin except for the maximal cases along the "exceptional spine" highlighted in red. Note, $D = 3$ is the exception to the exceptions in that maximal $\mathcal{N} = 16$ supergravity does have a 'trivial' $\mathcal{N} = 1$ twin, but it is not obtained from the double-copy procedure [10].

$j\backslash\mathcal{N}$	8	7	6	5	4	3	2	1
4	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
$\frac{7}{2}$	**16**	**14**	**12**	**10**	**8**	**6**	**4**	**2**
3	**119**	**90+1**	**65+1**	**44+1**	**27+1**	**14+1**	**5+1**	**1**
$\frac{5}{2}$	**544**	**350+14**	**208+12**	**110+10**	**48+8**	**14′+6**	**4**	–
2	**1700**	**910+90**	**429+65+1**	**165+44+1**	**42+27+1**	**14+1**	**1**	–
$\frac{3}{2}$	**3808**	**1638+350**	**572+208+12**	**132+110+10**	**48+8**	**6**	–	–
1	**6188**	**2002+910**	**429′+429+65**	**165+44+1**	**27+1**	**1**	–	–
$\frac{1}{2}$	**7072**	**1430+1638**	**572+208**	**110+10**	**8**	–	–	–
0	**4862**	**2002**	**429**	**44**	**1**	–	–	–
d.o.f	2^{16}	2×2^{14}	3×2^{12}	4×2^{10}	5×2^{8}	6×2^{6}	7×2^{4}	8×2^{2}
$\frac{7}{2}$		**1**	**1**	**1**	**1**	**1**	**1**	**1**
3		**14**	**12**	**10**	**8**	**6**	**4**	**2**
$\frac{5}{2}$		**90**	**65+1**	**44+1**	**27+1**	**14+1**	**5+1**	**1**
2		**350**	**208+12**	**110+10**	**48+8**	**14′+6**	**4**	–
$\frac{3}{2}$		**910**	**65+429**	**165+44+1**	**42+27+1**	**14+1**	**1**	–
1		**1638**	**208+572**	**132+110+10**	**48+8**	**6**	–	–
$\frac{1}{2}$		**2002**	**429+429′**	**165+44**	**27+1**	**1**	–	–
0		**1430**	**572**	**110**	**8**	–	–	–
d.o.f		2^{14}	2×2^{12}	3×2^{10}	4×2^{8}	5×2^{6}	6×2^{4}	7×2^{2}
3			**1**	**1**	**1**	**1**	**1**	**1**
$\frac{5}{2}$			**12**	**10**	**8**	**6**	**4**	**2**
2			**65**	**44+1**	**27+1**	**14+1**	**5+1**	**1**
$\frac{3}{2}$			**208**	**110+10**	**48+8**	**14′+6**	**4**	–
1			**429**	**165+44**	**42+27+1**	**14+1**	**1**	–
$\frac{1}{2}$			**572**	**132+110**	**48+8**	**6**	–	–
0			**429′**	**165**	**27**	**1**	–	–
d.o.f			2^{12}	2×2^{10}	3×2^{8}	4×2^{6}	5×2^{4}	6×2^{2}
$\frac{5}{2}$				**1**	**1**	**1**	**1**	**1**
2				**10**	**8**	**6**	**4**	**2**
$\frac{3}{2}$				**44**	**27+1**	**14+1**	**5+1**	**1**
1				**110**	**48+8**	**14′+6**	**4**	–
$\frac{1}{2}$				**165**	**42+27**	**14+1**	**1**	–
0				**132**	**48**	**6**	–	–
d.o.f				2^{10}	2×2^{8}	3×2^{6}	4×2^{4}	5×2^{2}
2					**1**	**1**	**1**	**1**
$\frac{3}{2}$					**8**	**6**	**4**	**2**
1					**27**	**14+1**	**5+1**	**1**
$\frac{1}{2}$					**48**	**14′+6**	**4**	–
0					**42**	**14**	**1**	–
d.o.f					2^{8}	2×2^{6}	3×2^{4}	4×2^{2}
$\frac{3}{2}$						**1**	**1**	**1**
1						**6**	**4**	**2**
$\frac{1}{2}$						**14**	**5+1**	**1**
0						**14′**	**4**	–
d.o.f						2^{6}	2×2^{4}	3×2^{2}
1							**1**	**1**
$\frac{1}{2}$							**4**	**2**
0							**5**	**1**
d.o.f							2^{4}	2×2^{2}
$\frac{1}{2}$								**1**
0								**2**
d.o.f								2^{2}

Table 2. The long massive $\mathcal{N}$-extended spin-$(j_{\max})$ multiplets for $1/2 \leq j_{\max} \leq 4$ and $1 \leq \mathcal{N} \leq 8$. The states are labelled by $\mathrm{Spin}(3)\times\mathrm{Sp}(\mathcal{N})$ representations.

To summarise, decomposing under $\mathcal{N}=3$,

$$[4,2]=[3,2]\oplus\mathbf{2}\times[3,3/2]\quad\longrightarrow\quad[3,2] \tag{3.8}$$

and the doublet of spin-3/2 multiplets are truncated out. On the other hand, decomposing under $\mathcal{N}=1$

$$[4,2]=[1,2]\oplus\mathbf{6}\times[1,3/2]\oplus\mathbf{14}\times[1,1]\oplus\mathbf{14'}\times[1,1/2]\quad\longrightarrow\quad[1,2]\oplus\mathbf{14}\times[1,1] \tag{3.9}$$

and the six (14) spin-3/2 (spin-1/2) multiplets are truncated out.

3.1.1 The $\mathcal{N}=3$ big twin

Let us first recall the key features of the $\mathcal{N}=3$ W-SCFTs constructed in [3]. From a field theory point of view the $\mathcal{N}=3$ theories are obtained by an S-fold projector, $S_b^{\{3,1\}}:=s_k\circ r_k$, generating a $\mathbb{Z}_k$ subgroup of the $\mathcal{N}=4$ R-symmetry and S-duality, $\mathrm{Spin}(6)\times\mathrm{SL}(2,\mathbb{Z})$. The R-symmetry operator, r_k, is straight-forwardly embeded in the R-symmetry group Spin(6). Consider the $\mathbb{Z}_k$ group

$$\mathbb{Z}_k\subset\mathrm{U}_a(1)\times\mathrm{U}_b(1)\times\mathrm{U}_c(1)\subset\mathrm{Spin}(6), \tag{3.10}$$

generated by a $(2\pi a/k,2\pi b/k,2\pi c/k)$ rotation on $\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2$, for a,b,c co-prime relative to k. Geometrically, it can be regarded as a rotation on the $\mathbb{R}^6$ transverse to a stack of D3-branes in $\mathbb{R}^{1,9}$. For (x,y,z) coordinates on $\mathbb{C}^3$ it is given by

$$(x,y,z)\mapsto(\zeta^a x,\zeta^b y,\zeta^c z),\qquad \zeta=e^{\frac{2\pi i}{k}}. \tag{3.11}$$

Here, r_k is given by $(a,b,c)=(1,1,-1)$. The corresponding action on the $\mathcal{N}=4$ supercharges is given by

$$r_k:\begin{cases} Q_{\alpha A}\mapsto\ e^{-\frac{i2\pi\sum_l\lambda_l^A}{k}}Q_{\alpha A}\\ \bar{Q}_{\dot\alpha}{}^{A}\mapsto e^{-\frac{i2\pi\sum_l\lambda_{Al}}{k}}\bar{Q}_{\dot\alpha}{}^{A}\end{cases} \tag{3.12}$$

Here, α ($\dot\alpha$) and upper (lower) A are the spinor (conjugate spinor) indices of the $\mathbf{2}$ ($\bar{\mathbf{2}}$) and the $\mathbf{4}$ ($\bar{\mathbf{4}}$) representations of $\mathrm{Spin}(1,3)\cong\mathrm{SL}(2,\mathbb{C})$ and $\mathrm{Spin}(6)\cong\mathrm{SU}(4)$, respectively. The weights of the $\mathbf{4}$ ($\bar{\mathbf{4}}$) are denoted by λ_A (λ^A). Explicitly,

$$r_k:\begin{cases} Q_a\mapsto e^{-\frac{i\pi}{k}}Q_a,\quad Q_4\mapsto\ \ e^{\frac{i3\pi}{k}}Q_4\\ \bar{Q}^a\mapsto\ \ e^{\frac{i\pi}{k}}\bar{Q}^a,\quad \bar{Q}^4\mapsto e^{-\frac{i3\pi}{k}}\bar{Q}^4\end{cases} \tag{3.13}$$

where $a=1,2,3$.

Consider now S-duality (assuming a simply-laced gauge group) acting on the coupling constant (complex structure) τ in usual fractional linear manner,[2]

$$\tau\mapsto\frac{a\tau+b}{c\tau+d},\qquad\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\mathrm{SL}(2,\mathbb{Z}). \tag{3.14}$$

[2]It is the projective $\mathrm{PSL}(2,\mathbb{Z})$ that acts faithfully on the upper-half plane, but since we will consider the S-duality action on the fermionic supercharges its double-cover $\mathrm{SL}(2,\mathbb{Z})$ is required.

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The corresponding action on the supercharges is given by,

$$Q_A \mapsto \sqrt{\frac{c\tau+d}{|c\tau+d|}}Q_A, \quad \bar{Q}^A \mapsto \sqrt{\frac{|c\tau+d|}{c\tau+d}}\bar{Q}^A, \tag{3.15}$$

where the central charge picks up a factor of $\frac{|c\tau+d|}{c\tau+d}$ under S-duality and the presence of the squareroot implies that the supercharges in the double-cover. Note, S-duality is only a symmetry (as opposed to a duality) if τ is preserved. This only happens for particular subgroups $\Gamma \subset \mathrm{SL}(2,\mathbb{Z})$ corresponding to certain values of τ, in which case $Q_A \mapsto \exp[i\pi/k]Q_A$ for specific values of k depending on τ, as summarised here:

Γ	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_4$	$\mathbb{Z}_2 \times \mathbb{Z}_3$
Generator	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$
τ	any	$e^{\frac{i\pi}{3}}$	i	$e^{\frac{i\pi}{3}}$
k	2	3	4	6

(3.16)

Corresponding to the $\mathbb{Z}_k$ R-symmetry operator (3.13) consider a $\mathbb{Z}_k \subset \mathrm{SL}(2,\mathbb{Z})$ S-duality subgroup generated by s_k,

$$s_k : \begin{cases} Q_A \mapsto \ \ e^{\frac{i\pi}{k}} Q_A \\ \bar{Q}^A \mapsto e^{-\frac{i\pi}{k}} \bar{Q}^A \end{cases} \tag{3.17}$$

The composite $s_k \circ r_k$ action is given by

$$S_b^{\{3,1\}} : \begin{cases} Q_a \mapsto Q_a, \quad Q_4 \mapsto \ \ e^{\frac{4\pi i}{k}} Q_4; \\ \bar{Q}^a \mapsto \bar{Q}^a, \quad \bar{Q}^4 \mapsto e^{-\frac{4\pi i}{k}} \bar{Q}^4. \end{cases} \tag{3.18}$$

For $k = 2$, corresponding to the usual orientifold case, we see all 16 supercharges are preserved. On the other hand for $k > 2$, only the 12 supercharges $Q_a, \bar{Q}^a$ are left invariant, reducing the $\mathcal{N} = 4$ algebra to the $\mathcal{N} = 3$ algebra, while the $\mathrm{SU}(4)_R$ R-symmetry is broken to $\mathrm{U}(3)_R$. For $k = 2$, τ can take arbitrary values and there is a perturbative limit, as expected for the standard orientifold. For $k > 2$, τ has a fixed value of order one and the S-fold is intrinsically non-perturbative.

The entire $\mathcal{N} = 4$ vector multiplet transforms non-trivially under $S_b^{\{3,1\}}$, as summarised in table 3. The spectrum is truncated to the $S_b^{\{3,1\}}$-invariant subsector. The lowest S_k-invariant operator, which has scaling dimension two, is the $\mathcal{N} = 3$ supercurrent multiplet, which can be written

$$J_a{}^b = \mathrm{tr}\left(V_a\overline{V}^b - \frac{1}{3}\delta_a{}^b V_c \overline{V}^c\right) \tag{3.19}$$

	SU(3)	$\mathrm{U}(1)_R$	$S_l^{\{3,1\}}$
F^+	$\mathbf{1}$	0	1
λ^a	$\mathbf{3}$	-1	1
λ^4	$\mathbf{1}$	3	-1
ϕ^{a4}	$\mathbf{3}$	2	-1
ϕ^{ab}	$\bar{\mathbf{3}}$	-2	1

Table 3. The charges carried by the component fields of the $\mathcal{N}=4$ super Yang-Mills multiplet under the $S_b^{\{3,1\}}$ S-fold operator (in units of $2\pi/k$) and the invariant $\mathrm{SU}(3)\times\mathrm{U}(1)_R\subset\mathrm{SU}(4)$ R-symmetry subgroup.

where the U(3) triplet V_a is the $\mathcal{N}=3$ spin-1 on-shell superfield [22, 23]. The physical components in terms of $\mathrm{Spin}(3)\times\mathrm{U}(3)_R$ representations are given by

$$\begin{aligned}
[3,2] = \;& (\mathbf{5},\mathbf{1})\\
&\oplus(\mathbf{4},\mathbf{3}_1+\bar{\mathbf{3}}_{-1})\\
&\oplus(\mathbf{3},\mathbf{1}_0+\mathbf{3}_{-2}+\bar{\mathbf{3}}_2+\mathbf{8}_0)\\
&\oplus(\mathbf{2},\mathbf{1_3}+\mathbf{1_{-3}}+\mathbf{6_{-1}}+\bar{\mathbf{6}}_\mathbf{1}+\mathbf{3_1}+\bar{\mathbf{3}}_\mathbf{-1})\\
&\oplus(\mathbf{1},\mathbf{3_{-2}}+\bar{\mathbf{3}}_\mathbf{2}+\mathbf{8_0})\\
=\;&(\mathbf{5},\mathbf{1})\oplus(\mathbf{4},\mathbf{6})\oplus(\mathbf{3},\mathbf{14}+\mathbf{1})\oplus(\mathbf{2},\mathbf{14}'+\mathbf{6})\oplus(\mathbf{1},\mathbf{14})
\end{aligned}\tag{3.20}$$

where in the last line we have collected the U(3) representations into Sp(3) representations corresponding to the automorphism algebra of the massive $\mathcal{N}=3$ supersymmetry algebra.

The $\mathcal{N}=3$ theories have string/M-theory embeddings that can be approached from a number of perspectives [3]. For example, D3-branes probing singularities in F-theory on an Abelian orbifold,

$$\underline{\mathbb{R}^{1,3}}\times(\mathbb{C}^3\times T^2)/\mathbb{Z}_k,\tag{3.21}$$

where the underline denotes the D3-brane world-volume directions. This corresponds to a limit of M2-branes in M-theory on $\underline{\mathbb{R}^{1,2}}\times(\mathbb{C}^3\times T^2)/\mathbb{Z}_k$. The complex structure of the F-theory T^2 is the coupling constant of the world-volume theory of the D3-branes. For (x,y,z,u) coordinates on $\mathbb{C}^4$, locally equivalent to the F-theory $\mathbb{C}^3\times T^2$, consider the $\mathbb{Z}_k$ group generated by

$$\sigma:(x,y,z,u)\mapsto(\zeta^a x,\zeta^b y,\zeta^c z,\zeta^d u)\tag{3.22}$$

where ζ is a primitive k^{th} root of unity. The singularities are isolated if and only if the weights (a,b,c,d) are all relatively prime to k. This action is embedded in the R-symmetry and S-duality groups,

$$\mathbb{Z}_k\subset\mathrm{U}_a(1)\times\mathrm{U}_b(1)\times\mathrm{U}_c(1)\times\mathrm{U}_d(1)\subset\mathrm{Spin}(6)\times\mathrm{SL}(2,\mathbb{Z}),\tag{3.23}$$

through

$$\sigma\mapsto\mathrm{diag}(R_a,R_b,R_c)\otimes R_d\tag{3.24}$$

where R_a is a rotation by $2\pi a/k$ on $\mathbb{R}^2 \cong \mathbb{C} \subset \mathbb{C}^4$. The corresponding action on the $(\mathbf{4},\mathbf{2})$ of $\mathrm{Spin}(6) \times \mathrm{SL}(2,\mathbb{Z})$ is given (in our conventions[3]) by,

$$\mathrm{diag}\left(\zeta^{\frac{a+b+c+d}{2}}, \zeta^{\frac{a-b-c+d}{2}}, \zeta^{\frac{-a+b-c+d}{2}}, \zeta^{\frac{-a-b+c+d}{2}}\right), \tag{3.25}$$

where the $S_b^{3,1}$ action given in (3.18), corresponds to $(a,b,c,d) = (1,1,-1,-1)$,

$$\sigma : (x,y,z,u) \mapsto (\zeta x, \zeta y, \zeta^{-1} z, \zeta^{-1} u). \tag{3.26}$$

The singularities are $\mathbb{Q}$-factorial (being quotient) Gorenstein terminal[4] and therefore do not admit any crepent resolution. The $\mathrm{SL}(2,\mathbb{Z})$ action is an involution of the torus only for $k = 2,3,4,6$, hence the restriction. Since the complex structure in the F-theory limit corresponds to the axion-dilaton, for $k > 2$ this is non-pertubative in $D = 10$.

3.1.2 The $\mathcal{N} = 1$ little twin

Let us now introduce the little twin S-fold operator $S_l^{\{3,1\}} := s_k \circ r_k$. The R-symmetry action is again given by,

$$r_k : \begin{cases} Q_a \mapsto e^{-\frac{i\pi}{k}} Q_a, & Q_4 \mapsto \ \ e^{\frac{i3\pi}{k}} Q_4 \\ \bar{Q}^a \mapsto \ \ e^{\frac{i\pi}{k}} \bar{Q}^a, & \bar{Q}^4 \mapsto e^{-\frac{i3\pi}{k}} \bar{Q}^4 \end{cases} \tag{3.27}$$

The S-duality operator, on the other hand, is now given by,

$$s_k : \begin{cases} Q_A \mapsto e^{-\frac{i3\pi}{k}} Q_A; \\ \bar{Q}^A \mapsto \ \ e^{\frac{i3\pi}{k}} \bar{Q}^A. \end{cases} \tag{3.28}$$

Hence, the composite action is given by

$$S_l^{\{3,1\}} : \begin{cases} Q_a \mapsto e^{\frac{-i4\pi}{k}} Q_a, & Q_4 \mapsto Q_4; \\ \bar{Q}^a \mapsto \ \ e^{\frac{i4\pi}{k}} \bar{Q}^a, & \bar{Q}^4 \mapsto \bar{Q}^4, \end{cases} \tag{3.29}$$

and we observe that for $k > 2$ only four of the superchrages are left invariant. For $k = 2$ all 16 charges survive as before. The R-symmetry is broken to $\mathrm{SU}(3) \times \mathrm{U}(1)_R$, but for $k > 2$ the remnant $\mathrm{SU}(3)$ is now a flavour symmetry rather than an R-symmetry, since only four supercharges are left invariant, reducing the $\mathcal{N} = 4$ algebra to the $\mathcal{N} = 1$ algebra. Note, this is the unique (up to trivial automorphisms) $\mathcal{N} = 1$ projection preserving an $\mathrm{SU}(3)$ subgroup of the $\mathcal{N} = 4$ R-symmetry.

Again, for $k \neq 3$ the entire $\mathcal{N} = 4$ vector multiplet transforms non-trivially under $S_l^{\{3,1\}}$, as summarised in table 4. To obtain the desired little twin one must set $k = 4$ (other values give further truncations). Using the $S_l^{\{3,1\}}$ charges it is straightforward to deduce the

[3] The weights of the $\mathbf{4}$ are given by $(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2})$ with an even number of negative signs.

[4] A singular variety is said to be Gorenstein if its canonical bundle (which may only be a coherent sheaf) is a line bundle. The quotients $\mathbb{C}^4/\mathbb{Z}_k$ are Gorenstein terminal if and only if there is a generator with weights given (up to permutations) by $(1,-1,a,-a)$ for $\gcd(a,k) = 1$ [24].

	SU(3)	$\mathrm{U}(1)_R$	$S_l^{\{3,1\}}$
F^+	$\mathbf{1}$	0	-3
λ^a	$\mathbf{3}$	-1	-1
λ^4	$\mathbf{1}$	3	-3
ϕ^{a4}	$\mathbf{3}$	2	-1
ϕ^{ab}	$\bar{\mathbf{3}}$	-2	1

Table 4. The charges carried by the component fields (in terms of on-shell field strengths) of the $\mathcal{N}=4$ super Yang-Mills multiplet under the $S_l^{\{3,1\}}$ S-fold operator (in units of $2\pi/k$) and the invariant $\mathrm{SU}(3)\times\mathrm{U}(1)_R\subset\mathrm{SU}(4)$ flavour/R-symmetry subgroup. Note, for $k=3$ both F and λ^4 are $S_l^{\{3,1\}}$-invariant and we restrict to $k=4$ to obtain the little twin.

quadratic $S_l^{\{3,1\}}$-invariant operators and, through their $\mathrm{SU}(3)\times\mathrm{U}(1)_R$ representations, to collect them into massive long $\mathcal{N}=1$ supermultiplets. In term of the on-shell superfields of [22, 23] we obtain a single spin-2 and 14 spin-1 supercurrents. This can be deduced directly from the $S_l^{\{3,1\}}$-invariant truncation of the $\mathcal{N}=4$ supercurrent,

$$J_{AB,CD}=V_{AB}V_{CD}-\frac{1}{12}\epsilon_{ABCD}\bar{V}^{EF}V_{EF},\qquad \bar{V}^{AB}=\frac{1}{12}\epsilon^{ABCD}V_{CD}. \tag{3.30}$$

The explicit projection in terms of off-shell component fields is given in appendix A.

The spin-2 supercurrent corresponds to the massive $\mathcal{N}=1$ super-Weyl multiplet, see table 2, which consists of the massive $\mathrm{Spin}(3)\times\mathrm{U}(1)_R$ states

$$\begin{aligned}[1,2]&=\mathbf{5_0}+\mathbf{4}_1+\mathbf{4}_{-1}+\mathbf{3}_0\\&=(\mathbf{5},\mathbf{1})+(\mathbf{4},\mathbf{2})+(\mathbf{3},\mathbf{1})\end{aligned} \tag{3.31}$$

where in the last line we have collected the U(1) representations into Sp(1) representations corresponding to the automorphism algebra of the massive $\mathcal{N}=1$ supersymmetry algebra.

The 14 spin-1 supercurrents, $J_a, J^a, J_a{}^b$ transform as the $\mathbf{3_{-2}},\bar{\mathbf{3}}_\mathbf{2},\mathbf{8_0}$ of the global U(3) and can be put in a $\mathbf{14}$ of Sp(3),

$$\begin{array}{ccc}\mathrm{Sp}(3)&\supset&\mathrm{U}(3)\\ \mathbf{14}&\longrightarrow&\mathbf{3_{-2}}+\bar{\mathbf{3}}_\mathbf{2}+\mathbf{8_0},\end{array} \tag{3.32}$$

although in this case Sp(3) is a flavour symmetry, rather than the supersymmetry algebra automorphism group. The spin-1 supercurrents correspond to the massive long $\mathcal{N}=1$ spin-1 multiplet, see table 2, which consists of the massive $\mathrm{Spin}(3)\times\mathrm{U}(1)_R$ states

$$\begin{aligned}[1,1]&=\mathbf{3_0}+\mathbf{2}_1+\mathbf{2}_{-1}+\mathbf{1}_0\\&=(\mathbf{3},\mathbf{1})+(\mathbf{2},\mathbf{2})+(\mathbf{1},\mathbf{1})\end{aligned} \tag{3.33}$$

where we have collected the $\mathrm{U}(1)_R$ representations into Sp(1) representations corresponding to the automorphism algebra of the massive $\mathcal{N}=1$ supersymmetry algebra. Including the flavour symmetry we have

$$14\times[1,1]=(\mathbf{3},\mathbf{1},\mathbf{14})+(\mathbf{2},\mathbf{2},\mathbf{14})+(\mathbf{1},\mathbf{1},\mathbf{14}) \tag{3.34}$$

which, with (3.31), reproduces the truncation given in (3.7).

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Note, geometrically the $S_l^{\{3,1\}}$ projection corresponds to $(\mathbb{C}^3 \times T^2)/\mathbb{Z}_k$, where the $\mathbb{Z}_k$ action is given by

$$(x, y, z, u) \mapsto (\zeta x, \zeta y, \zeta^{-1} z, \zeta^3 u). \tag{3.35}$$

For $k = 2$ all supercharges are left invariant and we return to the orientifold case. For, $k = 3, 6$ the singularities are not isolated. So, for $\mathcal{N} = 1$ supersymmetry and isolated singularities we must restrict to $k = 4$, in which case (3.35) reduces to the $\mathcal{N} = 3$ quotient given in (3.26). However, since the supercharges transform in the double-cover of the duality group they must be distinguished. For $k = 2, 3, 6$ the singularities are terminal[5] (but not Gorenstein) and therefore do not admit any crepent resolution. In fact, there is no isolated quotient singularity with $\mathcal{N} < 3$ supersymmetry that is Gorenstein terminal for any $k = 2, 3, 4, 6$. The actual string/F-theory embedding is rather more subtle; we shall return to this question in future work.

3.1.3 Dual supergravity theories

Ungauged $D = 5, \mathcal{N}_b = 6$ supergravity has a twin given by the $\mathcal{N}_l = 2$ quaternionic magic supergravity, which is coupled to 14 vector multiplets and is based on the Jordan algebra of 3×3 quaternionic Hermitian matrices, $\mathfrak{J}_3^{\mathbb{H}}$ [5, 26]. The bosonic sectors of the twins are determined by the common scalar coset $\mathrm{SU}^\star(6)/\,\mathrm{Sp}(3)$, where $\mathrm{SU}^\star(6)$ is the reduced structure group of $\mathfrak{J}_3^{\mathbb{H}}$. This is the $D = 5$ analog of the ungauged $D = 4, \{6, 2\}$ twins with common coset $\mathrm{SO}^\star(12)/\,\mathrm{U}(6)$. In this case, there are twins gaugings with the same U(4) gauge group [8, 27]. The gauged $\mathcal{N} = 6$ theory corresponds to the low energy limit of type II strings on a specific $\mathrm{AdS}_4 \times \mathbb{CP}^3$ geometry, but cannot viewed as spontaneously broken phase of a gauged $\mathcal{N} = 8$ supergravity [27]. The same applies to the gauged $\mathcal{N} = 2$ twin. Rather, they are consistent truncations of the SO(8) gauged $\mathcal{N} = 8$ theory.

An analogous discussion applies to the $D = 5$, $\{6, 2\}$ supergravity twins relevant here. Indeed, one can consistently truncate from SU(4) gauged $\mathcal{N} = 8$ supergravity on an AdS_5 background (geometrically obtained from type IIB supergravity on S^5) to both an $\mathrm{U}(3) \subset \mathrm{SU}(4)$ gauged $\mathcal{N} = 6$ supergravity or an $\mathrm{U}(1)_R \times \mathrm{U}(3) \subset \mathrm{SU}(4)$ gauged $\mathcal{N} = 2$ supergravity coupled to eight vector multiplets and 3+3 “self-dual” tensor multiplets, transforming as the $\mathbf{8}$ and $\mathbf{3} + \bar{\mathbf{3}}$ of $\mathrm{SU}(3) \subset \mathrm{SU}(2, 2|1) \times \mathrm{SU}(3) \subset \mathrm{SU}(2, 2|4)$ respectively [28]. Note, the $\mathcal{N} = 2$ multiplet structure is precisely reflected by the candidate little twin $\mathcal{N} = 1$ W-SCFT dual obtained from the S-folding in section 3.1.2, cf. appendix A. Moreover, the $\mathcal{N} = 6$ and $\mathcal{N} = 2$ truncations correspond to ‘twin gaugings’ of the twin $\mathcal{N} = 6$ and magic $\mathcal{N} = 2$ Poincaré supergravity theories [26, 28], appearing in table 1, which both have scalar coset $\mathrm{SU}^\star(6)/\,\mathrm{Sp}(3)$. As discussed in [1], the $\mathcal{N} = 6$ truncation should have a dual theory with superconformal group $\mathrm{SU}(2, 2|3)$: the $\mathrm{U}_1(1)$ projector used in [1], which is a linear combination of a U(1) R-symmetry and a (discretized) U(1) S-duality, to effect the truncation, eliminates all states in the dual theory not corresponding to $\mathcal{N} = 3$ operators. It corresponds directly to the dual big twin $\mathcal{N} = 3$ W-SCFT S-fold. Note, there is no conventional geometric symmetry that can effect this truncation and the use

[5] Isolated quotient singularities $\mathbb{C}^4/\mathbb{Z}^k$ are terminal if and only if, $s_p > k$ for $p = 1, 2, \ldots k - 1$, where $s_p := \langle pa \rangle + \langle pb \rangle + \langle pc \rangle + \langle pd \rangle$ and $\langle x \rangle$ is the unique integer in $\{0, 1, 2, \ldots k - 1\}$ congruent to x mod k [25].

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of S-duality (which for $k > 2$ fixes the string coupling to order one) makes it intrinsically non-perturbative [1, 3]. This provides the holographic dual of the $\mathcal{N} = 3$ W-SCFT [3]. Specifically, it is given by type IIB on $\mathrm{AdS}_5 \times S^5/\mathbb{Z}_k$ with a non-trivial S-duality bundle over the internal space. The corresponding F-theory construction is given by compactifying on $\mathrm{AdS}_5 \times (S^5 \times T^2)/\mathbb{Z}_k$. The analogous consistent truncation to the little $\mathcal{N} = 2$ gauged supergravity should be effected by the same procedure used in [1] for $\mathcal{N} = 6$, but with the little twin S-duality U(1) rotation (i.e. it is shifted as for the little twin W-SCFT, $\exp i\theta\pi \to \exp -3i\theta\pi$), and should have a dual theory with superconformal group $\mathrm{SU}(2, 2|1)$ that corresponds to the little $\mathcal{N} = 1$ W-SCFT. Again, there is no conventional geometric symmetry that can effect this truncation and the use of S-duality makes it intrinsically non-perturbative. The complete (non)-geometric picture will be developed in future work.

3.1.4 The double-copy construction

Following [10] the $\{3, 1\}$ twin theories may be generated by considering the product of Left and Right $\mathcal{N} = 2, 1, 0$ spin-1 theories. Assume the spin-1 theories have gauge groups G and $\tilde{G}$, with Lie algebras $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$, and are valued in the respective adjoint representations, A and $\tilde{A}$. The $\mathcal{N} = 4$ parent theory is given by,

$$[2, 1]^A \otimes [2, 1]^{\tilde{A}} = [4, 2]. \tag{3.36}$$

Consider a subgroup $G_0 \subset G$ corresponding to the positive eigenspace subspace of a Cartan involution $\theta : \mathfrak{g} \to \mathfrak{g}$. The adjoint representation decomposes as $A = A_0 \oplus \rho$, where ρ is a (not necessarily irreducible) representation of G_0. To obtain the $\mathcal{N} = 3$ twin, first decompose the Left factor into $\mathcal{N} = 1$ multiplets,

$$\left([1, 1]^A \oplus \mathbf{2}[1, \tfrac{1}{2}]^A\right) \otimes [2, 1]^{\tilde{A}} = [4, 2], \tag{3.37}$$

where the multiplicities are given as representations of $\mathrm{Sp}(1)_F$ in $\mathrm{Sp}(1)_R \times \mathrm{Sp}(1)_F \subset \mathrm{Sp}(2)_R$. Then let $\sigma := (-1)^F \circ \theta$, where $(-1)^F[\mathcal{N}, j] = (-1)^{2j}[\mathcal{N}, j]$, and truncate to the σ-invariant sector of the Left factor

$$\left([1, 1]^{A_0} \oplus \mathbf{2}[1, \tfrac{1}{2}]^{\rho}\right) \otimes [2, 1]^{\tilde{A}} = [3, 2], \tag{3.38}$$

where we have used the rule that adjoint and non-adjoint representations do not talk to one another in the double-copy and the remaining total global symmetry is $\mathrm{Sp}(3)_R \times \mathrm{Sp}(1)_F$.

To then obtain the $\mathcal{N} = 1$ twin, decompose the Right factor into $\mathcal{N} = 0$ multiplets and truncate to the $\tilde{\sigma} := (-1)^F \circ \tilde{\theta}$ invariant sector

$$\left([1, 1]^{A_0} \oplus \mathbf{2}[1, \tfrac{1}{2}]^{\rho}\right) \otimes \left([0, 1]^{\tilde{A}_0} \oplus \mathbf{4}[0, \tfrac{1}{2}]^{\tilde{\rho}} \oplus \mathbf{5}[0, 0]^{\tilde{A}_0}\right) \tag{3.39}$$

where the right multiplicities are given as representations of $\mathrm{Sp}(2)_{\tilde{F}} \subseteq \mathrm{Sp}(2)_{\tilde{R}}$. This yields

$$(\mathbf{1}, \mathbf{1})[1, 2] + ((\mathbf{1}, \mathbf{1}) + (\mathbf{2}, \mathbf{4}) + (\mathbf{1}, \mathbf{5}))[1, 1], \tag{3.40}$$

where the multiplicities are given as representations of $\mathrm{Sp}(1)_F \times \mathrm{Sp}(2)_{\tilde{F}}$ and can be collected into irreducible $\mathrm{Sp}(3)_F$ representations

$$[1, 2] + \mathbf{14}[1, 1], \tag{3.41}$$

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so that the total global symmetry is $\mathrm{Sp}(1)_R \times \mathrm{Sp}(3)_F$. Hence, the spectra and symmetries match those of the big $\mathcal{N}=3$ twin.

Note, the conventional Bern-Carrasco-Johansson (BCJ) double-copy [29–31] takes gauge theories into gravitational theories, whereas here we are generating the spectra and symmetries of *non-gravitational* W-SCFTs from the product of spin-1 SCFT "matter" multiplets. This is directly analogous to the BCJ double-copy of, for example, $\mathcal{N}=2$ hyper multiplet amplitudes, which generate the amplitudes of $\mathcal{N}=4$ Yang-Mills. However, for the hyper multiplets to have a local symmetry they must come coupled to an $\mathcal{N}=2$ Yang-Mills multiplet, which will generate the $\mathcal{N}=4$ gravitational sector when included in the double-copy. So the $\mathcal{N}=4$ Yang-Mills amplitudes generated by the hypers must be regarded as a subsector of the full double-copy theory including the gravitational degrees of freedom. It is tempting to apply the same logic in the present W-SCFT case: the product of the "matter" multiplets, given in (3.38) and (3.39), yields the non-gravitational twin W-SCFTs, but if they are to have local symmetries the "matter" multiplets entering in the Left and Right factors must themselves come coupled to W-SCFTs, which when included in the product will yield the gravitational sector in terms of W-supergravities, as described in [11] and section 4. So, in the end, we expect our double-copy constructed W-SCFTs to come coupled to W-supergravities.

Of course, this remains rather heuristic since we dealing with non-Lagrangian theories with no perturbative limit, although using the field-theoretic approach of [10, 32–34] the spectra and local/global symmetries can be determined from the product, even in the absence of a complete understanding of the factors. It may be possible to make further progress by studying the possible rational superconformal invariants, but we leave this for future work.

3.2 The $\{2,1\}$ twins

Before giving the S-fold construction of the twin pair, let us summarise their spectra. Consider the $\mathcal{N}=2$ and $\mathcal{N}=1$ super-Weyl multipets

$$[2,2] = (\mathbf{5},\mathbf{1}) + (\mathbf{4},\mathbf{4}) + (\mathbf{3},\mathbf{5}+\mathbf{1}) + (\mathbf{2},\mathbf{4}) + (\mathbf{1},\mathbf{1})\,; \tag{3.42}$$

$$[1,2] = (\mathbf{5},\mathbf{1}) + (\mathbf{4},\mathbf{2}) + (\mathbf{3},\mathbf{1})\,, \tag{3.43}$$

which are covariant under $\mathrm{Spin}(3)\times\mathrm{Sp}(2)$ and $\mathrm{Spin}(3)\times\mathrm{Sp}(1)$, respectively. Consequently, in order to equate their bosonic sectors, one must add at least five $[1,1]$ multiplets to the $\mathcal{N}=1$ theory, transforming in the $\mathbf{5}$ of $\mathrm{Sp}(2)$, giving the $\mathrm{Spin}(3)\times\mathrm{Sp}(1)\times\mathrm{Sp}(2)$-covariant result,

$$[2,2]_B = (\mathbf{5},\mathbf{1},\mathbf{1}) + (\mathbf{3},\mathbf{1},\mathbf{5}+\mathbf{1}) + (\mathbf{1},\mathbf{1},\mathbf{1})\,; \tag{3.44}$$

$$([1,2]\oplus\mathbf{5}[1,1])_B = (\mathbf{5},\mathbf{1},\mathbf{1}) + (\mathbf{3},\mathbf{1},\mathbf{5}+\mathbf{1}) + (\mathbf{1},\mathbf{1},\mathbf{5})\,. \tag{3.45}$$

The unique, minimal matching of the bosonic sectors is then given by adding one $[2,1]$ multiplet on the $\mathcal{N}=2$ side, and a further $[1,1]$ multiplet on the $\mathcal{N}=1$ side:

$$\begin{aligned}([2,2]\oplus[2,1])_B &= ([1,2]\oplus(\mathbf{5}+\mathbf{1})\,[1,1])_B \\ &= (\mathbf{5},\mathbf{1},\mathbf{1}) + (\mathbf{3},\mathbf{1},\mathbf{5}+\mathbf{1}+\mathbf{1}) + (\mathbf{1},\mathbf{1},\mathbf{5}+\mathbf{1})\,.\end{aligned} \tag{3.46}$$

Thus, the $\{2,1\}$ twin W-SCFT pair in $D=4$ is given by the $\mathcal{N}=2$ W-SCFT with $[2,2]\oplus[2,1]$ and the $\mathcal{N}=1$ W-SCFT with $[1,2]\oplus(\mathbf{5}+\mathbf{1})\,[1,1]$, where $\mathbf{5}+\mathbf{1}$ is a reducible representation of Sp(2).

The $\{2,1\}$ twin multiplets can be obtained via complementary truncations of $[3,2]$. First, decompose $[3,2]$ under $\mathrm{Sp}(1)\times\mathrm{Sp}(2)\subset\mathrm{Sp}(3)$:

$$\begin{array}{ll} (\mathbf{5},\mathbf{1}) & \to(\mathbf{5},\mathbf{1},\mathbf{1}) \\ (\mathbf{4},\mathbf{6}) & \to(\mathbf{4},\mathbf{1},\mathbf{4})+(\mathbf{4},\mathbf{2},\mathbf{1}) \\ (\mathbf{3},\mathbf{14}+\mathbf{1}) & \to(\mathbf{3},\mathbf{1},\mathbf{5})+(\mathbf{3},\mathbf{2},\mathbf{4})+(\mathbf{3},\mathbf{1},\mathbf{1})\;\;+(\mathbf{3},\mathbf{1},\mathbf{1}) \\ (\mathbf{2},\mathbf{14}'+\mathbf{6}) & \to(\mathbf{2},\mathbf{1},\mathbf{4})+(\mathbf{2},\mathbf{2},\mathbf{5})+(\mathbf{2},\mathbf{1},\mathbf{4})\;\;+(\mathbf{2},\mathbf{2},\mathbf{1}) \\ (\mathbf{1},\mathbf{14}) & \to(\mathbf{1},\mathbf{1},\mathbf{5})+(\mathbf{1},\mathbf{2},\mathbf{4})+(\mathbf{1},\mathbf{1},\mathbf{1})\,. \end{array} \tag{3.47}$$

The truncation to the Sp(1)-invariant subsector yields the $\mathcal{N}=2$ twin $[2,2]\oplus[2,1]$:

$$\begin{array}{lll} (\mathbf{5},\mathbf{1}) & \to(\mathbf{5},\mathbf{1},\mathbf{1}) & \\ (\mathbf{4},\mathbf{6}) & \to(\mathbf{4},\mathbf{1},\mathbf{4}) & \\ (\mathbf{3},\mathbf{14}+\mathbf{1}) & \to(\mathbf{3},\mathbf{1},\mathbf{5}+\mathbf{1}) & +(\mathbf{3},\mathbf{1},\mathbf{1}) \\ (\mathbf{2},\mathbf{14}'+\mathbf{6}) & \to(\mathbf{2},\mathbf{1},\mathbf{4}) & +(\mathbf{2},\mathbf{1},\mathbf{4}) \\ (\mathbf{1},\mathbf{14}) & \to(\mathbf{1},\mathbf{1},\mathbf{1}) & +(\mathbf{1},\mathbf{1},\mathbf{5})\,. \end{array} \tag{3.48}$$

Its $\mathcal{N}=1$ twin $[1,2]\oplus(\mathbf{5}+\mathbf{1})\,[1,1]$ is obtained by retaining the same bosonic sector, but truncating to the complementary fermionic sector, namely retaining only the fermions transforming as the $\mathbf{2}$ of Sp(1):

$$\begin{array}{lll} (\mathbf{5},\mathbf{1}) & \to(\mathbf{5},\mathbf{1},\mathbf{1}) & \\ (\mathbf{4},\mathbf{6}) & \to(\mathbf{4},\mathbf{2},\mathbf{1}) & \\ (\mathbf{3},\mathbf{14}+\mathbf{1}) & \to(\mathbf{3},\mathbf{1},\mathbf{1}) & +(\mathbf{3},\mathbf{1},\mathbf{5}+\mathbf{1}) \\ (\mathbf{2},\mathbf{14}'+\mathbf{6}) & \to & +(\mathbf{2},\mathbf{2},\mathbf{5}+\mathbf{1}) \\ (\mathbf{1},\mathbf{14}) & \to & +(\mathbf{1},\mathbf{1},\mathbf{5}+\mathbf{1})\,. \end{array} \tag{3.49}$$

To summerise, the $\mathcal{N}=2$ twin of the $\{2,1\}$ pair is given by decomposing $[3,2]$ into $\mathcal{N}=2$ multiplets

$$[3,2]=[2,2]\oplus\mathbf{2}\,[2,3/2]\oplus[2,1]\,, \tag{3.50}$$

and truncate out the Sp(2)-doublet $\mathbf{2}$ of long, massive spin-3/2 multiplets. On the other hand, in order to get the $\mathcal{N}=1$ twin of the $\{2,1\}$ pair, one decomposes $[3,2]$ into $\mathcal{N}=1$ multiplets

$$[3,2]=[1,2]\oplus\mathbf{4}\,[1,3/2]\oplus(\mathbf{5}+\mathbf{1})\,[1,1]\oplus\mathbf{4}\,[1,1/2]\,, \tag{3.51}$$

and truncates out the $\mathbf{4}$ Sp(2)-representations of long, massive spin-3/2 and spin-1/2 multiplets.

3.2.1 The $\mathcal{N}=2$ big twin

Let us now introduce the big $\mathcal{N}_l=2$ twin S-fold operator $S_b^{\{2,1\}}:=s_k\circ r_k$. The R-symmetry action in this case is given by,

$$r_k:\begin{cases} Q_i\mapsto e^{-\frac{i\pi}{k}}Q_i, & Q_3\mapsto\;\; e^{\frac{i2\pi}{k}}Q_3\quad Q_4\mapsto Q_4 \\ \bar{Q}^i\mapsto\;\; e^{\frac{i\pi}{k}}\bar{Q}^i, & \bar{Q}^3\mapsto e^{-\frac{i2\pi}{k}}\bar{Q}^3\quad \bar{Q}^4\mapsto\bar{Q}^4 \end{cases} \tag{3.52}$$

	SU(2)	$\mathrm{U}(1)_R$	$\mathrm{U}(1)_F$	$S_b^{\{2,1\}}$
F^+	$\mathbf{1}$	0	0	2
λ^i	$\mathbf{2}$	1	0	2
λ^3	$\mathbf{1}$	-1	1	-1
λ^4	$\mathbf{1}$	-1	-1	1
ϕ^{ij}	$\mathbf{1}$	2	0	2
ϕ^{i3}	$\mathbf{2}$	0	1	-1
ϕ^{i4}	$\mathbf{2}$	0	-1	1
ϕ^{34}	$\mathbf{1}$	-2	0	-2

Table 5. The charges carried by the component fields of the $\mathcal{N}=4$ super Yang-Mills multiplet under the $S_b^{\{2,1\}}$ S-fold operator (in units of π/k) and the invariant $\mathrm{SU}(2)\times\mathrm{U}(1)_R\times\mathrm{U}(1)_F\subset\mathrm{SU}(4)$ flavour/R-symmetry subgroup.

where $i=1,2$. The S-duality operator is given, as for the $\{3,1\}$ big twin, by,

$$s_k:\begin{cases} Q_A\mapsto\ \ e^{\frac{i\pi}{k}}Q_A; \\ \bar{Q}^A\mapsto e^{-\frac{i\pi}{k}}\bar{Q}^A. \end{cases} \tag{3.53}$$

Hence, the composite action is given by

$$S_b^{\{2,1\}}:\begin{cases} Q_i\mapsto Q_i,\ \ Q_3\mapsto\ \ e^{\frac{i3\pi}{k}}Q_3\ \ Q_4\mapsto\ \ e^{\frac{i\pi}{k}}Q_4 \\ \bar{Q}^i\mapsto\bar{Q}^i,\ \ \bar{Q}^3\mapsto e^{-\frac{i3\pi}{k}}\bar{Q}^3\ \ \bar{Q}^4\mapsto e^{-\frac{i\pi}{k}}\bar{Q}^4 \end{cases} \tag{3.54}$$

The R-symmetry is broken to $\mathrm{SU}(2)\times\mathrm{U}(1)_R\times\mathrm{U}(1)_F$ by r_k, where the first two factors make up the $\mathcal{N}=2$ R-symmetry $\mathrm{SU}(2)\times\mathrm{U}(1)_R$ of the preserved $\mathcal{N}=2$ superalgebra. The entire $\mathcal{N}=4$ vector multiplet transforms non-trivially under $S_b^{\{2,1\}}$, as summarised in table 5. The remaining $\mathrm{U}(1)_F$ would seem to be spurious since, from (3.48), we know that the maximal global symmetry of the big $\{2,1\}$ twin is $\mathrm{SU}(2)\times\mathrm{U}(1)_R\subset\mathrm{Sp}(2)$. However, the $S_b^{\{2,1\}}$-invariant sector is uncharged under the extra U(1) so that the non-trivial global symmetry is indeed $\mathrm{SU}(2)\times\mathrm{U}(1)_R$, as expected.

Using the $S_b^{\{2,1\}}$ charges it is straightforward to deduce the quadratic $S_b^{\{2,1\}}$-invariant operators and, through their $\mathrm{SU}(2)\times\mathrm{U}(1)_R$ representations, to collect them into massive long $\mathcal{N}=2$ supermultiplets. In term of the on-shell superfields of [22, 23] we obtain one spin-2 and one spin-1 supercurrent,

$$J=V\bar{V},\quad J_i{}^j=V_i\bar{V}^j-\frac{1}{2}\delta_i{}^jV_k\bar{V}^k, \tag{3.55}$$

where V is the $\mathcal{N}=2$ spin-1 on-shell superfield and V_i is the $\mathcal{N}=2$ spin-1/2 on-shell superfield [23], transforming as a doublet of the global symmetry U(2). Hence, the $S_b^{\{2,1\}}$ S-folding reproduces precisely the truncation (3.48) giving the big $\{2,1\}$ twin. The off-shell component field projection is given in appendix A.

3.2.2 The $\mathcal{N}=1$ S-fold construction

Let us now introduce the $\{2,1\}$ little twin S-fold operator $S_l^{\{2,1\}} := s_k \circ r_k$. As for the $\{3,1\}$ example, the R-symmetry action for the little is the same as that for the big twin,

$$r_k : \begin{cases} Q_i \mapsto e^{-\frac{i\pi}{k}} Q_i, & Q_3 \mapsto \ e^{\frac{i2\pi}{k}} Q_3 \quad Q_4 \mapsto Q_4 \\ \bar{Q}^i \mapsto \ e^{\frac{i\pi}{k}} \bar{Q}^i, & \bar{Q}^3 \mapsto e^{-\frac{i2\pi}{k}} \bar{Q}^3 \quad \bar{Q}^4 \mapsto \bar{Q}^4 \end{cases} \tag{3.56}$$

where $i=1,2$. The difference again lies solely in the S-duality operator,

$$s_k : \begin{cases} Q_A \mapsto e^{-\frac{2i\pi}{k}} Q_A; \\ \bar{Q}^A \mapsto \ e^{\frac{2i\pi}{k}} \bar{Q}^A. \end{cases} \tag{3.57}$$

Comparing with the $\{3,1\}$ case, we note that (i) the S-duality phase is $\exp[\pm i\pi/k]$ for all big twins while (ii) the R-symmetry operator is the same for each pair of big and little twins, and (iii) if $\mathcal{N}_b = n$ the S-duality on the corresponding little twin is given by $\exp[\mp i n\pi/k]$. Note: (i) simply reflects the fact that the supercharges transform uniformly under S-duality, so any change amounts to a trivial redefinition of the S-fold; (ii) follows from the requirement that each twin pair has the same global symmetry, which is determined by the subalgebra commuting with r_k alone; (iii) is a consequence of breaking the $\mathcal{N}=4$ R-symmetry to $\mathcal{N}=\mathcal{N}_b$, which implies a single supercharge carries charge $\mathcal{N}_b$ and so can always be chosen to be the $\mathcal{N}_l = 1$ supercharge.

Hence, the composite action for the little $\{2,1\}$ twin is given by

$$S_l^{\{2,1\}} : \begin{cases} Q_i \mapsto e^{-\frac{3i\pi}{k}} Q_i, & Q_3 \mapsto Q_3 \quad Q_4 \mapsto e^{-\frac{2i\pi}{k}} Q_4 \\ \bar{Q}^i \mapsto \ e^{\frac{3i\pi}{k}} \bar{Q}^i, & \bar{Q}^3 \mapsto \bar{Q}^3 \quad \bar{Q}^4 \mapsto \ e^{\frac{2i\pi}{k}} \bar{Q}^4 \end{cases} \tag{3.58}$$

Now only four supercharges survive, leaving an $\mathcal{N}=1$ superalgebra. Excluding $k=2$ the entire $\mathcal{N}=4$ vector multiplet transforms non-trivially under $S_l^{\{2,1\}}$, as summarised in table 6. Specialising to $k=3$ it is straightforward to deduce the quadratic $S_l^{\{2,1\}}$-invariant operators and, through their $\mathrm{U}(2)_F \times \mathrm{U}(1)_R$ representations, to collect them into massive long $\mathcal{N}=1$ supermultiplets, yielding one spin-2 and six spin-1 multiplets in the $\mathbf{5}+\mathbf{1}$ of $\mathrm{Sp}(2)$ as required. See appendix A.

3.2.3 Dual supergravity theories

Ungauged $D=5$, $\mathcal{N}_b = 4$ supergravity coupled to one vector multiplet has a twin given by the $\mathcal{N}_l = 2$ supergravity coupled to six vector multiplets, based on the semi-simple rank-3 Jordan algebra $\mathbb{R} \oplus \mathbf{\Gamma}_{1,5}$ [8]. The bosonic sectors of such twins are determined by the common scalar symmetric coset

$$\mathrm{SO}(1,1) \times \frac{\mathrm{SO}\,(1,5)}{\mathrm{SO}(5)}, \tag{3.59}$$

where $\mathrm{SO}(1,1) \times \mathrm{SO}(1,5)$ is the reduced structure group of $\mathbb{R} \oplus \mathbf{\Gamma}_{1,5}$. This is the $D=5$ analog of the $D=4$ $\{4,2\}$ supergravity twin pair [35], with common symmetric coset

$$\frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{U}(1)} \times \frac{\mathrm{SO}\,(2,6)}{\mathrm{SO}(2) \times \mathrm{SO}(6)}, \tag{3.60}$$

which is the R-map image of (3.59).

	SU(2)	$\mathrm{U}(1)_R$	$\mathrm{U}(1)_F$	$S_l^{\{2,1\}}$
F^+	$\mathbf{1}$	0	0	-4
λ^i	$\mathbf{2}$	1	0	-1
λ^3	$\mathbf{1}$	-1	1	-4
λ^4	$\mathbf{1}$	-1	-1	-2
ϕ^{ij}	$\mathbf{1}$	2	0	2
ϕ^{i3}	$\mathbf{2}$	0	1	-1
ϕ^{i4}	$\mathbf{2}$	0	-1	1
ϕ^{34}	$\mathbf{1}$	-2	0	-2

Table 6. The charges carried by the component fields of the $\mathcal{N} = 4$ super Yang-Mills multiplet under the $S_l^{\{2,1\}}$ S-fold operator (in units of π/k) and the invariant $\mathrm{SU}(2)_F \times \mathrm{U}(1)_R \times \mathrm{U}(1)_F \subset \mathrm{SU}(4)$ flavour/R-symmetry subgroup.

Following the discussion of section 3.1.3, together with the $S_b^{\{2,1\}}/S_l^{\{2,1\}}$ S-foldings and the observation that both the $\{2,1\}$ W-SCFT and the $\{4,2\}$ Poincaré supergravity twins are truncations of the $\{3,1\}$ W-SCFT and the $\{6,2\}$ Poincaré supergravity twins, respectively, we would anticipate analogous 'twin' truncations yielding the candidate bulk dual $\{4,2\}$ gauged twin supergravities. The big gauged $\mathcal{N} = 4$ twin corresponds to a further consistent truncation of the special case described in [28], in which SU(4) gauged $\mathcal{N} = 8$ supergravity is truncated down to Romans' gauged $\mathcal{N} = 4$ supergravity [36] coupled to a single vector multiplet. Note, in Romans' gauged $\mathcal{N} = 4$ supergravity the vectors of $\mathcal{N} = 4$ Poincaré supergravity sitting in the $\mathbf{5}$ of Sp(2) are replaced by three vectors and two "self-dual" two-forms in the $\mathbf{3}_0$ and $\mathbf{1}_2 + \mathbf{1}_{-2}$ of $\mathrm{U}(2) \subset \mathrm{Sp}(2)$, as required by the preservation of supersymmetry, which is precisely reflected by the multiplet structure of the candidate dual $\mathcal{N} = 2$ W-SCFT, cf. appendix A. Similarly, the little $\mathcal{N} = 2$ twin corresponds to a further consistent truncation of the $\mathcal{N} = 8 \to \mathcal{N} = 2$ case given in [28] (and described in section 3.1.3) down to $\mathrm{U}(1)_R \times \mathrm{U}(2)$ gauged $\mathcal{N} = 2$ supergravity coupled to 3+1 vector multiplets and 1+1 "self-dual" tensors multiplets in the $\mathbf{3}^0_0 + \mathbf{1}^0_0$ and $\mathbf{1}^0_2 + \mathbf{1}^0_{-2}$, as required by supersymmetry [37, 38], again reflecting the structure of the candidate dual $\mathcal{N} = 1$ W-SCFT. See appendix A. Note, the $\mathrm{U}(1)_R$ gauge factor is required for an AdS vacuum [38, 39]. As for the $\{6,2\}$ case discussed in section 3.1.3, we do not anticipate that these truncations can be obtained using purely (conventional) geometric symmetries, but require instead the twin $S_b^{\{2,1\}}/S_l^{\{2,1\}}$ S-foldings implemented on SU(4) gauged $\mathcal{N} = 8$ supergravity. Note, the need to invoke S-duality here implies that they are intrinsically non-perturbative. We would also expect to be able to obtain the gauged twin supergravities directly through twin gaugings of the $\{4,2\}$ Poincaré supergravities following [38, 40].

3.2.4 Double-copy construction

The "parent" [3, 2] multiplet is given by

$$[2,1]^A \otimes [1,1]^{\tilde{A}} = [3,2]\,. \tag{3.61}$$

To obtain the $\mathcal{N}=2$ twin of the $\{2,1\}$ pair, first decompose the Left factor into $\mathcal{N}=1$ multiplets:

$$\left([1,1]^{A} \oplus 2\,[1,1/2]^{A}\right) \otimes [1,1]^{\tilde{A}} = [3,2]\,, \tag{3.62}$$

and then truncate the Left factor to the σ-invariant sector. By using the rule that adjoint and non-adjoint representations do not talk to one another in the double copy, one obtains

$$\left([1,1]^{A_0} \oplus 2\,[1,1/2]^{\rho}\right) \otimes [1,1]^{\tilde{A}} = [2,2] \oplus [2,1]\,. \tag{3.63}$$

To obtain the $\mathcal{N}=1$ twin of the $\{2,1\}$ pair, one has then to decompose the Right factor into $\mathcal{N}=0$ multiplets and truncate to the $\tilde{\sigma}$-invariant sector:

$$\left([1,1]^{A_0} \oplus 2\,[1,1/2]^{\rho}\right) \otimes \left([0,1]^{\tilde{A}_0} + 2\,[0,1/2]^{\tilde{\rho}} + [0,0]^{\tilde{A}_0}\right) = [1,2] \oplus 6\,[1,1]\,, \tag{3.64}$$

where 6 must be specified as $\mathbf{5}+\mathbf{1}$ under Sp(2).

4 Twin W-supergravities

In [11] it was argued that W-supergravities, which possess a spin-4 field in place of the conventional graviton, follow from the effective field theory limit of asymmertric S-foldings of string theory. For recent developments on W-supergravities see [41, 42]. The string S-fold is effected by a T-duality twist and an S-duality twist combined with a new G-duality twist, which ensures that the S-fold only acts on the right movers, and a H-duality twist, which ensures that the S-fold is Lorentz covariant. The combined S-G-H-duality is an automorphism of the string theory, so the S-fold is a *bona fide* projection. The string S-fold reproduces the field theory S-fold on the right-moving sector and so the spectrum can be calculated using the product or "double-copy" of the corresponding W-SCFTs, where level matching forbids products amongst the right W-SCFT and the conventional massless states still present in the left-moving sector.

For example, the spectrum of $\mathcal{N}=7$ W-supergravity [11] follows from the product between the $\mathcal{N}=4$ left-moving sector and the S-folded $\mathcal{N}=3$ right-moving sector of type II strings on T^6, which at the lowest level reduces to

$$[4,2]_L \times [3,2]_R = [7,4]. \tag{4.1}$$

By considering various degrees of supersymmetry in the factors we can construct the spectra of the would-be W-supergravities [11], with all $0 \leq \mathcal{N} \leq 7$ as summarised in table 7. We can also generate (almost) arbitrary "matter" couplings for $\mathcal{N} \leq 6$. Note, although we have included the $[8,4]$ multiplet for completeness, it does not correspond to any W-supergravity as the S-fold always breaks some supersymmetry (at least not without some further, as yet to be determined, novel ingredients). It is also useful in that it provides a "parent" multiplet for the first example of twin W-supergravities.

Using table 7 and table 8, together with the branchings given in table 9, it is then straightforward to follow [10] (cf. section 3.1.4) to construct would-be twin W-supergravities, which have identical bosonic symmetries and spectra. To go beyond this would require a

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$[4,2]$	$\otimes$	$[4,2]$	$=$	$[8,4]$
$[4,2]$	$\otimes$	$[3,2]$	$=$	$[7,4]$
$[4,2]$	$\otimes$	$[2,2]$	$=$	$[6,4]$
$[4,2]$	$\otimes$	$[1,2]$	$=$	$[5,4]$
$[4,2]$	$\otimes$	$[0,2]$	$=$	$[4,4]$
$[3,2]$	$\otimes$	$[3,2]$	$=$	$[6,4]+[6,3]$
$[3,2]$	$\otimes$	$[2,2]$	$=$	$[5,4]+[5,3]$
$[3,2]$	$\otimes$	$[1,2]$	$=$	$[4,4]+[4,3]$
$[3,2]$	$\otimes$	$[0,2]$	$=$	$[3,4]+[3,3]$
$[2,2]$	$\otimes$	$[2,2]$	$=$	$[4,4]+[4,3]+[4,2]$
$[2,2]$	$\otimes$	$[1,2]$	$=$	$[3,4]+[3,3]+[3,2]$
$[2,2]$	$\otimes$	$[0,2]$	$=$	$[2,4]+[2,3]+[2,2]$
$[1,2]$	$\otimes$	$[1,2]$	$=$	$[2,4]+[2,3]+[2,2]+[2,1]$
$[1,2]$	$\otimes$	$[0,2]$	$=$	$[1,4]+[1,3]+[1,2]+[1,1]$
$[0,2]$	$\otimes$	$[0,2]$	$=$	$[0,4]+[0,3]+[0,2]+[0,1]+[0,0]$

Table 7. Products of massive spin-2 multiplets.

better understanding of the S-folded vertex operators, which we leave for future work. Let us give an example, generalising the prototypical case of the $\{6,2\}$ twins in conventional supergravity. As indicated, the maximally supersymmetric "parent" multiplet is given by

$$[8,4]_{\text{parent}} = [4,2]\times[4,2], \tag{4.2}$$

which corresponds to the parent $\mathcal{N}=8$ supergravity of the $\{6,2\}$ twin supergravities, although does not exist itself as a W-supergravity.

The big twin is given by branching the right theory down to $\mathcal{N}=2$ with $\mathrm{Sp}(2)_{\tilde{R}}\times\mathrm{Sp}(2)_{\tilde{F}}\subset\mathrm{Sp}(4)_{\tilde{R}}$,

$$[4,2]\times([2,2]+\mathbf{4}[2,3/2]+\mathbf{5}[2,1]) = [4,2]\times[2,2]+\mathbf{5}[4,2]\times[2,1] = [6,4]+\mathbf{5}[6,3], \tag{4.3}$$

where we have employed the rule that integer and half-integer multiplets to not talk to-one-another in the product, as described in section 3.1.4. This reflects the property of the scattering amplitude double-copy that adjoint and non-adjoint representations of the gauge group do not mix [43]. The multiplicities are given as representations of $\mathrm{Sp}(2)_{\tilde{F}}$ and the total global symmetry is $\mathrm{Sp}(6)_R\times\mathrm{Sp}(2)_{\tilde{F}}$.

Following [10] the little twin is given by further branching the left theory down to $\mathcal{N}=0$ with $\mathrm{Sp}(4)_F\equiv\mathrm{Sp}(4)_R$,

$$([0,2]+\mathbf{8}[0,3/2]+\mathbf{27}[0,1]+\mathbf{48}[0,1/2]+\mathbf{42}[0,0])\times([2,2]+\mathbf{4}[2,3/2]+\mathbf{5}[2,1])\,, \tag{4.4}$$

$[4,2]$	$\otimes$	$[3,3/2]$	$=$	$[7,7/2]$
$[4,2]$	$\otimes$	$[2,3/2]$	$=$	$[6,7/2]$
$[4,2]$	$\otimes$	$[2,1]$	$=$	$[6,3]$
$[3,2]$	$\otimes$	$[0,3/2]$	$=$	$[3,7/2]+[3,5/2]$
$[3,2]$	$\otimes$	$[0,1]$	$=$	$[3,3]+[3,2]$
$[3,3/2]$	$\otimes$	$[0,3/2]$	$=$	$[3,3]$
$[3,3/2]$	$\otimes$	$[0,1/2]$	$=$	$[3,2]$
$[2,2]$	$\otimes$	$[0,1]$	$=$	$[2,3]+[2,2]+[2,1]$
$[2,3/2]$	$\otimes$	$[0,3/2]$	$=$	$[2,3]+[2,2]$
$[2,3/2]$	$\otimes$	$[0,1/2]$	$=$	$[2,2]+[2,1]$
$[2,1]$	$\otimes$	$[0,2]$	$=$	$[2,3]$
$[2,1]$	$\otimes$	$[0,1]$	$=$	$[2,2]$

Table 8. Some relevant products of massive long multiplets.

$[4,2]$	$\to$	$[3,2]+\mathbf{2}\times[3,3/2]$
$[4,2]$	$\to$	$[2,2]+\mathbf{4}\times[2,3/2]+\mathbf{5}\times[2,1]$
$[4,2]$	$\to$	$[1,2]+\mathbf{6}\times[1,3/2]+\mathbf{14}\times[1,1]+\mathbf{14'}\times[1,1/2]$
$[4,2]$	$\to$	$[0,2]+\mathbf{8}\times[0,3/2]+\mathbf{27}\times[0,1]+\mathbf{48}\times[0,1/2]+\mathbf{42}\times[0,0]$
$[3,2]$	$\to$	$[2,2]+\mathbf{2}\times[2,3/2]+\mathbf{1}\times[2,1]$
$[3,2]$	$\to$	$[1,2]+\mathbf{4}\times[1,3/2]+(\mathbf{5}+\mathbf{1})\times[1,1]+\mathbf{4}\times[1,1/2]$
$[3,2]$	$\to$	$[0,2]+\mathbf{6}\times[0,3/2]+(\mathbf{14}+\mathbf{1})\times[0,1]+\mathbf{14'}+\mathbf{6}\times[0,1/2]+\mathbf{14}\times[0,0]$
$[2,2]$	$\to$	$[1,2]+\mathbf{2}\times[1,3/2]+\mathbf{1}\times[1,1]$
$[2,2]$	$\to$	$[0,2]+\mathbf{4}\times[0,3/2]+(\mathbf{5}+\mathbf{1})\times[0,1]+\mathbf{4}\times[0,1/2]+\mathbf{1}\times[0,0]$
$[1,2]$	$\to$	$[0,2]+\mathbf{2}\times[0,3/2]+\mathbf{1}\times[0,1]$

Table 9. Branchings of massive spin-2 multiplets under $\mathrm{Sp}(\mathcal{N}')\times\mathrm{Sp}(\mathcal{N}-\mathcal{N}')\subset\mathrm{Sp}(\mathcal{N})$. Multiplicities are given in terms of $\mathrm{Sp}(\mathcal{N}-\mathcal{N}')$ representations.

which yields,

$$\begin{aligned}[2,4] &+ ((\mathbf{1},\mathbf{1})+(\mathbf{1},\mathbf{5})+(\mathbf{27},\mathbf{1})+(\mathbf{8},\mathbf{4}))[2,3] \\ &+ ((\mathbf{1},\mathbf{1})+(\mathbf{27},\mathbf{5})+(\mathbf{27},\mathbf{1})+(\mathbf{8},\mathbf{4})+(\mathbf{48},\mathbf{4})+(\mathbf{42},\mathbf{1}))[2,2] \\ &+ ((\mathbf{27},\mathbf{1})+(\mathbf{48},\mathbf{4})+(\mathbf{42},\mathbf{5}))[2,1]\end{aligned} \tag{4.5}$$

where we have given the multiplicities as $\mathrm{Sp}(4)_F\times\mathrm{Sp}(2)_{\tilde{F}}$ representations. These may be collected into irreducible $\mathrm{Sp}(6)_F$ representations,

$$[2,4]+\mathbf{65}[2,3]+\mathbf{429}[2,2]+\mathbf{429'}[2,1], \tag{4.6}$$

so that the total global symmetry is $\mathrm{Sp}(6)_F\times\mathrm{Sp}(2)_{\tilde{R}}$. We see that the big and little twins thus have the same global symmetries. The bosonic spectra match. For instance, the big twin has 71 spin-3 states in the $(\mathbf{65}+\mathbf{1},\mathbf{1})+(\mathbf{1},\mathbf{5})$ of $\mathrm{Sp}(6)_R\times\mathrm{Sp}(2)_{\tilde{F}}$, while the little twin has 71 spin-3 states in the $(\mathbf{1},\mathbf{5}+\mathbf{1})+(\mathbf{65},\mathbf{1})$ of $\mathrm{Sp}(6)_F\times\mathrm{Sp}(2)_{\tilde{R}}$. Similarly, the spin-2 states sit in the $(\mathbf{429}+\mathbf{65}+\mathbf{1},\mathbf{1})+(\mathbf{65},\mathbf{5})$ and $(\mathbf{1},\mathbf{1})+(\mathbf{65},\mathbf{5}+\mathbf{1})+(\mathbf{429},\mathbf{1})$ of $\mathrm{Sp}(6)_R\times\mathrm{Sp}(2)_{\tilde{F}}$ and $\mathrm{Sp}(6)_F\times\mathrm{Sp}(2)_{\tilde{R}}$, respectively.

Using the same methodology we obtain the $\{5,1\},\{4,2\},\{3,1\}$ and $\{2,1\}$ twin W-supergravities, analogous to the conventional $D=4$ twins of table 1. There may also be further $D=4$ twins, since we also have parents with $\mathcal{N}=3$ factors, as well as twins in other dimensions, as suggested by table 1, but we leave the complete classification for future work.

5 Conclusions

We have argued that there are twin W-SCFTs using S-folds preserving $\mathcal{N}<3$ supersymmetry. The lowest level spectra may be deduced from the "double-copy" of massive long spin ≤ 1 multiplets. Similarly, at the level of spectra and symmetries there exist twin W-supergravities. There are a number of directions we will consider in future work. Perhaps most obviously is the need, given their intrinsically non-perturbative nature, of a more complete understanding of the twins, and the W-SCFTs with $\mathcal{N}<3$ in general. In particular, a string/F-theory embedding would lend further support to their existence and twiness beyond spectra alone. One might also consider their central charges. For instance, it is known (essentially using representation theory together with known properties of $\mathcal{N}=2$ theories alone) that the $\mathcal{N}=3$ theories obey $a=c$ [2]. This raises the possibility of relations (if any) amongst the central charges of the twins. We will also generalise to other dimensions, as suggested by the twin pyramid table 1. The $D=3,4$ levels of table 1 suggest the possibility of W-SCFTs in $D=2,3$. The $D=6$ layer, on the other hand, poses a puzzle as the unique $D=5$ superconformal group obstructs the existence of W-SCFT twins with distinct degrees of supersymmetry. The W-supergravities raise similar questions, especially with regard to their twinness beyond spectra/symmetries and further examples in $D=3,5,6$.

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A Supercurrent component projection

In terms of component fields the $\mathcal{N}=4$ supercurrent $J_{AB,CD}$ is given by [15, 44],

$$\begin{array}{ccccccccc} g_{\mu\nu}, & \psi_\mu^A & A_\mu{}^A{}_B & A_{\mu\nu}^{AB} & \chi^A & \chi_C^{AB} & \varphi & \varphi^{AB} & \varphi_{CD}^{AB}, \\ \mathbf{1}, & \mathbf{4}+\overline{\mathbf{4}} & \mathbf{15} & \mathbf{6}_{\mathbb{C}} & \mathbf{4}+\overline{\mathbf{4}} & \mathbf{20}+\overline{\mathbf{20}} & \mathbf{1}_{\mathbb{C}} & \mathbf{10}+\overline{\mathbf{10}} & \mathbf{20}' \end{array} \tag{A.1}$$

where we have indicated the corresponding SU(4) representations and

$$\begin{aligned} g_{\mu\nu} &= \frac{1}{2}\left(\eta_{\mu\nu}F^-_{\rho\sigma}F^{+\rho\sigma} - 4F^{-\rho}_\mu F^+_{\nu\rho} + \text{h.c.}\right) - \frac{1}{2}\bar\lambda_A\gamma_{(\mu}\overleftrightarrow{\partial}_{\nu)}\lambda^A \\ &\quad + \eta_{\mu\nu}\partial^\rho\phi^{AB}\partial_\rho\phi_{AB} - 2\partial_\mu\phi^{AB}\partial_\nu\phi_{AB} - \frac{1}{3}\left(\eta_{\mu\nu}\Box - \partial_\mu\partial_\nu\right)\phi^{AB}\phi_{AB} \end{aligned} \tag{A.2a}$$

$$\psi_\mu^A = -(\sigma F^-)\gamma_\mu\lambda^A + 2i\phi^{AB}\overleftrightarrow{\partial}_\mu\lambda_B + \frac{4}{3}i\sigma_{\mu\rho}\partial^\rho(\phi^{AB}\lambda_B) \tag{A.2b}$$

$$A_\mu{}_A{}^B = \phi_{AC}\overleftrightarrow{\partial}_\mu\phi^{CB} + \bar\lambda_A\gamma_\mu\lambda^B - \frac{1}{4}\delta_A{}^B\bar\lambda_C\gamma_\mu\lambda^C \tag{A.2c}$$

$$A_{\mu\nu}^{AB} = \bar\lambda^A\sigma_{\mu\nu}\lambda^B + 2i\phi^{AB}F^+_{\mu\nu} \tag{A.2d}$$

$$\chi^A = \sigma F^+\lambda^A \tag{A.2e}$$

$$\chi_E^{AB} = \frac{1}{2}\epsilon^{ABCD}\left(\phi_{CD}\lambda_E + \phi_{CE}\lambda_D\right) \tag{A.2f}$$

$$\varphi = F^-_{\mu\nu}F^{-\mu\nu} \tag{A.2g}$$

$$\varphi^{AB} = \bar\lambda^A\lambda^B \tag{A.2h}$$

$$\varphi_{CD}^{AB} = \phi^{AB}\phi_{CD} - \frac{1}{12}\delta_C{}^{[A}\delta_D{}^{B]}\phi^{EF}\phi_{EF} \tag{A.2i}$$

The $\{3,1\}$ twins. The fields transform under the $S_b^{\{3,1\}}$ S-fold (3.18) with weights (in units of $2\pi/k$),

$$\begin{array}{ccccc} \phi^{ab} & \phi^{a4} & \lambda^4 & \lambda^a & F^+ \\ 1 & -1 & -1 & 1 & 1 \end{array} \tag{A.3}$$

which project (A.2) onto a single spin-2 $\mathcal{N}=3$ supercurrent, with component field schematically given by

$$\left(g_{\mu\nu},\ \ \psi_\mu^a\ \psi_{\mu a},\ \ A_{\mu a}{}^b\ A_{\mu\nu}^{a4}\ A_{\mu\nu a4}\ A_{\mu 4}{}^4,\ \ \chi^4\ \chi_4\ \chi_4^{ab}\ \chi_{ab}^4\ \ \chi_4^{a4}\ \chi_{a4}^4,\ \ \varphi^{a4}\ \varphi_{a4}\ \varphi_{b4}^{a4}\right) \tag{A.4}$$

carrying U(3) representations given in (3.20). For the complete characterisation of the $\mathcal{N}=3$ Weyl supercurrent-multiplet see [12].

For $k=4$ the fields transform under the $S_l^{\{3,1\}}$ S-fold (3.29) with weights (in units of $\pi/2$),

$$\begin{array}{ccccc} \phi^{ab} & \phi^{a4} & \lambda^4 & \lambda^a & F^+ \\ -1 & 1 & -1 & 1 & -1 \end{array} \tag{A.5}$$

which project (A.2) onto a single spin-2 $\mathcal{N}=1$ supercurrent

$$\left(g_{\mu\nu},\ \ \psi_\mu^4,\ \ \psi_{\mu 4},\ \ A_{\mu 4}{}^4\right) \tag{A.6}$$

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and 14 spin-1 $\mathcal{N}=1$ supercurrents

$$(A_{\mu a}{}^{b},\ \chi_b^{a4},\ \varphi_{a4}^{b4}),\ (A_{\mu\nu}{}^{a4},\ \chi^{a},\ \varphi^{a4}),\ (A_{\mu\nu b4}\ \chi_b,\ \varphi_{b4}), \tag{A.7}$$

transforming in the $\mathbf{8}_0, \mathbf{3}_2, \overline{\mathbf{3}}_{-2}$ of U(3), respectively, as can be checked directly using the supersymmetry transformation rules of the $\mathcal{N}=4$ super Yang-Mills multiplet with the variational parameter ε^A restricted to ε^4.

The $\{2,1\}$ twins. The fields transform under the big twin $S_b^{\{2,1\}}$ S-fold (3.54) with weights (in units of π/k),

$$\begin{array}{cccccccc} \phi^{34} & \phi^{i4} & \phi^{i3} & \phi^{ij} & \lambda^4 & \lambda^3 & \lambda^i & F^+ \\ -2 & 1 & -1 & 2 & 1 & -1 & 2 & 2 \end{array} \tag{A.8}$$

which project (A.2) onto a single spin-2 and a single spin-1 $\mathcal{N}=2$ supercurrent, given schematically by

$$(g_{\mu\nu},\ \psi_\mu^i\ \psi_{\mu i},\ A_{\mu i}{}^{j}\ A_{\mu\nu}^{34}\ A_{\mu\nu 34}\ A_{\mu 3}{}^{3}+A_{\mu 4}{}^{4},\ \chi_3^{i3}+\chi_4^{i4}\ \chi_{i3}^{3}+\chi_{i4}^{4},\ \varphi_{34}^{34}) \tag{A.9}$$

and

$$(A_{\mu 3}{}^{3}-A_{\mu 4}{}^{4},\ \chi_3^{i3}-\chi_4^{i4}\ \chi_{i3}^{3}-\chi_{i4}^{4},\ \varphi_{i3}^{j3}+\varphi_{i4}^{j4}\ \varphi_{ij}\ \varphi^{ij}). \tag{A.10}$$

where the spin 3/2, 1, 1/2 and 0 fields are in the $\mathbf{2}_1+\mathbf{2}_{-1}$, $\mathbf{3}_0+\mathbf{1}_2+\mathbf{1}_{-2}+\mathbf{1}_0+\mathbf{1}_0$, $\mathbf{2}_1+\mathbf{2}_{-1}+\mathbf{2}_1+\mathbf{2}_{-1}$ and $\mathbf{3}_0+\mathbf{1}_2+\mathbf{1}_{-2}+\mathbf{1}_0$ of $\mathrm{U}(2)_R$, respectively, in agreement with the decomposition of (3.48) under $\mathrm{U}(2)\in\mathrm{Sp}(2)$. The precise linear combinations are uniquely determined by closure under the supersymmetry transformations given in [15] with the variational parameter ε^A restricted to ε^i.

For $k=3$ the fields transform under the little twin $S_l^{\{2,1\}}$ S-fold (3.58) with weights (in units of $\pi/3$),

$$\begin{array}{cccccccc} \phi^{34} & \phi^{i4} & \phi^{i3} & \phi^{ij} & \lambda^4 & \lambda^3 & \lambda^i & F^+ \\ -2 & 1 & -1 & 2 & -2 & 2 & -1 & 2 \end{array} \tag{A.11}$$

which project (A.2) onto a single spin-2 $\mathcal{N}=1$ supercurrent, given schematically by

$$(g_{\mu\nu},\ \psi_\mu^3\ \psi_{\mu 3},\ A_{\mu 3}{}^{3}) \tag{A.12}$$

and $5+1$ spin-1 $\mathcal{N}=1$ supercurrents

$$\begin{array}{l} (A_{\mu i}{}^{j}\quad A_{\mu\nu}^{34}\quad A_{\mu\nu 34}\quad A_{\mu 3}{}^{3}-3A_{\mu 4}{}^{4}, \\ \ \chi_j^{i3}\quad \chi_{j3}^{i}\quad \chi_3^{ij}\quad \chi_{ij}^{3}\quad \chi_4^{34}\quad \chi_{34}^{4}\quad \chi^4\quad \chi_4 \\ \ \varphi_{i3}^{j3}\quad \varphi^{34}\quad \varphi_{34}\quad \varphi_{34}^{34}), \end{array} \tag{A.13}$$

where the spin 3/2, 1, 1/2 and 0 fields are in the $\mathbf{1}_0^1+\mathbf{1}_0^{-1}$, $\mathbf{3}_0^0+\mathbf{1}_2^0+\mathbf{1}_{-2}^0+\mathbf{1}_0^0+\mathbf{1}_0^0$, $\mathbf{3}_0^1+\mathbf{1}_2^1+\mathbf{1}_{-2}^1+\mathbf{1}_0^1+$ c.c. and $\mathbf{3}_0^0+\mathbf{1}_2^0+\mathbf{1}_{-2}^0+\mathbf{1}_0^0$ of $\mathrm{U}(2)_F\times\mathrm{U}(1)_R$, respectively, in agreement with the decomposition of (3.49) under $\mathrm{U}(1)_R\times\mathrm{U}(2)_F\in\mathrm{Sp}(1)\times\mathrm{Sp}(2)$. The precise linear combinations are uniquely determined by closure under the supersymmetry transformations given in [15] with the variational parameter ε^A restricted to ε^3.

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