Dihedral flavor group as the key to understand quark and lepton flavor mixing

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ABSTRACT: We have studied the lepton and quark mixing patterns which can be derived from the dihedral group D_n in combination with CP symmetry. The left-handed lepton and quark doublets are assigned to the direct sum of a singlet and a doublet of D_n . A unified description of the observed structure of the quark and lepton mixing can be achieved if the flavor group D_n and CP are broken to $Z_2 \times CP$ in neutrino, charged lepton, up quark and down quark sectors, and the minimal group is D_{14} . We also consider another scenario in which the residual symmetry of the charged lepton and up quark sector is Z_2 while $Z_2 \times CP$ remains preserved by the neutrino and down quark mass matrices. Then D_7 can give the experimentally favored values of CKM and PMNS mixing matrices.

KEYWORDS: CP violation, Discrete Symmetries, Neutrino Physics

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1 Introduction

It is well established that the three generations of quarks are mixed with each other to form mass eigenstates in the standard model. The quark flavor mixing matrix appearing in the weak charged-current interactions is referred to as the Cabibbo-Kobayashi-Maskawa (CKM) matrix [1]. The quark mixing angles exhibit a strongly hierarchical structure, and the largest one is the Cabibbo mixing angle $\theta_c \simeq 13^\circ$ between the first and the second generation. Observation of neutrino oscillation implies that neutrinos have masses and non-zero mixing. Analogously there should be a lepton mixing matrix in the weak charged-current interactions, and it is usually called Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix [1]. However, the lepton mixing angles are less hierarchical, both solar and atmospheric mixing angles are large and the reactor angle $\theta_{13} \simeq 8.61^\circ$ is of a similar size to the Cabibbo angle [2]. As regards the CP violation phase, the CP violation in the quark sector has been precisely measured in a variety of meson decay processes. It is confirmed that the single complex phase in the CKM matrix is the dominant source of CP violation, and the angle α in the unitary triangle is determined to be $\alpha = (84.5^{+5.9}_{-5.2})^\circ$ [1]. If neutrinos are Majorana particles, there are additional sources of CP violation in the lepton sector, e.g., the Majorana phases in the PMNS matrix. At present CP conservation in neutrino oscillation is disfavored at 2σ level, and the exact values of the leptonic Dirac CP phase δ_{CP} is unknown although weak evidence for δ_{CP} around $3\pi/2$ is reported by T2K [3] and NO ν A [4]. Non-zero δ_{CP} is also preferred by global analysis of neutrino oscillation data [2, 5, 6].

The origin of the above flavor mixing structures of quarks and leptons is one of the most important problems in standard model. Many proposals have been advanced to explain this

puzzle in the literature. In particular, the non-abelian discrete flavor symmetries appear to be particularly suitable to reproduce the large flavor mixing angles of the leptons. In this paradigm, the three generations of left-handed lepton doublets are usually assumed to transform as a three-dimensional representation of the discrete flavor group G_f which is subsequently broken to different subgroups in the charged lepton and neutrino sectors, the mismatch between the two subgroups allows one to predict the PMNS matrix up to permutations of rows and columns. In a similar way, the mismatch of residual symmetries in up and down quark sectors can be employed to determine the CKM mixing matrix [7–12]. However, no finite group has been found that can predict the correct values of the three different quark mixing angles at leading order, and only the Cabibbo angle can be generated [11, 12].

A recent progress is to extend the discrete flavor symmetry with CP symmetry [13–15]. This approach turns out to be quite powerful and it allows for precise predictions of both lepton mixing angles and CP violating phases. It can lead to very predictive scenarios, where all the mixing angles and CP phases are related to a small number of input parameters [16–24]. Many models and analyses of CP and flavor symmetries have been studied so far, e.g. A_4 [25, 26], S_4 [27–32], $\Delta(27)$ [33, 34], $\Delta(48)$ [35, 36], A_5 [37–41], $\Delta(96)$ [42] and the infinite group series $\Delta(3n^2)$ [43, 44], $\Delta(6n^2)$ [43, 45] and $D_{9n,3n}^{(1)}$ [46]. Flavor and CP symmetries can also constrain the CP violation in leptogenesis [47–49]. Moreover, after including CP symmetry, we can achieve a unified description of quark and lepton flavor mixing from a single flavor symmetry group if the residual symmetries of the charged lepton, neutrino, up quark and down quark sectors are different $Z_2 \times CP$ subgroups, and the minimal flavor symmetry is $\Delta(294)$ [50]. One could reduce the residual subgroups of the charged lepton and up quark sectors to Z_2 while distinct $Z_2 \times CP$ residual symmetries remain preserved by the neutrino and up quark mass matrices. Then either PMNS or CKM matrices depend on only three real free parameters, and the $\Delta(294)$ is still the smallest flavor symmetry group to generate the experimentally preferred quark and lepton mixing patterns [51]. There are other proposals to explain lepton and quark mixing from flavor symmetry and CP symmetry, see [51–53] for different perspectives. We observe that the group order of the required flavor symmetry is a bit large.

The dihedral group D_n with general n is the group of symmetries of a regular polygon and it doesn't have irreducible three-dimensional representation. It is found that the phenomenologically acceptable Cabibbo angle can be accommodated by the dihedral group D_7 [7, 8]. In the present work, we shall extend the dihedral flavor group to involve also CP as symmetry.¹ Both left-handed lepton doublets and quark doublets are assumed to transform as a reducible three-dimensional representation which is the direct sum of a singlet and a doublet representation of D_n . We shall analyze the mixing patterns for leptons and quarks arising from the breaking of D_n and CP symmetry into $Z_2 \times CP$ in all the relevant quark and lepton sectors. It is remarkable that the D_{14} group of order 28 can give an acceptable prediction for the PMNS and the CKM matrices at leading order. The possible mixing patterns are also studied for the second scenario in which the residual symmetries of the charged lepton and up quark mass matrices are reduced to Z_2 . Then we find that both PMNS and CKM mixing matrices can be accommodated by the dihedral group D_7 .

¹The interplay of D_3 and CP has been studied in [54].

The structure of the paper is as follows: in section 2 we show the general constraints on the mass matrix and how to extract the mixing matrix if a residual symmetry $Z_2^{g_\psi} \times X_\psi$ or $Z_2^{g_\psi}$ is preserved by the fermion fields $\psi \in \{\nu, e, u, d\}$. In section 3 we present the mathematical properties of the dihedral group D_n which is employed as flavor symmetry, and the CP transformations compatible with the D_n flavor group are discussed. In section 4 we analyze the predictions for lepton and quark mixing if the flavor symmetry D_n and CP symmetry are broken to different $Z_2 \times CP$ subgroups in the charged lepton (up quark) and neutrino (down quark) sectors. In section 5 we study the scenario that the residual symmetries of charged lepton and up quark sectors are Z_2 subgroups while the neutrino and down quark mass matrices are invariant under $Z_2 \times CP$. Finally we summarize our main results and conclude in section 6. We show the constraints on the unitary transformation U_ψ for a generic $Z_2^{g_\psi} \times X_\psi$ residual symmetry in appendix A.

2 Lepton and quark mixing from residual symmetry

In the following, we briefly review how the lepton mixing can be predicted from a flavor symmetry group G_f and a CP symmetry which are broken down to two different subgroups of the structure $Z_2 \times CP$ in the charged lepton and neutrino sectors [31, 50, 51]. The quark CKM mixing matrix can be derived in an analogous way using this method. We assume that the three generations of left-handed leptons doublets $L \equiv (\nu, e)^T$ and quark doublets $Q \equiv (u, d)^T$ transform as a three-dimensional representation ρ of G_f . Notice that the following results for mixing matrix hold true no matter whether ρ is a reducible or irreducible representation of G_f . For each type of fermionic field $\psi \in \{\nu, e, u, d\}$, the corresponding residual symmetry is denoted as $Z_2^{g_\psi} \times X_\psi$, where g_ψ is the generator of the residual flavor symmetry $Z_2^{g_\psi}$ and it fulfills $g_\psi^2 = 1$. The residual CP transformation X_ψ should be a symmetric and unitary matrix otherwise the mass spectrum would be partially or completely degenerate [16, 17]. The restricted consistency condition between the residual flavor and CP symmetries reads [13, 16–19, 27]

$$X_\psi \rho^*(\psi) X_\psi^{-1} = \rho(\psi). \quad (2.1)$$

Under the action of residual symmetry, the field ψ transforms as

$$\psi(x) \xrightarrow{G_f} \rho(g_\psi) \psi(x), \quad \psi(x) \xrightarrow{CP} i X_\psi \gamma^0 \mathcal{C} \bar{\psi}^T(\mathcal{P}x), \quad (2.2)$$

where $\mathcal{C}$ is the charge conjugation matrix with $\mathcal{P}x = (t, -\vec{x})$. For the residual symmetry to hold, the mass matrix should satisfy the following conditions

$$\rho^\dagger(g_\psi) m_\psi^\dagger m_\psi \rho(g_\psi) = m_\psi^\dagger m_\psi, \quad X_\psi^\dagger m_\psi^\dagger m_\psi X_\psi = (m_\psi^\dagger m_\psi)^*, \quad (2.3)$$

if ψ is Dirac field, where the mass matrix m_ψ is defined in the right-left basis $\psi^c m_\psi \psi$. If ψ (neutrino in the standard model) is Majorana field, the invariance conditions are

$$\rho^T(g_\psi) m_\psi \rho(g_\psi) = m_\psi, \quad X_\psi^T m_\psi X_\psi = m_\psi^*, \quad (2.4)$$

where the Majorana mass matrix m_ψ is defined as $\frac{1}{2}\psi^T m_\psi \psi$. We can diagonalize the mass matrices $m_\psi^\dagger m_\psi$ and m_ψ with a unitary transformation U_ψ ,

$$U_\psi^\dagger m_\psi^\dagger m_\psi U_\psi = \text{diag}(m_1^2, m_2^2, m_3^2), \quad U_\psi^T m_\psi U_\psi = \text{diag}(m_1, m_2, m_3). \quad (2.5)$$

From eqs. (2.3), (2.4) we can derive that the residual symmetry $Z_2^{g_\psi} \times X_\psi$ leads to the following constraints on the unitary matrix U_ψ [31, 50, 51],

$$U_\psi^\dagger \rho(g_\psi) U_\psi = \pm P_\psi^T \text{diag}(1, -1, -1) P_\psi \quad (2.6)$$

and

$$U_\psi^\dagger X_\psi U_\psi^* \equiv Q_\psi^2 = \begin{cases} \text{diag}(e^{i\alpha_1}, e^{i\alpha_2}, e^{i\alpha_3}), & \text{for Dirac field } \psi, \\ \text{diag}(\pm 1, \pm 1, \pm 1), & \text{for Majorana field } \psi, \end{cases} \quad (2.7)$$

where $\alpha_{1,2,3}$ are real free parameters, P_ψ is a generic three dimensional permutation matrix and it can take six possible forms generated by

$$P_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad P_{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad P_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \quad (2.8)$$

As shown in [31, 50, 51, 55], one can solve the constraint equations of eqs. (2.6), (2.7) by performing Takagi factorization for the residual CP transformation X_ψ with the properties given in appendix A as

$$X_\psi = \Sigma_\psi \Sigma_\psi^T, \quad \Sigma_\psi^\dagger \rho(g_\psi) \Sigma_\psi = \pm \text{diag}(1, -1, -1). \quad (2.9)$$

Then the residual symmetry $Z_2^{g_\psi} \times X_\psi$ would determine the unitary transformation U_ψ to be of the form [31, 50, 51, 55]

$$U_\psi = \Sigma_\psi S_{23}(\theta_\psi) P_\psi Q_\psi^\dagger, \quad (2.10)$$

with

$$S_{23}(\theta_\psi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta_\psi & \sin \theta_\psi \\ 0 & -\sin \theta_\psi & \cos \theta_\psi \end{pmatrix}, \quad (2.11)$$

where the free parameter θ_ψ can be limited in the range $0 \leq \theta_\psi < \pi$ without loss of generality. Since both quark and charged lepton carry non-zero electric charges, only neutrinos in standard model can be Majorana particle. From eq. (2.10) we can see that the difference between Dirac and Majorana neutrinos is in the diagonal matrix Q_ψ . In the case of Dirac neutrinos, Q_ψ is a general phase matrix and it can be rotated away by appropriate field redefinition. For Majorana neutrinos, the diagonal entries of Q_ψ are ± 1 and $\pm i$ and the Majorana phases can be predicted up to π .

If only a residual flavor symmetry $Z_2^{g_\psi}$ is preserved by the mass matrix of ψ , the unitary transformation U_ψ would be only subject to the constraint in eq. (2.6). Since g_ψ is of order

two and its representation matrix $\rho(g_\psi)$ has two degenerate eigenvalues, the residual flavor symmetry $Z_2^{g_\psi}$ can only distinguish one generation from the other two ones. As a result, only one column of U_ψ is fixed and it takes the following form [51]

$$U_\psi = \Sigma_\psi U_{23}^\dagger(\theta_\psi, \delta_\psi) P_\psi Q_\psi^\dagger, \quad (2.12)$$

where Σ_ψ diagonalizes $\rho(g_\psi)$ as in eq. (2.9), Q_ψ is an arbitrary diagonal phase matrix, and $U_{23}(\theta_\psi, \delta_\psi)$ is a block diagonal unitary rotation in the (23)-plane with

$$U_{23}(\theta_\psi, \delta_\psi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta_\psi & \sin \theta_\psi \\ 0 & -\sin \theta_\psi & \cos \theta_\psi \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\delta_\psi} & 0 \\ 0 & 0 & e^{-i\delta_\psi} \end{pmatrix}, \quad (2.13)$$

where the angles θ_ψ and δ_ψ need only be defined over the intervals $[0, \pi/2]$ and $[0, \pi)$ respectively. In comparison with the residual symmetry $Z_2^{g_\psi} \times X_\psi$, the Majorana phases cannot be predicted for Majorana neutrinos in this case.

As a result, if the flavor and CP symmetries are broken to $Z_2^{g_\nu} \times X_\nu$ and $Z_2^{g_e} \times X_e$ in the neutrino and charged lepton sectors respectively, the lepton mixing matrix would be given by

$$U \equiv U_e^\dagger U_\nu = Q_e P_e^T S_{23}^T(\theta_e) \Sigma_e^\dagger \Sigma_\nu S_{23}(\theta_\nu) P_\nu Q_\nu^\dagger. \quad (2.14)$$

In this scenario, all the mixing angles and CP phases only depend on two real rotation angles θ_e and θ_ν in the interval between 0 and π . If neutrinos are Majorana particles, Q_ν is a diagonal matrix with elements ± 1 and $\pm i$, and without loss of generality it can be parameterized as

$$Q_\nu = \text{diag}(1, i^{k_1}, i^{k_2}), \quad (2.15)$$

with $k_{1,2} = 0, 1, 2, 3$. Analogously the residual symmetries $Z_2^{g_u} \times X_u$ and $Z_2^{g_d} \times X_d$ in the up type quark and down type quark sectors allow us to pin down the CKM mixing matrix as

$$V \equiv U_u^\dagger U_d = Q_u P_u^T S_{23}^T(\theta_u) \Sigma_u^\dagger \Sigma_d S_{23}(\theta_d) P_d Q_d^\dagger. \quad (2.16)$$

In the following, we shall consider a second scenario in which the residual symmetry of the charged lepton sector is degraded to $Z_2^{g_e}$ such that the neutrino and charged lepton mass matrices exhibit the residual symmetries $Z_2^{g_e}$ and $Z_2^{g_\nu} \times X_\nu$ respectively. Then the lepton mixing matrix is given by

$$U = Q_e P_e^T U_{23}(\theta_e, \delta_e) \Sigma_e^\dagger \Sigma_\nu S_{23}(\theta_\nu) P_\nu Q_\nu^\dagger, \quad (2.17)$$

which depends on three free parameters θ_e , δ_e and θ_ν . In the same fashion, if the residual symmetries $Z_2^{g_u}$ and $Z_2^{g_d} \times X_d$ are preserved in the up and down quark sectors respectively, the quark mixing matrix would be determined by

$$V = Q_u P_u^T U_{23}(\theta_u, \delta_u) \Sigma_u^\dagger \Sigma_d S_{23}(\theta_d) P_d Q_d^\dagger. \quad (2.18)$$

One can also straightforwardly extract the expression of the CKM mixing matrix for the residual symmetry $Z_2^{g_u} \times X_u$ and $Z_2^{g_d}$. Before closing this section, we note that the above schemes can be extended to the grand unification theory if both left-handed quarks and leptons could be assigned to the same representation. In the following sections, we assume neutrinos are Majorana particles.

3 Dihedral group and CP symmetry

The dihedral group D_n is the symmetry group of an n -sided regular polygon for $n > 1$. A regular polygon with n sides has $2n$ different symmetries: n rotational symmetries and n reflection symmetries, therefore the group order of D_n is $2n$. All D_n are non-abelian permutation groups for $n > 2$, D_1 is isomorphic to Z_2 and D_2 is isomorphic to $Z_2 \times Z_2$. The group D_n is the semidirect product $Z_n \rtimes Z_2$ of the cyclic groups Z_n and Z_2 . The dihedral group can be conveniently defined by two generators R and S which obey the relations [8, 56, 57],

$$R^n = S^2 = (RS)^2 = 1, \quad (3.1)$$

where R refers to rotation and S is the reflection. As a consequence, all the group elements of D_n can be expressed as

$$g = S^\alpha R^\beta \quad (3.2)$$

where $\alpha = 0, 1$ and $\beta = 0, 1, \dots, n-1$. Then it is straightforward to determine the conjugacy classes of the dihedral group. Depending on whether the group index n is even or odd, the $2n$ group elements of D_n can be classified into three or five types of conjugacy classes.

- n is odd

$$\begin{aligned} 1C_1 &= \{1\}, \\ 2C_m^{(\rho)} &= \{R^\rho, R^{-\rho}\}, \quad \text{with } \rho = 1, \dots, \frac{n-1}{2}, \\ nC_2 &= \{S, SR, SR^2, \dots, SR^{n-1}\}, \end{aligned} \quad (3.3)$$

where m is minimal integer such that the identity $m\rho = 0 \pmod{n}$ is satisfied, and kC_l denotes a conjugacy class of k elements whose order are l .

- n is even

$$\begin{aligned} 1C_1 &= \{1\}, \\ 1C_2 &= \{R^{n/2}\}, \\ 2C_m^{(\rho)} &= \{R^\rho, R^{-\rho}\}, \quad \text{with } \rho = 1, \dots, \frac{n-2}{2}, \\ \frac{n}{2}C_2 &= \{S, SR^2, SR^4, \dots, SR^{n-4}, SR^{n-2}\}, \\ \frac{n}{2}C_2 &= \{SR, SR^3, \dots, SR^{n-3}, SR^{n-1}\}, \end{aligned} \quad (3.4)$$

The group structure of D_n is simple, and the subgroups of D_n group turn out to be either dihedral or cyclic group. The explicit expressions of all the subgroups are

$$\begin{aligned} Z_j &= \langle R^{\frac{n}{j}} \rangle && \text{with } j|n, \\ Z_2^{(m)} &= \langle SR^m \rangle && \text{with } m = 0, 1, \dots, n-1, \\ D_j^{(m)} &= \langle R^{\frac{n}{j}}, SR^m \rangle && \text{with } j|n, m = 0, 1, \dots, \frac{n}{j} - 1. \end{aligned} \quad (3.5)$$

Hence the total number of cyclic subgroups generated by certain power of R is equal to the number of positive divisors of n , and the total number of dihedral subgroups is the sum of positive divisors of n .

The group D_n only has real one-dimensional and two-dimensional irreducible representations. The number of irreducible representations is dependent on the parity of the group index n .

- n is odd

If the index n is an odd integer, the group D_n has two singlet representations $\mathbf{1}_i$ and $\frac{n-1}{2}$ doublet representations $\mathbf{2}_j$, where the indices i and j are $i = 1, 2$ and $j = 1, \dots, \frac{n-1}{2}$. We observe that the sum of the squares of the dimensions of the irreducible representations is

$$1^2 + 1^2 + 2^2 \times \frac{n-1}{2} = 2n, \quad (3.6)$$

which is exactly the number of elements in D_n group. In the one-dimensional representations, we have

$$\mathbf{1}_1 : R = S = 1, \quad \mathbf{1}_2 : R = 1, S = -1. \quad (3.7)$$

For the two-dimensional representations, the generators R and S are represented by

$$\mathbf{2}_j : R = \begin{pmatrix} e^{2\pi i \frac{j}{n}} & 0 \\ 0 & e^{-2\pi i \frac{j}{n}} \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (3.8)$$

with $j = 1, \dots, \frac{n-1}{2}$.

- n is even

For the case that the index n is an even integer, the group D_n has four singlet representations $\mathbf{1}_i$ with $i = 1, 2, 3, 4$ and $\frac{n}{2} - 1$ doublet representations $\mathbf{2}_j$ with $j = 1, \dots, \frac{n}{2} - 1$. It can be checked that the squared dimensions of the inequivalent irreducible representations add up to the group order as well,

$$1^2 + 1^2 + 1^2 + 1^2 + 2^2 \times \left(\frac{n}{2} - 1\right) = 2n. \quad (3.9)$$

The generators R and S for the one-dimensional representations are given by

$$\begin{aligned} \mathbf{1}_1 : R = S = 1, & \quad \mathbf{1}_2 : R = 1, S = -1, \\ \mathbf{1}_3 : R = -1, S = 1, & \quad \mathbf{1}_4 : R = S = -1. \end{aligned} \quad (3.10)$$

The explicit forms of these generators in the irreducible two-dimensional representations are

$$\mathbf{2}_j : R = \begin{pmatrix} e^{2\pi i \frac{j}{n}} & 0 \\ 0 & e^{-2\pi i \frac{j}{n}} \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (3.11)$$

with $j = 1, \dots, \frac{n}{2} - 1$. Notice that the doublet representation $\mathbf{2}_j$ and the complex conjugate $\bar{\mathbf{2}}_j$ are unitarily equivalent, and they are related through change of basis, i.e.,

$R^* = URU^{-1}$ and $S^* = USU^{-1}$ where $U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Hence all the two-dimensional

representations of D_n are real representations, although the representation matrix of R is complex in the chosen basis. Moreover, if $a = (a_1, a_2)^T$ is a doublet transforming as $\mathbf{2}_j$, the complex conjugate $\bar{a} = (a_1^*, a_2^*)^T$ doesn't transform as $\mathbf{2}_j$, but rather $(a_2^*, a_1^*)^T$ transform as $\mathbf{2}_j$ under D_n .

In order to consistently combine a flavor symmetry G_f with the CP symmetry, the subsequent action of the CP transformation, an element of the flavor group and the CP transformation should be equivalent to the action of another element of the flavor group. In other word, the so-called consistency condition has to be fulfilled [13–15, 58]

$$X_{\mathbf{r}} \rho_{\mathbf{r}}^*(g) X_{\mathbf{r}}^\dagger = \rho_{\mathbf{r}}(g'), \quad g, g' \in G_f, \quad (3.12)$$

where $\rho_{\mathbf{r}}(g)$ is the representation matrix of the element g in the representation $\mathbf{r}$, and $X_{\mathbf{r}}$ is the CP transformation. Here g and g' are generally different group elements, consequently the CP transformation $X_{\mathbf{r}}$ is related to an automorphism which maps g into g' . In addition, ref. [15] showed that physical CP transformations should be a class-inverting automorphism of G_f , i.e. g^{-1} and g' which is the image of g under the automorphism should be in the same conjugacy class.

We find that the D_n groups really have a class-inverting outer automorphism $\mathbf{u}$, and its action on the generators is

$$R \xrightarrow{\mathbf{u}} R^{-1}, \quad S \xrightarrow{\mathbf{u}} S. \quad (3.13)$$

The CP transformation corresponding to $\mathbf{u}$ is denoted by $X_{\mathbf{r}}^0$, its concrete form is determined by the following consistency conditions,

$$\begin{aligned} X_{\mathbf{r}}^0 \rho_{\mathbf{r}}^*(R) X_{\mathbf{r}}^{0\dagger} &= \rho_{\mathbf{r}}(\mathbf{u}(R)) = \rho_{\mathbf{r}}(R^{-1}), \\ X_{\mathbf{r}}^0 \rho_{\mathbf{r}}^*(S) X_{\mathbf{r}}^{0\dagger} &= \rho_{\mathbf{r}}(\mathbf{u}(S)) = \rho_{\mathbf{r}}(S). \end{aligned} \quad (3.14)$$

In our working basis shown above, $X_{\mathbf{r}}^0$ is fixed to be a unit matrix up to an overall irrelevant phase,

$$X_{\mathbf{r}}^0 = \mathbb{1}. \quad (3.15)$$

Furthermore, including the inner automorphisms, the full set of CP transformations compatible with D_n flavor symmetry are

$$X_{\mathbf{r}} = \rho_{\mathbf{r}}(g) X_{\mathbf{r}}^0 = \rho_{\mathbf{r}}(g), \quad g \in D_n. \quad (3.16)$$

Hence the CP transformations compatible with D_n are of the same form as the flavor symmetry transformations in the chosen basis.

4 Mixing patterns from D_n and CP symmetry breaking to two distinct $Z_2 \times CP$ subgroups

In this section, we shall consider the dihedral group D_n as flavor symmetry G_f which is combined with CP symmetry. The three generations of left-handed lepton and quark

doublets are assumed to transform as a direct sum of one-dimensional representation $\mathbf{1}_i$ and two-dimensional representation $\mathbf{2}_j$ of D_n ,

$$L \sim (\mathbf{1}_i, \mathbf{2}_j)^T, \quad Q \sim (\mathbf{1}_i, \mathbf{2}_j)^T. \quad (4.1)$$

Notice that usually the three lepton doublets are assigned to an irreducible three-dimensional representation of G_f in order to obtain at least two non-vanishing lepton mixing angles. In the following, we shall show that the singlet plus doublet assignment can also accommodate the experimental data on mixing angles after the CP symmetry is considered. Moreover, one can also assign either the second or the third generation lepton (quark) doublet to a one-dimensional representation of D_n while the remaining two generations transform as a two-dimensional representation of D_n . In the same fashion as section 2, we can see these alternative assignments don't give rise to new predictions for the quark and lepton mixing matrices.

Here we assume that the discrete flavor group D_n in combination with CP symmetry is broken down to $Z_2 \times CP$ in both charged lepton and neutrino sectors, then one entry of the mixing matrix is completely fixed by the residual symmetry. Considering all possible residual symmetries $Z_2^{g_e} \times X_e$ and $Z_2^{g_\nu} \times X_\nu$, we find the fixed element can only be 0, 1 or $\cos \varphi_1$, where the value of φ_1 is determined by the choice of the residual symmetry. Obviously only the mixing pattern with the fixed element $\cos \varphi_1$ could be in agreement with experimental data for certain values of φ_1 characterizing the residual symmetry. As a consequence, we find the viable residual symmetries in the lepton sector are $Z_2^{g_e} = Z_2^{SR^{z_e}}$, $X_e = \{R^{-z_e+x}, SR^x\}$, $Z_2^{g_\nu} = Z_2^{SR^{z_\nu}}$ and $X_\nu = \{R^{-z_\nu+y}, SR^y\}$, where $z_e, z_\nu = 0, 1, \dots, n-1$, $x = y = 0$ for odd n and $x, y = 0, n/2$ if the group index n is even. Following the procedures presented in appendix A, we can determine the Takagi factorization of the residual symmetry $Z_2^{SR^z} \times \{R^{-z+x}, SR^x\}$ as follow,

$$\Sigma = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2}i^\rho & 0 & 0 \\ 0 & -e^{\frac{-i\pi jz}{n}} & e^{\frac{-i\pi jz}{n}} \\ 0 & e^{\frac{i\pi jz}{n}} & e^{\frac{i\pi jz}{n}} \end{pmatrix} P_{12}^{\sigma+1} P_{13}^\sigma, \quad (4.2)$$

with $\sigma = 0, 1, z, z+1$ for $i = 1, 2, 3, 4$ respectively, and the parameter ρ depends on the values of x and i which is the index of the one-dimensional representation $\mathbf{1}_i$,

$$\rho = \begin{cases} 0, & \text{for } x = 0, i \in \{1, 2\}, \\ z, & \text{for } x = 0, i \in \{3, 4\}, \\ -j, & \text{for } x = n/2, i \in \{1, 2\}, \\ z - j - n/2, & \text{for } x = n/2, i \in \{3, 4\}. \end{cases} \quad (4.3)$$

As a consequence, the lepton mixing matrix reads

$$U_I = \begin{pmatrix} \cos \varphi_1 & -c_\nu \sin \varphi_1 & s_\nu \sin \varphi_1 \\ c_e \sin \varphi_1 & s_\nu s_e e^{i\varphi_2} + c_\nu c_e \cos \varphi_1 & c_\nu s_e e^{i\varphi_2} - s_\nu c_e \cos \varphi_1 \\ -s_e \sin \varphi_1 & s_\nu c_e e^{i\varphi_2} - c_\nu s_e \cos \varphi_1 & c_\nu c_e e^{i\varphi_2} + s_\nu s_e \cos \varphi_1 \end{pmatrix}, \quad (4.4)$$

n	i	x	y	$ z_\nu - z_e $	φ_1	φ_2
odd	1, 2	0	0	even or odd	$j(z_\nu - z_e)\pi/n$	0
		0	0			0
		$n/2$	$n/2$			0
		0	$n/2$			$-j\pi/2$
		$n/2$	0			$j\pi/2$
even	3, 4	0	0	even	$j(z_\nu - z_e)\pi/n$	$(z_\nu - z_e)\pi/2$
				odd	$j(z_\nu - z_e)\pi/n + \pi/2$	$(z_\nu - z_e)\pi/2 + \pi/2$
		$n/2$	$n/2$	even	$j(z_\nu - z_e)\pi/n$	$(z_\nu - z_e)\pi/2$
				odd	$j(z_\nu - z_e)\pi/n + \pi/2$	$(z_\nu - z_e)\pi/2 + \pi/2$
		0	$n/2$	even	$j(z_\nu - z_e)\pi/n$	$(z_\nu - z_e)\pi/2 - (2j+n)\pi/4$
				odd	$j(z_\nu - z_e)\pi/n + \pi/2$	$(z_\nu - z_e)\pi/2 - (2j+n)\pi/4 + \pi/2$
		$n/2$	0	even	$j(z_\nu - z_e)\pi/n$	$(z_\nu - z_e)\pi/2 + (2j+n)\pi/4$
				odd	$j(z_\nu - z_e)\pi/n + \pi/2$	$(z_\nu - z_e)\pi/2 + (2j+n)\pi/4 + \pi/2$

Table 1. The values of the parameters φ_1 and φ_2 for the residual symmetries $Z_2^{g_e} = Z_2^{SR^{ze}}$, $X_e = \{R^{-z_e+x}, SR^x\}$, $Z_2^\nu = Z_2^{SR^{z_\nu}}$ and $X_\nu = \{R^{-z_\nu+y}, SR^y\}$, where i and j are the indices of the representations $\mathbf{1}_i$ and $\mathbf{2}_j$.

up to permutations of rows and columns, where

$$s_e \equiv \sin \theta_e, \quad s_\nu \equiv \sin \theta_\nu, \quad c_e \equiv \cos \theta_e, \quad c_\nu \equiv \cos \theta_\nu. \quad (4.5)$$

The parameters φ_1 and φ_2 are discrete group parameters characterizing the residual symmetry, and all possible values of φ_1 and φ_2 are summarized in table 1. We can see that φ_1 and φ_2 can take the following discrete values

$$\begin{aligned} \varphi_1 \pmod{2\pi} &= 0, \frac{1}{n}\pi, \frac{2}{n}\pi, \dots, \frac{2n-1}{n}\pi, \\ \varphi_2 \pmod{2\pi} &= 0, \frac{1}{2}\pi, \pi, \frac{3}{2}\pi. \end{aligned} \quad (4.6)$$

In particular, φ_2 would be 0 or π if the group index n is odd or $x = y = 0, n/2$ for even n . It is easy to check that the mixing matrix in eq. (4.4) has the following symmetry properties,

$$\begin{aligned} U_I(\varphi_1, \varphi_2, \pi + \theta_e, \theta_\nu) &= \text{diag}(1, -1, -1)U_I(\varphi_1, \varphi_2, \theta_e, \theta_\nu), \\ U_I(\varphi_1, \varphi_2, \pi - \theta_e, \theta_\nu) &= \text{diag}(1, -1, 1)U_I(\varphi_1, \varphi_2, \theta_e, \pi - \theta_\nu)\text{diag}(1, -1, 1), \\ U_I(\varphi_1, \varphi_2, \theta_e, \pi + \theta_\nu) &= U_I(\varphi_1, \varphi_2, \theta_e, \theta_\nu)\text{diag}(1, -1, -1), \\ U_I(\pi + \varphi_1, \varphi_2, \theta_e, \theta_\nu) &= \text{diag}(-1, 1, -1)U_I(\varphi_1, \varphi_2, \pi - \theta_e, \theta_\nu), \\ U_I(\pi - \varphi_1, \varphi_2, \theta_e, \theta_\nu) &= \text{diag}(-1, 1, 1)U_I(\varphi_1, \varphi_2, \theta_e, \pi - \theta_\nu)\text{diag}(1, 1, -1), \\ U_I(\varphi_1, \pi + \varphi_2, \theta_e, \theta_\nu) &= U_I(\varphi_1, \varphi_2, \theta_e, \pi - \theta_\nu)\text{diag}(1, -1, 1), \\ U_I(\varphi_1, \pi - \varphi_2, \theta_e, \theta_\nu) &= U_I^*(\varphi_1, \varphi_2, \theta_e, \pi - \theta_\nu)\text{diag}(1, -1, 1), \end{aligned} \quad (4.7)$$

where the above diagonal matrices can be absorbed into the lepton fields. Therefore the parameters φ_1 and φ_2 can be limited in the ranges $0 \leq \varphi_1 \leq \pi/2$ and $0 \leq \varphi_2 < \pi$

respectively, and the values of φ_2 are 0 and $\pi/2$ in the fundamental region. In the case of $\varphi_2 = 0$, the mixing matrix in eq. (4.4) is real such that all CP phases are conserved. The evidence of CP violation in neutrino oscillation has been reported by T2K [3] and NO ν A [4] and nontrivial δ_{CP} is also preferred by global data analysis [2]. Hence we shall focus on $\varphi_2 = \pi/2$ in the following.

Including the row and column permutations encoded in P_e and P_ν , we find that the 36 possible permutations of rows and columns give rise to nine independent mixing patterns,

$$\begin{aligned} U_{I,1} &= U_I, & U_{I,2} &= U_I P_{12}, & U_{I,3} &= U_I P_{13}, \\ U_{I,4} &= P_{12} U_I, & U_{I,5} &= P_{12} U_I P_{12}, & U_{I,6} &= P_{12} U_I P_{13}, \\ U_{I,7} &= P_{23} P_{12} U_I, & U_{I,8} &= P_{23} P_{12} U_I P_{12}, & U_{I,9} &= P_{23} P_{12} U_I P_{13}. \end{aligned} \quad (4.8)$$

That is to say, the fixed element $\cos \varphi_1$ can be in any position of the mixing matrix. For each mixing pattern, we can straightforwardly extract the expressions of the mixing angles $\sin^2 \theta_{13}$, $\sin^2 \theta_{12}$, $\sin^2 \theta_{23}$ and the CP invariants J_{CP} , I_1 , and I_2 , as shown in table 2. We see that the mixing matrices $U_{I,7}$, $U_{I,8}$ and $U_{I,9}$ can be obtained from $U_{I,4}$, $U_{I,5}$ and $U_{I,6}$ through an exchange of the second and third rows. Therefore they lead to the same solar mixing angle, reactor mixing angle and Majorana CP phases while atmospheric angle changes from θ_{23} to $\pi/2 - \theta_{23}$ and the Dirac CP phase changes from δ_{CP} to $\pi + \delta_{CP}$. For the mixing pattern $U_{I,4}$ and the concerned value $\varphi_2 = \pi/2$, the smallness of θ_{13} requires

$$\theta_\nu \simeq \theta_e \simeq 0, \quad \text{or} \quad \theta_\nu \simeq \theta_e \simeq \frac{\pi}{2}. \quad (4.9)$$

Consequently we have

$$\sin^2 \theta_{23} \simeq 0, \quad \text{or} \quad \sin^2 \theta_{12} \simeq 1, \quad (4.10)$$

which is obviously not compatible with the measured values of θ_{12} and θ_{23} [2, 5, 6]. Therefore this mixing pattern can not accommodate the experimental data. Analogously the mixing patterns $U_{I,5}$, $U_{I,7}$ and $U_{I,8}$ are not viable as well. Furthermore, all the mixing angles and CP phases are expressed in terms of two continuous parameters θ_e and θ_ν as well as the discrete parameter φ_1 such that strong correlations among the mixing parameters are expected. Eliminating the rotation angles θ_e and θ_ν , we can find the following sum rules among mixing angles and Dirac CP violation phase,

$$U_{I,1}: \quad \cos^2 \theta_{13} \cos^2 \theta_{12} = \cos^2 \varphi_1, \quad \sin^2 \theta_{23} = \frac{1}{2} \pm \sqrt{1 - 4x^2} \left(\frac{1}{2} - \cot^2 \varphi_1 \tan^2 \theta_{13} \right), \quad (4.11a)$$

$$U_{I,2}: \quad \cos^2 \theta_{13} \sin^2 \theta_{12} = \cos^2 \varphi_1, \quad \sin^2 \theta_{23} = \frac{1}{2} \pm \sqrt{1 - 4x^2} \left(\frac{1}{2} - \cot^2 \varphi_1 \tan^2 \theta_{13} \right), \quad (4.11b)$$

$$U_{I,6}: \quad \cos^2 \theta_{13} \sin^2 \theta_{23} = \cos^2 \varphi_1, \quad \sin^2 \theta_{12} = \frac{1}{2} \pm \sqrt{1 - 4x^2} \left(\frac{1}{2} - \cot^2 \varphi_1 \tan^2 \theta_{13} \right), \quad (4.11c)$$

$$U_{I,9}: \quad \cos^2 \theta_{13} \cos^2 \theta_{23} = \cos^2 \varphi_1, \quad \sin^2 \theta_{12} = \frac{1}{2} \pm \sqrt{1 - 4x^2} \left(\frac{1}{2} - \cot^2 \varphi_1 \tan^2 \theta_{13} \right), \quad (4.11d)$$

where “+” is for $\theta_e \in (\pi/4, 3\pi/4)$ and “−” for $\theta_e \in [0, \pi/4] \cup [3\pi/4, \pi]$, and the parameter x is defined as

$$x = \frac{J_{CP}}{\sin \theta_{13} \cos \varphi_1 \sqrt{\sin^2 \varphi_1 - \sin^2 \theta_{13}}}. \quad (4.12)$$

PMNS	φ_1/π	$n_{\min}$	Mixing Parameters
$U_{I,1}$	$[0.181, 0.206]$	10	$\sin^2 \theta_{13} = s_\nu^2 \sin^2 \varphi_1$
			$\sin^2 \theta_{12} = \frac{c_\nu^2 \sin^2 \varphi_1}{1 - s_\nu^2 \sin^2 \varphi_1}$
			$\sin^2 \theta_{23} = \frac{s_\nu^2 c_e^2 \cos^2 \varphi_1 + s_e^2 c_\nu^2 - \mathcal{F} \cos \varphi_2}{1 - s_\nu^2 \sin^2 \varphi_1}$
			$I_1 = I_2 = 0$
$U_{I,2}$	$[0.301, 0.327]$	16	$\sin^2 \theta_{13} = s_\nu^2 \sin^2 \varphi_1$
			$\sin^2 \theta_{12} = \frac{\cos^2 \varphi_1}{1 - s_\nu^2 \sin^2 \varphi_1}$
			$\sin^2 \theta_{23} = \frac{s_\nu^2 c_e^2 \cos^2 \varphi_1 + s_e^2 c_\nu^2 - \mathcal{F} \cos \varphi_2}{1 - s_\nu^2 \sin^2 \varphi_1}$
			$I_1 = I_2 = 0$
$U_{I,3}$	$[0.450, 0.454]$	30	$\sin^2 \theta_{13} = \cos^2 \varphi_1$
			$\sin^2 \theta_{12} = c_\nu^2$
			$\sin^2 \theta_{23} = c_e^2$
			$I_1 = I_2 = 0$
$U_{I,4}$	$[0.339, 0.424]$	—	$\sin^2 \theta_{13} = s_\nu^2 c_e^2 \cos^2 \varphi_1 + s_e^2 c_\nu^2 - \mathcal{F} \cos \varphi_2$
			$\sin^2 \theta_{12} = 1 - \frac{c_e^2 \sin^2 \varphi_1}{1 - s_\nu^2 c_e^2 \cos^2 \varphi_1 - s_e^2 c_\nu^2 + \mathcal{F} \cos \varphi_2}$
			$\sin^2 \theta_{23} = \frac{s_\nu^2 \sin^2 \varphi_1}{1 - s_\nu^2 c_e^2 \cos^2 \varphi_1 - s_e^2 c_\nu^2 + \mathcal{F} \cos \varphi_2}$
			$ I_1 = 2s_e s_\nu c_e^2 \sin^2 \varphi_1 \sin \varphi_2 (s_e s_\nu \cos \varphi_2 + c_e c_\nu \cos \varphi_1) $
			$ I_2 = 2s_e c_e^2 c_\nu \sin^2 \varphi_1 \sin \varphi_2 (s_e c_\nu \cos \varphi_2 - s_\nu c_e \cos \varphi_1) $
$U_{I,5}$	$[0.266, 0.349]$	—	$\sin^2 \theta_{13} = s_\nu^2 c_e^2 \cos^2 \varphi_1 + s_e^2 c_\nu^2 - \mathcal{F} \cos \varphi_2$
			$\sin^2 \theta_{12} = \frac{c_e^2 \sin^2 \varphi_1}{1 - s_\nu^2 c_e^2 \cos^2 \varphi_1 - s_e^2 c_\nu^2 + \mathcal{F} \cos \varphi_2}$
			$\sin^2 \theta_{23} = \frac{s_\nu^2 \sin^2 \varphi_1}{1 - s_\nu^2 c_e^2 \cos^2 \varphi_1 - s_e^2 c_\nu^2 + \mathcal{F} \cos \varphi_2}$
			$ I_1 = 2s_e s_\nu c_e^2 \sin^2 \varphi_1 \sin \varphi_2 (s_e s_\nu \cos \varphi_2 + c_e c_\nu \cos \varphi_1) $
			$ I_2 = \mathcal{F} \sin \varphi_2 (s_e^2 - c_e^2 \cos^2 \varphi_1 - 2c_e s_e \cot 2\theta_\nu \cos \varphi_1 \cos \varphi_2) $
$U_{I,6}$	$[0.215, 0.276]$	4	$\sin^2 \theta_{13} = c_e^2 \sin^2 \varphi_1$
			$\sin^2 \theta_{12} = \frac{s_e^2 s_\nu^2 + c_e^2 c_\nu^2 \cos^2 \varphi_1 + \mathcal{F} \cos \varphi_2}{1 - c_e^2 \sin^2 \varphi_1}$
			$\sin^2 \theta_{23} = \frac{\cos^2 \varphi_1}{1 - c_e^2 \sin^2 \varphi_1}$
			$ I_1 = \mathcal{F} \sin \varphi_2 (2c_e s_e \cot 2\theta_\nu \cos \varphi_1 \cos \varphi_2 + c_e^2 \cos^2 \varphi_1 - s_e^2) $
			$ I_2 = 2s_e c_e^2 c_\nu \sin^2 \varphi_1 \sin \varphi_2 (s_\nu c_e \cos \varphi_1 - s_e c_\nu \cos \varphi_2) $

Table 2. The predictions of the lepton mixing parameters for the mixing patterns in eq. (4.8). The absolute value of the Jarlskog invariant J_{CP} [59] is the same for all the nine mixing patterns, i.e. $|J_{CP}| = \frac{1}{2} |\sin 2\theta_e \sin 2\theta_\nu \sin^2 \varphi_1 \cos \varphi_1 \sin \varphi_2|$. The symbols I_1 and I_2 are Majorana CP invariants [60–63]. The parameter $\mathcal{F}$ is defined as $\mathcal{F} = \frac{1}{2} \sin 2\theta_e \sin 2\theta_\nu \cos \varphi_1$. The allowed region of φ_1 in the second column is obtained by requiring the fixed element $\cos \varphi_1$ is in the experimentally preferred 3σ range [2]. The notation $n_{\min}$ denotes the minimal value of n which can accommodate the measured values of the lepton mixing angles for $\varphi_2 = \pi/2$. Notice that only the value of $\varphi_2 = \pi/2$ in its fundamental interval can generate a non-trivial Dirac CP phase in the scenario of section 4.

The current and upcoming neutrino experiments will be able to significantly reduce the experimental errors on θ_{12} and θ_{23} , and the next generation long-baseline experiments are expected to considerably improve the sensitivity to the Dirac phase δ_{CP} if running in both the neutrino and the anti-neutrino modes. Thus the above sum rules could be tested and possibly distinguished from each other in future, or to be ruled out entirely.

It is notable that the experimental data on lepton mixing angles can be explained by small dihedral group D_4 which is the symmetry group of a square. For instance, if the residual symmetry is specified by $x = 2$, $y = 0$, $z_\nu = 1$, $z_e = 0$ and the three left-handed lepton doublets are assigned to $\mathbf{1}_1 \oplus \mathbf{2}_1$, we have $\varphi_1 = \pi/4$ and $\varphi_2 = \pi/2$, the mixing angles for the mixing matrices $U_{I,6}$ and $U_{I,9}$ can be quite close to their best fit values for certain choices of parameters θ_ν and θ_e . As a measure for the goodness of fit, we perform a global fit using the χ^2 function which is defined in the usual way.² The numerical results are listed in table 3. We can see that the deviation of θ_{23} from maximal value can be accommodated and the Dirac phase δ_{CP} could be around 1.5π . Moreover, we display the contour regions for $\sin^2 \theta_{ij}$, $|\sin \delta_{CP}|$, $|\sin \alpha_{21}|$ and $|\sin \alpha_{31}|$ in figure 1. As one can clearly see, the rotation angles θ_e and θ_ν are strongly constrained to accommodate the three lepton mixing angles θ_{ij} within the experimentally preferred 3σ intervals (black areas in the figure). Therefore the allowed ranges of the mixing angles and CP phases should be rather narrow around the numerical values in table 3 and the present approach is very predictive.

The upcoming reactor neutrino oscillation experiments, such as JUNO [64] and RENO-50 [65], expect to make very precise measurements of the solar neutrino mixing angle θ_{12} . They will be capable of reducing the error of θ_{12} to about 0.1° or around 0.3%. Future long baseline experiments DUNE [66–69], T2HK [70, 71], T2HKK [72] can make very precise measurements of the oscillation parameters θ_{12} , θ_{23} and δ_{CP} . Therefore future neutrino facilities have the potential to test the predictions of our models. Furthermore, we expect that a more ambitious facility such as the neutrino factory [73–75] could provide a more stringent test of our approach.

The neutrinoless double beta ($0\nu\beta\beta$) decay is the unique probe for the Majorana nature of neutrinos, and it explicitly depends on the values of the Majorana CP violation phases. The $0\nu\beta\beta$ decay experiments can provide valuable information on the neutrino mass spectrum and constrain the Majorana phases. The $0\nu\beta\beta$ decay rate is proportional to the effective Majorana mass $|m_{ee}|$ which is the (11) element of the neutrino mass matrix in the charged lepton diagonal basis,

$$|m_{ee}| = |m_1 \cos^2 \theta_{12} \cos^2 \theta_{13} + m_2 \sin^2 \theta_{12} \cos^2 \theta_{13} e^{i\alpha_{21}} + m_3 \sin^2 \theta_{13} e^{i(\alpha_{31} - 2\delta_{CP})}|. \quad (4.13)$$

For the mixing pattern $U_{I,6}$ with $\varphi_1 = \pi/4$ and $\varphi_2 = \pi/2$, we show the prediction of the effective Majorana mass $|m_{ee}|$ as a function of the lightest neutrino mass m_{lightest} in figure 2. If the neutrino mass spectrum is IO, the effective mass $|m_{ee}|$ is around 0.046 eV or in the narrow interval [0.020 eV, 0.026 eV]. Since the next generation $0\nu\beta\beta$ decay experiments will be able to explore the whole region of the IO parameter space such that these predictions

²The information of δ_{CP} is not included in the χ^2 function because it is measured with large uncertainties at present and the indication of its preferred value from global data analyses is rather weak [2].

Figure 1. The contour plots of $\sin^2 \theta_{ij}$ and the CP violation phases $|\sin \delta_{CP}|$, $|\sin \alpha_{21}|$ and $|\sin \alpha_{31}|$ in the plane θ_ν versus θ_e for the mixing pattern $U_{I,6}$ with $\varphi_1 = \pi/4$ and $\varphi_2 = \pi/2$ which can be achieved from the D_4 flavor group. The upper-left panel is the contour plots of $\sin^2 \theta_{ij}$. The red, green and blue areas denote the 3σ contour regions of $\sin^2 \theta_{13}$, $\sin^2 \theta_{12}$ and $\sin^2 \theta_{23}$ respectively. The dashed contour lines stand for the corresponding experimental best fit values. The 3σ ranges and the best fit values of the mixing angles are taken from [2]. The best fitting values of $\theta_{e,\nu}$ are indicated with yellow pentagrams. The black areas represent the regions for which all the three lepton mixing angles are compatible with experimental data at 3σ level.

could be tested. In the case of NO spectrum, the effective Majorana mass has a lower limit $|m_{ee}| \geq 0.0011$ eV.

Regarding the quark mixing, it is described by the CKM matrix which is parameterized by three mixing angles θ_{ij}^q and one CP-violating phase δ_{CP}^q [1]. The full fit values of quark

Figure 2. The possible values of the effective Majorana mass m_{ee} as a function of the lightest neutrino mass m_{lightest} for the mixing pattern $U_{I,6}$ with $n = 4$ and $\varphi_1 = \pi/4$, $\varphi_2 = \pi/2$. The red (blue) dashed lines indicate the most general allowed regions for IO (NO) neutrino mass spectrum obtained by varying the mixing parameters over their 3σ ranges [2]. The present most stringent upper limits $|m_{ee}| < 0.120$ eV from EXO-200 [76, 77] and KamLAND-ZEN [78] is represented by horizontal grey band. The vertical grey exclusion band denotes the current bound coming from the cosmological data of $\sum m_i < 0.130$ eV at 95% confidence level obtained by the Planck collaboration [79], the dashed and solid black lines are for IO and NO respectively.

mixing angles and Jarlskog invariant, given by the UTfit collaboration [80], read as

$$\begin{aligned} \sin \theta_{12}^q &= 0.22500 \pm 0.00100, & \sin \theta_{13}^q &= 0.003675 \pm 0.000095, \\ \sin \theta_{23}^q &= 0.04200 \pm 0.00059, & J_{CP}^q &= (3.120 \pm 0.090) \times 10^{-5}. \end{aligned} \quad (4.14)$$

Similar to the lepton sector, the precisely measured CKM mixing matrix can only be explained by the residual symmetry $Z_2^{g_u} = Z_2^{SR^{z_u}}$, $X_u = \{R^{-z_u+x}, SR^x\}$, $Z_2^{g_d} = Z_2^{SR^{z_d}}$ and $X_d = \{R^{-z_d+y}, SR^y\}$ where $z_u, z_d = 0, 1, \dots, n-1$, $x = y = 0$ for odd n and $x, y = 0, n/2$ for even n . The CKM matrix is predicted to be of the same form as eq. (4.4) and it reads

$$V_I = \begin{pmatrix} \cos \varphi_1 & -c_d \sin \varphi_1 & s_d \sin \varphi_1 \\ c_u \sin \varphi_1 & s_d s_u e^{i\varphi_2} + c_d c_u \cos \varphi_1 & c_d s_u e^{i\varphi_2} - s_d c_u \cos \varphi_1 \\ -s_u \sin \varphi_1 & s_d c_u e^{i\varphi_2} - c_d s_u \cos \varphi_1 & c_d c_u e^{i\varphi_2} + s_d s_u \cos \varphi_1 \end{pmatrix}. \quad (4.15)$$

The values of the parameters φ_1 and φ_2 are summarized in table 1 where z_e and z_ν should be replaced by z_u and z_d respectively. The parameter φ_2 can be $0, \pi/2, \pi$ and $3\pi/2$, and it should be $\pi/2$ or $3\pi/2$ to explain the observed CP violation in the quark sector. We have numerically analyzed all the D_n groups with $n \leq 40$ and find that the experimentally measured quark mixing matrix can only be described by the following two permutations

$$V_{I,1} = V_I, \quad V_{I,2} = V_I P_{12}. \quad (4.16)$$

	φ_1	φ_2	MO	θ_e^{bf}/π	$\theta_\nu^{\text{bf}}/\pi$	χ_{min}^2	$\sin^2 \theta_{13}$	$\sin^2 \theta_{12}$	$\sin^2 \theta_{23}$	δ_{CP}/π	$\alpha_{21}/\pi \pmod{1}$	$\alpha_{31}/\pi \pmod{1}$
$U_{I,6}$	$\frac{\pi}{4}$	$\frac{\pi}{2}$	NO	0.432	0.185	13.781	0.0224	0.310	0.511	1.459	0.790	0.854
				0.568	0.815					0.541	0.210	0.146
				0.432	0.815							
				0.568	0.185							
			IO	0.432	0.185	15.301	0.0227	0.310	0.512	1.459	0.789	0.853
				0.568	0.815					0.541	0.211	0.147
				0.432	0.815							
				0.568	0.185							
$U_{I,9}$	$\frac{\pi}{4}$	$\frac{\pi}{2}$	NO	0.432	0.815	24.186	0.0223	0.310	0.489	1.541	0.209	0.146
				0.568	0.185					0.459	0.791	0.854
				0.432	0.185							
				0.568	0.815							
			IO	0.432	0.815	27.017	0.0226	0.310	0.488	1.541	0.211	0.147
				0.568	0.185					0.459	0.789	0.853
				0.432	0.185							
				0.568	0.815							

Table 3. Results of the χ^2 analysis for the phenomenologically viable mixing patterns $U_{I,6}$ and $U_{I,9}$ with $\varphi_1 = \pi/4$ and $\varphi_2 = \pi/2$ which can be achieved from the D_4 flavor group. Since the global fit results of the mixing angles slightly differ for normal ordering (NO) and inverted ordering (IO) neutrino mass spectrums, we consider these two cases separately. The χ^2 function reaches a global minimum χ_{min}^2 at the best fit values $(\theta_e, \theta_\nu) = (\theta_e^{\text{bf}}, \theta_\nu^{\text{bf}})$. We display the values of the mixing angles and CP violation phases at the best fitting points.

Accordingly the smallest value of group index n which can accommodate the experimental data is $n = 14$. The expressions of the mixing parameters can be read from table 2 by replacing θ_e and θ_ν with θ_u and θ_d respectively.

For the mixing pattern $V_{I,1}$, the correlations in eq. (4.11a) are also satisfied for the quark mixing angles θ_{ij}^q and Jarlskog invariant J_{CP}^q . Moreover we can express the quark CP violation phase δ_{CP}^q in terms of mixing angles,

$$\sin \delta_{CP}^q \simeq \frac{\sin 2\varphi_1}{\sin 2\theta_{12}^q \cos^2 \theta_{13}^q \cos \theta_{23}^q}. \quad (4.17)$$

If we assign the quark doublets to $\mathbf{1}_1 \oplus \mathbf{2}_1$ of the D_{14} group and choose the residual symmetry $x = 7$, $y = 0$, $z_d = 1$, $z_u = 0$, we have $\varphi_1 = \pi/14$, $\varphi_2 = \pi/2$ and the observed quark flavor mixing parameters can be accommodated for certain choices of the free parameters $\theta_{u,d}$,

$$\begin{aligned} \theta_u &= 0.01237\pi, & \theta_d &= 0.99473\pi, & \sin \theta_{12}^q &= 0.22249, \\ \sin \theta_{13}^q &= 0.00369, & \sin \theta_{23}^q &= 0.04206, & J_{CP}^q &= 3.104 \times 10^{-5}. \end{aligned} \quad (4.18)$$

We see that $\sin \theta_{13}^q$, $\sin \theta_{23}^q$ and J_{CP}^q are compatible with the global fitting results of the UTfit collaboration [80]. The mixing angle $\sin \theta_{12}^q$ is about 1% smaller than its measured value and it could be brought into agreement with experimental data in an explicit model since small higher order corrections are generally expected to arise.

For the mixing pattern $V_{I,2}$, the sum rules in eqs. (4.11b), (4.17) are fulfilled. The breaking of the flavor group D_{14} and CP symmetry to the residual symmetry with $x = 7$, $y = 0$, $z_d = 6$, $z_u = 0$ gives $\varphi_1 = 3\pi/7$, $\varphi_2 = \pi/2$, we can find values of $\theta_{u,d}$ for which the hierarchical quark mixing angles as well as realistic CP violation phase can be achieved, e.g.

$$\begin{aligned} \theta_u &= 0.01326\pi, & \theta_d &= 0.00117\pi, & \sin\theta_{12}^q &= 0.22252, \\ \sin\theta_{13}^q &= 0.00357, & \sin\theta_{23}^q &= 0.04166, & J_{CP}^q &= 3.223 \times 10^{-5}. \end{aligned} \quad (4.19)$$

We see that $\sin\theta_{23}^q$ is in the experimentally preferred region, the relative deviations of $\sin\theta_{12}^q$, $\sin\theta_{13}^q$ and J_{CP}^q from their best fit values are about 1%. This tiny discrepancy should be easily resolved by higher order corrections or renormalization group evolution effects.

Furthermore, the flavor group D_{14} in combination with CP symmetry can reproduce the experimentally favored values of lepton mixing angles if it is broken down to $Z_2^{g_e} \times X_e$ and $Z_2^{g_\nu} \times X_\nu$ in charged lepton sector and neutrino sector respectively. We could choose the residual symmetry specified by $x = 7$, $y = 0$, $z_\nu = 4$ and $z_e = 0$, then the discrete parameters are $\varphi_1 = 2\pi/7$ and $\varphi_2 = \pi/2$. The mixing pattern $U_{I,9}$ can accommodate the three lepton mixing angles very well, and the best fit values of the mixing parameters are

$$\begin{aligned} \theta_e^{\text{bf}} &= 0.439\pi, & \theta_\nu^{\text{bf}} &= 0.814\pi, & \chi_{\min}^2 &= 1.841, & \sin^2\theta_{13} &= 0.0224, & \sin^2\theta_{12} &= 0.310, \\ \sin^2\theta_{23} &= 0.602, & \delta_{CP}/\pi &= 1.532, & \alpha_{21}/\pi &= 0.167 \pmod{1}, & \alpha_{31}/\pi &= 0.116 \pmod{1}. \end{aligned} \quad (4.20)$$

We would like to remind the readers that the smallest flavor group is $\Delta(294)$ which can accommodate quark and lepton flavor mixing simultaneously if both left-handed quarks and leptons are assigned to an irreducible triplet of G_f and the residual symmetries are $Z_2 \times CP$ [50]. Hence the singlet plus doublet assignment seems better than the triplet assignment after including CP symmetry, the order of the flavor symmetry group can be reduced considerably, i.e. 28 versus 294 in this scheme. In particular, the simple dihedral group D_n allows for a unified description of quark and lepton mixing. The dihedral group together with the residual symmetry $Z_2 \times CP$ indicated above provides an interesting opportunity for model building.

The dihedral group D_n is a subgroup of $\Delta(6n^2)$ [50], therefore all the results of D_n can be obtained from the $\Delta(6n^2)$ flavor symmetry. We find that the mixing pattern U_I in eq. (4.4) coincides with the mixing matrix for the **case I** of ref. [50] up to the redefinition of the free parameters θ_e and θ_ν , while the allowed values of φ_1 and φ_2 are different. For the phenomenologically viable flavor groups D_{14} and $\Delta(294)$, only the mixing pattern $U_{I,9}$ can agree well with the experimental data on lepton mixing angles with non-trivial Dirac CP phase δ_{CP} , and the results of χ^2 analysis are shown in table 4. We see that D_{14} and $\Delta(294)$ give the same lepton mixing angles as well as the same $\chi_{\min}^2$ while the predictions for the CP violation phases depend on the values of φ_2 . As a result, at least precise measurement of δ_{CP} is necessary in order to distinguish D_{14} from $\Delta(294)$.

Results for $\Delta(294)$											
	φ_1	φ_2	θ_e^{bf}/π	$\theta_\nu^{\text{bf}}/\pi$	χ^2_{min}	$\sin^2 \theta_{13}$	$\sin^2 \theta_{12}$	$\sin^2 \theta_{23}$	$ \sin \delta_{CP} $	$ \sin \alpha_{21} $	$ \sin \alpha_{31} $
$U_{I,9}$	$\frac{2\pi}{7}$	$\frac{\pi}{7}$	0.439	0.778	1.841	0.0224	0.310	0.602	0.462	0.223	0.144
		$\frac{2\pi}{7}$	0.439	0.789					0.820	0.397	0.265
		$\frac{3\pi}{7}$	0.439	0.805					0.991	0.489	0.341
Results for D_{14}											
$U_{I,9}$	$\frac{2\pi}{7}$	$\frac{\pi}{2}$	0.439	0.814	1.841	0.0224	0.310	0.602	0.995	0.501	0.356

Table 4. Results of the χ^2 analysis for the permutations of the mixing patterns U_I obtained from the $\Delta(294)$ and D_{14} flavor groups. Notice that only the mixing matrix $U_{I,9}$ can accommodate the experimental data. We display the values of $\sin^2 \theta_{ij}$, $|\sin \delta_{CP}|$, $|\sin \alpha_{21}|$ and $|\sin \alpha_{31}|$ at the best fitting point $(\theta_e, \theta_\nu) = (\theta_e^{\text{bf}}, \theta_\nu^{\text{bf}})$.

5 Mixing patterns from D_n and CP symmetry breaking to Z_2 and $Z_2 \times CP$ subgroups

Similar to section 4, the left-handed lepton and quark doublets are assigned to the reducible representation $\mathbf{1}_i \oplus \mathbf{2}_j$, as shown in eq. (4.1). In this section, we consider the scenario that the residual symmetries of the neutrino and charged lepton mass matrices are $Z_2^{g_\nu} \times X_\nu$ and $Z_2^{g_e}$ respectively arising from the flavor group D_n and CP. Considering all possible choices for g_ν , X_ν and g_e , we find only the residual symmetry $Z_2^{g_e} = Z_2^{SR^{z_e}}$, $Z_2^{g_\nu} = Z_2^{SR^{z_\nu}}$ and $X_\nu = \{R^{-z_\nu+x}, SR^x\}$ can lead to mixing pattern in agreement with the present data, where $z_e, z_\nu = 0, 1, \dots, n-1$, $x = 0$ if n is an odd integer and $x = 0, n/2$ for even n . Using the general formula of eq. (2.17), we can get the lepton mixing matrix as follow,

$$U_{II} = \begin{pmatrix} \cos \varphi_1 & -c_\nu \sin \varphi_1 & s_\nu \sin \varphi_1 \\ c_e \sin \varphi_1 & s_\nu s_e e^{i\delta} + c_\nu c_e \cos \varphi_1 & c_\nu s_e e^{i\delta} - s_\nu c_e \cos \varphi_1 \\ -s_e \sin \varphi_1 & s_\nu c_e e^{i\delta} - c_\nu s_e \cos \varphi_1 & c_\nu c_e e^{i\delta} + s_\nu s_e \cos \varphi_1 \end{pmatrix}, \quad (5.1)$$

where $\delta = \varphi_2 + 2\delta_e$, $s_e = \sin \theta_e$, $s_\nu = \sin \theta_\nu$, $c_e = \cos \theta_e$, $c_\nu = \cos \theta_\nu$, the permutation matrices $P_{e,\nu}$ and phase matrices $Q_{e,\nu}$ are omitted. The parameters φ_1 and φ_2 are determined by residual symmetry, and their admissible values are summarized in table 5. We can see that φ_1 takes the following discrete values

$$\varphi_1 \pmod{2\pi} = 0, \frac{1}{n}\pi, \frac{2}{n}\pi, \dots, \frac{2n-1}{n}\pi. \quad (5.2)$$

The second discrete parameter φ_2 appears in U_{II} through the combination $\delta = \varphi_2 + 2\delta_e$, the value of φ_2 is irrelevant since it can be absorbed into the continuous free parameter δ_e . Comparing eq. (5.1) with eq. (4.4), we see that U_{II} can be obtained from U_I by replacing φ_2 with δ . Therefore the parameter φ_1 can be limited in the interval $0 \leq \varphi_1 \leq \pi/2$, and the variation ranges of the free parameters θ_e , θ_ν and δ can be taken to be $0 \leq \theta_e \leq \pi/2$, $0 \leq \theta_\nu < \pi$ and $0 \leq \delta < \pi$ respectively.

In this approach, we can not make any prediction for the lepton masses, consequently the lepton mixing matrix is determined up to independent row and column permutations.

n	i	x	$ z_\nu - z_e $	φ_1	φ_2
odd		0			$-jz_e\pi/n$
even	1, 2	0	odd or even	$j(z_\nu - z_e)\pi/n$	$-jz_e\pi/n$
		$n/2$			$-jz_e\pi/n - j\pi/2$
	3, 4	0	even	$j(z_\nu - z_e)\pi/n$	$-jz_e\pi/n + z_\nu\pi/2$
			odd	$j(z_\nu - z_e)\pi/n + \pi/2$	$-jz_e\pi/n + (z_\nu + 1)\pi/2$
		$n/2$	even	$j(z_\nu - z_e)\pi/n$	$-jz_e\pi/n + (z_\nu - j - n/2)\pi/2$
			odd	$j(z_\nu - z_e)\pi/n + \pi/2$	$-jz_e\pi/n + (z_\nu - j - n/2 + 1)\pi/2$

Table 5. The values of the parameters φ_1 and φ_2 for the residual symmetry $Z_2^{g_e} = Z_2^{SR^e}$, $Z_2^{g_\nu} = Z_2^{SR^{z_\nu}}$ and $X_\nu = \{R^{-z_\nu+x}, SR^x\}$, where i and j are the indices of the D_n representations $\mathbf{1}_i$ and $\mathbf{2}_j$ respectively.

The residual symmetry fixes one element of the PMNS mixing matrix is $\cos \varphi_1$ and it can be any entry. As a result, the 36 possible permutations of rows and columns give rise to nine independent mixing patterns

$$\begin{aligned}
 U_{II,1} &= U_{II}, & U_{II,2} &= U_{II}P_{12}, & U_{II,3} &= U_{II}P_{13}, \\
 U_{II,4} &= P_{12}U_{II}, & U_{II,5} &= P_{12}U_{II}P_{12}, & U_{II,6} &= P_{12}U_{II}P_{13}, \\
 U_{II,7} &= P_{23}P_{12}U_{II}, & U_{II,8} &= P_{23}P_{12}U_{II}P_{12}, & U_{II,9} &= P_{23}P_{12}U_{II}P_{13}.
 \end{aligned} \tag{5.3}$$

For each mixing pattern $U_{II,i}$ ($i = 1, 2, \dots, 9$), the expressions of the mixing parameters can be obtained from those of $U_{I,i}$ in table 2 by replacing φ_2 with δ . From the modulus of the fixed element, we can obtain the following sum rules among the mixing angles and Dirac CP phase,

$$\begin{aligned}
 U_{II,1} : \quad & \cos^2 \theta_{12} \cos^2 \theta_{13} = \cos^2 \varphi_1, & U_{II,2} : \quad & \sin^2 \theta_{12} \cos^2 \theta_{13} = \cos^2 \varphi_1, \\
 U_{II,6} : \quad & \sin^2 \theta_{23} \cos^2 \theta_{13} = \cos^2 \varphi_1, & U_{II,9} : \quad & \cos^2 \theta_{23} \cos^2 \theta_{13} = \cos^2 \varphi_1, \\
 U_{II,4} : \quad & \cos \delta_{CP} = \frac{2(\cos^2 \varphi_1 - \sin^2 \theta_{12} \cos^2 \theta_{23} - \sin^2 \theta_{13} \cos^2 \theta_{12} \sin^2 \theta_{23})}{\sin 2\theta_{12} \sin \theta_{13} \sin 2\theta_{23}}, \\
 U_{II,5} : \quad & \cos \delta_{CP} = -\frac{2(\cos^2 \varphi_1 - \cos^2 \theta_{12} \cos^2 \theta_{23} - \sin^2 \theta_{13} \sin^2 \theta_{12} \sin^2 \theta_{23})}{\sin 2\theta_{12} \sin \theta_{13} \sin 2\theta_{23}}, \\
 U_{II,7} : \quad & \cos \delta_{CP} = -\frac{2(\cos^2 \varphi_1 - \sin^2 \theta_{12} \sin^2 \theta_{23} - \sin^2 \theta_{13} \cos^2 \theta_{12} \cos^2 \theta_{23})}{\sin 2\theta_{12} \sin \theta_{13} \sin 2\theta_{23}}, \\
 U_{II,8} : \quad & \cos \delta_{CP} = \frac{2(\cos^2 \varphi_1 - \cos^2 \theta_{12} \sin^2 \theta_{23} - \sin^2 \theta_{13} \sin^2 \theta_{12} \cos^2 \theta_{23})}{\sin 2\theta_{12} \sin \theta_{13} \sin 2\theta_{23}}.
 \end{aligned} \tag{5.4}$$

In order to show concrete examples and find new interesting mixing, we have numerically scanned over the free parameters θ_e , θ_ν and δ and all possible values of the discrete parameter φ_1 for each integer group index n . We find the smallest dihedral group which can accommodate the data is $D_3 \cong S_3$. Note that D_3 group with $n = 3$ is the symmetry group of an equilateral triangle, and then φ_1 can be either 0 or $\pi/3$ in the fundamental region of $\varphi_1 \in [0, \pi/2]$. Only the value $\varphi_1 = \pi/3$ can generate a viable mixing pattern, and it can

	$\theta_{13}/^\circ$	$\theta_{12}/^\circ$	$\theta_{23}/^\circ$	δ_{CP}/π	$\alpha_{21}/\pi \pmod{1}$	$\alpha_{31}/\pi \pmod{1}$
$U_{II,4} _{\varphi_1=\pi/3}$	8.220–8.981	33.258–36.271	40.9–46.246	0–0.299 $\oplus 1.701-2$	0–0.139 $\oplus 0.861-1$	0–0.085 $\oplus 0.915-1$
$U_{II,5} _{\varphi_1=\pi/3}$	8.220–8.981	31.628–36.271	45.453–52.180	0–0.505 $\oplus 1.495-2$	0–0.263 $\oplus 0.737-1$	0–0.172 $\oplus 0.828-1$
$U_{II,7} _{\varphi_1=\pi/3}$	8.220–8.981	32.032–36.271	43.766–52.180	0.617–1.383	0–0.147 $\oplus 0.853-1$	0–0.092 $\oplus 0.908-1$
$U_{II,8} _{\varphi_1=\pi/3}$	8.220–8.981	31.628–36.271	40.9–44.542	0.650–1.350	0–0.202 $\oplus 0.798-1$	0–0.124 $\oplus 0.877-1$

Table 6. The allowed ranges of the mixing parameters for the mixing patterns $U_{II,4}$, $U_{II,5}$, $U_{II,7}$ and $U_{II,8}$. Here we choose $\varphi_1 = \pi/3$ which is the unique viable value of φ_1 in the D_3 flavor group.

be achieved from the residual symmetry $g_e = S$, $g_\nu = SR$, $X_\nu = \{R^2, S\}$ under the lepton doublets assignment $\mathbf{1}_1 \oplus \mathbf{2}_1$. Accordingly the fixed element is $\cos \varphi_1 = 1/2$ and it can be the (21), (22), (31) or (32) entry of the lepton mixing matrix. Hence only the mixing patterns $U_{II,4}$, $U_{II,5}$, $U_{II,7}$ and $U_{II,8}$ can be compatible with experimental data. Requiring all the three mixing angles θ_{12} , θ_{13} and θ_{23} in the 3σ intervals of global fit [2], we can obtain the allowed regions of the mixing angles and CP violation phases and the numerical results are summarized in table 6. We can find that for different viable mixing patterns, the predictions for θ_{13} are almost same while the allowed regions of θ_{12} , θ_{23} , δ_{CP} , α_{21} and α_{31} are different. The neutrino oscillation experiments, such as JUNO [64], RENO-50 [65], and the future long baseline experiments DUNE [66–69], T2HK [70, 71], T2HKK [72] allow a measurement of θ_{12} , θ_{23} and δ_{CP} with significantly improved sensitivities. Thus a discrimination between the above possible cases in table 6 will be possible.

Subsequently we extend the above scheme to the quark sector, the flavor symmetry D_n and CP symmetry are broken down to $Z_2^{g_u}$ and $Z_2^{g_d} \times X_d$ in the up quark and down quark sectors respectively. The CKM mixing matrix can be correctly reproduced if the residual symmetry is $g_u = SR^{z_u}$, $g_d = SR^{z_d}$ and $X_d = \{R^{-z_d+x}, SR^x\}$ with $z_u, z_d = 0, 1, \dots, n-1$, $x = 0$ for odd n and $x = 0, n/2$ for even n . The CKM matrix is determined to be

$$V_{II} = \begin{pmatrix} \cos \varphi_1 & -c_d \sin \varphi_1 & s_d \sin \varphi_1 \\ c_u \sin \varphi_1 & s_u s_d e^{i\delta} + c_u c_d \cos \varphi_1 & s_u c_d e^{i\delta} - s_d c_u \cos \varphi_1 \\ -s_u \sin \varphi_1 & s_d c_u e^{i\delta} - s_u c_d \cos \varphi_1 & c_u c_d e^{i\delta} + s_u s_d \cos \varphi_1 \end{pmatrix}, \quad (5.5)$$

up to permutations of rows and columns, where $c_u = \cos \theta_u$, $c_d = \cos \theta_d$, $s_u = \sin \theta_u$, $s_d = \sin \theta_d$ and $\delta = \varphi_2 + 2\delta_u$. The values of the discrete parameters φ_1 and φ_2 can be read from table 5 by substituting z_e and z_ν with z_u and z_d respectively. Furthermore, it is straightforward to check that the same mixing pattern would be obtained if the residual symmetry is instead $Z_2^{g_u} = Z_2^{SR^{z_u}}$, $X_u = \{R^{-z_u+x}, SR^x\}$ and $Z_2^{g_d} = Z_2^{SR^{z_d}}$.

Similar to the lepton mixing matrices in eq. (5.3), the row and columns permutations of V_{II} can give rise to nine mixing patterns $V_{II,i}$ ($i = 1, \dots, 9$). The mixing matrix $V_{II,i}$ can be obtained from $U_{II,i}$ by replacing θ_e , θ_ν and δ_e with θ_u , θ_d and δ_u respectively. We have considered all possible values of the discrete parameters φ_1 for each group index n

	n	φ_1	θ_u/π	θ_d/π	δ/π	$\sin \theta_{13}^q$	$\sin \theta_{12}^q$	$\sin \theta_{23}^q$	$J_{CP}^q/10^{-5}$
$V_{II,1}$	14, 28	$\pi/14$	0.98405	0.00526	0.71399	0.00367	0.22249	0.04200	3.120
$V_{II,2}$	7, 14, 21, 28, 35	$3\pi/7$	0.01347	0.00120	0.37658	0.00368	0.22252	0.04200	3.120
$V_{II,4}$	7, 14, 21, 28, 35	$3\pi/7$	0.00365	0.01372	0.09649	0.00368	0.22277	0.04201	3.118
$V_{II,5}$	27	$2\pi/27$	0.94259	0.05847	0.99395	0.00368	0.22688	0.04212	3.119
$V_{II,6}$	37	$18\pi/37$	0.50117	0.07226	0.36627	0.00367	0.22500	0.04244	3.120
	39	$19\pi/39$	0.50117	0.07225	0.41240	0.00368	0.22500	0.04027	3.120
$V_{II,8}$	37	$18\pi/37$	0.42770	0.49723	0.12558	0.00369	0.22499	0.04316	3.108
	39	$19\pi/39$	0.42770	0.49643	0.10190	0.00368	0.22502	0.04164	3.114

Table 7. Numerical results of the quark mixing parameters for the permutations of the mixing matrix V_{II} in eq. (5.5), where the residual symmetry is $Z_2^{g_u} = Z_2^{SR^u}$, $Z_2^{g_d} = Z_2^{SR^d}$, $X_d = \{R^{-z_d+x}, SR^x\}$. We have analyzed all the D_n groups with $n \leq 40$. Here we show the values of $\sin \theta_{ij}^q$ and J_{CP}^q which are compatible with the experimental data for certain choices of θ_u , δ_u , θ_d and φ_1 .

with $n \leq 40$. We scan over the free parameters θ_u , δ_u and θ_d in the range from 0 and π to determine whether a good fit to the experimental data can be achieved. For the D_n groups with $n \leq 40$, we find six permutations $V_{II,1}$, $V_{II,2}$, $V_{II,4}$, $V_{II,5}$, $V_{II,6}$, and $V_{II,8}$ can describe the measured values of the quark mixing parameters shown in eq. (4.14). The values of n , φ_1 and the resulting predictions for $\sin \theta_{ij}^q$ and J_{CP}^q at certain benchmark values of θ_u , δ_u , θ_d are summarized in table 7. We see that the smallest group index n which can accommodate the experimental data is $n = 7$ and accordingly the mixing patterns are $V_{II,2}$ and $V_{II,4}$.

Furthermore, we notice that the D_7 group and CP symmetry can also generate phenomenologically viable lepton mixing patterns if the residual symmetries of the charged lepton and neutrino mass matrices are $Z_2^{g_e} = Z_2^{SR^{ze}}$ and $Z_2^{g_\nu} = Z_2^{SR^{z\nu}}$, $X_\nu = \{R^{-z_\nu+x}, SR^x\}$ respectively. We find that only the mixing patterns $U_{II,4}$, $U_{II,5}$, $U_{II,8}$ and $U_{II,9}$ can agree well with the experimental data on lepton mixing angles, and the discrete parameter φ_1 can be $2\pi/7$ or $3\pi/7$. The continuous parameters θ_e , δ and θ_ν are freely varied between 0 and π , and the current 3σ bounds of $\sin^2 \theta_{ij}$ [2] are imposed. The allowed regions of the lepton mixing angles and CP phases are reported in table 8. As an example, we display the correlations among the different mixing parameters for $U_{II,8}|_{\varphi_1=2\pi/7}$ in figure 3. Comparing with the scenario of $Z_2 \times CP$ residual symmetry in both neutrino and charged lepton sectors, we see that the admissible region of the Dirac phase δ_{CP} is generally more larger. It is remarkable that the D_7 flavor symmetry with group order 14 already can give experimentally favored values of PMNS and CKM matrix. For the irreducible triplet assignment of quark and lepton doublets, we would like to mention that $\Delta(294)$ is the minimal flavor group to generate realistic quark and lepton flavor mixing patterns in the present scheme [51].

6 Summary and conclusions

A compelling theory of flavor mixing is still missing. The discrete flavor symmetry and CP symmetry through the mismatch of residual symmetries is a powerful approach to explain the observed flavor mixing structure of quarks and leptons. In previous work, we find that realistic CKM and PMNS matrices can be achieved if the residual symmetry in the neutrino

Figure 3. Correlations between different mixing parameters for the mixing pattern $U_{II,8}$ with $\varphi_1 = 2\pi/7$ which can be achieved from the D_7 flavor group, and the three lepton mixing angles are required to lie in their 3σ ranges [2].

case	$\theta_{13}/^\circ$	$\theta_{12}/^\circ$	$\theta_{23}/^\circ$	δ_{CP}/π	$\alpha_{21}/\pi \pmod{1}$	$\alpha_{31}/\pi \pmod{1}$
$U_{II,4} _{\varphi_1=\frac{3\pi}{7}}$	8.472 – 8.981	31.628 – 32.195	51.502 – 52.180	0.912 – 1.088	0 – 0.101 $\oplus 0.899 - 1$	0 – 0.082 $\oplus 0.918 - 1$
$U_{II,5} _{\varphi_1=\frac{2\pi}{7}}$	8.220 – 8.981	31.628 – 36.271	40.9 – 48.673	0.353 – 1.646	0 – 0.218 $\oplus 0.782 - 1$	0 – 0.163 $\oplus 0.837 - 1$
$U_{II,8} _{\varphi_1=\frac{2\pi}{7}}$	8.220 – 8.981	31.628 – 36.271	41.307 – 52.180	0 – 2	0 – 0.218 $\oplus 0.782 - 1$	0 – 0.163 $\oplus 0.837 - 1$
$U_{II,9} _{\varphi_1=\frac{2\pi}{7}}$	8.223 – 8.981	31.628 – 36.271	50.860 – 50.952	0 – 2	0 – 0.181 $\oplus 0.819 - 1$	0 – 0.132 $\oplus 0.868 - 1$

Table 8. The allowed ranges of the mixing parameters for the viable mixing patterns $U_{II,4}$, $U_{II,5}$, $U_{II,8}$ and $U_{II,9}$, and the flavor symmetry group is D_7 such that the viable values of φ_1 are $2\pi/7$ and $3\pi/7$.

and down quark sectors is $Z_2 \times CP$, and a subgroup $Z_2 \times CP$ or Z_2 is preserved by the charged lepton and up quark mass matrices [50, 51]. If the three generations of left-handed quark and lepton doublets transform as an irreducible three-dimensional representation of the flavor symmetry group, the minimal group turns out to be $\Delta(294)$ [50, 51]. The motivation of the present work is to find a smaller flavor group which can give a unified description of quark and lepton flavor mixing.

In this paper, we perform a detailed analysis of the dihedral group D_n as flavor symmetry in combination with CP symmetry. We have identified the most general form of the CP transformations compatible with D_n . Since the group D_n only has one-dimensional and two-dimensional irreducible representations, the left-handed quark and lepton fields are assigned to the direct sum of a singlet and a doublet of D_n . If the symmetries D_n and CP are broken in such a way that neutrino and charged lepton sectors remain invariant under two different $Z_2 \times CP$ subgroups, all the lepton mixing angles and CP phases would depend on only two real free parameters θ_e and θ_ν . The measured values of the lepton mixing angles can be explained by small group D_4 which is the symmetry group of a square, see table 3 for numerical results. In the same way as presented for leptons, viable quark mixing can be derived under the assumption that the residual symmetries of the up and down quark sectors are $Z_2 \times CP$ as well. Moreover, we find that the flavor group D_{14} can give the experimentally favored CKM and PMNS mixing matrices.

Furthermore, we consider a second scenario in which the residual symmetries of the charged lepton and up quark sectors are Z_2 instead of $Z_2 \times CP$ while the neutrino and down quark mass matrices remain invariant under a $Z_2 \times CP$ subgroup. The resulting lepton and quark mixing matrices would depend on three free parameters θ_e , θ_ν , δ_e and θ_u , θ_d , δ_u respectively. The observed patterns of quark and lepton flavor mixing can be accommodated by the D_7 group.

In the approach with only flavor symmetry (without CP), in order to achieve at least two non-vanishing mixing angles, the left-handed leptons are usually assumed to transform as an irreducible three-dimensional representation under the flavor symmetry group. An important lesson what we learn is that the singlet plus doublet assignment also allows one

to understand the experimental data on lepton mixing angles after the CP symmetry is considered. We conclude that dihedral group and CP symmetry provide new opportunity for building models of quark and lepton mixing. It is interesting to implement the presented symmetry breaking patterns here in a concrete model, and as usual the assumed residual symmetries could be dynamically realized through non-vanishing vacuum expectation values of some flavons.

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A Flavor mixing from the residual symmetry $Z_2 \times CP$

The most general constraints on the unitary transformation U_ψ imposed by the residual symmetry $Z_2^{g_\psi} \times X_\psi$ have been explicitly derived in [31, 55]. For the sake of completeness, we recapitulate the derivation of U_ψ in such a context in the following. Since the element g_ψ is of order two, the eigenvalues of the representation matrix $\rho(g_\psi)$ is either +1 or -1. Assuming $\Sigma_{\psi 1}$ is a diagonalization matrix of $\rho(g_\psi)$ and it satisfies

$$\Sigma_{\psi 1}^\dagger \rho(g_\psi) \Sigma_{\psi 1} = \pm \text{diag}(1, -1, -1) \equiv \hat{\rho}(g_\psi). \quad (\text{A.1})$$

Inserting eq. (A.1) into the restricted consistency condition of eq. (2.1), we have

$$(\Sigma_{\psi 1}^\dagger X_\psi \Sigma_{\psi 1}^*) \hat{\rho}(g_\psi) = \hat{\rho}(g_\psi) (\Sigma_{\psi 1}^\dagger X_\psi \Sigma_{\psi 1}^*). \quad (\text{A.2})$$

Therefore the combination $\Sigma_{\psi 1}^\dagger X_\psi \Sigma_{\psi 1}^*$ is a block diagonal and symmetric matrix and its general form can be written as

$$\Sigma_{\psi 1}^\dagger X_\psi \Sigma_{\psi 1}^* = \begin{pmatrix} e^{i\xi_1} & 0 \\ 0 & u_{2 \times 2}^\psi \end{pmatrix}, \quad (\text{A.3})$$

where ξ_1 is a real number and $u_{2 \times 2}^\psi$ is a two-dimensional symmetric unitary matrix. We denote the Takagi factorization of $u_{2 \times 2}^\psi$ as $\sigma_{2 \times 2}^\psi$ fulfilling $u_{2 \times 2}^\psi = \sigma_{2 \times 2}^\psi \sigma_{2 \times 2}^{\psi T}$, where $\sigma_{2 \times 2}^\psi$ is a two-dimensional unitary matrix. Then we can obtain the Takagi factorization of X_ψ as

$$X_\psi = \left[\Sigma_{\psi 1} \begin{pmatrix} e^{i\xi_1/2} & 0 \\ 0 & \sigma_{2 \times 2}^\psi \end{pmatrix} \right] \left[\Sigma_{\psi 1} \begin{pmatrix} e^{i\xi_1/2} & 0 \\ 0 & \sigma_{2 \times 2}^\psi \end{pmatrix} \right]^T \equiv \Sigma_\psi \Sigma_\psi^T \quad (\text{A.4})$$

with

$$\Sigma_\psi = \Sigma_{\psi 1} \begin{pmatrix} e^{i\xi_1/2} & 0 \\ 0 & \sigma_{2 \times 2}^\psi \end{pmatrix}. \quad (\text{A.5})$$

It is straightforward to check that the remnant flavor transformation $\rho(g_\psi)$ is diagonalized by Σ_ψ ,

$$\Sigma_\psi^\dagger \rho(g_\psi) \Sigma_\psi = \pm \text{diag}(1, -1, -1). \quad (\text{A.6})$$

As regards the constraint arising from the residual CP transformation X_ψ , plugging the Takagi factorization of X_ψ in eq. (A.4) into eq. (2.7), we find

$$(\Sigma_\psi^T U_\psi^* Q_\psi^\dagger)^T (\Sigma_\psi^T U_\psi^* Q_\psi^\dagger) = 1, \quad (\text{A.7})$$

which implies that the combination $\Sigma_\psi^T U_\psi^* Q_\psi^{-1}$ is a real orthogonal matrix denoted by $O_{3 \times 3}$. Consequently the unitary transformation U_ψ takes the following form

$$U_\psi = \Sigma_\psi O_{3 \times 3} Q_\psi^\dagger. \quad (\text{A.8})$$

Subsequently we consider the constraint of the residual flavor symmetry $Z_2^{g_\psi}$. Plugging the expression of U_ψ in eq. (A.8) into eq. (2.6) and considering the relation in eq. (A.6), we find

$$(P_\psi Q_\psi O_{3 \times 3}^T) \text{diag}(1, -1, -1) = \text{diag}(1, -1, -1) (P_\psi Q_\psi O_{3 \times 3}^T). \quad (\text{A.9})$$

Therefore $O_{3 \times 3} P_\psi^T$ should be a rotation matrix in the (23)-plane, i.e.

$$O_{3 \times 3} P_\psi^T = S_{23}(\theta_\psi), \quad (\text{A.10})$$

with

$$S_{23}(\theta_\psi) \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta_\psi & \sin \theta_\psi \\ 0 & -\sin \theta_\psi & \cos \theta_\psi \end{pmatrix}. \quad (\text{A.11})$$

Hence the residual symmetry $Z_2^{g_\psi} \times X_\psi$ would determine the unitary transformation U_ψ to be of the form [31, 50, 51, 55]

$$U_\psi = \Sigma_\psi S_{23}(\theta_\psi) P_\psi Q_\psi^\dagger. \quad (\text{A.12})$$

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