On exact correlation functions of chiral ring operators in $2d \mathcal{N} = (2, 2)$ SCFTs via localization

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ABSTRACT: We study the extremal correlation functions of (twisted) chiral ring operators via superlocalization in $\mathcal{N} = (2, 2)$ superconformal field theories (SCFTs) with central charge $c \geq 3$, especially for SCFTs with Calabi-Yau geometric phases. We extend the method in arXiv:1602.05971 with mild modifications, so that it is applicable to disentangle operators mixing on S^2 in nilpotent (twisted) chiral rings of $2d$ SCFTs. With the extended algorithm and technique of localization, we compute exactly the extremal correlators in $2d \mathcal{N} = (2, 2)$ (twisted) chiral rings as non-holomorphic functions of marginal parameters of the theories. Especially in the context of Calabi-Yau geometries, we give an explicit geometric interpretation to our algorithm as the Griffiths transversality with projection on the Hodge bundle over Calabi-Yau complex moduli. We also apply the method to compute extremal correlators in Kähler moduli, or say twisted chiral rings, of several interesting Calabi-Yau manifolds. In the case of complete intersections in toric varieties, we provide an alternative formalism for extremal correlators via localization onto Higgs branch. In addition, as a spinoff we find that, from the extremal correlators of the top element in twisted chiral rings, one can extract chiral correlators in A-twisted topological theories.

KEYWORDS: Conformal Field Theory, Field Theories in Lower Dimensions, Supersymmetric Gauge Theory, Supersymmetry and Duality

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1 Introduction

Recently, a series of papers [1–4] initiated a systematic study on the correlation functions of operators in chiral rings of four-dimensional $\mathcal{N} = 2$ superconformal field theories with exactly marginal couplings. In these papers, the so-called extremal correlators,

$$\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \bar{\mathcal{O}}_J(y) \rangle_{\mathbb{R}^4}$$

containing arbitrarily many chiral primary and one anti-chiral primary operators in the chiral rings, are exactly calculated. Because of the insertion of an anti-chiral operator, these correlators are *non-holomorphic* functions of the marginal couplings, and thus hard to compute in comparison of the chiral correlators $\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle_{\mathbb{R}^4}$ with topological twist. On the other hand, these correlators are known to satisfy the four-dimensional version of tt^* -equations [6]. The equations are, nevertheless, insufficient to determine them all as in the two-dimensional situation. Therefore people in [1–3] resorted to additional input data via supersymmetric localization [5] on $\mathcal{N} = 2$ gauge theories. With the technique of superlocalization, one is able to compute exact partition functions $\mathcal{Z}[S^4]$ on 4-sphere of the $\mathcal{N} = 2$ SCFTs with Lagrangian descriptions, from which the extremal correlation functions $\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \bar{\mathcal{O}}_J(y) \rangle_{S^4}$ on S^4 can be extracted. In the paper [4], an algorithm was further developed to successfully disentangle the operators mixing from S^4 to $\mathbb{R}^4$. Therefore they are able to find all $\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \bar{\mathcal{O}}_J(y) \rangle_{\mathbb{R}^4}$ on $\mathbb{R}^4$, which also solve the tt^* -equations automatically.

In this paper, we consider the extremal correlators

$$\langle \phi_1(x_1) \cdots \phi_n(x_n) \bar{\phi}_J(y) \rangle_{\mathbb{R}^2}$$

in the (twisted) chiral rings of two-dimensional $\mathcal{N} = (2, 2)$ SCFTs with exactly marginal coupling constants $\{\tau, \bar{\tau}\}$. The fields ϕ 's and $\bar{\phi}$'s are primary (twisted) chiral operators and their Hermitian conjugates. Some of these correlators, e.g. the Zamolodchikov metrics, as well as the tt^* -equations they satisfy have been intensively studied in [8, 9], where the input data mainly concerned about OPE coefficients computed from topological twisted theories. We here in this note will instead provide an alternative, analogue to the method in [4], to apply the $2d$ supersymmetric localization as the input data to exactly compute these extremal correlators, both in perturbative and non-perturbative series, *with no need* of knowledge on OPE coefficients.

Compared to $4d$ $\mathcal{N} = 2$ SCFTs, the nilpotent (twisted) chiral rings in $2d$ $\mathcal{N} = (2, 2)$ SCFTs are finite and not freely generated. Therefore the OPE of (twisted) chiral primaries are related to each others due to the specific equivalence relations in the (twisted) chiral rings, and the products of sufficiently many of them will eventually turn out to be zero, modulo non-singular superdescendants. This feature will impose many constraints on the $2d$ correlators and their tt^* -equations. Therefore, while the methodology in this paper is inspired by and similar to the work of [4], we are still motivated to establish the algorithm applicable to disentangle operators mixing from S^2 to $\mathbb{R}^2$ for $2d$ nilpotent (twisted) chiral rings, and develop the full details of exact determination of the extremal correlators. Furthermore, the two-dimensional SCFTs we consider beautifully interplay with geometries

and topology. A given $2d$ $\mathcal{N} = (2, 2)$ SCFT $\mathcal{S}$, with center charge $c \geq 3$, usually has geometric phases related to a Calabi-Yau manifold $\mathcal{Y}$. Their moduli spaces $\mathcal{M}(\mathcal{Y})$ and $\mathcal{M}(\mathcal{S})$ coincides with each other. Therefore the extremal correlators exactly encodes the information of the metrics of $\mathcal{M}(\mathcal{Y})$ and various vector bundles over it. One will see that, from the mathematical side, our algorithm developed in this paper admits a geometric interpretation as Griffiths transversality, and also reconstructs the tt^* -equations on Calabi-Yau complex moduli. Furthermore, via localization onto Higgs branch, we also relate the extremal correlators of a theory $\mathcal{Y}$ to the periods of its mirror $\tilde{\mathcal{Y}}$ in the case of complete intersections in toric varieties. We wish that the exact computation of the extremal correlators would lead a detailed investigation on the structures of partition functions as well as the extremal correlators, integrability in $2d$ $\mathcal{N} = (2, 2)$ SCFTs, test of resurgence, and provide further implications to $2d/4d$ correspondence and so forth.

We plan the rest of the paper as follows. In section 2 we review some basics on $\mathcal{N} = (2, 2)$ SCFTs, their (twisted) chiral rings, and tt^* -equations the extremal correlators have to satisfy. In section 3, we review the method of supersymmetric localization on S^2 for SCFTs with irrelevant operator deformations, and establish the main algorithm to disentangle operators mixing from S^2 to $\mathbb{R}^2$. In section 4 we explain how the algorithm could naturally arise as Griffiths transversality in Calabi-Yau complex moduli. We also use this observation to reconstruct tt^* -equations and constraints that the extremal correlators have to satisfy for chiral rings containing only marginal generators. At last, in section 5, we apply the method to several interesting Calabi-Yau manifolds and compute their extremal correlators in twisted chiral rings as well as chiral correlators in their A-twisted topological theories as a byproduct. We also provide a different formulation of these correlators via localization onto Higgs branch in the case of complete intersections in toric varieties.

2 Preliminaries

2.1 Chiral rings in $\mathcal{N} = (2, 2)$ SCFTs

We start from recalling some properties of $\mathcal{N} = (2, 2)$ superconformal algebra. Our notation follows the paper [7]. In an Euclidean $\mathcal{N} = (2, 2)$ SCFT, we have left moving currents, $T(z)$, $G^\pm(z)$ and $J(z)$, and right ones, $\bar{T}(\bar{z})$, $\bar{G}^\pm(\bar{z})$ and $\bar{J}(\bar{z})$, corresponding to the holomorphic and anti-holomorphic part of energy momentum tensor, supercurrents and U(1) R-currents respectively. We from now on focus on the holomorphic part and NS sectors of the $\mathcal{N} = (2, 2)$ SCFTs. Among these operators, of particular importance is the anticommuting algebra of supercurrents:

$$\{G_r^-, G_s^+\} = 2L_{r+s} - (r-s)J_{r+s} + \frac{c}{3}\left(r^2 - \frac{1}{4}\right)\delta_{r+s,0}, \quad (2.1)$$

where L_n , J_m and $G_r^\pm$ are the modes expansion of currents $T(z)$, $J(z)$ and $G^\pm(z)$. For any states $|\phi\rangle$, unitarity requires

$$\left\|G_{\pm 1/2}^\pm|\phi\rangle\right\| \geq 0. \quad (2.2)$$

By superalgebra, the conformal weight h_ϕ is bounded by its R-charge q_ϕ

$$h_\phi \geq \frac{1}{2} |q_\phi| \quad (2.3)$$

The (anti)chiral primary states are those states saturating the above inequality. We define them as follows:

$$\begin{aligned} \text{chiral primary states:} \quad & L_n |\phi_c\rangle = G_{n-1/2}^\pm |\phi_c\rangle = G_{-1/2}^+ |\phi_c\rangle = 0 \quad \text{for } n \geq 1 \\ \text{antichiral primary states:} \quad & L_n |\phi_a\rangle = G_{n-1/2}^\pm |\phi_a\rangle = G_{-1/2}^- |\phi_a\rangle = 0 \quad \text{for } n \geq 1. \end{aligned} \quad (2.4)$$

With the aid of superconformal algebra, one can easily derive, from

$$\begin{aligned} \{G_{1/2}^-, G_{-1/2}^+\} |\phi_c\rangle &= (2L_0 - J_0) |\phi_c\rangle = 0, \\ \{G_{-1/2}^-, G_{1/2}^+\} |\phi_a\rangle &= (2L_0 + J_0) |\phi_a\rangle = 0. \end{aligned}$$

that the conformal dimension and U(1) R-charge of any (anti)chiral primary states are related by

$$h_{\phi_c} = \frac{1}{2} q_{\phi_c}, \quad h_{\phi_a} = -\frac{1}{2} q_{\phi_a}. \quad (2.5)$$

Besides, the unitarity requires further

$$\|G_{-3/2}^+ |\phi_c\rangle\| \geq 0, \quad \|G_{-3/2}^- |\phi_a\rangle\| \geq 0. \quad (2.6)$$

These two inequalities constrains the conformal dimension of (anti)chiral primary states

$$h \leq \frac{c}{6}. \quad (2.7)$$

This bound fundamentally distinguishes the $2d$ chiral ring structure from that in $4d$, say the number of chiral ring operators is *finitely* many in $2d$ $\mathcal{N} = (2, 2)$ SCFTs.

We next consider the OPE of chiral primary operators $\phi_i(z)$, which is associated to the chiral primary states $|\phi_i\rangle$ due to operator-state correspondence. In general OPE, one has to worry about the appearance of singularities when one operator $\phi_i(z)$ is approaching another $\phi_j(0)$,

$$\phi_i(z)\phi_j(0) \sim \sum_k \frac{C_{ij}^k}{z^{h_i+h_j-h_k}} \mathcal{O}_k(0). \quad (2.8)$$

However, for the OPE of two chiral primary fields, their additive U(1) R-charge guarantees that their OPE is actually non-singular and the leading constant coefficient terms must be also chiral primary fields [10], i.e.

$$\phi_i(z)\phi_j(0) = \sum_k C_{ij}^k \phi_k(0) + \text{nonsingular superdescendants}, \quad (2.9)$$

where C_{ij}^k is the z -independent OPE coefficient. Therefore, modulo secondary fields, the chiral primary fields $\{\phi_i\}$ have a ring structure respect to their OPE coefficient, and form the so-called chiral ring $\mathcal{R}_z$. Since we have argued the number of chiral primary fields is

finite, the chiral ring $\mathcal{R}_z$ is finitely generated but *nilpotent*. It is crucially different from the structure of 4d chiral ring, which is finitely and freely generated. This difference will be explicitly elucidated later when we compute the correlators of chiral primary fields.

One can also define antichiral rings $\overline{\mathcal{R}}_z$ in holomorphic sector in a similar fashion, as well as (anti)chiral rings $\mathcal{R}_{\bar{z}}$ ($\overline{\mathcal{R}}_{\bar{z}}$) in anti-holomorphic sector. For a non-chiral CFT, all states must be tensor products of holomorphic and anti-holomorphic sectors. We thus end up with four different combinations to define the (twisted) chiral primary fields and their hermitian conjugates, i.e.

$$\begin{aligned} \text{(anti)chiral primary fields:} \quad & \phi_i \in \mathcal{R}_z \otimes \mathcal{R}_{\bar{z}}, \quad \bar{\phi}_i \in \overline{\mathcal{R}}_z \otimes \overline{\mathcal{R}}_{\bar{z}} \\ \text{twisted (anti)chiral primary fields:} \quad & \sigma_a \in \mathcal{R}_z \otimes \overline{\mathcal{R}}_{\bar{z}}, \quad \bar{\sigma}_a \in \overline{\mathcal{R}}_z \otimes \mathcal{R}_{\bar{z}}, \end{aligned} \quad (2.10)$$

where we somewhat abuse the notation ϕ_i and the name “chiral” that should not be confused with those defined in holomorphic sector.

Throughout the paper, theories we consider will only contain scalar operators in their (twisted) chiral rings. Therefore, the conformal weight $(h, \bar{h})$ of ϕ and $(\tilde{h}, \tilde{\bar{h}})$ of σ must obey

$$h = \bar{h}, \quad \tilde{h} = \tilde{\bar{h}}. \quad (2.11)$$

On the other hand, for the U(1) R-charge $(q, \bar{q})$ of ϕ and $(\tilde{q}, \tilde{\bar{q}})$ of σ , we have

$$h = \frac{q}{2}, \quad \bar{h} = \frac{\bar{q}}{2}; \quad \tilde{h} = \frac{\tilde{q}}{2}, \quad \tilde{\bar{h}} = -\frac{\tilde{\bar{q}}}{2}. \quad (2.12)$$

Therefore it is convenient to define the so-called U(1)_V and U(1)_A currents as linear combination of $J(z)$ and $\bar{J}(\bar{z})$

$$\begin{aligned} J_V &= J(z) + \bar{J}(\bar{z}) \\ J_A &= J(z) - \bar{J}(\bar{z}), \end{aligned} \quad (2.13)$$

associated to the U(1)_V-charge q_V and U(1)_A-charge q_A . Under the V-A notation, we see that the (twisted) chiral primary fields ϕ and $\tilde{\phi}$ have

$$\begin{aligned} \Delta_\phi &= h + \bar{h} = \frac{q + \bar{q}}{2} = \frac{q_V}{2}, & q - \bar{q} &= q_A = 0 \\ \Delta_\sigma &= \tilde{h} + \tilde{\bar{h}} = \frac{\tilde{q} - \tilde{\bar{q}}}{2} = \frac{q_A}{2}, & \tilde{q} + \tilde{\bar{q}} &= q_V = 0, \end{aligned} \quad (2.14)$$

where Δ denotes the dimension of operators. In the language of field theories with Lagrangian description, we give the following important examples as (twisted) chiral primary fields: the $\mathcal{N} = (2, 2)$ chiral multiplet with dimension one,

$$\Phi = (\phi, \psi, \mathcal{O}), \quad (2.15)$$

has its bottom component ϕ as a chiral primary field with charge $(q_V, q_A) = (2, 0)$ and dimension $\Delta = 1$. Its top component $\mathcal{O}$ has neutral R-charge and dimension 2, thus serves

as a *marginal* secondary field to perturb the CFT. Similarly for a $\mathcal{N} = (2, 2)$ twisted chiral multiplet, for example the field strength of U(1) vector multiplet,

$$\Sigma = \left(\sigma, \tilde{\lambda}, \tilde{\mathcal{O}} \right), \quad (2.16)$$

its bottom component σ is a twisted chiral primary field with charge $(0, 2)$ and dimension 1, and top component $\tilde{\mathcal{O}}$ with neutral R-charge and dimension 2 is also marginal to perturb the CFT.

2.2 Conformal manifolds of $\mathcal{N} = (2, 2)$ SCFTs

Now we turn to discuss how to perturb a CFT by its marginal operators. Suppose that for a d dimensional CFT $\mathcal{S}_0$, there are exactly marginal operators $\mathcal{O}_i$. One can use these operators to deform the original theory $\mathcal{S}_0$ to

$$\mathcal{S}_0 \longrightarrow \mathcal{S} \equiv \mathcal{S}_0 + \lambda^i \int d^d x \mathcal{O}_i, \quad (2.17)$$

where $\{\lambda^i\}$ are exactly marginal couplings. Since the operators are exactly marginal, the coupling constants λ^i are all dimensionless and their β -functions vanish,

$$\beta_{\lambda^i} = 0. \quad (2.18)$$

Therefore the deformed theory $\mathcal{S}$ is still conformal. We then in fact consider a family of CFTs, parametrized by the exactly marginal couplings $\{\lambda^i\}$. Put in other words, the conformal theory $\mathcal{S}$ has a moduli space, a.k.a the conformal manifold $\mathcal{M}(\mathcal{S})$, whose local coordinates are $\{\lambda^i\}$. One can further define the Zamolodchikov metric g_{ij} [7] on $\mathcal{M}(\mathcal{S})$ via the correlators of $\mathcal{O}_i$,

$$\langle \mathcal{O}_i(x) \mathcal{O}_j(y) \rangle_{\mathbb{R}^2} = \frac{g_{ij}(\lambda)}{|x - y|^{2d}}, \quad (2.19)$$

where we evaluate the correlation function in the CFT with couplings $\{\lambda^i\}$.

In the case of $\mathcal{N} = (2, 2)$ SCFTs we consider, there are two types of exactly marginal operators (and their hermitian conjugates): the top components $\mathcal{O}_i$ of chiral primary multiplet Φ_i and $\tilde{\mathcal{O}}_a$ of twisted chiral primary multiplet Σ_a . We formulate the marginal deformation in superspace,

$$\mathcal{S} \equiv \mathcal{S}_0 + \tau^i \int d^2 x d^2 \theta \Phi_i + \tilde{\tau}^a \int d^2 x d^2 \tilde{\theta} \Sigma_a + \text{c.c.} \quad (2.20)$$

where $d^2 \theta$ ($d^2 \tilde{\theta}$) is the measure of (twisted) chiral sub-superspace. It is known [14, 17] that the moduli space of $\mathcal{N} = (2, 2)$ SCFTs is locally a direct product of two Kähler manifolds spanned by the chiral and twisted chiral descendants $\mathcal{O}_i$ and $\tilde{\mathcal{O}}_a$,

$$\mathcal{M}(\mathcal{S}) \simeq \mathcal{M}_c(\tau, \bar{\tau}) \times \mathcal{M}_{tc}(\tilde{\tau}, \bar{\tilde{\tau}}). \quad (2.21)$$

The corresponding Zamolodchikov metrics can be found by computing the correlators

$$\langle \mathcal{O}_i(x) \bar{\mathcal{O}}_j(y) \rangle_{\mathbb{R}^2} = \frac{g_{i\bar{j}}(\tau, \bar{\tau})}{|x - y|^4}, \quad \langle \tilde{\mathcal{O}}_a(x) \bar{\tilde{\mathcal{O}}}_b(y) \rangle_{\mathbb{R}^2} = \frac{\tilde{g}_{a\bar{b}}(\tilde{\tau}, \bar{\tilde{\tau}})}{|x - y|^4}. \quad (2.22)$$

Or instead, noticing that

$$\begin{aligned}\mathcal{O}_i &= \frac{1}{2} G_{-1/2}^- \bar{G}_{-1/2}^- \phi_i, & \bar{\mathcal{O}}_j &= \frac{1}{2} G_{-1/2}^+ \bar{G}_{-1/2}^+ \bar{\phi}_j; \\ \tilde{\mathcal{O}}_a &= \frac{1}{2} G_{-1/2}^- \bar{G}_{-1/2}^+ \sigma_a, & \bar{\tilde{\mathcal{O}}}_b &= \frac{1}{2} G_{-1/2}^+ \bar{G}_{-1/2}^- \bar{\sigma}_b;\end{aligned}\quad (2.23)$$

by conformal Ward identities [7], we can directly evaluate the correlators of (twisted) chiral primary fields,

$$\langle \phi_i(x) \bar{\phi}_j(y) \rangle_{\mathbb{R}^2} = \frac{g_{i\bar{j}}(\tau, \bar{\tau})}{|x-y|^2}, \quad \langle \sigma_a(x) \bar{\sigma}_b(y) \rangle_{\mathbb{R}^2} = \frac{\tilde{g}_{a\bar{b}}(\tilde{\tau}, \bar{\tilde{\tau}})}{|x-y|^2}, \quad (2.24)$$

where the “1/2” is to normalize the superalgebra to avoid unwanted numerical factors.

Let us briefly remark that, by a simple dimension count, the operator ϕ_i (σ_a) has conformal weight $(\frac{1}{2}, \frac{1}{2})$. The unitarity bound for chiral ring elements requires the center charge of our SCFTs, $c \geq 3$. Equivalently, only $\mathcal{N} = (2, 2)$ SCFTs with $c \geq 3$ have exactly marginal operators and thus moduli spaces. Throughout the paper, we only discuss theories subject to this condition, and require all chiral operators having *integer* dimensions. Therefore the correlators $\langle \phi_i(x) \bar{\phi}_j(y) \rangle_{\mathbb{R}^2}$ as well as $\langle \sigma_a(x) \bar{\sigma}_b(y) \rangle_{\mathbb{R}^2}$ are the first nontrivial extremal correlation functions to compute. The operators ϕ_i ’s (σ_a ’s) are also the first non-trivial elements with lowest conformal weight in the (twisted) chiral rings of $\mathcal{N} = (2, 2)$ SCFTs. We will review more details of the ring structure right soon.

2.3 Extremal correlators

As we have seen that the Zamolodchikov metric is one of interesting objects to compute, we can in fact consider more general “extremal correlators” in chiral rings (all discussions below equally work for twisted chiral rings). These are correlators of the form

$$\langle \phi_1(x_1) \dots \phi_n(x_n) \bar{\phi}_J(y) \rangle_{\mathbb{R}^2} \quad (2.25)$$

where ϕ_i are chiral primaries and $\bar{\phi}_J$ is antichiral. The selection rule respect to $U(1)_V$ symmetry requires the above correlator vanish unless

$$q_J = - \sum_i q_i, \quad \text{or equivalently} \quad \Delta_J = \sum_i \Delta_i. \quad (2.26)$$

In comparison of chiral correlators, which contain only chiral operators and holomorphically depend on marginal parameters $\{\tau\}$, the extremal correlators are in general non-holomorphic functions of marginal couplings $\{\tau, \bar{\tau}\}$. The main part of this paper is devoted to compute these extremal correlators both perturbatively and non-perturbatively.

To compute the extremal correlators, it is instrument to apply a standard conformal transformation,

$$x'_i = \frac{x_i - y}{|x_i - y|^2}. \quad (2.27)$$

to hide the coordinates “ y ” of the antichiral field to “ ∞ ”,

$$\langle \phi_1(x_1) \dots \phi_n(x_n) \bar{\phi}_J(y) \rangle_{\mathbb{R}^2} = \frac{\langle \phi_1(x'_1) \dots \phi_n(x'_n) \bar{\phi}_J(\infty) \rangle_{\mathbb{R}^2}}{|x_1 - y|^{2\Delta_1} \dots |x_n - y|^{2\Delta_n}}. \quad (2.28)$$

Next, one can show the numerator is actually spacetime independent. Notice that, by acting $\partial_{z'_i}$ on $\langle \phi_1(x'_1) \cdots \phi_n(x'_n) \bar{\phi}_J(\infty) \rangle_{\mathbb{R}^2}$,

$$\partial_{z'_i} \phi_i(x'_i) = G_{-1/2}^+ G_{-1/2}^- \phi_i(x'_i). \quad (2.29)$$

By superconformal Ward identity, one can rewrite $G_{-1/2}^+$ acting on each of other operators. $G_{-1/2}^+$ annihilates all other chiral primaries, while, acting on $\bar{\phi}_J(y)$, the correlator

$$\langle \phi_1(x'_1) \cdots \phi_n(x'_n) G_{-1/2}^+ \bar{\phi}_J(y) \rangle \quad (2.30)$$

decays as $|y|^{-2\Delta_J-1}$. Therefore, when putting “ y ” to infinity, this contribution will decay as $|y|^{-1}$ to zero as well. Overall we single out the spacetime dependent part from the extremal correlator, and show that

$$\langle \phi_1(x_1) \cdots \phi_n(x_n) \bar{\phi}_J(\infty) \rangle_{\mathbb{R}^2} = \langle \phi_1(0) \cdots \phi_n(0) \bar{\phi}_J(\infty) \rangle_{\mathbb{R}^2}, \quad (2.31)$$

only depends on the marginal couplings $\{\tau, \bar{\tau}\}$.

Now one can apply OPE to these chiral primaries,

$$\phi_1(0) \cdots \phi_n(0) = C_{12}^i C_{i3}^j \cdots C_{ln}^k \phi_k(0) \equiv \phi_I(0) \quad (2.32)$$

modulo secondary operators which will not contribute to the correlator, and have

$$\langle \phi_1(x_1) \cdots \phi_n(x_n) \bar{\phi}_J(y) \rangle_{\mathbb{R}^2} = \frac{\langle \phi_I(0) \bar{\phi}_J(\infty) \rangle_{\mathbb{R}^2}}{|x_1 - y|^{2\Delta_1} \cdots |x_n - y|^{2\Delta_n}} \equiv \frac{g_{I\bar{J}}(\tau, \bar{\tau})}{|x_1 - y|^{2\Delta_1} \cdots |x_n - y|^{2\Delta_n}}. \quad (2.33)$$

Therefore, similar to Zamolodchikov metric, the computation of extremal correlators is to determine the $g_{I\bar{J}}$, which is referred as “chiral ring data” of the SCFT [4].

2.4 tt^* -geometries

In this subsection, we will briefly review tt^* -equations of chiral ring data. For more details and derivations, we refer readers to [7, 9].

Given a chiral ring $\mathcal{R}$, we can grade it by the dimensions or R-charges of the operators in it,

$$\mathcal{R} = \bigoplus_{\Delta_I=0}^{N=c/3} \mathcal{R}_I. \quad (2.34)$$

Since we work in NS sector, the vacuum state is unique, or say $\mathcal{R}_0$ contains only the unit operator $\mathbf{1}$. As required before, the next level $\mathcal{R}_1$ contains chiral primaries with dimension $\Delta_1 = 1$, whose descendants gives the marginal operators to span the moduli space $\mathcal{M}(\mathcal{S})$ of SCFT $\mathcal{S}$. $\mathcal{R}_I$ contains chiral primaries with dimension $\Delta_I = I$, and so on. At last the top sub-ring $\mathcal{R}_N$ also contains only one operator with the highest dimension $c/3$ [10].

From the geometric perspective, one can interpret the (anti)chiral primaries ϕ_i and $\bar{\phi}_j$ in $\mathcal{R}_1$ and $\bar{\mathcal{R}}_1$ as sections on tangent bundle $\mathcal{TM}(\mathcal{S})$. Their correlator

$$\langle \phi_i(0) \bar{\phi}_j(\infty) \rangle_{\mathbb{R}^2} = g_{i\bar{j}}(\tau, \bar{\tau}) \quad (2.35)$$

designates a Hermitian metric on $\mathcal{TM}(\mathcal{S})$. Similarly operators ϕ_I and $\bar{\phi}_J$ living in $\mathcal{R}_I$ and $\bar{\mathcal{R}}_J$ can be also understood as sections living on certain vector bundles $\mathcal{V}_I$ and $\mathcal{V}_J$ over moduli space $\mathcal{M}(\mathcal{S})$. The extremal correlators

$$\langle \phi_I(0) \bar{\phi}_J(\infty) \rangle_{\mathbb{R}^2} = g_{I\bar{J}}(\tau, \bar{\tau}) \delta_{\Delta_I \Delta_{\bar{J}}}, \quad (2.36)$$

analogously define Hermitian metrics on various bundle $\mathcal{V}_I$'s. Here the appearance of $\delta_{\Delta_I \Delta_{\bar{J}}}$ is imposed by the selection rule (2.26), which implies the total vector bundle

$$\mathcal{V} = \bigoplus_{\Delta_I=0}^{N=c/3} \mathcal{V}_I \quad (2.37)$$

is also graded by dimensions of the operators.

Now we are ready to discuss the tt^* -equations. Roughly speaking, tt^* -equations interpolate metrics $g_{I\bar{J}}$ defined on different bundles $\mathcal{V}_I$'s via the OPE coefficients of these (anti)chiral primaries. More specifically, let us consider the metric $g_{I\bar{J}}$ varied along certain direction τ^i (or $\bar{\tau}^{\bar{j}}$) of the moduli space $\mathcal{M}(\mathcal{S})$. It is equivalent, from the action (2.20), to compute

$$\delta_\tau g_{I\bar{J}} \approx \delta\tau^i \partial_i g_{I\bar{J}} = \left\langle \delta\tau^i \int d^2x \mathcal{O}_i(x) \phi_I(0) \bar{\phi}_J(\infty) \right\rangle. \quad (2.38)$$

However the correlator is divergent when evaluating the integration. Therefore one has to renormalize it by adding counter terms. This process might lead the computation with ambiguities. It can be shown [11] that the renormalization process is equivalent to introduce connections ∇_τ and $\nabla_{\bar{\tau}}$ on the vector bundle $\mathcal{V}$, and the variation of correlators along moduli space has to be modified as

$$\delta_\tau g_{I\bar{J}} = \delta\tau^i \nabla_i g_{I\bar{J}} = \left\langle \delta\tau^i \int d^2x \mathcal{O}_i(x) \phi_I(0) \bar{\phi}_J(\infty) \right\rangle_{\text{renormalized}}. \quad (2.39)$$

In this sense, the physical process of renormalization in fact reflects non-trivial geometries of the moduli space $\mathcal{M}(\mathcal{S})$ and the bundle $\mathcal{V}$ over it.

The geometries are encoded in tt^* equations which determine the curvature of the vector bundle $\mathcal{V}$ via the dynamics of the SCFT. In concrete, one can establish the equations

$$[\nabla_i, \nabla_{\bar{j}}]_L^K = g_{i\bar{j}} \delta_L^K \left(1 - \frac{6}{c} \Delta_K \right) - [C_i, \bar{C}_{\bar{j}}]_L^K. \quad (2.40)$$

The C_i and $\bar{C}_{\bar{j}}$ should be understood as OPE coefficient in matrix form, i.e.

$$(C_i)_L^K \equiv C_{iL}^K, \quad (\bar{C}_{\bar{j}})_L^K \equiv g_{L\bar{N}} \bar{C}_{\bar{j}\bar{M}}^{\bar{N}} g^{\bar{M}K}, \quad (2.41)$$

where indexes $i, \bar{j}$ run for marginal operators, and $g^{\bar{M}L}$ stands for inverse of the metric. The tt^* -equations here is derived in NS sector [7], different from that in Ramond sector [9] by the first diagonal piece. We will come back in later section to comment more on this term as a matter of normalization, see section 4.1.

To see how the tt^* -equations relate metrics in various bundle $\mathcal{V}_I$, one can choose a holomorphic gauge as

$$\nabla_i = \partial_i - (\partial_i g_{I\bar{J}}) g^{\bar{J}K}, \quad \nabla_{\bar{j}} = \partial_{\bar{j}}. \quad (2.42)$$

The holomorphic gauge can be always achieved and thus the metrics of the vector bundle $\mathcal{V}$ are constrained via tt^* -equations (2.40). The metrics, or say the chiral ring data, are solutions to the tt^* -equations. Nevertheless in the paper we will not solve them from the equations. Instead, in next section, we will show that these chiral ring data can be directly computed via supersymmetric localization, and the results will automatically solve the tt^* -equations.

In the end of this subsection, for completeness, we would like to make some remarks. Above discussion on chiral rings can be identically repeated to twisted chiral rings. For $\mathcal{N} = (2, 2)$ SCFTs, the correlator of (anti)chiral and twisted (anti)chiral primaries always vanishes even when they have same dimensions, because their R-charges are different. It thereby implies the factorization of the moduli space. However this result breaks down in $\mathcal{N} = (4, 4)$ SCFTs, whose moduli space is neither factorisable, nor even Kählerian. More details on this issue are discussed in [12].

3 Chiral ring data via superlocalization on S^2

In this section, we will establish an algorithm to systematically compute (twisted) chiral ring data¹ $g_{I\bar{J}}$ of $\mathcal{N} = (2, 2)$ SCFTs with UV Lagrangian descriptions in the fashion of [4]. The general idea is sketched as follows: for a chiral ring, $\mathcal{R} = \bigoplus_{\Delta_I=0}^{N=c/3} \mathcal{R}_I$, every element in the ring can be uniquely represented by

$$\phi_{\{n_{\hat{I}_\alpha}\}} = \prod_{\alpha=1} \prod_{\hat{I}_\alpha} \phi_{\hat{I}_\alpha}^{n_{\hat{I}_\alpha}} \quad (3.1)$$

where the collection of $\{\hat{I}_1\} \equiv \{i\}$ labels the primitive generators² in $\mathcal{R}_1$ corresponding to marginal deformation, $\{\hat{I}_\alpha | \alpha \geq 2\}$ enumerates the primitive generators in $\mathcal{R}_\alpha$ for a given dimension $\Delta = \alpha$, and $n_{\hat{I}_\alpha}$ specifies the power of a given generator. We can deform the SCFT by introducing not only marginal, but also irrelevant deformations respect to all chiral ring generators and their complex conjugates,

$$\mathcal{S}_0 \rightarrow \mathcal{S}_{\text{deform}} = \mathcal{S}_0 + \frac{\tau^i}{2} \int d^2x d^2\theta \Phi_i + \frac{\tau^{\hat{I}_\alpha}}{2} \int d^2x d^2\theta \Phi_{\hat{I}_\alpha} + \text{h.c.}, \quad (3.2)$$

where the couplings are normalized with a factor of 4π , and $\{\Phi_{\hat{I}_\alpha}\}$ denote the corresponding supermultiplets of chiral primaries $\{\phi_{\hat{I}_\alpha}\}$. Such deformations surely break $\mathcal{N} = (2, 2)$ superconformal symmetries, while leaving a $\mathfrak{su}(2|1)$ sub-superalgebra intact. It is exactly the

¹In this section, we actually will consider twisted chiral primaries in details. All discussions on chiral primaries equally work for twisted chiral primaries, and vice versa.

²Here, by primitive generators, we mean the linearly independent chiral primary operators that spans the whole chiral ring, which later correspond to the generators of cohomology of Calabi-Yau manifolds, see section 4 and also [23].

most general $\mathcal{N} = (2, 2)$ massive supersymmetries that can be preserved on S^2 . Therefore we are able to place the deformed theory S_{deform} on S^2 and compute its partition function

$$Z[S^2](\tau^i, \bar{\tau}^{\bar{j}}, \tau^{\hat{I}_\alpha}, \bar{\tau}^{\hat{J}_\beta})$$

via localization techniques. Once we find $Z[S^2]$, by varying its parameters and utilizing supersymmetric Ward identities, one can obtain the extremal correlators of the chiral ring generators on S^2 ,

$$\left\langle \phi_{\hat{I}_\alpha}(N) \bar{\phi}_{\hat{J}_\beta}(S) \right\rangle_{S^2} = -\frac{1}{Z[S^2]} \partial_{\tau^{\hat{I}_\alpha}} \partial_{\bar{\tau}^{\hat{J}_\beta}} Z[S^2] \Big|_{\tau^{\hat{I}_\gamma} = \bar{\tau}^{\hat{J}_\gamma} = 0, \gamma \geq 2}, \quad (3.3)$$

where “N” and “S” denote the north and south poles of the S^2 . Finally, as the most important step, a Gram-Schmidt orthogonalization needs to be performed to extract extremal correlator,

$$g_{\hat{I}_\alpha \hat{J}_\beta} = \left\langle \phi_{\hat{I}_\alpha}(0) \bar{\phi}_{\hat{J}_\beta}(\infty) \right\rangle_{\mathbb{R}^2}, \quad (3.4)$$

on flat space from $\left\langle \phi_{\hat{I}_\alpha}(N) \bar{\phi}_{\hat{J}_\beta}(S) \right\rangle_{S^2}$.

Most of materials in this section can be regarded as a $2d$ version of discussion parallel to that for $4d$ $\mathcal{N} = 2$ SCFTs in [4]. As we will point out in section 3.3, the algorithm needs to be modified somewhat due to the nilpotency of the $2d$ chiral rings.

3.1 Placing deformed theories on S^2

The general methodology of putting theories on curved space supersymmetrically is developed in [13]. Discussion specific to $2d$ $\mathcal{N} = (2, 2)$ theories is also explained in [14–16], as well as in [17] with an emphasis on spurious field analysis. We will follow [14] with mild modification to place irrelevant operator deformations onto S^2 as well.

We have seen that a $\mathcal{N} = (2, 2)$ SCFT has $U(1)_V \times U(1)_A$ R-symmetries. Correspondingly elements in chiral ring $\mathcal{R}$ and twisted chiral ring $\tilde{\mathcal{R}}$ take non-vanishing $U(1)_V$ and $U(1)_A$ charge respectively. The deformations (3.2) are from F-terms and will break part of R-symmetries unless they are marginal. More explicitly, since superspace measure $d^2\theta$ takes $(-2, 0)$ R-charges, an irrelevant deformation

$$\frac{\tau^{\hat{I}_\alpha}}{2} \int d^2x d^2\theta \Phi_{\hat{I}_\alpha} \quad (3.5)$$

inevitably breaks the $U(1)_V$ R-symmetry but keeps the $U(1)_A$ intact. The remaining massive superalgebra is labeled as $\mathfrak{su}(2|1)_B$. Similarly, an irrelevant deformation from a twisted chiral primary multiplet Σ_{A_I} ,

$$\frac{\tilde{\tau}^{\hat{I}_\alpha}}{2} \int d^2x d^2\tilde{\theta} \Sigma_{\hat{I}_\alpha}, \quad (3.6)$$

will break $U(1)_A$ while preserving $U(1)_V$, whose remaining massive superalgebra is denoted as $\mathfrak{su}(2|1)_A$. The $\mathfrak{su}(2|1)_A$ and $\mathfrak{su}(2|1)_B$ are two *inequivalent* sub-superalgebras in $\mathcal{N} = (2, 2)$ superconformal algebra. Interestingly they correspond to two inequivalent ways to place the deformed theories on S^2 , that we will discuss in some details.

3.1.1 Deformations respect to $\mathfrak{su}(2|1)_A$

The $\mathfrak{su}(2|1)_A$ type deformation allows us to use twisted chiral primaries $\{\Sigma_a, \Sigma_{\hat{I}_\alpha}\}$ to deform the original action while preserving $U(1)_V$ R-symmetry,

$$\mathcal{S}_A = \mathcal{S}_0 + \frac{\tilde{\tau}^a}{2} \int d^2x d^2\theta \Sigma_a + \frac{\tilde{\tau}^{\hat{I}_\alpha}}{2} \int d^2x d^2\theta \Sigma_{\hat{I}_\alpha} + \text{h.c.} \quad (3.7)$$

For twisted chiral superfield with dimension $\Delta_\Sigma = \tilde{\omega}$,

$$\Sigma = \left(\sigma, \tilde{\lambda}, \tilde{\mathcal{O}} \right), \quad (3.8)$$

its supersymmetric transformation on S^2 respect to $\mathfrak{su}(2|1)_A$ is cast as

$$\begin{aligned} \delta\sigma &= \zeta \cdot \tilde{\lambda}, \\ \delta\tilde{\lambda} &= i\gamma^\mu \tilde{\zeta} D_\mu \sigma + \zeta \tilde{\mathcal{O}} - \frac{\tilde{\omega}}{R} \zeta \sigma \\ \delta\tilde{\mathcal{O}} &= i\tilde{\zeta} \cdot \gamma^\mu D_\mu \tilde{\lambda} + \frac{\tilde{\omega}}{R} \zeta \cdot \tilde{\lambda}, \end{aligned} \quad (3.9)$$

where ζ and $\tilde{\zeta}$ are Killing spinors parameterizing the $\mathfrak{su}(2|1)_A$ superalgebra, D_μ is the covariant derivative on S^2 with radius R , and more about notations are summarized in appendix A. Now placing Σ from flat $\mathbb{R}^2$ to S^2 ,

$$\int_{\mathbb{R}^2} d^2x d^2\theta \Sigma = \int_{\mathbb{R}^2} d^2x \tilde{\mathcal{O}}(x) \rightarrow \int_{S^2} d^2x \sqrt{g} \tilde{\mathcal{O}}(x), \quad (3.10)$$

where g is the determinant of metric on S^2 . Apparently from eq. (3.9), the above F -term is *not* supersymmetric invariant,

$$\delta \left(\int_{S^2} d^2x \sqrt{g} \tilde{\mathcal{O}}(x) \right) = \int_{S^2} d^2x \sqrt{g} \frac{\tilde{\omega} - 1}{R} \zeta \cdot \tilde{\lambda}, \quad (3.11)$$

unless $\tilde{\omega} = 1$ corresponding to a marginal deformation. However, by compensating an additional piece proportional to σ , the modified F-term

$$\int_{S^2} d^2x \sqrt{g} \left(\tilde{\mathcal{O}}(x) - \frac{\tilde{\omega} - 1}{R} \sigma(x) \right) \quad (3.12)$$

is supersymmetric invariant on S^2 . Therefore for a deformed theory (3.7), we can place it on S^2 with order of $1/R$ modifications as

$$\mathcal{S}_A[S^2] = \mathcal{S}_0[S^2] + \frac{\tilde{\tau}^{\hat{I}_\alpha}}{2} \int_{S^2} d^2x \sqrt{g} \left(\tilde{\mathcal{O}}_{\hat{I}_\alpha}(x) - \frac{\tilde{\omega}_{\hat{I}_\alpha} - 1}{R} \sigma_{\hat{I}_\alpha}(x) \right) + \text{h.c.} \quad (3.13)$$

where I_α runs for all marginal and irrelevant couplings.

3.1.2 Deformations respect to $\mathfrak{su}(2|1)_B$

Parallel to above discussion, the $\mathfrak{su}(2|1)_B$ superalgebra allows us to preserve $U(1)_A$ R-symmetries, which makes deformations by chiral primary multiplets $\{\Phi_i, \Phi_{\hat{I}_\alpha}\}$ feasible,

$$\mathcal{S}_B = \mathcal{S}_0 + \frac{\tau^i}{2} \int d^2x d^2\theta \Phi_i + \frac{\tau^{\hat{I}_\alpha}}{2} \int d^2x d^2\theta \Phi_{\hat{I}_\alpha} + \text{h.c.} \quad (3.14)$$

The supersymmetric transformation of a chiral superfield

$$\Phi = (\phi, \psi, \mathcal{O}) \quad (3.15)$$

with dimension $\Delta_\Phi = \omega$, can be written down,

$$\begin{aligned} \delta\phi &= \tilde{\epsilon} \cdot \psi, \\ \delta\psi &= i\gamma^\mu \epsilon D_\mu \phi + \tilde{\epsilon} \mathcal{O} - \frac{\omega}{R} \tilde{\epsilon} \phi \\ \delta\mathcal{O} &= i\epsilon \cdot \gamma^\mu D_\mu \psi + \frac{\omega}{R} \tilde{\epsilon} \cdot \psi \end{aligned} \quad (3.16)$$

When placing the chiral primary on S^2 , one can check,

$$\delta \left(\int_{S^2} d^2x \sqrt{g} \mathcal{O}(x) \right) = \int_{S^2} d^2x \sqrt{g} \frac{\omega - 1}{R} \tilde{\epsilon} \cdot \psi, \quad (3.17)$$

is not supersymmetric invariant, unless $\omega = 1$ for marginal deformations. Therefore we modify the F -term by

$$\int_{S^2} d^2x \sqrt{g} \left(\mathcal{O}(x) - \frac{\omega - 1}{R} \phi(x) \right), \quad (3.18)$$

and thus the deformed action (3.14), corrected as

$$\mathcal{S}_B[S^2] = \mathcal{S}_0[S^2] + \frac{\tau^{\hat{I}_\alpha}}{2} \int_{S^2} d^2x \sqrt{g} \left(\mathcal{O}_{\hat{I}_\alpha}(x) - \frac{\omega_{\hat{I}_\alpha} - 1}{R} \phi_{\hat{I}_\alpha}(x) \right) + \text{h.c.}, \quad (3.19)$$

is supersymmetric on S^2 with respect to $\mathfrak{su}(2|1)_B$.

3.2 The $\mathfrak{su}(2|1)_A$ deformed partition functions on S^2

Our discussion is focused on computing the partition function Z_A on S^2 respect to $\mathfrak{su}(2|1)_A$ superalgebra, for one is always able to choose a “twisted basis” [19] to realize $\mathfrak{su}(2|1)_B$ deformation in terms of $\mathfrak{su}(2|1)_A$ superalgebra. Besides, under the assumption of mirror symmetry, for a theory $\mathcal{S}$ with $\mathfrak{su}(2|1)_B$ deformation, one can always find a mirror $\tilde{\mathcal{S}}$, such that $Z_B(\mathcal{S}) = Z_A(\tilde{\mathcal{S}})$. We will come back to this point in later sections.

The details of localization computation can be found in [15, 16]. The partition function $Z_A(\mathcal{S})$ captures the data of the twisted chiral ring $\tilde{\mathcal{R}}$ of a theory $\mathcal{S}$. We will adopt it by adding irrelevant deformations that correspond to all primitive generators in $\tilde{\mathcal{R}}$ with dimension greater than one.

An $\mathcal{N} = (2, 2)$ SCFT $\mathcal{S}$ considered here has a UV Lagrangian description, more concretely, realized as a gauged linear sigma model (GLSM) with gauge group $U(1)^s \times G$, where G is a product of simple groups. The action of the theory,

$$\mathcal{S} = \int d^2x d^4\theta \mathcal{L}_{\text{gauge}}(V) + \mathcal{L}_{\text{matter}}(\Phi, V) + \int d^2x d^2\theta \mathcal{W}(\Phi) + \int d^2x d^2\tilde{\theta} \widetilde{\mathcal{W}}(\Sigma) + \text{h.c.} \quad (3.20)$$

contains gauge multiplets V , matter multiplets Φ 's, superpotential $\mathcal{W}$ of matters, and twisted superpotential $\widetilde{\mathcal{W}}$ of field strength Σ , or say FI-terms, of $U(1)^s$ factors of gauge group. When placed on S^2 , $\mathcal{S}$ will receive corrections in terms of $\mathcal{O}(1/R)$,

$$\mathcal{S} \rightarrow \mathcal{S}_A = \mathcal{S} + \mathcal{O}\left(\frac{1}{R}\right)$$

as we have seen in eq. (3.13), and the modified action is invariant respect to supercharges $\mathcal{Q} \in \mathfrak{su}(2|1)_A$. We are allowed to add arbitrary $\mathcal{Q}$ -exact terms $\mathcal{Q}\mathcal{V}$ with suitable asymptotic behaviors to evaluate the partition function

$$Z_A(\mathcal{S}) = \int \mathcal{D}\varphi e^{-\mathcal{S}_A[\varphi] + t\mathcal{Q}\mathcal{V}} \quad (3.21)$$

without changing the final result. Therefore, by putting $t \rightarrow \infty$, we evaluate the above path integral on the locus $\mathcal{M}_0 = \{\varphi_0 \mid \mathcal{Q}\mathcal{V}(\varphi_0) = 0\}$,

$$Z_A(\mathcal{S}) = \int d\varphi_0 Z_{\text{cl}}(\varphi_0) Z_{1\text{-loop}}(\varphi_0), \quad (3.22)$$

where we have chosen to localize the theory onto the Coulomb branch, and the term $Z_{1\text{-loop}}$ corresponds to the one-loop fluctuation of $\mathcal{Q}\mathcal{V}$ around $\mathcal{M}_0$. We now spell out the detailed expression of $Z_A(\mathcal{S})$ [15, 16],

$$Z_A(\mathcal{S}) = \frac{1}{|\mathcal{W}|} \sum_{\mathbf{m}=\{\mathbf{m}_i, \tilde{\mathbf{m}}_l\}} \int \left[\prod_{i=1}^{\text{rank}(G)} \prod_{l=1}^s \frac{d\sigma_i}{2\pi} \frac{d\tilde{\sigma}_l}{2\pi} \right] Z_{\text{cl}}(\sigma, \mathbf{m}) Z_{\text{gauge}}(\{\sigma_i\}, \{\mathbf{m}_i\}) \prod_{\Phi} Z_{\Phi}(\sigma, \mathbf{m}), \quad (3.23)$$

where we have scaled the radius of sphere $R = 1$, $|\mathcal{W}|$ is the order of the Weyl group G and $\sigma = \{\sigma_i, \tilde{\sigma}_l\}$. $\sigma_i \in \mathbb{R}^{\text{rank}(G)}$ is in the Cartan subalgebra of G and $\mathbf{m}_i \in \mathbb{Z}^{\text{rank}(G)}$ is the GNO quantized magnetic charge of the Cartan part of G . Similarly $\tilde{\sigma}_l$ and $\tilde{\mathbf{m}}_l$ parametrize $\mathbb{R}^s$ and $\mathbb{Z}^s$ corresponding to the $U(1)^s$ factors of G .

Z_{gauge} and Z_{Φ} are 1-loop fluctuations of gauge and matter multiplets around $\mathcal{Q}\mathcal{V}$,

$$Z_{\text{gauge}}(\{\sigma_i\}, \{\mathbf{m}_i\}) = \prod_{\alpha \in \Delta_+} \left(\frac{(\alpha, \mathbf{m})^2}{4} + (\alpha, \sigma)^2 \right),$$

$$Z_{\Phi}(\sigma, \mathbf{m}) = \prod_{\rho \in R_{\Phi}} \frac{\Gamma(\frac{1}{2}\mathbf{q}_{\Phi} - (\rho, i\sigma + \frac{1}{2}\mathbf{m}) - \sum_l Q_{\Phi}^l(i\tilde{\sigma}_l + \frac{1}{2}\tilde{\mathbf{m}}_l))}{\Gamma(1 - \frac{1}{2}\mathbf{q}_{\Phi} + (\rho, i\sigma - \frac{1}{2}\mathbf{m}) + \sum_l Q_{\Phi}^l(i\tilde{\sigma}_l - \frac{1}{2}\tilde{\mathbf{m}}_l))}, \quad (3.24)$$

where $(\cdot, \cdot)$ is the standard inner product of Lie algebra $\mathfrak{g}$ of G , $\alpha \in \Delta_+$ are positive roots over $\mathfrak{g}$, Q_{Φ}^l is the gauge charge of Φ for $U(1)^s$ factors, ρ is the weight of the representation R_{Φ} of G , and $\mathbf{q}_{\Phi}$ is the $U(1)_V$ R-charge of Φ .

$Z_{\text{cl}}(\sigma, \mathbf{m})$ is the classical piece. For all gauge, matters and superpotential sectors are $\mathcal{Q}$ -exact, so absorbed in $\mathcal{V}$, Z_{cl} interestingly only matters with twisted superpotentials $\widetilde{\mathcal{W}}$ and exactly encodes the information of twisted chiral ring of the theory $\mathcal{S}$. If $\mathcal{S}$ contains *only* marginal deformations, the twisted potentials are FI-terms corresponding to the $U(1)^s$ factors,³

$$\widetilde{\mathcal{W}}(\Sigma) + \text{h.c.} = \frac{1}{2} \sum_{l=1}^s \left(\tilde{\tau}^l \Sigma_l - \bar{\tilde{\tau}}^l \bar{\Sigma}_l \right), \quad (3.25)$$

where we use $\Sigma_l \equiv (\tilde{\sigma}_l, \tilde{\lambda}_l, \tilde{\mathcal{O}}_l)$ to denote the twisted super-field strength of the $U(1)^s$ gauge multiplets, and $\tilde{\tau}^l \equiv \frac{\tilde{\theta}^l}{2\pi} + i \tilde{r}^l$ are their complex FI-couplings. Evaluating it at locus $\mathcal{M}_0$ gives

$$Z_{\text{cl}}(\tilde{\sigma}, \tilde{\mathbf{m}}) = \exp \left(-4\pi i \sum_{l=1}^s \tilde{r}^l \tilde{\sigma}_l - i \sum_{l=1}^s \tilde{\theta}^l \tilde{\mathbf{m}}_l \right). \quad (3.26)$$

Now we introduce irrelevant deformations, see eq. (3.7), for generators with dimension greater than one in twisted chiral ring. Since all of them are twisted superpotentials, only the term Z_{cl} in eq. (3.23) needs modifications. In case of gauge group $U(N)$, we spell out the deformed partition function. Following appendix B, all generators are of the form

$$\text{Tr}(\Sigma), \text{Tr}(\Sigma^2), \dots, \text{Tr}(\Sigma^N),$$

with $\Sigma = (\sigma, \lambda, iG - D)$ taking values on $\mathfrak{u}(N)$ Lie algebra. We therefore introduce the deformations as the twisted superpotential

$$\widetilde{\mathcal{W}}(\Sigma) + \text{h.c.} = \frac{1}{2} \left(\tilde{\tau}_1 \text{Tr} \Sigma - \bar{\tilde{\tau}}_1 \text{Tr} \bar{\Sigma} \right) + \frac{1}{2} \sum_{n=2}^N \left(\tilde{\tau}_n \text{Tr} \Sigma^n - \bar{\tilde{\tau}}_n \text{Tr} \bar{\Sigma}^n \right), \quad (3.27)$$

where $\tilde{\tau}_1$ is marginal and singled out, and $\tilde{\tau}_n \text{Tr} \Sigma^n$ are irrelevant deformations⁴ with dimension $\Delta_n = n$. Their F -terms are

$$\text{Tr} \Sigma^n \Big|_{F\text{-term}} = n \text{Tr} \{ (iG - D) \sigma^{n-1} \} + \text{fermi}. \quad (3.28)$$

When placed on S^2 , following eq. (3.13), we correct them by

$$n \text{Tr} \{ (iG - D) \sigma^{n-1} \} + \text{fermi}. - (n-1) \text{Tr} \sigma^n. \quad (3.29)$$

Localizing on Coulomb branch implies that we set the locus $\mathcal{M}_0$ at

$$\begin{aligned} \sigma &= \text{diag} \left\{ \sigma_1 + \frac{i}{2} m_1, \sigma_2 + \frac{i}{2} m_2, \dots, \sigma_n + \frac{i}{2} m_n \right\} \equiv \sigma_c, \\ D &= -\text{diag} \{ \sigma_1, \sigma_2, \dots, \sigma_n \}, \quad G = \text{diag} \left\{ \frac{m_1}{2}, \frac{m_2}{2}, \dots, \frac{m_n}{2} \right\} \\ \bar{\sigma} &= \sigma_c^\dagger \equiv \bar{\sigma}_c, \quad \lambda = 0 \end{aligned} \quad (3.30)$$

³The unusual sign in front of $\tilde{\tau}$ and $\bar{\tilde{\tau}}$ are due to analytical continuation of the theory from Minkowskian space.

⁴We here use notation “ $\tilde{\tau}_n$ ” with subscripts as the couplings of the deformations to avoid confusion with the powers of $\tilde{\tau}$.

Therefore we have

$$Z_{\text{cl}}(\sigma_c, \bar{\sigma}_c) = \exp \left\{ -2\pi \sum_{n=1}^N (\tilde{\tau}_n \text{Tr } \sigma_c^n - \bar{\tilde{\tau}}_n \text{Tr } \bar{\sigma}_c^n) \right\} \quad (3.31)$$

Overall our full deformed partition function for $U(N)$ gauge group is

$$Z_A(\tilde{\tau}, \bar{\tilde{\tau}}) = \frac{1}{N!} \sum_{\{m_i\} \in \mathbb{Z}} \int_{-\infty}^{\infty} \prod_{i=1}^N \frac{d\sigma_i}{2\pi} Z_{\text{cl}}(\sigma_c, \bar{\sigma}_c) Z_{\text{gauge}}(\sigma_c, \bar{\sigma}_c) \prod_{\Phi} Z_{\Phi}(\sigma_c, \bar{\sigma}_c), \quad (3.32)$$

with

$$\begin{aligned} Z_{\text{cl}}(\sigma_c, \bar{\sigma}_c) &= \exp \left\{ -2\pi \sum_{n=1}^N (\tilde{\tau}_n \text{Tr } \sigma_c^n - \bar{\tilde{\tau}}_n \text{Tr } \bar{\sigma}_c^n) \right\}, \\ Z_{\text{gauge}}(\sigma_c, \bar{\sigma}_c) &= \prod_{i < j} \left(\frac{(m_i - m_j)^2}{4} + (\sigma_i - \sigma_j)^2 \right) = \prod_{i < j} |\sigma_{ci} - \sigma_{cj}|^2, \\ Z_{\Phi}(\sigma_c, \bar{\sigma}_c) &= \prod_{\rho \in R_{\Phi}} \frac{\Gamma(\frac{1}{2} \mathbf{q}_{\Phi} - (\rho, i\bar{\sigma}_c))}{\Gamma(1 - \frac{1}{2} \mathbf{q}_{\Phi} + (\rho, i\sigma_c))}. \end{aligned}$$

Eq. (3.32) serves as our main formula that will be used in section 5. Different from 4d situation, where the Nekrasov's partition functions are not known yet for deformed theories [4], meanwhile our 2d deformed partition function here is *exact*, because 2d localization onto Coulomb branch has no instanton correction! It would be very interesting to evaluate the 2d *deformed* partition function through localization onto Higgs branch, which in principle could be written as discrete sum of vortex and anti-vortex partition functions as 2d version of Nekrasov partition function. We wish that it might shed light on how to compute the 4d deformed partition function exactly.

3.3 Twisted chiral primary correlators from S^2 to $\mathbb{R}^2$

From action (3.13) and partition function (3.23), one can extract exact correlation functions of the twisted chiral primaries,

$$\begin{aligned} \frac{1}{Z_A[S^2]} \partial_{\tilde{\tau}^{\hat{I}_{\alpha}}} \partial_{\bar{\tilde{\tau}}^{\hat{J}_{\beta}}} Z_A[S^2] \Big|_{\tilde{\tau}^{\hat{I}_{\gamma}} = \bar{\tilde{\tau}}^{\hat{I}_{\gamma}} = 0, \gamma \geq 2} &= \left\langle \left(\frac{1}{2} \int_{S^2} d^2x \sqrt{g} \left(\mathcal{O}_{\hat{I}_{\alpha}}(x) - (\omega_{\hat{I}_{\alpha}} - 1) \sigma_{\hat{I}_{\alpha}}(x) \right) \right. \right. \\ &\quad \left. \left. \left(\frac{1}{2} \int_{S^2} d^2y \sqrt{g} \left(\bar{\mathcal{O}}_{\hat{J}_{\beta}}(y) - (\omega_{\hat{J}_{\beta}} - 1) \bar{\sigma}_{\hat{J}_{\beta}}(y) \right) \right) \right\rangle_{S^2} \\ &= -4\pi^2 \left\langle \sigma_{\hat{I}_{\alpha}}(N) \bar{\sigma}_{\hat{J}_{\beta}}(S) \right\rangle_{S^2} \end{aligned} \quad (3.33)$$

where $\sigma_{\hat{I}}$ and $\mathcal{O}_{\hat{I}}$ are bottom and top terms of the twisted chiral primary multiplets $\Sigma_{\hat{I}}$ with dimension $\Delta_{\hat{I}} = \omega_{\hat{I}}$, index $\hat{I}, \hat{J}$ labels all twisted chiral ring generators. The second equality is due to the $\mathfrak{su}(2|1)_A$ supersymmetric Ward identity [14]. In fact, taking derivative respect to $\tilde{\tau}_n$ on eq. (3.32), we have

$$\partial_{\tilde{\tau}_n} Z_A = \frac{1}{N!} \sum_{\{m_i\} \in \mathbb{Z}} \int_{-\infty}^{\infty} \prod_{i=1}^N \frac{d\sigma_i}{2\pi} (-2\pi \text{Tr } \sigma_c^n) Z_{\text{cl}}(\sigma_c, \bar{\sigma}_c) Z_{\text{gauge}}(\sigma_c, \bar{\sigma}_c) \prod_{\Phi} Z_{\Phi}(\sigma_c, \bar{\sigma}_c).$$

Noticing that σ_c in eq. (3.30) is exactly the BPS solution evaluated at north pole [15, 21], we indeed have

$$\langle -2\pi \text{Tr } \sigma^n(N) \rangle_{S^2} = \frac{1}{Z_A[S^2]} \partial_{\tilde{\tau}_n} Z_A[S^2] \Big|_{\tilde{\tau}_n = \bar{\tilde{\tau}}_n = 0, n \geq 2} \quad (3.34)$$

Similarly,

$$\langle 2\pi \text{Tr } \bar{\sigma}^n(S) \rangle_{S^2} = \frac{1}{Z_A[S^2]} \partial_{\bar{\tilde{\tau}}_n} Z_A[S^2] \Big|_{\tilde{\tau}_n = \bar{\tilde{\tau}}_n = 0, n \geq 2},$$

for $\bar{\sigma}_c$ in eq. (3.30) is evaluated at south pole.

It has been throughout analyzed in [14, 17] that eq. (3.33) is the consequence of the unique regularization scheme respect to $\mathfrak{su}(2|1)_A$ supersymmetries on S^2 . Alternatively, one can understand it as that, to regularize the partition function Z_A as well as correlators on S^2 respect to $\mathfrak{su}(2|1)_A$, one has to introduce counter terms combined with the $\mathcal{N} = (2, 2)$ supergravity multiplet $\mathfrak{R}$ preserving $\mathfrak{su}(2|1)_A$,

$$\Gamma_{\text{c.t.}} = \frac{1}{2} \int d^2x d^2\tilde{\theta} \mathcal{E} \mathfrak{R} \mathcal{F}(\tilde{\tau}) + \text{h.c.}, \quad (3.35)$$

where the supergravity multiplet $\mathfrak{R}$ has dimension $\Delta_{\mathfrak{R}} = 1$ containing Ricci scalar curvature as the top component, $\mathcal{E}$ is the density in curved twisted superspace, and $\mathcal{F}(\tilde{\tau})$ is a holomorphic function in terms of couplings $\{\tilde{\tau}^{\hat{I}}\}$.

More importantly, it is the multiplet $\mathfrak{R}$ that mixes twisted chiral operator Σ with other lower dimensional operators [4], i.e.

$$\Sigma_{\Delta} \longrightarrow \Sigma_{\Delta} + \alpha_1(\tilde{\tau}, \bar{\tilde{\tau}}; \Delta) \mathfrak{R} \Sigma_{\Delta-1} + \alpha_2(\tilde{\tau}, \bar{\tilde{\tau}}; \Delta) \mathfrak{R}^2 \Sigma_{\Delta-2} + \cdots + \alpha_{\Delta}(\tilde{\tau}, \bar{\tilde{\tau}}; \Delta) \mathfrak{R}^{\Delta} \mathbb{1}, \quad (3.36)$$

where Δ denotes the dimension of Σ_{Δ} , and $\alpha_i(\tilde{\tau}, \bar{\tilde{\tau}}; \Delta)$ are certain coefficients presenting mixing. Similar to the $4d$ case [4], the mixing only happens among twisted chiral primaries themselves, but not to twisted chiral mixing with twisted antichiral or (anti)chiral primaries. It is because only twisted chiral primaries are $\mathfrak{su}(2|1)_A$ supersymmetric on the north pole of S^2 , whereas twisted anti-chiral primaries are supersymmetric on the south pole of S^2 , meanwhile (anti)chiral primaries respect to $\mathfrak{su}(2|1)_B$ supersymmetries instead and thus are nowhere to be put on S^2 in $\mathfrak{su}(2|1)_A$ regularization scheme. It explains the phenomenon that we observe *nonzero* correlation functions between operators with different dimensions on S^2 , see for example eq. (3.34) as a result of $\Sigma_{\hat{I}_{\alpha}}$ mixing with identity operator. Therefore, to find correct correlation functions on $\mathbb{R}^2$, we need to perform Gram-Schmidt orthogonalization to disentangle twisted (anti)chiral operators with those of lower dimensions. We will see that the Gram-Schmidt procedure is adequate to disentangle operators mixing from S^2 to $\mathbb{R}^2$, as it admits a natural geometric interpretation in Calabi-Yau geometries we will discuss in section 4, and pass both perturbative and non-perturbative checks in the examples we give in section 5. However it would be interesting and important to investigate more detailed structures of the mixing coefficients $\alpha_i(\tilde{\tau}, \bar{\tilde{\tau}}; \Delta)$ in terms of conformal anomalies as analyzed in [17] for $\alpha_1(\tilde{\tau}, \bar{\tilde{\tau}}; 1)$ corresponding to the mixing between marginal primaries and identity operator. We will leave the answer to this question in our subsequent work [18].

We now explain the algorithm to disentangle operators in somewhat detail. From eq. (3.36), it is seen that the mixing only happens to twisted chiral operators themselves with dimension separation by multiples of one, therefore we disentangle them by induction on operators' dimensions.

Since there will be many indexes appearing, we would like to summarize the notations we will use first. For a twisted chiral primary σ_{a_α} , “ α ” labels the operator's dimension, and “ a_α ” enumerate the number of the operators with dimension $\Delta = \alpha$. We will see soon that, at each dimension $\Delta = \alpha$, operators σ_{a_α} 's are in general not linear independent. We thus collect those linear independent operators and label them by σ_{I_α} . At last $\sigma_{\hat{I}_\alpha}$'s still denote all twisted ring generators as before, with $\hat{I}_\alpha$ enumerating all primitive generators with dimension $\Delta = \alpha$. For $\sigma_{\hat{I}_\alpha}$'s must be linear independent for given α , we have $\{\hat{I}_\alpha\} \subset \{I_\alpha\}$. With these notation, we start the orthogonalization from dimension zero.

$\Delta = 0$. The unit operator $\mathbb{1}$ is dimension zero and need no change.

$\Delta = 1$. We have marginal twisted chiral primaries $\sigma_{a_1} \equiv \sigma_i$. For every primary specify a direction to marginally deform the SCFT, they are all linear independent, i.e. the index sets $\{I_1\}$, $\{a_1\}$ and $\{i\}$ identical. They are required to be orthogonal to unit operator $\mathbb{1}$. We thus define

$$\hat{\sigma}_{a_1} \equiv \sigma_{a_1} - \frac{\langle \sigma_{a_1}(N) \bar{\mathbb{1}}(S) \rangle_{S^2}}{\langle \mathbb{1}(N) \bar{\mathbb{1}}(S) \rangle_{S^2}} \mathbb{1}, \quad \hat{\sigma}_{b_1} \equiv \bar{\sigma}_{b_1} - \frac{\langle \mathbb{1}(N) \bar{\sigma}_{b_1}(S) \rangle_{S^2}}{\langle \mathbb{1}(N) \bar{\mathbb{1}}(S) \rangle_{S^2}} \bar{\mathbb{1}} \quad (3.37)$$

One can check indeed

$$\langle \hat{\sigma}_{a_1}(N) \bar{\mathbb{1}}(S) \rangle_{S^2} = \langle \mathbb{1}(N) \hat{\sigma}_{b_1}(S) \rangle_{S^2} = 0, \quad (3.38)$$

and the twisted chiral ring data

$$\begin{aligned} g_{a_1 \bar{b}_1}^{(1)} &\equiv \langle \sigma_{a_1}(0) \bar{\sigma}_{b_1}(\infty) \rangle_{\mathbb{R}^2} = \langle \hat{\sigma}_{a_1}(N) \hat{\sigma}_{b_1}(S) \rangle_{S^2} \\ &= \langle \sigma_{a_1}(N) \bar{\sigma}_{b_1}(S) \rangle_{S^2} - \frac{\langle \sigma_{a_1}(N) \bar{\mathbb{1}}(S) \rangle_{S^2} \langle \mathbb{1}(N) \bar{\sigma}_{b_1}(S) \rangle_{S^2}}{\langle \mathbb{1}(N) \bar{\mathbb{1}}(S) \rangle_{S^2}} \\ &= -\partial_i \partial_{\bar{j}} \log Z_A[S^2] \equiv g_{I_1 \bar{J}_1}^{(1)}, \end{aligned} \quad (3.39)$$

is exactly the Zamolodchikov metric of moduli space $\mathcal{M}$ of the SCFT.

$\Delta = 2$. We define

$$\hat{\sigma}_{a_2} \equiv \sigma_{a_2} - \frac{\langle \sigma_{a_2}(N) \bar{\mathbb{1}}(S) \rangle}{\langle \mathbb{1}(N) \bar{\mathbb{1}}(S) \rangle} \mathbb{1} - \sum_{I_1, \bar{J}_1} g^{(1) \bar{J}_1 I_1} \langle \sigma_{a_2}(N) \hat{\sigma}_{J_1}(S) \rangle \hat{\sigma}_{I_1}, \quad (3.40)$$

where $g^{(1) \bar{J}_1 I_1}$ is the inverse of metric $g_{I_1 \bar{J}_1}^{(1)}$, and so can be defined for $\hat{\sigma}_{b_2}$. One can firmly check that $\hat{\sigma}_{a_2}$ is orthogonal to all operators with dimension less than two, say $\{1, \hat{\sigma}_{I_1}\}$. The twisted chiral ring data on vector bundle $\mathcal{V}_2$ over $\mathcal{M}$ is computed by,

$$g_{a_2 \bar{b}_2}^{(2)} \equiv \langle \sigma_{a_2}(0) \bar{\sigma}_{b_2}(\infty) \rangle_{\mathbb{R}^2} = \langle \hat{\sigma}_{a_2}(N) \hat{\sigma}_{b_2}(S) \rangle_{S^2}. \quad (3.41)$$

The eq. (3.39) and (3.41) automatically satisfy the tt^* -equation (2.40) [4], where we choose the “shift” basis⁵ for OPE coefficient C_{IJ}^K , i.e.

$$\sigma_{I_1}(x)\sigma_{J_1}(0) = \sigma_{I_1}\sigma_{J_1}(0) \equiv \sigma_{c_2}(0). \quad (3.42)$$

We will stick to this basis for all operators with arbitrary dimensions. However as emphasized before, the $2d$ (twisted) chiral ring is finite and nilpotent. Therefore with this “shift” basis, we will obtain too many operators which turns out to be *linear dependent*. From level $\Delta = 3$ we may encounter this problem. We thus need to collect those linear independent to continue orthogonalization.

$\Delta = 3$. We want to continue the disentanglement as we have done at $\Delta = 0, 1, 2$. However the metric (3.41) may be singular in general. In section 4 and 5 we will give such examples. The singular $g_{a_2\bar{b}_2}$ implies that not all $\hat{\sigma}_{a_2}$ of dimension two are linear independent. So we collect some of them to form a *maximal* linear independent set, which includes all primitive generators of dimension two, and those generated by generators of dimension one. Assume we have picked such a set

$$\mathcal{A}_2 = \{\hat{\sigma}_{I_2}\},$$

and computed the corresponding metric $g_{I_2\bar{J}_2}^{(2)}(\mathcal{A}_2)$. Now $g_{I_2\bar{J}_2}^{(2)}(\mathcal{A}_2)$ is non-singular and invertible. We can use its inverse $g^{(2)\bar{J}_2 I_2}(\mathcal{A}_2)$ to continue our orthogonalization for all operators σ_{a_3} of $\Delta = 3$,

$$\begin{aligned} \hat{\sigma}_{a_3} \equiv \sigma_{a_3} &- \frac{\langle \sigma_{a_3}(N) \bar{\mathbb{1}}(S) \rangle_{S^2}}{\langle \mathbb{1}(N) \bar{\mathbb{1}}(S) \rangle_{S^2}} \mathbb{1} - \sum_{I_1, \bar{J}_1} g^{(1)\bar{J}_1 I_1} \langle \sigma_{a_3}(N) \hat{\sigma}_{J_1}(S) \rangle_{S^2} \hat{\sigma}_{I_1} \\ &- \sum_{I_2, \bar{J}_2 \in \mathcal{A}_2} g^{(2)\bar{J}_2 I_2}(\mathcal{A}_2) \langle \sigma_{a_3}(N) \hat{\sigma}_{J_2}(S) \rangle_{S^2} \hat{\sigma}_{I_2}. \end{aligned} \quad (3.43)$$

Now $\hat{\sigma}_{a_3}$ is orthogonal to all lower dimensional operators, $\{1, \sigma_{a_1}, \sigma_{a_2}\}$.

At last we need show such construction does *not* depend on the choice of $\mathcal{A}_2$. If we choose another maximal linear independent set $\mathcal{A}'_2 = \{\hat{\sigma}_{I'_2}\}$, one can always find the linear transformation $\mathcal{T}$ relating them, and their hermitian conjugates,

$$\hat{\sigma}_{I'_2} = \mathcal{T}_{I'_2}^{I_2} \hat{\sigma}_{I_2}, \quad \hat{\sigma}_{J'_2} = \overline{\mathcal{T}_{J'_2}^{J_2}} \hat{\sigma}_{J_2}, \quad (3.44)$$

where $\overline{\mathcal{T}_{J'_2}^{J_2}}$ is the complex conjugate of $\mathcal{T}_{J'_2}^{J_2}$. Correspondingly the inverse of metric transforms as

$$g^{(2)\bar{J}'_2 I'_2}(\mathcal{A}'_2) = (\mathcal{T}^{-1})_{I'_2}^{I_2} \overline{(\mathcal{T}^{-1})_{J'_2}^{J_2}} g^{(2)\bar{J}_2 I_2}(\mathcal{A}_2). \quad (3.45)$$

Therefore we show that eq. (3.43) is indeed independent of the choice of $\mathcal{A}$. Based on the new set of $\{\hat{\sigma}_{a_3}\}$, we can compute the twisted chiral ring data on $\mathcal{V}_3$ over $\mathcal{M}$,

$$g_{a_3\bar{b}_3}^{(3)} \equiv \langle \sigma_{a_3}(0) \bar{\sigma}_{b_3}(\infty) \rangle_{\mathbb{R}^2} = \langle \hat{\sigma}_{a_3}(N) \hat{\sigma}_{b_3}(S) \rangle_{S^2}, \quad (3.46)$$

which can be shown again satisfying the tt^* -equation (2.40) up to $\Delta = 3$

⁵It is named “diagonal” basis in [4] for $4d$ chiral ring is freely generated. The nilpotency of $2d$ chiral rings actually realizes the OPE coefficients C as shift matrices.

$\Delta = n$. For generic n , by induction, we have orthogonalized operators up to $\Delta = n - 1$. Among operators $\hat{\sigma}_{a_\alpha}$ of $\Delta = \alpha$, we can collect a maximal linear independent set $\mathcal{A}_\alpha$ for $\alpha = 0, 1, \dots, n - 1$, where $\mathcal{A}_0 = \{\mathbb{1}\}$ and $\mathcal{A}_1 = \{\hat{\sigma}_{I_1}\}$. We compute their metrics $g_{I_\alpha \bar{J}_\alpha}$ and inverse $g^{\bar{J}_\alpha I_\alpha}$, and define

$$\hat{\sigma}_{a_n} \equiv \sigma_{a_n} - \sum_{\alpha=0}^{n-1} \sum_{I_\alpha, \bar{J}_\alpha \in \mathcal{A}_\alpha} g^{(\alpha) \bar{J}_\alpha I_\alpha}(\mathcal{A}_\alpha) \langle \sigma_{a_n}(N) \hat{\sigma}_{J_\alpha}(S) \rangle_{S^2} \hat{\sigma}_{I_\alpha}, \quad (3.47)$$

where $g^{(0) \bar{J}_0 I_0} \equiv (\langle \mathbb{1}(N) \bar{\mathbb{1}}(S) \rangle_{S^2})^{-1}$. It allows us to compute the twisted chiral ring data at level $\Delta = n$ for bundle $\mathcal{V}_n$ over $\mathcal{M}$,

$$\begin{aligned} g_{a_n \bar{b}_n}^{(n)} &\equiv \langle \sigma_{a_n}(0) \bar{\sigma}_{b_n}(\infty) \rangle_{\mathbb{R}^2} = \langle \hat{\sigma}_{a_n}(N) \hat{\bar{\sigma}}_{b_n}(S) \rangle_{S^2} \\ &= \langle \sigma_{a_1}(N) \bar{\sigma}_{b_1}(S) \rangle_{S^2} - \sum_{\alpha=0}^{n-1} \sum_{I_\alpha, \bar{J}_\alpha \in \mathcal{A}_\alpha} g^{(\alpha) \bar{J}_\alpha I_\alpha}(\mathcal{A}_\alpha) \langle \sigma_{a_n}(N) \hat{\sigma}_{J_\alpha}(S) \rangle_{S^2} \langle \hat{\sigma}_{I_\alpha}(N) \bar{\sigma}_{b_n}(S) \rangle_{S^2} \end{aligned} \quad (3.48)$$

In practice, since we have showed that eq. (3.47) and (3.48) are independent of the choice of $\mathcal{A}_\alpha$, one can freely choose any convenient set of $\{\mathcal{A}_\alpha\}$ to perform the orthogonalization. We therefore for convenience pick up the original operators $\{\sigma_{I_\alpha}\}$, instead of $\{\hat{\sigma}_{I_\alpha}\}$ in all sets of $\{\mathcal{A}_\alpha\}$, to perform the orthogonalization.

Now we summarize the algorithm. For a given set of primitive generators in twisted chiral ring, $\mathcal{G} = \{\mathbb{1}, \sigma_{\hat{I}_1}, \sigma_{\hat{I}_2}, \dots, \sigma_{\hat{I}_A}\}$, where A labels the maximal dimension of the generators, we choose the “shift” OPE basis as before

$$\sigma_{\hat{I}_\alpha}(x) \sigma_{\hat{J}_\beta}(0) = \sigma_{\hat{I}_\alpha} \sigma_{\hat{J}_\beta}(0), \quad (3.49)$$

where on r.h.s., $\sigma_{\hat{I}_\alpha} \sigma_{\hat{J}_\beta}$ stands for an element with dimension $\Delta = \alpha + \beta$ in the ring. Under this basis, we collect all possible elements and arrange them by dimensions,

$$\left\{ \mathbb{1}; \left\{ \sigma_{\hat{I}_1} \right\}; \left\{ \sigma_{\hat{I}_1} \sigma_{\hat{J}_1}, \sigma_{\hat{I}_2} \right\}; \left\{ \sigma_{\hat{I}_1} \sigma_{\hat{J}_1} \sigma_{\hat{K}_1}, \sigma_{\hat{I}_2} \sigma_{\hat{J}_1}, \sigma_{\hat{I}_3} \right\}; \dots \right\}, \quad (3.50)$$

which can be uniquely expressed by the primitive generators in $\mathcal{G}$. In above we do not take account of any equivalent relations among the generators, and treat all elements as freely generated, which is the reason that many of them are linear dependent. For any two of them,

$$\sigma_{a_\alpha} \equiv \sigma_{\{n_{\hat{I}_\gamma}\}} = \prod_{\gamma=1}^A \prod_{\hat{I}_\gamma} \sigma_{\hat{I}_\gamma}^{n_{\hat{I}_\gamma}}, \quad \sigma_{b_\beta} \equiv \sigma_{\{n'_{\hat{I}_\gamma}\}} = \prod_{\gamma=1}^A \prod_{\hat{I}_\gamma} \sigma_{\hat{I}_\gamma}^{n'_{\hat{I}_\gamma}}, \quad (3.51)$$

we can compute their correlator from Z_A , the partition function (3.23),

$$\begin{aligned} M_{a_\alpha \bar{b}_\beta} &\equiv \langle \sigma_{a_\alpha}(N) \bar{\sigma}_{b_\beta}(S) \rangle_{S^2} \\ &= \frac{1}{Z_A} \prod_{\gamma=0}^A \prod_{\hat{I}_\gamma, \bar{\hat{J}}_\gamma} \left(-\frac{1}{2\pi} \frac{\partial}{\partial \tau^{\hat{I}_\gamma}} \right)^{n_{\hat{I}_\gamma}} \left(\frac{1}{2\pi} \frac{\partial}{\partial \bar{\tau}^{\bar{\hat{J}}_\gamma}} \right)^{n'_{\bar{\hat{J}}_\gamma}} Z_A \Big|_{\tau^{\hat{K}_\delta} = \bar{\tau}^{\bar{\hat{K}}_\delta} = 0, \delta \geq 2} \end{aligned} \quad (3.52)$$

where we use matrix $M_{a_\alpha \bar{b}_\beta}$ to relabel all these correlators, where as before α, β denote the dimension of operators, and a_α, b_β enumerate all operators of dimension α and β . Since we

have argued that, in general, M is a singular matrix and not invertible, we have to *remove all but one* of the rows and columns corresponding to those linear dependent operators. One can perform this operation level by level respect to the dimensions, $\Delta = n - 1$, of the operators, and finally obtain a matrix

$$\widetilde{M}_{n-1, I_\alpha \bar{J}_\beta} = \langle \sigma_{I_\alpha}(N) \bar{\sigma}_{J_\beta}(S) \rangle_{S^2}, \quad (3.53)$$

where $I_\alpha, \bar{J}_\beta$ denote only those linear independent operators up to dimension Δ . Since we also showed that orthogonalization does not depend on the choice of $\{\mathcal{A}_\alpha\}$, we can use $\widetilde{M}_{n-1}$, instead of $g_{I_\alpha \bar{J}_\beta}$, in eq. (3.47), i.e.

$$\hat{\sigma}_{a_n} = \sigma_{a_n} - \sum_{\alpha, \beta=0}^{n-1} \sum_{I_\alpha, \bar{J}_\beta} \widetilde{M}_{n-1}^{\bar{J}_\beta I_\alpha} \langle \sigma_{a_n}(N) \bar{\sigma}_{J_\beta}(S) \rangle_{S^2} \sigma_{I_\alpha}, \quad (3.54)$$

where $\widetilde{M}_{n-1}^{\bar{J}_\beta I_\alpha}$ is the inverse of $\widetilde{M}_{n-1}$. Similarly, instead of using eq. (3.48), for any two elements σ_{c_n} and $\bar{\sigma}_{d_m}$ with dimension n and m , their correlator can be expressed in terms of $\widetilde{M}_{n-1}^{\bar{J}_\beta I_\alpha}$ from eq. (3.54) as well. Finally we have

$$\begin{aligned} g_{c_n \bar{d}_m}^{(n)} &\equiv \delta_{nm} \langle \sigma_{c_n}(0) \bar{\sigma}_{d_n}(\infty) \rangle_{\mathbb{R}^2} = \delta_{nm} \langle \hat{\sigma}_{c_n}(N) \hat{\sigma}_{d_n}(S) \rangle_{S^2} \\ &= \delta_{nm} \left(\langle \sigma_{c_n}(N) \bar{\sigma}_{d_n}(S) \rangle_{S^2} \right. \\ &\quad \left. - \sum_{\alpha, \beta=0}^{n-1} \sum_{I_\alpha, \bar{J}_\beta} \langle \sigma_{c_n}(N) \bar{\sigma}_{J_\beta}(S) \rangle_{S^2} \widetilde{M}_{n-1}^{\bar{J}_\beta I_\alpha} \langle \sigma_{I_\alpha}(N) \bar{\sigma}_{d_n}(S) \rangle_{S^2} \right). \end{aligned} \quad (3.55)$$

Eq. (3.55) is the main formula that will be applied later. It automatically satisfies the tt^* -equations as eq. (3.41) and (3.46). However since we choose the OPE basis eq. (3.49) regardless of the equivalence relations in the rings, there would be additional constraints imposed on these metrics $g_{c_n \bar{d}_m}$. We will discuss these constraints next section in the context of Calabi-Yau manifolds.

4 Chiral ring data in Calabi-Yau manifolds

In this section, we will consider d -(complex) dimensional compact Calabi-Yau manifolds as our important examples of $\mathcal{N} = (2, 2)$ SCFTs with center charge $c = 3d$. Discussion in the first two subsections is focused on the complex moduli space of CY-manifolds with one complex dimension, or say the chiral ring generated by a single marginal chiral primary, then generalized to complex moduli of higher dimensions. We will reconstruct tt^* -equations and their solutions, the chiral ring data, via variation of the Hodge structure of the horizontal cohomology classes in Calabi-Yau manifolds. In fact the equivalence between geometries of Hodge structure and tt^* -equations has been long time known [9]. The chiral ring data are uniquely determined by tt^* -equations if we have full OPE data C_{IJ}^K which can be obtained from topological twist [25–29].

On the other hand, in this note, we resort to an alternative route to find chiral ring data, i.e. extracting them directly from partition functions with deformation eq. (3.32). In this scenario, it is *not* necessary to know any OPE data, which allows us to simply work under the “shift” OPE basis (3.49). However the price we pay is that we are blind for equivalence relations in the chiral rings, and we have to collect linear independent operators out of excessively many linear dependent ones. The algorithm developed in section 3 resolves this problem. It therefore must be compatible with tt^* -equations as well as the geometries of Hodge structure in the context of CY-manifolds.

Indeed, in this section, we will show that the Gram-Schmidt orthogonalization in the algorithm admits a natural geometric interpretation as Griffith transversality with projections [22–24] on the Hodge structure of CY-manifolds, and there will be more constraints imposed on the chiral ring data if we use the “shift” OPE basis (3.49). On the contrary, Griffiths transversality with projection on generic complex moduli of higher dimensions can be reciprocally defined with help of the algorithm as a spinoff in section 4.3.

4.1 Variation of Hodge structure

For a given d -dimensional Calabi-Yau manifold $\mathcal{Y}$, its metric deformations fall into two families: Kähler class and complex structure deformations, both of which form the moduli spaces of $\mathcal{Y}$. The Kähler and complex moduli, labeled as $\mathcal{M}_K(\mathcal{Y})$ and $\mathcal{M}_C(\mathcal{Y})$, themselves are Kähler manifolds⁶ with dimensions $h^{1,1}(\mathcal{Y})$ and $h^{d-1,1}(\mathcal{Y})$, the non-trivial Hodge numbers of $\mathcal{Y}$. On the other hand, a supersymmetric sigma model $\mathcal{S}$ with the target space of $\mathcal{Y}$ is a $\mathcal{N} = (2, 2)$ SCFT with center charge $c = 3d$. The moduli space $\mathcal{M}_c(\mathcal{S})$ spanned by marginal primaries coincides with $\mathcal{M}_C(\mathcal{Y})$, and $\mathcal{M}_{tc}(\mathcal{S})$ with $\mathcal{M}_K(\mathcal{Y})$.

In fact the identification between $\mathcal{M}_c$ and $\mathcal{M}_C$ can be extended to the chiral ring bundle $\mathcal{V}$ over $\mathcal{M}_c$ with the fiber of chiral ring $\mathcal{R}$, and the Hodge bundle $\mathcal{H}$ over $\mathcal{M}_C$, via a bundle map φ ,

$$\varphi: \mathcal{V} \simeq \mathcal{H}, \quad (4.1)$$

where $\mathcal{H}$ has fibers $H^d(\mathcal{Y}, \mathbb{C})$, i.e. the horizontal cohomology classes of $\mathcal{Y}$. We will show, in the case of $\mathcal{M}_C$ being one-dimensional, that the chiral ring data in $\mathcal{V}$ can be computed via the canonical non-degenerate bilinear form on the fiber $H^d(\mathcal{Y}, \mathbb{C})$ once after we specify the isomorphism φ . Below we will explain in concrete how the φ can be introduced by identifying elements in the chiral ring $\mathcal{R}$ and $H^d(\mathcal{Y}, \mathbb{C})$ on a fixed fiber of $\mathcal{V}$ and $\mathcal{H}$.

A given point τ on $\mathcal{M}_C$ specifies a complex structure of $\mathcal{Y}$, we therefore have a Hodge decomposition of the fiber

$$H^d(\mathcal{Y}_\tau, \mathbb{C}) \simeq \bigoplus_{\alpha+\beta=d} H^{\beta,\alpha}(\mathcal{Y}_\tau), \quad (4.2)$$

respect to the complex structure. A holomorphic d -form $\Omega(\tau)$ spanning $H^{d,0}(\mathcal{Y}_\tau)$ is a natural section over a line bundle $\mathcal{L} \subset \mathcal{H}$. We want to consider how $\Omega(\tau)$ varies respect to τ . It turns out that moving on the moduli space $\mathcal{M}_C$ will change complex structure,

⁶We will not consider CY-manifolds like $K3$ surface which corresponds to a $\mathcal{N} = (4, 4)$ SCFT with a homogenous moduli space.

so that $\partial_\tau \Omega$ will not be in $H^{d,0}(\mathcal{Y}_\tau)$. In general, for any element in $H^{\beta,\alpha}(\mathcal{Y}_\tau)$, its variation respect to τ will not be an element in $H^{\beta,\alpha}(\mathcal{Y}_\tau)$ anymore. The inherent conflict between the basis of $H^{\beta,\alpha}(\mathcal{Y}_\tau)$ and that varying holomorphically on τ has been recognized since the work of Griffiths [22]. To circumvent this conflict, instead of working on $H^{\beta,\alpha}(\mathcal{Y}_\tau)$, one considers the Hodge filtration $\mathcal{F}^\bullet(\mathcal{Y}) = \{\mathcal{F}^\beta(\mathcal{Y})\}_{\beta=0}^d$ as

$$\mathcal{F}^\beta = \bigoplus_{a \geq \beta} H^{a,d-a}(\mathcal{Y}), \quad H^d(\mathcal{Y}, \mathbb{C}) = \mathcal{F}^0 \supset \mathcal{F}^1 \supset \dots \supset \mathcal{F}^{d+1} = 0. \quad (4.3)$$

Now the variation ∂_τ equipping on $\mathcal{F}^\beta$ is a flat connection, or say the Gauss-Manin connection, with the important property, see also [24],

$$\partial_\tau : \mathcal{F}^\beta \longrightarrow \mathcal{F}^{\beta-1}, \quad (4.4)$$

which is called Griffiths transversality. The d -form $\Omega(\tau) \in \mathcal{F}^d$ varies holomorphically on the pages $\{\mathcal{F}^\beta\}$. To project elements in $\mathcal{F}^\beta$ back to states in $H^{\beta,\alpha}(\mathcal{Y}_\tau)$, we introduce the anti-holomorphic filtration, $\overline{\mathcal{F}}^\bullet(\mathcal{Y}) = \{\overline{\mathcal{F}}^\alpha(\mathcal{Y})\}_{\alpha=0}^d$ by the virtue of $\overline{H^{\beta,\alpha}(\mathcal{Y}_\tau)} = H^{\alpha,\beta}(\mathcal{Y}_\tau)$, and we thus have

$$H^{\beta,\alpha}(\mathcal{Y}_\tau) = \mathcal{F}^\beta(\mathcal{Y}) \cap \overline{\mathcal{F}}^\alpha(\mathcal{Y}). \quad (4.5)$$

Eq. (4.5) accompanied by the canonical non-degenerate pair $\langle \cdot, \cdot \rangle$ on compact $\mathcal{Y}$ will project $\partial_\tau \Omega$ to various (β, α) -pure states in $H^{\beta,\alpha}(\mathcal{Y}_\tau)$, where

$$\langle \cdot, \cdot \rangle : H^{\beta,\alpha}(\mathcal{Y}_\tau) \times H^{d-\beta,d-\alpha}(\mathcal{Y}_\tau) \longrightarrow \mathbb{R}. \quad (4.6)$$

Especially for the d -form $\Omega(\tau)$ and its complex conjugate $\overline{\Omega}(\bar{\tau})$, we have the following result [30],

$$e^{-K(\tau, \bar{\tau})} = \langle \Omega(\tau), \overline{\Omega}(\bar{\tau}) \rangle \equiv i^{d^2} c_d \int_{\mathcal{Y}} \Omega(\tau) \wedge \overline{\Omega}(\bar{\tau}), \quad (4.7)$$

where $K(\tau, \bar{\tau})$ is the Kähler potential of moduli space $\mathcal{M}_C(\mathcal{Y})$, and c_d is a specific constant, the degree of $\mathcal{M}_C$. With eq. (4.4), (4.5) and (4.7), we are able to project, for example,

$$\partial_\tau \Omega \in H^{d,0} \oplus H^{d-1,1},$$

onto $H^{d-1,1}$. Explicitly, we decompose

$$\partial_\tau \Omega = \omega \Omega + X^{(d-1,1)},$$

where $X^{(d-1,1)} \in H^{d-1,1}$ and ω is about to be determined. Wedging $\overline{\Omega}(\bar{\tau})$ and by the virtue of its anti-holomorphy, we find

$$X^{(d-1,1)} = \partial_\tau \Omega - \frac{\langle \partial_\tau \Omega, \overline{\Omega} \rangle}{\langle \Omega, \overline{\Omega} \rangle} \Omega = (\partial_\tau + \partial_\tau K) \Omega \equiv D_\tau \Omega \in H^{d-1,1}(\mathcal{Y}_\tau), \quad (4.8)$$

where

$$D_\tau : \mathcal{L} \rightarrow \mathcal{L} \otimes \mathcal{T}^* \mathcal{M}_C$$

is defined as a covariant derivative from bundle $\mathcal{L}$ to $\mathcal{L} \otimes \mathcal{T}^* \mathcal{M}_C$ with fiber $H^{d-1,1}(\mathcal{Y}_\tau)$. One can further express the metric of $\mathcal{M}_C$ as

$$g_{\tau\bar{\tau}} = \partial_\tau \partial_{\bar{\tau}} K = -\frac{\langle D_\tau \Omega, \overline{D_\tau \Omega} \rangle}{\langle \Omega, \overline{\Omega} \rangle} = -i^{d^2} c_d e^K \int_{\mathcal{Y}} D_\tau \Omega \wedge \overline{D_\tau \Omega}. \quad (4.9)$$

Eq. (4.8) and (4.9) imply that the bundle map φ , on a fixed fiber, is specified as

$$\varphi : \mathbf{1} \mapsto \Omega, \quad \phi \mapsto D_\tau \Omega, \quad (4.10)$$

up to a scaling factor, where ϕ is the marginal primary in the chiral ring $\mathcal{R}$, meanwhile the OPE in $\mathcal{R}$ should be identified to the covariant derivative D_τ , see also [23].

Acting an additional ∂_τ onto $\partial_\tau \Omega$, by Griffith transversality we have

$$\partial_\tau^2 \Omega = \omega_1 \Omega + \omega_2 D_\tau \Omega + X^{(d-2,2)},$$

such that $X^{(d-2,2)} \in H^{d-2,2}$. Successively wedging $\overline{\Omega}$ and $\overline{D_\tau \Omega}$, after some algebra, one arrives

$$\begin{aligned} X^{(d-2,2)} &= \partial_\tau^2 \Omega - \frac{\langle \partial_\tau^2 \Omega, \overline{\Omega} \rangle}{\langle \Omega, \overline{\Omega} \rangle} \Omega - \frac{\langle \partial_\tau^2 \Omega, \overline{D_\tau \Omega} \rangle}{\langle D_\tau \Omega, \overline{D_\tau \Omega} \rangle} D_\tau \Omega \\ &= (\partial_\tau + \partial_\tau K - \Gamma_{\tau\tau}^\tau) D_\tau \Omega \equiv D_\tau (D_\tau \Omega) \in H^{d-2,2}(\mathcal{Y}_\tau), \end{aligned} \quad (4.11)$$

where $\Gamma_{\tau\tau}^\tau$ is the Christoffel connection respect to metric $g_{\tau\bar{\tau}}$, and

$$D_\tau : \mathcal{L} \otimes \mathcal{T}^* \mathcal{M}_C \rightarrow \mathcal{L} \otimes \text{Sym}_2 \mathcal{T}^* \mathcal{M}_C \text{ with fiber } H^{d-2,2}(\mathcal{Y}_\tau).$$

The state $D_\tau^2 \Omega$ in $H^{d-2,2}(\mathcal{Y}_\tau)$ is identified to $\phi \cdot \phi \equiv \phi^2$ according to our OPE basis.

One can repeat the process by successively acting ∂_τ and projection to find more pure states in $H^{d-\alpha,\alpha}(\mathcal{Y}_\tau)$ with $\alpha = 0, 1, \dots, d$,

$$X^{(d-\alpha,\alpha)} = \partial_\tau^\alpha \Omega - \sum_{\beta=0}^{\alpha-1} \frac{\langle \partial_\tau^\alpha \Omega, \overline{D_\tau^\beta \Omega} \rangle}{\langle D_\tau^\beta \Omega, \overline{D_\tau^\beta \Omega} \rangle} D_\tau^\beta \Omega \equiv D_\tau^\alpha \Omega \in H^{d-\alpha,\alpha}(\mathcal{Y}_\tau), \quad (4.12)$$

which hence specifies the bundle map

$$\varphi|_{\mathcal{R}} : \{ \mathbf{1}, \phi, \phi^2, \dots, \phi^d \} \mapsto \{ \Omega, D_\tau \Omega, D_\tau^2 \Omega, \dots, D_\tau^d \Omega \}. \quad (4.13)$$

Comparing eq. (4.8), (4.11) as well as eq. (4.12) to eq. (3.37), (3.40) and (3.47), we find that the Griffiths transversality with projection naturally gives rise to the Gram-Schmidt orthogonalization we developed in last section. The reason behind is actually simple: first, from the geometric perspective, the bundle map φ gives correct correspondence between chiral ring operators ϕ^α and cohomology states $D_\tau^\alpha \Omega$ in $H^{d-\alpha,\alpha}(\mathcal{Y}_\tau)$, both of which are graded in dimension and degree α . Therefore the correct chiral ring data will be expressed in terms of these states. On the other hand, the partition function Z_B computed via localization respect to $\mathfrak{su}(2|1)_B$ is exactly eq. (4.7) [19, 20],

$$Z_B[S^2] = e^{-K(\tau,\bar{\tau})} = \langle \Omega(\tau), \overline{\Omega}(\bar{\tau}) \rangle \equiv i^{d^2} c_d \int_{\mathcal{Y}} \Omega(\tau) \wedge \overline{\Omega}(\bar{\tau}). \quad (4.14)$$

Overall, to produce correct correlators from $Z_B[S^2]$, one has to apply Griffiths transversality with projection respect to the states' degree α , which is nothing more than the Gram-Schmidt orthogonalization respect to the operators' dimension α .

4.2 tt^* -equations of chiral ring data on complex moduli

Let us now work out in detail the tt^* -equations of chiral ring data on one-dimensional complex moduli. It turns out to be the Toda chain equations with constraints. The derivation is only based on the orthogonality of the pure states $D_\tau^\alpha \Omega \in H^{d-\alpha, \alpha}(\mathcal{Y}_\tau)$, or say the chiral primaries. In this sense, the tt^* -equations that the chiral ring data need to satisfy are universal for both $2d$ and $4d$ cases [4]. However with the help of geometry, we will see that there are more constraints that the $2d$ ones must obey, which as we emphasized is due to the nilpotency of the $2d$ chiral ring.

For simplicity, we label the chiral ring data as

$$g^{(\alpha)} \equiv (-1)^\alpha \frac{\langle \phi_\alpha, \overline{\phi_\alpha} \rangle}{\langle \phi_0, \overline{\phi_0} \rangle}, \quad \text{with } \phi_\alpha \equiv D_\tau^\alpha \Omega, \quad \alpha = 0, 1, \dots, d. \quad (4.15)$$

Before establishing equations on g_α 's, we first prove a useful lemma that

$$\text{lemma : } \partial_{\bar{\tau}} \phi_\alpha \in \bigoplus_{\beta=1}^{\alpha} H^{d-\alpha+\beta, \alpha-\beta}(\mathcal{Y}_\tau), \quad \text{for } \phi_\alpha \in H^{d-\alpha, \alpha}(\mathcal{Y}_\tau). \quad (4.16)$$

It can be shown from eq. (4.12) by the holomorphicity of $\partial_\tau^q \Omega$ and induction on q . Next we show that

$$\partial_\tau \phi_\alpha \in H^{d-\alpha, \alpha}(\mathcal{Y}_\tau) \oplus H^{d-\alpha-1, \alpha+1}(\mathcal{Y}_\tau). \quad (4.17)$$

From eq. (4.12) and Griffiths transversality, $\partial_\tau \phi_\alpha \in \bigoplus_{\beta=0}^{\alpha+1} H^{d-\beta, \beta}$. Further by wedging $\overline{\phi_\beta}$ for $\beta = 0, 1, \dots, \alpha - 1$, we have

$$\langle \partial_\tau \phi_\alpha, \overline{\phi_\beta} \rangle = -\langle \phi_\alpha, \partial_\tau \overline{\phi_\beta} \rangle = 0 \quad \text{for } \beta = 0, 1, \dots, \alpha - 1,$$

where the second equality is due to lemma (4.16). Therefore we can express $\phi_{\alpha+1}$ in eq. (4.12) in terms of ϕ_α ,

$$\partial_\tau \phi_\alpha = \phi_{\alpha+1} + \Gamma_\alpha \phi_\alpha, \quad (4.18)$$

and further determine Γ_α as

$$\Gamma_\alpha = \frac{\langle \partial_\tau \phi_\alpha, \overline{\phi_\alpha} \rangle}{\langle \phi_\alpha, \overline{\phi_\alpha} \rangle} = \partial_\tau \log \langle \phi_\alpha, \overline{\phi_\alpha} \rangle = -\partial_\tau K + g^{(\alpha)-1} \partial_\tau g^{(\alpha)}, \quad (4.19)$$

where lemma (4.16) is used in the second equality. For $\alpha = 1$, Γ_1 is the standard connection on $\mathcal{L} \otimes \mathcal{T}^* \mathcal{M}_C$, see eq. (4.11), and for arbitrary α , it serves as the connection on subbundle $\mathcal{L} \otimes \text{Sym}_\alpha \mathcal{T}^* \mathcal{M}_C$. It will be seen more explicitly when treating higher dimensional moduli. We thus define the covariant derivative $D_\tau : \mathcal{L} \otimes \text{Sym}_\alpha \mathcal{T}^* \mathcal{M}_C \rightarrow \mathcal{L} \otimes \text{Sym}_{\alpha+1} \mathcal{T}^* \mathcal{M}_C$

$$\phi_{\alpha+1} = D_\tau \phi_\alpha \equiv (\partial_\tau - \Gamma_\alpha) \phi_\alpha. \quad (4.20)$$

With eq. (4.20), we have

$$\langle \phi_{\alpha+1}, \overline{\phi_{\alpha+1}} \rangle = -\langle \partial_{\bar{\tau}} D_\tau \phi_\alpha, \overline{\phi_\alpha} \rangle = \left(\sum_{\beta=0}^{\alpha} \partial_{\bar{\tau}} \Gamma_\beta \right) \langle \phi_\alpha, \overline{\phi_\alpha} \rangle, \quad (4.21)$$

where the last equality is obtained by repeatedly computing the commutator $[\partial_{\bar{\tau}}, D_{\tau}]$ and applying lemma (4.16). One can further rewrite eq. (4.21) as

$$\partial_{\bar{\tau}} \Gamma_{\alpha} = \partial_{\bar{\tau}} \partial_{\tau} \log \langle \phi_{\alpha}, \overline{\phi_{\alpha}} \rangle = \frac{\langle \phi_{\alpha+1}, \overline{\phi_{\alpha+1}} \rangle}{\langle \phi_{\alpha}, \overline{\phi_{\alpha}} \rangle} - \frac{\langle \phi_{\alpha}, \overline{\phi_{\alpha}} \rangle}{\langle \phi_{\alpha-1}, \overline{\phi_{\alpha-1}} \rangle}, \quad (4.22)$$

or in terms of $g^{(\alpha)}$'s by eq. (4.15)

$$\begin{aligned} \partial_{\bar{\tau}} \partial_{\tau} \log Z &= -g^{(1)}, \\ \partial_{\bar{\tau}} \partial_{\tau} \log g^{(\alpha)} &= \frac{g^{(\alpha)}}{g^{(\alpha-1)}} - \frac{g^{(\alpha+1)}}{g^{(\alpha)}} + g^{(1)}, \quad \text{for } 1 \leq \alpha \leq d-1, \\ \partial_{\bar{\tau}} \partial_{\tau} \log g^{(d)} &= \frac{g^{(d)}}{g^{(d-1)}} + g^{(1)}, \quad \text{with } g^{(0)} = 1 \end{aligned} \quad (4.23)$$

i.e. the celebrated Toda chain equations as one-dimensional tt^* -equations.

Now let us figure out the constraints imposed on g_{α} 's. First noticing that $\phi_d \in H^{0,d}(\mathcal{Y}_{\tau})$ is linear dependent on the anti-holomorphic d -form $\overline{\Omega} = \overline{\phi_0}$, we thus write

$$\phi_d = \mathcal{C}^{(d)} e^K \overline{\phi_0}, \quad (4.24)$$

where e^K is for convenient normalization. $\mathcal{C}^{(d)}$ is determined by wedging ϕ_0 as

$$\mathcal{C}^{(d)}(\tau) = \langle \phi_0, \phi_d \rangle = \langle \phi \cdot \phi \cdot \dots \cdot \phi \rangle_{S^2} \quad (4.25)$$

$\mathcal{C}^{(d)}$ is actually the d -point chiral correlation function computed via B-twist on S^2 [23]. Its holomorphicity on τ can be shown by acting $\partial_{\bar{\tau}}$ and use lemma (4.16). In terms of $\mathcal{C}^{(d)}$, one can relate ϕ_{α} with $\overline{\phi_{d-\alpha}}$ as

$$\phi_{\alpha} = \mathcal{C}^{(d)} e^K \left(g^{(d-\alpha)} \right)^{-1} \overline{\phi_{d-\alpha}}, \quad \overline{\phi_{\alpha}} = \overline{\mathcal{C}^{(d)}} e^K \left(g^{(d-\alpha)} \right)^{-1} \phi_{d-\alpha} \quad (4.26)$$

Therefore we have the additional constraints imposed on $g^{(\alpha)}$'s

$$g^{(\alpha)} g^{(d-\alpha)} = e^{2K} \overline{\mathcal{C}^{(d)}} \mathcal{C}^{(d)} \quad \text{for } \alpha = 1, 2, \dots, d. \quad (4.27)$$

Eq. (4.23) together with constraints (4.27) will completely determine the *full* chiral ring data of one-dimensional complex moduli under the “shift” OPE basis (3.49). The constraints (4.27) in turn will give consistency check of our computation in next section.

In the end of this subsection, we make some remarks on the tt^* -equations for states in NS and Ramond sectors. If we do not normalize the vacuum state $\mathbb{1}$ with $\langle \mathbb{1}, \overline{\mathbb{1}} \rangle = 1$, the tt^* -equations (4.22) is actually derived in Ramond sector [9]. Meanwhile the tt^* -equations in NS sector need to be modified by an additional piece in eq. (2.40), see also [7], where they derived the equations from the OPE of SCFTs without twisting the theories.

On the other hand, it has been shown in the work of [20] that, if one places the states ϕ_{α} and its complex conjugate $\overline{\phi_{\alpha}}$ on the north and south poles of the sphere and drags the sphere to an infinitely long cigar, the deformed partition function $Z_B[\text{cigar}]$ will exactly realize Cecotti and Vafa's topological-antitopological twist construction. Therefore

the unnormalized correlators $\langle \phi_\alpha, \overline{\phi_\alpha} \rangle$ are computed in Ramond sector and thus satisfy eq. (4.22). Furthermore, it is also known that partition function $Z_B[S^2] = Z_B[\text{cigar}]$ on round sphere. And computing $Z_B[S^2]$ treats fermionic states in NS sector. So we should expect that the tt^* -equations (2.40) in NS sector would be obtained with appropriate normalizations of states ϕ_α .

Indeed, the additional diagonal piece in eq. (2.40) only matters with normalization. The ambiguous normalization is encoded in the non-trivial curvature of line bundle $\mathcal{L}$ [4, 7, 23]. If the states are normalized as eq. (4.15), we will reach tt^* -equations (4.23), where the additional piece g_1 is the curvature of $\mathcal{L}$. To normalize the states in standard NS sector, we require the unique vacuum state ϕ_0 and highest chiral state ϕ_d normalized as

$$|\mathbb{1}\rangle \equiv e^{\frac{1}{2}K} \phi_0, \quad \text{and} \quad |d\rangle \equiv e^{-\frac{1}{2}K} \phi_d,$$

so that [7]

$$\langle \bar{\mathbb{1}} | \mathbb{1} \rangle = 1, \quad \text{and} \quad \langle \bar{d} | d \rangle = \overline{\mathcal{C}^{(d)}} \mathcal{C}^{(d)}.$$

All other states ϕ_α are uniformly placed between $[-\frac{K}{2}, \frac{K}{2}]$ respect to their degree q ,

$$|\alpha\rangle \equiv e^{\frac{1}{2}(1-\frac{2}{d}\alpha)K} \phi_\alpha.$$

With these normalizations, we restore the tt^* -equations (2.40) in NS sector.

4.3 Chiral ring data in complex moduli of higher dimensions

Now we generalize the previous results to the case of complex moduli of higher dimensions. The equations we will construct for chiral ring data are essentially the tt^* -equation in “shift” OPE basis (3.49). We assume that *all* primitive generators belong to $H^{d-1,1}(\mathcal{Y}_\tau)$. Similar to one-dimensional situation, we start from a holomorphic d -form

$$\phi_0 \equiv \Omega(\tau^i), \tag{4.28}$$

parametrized by moduli coordinates $\{\tau^i\}$ and spanning $H^{d,0}(\mathcal{Y}_\tau)$. States in $H^{d-\alpha,\alpha}(\mathcal{Y}_\tau)$ can be further build out via Griffiths transversality with projection. For simplicity, we consider *unnormalized* chiral ring data and construct their equations in Ramond sector by degree α .

$\alpha = 1$. Obviously, for states ϕ_i with degree one, they span the cotangent space of $\mathcal{M}_C$, where

$$\phi_i \equiv D_i \phi_0 = \partial_i \phi_0 - \frac{\langle \partial_i \phi_0, \overline{\phi_0} \rangle}{\langle \phi_0, \overline{\phi_0} \rangle} \phi_0 = (\partial_i + \partial_i K) \phi_0, \tag{4.29}$$

analogue to eq. (4.8). The unnormalized chiral ring data

$$G_{i\bar{j}}^{(1)} \equiv -\langle \phi_i, \overline{\phi_j} \rangle = \langle \partial_{\bar{j}} D_i \phi_0, \overline{\phi_0} \rangle = g_{i\bar{j}}^{(1)} \cdot \langle \phi_0, \overline{\phi_0} \rangle \equiv g_{i\bar{j}}^{(1)} G_{0\bar{0}}^{(0)},$$

where $g_{i\bar{j}}^{(1)}$ is the metric of $\mathcal{M}_C$. Instead, we can rewrite it as

$$\partial_{\bar{j}} \left(\partial_i G_{0\bar{0}}^{(0)} G^{(0)\bar{0}0} \right) = -G_{i\bar{j}}^{(1)} G^{(0)\bar{0}0}, \tag{4.30}$$

where $G^{(0)\bar{0}0}$ is the inverse of $G_{0\bar{0}}^{(0)}$. Eq. (4.33) is the tt^* -equation at degree $\alpha = 0$ in Ramond sector, see also eq. (2.40).

$\alpha = 2$. Because of the assumption that there is no primitive generators in $H^{d-2,2}(\mathcal{Y}_\tau)$, all states herein are generated by those with degree one. Similar to eq. (4.11) and eq. (4.20), let us spell them out,

$$\begin{aligned}\phi_{ij} &\equiv D_i D_j \phi_0 = \partial_i \partial_j \phi_0 - \frac{\langle \partial_i \partial_j \phi_0, \overline{\phi_0} \rangle}{\langle \phi_0, \overline{\phi_0} \rangle} \phi_0 + \sum_{k, \bar{l}} G^{(1)\bar{l}k} \langle \partial_i \partial_j \phi_0, \overline{\phi_l} \rangle \phi_k \\ &= \partial_i \phi_j - \partial_i G_{j\bar{l}}^{(1)} G^{(1)\bar{l}k} \phi_k,\end{aligned}\quad (4.31)$$

where $G^{(1)\bar{l}k}$ is the inverse of $G_{k\bar{l}}^{(1)}$ and the second line is obtained by lemma (4.16) similar to eq. (4.20). Apparently ϕ_{ij} is symmetric respect to i, j , so the covariant derivative D_i defines a map

$$D_i : \mathcal{L} \otimes \mathcal{T}^* \mathcal{M}_C \rightarrow \mathcal{L} \otimes \text{Sym}_2 \mathcal{T}^* \mathcal{M}_C \text{ with fiber } H^{d-2,2}(\mathcal{Y}_\tau).$$

The unnormalized chiral ring data

$$G_{ik, \bar{j}\bar{l}}^{(2)} \equiv \langle \phi_{ik}, \overline{\phi_{jl}} \rangle = -\langle \partial_j D_i \phi_k, \overline{\phi_l} \rangle = -\partial_j \left(\partial_i G_{k\bar{n}}^{(1)} G^{(1)\bar{n}m} \right) G_{m\bar{l}}^{(1)} - \partial_j \left(\partial_k G_{0\bar{0}}^{(0)} G^{(0)\bar{0}0} \right) G_{i\bar{l}}^{(1)}, \quad (4.32)$$

where the computation is similar to eq. (4.21). Using eq. (4.33), we obtain the tt^* -equations at degree $\alpha = 1$ in Ramond sector,

$$\partial_{\bar{j}} \left(\partial_i G_{k\bar{n}}^{(1)} G^{(1)\bar{n}m} \right) = -G_{ik, \bar{j}\bar{l}}^{(2)} G^{(1)\bar{l}m} + G_{kj}^{(1)} G^{(0)\bar{0}0} \delta_i^m. \quad (4.33)$$

$\alpha = 3$. When constructing states $\phi_{ijk} \in H^{d-3,3}(\mathcal{Y}_\tau)$, we encounter the problem that one needs to figure out the “inverse” of $G_{ik, \bar{j}\bar{l}}^{(2)}$, see also [23], which in terms of the geometric data on $\mathcal{M}_C$ is

$$G_{ik, \bar{j}\bar{l}}^{(2)} G^{(0)\bar{0}0} = g_{i\bar{j}}^{(1)} g_{k\bar{l}}^{(1)} + g_{i\bar{l}}^{(1)} g_{k\bar{j}}^{(1)} - R_{i\bar{j}k\bar{l}}^{(2)}. \quad (4.34)$$

However, analogue to the discussion in section 3.3, $\{\phi_{ij}\}$ are not necessarily linearly independent. Therefore $G_{ik, \bar{j}\bar{l}}$ might be singular and not invertible. The resolution is still to pick up a maximal set of linearly independent states, denoted as $\{\phi_{I_2}\} \subset \{\phi_{ij}\}$. Therefore the unnormalized metric on $\mathcal{L} \otimes \text{Sym}_2 \mathcal{T}^* \mathcal{M}_C$

$$G_{I_2 \bar{J}_2}^{(2)} \equiv \langle \phi_{I_2}, \overline{\phi_{J_2}} \rangle$$

can be defined, and its inverse $G^{(2)\bar{J}_2 I_2}$ exists. With the aid of $G^{(2)\bar{J}_2 I_2}$, we are able to obtain states projected onto $H^{d-3,3}(\mathcal{Y}_\tau)$,

$$\begin{aligned}\phi_{ijk} &\equiv D_i D_j D_k \phi_0 \equiv \partial_i \partial_j \partial_k \phi_0 - \sum_{\alpha=0}^2 \sum_{I_\alpha, \bar{J}_\alpha} G^{(2)\bar{J}_\alpha I_\alpha} \langle \partial_i \partial_j \partial_k \phi_0, \overline{\phi_{J_\alpha}} \rangle \phi_{I_\alpha} \\ &= \partial_i \phi_{jk} - \partial_i G_{jk, \bar{J}_2}^{(2)} G^{(2)\bar{J}_2 I_2} \phi_{I_2},\end{aligned}\quad (4.35)$$

where $\{I_0\} \equiv \{0\}$, $\{I_1\} \equiv \{i\}$, and $G_{jk, \bar{J}_2}^{(2)} = \langle \phi_{jk}, \overline{\phi_{J_2}} \rangle$. It can be shown similar to the argument in section 3.3 that ϕ_{ijk} is well-defined respect to different choice of the maximal set $\{\phi_{I_2}\}$. ϕ_{ijk} is symmetric respect to i, j and k , and D_i defines a map

$$D_i : \mathcal{L} \otimes \text{Sym}_2 \mathcal{T}^* \mathcal{M}_C \rightarrow \mathcal{L} \otimes \text{Sym}_3 \mathcal{T}^* \mathcal{M}_C \text{ with fiber } H^{d-3,3}(\mathcal{Y}_\tau).$$

The unnormalized chiral ring data

$$\begin{aligned} G_{ikm, \bar{j}\bar{J}_2}^{(3)} &\equiv -\langle \phi_{ikm}, \overline{\phi_{jJ_2}} \rangle = \langle \partial_{\bar{j}} D_i \phi_{km}, \overline{\phi_{J_2}} \rangle \\ &= -\partial_{\bar{j}} \left(\partial_i G_{km, \bar{L}_2}^{(2)} G_{K_2 \bar{J}_2}^{(2)} \right) G_{K_2 \bar{J}_2}^{(2)} - \partial_{\bar{j}} \left(\partial_k G_{m\bar{n}}^{(1)} G^{(1)\bar{n}p} \right) G_{ip, \bar{J}_2}^{(2)} \\ &\quad - \partial_{\bar{j}} \left(\partial_m G_{00}^{(0)} G^{(0)\bar{0}0} \right) G_{ik, \bar{J}_2}^{(2)}. \end{aligned}$$

Applying eq. (4.30) and (4.33), we obtain

$$\partial_{\bar{j}} \left(\partial_i G_{km, \bar{J}_2}^{(2)} G^{(2)\bar{J}_2 K_2} \right) = -G_{ikm, \bar{j}\bar{J}_2}^{(3)} G^{(2)\bar{J}_2 K_2} + G_{km, \bar{j}\bar{l}}^{(2)} G^{(1)\bar{l}p} \delta_{ip}^{K_2} + G_{m\bar{j}}^{(1)} G^{(0)\bar{0}0} \delta_{ik}^{K_2} \quad (4.36)$$

Choosing index $\{km\} \subset \{I_2\}$ and “shift” OPE basis (3.49), we reconstruct eq. (2.40) up to degree $\alpha = 2$. One can continue this procedure and reconstruct the tt^* -equations to all degrees. We will not go through the details.

Now we turn to study the constraints imposed on the chiral ring data $G_{a_\alpha \bar{b}_\alpha}$, where $\{a_\alpha\}$ and $\{\bar{b}_\alpha\}$ enumerate all states with degree α . Resembling one-dimensional case, first we have

$$\phi_{a_d} = \mathcal{C}_{a_d}^{(d)} e^K \overline{\phi_0},$$

from which we compute

$$\mathcal{C}_{a_d}^{(d)}(\tau) = \langle \phi_0, \phi_{a_d} \rangle. \quad (4.37)$$

It contains various d -point chiral correlators computed in B-twisted models. The holomorphicity is guaranteed by lemma (4.16) as well. Further because of the symmetry of horizontal cohomology classes $H^d(\mathcal{Y}_\tau)$, we have

$$\phi_{a_\alpha} = \mathcal{C}_{a_\alpha I_{d-\alpha}}^{(d)} G^{(d-\alpha)\bar{J}_{d-\alpha} I_{d-\alpha}} \overline{\phi_{J_{d-\alpha}}}, \quad \overline{\phi_{b_\alpha}} = \overline{\mathcal{C}_{b_\alpha J_{d-\alpha}}^{(d)}} G^{(d-\alpha)\bar{J}_{d-\alpha} I_{d-\alpha}} \phi_{I_{d-\alpha}},$$

where the index $a_\alpha I_{d-\alpha}$ specifies an element in $\{a_d\}$. Overall we have the constraints

$$G_{a_\alpha \bar{b}_\alpha}^{(\alpha)} = \mathcal{C}_{a_\alpha I_{d-\alpha}}^{(d)} \overline{\mathcal{C}_{b_\alpha J_{d-\alpha}}^{(d)}} G^{(d-\alpha)\bar{J}_{d-\alpha} I_{d-\alpha}}$$

or

$$g_{a_\alpha \bar{b}_\alpha}^{(\alpha)} = \mathcal{C}_{a_\alpha I_{d-\alpha}}^{(d)} \overline{\mathcal{C}_{b_\alpha J_{d-\alpha}}^{(d)}} e^{2K} g^{(d-\alpha)\bar{J}_{d-\alpha} I_{d-\alpha}} \quad (4.38)$$

For example, in $d = 3$ the CY-threefold case, putting $\alpha = 2$ we obtain the constraint on eq. (4.34), see also [24],

$$g_{ik, \bar{j}\bar{l}}^{(2)} = g_{i\bar{j}}^{(1)} g_{k\bar{l}}^{(1)} + g_{i\bar{l}}^{(1)} g_{k\bar{j}}^{(1)} - R_{i\bar{j}k\bar{l}}^{(2)} = \mathcal{C}_{ikm}^{(3)} \overline{\mathcal{C}_{j\bar{l}n}^{(3)}} e^{2K} g^{(2)\bar{n}m}. \quad (4.39)$$

Eq. (4.37) and (4.27) will serve as consistency checks of computation in next section.

At last, we comment on when there are primitive generators $\phi_{\hat{I}_\alpha}$ of degree $\alpha \geq 2$ in $H^d(\mathcal{Y}_\tau)$. In this situation, we are *unable* to establish the tt^* -equations of chiral ring data including $\phi_{\hat{I}_\alpha}$ only from Griffiths transversality, because we have insufficient *input* data. Recall, from eq. (4.10), that Griffiths transversality establishes the relation between the OPE $\phi \cdot \mathbb{1}$ and $D_\tau \Omega$, and so forth. Therefore only when bridging the OPE $\phi_{\hat{I}_\alpha} \cdot \mathbb{1}$ and some operations acting on Ω , can we establish corresponding equations on $\phi_{\hat{I}_\alpha}$. Fortunately the localization method discussed in section 3 indeed provides enough input data to compute all chiral ring data, where the partition function with irrelevant deformation eq. (3.32) can be regarded as the generating function of all (twisted) chiral ring data. Explicit examples and computation will be given in next section.

5 Examples

We will compute the twisted chiral ring data of compact Calabi-Yau manifolds. All of the examples, collected from [31, 32], have GLSM descriptions at UV regime and flow to the CY geometric phase in deep infrared. The twisted chiral ring data encode geometries of Kähler moduli $\mathcal{M}_K(\mathcal{Y})$ as well as the data of vertical cohomology classes of $\mathcal{Y}$. Our algorithm in section 3 is designed for twisted chiral operator deformations and thus can be directly applied to these examples. On the other hand, the constraints imposed on chiral ring data are derived for complex moduli $\mathcal{M}_C(\mathcal{Y})$ and horizontal cohomology classes of $\mathcal{Y}$ in section 4. With the property of mirror symmetry, they equally work for twisted chiral ring data as well.

The Kähler moduli $\mathcal{M}_K(\mathcal{Y})$ are parameterized by marginal FI parameters $\{\tilde{\tau}, \bar{\tilde{\tau}}\}$,

$$\tilde{\tau} = \frac{\theta}{2\pi} + i r,$$

of the given GLSM at UV regime. The twisted chiral ring data, $\tilde{g}_\alpha(\tilde{\tau}, \bar{\tilde{\tau}})$, as well as partition function $Z_A(\tilde{\tau}, \bar{\tilde{\tau}})$, are non-holomorphic functions of $\{\tilde{\tau}, \bar{\tilde{\tau}}\}$. Sometimes, it is convenient to perform all computation in the large volume limit of $\mathcal{M}_K(\mathcal{Y})$,

$$r \gg 0.$$

In this region, one can instead expand Z_A and $\tilde{g}^{(\alpha)}$ in terms of the flat coordinates $\{t, \bar{t}\}$ of $\mathcal{M}_K(\mathcal{Y})$ [31], where the expression will be greatly simplified. The charts $\{t, \bar{t}\}$ is related to $\{\tilde{\tau}, \bar{\tilde{\tau}}\}$ via the “mirror map”,

$$t = f(\tilde{\tau}), \quad \bar{t} = \bar{f}(\bar{\tilde{\tau}}). \quad (5.1)$$

Therefore one express

$$\tilde{g}^{(\alpha)}(\tilde{\tau}, \bar{\tilde{\tau}}) = \left(\frac{\partial f}{\partial \tilde{\tau}} \right)^\alpha \left(\frac{\partial \bar{f}}{\partial \bar{\tilde{\tau}}} \right)^\alpha g^{(\alpha)}(t, \bar{t}), \quad (5.2)$$

where α , as before, labels the degree of twisted chiral ring data, and we have omitted other indexes for brevity. We will compute

$$g^{(\alpha)}(t, \bar{t}) = \mathcal{O}_{\text{pert.}} \left(\frac{1}{\text{Im } t} \right) + \mathcal{O}_{\text{inst.}} (e^{2\pi i t} + \text{c.c.}), \quad (5.3)$$

where $\mathcal{O}_{\text{pert.}}$ and $\mathcal{O}_{\text{inst.}}$ respectively collect the perturbative series and non-perturbative corrections of $g_\alpha(t, \bar{t})$.

5.1 The sextic fourfold: $X_6 \subset \mathbb{P}^5$

The first example we consider is the Fermat sextic fourfold $X_6 \subset \mathbb{P}^5$ [32], defined by a degree six hypersurface in $\mathbb{P}^5$. It can be realized as an $U(1)$ Abelian $\mathcal{N} = (2, 2)$ GLSM with matter content summarized in table 1. The model has one-dimensional Kähler moduli $\mathcal{M}_K(X_6)$

Field	U(1)	U(1) _V	U(1) _A
Φ_i	+1	2 $\mathfrak{q}$	0
P	−6	2 − 12 $\mathfrak{q}$	0

Table 1. The U(1) gauge charge, U(1)_V and U(1)_A R-charge of matter fields P , Φ_i for $i = 1, 2, \dots, 6$.

spanned by the twisted chiral primary Σ , as the field strength of U(1) vector multiplet V , see also in appendix B, associated to the marginal FI-coupling

$$\tilde{\tau} = \frac{\theta}{2\pi} + i r.$$

The model also has a superpotential $W = PW_6(\Phi)$ where $W_6(\Phi)$ is a homogeneous degree six polynomial of Φ_i . Although W is supersymmetric exact respect to $\mathfrak{su}(2|1)_A$, it restricts the U(1)_V R-charge of the matter contents up to an arbitrary number $\mathfrak{q}$. For convergent reason, we require $0 < \mathfrak{q} < \frac{1}{6}$ to compute the partition function in the same way of [31], using eq. (3.23) and (3.32),

$$Z_{X_6} = e^{-K(\tilde{\tau}, \bar{\tilde{\tau}})} = \sum_{m \in \mathbb{Z}} e^{-i\theta m} \int_{-\infty}^{+\infty} \frac{d\sigma}{2\pi} e^{-4\pi i r \sigma} \frac{\Gamma(\mathfrak{q} - i\sigma - \frac{1}{2}m)^6 \Gamma(1 - 6\mathfrak{q} + 6i\sigma + 3m)}{\Gamma(1 - \mathfrak{q} + i\sigma - \frac{1}{2}m)^6 \Gamma(6\mathfrak{q} - 6i\sigma + 3m)}, \quad (5.4)$$

where $K(\tilde{\tau}, \bar{\tilde{\tau}})$ is the Kähler potential of $\mathcal{M}_K(X_6)$.

5.1.1 Calabi-Yau phase

In the Calabi-Yau phase, $r \gg 0$, the integral is evaluated as,

$$Z_{X_6}^{\text{CY}} = (\bar{z}z)^{\mathfrak{q}} \oint \frac{d\epsilon}{2\pi i} (\bar{z}z)^{-\epsilon} \frac{\pi^5 \sin(6\pi\epsilon)}{\sin^6(\pi\epsilon)} \left| \sum_{k=0}^{\infty} z^k \frac{\Gamma(1 + 6k - 6\epsilon)}{\Gamma(1 + k - \epsilon)^6} \right|^2 \quad (5.5)$$

where $z = e^{2\pi i \tilde{\tau}}$ and the complex conjugate does not act on ϵ . We will expand Z_A respect to flat coordinates as mentioned before. Therefore, following [31, 32], we perform a Kähler transformation,

$$K(\tilde{\tau}, \bar{\tilde{\tau}}) \longrightarrow K(\tilde{\tau}, \bar{\tilde{\tau}}) + \log T(z) + \log \overline{T(z)}, \quad \text{with} \quad T(z) = z^{\mathfrak{q}} \sum_{k=0}^{\infty} z^k \frac{\Gamma(1 + 6k)}{\Gamma(1 + k)^6} \quad (5.6)$$

and read off the coefficient of $\log^3 \bar{z}$, which defines the mirror map (5.1),

$$2\pi i t = 2\pi i f(\tilde{\tau}) = \log z + 6264z + 67484340z^2 + 1272752107200z^3 + \dots$$

Inverse the map and expand Z_A in terms of $\{t, \bar{t}\}$, we have

$$Z_{X_6}^{\text{CY}}(t, \bar{t}) = \frac{1}{4} \xi^{-4} + 840 \zeta(3) \xi^{-1} + 30248 (\bar{q} + q) (\xi^{-2} + 2\xi^{-1}) + 609638400 \bar{q}q + \mathcal{O}(q^2) + \text{c.c.}, \quad (5.7)$$

where we have set

$$\xi \equiv \frac{1}{4\pi \operatorname{Im} t}, \quad \text{and} \quad q \equiv e^{2\pi i t}.$$

The first line of eq. (5.7) includes all perturbative contributions, and the second line collects the non-perturbative ones starting from one-instanton correction to the Kähler potential. In the case of X_6 as a CY-fourfold with center charge $c = 12$, we have five twisted chiral ring data, see eq. (4.15),

$$g^{(\alpha)}(t, \bar{t}) \equiv \langle \sigma^\alpha(0) \bar{\sigma}^\alpha(\infty) \rangle_{\mathbb{R}^2}, \quad \text{for } \alpha = 0, 1, 2, 3, 4, \quad (5.8)$$

with $g^{(0)} \equiv 1$ by normalization, where σ is the bottom component of twisted chiral primary Σ . Using eq. (5.7), and (3.55), we are able to compute all of them in the large volume limit $\operatorname{Im} t \sim \operatorname{Im} \tilde{\tau} \gg 0$,

$$\begin{aligned} g^{(1)} &= \frac{4\xi^2 (1 - 1680 \zeta(3) \xi^3)^2}{(1 + 3360 \zeta(3) \xi^3)^2} - 241920 (\bar{q} + q) (\xi^3 + \mathcal{O}(\xi^4)) \\ &\quad + 12192768000 \bar{q} q (\xi^4 + \mathcal{O}(\xi^5)) + \mathcal{O}(q^2) + \text{c.c.}, \\ g^{(2)} &= \frac{24\xi^4}{1 + 3360 \zeta(3) \xi^3} + 241920 (\bar{q} + q) (\xi^4 + \mathcal{O}(\xi^5)) \\ &\quad + 2438553600 \bar{q} q (\xi^4 + \mathcal{O}(\xi^6)) + \mathcal{O}(q^2) + \text{c.c.}, \\ g^{(3)} &= \frac{144\xi^6}{(1 - 1680 \zeta(3) \xi^3)^2} + 2903040 (\bar{q} + q) (\xi^6 + \mathcal{O}(\xi^7)) \\ &\quad + 58525286400 \bar{q} q (\xi^6 + \mathcal{O}(\xi^7)) + \mathcal{O}(q^2) + \text{c.c.}, \\ g^{(4)} &= \frac{576\xi^8}{(1 + 3360 \zeta(3) \xi^3)^2} + 11612160 (\bar{q} + q) (\xi^8 + \mathcal{O}(\xi^9)) \\ &\quad + 234101145600 \bar{q} q (\xi^8 + \mathcal{O}(\xi^9)) + \mathcal{O}(q^2) + \text{c.c.}, \end{aligned} \quad (5.9)$$

up to one-instanton correction. The perturbative part of $g^{(\alpha)}$ is of closed forms, for the partition function (5.7) on perturbative part is closed.

To restore the twisted chiral ring data in original $\tilde{\tau}$ -coordinates, we have

$$\tilde{g}^{(\alpha)}(\tilde{\tau}, \bar{\tilde{\tau}}) = \left(\frac{dt}{d\tilde{\tau}} \right)^\alpha \left(\frac{d\bar{t}}{d\bar{\tilde{\tau}}} \right)^\alpha g^{(\alpha)}(t(\tilde{\tau}), \bar{t}(\bar{\tilde{\tau}})) \quad (5.10)$$

Now we can give some consistency checks of the eq. (5.9). First they satisfy the Toda chain eq. (4.23) as designed by the algorithm (3.55). Second, we check the following consistency conditions (4.27) up to four-instanton corrections,

$$g^{(4)} = g^{(1)} g^{(3)} = \left(g^{(2)} \right)^2, \quad \text{and} \quad g^{(\alpha)} = 0, \quad \text{for } \alpha \geq 5. \quad (5.11)$$

Finally it is interesting to compute the 4-point chiral correlator (4.25) in A-twisted topological theory on S^2 , by use of eq. (4.27),

$$\begin{aligned} \overline{\mathcal{C}^{(4)}} \mathcal{C}^{(4)} &= \left| \langle \sigma \cdot \sigma \cdot \sigma \cdot \sigma \rangle_{S^2} \right|^2 = g^{(\alpha)} Z_A^2 \\ &= 36 \left| 1 + 20160 q + 689472000 q^2 + 24691154100480 q^3 \right. \\ &\quad \left. + 903369974818590720 q^4 + \mathcal{O}(q^5) \right|^2 \end{aligned}$$

Therefore we have

$$\begin{aligned} \mathcal{C}^{(4)}(q) = & 6 \left(1 + 20160 q + 689472000 q^2 + 24691154100480 q^3 \right. \\ & \left. + 903369974818590720 q^4 + \mathcal{O}(q^5) \right), \end{aligned} \quad (5.12)$$

which perfectly matches the results in [23]. It is worth of noticing that all twisted chiral ring data $g^{(\alpha)}$ are invariant respect to Kähler transformation, e.g. eq. (5.6), meanwhile $\mathcal{C}^{(4)}$ does depend on it. Our choice eq. (5.6) is consistent with [23].

5.1.2 Landau-Ginzburg phase and localization on Higgs branch

We can also compute the correlators in Landau-Ginzburg phase [33], $r \ll 0$. The integral (5.4) in the limit of $r \ll 0$ can be recast as [31]

$$Z_{X_6}^{\text{LG}} = \sum_{\alpha=0}^4 Z_{\text{cl}}^{(\alpha)} Z_{1\text{-loop}}^{(\alpha)} \overline{Z_{\text{vortex}}^{(\alpha)}(z)} Z_{\text{vortex}}^{(\alpha)}(z) \quad (5.13)$$

with

$$\begin{aligned} Z_{\text{cl}}^{(\alpha)} &= e^{4\pi r \cdot \frac{\alpha}{6}} = (\bar{z}z)^{-\frac{\alpha}{6}}, \\ Z_{1\text{-loop}}^{(\alpha)} &= \frac{(-1)^\alpha}{6} \frac{\Gamma\left(\frac{1+\alpha}{6}\right)^6}{\Gamma(1+\alpha)^2 \Gamma\left(\frac{5-\alpha}{6}\right)^6}, \\ Z_{\text{vortex}}^{(\alpha)}(z) &= {}_5F_4 \left(\left\{ \frac{1+\alpha}{6}, \dots, \frac{1+\alpha}{6} \right\}; \left\{ \frac{2+\alpha}{6}, \dots, \hat{1}, \dots, \frac{6+\alpha}{6} \right\}; \frac{1}{6^6 z} \right), \end{aligned}$$

where $\mathfrak{q}$ has been set to $\frac{1}{6}$ and $\hat{1}$ indicates the term one needs to omit. We can use eq. (3.55) to compute g_α as before. However notice that Z_A in Landau-Ginzburg phase should be interpreted as the partition function evaluated onto Higgs branch [15], where $Z_{\text{vortex}}^{(\alpha)}$ is the U(1) vortex partition function in Ω background [34]. It would be interesting if we can propose a different expression of (twisted) chiral ring data in terms of the vortex and anti-vortex partition functions.

Indeed for the U(1) Abelian case, it is not hard to reformulate the twisted chiral ring data in terms of $Z_{\text{vortex}}^{(\alpha)}$ and its complex conjugate. For convenience, we define the *unnormalized* correlators, referred to eq. (4.15),

$$G^{(\alpha)} \equiv \tilde{g}^{(\alpha)} Z_{X_6}^{\text{LG}},$$

which satisfy the Toda chain equations (4.22). We define further

$$c_\alpha \equiv Z_{1\text{-loop}}^{(\alpha)}, \quad \text{and} \quad \mathcal{F}^{(\alpha)}(z) \equiv z^{-\frac{\alpha}{6}} Z_{\text{vortex}}^{(\alpha)}(z)$$

and rewrite

$$Z_{X_6}^{\text{LG}} = \sum_{\alpha=0}^4 c_\alpha \overline{\mathcal{F}^{(\alpha)}} \mathcal{F}^{(\alpha)} = G^{(0)} \equiv \mathfrak{D}_0.$$

Applying eq. (4.22) and a little algebra, it is easy to find

$$G^{(1)} = -\mathfrak{D}_0 \partial_{\bar{\tau}} \partial_{\tau} \log \mathfrak{D}_0 = -\frac{\mathfrak{D}_1}{\mathfrak{D}_0},$$

with

$$\mathfrak{D}_1 = \frac{1}{2} \sum_{\alpha, \beta=0}^4 c_{\alpha} c_{\beta} \left| \mathcal{F}^{(\alpha)} \partial_{\tau} \mathcal{F}^{(\beta)} - \mathcal{F}^{(\beta)} \partial_{\tau} \mathcal{F}^{(\alpha)} \right|^2 \equiv \sum_{0 \leq \alpha < \beta \leq 4} c_{\alpha} c_{\beta} \left| \mathcal{W}(\mathcal{F}^{(\alpha)}, \mathcal{F}^{(\beta)}) \right|^2,$$

where

$$\mathcal{W}(\mathcal{F}^{(\alpha)}, \mathcal{F}^{(\beta)}) = \begin{vmatrix} \mathcal{F}^{(\alpha)} & \mathcal{F}^{(\beta)} \\ \partial_{\tau} \mathcal{F}^{(\alpha)} & \partial_{\tau} \mathcal{F}^{(\beta)} \end{vmatrix},$$

is the Wronskian respect to $\mathcal{F}^{(\alpha)}$. In general, define

$$\mathfrak{D}_n \equiv \sum_{0 \leq \alpha_0 < \dots < \alpha_n \leq 4} c_{\alpha_0} \dots c_{\alpha_n} \left| \mathcal{W}(\mathcal{F}^{(\alpha_0)}, \dots, \mathcal{F}^{(\alpha_n)}) \right|^2, \quad (5.14)$$

with

$$\mathcal{W}(\mathcal{F}^{(\alpha_0)}, \dots, \mathcal{F}^{(\alpha_n)}) = \begin{vmatrix} \mathcal{F}^{(\alpha_0)} & \mathcal{F}^{(\alpha_1)} & \dots & \mathcal{F}^{(\alpha_n)} \\ \partial_{\tau} \mathcal{F}^{(\alpha_0)} & \partial_{\tau} \mathcal{F}^{(\alpha_1)} & \dots & \partial_{\tau} \mathcal{F}^{(\alpha_n)} \\ \vdots & \vdots & \ddots & \vdots \\ \partial_{\tau}^n \mathcal{F}^{(\alpha_0)} & \partial_{\tau}^n \mathcal{F}^{(\alpha_1)} & \dots & \partial_{\tau}^n \mathcal{F}^{(\alpha_n)} \end{vmatrix}$$

the n -th Wronskian. One then can show an identity,

$$\partial_{\bar{\tau}} \partial_{\tau} \log \mathfrak{D}_n = \frac{\mathfrak{D}_{n+1} \mathfrak{D}_{n-1}}{\mathfrak{D}_n^2}. \quad (5.15)$$

A useful trick to derive this identity is to rewrite the Wronskian in terms of Pfaffian [35], and prove it by induction on n . With the aid of eq. (5.15), one can solve the Toda chain eq. (4.22) as

$$G^{(\alpha)} = (-1)^{\alpha} \frac{\mathfrak{D}_{\alpha}}{\mathfrak{D}_{\alpha-1}}, \quad \text{for } \alpha = 1 \dots 4 \quad (5.16)$$

In this sense, the twisted chiral ring data are expressed in *closed forms*. One still needs to check if eq. (5.16) satisfies the additional constraints (4.27), which we indeed confirm correctly up to $\mathcal{O}(1/z^{10})$. Especially for the nilpotency of twisted chiral ring data,

$$G^{(\alpha)} = 0, \quad \text{for } \alpha \geq 5,$$

are automatically guaranteed. In fact

$$\begin{aligned} \mathcal{F}^{(\alpha)}(z) &= z^{-\frac{\alpha}{6}} Z_{\text{vortex}}^{(\alpha)}(z) \\ &= \left(\frac{1}{z} \right)^{\frac{\alpha}{6}} {}_5F_4 \left(\left\{ \frac{1+\alpha}{6}, \dots, \frac{1+\alpha}{6} \right\}; \left\{ \frac{2+\alpha}{6}, \dots, \hat{1}, \dots, \frac{6+\alpha}{6} \right\}; \frac{1}{6^6 z} \right), \end{aligned}$$

in the context of mirror symmetries, are the periods of the mirror manifold of X_6 , and saturates the Picard-Fuchs equation [32],

$$\left(\Theta^5 - 6z \prod_{k=1}^5 (6\Theta + k) \right) \mathcal{F}^{(\alpha)}(z) = 0 \quad \text{with} \quad \Theta \equiv z \frac{d}{dz}. \quad (5.17)$$

Field	$U(1)_{\Sigma_0}$	$U(2)_{\Sigma_1}$	$U(1)_V$	$U(1)_A$
Φ_a	+1	1	$2\mathbf{q}_\phi$	0
P^i	-1	2	$2 - 2\mathbf{q}_x - 2\mathbf{q}_\phi$	0
X_i	0	$\overline{\mathbf{2}}$	$2\mathbf{q}_x$	0

Table 2. The $U(1)_{\Sigma_0}$ and $U(2)_{\Sigma_1}$ gauge group representations, $U(1)_V$ and $U(1)_A$ R-charge of matter fields Φ_a , P_i and X_i , with $a = 1, 2, \dots, 8$ and $i = 1, 2, \dots, 6$, in the PAX GLSM for GN-CY₃.

Therefore, any $\mathfrak{D}_\alpha$ with $\alpha \geq 5$ contains terms of $\Theta^5 \mathcal{F}^{(\alpha)}$, which is linear dependent on its lower derivatives due to eq. (5.17), and thus the Wronskian vanishes.

To end this subsection, we make some comments on the (twisted) chiral ring data in the formulation of localization onto Higgs branch. Firstly our derivation of eq. (5.15) and (5.16) is actually valid for any $U(1)$ Abelian GLSM. The $U(1)$ gauge group ensures that the (twisted) chiral ring is only generated by a single primitive generator with dimension one. The chiral ring data are thus dominated by the Toda chain eq. (4.23) *universally*. On the other hand, localization formula onto Higgs branch [15, 16] tells us that the partition function,

$$Z_A = \sum_{\alpha=0}^n c_\alpha \overline{\mathcal{F}^{(\alpha)}(z)} \mathcal{F}^{(\alpha)}(z), \quad (5.18)$$

is always of finite sum of factorizable building blocks, the vortex/anti-vortex partition functions, dressed up with factors of one-loop contributions as coefficients. Therefore, applying Toda chain eq. (4.22) and identity (5.15), we can determine all chiral ring data as ratios of sum of factorizable blocks $\mathcal{F}^{(\alpha)}(z)$, see e.g. eq. (5.16). In addition, the nilpotency of the chiral ring must be guaranteed, for $\mathcal{F}^{(\alpha)}(z)$ are in general some hypergeometric functions which will be linearly expressed by enough numbers of operator $\Theta = z \frac{d}{dz}$ acting on themselves. The argument above does not resort to any details of $\mathcal{F}^{(\alpha)}(z)$, while the constraints (4.27) does depend on the expression of them in concrete.

5.2 The determinantal Gulliksen-Negård CY-threefold

Our second example is the PAX/PAXY gauged linear sigma models introduced in [36] to describe the determinantal Calabi-Yau varieties. The gauge group is $U(1) \times U(2)$, associated to two $U(1)$ field strengths Σ_0 and $\text{Tr } \Sigma_1$ as twisted chiral primaries. The matter fields, summarized in table 2, contain 8 chiral multiplets Φ_a of gauge charge +1 under $U(1)_{\Sigma_0}$, 4 chiral multiplets P^i in the bifundamental representation of $U(2)_{\Sigma_1} \times U(1)_{\Sigma_0}$, and the 4 chiral multiplets X_i in the antifundamental representation of $U(2)_{\Sigma_1}$, subject to the superpotential,

$$W = \text{Tr} (PA(\Phi)X),$$

where $A(\Phi) = A^a \Phi_a$ and A^a are 8 constant 4×4 matrices.

The theory has two FI marginal couplings

$$\tilde{\tau}_i = \frac{\theta_i}{2\pi} + i r_i, \quad \text{for } i = 0, 1,$$

corresponding to the two $U(1)$ factors of the gauge group. Therefore the Kähler moduli space is two-dimensional. The dimensions of the vertical cohomology classes, $\bigoplus_{i=0}^3 H^{(i,i)}$, of “PAX” CY_3 model are

$$\{1, 2, 2, 1\}, \quad (5.19)$$

due to the symmetries of the cohomology ring. The twisted chiral ring thus has 6 elements with two primitive generators of dimension one.

The model has three phases respect to its FI-couplings. We restrict ourselves in the region of $r_0 + 2r_1 \gg 0$ and $r_1 \gg 0$ to compute the twisted chiral ring data. In this phase, it is convenient to use the linear combination of σ_0 and $\text{Tr } \sigma_1$, which are the bottom components of Σ_0 and $\text{Tr } \Sigma_1$ conjugate to $\tilde{\tau}_0$ and $\tilde{\tau}_1$, as the generators of the twisted chiral ring,

$$\chi_1 \equiv \sigma_0, \quad \text{and} \quad \chi_2 \equiv \text{Tr } \sigma_1 - 2\sigma_0. \quad (5.20)$$

χ_1 and χ_2 are thus conjugate to $\tilde{\tau}_0 + 2\tilde{\tau}_1$ and $\tilde{\tau}_1$ respectively. According the “shift” OPE basis (3.49), we are about to compute the following twisted chiral ring data,

$$\begin{aligned} g^{(0)} &= \langle \mathbb{1} \, \overline{\mathbb{1}} \rangle_{\mathbb{R}^2} \equiv 1, \\ g^{(1)} &= \begin{pmatrix} \langle \chi_1 \, \overline{\chi_1} \rangle_{\mathbb{R}^2}, \langle \chi_1 \, \overline{\chi_2} \rangle_{\mathbb{R}^2} \\ \langle \chi_2 \, \overline{\chi_1} \rangle_{\mathbb{R}^2}, \langle \chi_2 \, \overline{\chi_2} \rangle_{\mathbb{R}^2} \end{pmatrix}, \\ g^{(2)} &= \begin{pmatrix} \langle \chi_1^2 \, \overline{\chi_1^2} \rangle_{\mathbb{R}^2}, \langle \chi_1^2 \, \overline{\chi_1 \chi_2} \rangle_{\mathbb{R}^2}, \langle \chi_1^2 \, \overline{\chi_2^2} \rangle_{\mathbb{R}^2} \\ \langle \chi_1 \chi_2 \, \overline{\chi_1^2} \rangle_{\mathbb{R}^2}, \langle \chi_1 \chi_2 \, \overline{\chi_1 \chi_2} \rangle_{\mathbb{R}^2}, \langle \chi_1 \chi_2 \, \overline{\chi_2^2} \rangle_{\mathbb{R}^2} \\ \langle \chi_2^2 \, \overline{\chi_1^2} \rangle_{\mathbb{R}^2}, \langle \chi_2^2 \, \overline{\chi_1 \chi_2} \rangle_{\mathbb{R}^2}, \langle \chi_2^2 \, \overline{\chi_2^2} \rangle_{\mathbb{R}^2} \end{pmatrix}, \\ g^{(3)} &= \begin{pmatrix} \langle \chi_1^3 \, \overline{\chi_1^3} \rangle_{\mathbb{R}^2}, \langle \chi_1^3 \, \overline{\chi_1^2 \chi_2} \rangle_{\mathbb{R}^2}, \langle \chi_1^3 \, \overline{\chi_1 \chi_2^2} \rangle_{\mathbb{R}^2}, \langle \chi_1^3 \, \overline{\chi_2^3} \rangle_{\mathbb{R}^2} \\ \langle \chi_1^2 \chi_2 \, \overline{\chi_1^3} \rangle_{\mathbb{R}^2}, \langle \chi_1^2 \chi_2 \, \overline{\chi_1^2 \chi_2} \rangle_{\mathbb{R}^2}, \langle \chi_1^2 \chi_2 \, \overline{\chi_1 \chi_2^2} \rangle_{\mathbb{R}^2}, \langle \chi_1^2 \chi_2 \, \overline{\chi_2^3} \rangle_{\mathbb{R}^2} \\ \langle \chi_1 \chi_2^2 \, \overline{\chi_1^3} \rangle_{\mathbb{R}^2}, \langle \chi_1 \chi_2^2 \, \overline{\chi_1^2 \chi_2} \rangle_{\mathbb{R}^2}, \langle \chi_1 \chi_2^2 \, \overline{\chi_1 \chi_2^2} \rangle_{\mathbb{R}^2}, \langle \chi_1 \chi_2^2 \, \overline{\chi_2^3} \rangle_{\mathbb{R}^2} \\ \langle \chi_2^3 \, \overline{\chi_1^3} \rangle_{\mathbb{R}^2}, \langle \chi_2^3 \, \overline{\chi_1^2 \chi_2} \rangle_{\mathbb{R}^2}, \langle \chi_2^3 \, \overline{\chi_1 \chi_2^2} \rangle_{\mathbb{R}^2}, \langle \chi_2^3 \, \overline{\chi_2^3} \rangle_{\mathbb{R}^2} \end{pmatrix}. \quad (5.21) \end{aligned}$$

It is worth to mention that, for CY_3 ’s, there are *no* primitive generators of dimension two or three. Therefore the twisted operators, $\text{Tr}(\sigma_1^2)$ of dimension two and $\text{Tr}(\sigma_1^3)$, $\text{Tr } \sigma_1 \text{Tr}(\sigma_1^2)$ as well as $\sigma_0 \text{Tr}(\sigma_1^2)$ of dimension three, must linearly depend on certain powers of χ_1 and χ_2 . It can be also observed due to the symmetry (5.19) of the vertical cohomology of GN- CY_3 . We thus do not include these operators but only the marginal ones. Later in the third example of the complete intersection in Grassmannian $G(2, 8)$, say a CY_4 , we do have a primitive generator of dimension two. Therefore the operator $\text{Tr}(\sigma^2)$ in that case must be considered.

Now we literally follow [31] to compute the partition function in the large volume limit. In the region of $r_0 + 2r_1 \gg 0$ and $r_1 \gg 0$, the partition function is written as,

$$Z_{\text{GN}} = -\frac{1}{2} \oint \frac{d\epsilon_0 d\epsilon_1 d\epsilon_2}{(2\pi i)^3} \frac{\pi^8 \sin^4(\pi\epsilon_0 + \pi\epsilon_1) \sin^4(\pi\epsilon_0 + \pi\epsilon_2)}{\sin^8(\pi\epsilon_0) \sin^4(\pi\epsilon_1) \sin^4(\pi\epsilon_2)} (z_1 \bar{z}_1)^{\epsilon_0} (z_2 \bar{z}_2)^{\epsilon_1 + \epsilon_2} \left| \sum_{K_1, K_2=0}^{\infty} z_1^{K_1} z_2^{K_2} \sum_{k=0}^{K_2} (2k - K_2 + \epsilon_1 - \epsilon_2) \frac{\Gamma(1 + K_1 + k + \epsilon_0 + \epsilon_1)^4}{\Gamma(1 + K_1 + \epsilon_0)^4 \Gamma(1 + k + \epsilon_1)^4} \frac{\Gamma(1 + K_1 + K_2 - k + \epsilon_0 + \epsilon_2)^4}{\Gamma(1 + K_1 + \epsilon_0)^4 \Gamma(1 + K_2 - k + \epsilon_2)^4} \right|^2, \quad (5.22)$$

with

$$z_1 = e^{2\pi i(\tilde{\tau}_0 + 2\tilde{\tau}_1)}, \quad \text{and} \quad z_2 = e^{2\pi i\tilde{\tau}_1}. \quad (5.23)$$

Evaluating the multiple residues of the integral, one finds the Kähler transformation,

$$T(z_1, z_2) = 1 + z_1 + 2z_2 + z_1^2 + 3z_2^2 - 54z_1^2 z_2 - 14z_1 z_2^2 + 351z_1^2 z_2^2 + \dots, \quad (5.24)$$

to simplify the result, and further reads off the mirror map from the coefficients of $\log^2 \bar{z}_1$ and $\log^2 \bar{z}_2$,

$$\begin{aligned} 2\pi i t_1 &= \log z_1 + 4z_2 - 20z_1 z_2 + 2z_2^2 - 92z_1^2 z_2 - 72z_1 z_2^2 + 38z_1^2 z_2^2 + \dots, \\ 2\pi i t_2 &= \log(-z_2) + 4z_1 + 16z_1 z_2 + 2z_1^2 - 128z_1^2 z_2 + 36z_1 z_2^2 - 1080z_1^2 z_2^2 + \dots \end{aligned} \quad (5.25)$$

Solving z_1, z_2 in terms of t_1 and t_2 , one arrives at the partition function up to one-instanton correction,⁷

$$\begin{aligned} Z_{\text{GN}} &= \frac{10}{3} \xi_1^{-3} + 10 \xi_1^{-2} \xi_2^{-1} + 8 \xi_1^{-1} \xi_2^{-2} + \frac{4}{3} \xi_2^{-3} + 128 \zeta(3) \\ &\quad + 112 q_1 + 384 q_1 q_2 + (56 q_1 + 192 q_1 q_2) \xi_1^{-1} + 192 q_1 q_2 \xi_2^{-1} + \text{c.c.} + \mathcal{O}(q_1^2, q_2^2), \end{aligned} \quad (5.26)$$

where as before we set

$$\xi_i \equiv \frac{1}{4\pi \text{Im } t_i}, \quad \text{and} \quad q_i \equiv e^{2\pi i t_i}, \quad \text{for } i = 1, 2. \quad (5.27)$$

From Z_{GN} , we can compute all correlators on S^2 ,

$$M_{a_\alpha \bar{b}_\beta} = \frac{1}{Z_{\text{GN}}} \prod_{i, \bar{j}} \left(-\frac{1}{2\pi} \frac{\partial}{\partial t^i} \right)^{n_i} \left(\frac{1}{2\pi} \frac{\partial}{\partial \bar{t}^{\bar{j}}} \right)^{n'_{\bar{j}}} Z_{\text{GN}}. \quad (5.28)$$

with

$$\sum_i n_i = \alpha, \quad \text{and} \quad \sum_{\bar{j}} n'_{\bar{j}} = \beta$$

The rows and columns of M are indexed by the degree of operators following

$$\{\mathbb{1}; \chi_1, \chi_2; \chi_1^2, \chi_1 \chi_2, \chi_2^2; \chi_1^3, \chi_1^2 \chi_2, \chi_1 \chi_2^2, \chi_2^3\}. \quad (5.29)$$

⁷Our normalization is different from [31] by a factor of $-\frac{1}{8\pi^3}$.

One can check that the 10×10 matrix M have $\text{rank}(M) = 6$, and the rank of its submatrices up to degree $\{0, 1, 2, 3\}$ is exactly $\{1, 3, 5, 6\}$ consistent with the dimension of dimensions of vertical cohomology (5.19).

Based on eq. (5.28) and algorithm (3.55), we can extract all twisted chiral ring data. For simplicity, we compute the unnormalized correlator,

$$G^{(\alpha)} = g^{(\alpha)} Z_{\text{GN}} = G_{\text{pert.}}^{(\alpha)} + G_{\text{np.}}^{(\alpha)}. \quad (5.30)$$

by implementing eq. (3.55), where we have separated them by perturbative and non-perturbative two parts. Since $G_{\text{np.}}^{(\alpha)}$ are kind of lengthy even for one-instanton correction, we only present the perturbative $G_{\text{pert.}}^{(\alpha)}$ here and leave $G_{\text{np.}}^{(\alpha)}$ in appendix C.

The (unnormalized) metric $G^{(1)}$ on $\mathcal{M}_K(\mathcal{V}_{\text{GN}})$ can be obtained by eq. (3.39),

$$G_{\text{pert.}}^{(1)} = \frac{1}{D^{(1)}} \begin{pmatrix} N_{1\bar{1}}^{(1)}, N_{1\bar{2}}^{(1)} \\ N_{2\bar{1}}^{(1)}, N_{2\bar{2}}^{(1)} \end{pmatrix}, \quad (5.31)$$

with

$$\begin{aligned} N_{1\bar{1}}^{(1)} &= 56\xi_1^4 + 200\xi_2\xi_1^3 + 300\xi_2^2\xi_1^2 + 200\xi_2^3\xi_1 + 50\xi_2^4 - 3840\xi_2^3\xi_1^4\zeta(3) - 3840\xi_2^4\xi_1^3\zeta(3) \\ N_{2\bar{2}}^{(1)} &= 8\xi_1^4 + 64\xi_2\xi_1^3 + 192\xi_2^2\xi_1^2 + 200\xi_2^3\xi_1 + 70\xi_2^4 - 1536\xi_2^3\xi_1^4\zeta(3) - 3072\xi_2^4\xi_1^3\zeta(3) \\ N_{1\bar{2}}^{(1)} &= N_{2\bar{1}}^{(1)} \\ &= 16\xi_1^4 + 80\xi_2\xi_1^3 + 180\xi_2^2\xi_1^2 + 160\xi_2^3\xi_1 + 50\xi_2^4 - 3072\xi_2^3\xi_1^4\zeta(3) - 3840\xi_2^4\xi_1^3\zeta(3) \\ D^{(1)} &= \xi_1\xi_2 (2\xi_1^3 + 12\xi_2\xi_1^2 + 15\xi_2^2\xi_1 + 5\xi_2^3 + 192\xi_2^3\xi_1^3\zeta(3)) \end{aligned} \quad (5.32)$$

$G^{(2)}$ are also straightforwardly computed either from eq. (3.55) or (4.32),

$$G_{\text{pert.}}^{(2)} = \frac{1}{D^{(2)}} \begin{pmatrix} N_{1\bar{1}\bar{1}\bar{1}}^{(2)}, N_{1\bar{1}\bar{1}\bar{2}}^{(2)}, N_{1\bar{1}\bar{2}\bar{2}}^{(2)} \\ N_{1\bar{2}\bar{1}\bar{1}}^{(2)}, N_{1\bar{2}\bar{1}\bar{2}}^{(2)}, N_{1\bar{2}\bar{2}\bar{2}}^{(2)} \\ N_{2\bar{2}\bar{1}\bar{1}}^{(2)}, N_{2\bar{2}\bar{1}\bar{2}}^{(2)}, N_{2\bar{2}\bar{2}\bar{2}}^{(2)} \end{pmatrix}, \quad (5.33)$$

with

$$\begin{aligned} N_{1\bar{1}\bar{1}\bar{1}}^{(2)} &= 200\xi_1\xi_2 (\xi_1 + \xi_2) (8\xi_1^3 + 18\xi_2\xi_1^2 + 15\xi_2^2\xi_1 + 5\xi_2^3 + 192\xi_2^3\xi_1^3\zeta(3)) \\ N_{1\bar{2}\bar{1}\bar{2}}^{(2)} &= 4\xi_1\xi_2 (228\xi_1^4 + 800\xi_2\xi_1^3 + 1200\xi_2^2\xi_1^2 + 900\xi_2^3\xi_1 + 275\xi_2^4 \\ &\quad + 11520\xi_2^3\xi_1^4\zeta(3) + 7680\xi_2^4\xi_1^3\zeta(3)) \\ N_{2\bar{2}\bar{2}\bar{2}}^{(2)} &= 16\xi_1\xi_2 (12\xi_1^4 + 68\xi_2\xi_1^3 + 174\xi_2^2\xi_1^2 + 180\xi_2^3\xi_1 + 65\xi_2^4 \\ &\quad + 1152\xi_2^3\xi_1^4\zeta(3) - 384\xi_2^4\xi_1^3\zeta(3)) \end{aligned}$$

$$\begin{aligned}
 N_{11\bar{1}\bar{2}}^{(2)} &= N_{12\bar{1}\bar{1}}^{(2)} = 40\xi_1\xi_2(30\xi_1^4 + 100\xi_2\xi_1^3 + 135\xi_2^2\xi_1^2 + 90\xi_2^3\xi_1 + 25\xi_2^4 \\
 &\quad + 1152\xi_2^3\xi_1^4\zeta(3) + 960\xi_2^4\xi_1^3\zeta(3)) \\
 N_{11\bar{2}\bar{2}}^{(2)} &= N_{22\bar{1}\bar{1}}^{(2)} = 160\xi_1\xi_2(3\xi_1^4 + 11\xi_2\xi_1^3 + 18\xi_2^2\xi_1^2 + 15\xi_2^3\xi_1 + 5\xi_2^4 \\
 &\quad + 288\xi_2^3\xi_1^4\zeta(3) + 192\xi_2^4\xi_1^3\zeta(3)) \\
 N_{12\bar{2}\bar{2}}^{(2)} &= N_{22\bar{1}\bar{2}}^{(2)} = 8\xi_1\xi_2(48\xi_1^4 + 200\xi_2\xi_1^3 + 390\xi_2^2\xi_1^2 + 360\xi_2^3\xi_1 + 125\xi_2^4 \\
 &\quad + 4608\xi_2^3\xi_1^4\zeta(3) + 1920\xi_2^4\xi_1^3\zeta(3)) \\
 D^{(2)} &= (6\xi_1^2 + 10\xi_2\xi_1 + 5\xi_2^2)(2\xi_1^3 + 12\xi_2\xi_1^2 + 15\xi_2^2\xi_1 + 5\xi_2^3 - 384\xi_2^3\xi_1^3\zeta(3)) . \quad (5.34)
 \end{aligned}$$

Now before proceeding further to compute $G^{(3)}$, one will find that rank of $G^{(2)}$ is *two*,⁸ which implies that there are only two independent operators of dimension two among χ_1^2 , $\chi_1\chi_2$ and χ_2^2 . It is surely consistent with,

$$\dim H_{\text{prim}}^{(2,2)}(\mathcal{Y}_{\text{GN}}) = 2 . \quad (5.35)$$

Therefore we have to remove one of them, and compute the inverse of $G^{(2)}$ to perform further orthogonalization. Let us, for example, remove operator χ_2^2 and its corresponding row and column in M from eq. (5.28), and $G^{(3)}$ is given by eq. (3.55),

$$G^{(3)} = \frac{1}{Z_{\text{GN,pert.}}} \begin{pmatrix} 20 \cdot 20, & 20 \cdot 20, & 20 \cdot 20, & 16 \cdot 8 \\ 20 \cdot 20, & 20 \cdot 20, & 20 \cdot 16, & 20 \cdot 8 \\ 20 \cdot 16, & 20 \cdot 16, & 16 \cdot 16, & 16 \cdot 8 \\ 20 \cdot 8, & 20 \cdot 8, & 16 \cdot 8, & 8 \cdot 8 \end{pmatrix} , \quad (5.36)$$

with

$$Z_{\text{GN,pert.}} = \frac{10}{3} \xi_1^{-3} + 10 \xi_1^{-2} \xi_2^{-1} + 8 \xi_1^{-1} \xi_2^{-2} + \frac{4}{3} \xi_2^{-3} + 128 \zeta(3) , \quad (5.37)$$

the perturbative part of partition function Z_{GN} . Indeed one can verify that our computation (5.36) is independent on the choice of the removed operator.

To restore the twisted chiral ring data in τ -coordinates, as before, we have

$$\begin{aligned}
 \tilde{g}_{i\bar{j}}^{(1)}(\tilde{\tau}, \bar{\tilde{\tau}}) &= \left(\frac{dt_{i'}}{d\tilde{\tau}_i} \right) \left(\frac{d\bar{t}_{j'}}{d\bar{\tilde{\tau}}_j} \right) g_{i'\bar{j}'}^{(1)}(t(\tilde{\tau}), \bar{t}(\bar{\tilde{\tau}})) , \\
 \tilde{g}_{ik\bar{j}\bar{l}}^{(2)}(\tilde{\tau}, \bar{\tilde{\tau}}) &= \left(\frac{dt_{i'}}{d\tilde{\tau}_i} \right) \left(\frac{dt_{k'}}{d\tilde{\tau}_k} \right) \left(\frac{d\bar{t}_{j'}}{d\bar{\tilde{\tau}}_j} \right) \left(\frac{d\bar{t}_{l'}}{d\bar{\tilde{\tau}}_l} \right) g_{i'k'\bar{j}'\bar{l}'}^{(2)}(t(\tilde{\tau}), \bar{t}(\bar{\tilde{\tau}})) , \\
 \tilde{g}_{ikm\bar{j}\bar{l}\bar{n}}^{(3)}(\tilde{\tau}, \bar{\tilde{\tau}}) &= \left(\frac{dt_{i'}}{d\tilde{\tau}_i} \right) \left(\frac{dt_{k'}}{d\tilde{\tau}_k} \right) \left(\frac{dt_{m'}}{d\tilde{\tau}_m} \right) \left(\frac{d\bar{t}_{j'}}{d\bar{\tilde{\tau}}_j} \right) \left(\frac{d\bar{t}_{l'}}{d\bar{\tilde{\tau}}_l} \right) \left(\frac{d\bar{t}_{n'}}{d\bar{\tilde{\tau}}_n} \right) g_{i'k'm'\bar{j}'\bar{l}'\bar{n}'}^{(3)}(t(\tilde{\tau}), \bar{t}(\bar{\tilde{\tau}})) .
 \end{aligned} \quad (5.38)$$

Now we give some consistency checks of above results. First the rank of $G^{(3)}$ is one, which implies that all χ_1^3 , $\chi_1^2\chi_2$, $\chi_1\chi_2^2$ and χ_2^3 are linear dependent, corresponding to the

⁸We notice that, including instanton correction, the rank of $G^{(2)}$ will turn out to be three but in higher order of q . We believe that the rank of $G^{(2)}$ will remain *two* if adding full instanton corrections.

unique top element of twisted chiral ring of $H^{(3,3)}(\mathcal{Y}_{\text{GN}})$. Further applying eq. (3.55), we have

$$G_{\text{pert.}}^{(\alpha)} = 0, \quad \text{for } \alpha \geq 4,$$

showing the nilpotency of the twisted chiral ring. We next check that, using eq. (5.31), (5.33) and (5.36), the constraints eq. (4.38) and (4.39) are also satisfied as designed. More interestingly, we can read off the topological correlators in A-twisted theory. For example, let $\alpha = d$ in eq. (4.38), we have

$$G^{(3)} Z_{\text{GN}} = \mathcal{C}_{a_3 0}^{(d)} \overline{\mathcal{C}_{b_3 0}^{(d)}}. \quad (5.39)$$

Comparing to eq. (5.36), the diagonal entries of eq. (5.36) imply that

$$\langle \chi_1^3 \rangle_{S^2, \text{pert.}} = 20, \quad \langle \chi_1^2 \chi_2 \rangle_{S^2, \text{pert.}} = 20, \quad \langle \chi_1 \chi_2^2 \rangle_{S^2, \text{pert.}} = 16, \quad \text{and} \quad \langle \chi_2^3 \rangle_{S^2, \text{pert.}} = 8,$$

which are the classical intersection numbers of hyperplanes in $G(2, 4) \supset \mathcal{Y}_{\text{GN}}$.

All above checks are surely correct after including instanton corrections. We perform up to 3-instanton computation. Especially we can compute the topological correlators in the A-twist theory [31] up to the order of 3-instantons,

$$\begin{aligned} \langle \chi_1^3 \rangle_{S^2} &= 4(5 + 14q_1 + 14q_1^2 + 14q_1^3 + 48q_1q_2 + 1792q_1^2q_2 + 1296q_1^3q_2 + 14q_1q_2^2 \\ &\quad + 5136q_1^2q_2^2 + 155358q_1^3q_2^2 + 1792q_1^2q_2^3 + 357312q_1^3q_2^3 + \mathcal{O}(q_1^4, q_2^4)), \\ \langle \chi_1^2 \chi_2 \rangle_{S^2} &= 4(5 + 48q_1q_2 + 896q_1^2q_2 + 432q_1^3q_2 + 28q_1q_2^2 + 5136q_1^2q_2^2 + 103572q_1^3q_2^2 \\ &\quad + 2688q_1^2q_2^3 + 357312q_1^3q_2^3 + \mathcal{O}(q_1^4, q_2^4)), \\ \langle \chi_1 \chi_2^2 \rangle_{S^2} &= 16(1 + 12q_1q_2 + 112q_1^2q_2 + 36q_1^3q_2 + 14q_1q_2^2 + 1284q_1^2q_2^2 + 17262q_1^3q_2^2 \\ &\quad + 1008q_1^2q_2^3 + 89328q_1^3q_2^3 + \mathcal{O}(q_1^4, q_2^4)), \\ \langle \chi_2^3 \rangle_{S^2} &= 8(1 + 24q_1q_2 + 112q_1^2q_2 + 24q_1^3q_2 + 56q_1q_2^2 + 2568q_1^2q_2^2 + 23016q_1^3q_2^2 \\ &\quad + 3024q_1^2q_2^3 + 178656q_1^3q_2^3 + \mathcal{O}(q_1^4, q_2^4)). \end{aligned} \quad (5.40)$$

One can further apply eq. (5.38) to obtain these chiral correlators in $\tilde{\tau}$ -coordinates. The results also match with those in [38] up to a Kähler transformation, see more details in appendix C.

5.3 The complete intersection in Grassmannian: $X_{18} \subset G(2, 8)$

Our final example is the complete intersection of eight hyperplanes with degree one, a Calabi-Yau fourfold X_{18} , in Grassmannian $G(2, 8)$ [32], see also [39]. It is endowed with a GLSM description with a $U(2)$ gauge group. The matter content are summarized in table 3, and constrained by a superpotential

$$W = \sum_{\substack{a, i, j=8 \\ \alpha, \beta=1, 2}} A_{ij}^a P_a \Phi_\alpha^i \Phi_\beta^j \epsilon^{\alpha\beta},$$

where A^a are eight 8×8 constant anti-symmetric matrices, and $\epsilon^{\alpha\beta}$ is the $SU(2) \subset U(2)_\Sigma$ invariant antisymmetric tensor. The dimensions of the vertical cohomology classes,

Field	$U(1) \subset U(2)_\Sigma$	$SU(2) \subset U(2)_\Sigma$	$U(1)_V$	$U(1)_A$
Φ^i	+1	2	2q	0
P_a	-2	1	$2 - 4q$	0

Table 3. The $U(2)_\Sigma$ gauge representation, $U(1)_V$ and $U(1)_A$ R-charge of matter fields P_a , Φ^i for $a, i = 1, 2, \dots, 8$.

$\bigoplus_{i=0}^4 H^{(i,i)}(X_{18})$, are listed below,

$$\{1, 1, 2, 1, 1\}. \quad (5.41)$$

Therefore its twisted chiral ring is generated by two primitive generators, the marginal twisted primary ψ , and further a primitive generator χ of degree two,

$$\psi \equiv \text{Tr } \sigma, \quad \text{and} \quad \chi \equiv \text{Tr } (\sigma^2), \quad (5.42)$$

where σ is the bottom component of the twisted chiral multiplet Σ . We therefore need to compute the following twisted chiral ring data,

$$\begin{aligned} g^{(0)} &= \langle \mathbf{1} \, \bar{\mathbf{1}} \rangle_{\mathbb{R}^2} \equiv 1, & g^{(1)} &= \langle \psi \, \bar{\psi} \rangle_{\mathbb{R}^2}, \\ g^{(2)} &= \left(\begin{array}{cc} \langle \psi^2 \bar{\psi}^2 \rangle_{\mathbb{R}^2}, & \langle \psi^2 \bar{\chi} \rangle_{\mathbb{R}^2} \\ \langle \chi \bar{\psi}^2 \rangle_{\mathbb{R}^2}, & \langle \chi \bar{\chi} \rangle_{\mathbb{R}^2} \end{array} \right), & g^{(3)} &= \left(\begin{array}{cc} \langle \psi^3 \bar{\psi}^3 \rangle_{\mathbb{R}^2}, & \langle \psi^3 \bar{\psi} \bar{\chi} \rangle_{\mathbb{R}^2} \\ \langle \psi \chi \bar{\psi}^3 \rangle_{\mathbb{R}^2}, & \langle \psi \chi \bar{\psi} \bar{\chi} \rangle_{\mathbb{R}^2} \end{array} \right), \\ g^{(4)} &= \left(\begin{array}{ccc} \langle \psi^4 \bar{\psi}^4 \rangle_{\mathbb{R}^2}, & \langle \psi^4 \bar{\psi}^2 \bar{\chi} \rangle_{\mathbb{R}^2}, & \langle \psi^4 \bar{\chi}^2 \rangle_{\mathbb{R}^2} \\ \langle \psi^2 \chi \bar{\psi}^4 \rangle_{\mathbb{R}^2}, & \langle \psi^2 \chi \bar{\psi}^2 \bar{\chi} \rangle_{\mathbb{R}^2}, & \langle \psi^2 \chi \bar{\chi}^2 \rangle_{\mathbb{R}^2} \\ \langle \chi^2 \bar{\psi}^4 \rangle_{\mathbb{R}^2}, & \langle \chi^2 \bar{\psi}^2 \bar{\chi} \rangle_{\mathbb{R}^2}, & \langle \chi^2 \bar{\chi}^2 \rangle_{\mathbb{R}^2} \end{array} \right). \end{aligned}$$

Since now there exists a primitive generator of dimension two, we have to deform the original GLSM by an additional twisted superpotential as eq. (3.27),

$$\widetilde{W}_{\text{deform.}} + \text{h.c.} = \frac{1}{2} \left(\tilde{\tau}_2 \text{Tr } \Sigma^2 - \bar{\tilde{\tau}}_2 \text{Tr } \bar{\Sigma}^2 \right), \quad (5.43)$$

and obtain the deformed partition function $Z_{X_{18}}$ by use of eq. (3.32),

$$\begin{aligned} Z_{X_{18}} &= \frac{1}{2} \sum_{m_1, m_2 \in \mathbb{Z}} e^{-i\theta(m_1+m_2)} \int_{-\infty}^{\infty} \frac{d\sigma_1}{2\pi} \frac{d\sigma_2}{2\pi} e^{-4\pi i r(\sigma_1+\sigma_2)} e^{-2\pi \tilde{\tau}_2 \sum_{j=1}^2 \left(\sigma_j + i \frac{m_j}{2} \right)^2 - \text{c.c.}} \\ &\times \left(\frac{(m_1 - m_2)^2}{4} + (\sigma_1 - \sigma_2)^2 \right) \left(\frac{\Gamma \left(1 - 2q + i(\sigma_1 + \sigma_2) + \frac{1}{2}(m_1 + m_2) \right)}{\Gamma \left(2q - i(\sigma_1 + \sigma_2) + \frac{1}{2}(m_1 + m_2) \right)} \right)^8 \\ &\times \left(\frac{\Gamma \left(q - i\sigma_1 - \frac{1}{2}m_1 \right) \Gamma \left(q - i\sigma_2 - \frac{1}{2}m_2 \right)}{\Gamma \left(1 - q + i\sigma_1 - \frac{1}{2}m_1 \right) \Gamma \left(1 - q + i\sigma_2 - \frac{1}{2}m_2 \right)} \right)^8, \end{aligned} \quad (5.44)$$

where $\tilde{\tau} = \frac{\theta}{2\pi} + ir$ is the marginal parameter as before, and $\tilde{\tau}_2$ is the irrelevant parameter introduced to probe the twisted chiral primary $\text{Tr } \Sigma^2$ of degree two. We will calculate $Z_{X_{18}}$ in the CY geometric phase $r \gg 0$, and thus set

$$q - i\sigma_i \equiv \kappa_i,$$

and close the contour in the left half-planes of κ_i . During the evaluation, we will keep⁹ $\mathbf{q} \rightarrow 0^+$ to simplify the computation. With $\mathbf{q} = 0$ and the standard method in [31, 32], one can rewrite eq. (5.44) as a sum of multi-residues,

$$\begin{aligned}
 Z_{X_{18}} = & -\frac{1}{2} \oint \frac{d\epsilon_1 d\epsilon_2}{(2\pi i)^2} \frac{\pi^8 \sin^8(\pi\epsilon_1 + \pi\epsilon_2)}{\sin^8(\pi\epsilon_1) \sin^8(\pi\epsilon_2)} (z\bar{z})^{-\epsilon_1 - \epsilon_2} (UV)^{\epsilon_1^2 + \epsilon_2^2} \\
 & \times \left(\sum_{J=0}^{\infty} z^J \sum_{j=0}^J U^{j^2 - 2j\epsilon_1 + (J-j)^2 - 2(J-j)\epsilon_2} \frac{(2j - J - \epsilon_1 + \epsilon_2) \Gamma(1 + J - \epsilon_1 + \epsilon_2)^8}{\Gamma(1 + j - \epsilon_1)^8 \Gamma(1 + J - j - \epsilon_2)^8} \right) \\
 & \times \left(\sum_{L=0}^{\infty} \bar{z}^L \sum_{l=0}^L V^{l^2 - 2l\epsilon_1 + (L-l)^2 - 2(L-l)\epsilon_2} \frac{(2l - L - \epsilon_1 + \epsilon_2) \Gamma(1 + L - \epsilon_1 + \epsilon_2)^8}{\Gamma(1 + l - \epsilon_1)^8 \Gamma(1 + L - l - \epsilon_2)^8} \right), \quad (5.45)
 \end{aligned}$$

with

$$z = e^{2\pi i \tilde{\tau}}, \quad U = e^{2\pi \tilde{\tau}_2} \quad \text{and} \quad V = e^{-2\pi \bar{\tilde{\tau}}_2}. \quad (5.46)$$

There are two technical issues to clarify before evaluating eq. (5.46). First there also exists a mirror map $t = f(\tilde{\tau})$ to simplify $Z_{X_{18}}$ in large volume limit. However, since now that we include irrelevant parameters in the partition function, there seems no straightforward way to see how operator χ transforms respect to the change of coordinates from “ $\tilde{\tau}$ ” to “ t ”, while it does transform tensorially as a section living on the vector bundle over $\mathcal{M}_K(\mathcal{Y}_{X_{18}})$. Therefore, we directly evaluate the integral in the $\tilde{\tau}$ -coordinates. Secondly we can find a “Kähler transformation”,

$$T(z, U) = 1 - 6zU + 256z^2U^2 - 22z^2U^4 + 2zU \log U + 8z^2U^4 \log U + \dots, \quad (5.47)$$

to simplify the computation as we did before. But note that the variables U and V are actually not complex conjugate to each other, $T(z, U)$ and $\bar{T}(\bar{z}, V)$ are thus not either. Recall that the twisted chiral ring data are invariant respect to Kähler transformations. We checked our result with and without use of the “Kähler transformation” $T(z, U)$, and fortunately the results agree up to two-instanton.

With eq. (5.47), we evaluate

$$Z_{X_{18}} \longrightarrow \frac{Z_{X_{18}}}{T(z, U) \bar{T}(\bar{z}, V)}, \quad (5.48)$$

and spell out the perturbative part of $Z_{X_{18}}$,

$$Z_{X_{18}} = \frac{11}{2} \xi^{-4} + 24 \log(UV) \xi^{-2} + 672 \zeta(3) \xi^{-1} + 10 \log(UV) + Z_{\text{np.}}(z, \bar{z}, U, V), \quad (5.49)$$

where

$$\xi \equiv \frac{1}{4\pi \text{Im } \tilde{\tau}} \quad \text{and} \quad z = e^{2\pi i \tilde{\tau}}. \quad (5.50)$$

⁹The correct R-charge should be $\mathbf{q} \rightarrow \frac{1}{2}^-$. However we check up to the order of two-instanton, the twisted chiral ring data is independent on $\mathbf{q}$.

and $Z_{\text{np.}}(z, \bar{z}, U, V)$ denotes the non-perturbative contributions that is too lengthy to present. From $Z_{X_1^8}$ and eq. (3.52), we compute all correlators on S^2 ,

$$M_{a_\alpha \bar{b}_\beta} = \frac{1}{Z_{X_1^8}} \left(-\frac{1}{2\pi} \frac{\partial}{\partial \bar{\tau}} \right)^{n_1} \left(-\frac{1}{2\pi} \frac{\partial}{\partial \bar{\tau}_2} \right)^{n_2} \left(\frac{1}{2\pi} \frac{\partial}{\partial \bar{\tau}} \right)^{m_1} \left(\frac{1}{2\pi} \frac{\partial}{\partial \bar{\tau}_2} \right)^{m_2} Z_{X_1^8} \Big|_{\bar{\tau}_2 = \bar{\tau}_2 = 0}, \quad (5.51)$$

with $n_1 + 2n_2 = \alpha$ and $m_1 + 2m_2 = \beta$. We index the rows and columns of M by the degrees of operators as,

$$\{ \mathbb{1}; \psi; \psi^2, \chi; \psi^3, \psi\chi; \psi^4, \psi^2\chi, \chi^2 \}. \quad (5.52)$$

One can check that the rank of M is 6, and that of its sub-matrices up to the degrees of operators $\{0, 1, 2, 3, 4\}$ are precisely $\{1, 2, 4, 5, 6\}$ matching with the dimensions of vertical cohomology classes (5.41) of X_{1^8} .

Implementing algorithm (3.55), we find up to one-instanton,

$$\begin{aligned} g^{(1)} &= 4\xi^2 \left(\frac{F_1(\xi)}{F_2(\xi)} \right)^2 + (-80\xi^2(z + \bar{z}) + \mathcal{O}(\xi^3)) + \mathcal{O}(z^2, \bar{z}^2), \\ g^{(2)} &= \frac{1}{F_2(\xi)} \begin{pmatrix} 264\xi^4, & 96\xi^4 \\ 96\xi^4, & 40\xi^4 \end{pmatrix} + g_{1-\text{inst.}}^{(2)} + \mathcal{O}(z^2, \bar{z}^2), \\ g^{(3)} &= \frac{1}{F_1^2(\xi)} \begin{pmatrix} 17424\xi^6, & 6336\xi^6 \\ 6336\xi^6, & 2304\xi^6 \end{pmatrix} + g_{1-\text{inst.}}^{(3)} + \mathcal{O}(z^2, \bar{z}^2), \\ g^{(4)} &= \frac{1}{F_2^2(\xi)} \begin{pmatrix} 69696\xi^8, & 25344\xi^8, & 10560\xi^8 \\ 25344\xi^8, & 9216\xi^8, & 3840\xi^8 \\ 10560\xi^8, & 3840\xi^8, & 1600\xi^8 \end{pmatrix} + g_{1-\text{inst.}}^{(4)} + \mathcal{O}(z^2, \bar{z}^2), \end{aligned} \quad (5.53)$$

where

$$\begin{aligned} F_1(\xi) &= 11 - 672\zeta(3)\xi^3, & F_2(\xi) &= 11 + 1344\zeta(3)\xi^3, \\ g_{1-\text{inst.}}^{(2)} &= -\frac{32}{11}\xi^4 \begin{pmatrix} 435z - 11514\bar{z}z + \text{c.c.}, & 162z + 210\bar{z} - 11088\bar{z}z \\ 210z + 162\bar{z} - 11088\bar{z}z, & 77z - 2674\bar{z}z + \text{c.c.} \end{pmatrix} + \mathcal{O}(\xi^5), \\ g_{1-\text{inst.}}^{(3)} &= -\frac{64}{121}\xi^6 \begin{pmatrix} 23265z - 994050\bar{z}z + \text{c.c.}, & 8460z + 10296\bar{z} - 879840\bar{z}z \\ 10296z + 8460\bar{z} - 879840\bar{z}z, & 3744z - 194688\bar{z}z + \text{c.c.} \end{pmatrix} \\ &\quad + \mathcal{O}(\xi^7), \\ g_{1-\text{inst.}}^{(4)} &= -\frac{512}{121}\xi^8 \begin{pmatrix} g_{11}^{(4)} & g_{12}^{(4)} & g_{13}^{(4)} \\ g_{21}^{(4)} & g_{22}^{(4)} & g_{23}^{(4)} \\ g_{31}^{(4)} & g_{32}^{(4)} & g_{33}^{(4)} \end{pmatrix} + \mathcal{O}(\xi^9), \end{aligned}$$

$$\begin{aligned}
g_{11}^{(4)} &= 14355z + 14355\bar{z} - 1513800\bar{z}z, & g_{12}^{(4)} &= \overline{g_{21}^{(4)}} = 5220z + 6138\bar{z} - 647280\bar{z}z, \\
g_{13}^{(4)} &= \overline{g_{31}^{(4)}} = 2175z + 2541\bar{z} - 267960\bar{z}z, & g_{22}^{(4)} &= 2232z + 2232\bar{z} - 276768\bar{z}z, \\
g_{23}^{(4)} &= \overline{g_{32}^{(4)}} = 930z + 924\bar{z} - 114576\bar{z}z, & g_{33}^{(4)} &= 385z + 385\bar{z} - 47432\bar{z}z.
\end{aligned}$$

In the evaluation, the rank of $g^{(3)}$ is one as expected. Therefore we have to remove, for example, the row and column corresponding to “ $\psi\chi$ ” to define $\widetilde{M}$ in eq. (3.53) and compute $g^{(4)}$.

Now we give some consistency checks. First the nilpotency of twisted chiral ring is confirmed up to two-instanton order, i.e.

$$\langle \psi^m \chi^n \overline{\psi^p \chi^q} \rangle_{\mathbb{R}^2} = 0 \quad \text{for} \quad m + 2n = p + 2q \geq 5. \quad (5.54)$$

Second the constraints eq. (4.38) get satisfied. Let $\alpha = d$ in eq. (4.38), we find the A-twisted chiral correlators

$$\begin{aligned}
\langle \psi \cdot \psi \cdot \psi \cdot \psi \rangle_{S^2} &= 132 - 13920z + 1912032z^2, \\
\langle \psi \cdot \psi \cdot \chi \rangle_{S^2} &= 48 - 5952z + 791424z^2, \\
\langle \chi \cdot \chi \rangle_{S^2} &= 20 - 2464z + 327776z^2.
\end{aligned} \quad (5.55)$$

The last chiral correlator can be normalized, by a Kähler transformation, to certain constants as in standard topological field theories. However because of the lack of knowledge on how the operator χ transforms under mirror map, we cannot reproduce the results of chiral correlators in large volume limit as we did in last subsection. It would be interesting to investigate this point further.

6 Discussions

In this paper, we have provided a general method to extract (twisted) chiral ring data directly from deformed partition functions in two-dimensional $\mathcal{N} = (2, 2)$ SCFT. In the context of Calabi-Yau complex moduli, the method is endowed with explicit geometric interpretations. In the examples, we also developed alternative formulas, via localization onto Higgs branch, for (twisted) chiral ring data on the complete intersections in toric varieties.

There would be several interesting directions deserving further studies. First, as we have seen, in the case of complete intersections in toric varieties, the (twisted) chiral ring data can be formulated in terms of sum of factorized vortex and anti-vortex partition functions, or say the periods of their mirror manifolds. It would be instructive to see how this formalism could be generalized to the case when there are more marginal generators and additional primitive generators of dimension greater than one. On the other hand, deforming the partition function by irrelevant operators could help us extract additional information on primitive generators of dimension greater than one. In this sense the deformed partition function is the generating function of all (twisted) chiral ring data. Therefore if one has

(twisted) chiral ring data as input data, it would be possible to reconstruct the deformed partition function reciprocally. Combining these two observations, one may reformulate the partition function of a theory $\mathcal{S}$ in terms of the periods of its mirror $\tilde{\mathcal{S}}$ with certain patterns, as people already knew or conjectured for threefolds [31], fourfolds [32] and perturbative part of partition functions of general n -folds [40].

Secondly it has been known since long time ago that tt^* -equations of (twisted) chiral correlators were valid even for off-critical theories [9]. It would be interesting to extract these correlators directly from the partition functions of off-critical theories. Correspondingly, for critical theories with center charge $c \geq 3$, we have an explanation on the counter terms and operators mixing from the perspective of supergravity, see section 3.3. It would be very nice to extend this picture to off-critical theories, e.g. minimal $\mathcal{N} = (2, 2)$ SCFT perturbed by relevant operators, so that we could have a better understanding how the correlators from S^2 to $\mathbb{R}^2$ are related to each other.

At last, we should mention that the work was originally motivated from an attempt to understand semi-local vortex strings and $2d/4d$ correspondence and so on. It has been well known and studied on the correspondence between the BPS spectrum in $4d$ $\mathcal{N} = 2$ gauge theories on the bulk and that in $2d$ $\mathcal{N} = (2, 2)$ GLSM for vortex strings [41–45]. Especially, for $U(2)$ gauge theory with 4 hypermultiplets, the corresponding worldsheet theory, describing the low-dynamics of critical semi-local vortex string, is a non-compact Calabi-Yau threefold, the resolved conifold $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ [46, 47]. It is straightforward to apply our algorithm (3.55) to non-compact CY cases, so long as one could correctly resolve the singularities in evaluation of the partition functions [32]. It would be curious to see if our computations on twisted chiral ring data have interesting implications or applications in the thin string regime, $r \sim 0$ [48]. We expect to answer some of these questions in subsequent works.

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A Notations and $\mathfrak{su}(2|1)$ superalgebra on S^2

Most of our notations follow from [15]. For self-consistency, we list some of them, and briefly discuss how to obtain the $\mathfrak{su}(2|1)$ superalgebra on S^2 .

Gamma matrices.

$$\begin{aligned} \gamma_\mu &= (\sigma_1, \sigma_2), & \sigma_{1,2} & \text{ are Pauli matrices, for } \mu = 1, 2 \\ \{\gamma_\mu, \gamma_\nu\} &= 2\delta_{\mu\nu}, & \gamma_3 &= -i\gamma_1\gamma_2 = \sigma_3, \quad \gamma_\pm \equiv \frac{1}{2}(1 \pm \gamma_3) \end{aligned}$$

Charge conjugation.

$$\mathcal{C} \equiv \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \mathcal{C}^\dagger = -\mathcal{C}^T$$

satisfying

$$\mathcal{C}^2 = 1, \quad \mathcal{C}\gamma_i\mathcal{C}^{-1} = -\gamma_i^T, \quad \text{for } i = 1, 2, 3$$

Spinors. Throughout the paper, we take Killing spinors

$$\epsilon \equiv \begin{pmatrix} \epsilon_+ \\ \epsilon_- \end{pmatrix}, \quad \tilde{\epsilon} \equiv \begin{pmatrix} \tilde{\epsilon}_+ \\ \tilde{\epsilon}_- \end{pmatrix},$$

as two *independent* $\mathbb{C}$ -valued spinors. For convenience we also define

$$\zeta \equiv \gamma_+\epsilon + \gamma_-\tilde{\epsilon} = \begin{pmatrix} \epsilon_+ \\ \tilde{\epsilon}_- \end{pmatrix}, \quad \tilde{\zeta} \equiv \gamma_+\tilde{\epsilon} + \gamma_-\epsilon = \begin{pmatrix} \tilde{\epsilon}_+ \\ \epsilon_- \end{pmatrix}.$$

For fermionic fields ψ and $\bar{\psi}$,

$$\psi \equiv \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}, \quad \bar{\psi} \equiv \begin{pmatrix} \bar{\psi}_+ \\ \bar{\psi}_- \end{pmatrix},$$

they are considered as two independent two-component Grassmannian spinors and thus anticommuting. Given two spinors ϵ and ψ , the (Euclidean) Lorentz scalar is defined

$$\epsilon \cdot \psi \equiv \epsilon^T \mathcal{C} \psi \quad \text{or} \quad \epsilon^\dagger \psi.$$

Supersymmetries on curved space. We first spell out the supersymmetries of (twisted) chiral multiplets on flat space.¹⁰ For the chiral multiplet $\Phi = (\phi, \psi, \mathcal{O})$ with dimension ω ,

$$\begin{aligned} \delta\phi &= \tilde{\epsilon} \cdot \psi \\ \delta\psi &= i\gamma^\mu \epsilon D_\mu \phi + \tilde{\epsilon} \mathcal{O} \\ \delta\mathcal{O} &= i\epsilon \cdot \gamma^\mu D_\mu \psi \end{aligned}$$

and the twisted chiral multiplet $\Sigma = (\tilde{\sigma}, \tilde{\lambda}, \tilde{\mathcal{O}})$ with dimension $\tilde{\omega}$

$$\begin{aligned} \delta\tilde{\sigma} &= \zeta \cdot \tilde{\lambda}, \\ \delta\tilde{\lambda} &= i\gamma^\mu \tilde{\zeta} D_\mu \tilde{\sigma} + \zeta \tilde{\mathcal{O}} \\ \delta\tilde{\mathcal{O}} &= i\tilde{\zeta} \cdot \gamma^\mu D_\mu \tilde{\lambda}, \end{aligned}$$

where D_μ is ordinary partial derivative so far. To place the fields on curved space, ϵ and $\tilde{\epsilon}$ are spacetime dependent. The above transformations are also required Weyl covariant, see

¹⁰The superalgebra quoted here is from [15]. Curious readers who care about its relation with eq. (2.1) may consult the appendix of [16].

also [16]. Therefore they receive compensations respect to their dimensions,

$$\begin{aligned}\delta\phi &= \tilde{\epsilon} \cdot \psi \\ \delta\psi &= i\gamma^\mu \epsilon D_\mu \phi + \tilde{\epsilon} \mathcal{O} + i\omega \gamma^\mu D_\mu \epsilon \phi \\ \delta\mathcal{O} &= i\epsilon \cdot \gamma^\mu D_\mu \psi + i\omega D_\mu \epsilon \cdot \gamma^\mu \psi\end{aligned}$$

and

$$\begin{aligned}\delta\tilde{\sigma} &= \zeta \cdot \tilde{\lambda}, \\ \delta\tilde{\lambda} &= i\gamma^\mu \tilde{\zeta} D_\mu \tilde{\sigma} + \zeta \tilde{\mathcal{O}} + i\tilde{\omega} \gamma^\mu D_\mu \tilde{\zeta} \tilde{\sigma} \\ \delta\tilde{\mathcal{O}} &= i\tilde{\zeta} \cdot \gamma^\mu D_\mu \tilde{\lambda} + i\tilde{\omega} D_\mu \tilde{\zeta} \cdot \gamma^\mu \tilde{\lambda},\end{aligned}$$

where D_μ is necessary to be improved as covariant derivative respect to the spin connections of curved space.

Killing spinor equations. On S^2 with radius R , there are two different ways to put $\mathcal{N} = (2, 2)$ supersymmetries, $\mathfrak{su}(2|1)_A$ and $\mathfrak{su}(2|1)_B$. They can be realized by imposing different Killing equations on spinors ϵ and $\tilde{\epsilon}$. For $\mathfrak{su}(2|1)_A$,

$$D_\mu \epsilon = \frac{i}{2R} \gamma_\mu \epsilon, \quad D_\mu \tilde{\epsilon} = \frac{i}{2R} \gamma_\mu \tilde{\epsilon},$$

and for $\mathfrak{su}(2|1)_B$, we require

$$D_\mu \epsilon = \frac{i}{2R} \gamma_\mu \tilde{\epsilon}, \quad D_\mu \tilde{\epsilon} = \frac{i}{2R} \gamma_\mu \epsilon,$$

or in terms of ζ and $\tilde{\zeta}$

$$D_\mu \zeta = \frac{i}{2R} \gamma_\mu \tilde{\zeta}, \quad D_\mu \tilde{\zeta} = \frac{i}{2R} \gamma_\mu \zeta.$$

Applying the above equation, we can obtain $\mathfrak{su}(2|1)_A$ and $\mathfrak{su}(2|1)_B$ superalgebra of (twisted) chiral fields respectively, i.e. eq. (3.9, 3.16).

B Irrelevant deformations in GLSM with $U(N)$ gauge group

In this section, we show how to introduce irrelevant deformations for twisted chiral rings, or say a type of generic twisted superpotentials, in GLSM with $U(N)$ gauge group, see also section 3.2.

We first recall the $\mathcal{N} = (2, 2)$ SUSY of vector multiplet $V = (A_\mu, \sigma_1, \sigma_2, \lambda, \bar{\lambda}, D)$ on flat space,

$$\begin{aligned}\delta A_\mu &= -\frac{i}{2} \tilde{\epsilon} \cdot \gamma_\mu \lambda - \frac{i}{2} \epsilon \cdot \gamma_\mu \bar{\lambda} \\ \delta \sigma_1 &= \frac{1}{2} \tilde{\epsilon} \cdot \lambda - \frac{1}{2} \epsilon \cdot \bar{\lambda}, \quad \delta \sigma_2 = -\frac{i}{2} \tilde{\epsilon} \cdot \gamma_3 \lambda - \frac{i}{2} \epsilon \cdot \gamma_3 \bar{\lambda} \\ \delta \lambda &= i\gamma_3 \epsilon (G + i[\sigma_1, \sigma_2]) - \epsilon D + i\gamma^\mu \epsilon D_\mu \sigma_1 - \gamma_3 \gamma^\mu \epsilon D_\mu \sigma_2 \\ \delta \bar{\lambda} &= i\gamma_3 \tilde{\epsilon} (G - i[\sigma_1, \sigma_2]) + \bar{\epsilon} D - i\gamma^\mu \tilde{\epsilon} D_\mu \sigma_1 - \gamma_3 \gamma^\mu \tilde{\epsilon} D_\mu \sigma_2 \\ \delta D &= -\frac{i}{2} \tilde{\epsilon} \cdot \gamma^\mu D_\mu \lambda + \frac{i}{2} \epsilon \cdot \gamma^\mu D_\mu \bar{\lambda} + \frac{i}{2} [\tilde{\epsilon} \cdot \lambda, \sigma_1] + \frac{i}{2} [\epsilon \cdot \bar{\lambda}, \sigma_1] \\ &\quad + \frac{1}{2} [\tilde{\epsilon} \cdot \gamma_3 \lambda, \eta] - \frac{1}{2} [\epsilon \cdot \gamma_3 \bar{\lambda}, \sigma_2]\end{aligned}$$

where

$$G \equiv \frac{1}{2} \epsilon^{\mu\nu} (\partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]),$$

is the $2d$ field strength. In the case of $U(1)$ vector multiplet V , one can define a twisted chiral multiplet $\Sigma = (\sigma, \tilde{\lambda}, \tilde{\mathcal{O}})$,

$$\begin{aligned}\sigma &\equiv \sigma_1 + i\sigma_2 \\ \tilde{\lambda} &\equiv \begin{pmatrix} \lambda_+ \\ -\bar{\lambda}_- \end{pmatrix} \\ \tilde{\mathcal{O}} &\equiv -D + iG,\end{aligned}$$

Σ serving as the superfield strength of V . Its superalgebra can be found from that of vector multiplet,

$$\begin{aligned}\delta\sigma &= \zeta \cdot \tilde{\lambda}, \\ \delta\tilde{\lambda} &= i\gamma^\mu \tilde{\zeta} D_\mu \sigma + \zeta \tilde{\mathcal{O}} \\ \delta\tilde{\mathcal{O}} &= i\tilde{\zeta} \cdot \gamma^\mu D_\mu \tilde{\lambda}.\end{aligned}$$

For non-Abelian vector multiplet, one can find

$$\begin{aligned}\delta\sigma &= \zeta \cdot \tilde{\lambda}, \\ \delta\tilde{\lambda} &= i\gamma^\mu \tilde{\zeta} D_\mu \sigma + \zeta \tilde{\mathcal{O}} - \frac{i}{2} \gamma_3 \zeta [\sigma, \bar{\sigma}] \\ \delta\tilde{\mathcal{O}} &= i\tilde{\zeta} \cdot \gamma^\mu D_\mu \tilde{\lambda} - \frac{i}{2} [\tilde{\zeta} \cdot \gamma_3 \bar{\lambda}, \sigma] - \frac{i}{2} [\zeta \cdot \gamma_3 \tilde{\lambda}, \bar{\sigma}].\end{aligned}$$

where $\bar{\sigma} \equiv \sigma_1 - i\sigma_2$ and $\bar{\tilde{\lambda}} \equiv (\bar{\lambda}_+, -\lambda_-)^T$. The superalgebra of $\Sigma = (\sigma, \tilde{\lambda}, \tilde{\mathcal{O}})$ does *not* close anymore. In addition, we actually want gauge invariant twisted chiral multiplet. We therefore consider taking trace of above multiplet Σ . Indeed $\text{Tr}(\Sigma)$ is a gauge invariant twisted chiral multiplet. If our gauge group contains $U(1)$ factors, it reproduce the result of Abelian case. For non-Abelian gauge group, one can consider further $\frac{1}{2}\text{Tr}(\Sigma^2)$, $(\text{Tr}(\Sigma))^2, \dots$, etc.. One can check by above algebra that they are all gauge invariant twisted chiral operators. Here $\frac{1}{2}\text{Tr}(\Sigma^2)$, for example, has components,

$$\frac{1}{2}\text{Tr}(\Sigma^2) = \left(\frac{1}{2}\text{Tr}(\sigma^2), \text{Tr}(\tilde{\lambda}\sigma), \text{Tr}(\tilde{\mathcal{O}}\sigma - \frac{1}{2}\tilde{\lambda} \cdot \tilde{\lambda}) \right),$$

whose superalgebra,

$$\begin{aligned}\delta\left(\frac{1}{2}\text{Tr}(\sigma^2)\right) &= \zeta \cdot \text{Tr}(\tilde{\lambda}\sigma), \\ \delta\left(\text{Tr}(\tilde{\lambda}\sigma)\right) &= i\gamma^\mu \tilde{\zeta} D_\mu \left(\frac{1}{2}\text{Tr}(\sigma^2)\right) + \zeta \left(\text{Tr}\left(\tilde{\mathcal{O}} - \frac{1}{2}\tilde{\lambda} \cdot \tilde{\lambda}\right)\right) \\ \delta\left(\text{Tr}\left(\tilde{\mathcal{O}}\sigma - \frac{1}{2}\tilde{\lambda} \cdot \tilde{\lambda}\right)\right) &= i\tilde{\zeta} \cdot \gamma^\mu D_\mu \left(\text{Tr}(\tilde{\lambda}\sigma)\right),\end{aligned}$$

shows it indeed a $\mathcal{N} = (2, 2)$ twisted chiral multiplet. In the case of GLSM with $U(N)$ gauge group, the set $\{\text{Tr}(\Sigma), \text{Tr}(\Sigma^2), \text{Tr}(\Sigma^3), \dots, \text{Tr}(\Sigma^N)\}$ exhausts all primitive generators of the twisted chiral ring. One can further put them all on S^2 with respect to their dimensions as discussed in appendix A, see also section 3.2.

C One-instanton correction to twisted chiral ring data

In this appendix, we collect the non-perturbative contribution, up to one-instanton, of twisted chiral ring data $G^{(\alpha)}$ in GN-CY₃ in case of need. The full perturbative expression of $G^{(\alpha)}$ are given in eq. (5.31), (5.33) and (5.36). Meanwhile, the one instanton corrections to $G^{(\alpha)}$ are expanded in terms of $\mathcal{O}(\xi_2)$ while fixing ξ_1 ,

$$G_{1-\text{inst.}}^{(1)} = \begin{pmatrix} I_{1\bar{1}}^{(1)} & I_{1\bar{2}}^{(1)} \\ I_{2\bar{1}}^{(1)} & I_{2\bar{2}}^{(1)} \end{pmatrix} + \mathcal{O}(\xi_2) \quad (\text{C.1})$$

with

$$\begin{aligned} I_{1\bar{1}}^{(1)} &= 56q_1 - 960q_1q_2 + 56\bar{q}_1 - 960\bar{q}_1\bar{q}_2 \\ I_{1\bar{2}}^{(1)} &= \overline{I_{1\bar{2}}^{(1)}} = -384q_1q_2 - 960\bar{q}_1\bar{q}_2 \\ I_{2\bar{2}}^{(1)} &= -384q_1q_2 - 384\bar{q}_1\bar{q}_2, \end{aligned} \quad (\text{C.2})$$

$$G_{1-\text{inst.}}^{(2)} = \xi_2 \begin{pmatrix} I_{1\bar{1}\bar{1}\bar{1}}^{(2)}, I_{1\bar{1}\bar{1}\bar{2}}^{(2)}, I_{1\bar{1}\bar{2}\bar{2}}^{(2)} \\ I_{1\bar{2}\bar{1}\bar{1}}^{(2)}, I_{1\bar{2}\bar{1}\bar{2}}^{(2)}, I_{1\bar{2}\bar{2}\bar{2}}^{(2)} \\ I_{2\bar{2}\bar{1}\bar{1}}^{(2)}, I_{2\bar{2}\bar{1}\bar{2}}^{(2)}, I_{2\bar{2}\bar{2}\bar{2}}^{(2)} \end{pmatrix} + \mathcal{O}(\xi_2^2), \quad (\text{C.3})$$

with

$$\begin{aligned} I_{1\bar{1}\bar{1}\bar{1}}^{(2)} &= -\frac{280}{3}q_1 + 1280q_1q_2 + \frac{784}{3}q_1\bar{q}_1 - 896q_1q_2\bar{q}_1 + 6144q_1q_2\bar{q}_1\bar{q}_2 + \text{c.c.} \\ I_{1\bar{1}\bar{1}\bar{2}}^{(2)} &= \overline{I_{1\bar{2}\bar{1}\bar{1}}^{(2)}} = -56q_1 + 960q_1q_2 + 1280\bar{q}_1\bar{q}_2 - 896q_1\bar{q}_1\bar{q}_2 + 12288q_1q_2\bar{q}_1\bar{q}_2 \\ I_{1\bar{1}\bar{2}\bar{2}}^{(2)} &= \overline{I_{2\bar{2}\bar{1}\bar{1}}^{(2)}} = -384q_1q_2 + 1280\bar{q}_1\bar{q}_2 - 896q_1\bar{q}_1\bar{q}_2 + 12288q_1q_2\bar{q}_1\bar{q}_2 \\ I_{1\bar{2}\bar{1}\bar{2}}^{(2)} &= 960q_1q_2 + 960\bar{q}_1\bar{q}_2 + 12288q_1q_2\bar{q}_1\bar{q}_2 \\ I_{1\bar{2}\bar{2}\bar{2}}^{(2)} &= \overline{I_{2\bar{2}\bar{1}\bar{2}}^{(2)}} = 384q_1q_2 + 960\bar{q}_1\bar{q}_2 + 12288q_1q_2\bar{q}_1\bar{q}_2 \\ I_{2\bar{2}\bar{2}\bar{2}}^{(2)} &= 384q_1q_2 + 384\bar{q}_1\bar{q}_2 + 12288q_1q_2\bar{q}_1\bar{q}_2, \end{aligned} \quad (\text{C.4})$$

and

$$G_{1-\text{inst.}}^{(3)} = \xi_2^3 \begin{pmatrix} I_{1\bar{1}\bar{1}\bar{1}\bar{1}\bar{1}}^{(3)}, I_{1\bar{1}\bar{1}\bar{1}\bar{1}\bar{2}}^{(3)}, I_{1\bar{1}\bar{1}\bar{1}\bar{2}\bar{2}}^{(3)}, I_{1\bar{1}\bar{1}\bar{2}\bar{2}\bar{2}}^{(3)} \\ I_{1\bar{1}\bar{2}\bar{1}\bar{1}\bar{1}}^{(3)}, I_{1\bar{1}\bar{2}\bar{1}\bar{1}\bar{2}}^{(3)}, I_{1\bar{1}\bar{2}\bar{1}\bar{2}\bar{2}}^{(3)}, I_{1\bar{1}\bar{2}\bar{2}\bar{2}\bar{2}}^{(3)} \\ I_{1\bar{2}\bar{2}\bar{1}\bar{1}\bar{1}}^{(3)}, I_{1\bar{2}\bar{2}\bar{1}\bar{1}\bar{2}}^{(3)}, I_{1\bar{2}\bar{2}\bar{1}\bar{2}\bar{2}}^{(3)}, I_{1\bar{2}\bar{2}\bar{2}\bar{2}\bar{2}}^{(3)} \\ I_{2\bar{2}\bar{2}\bar{1}\bar{1}\bar{1}}^{(3)}, I_{2\bar{2}\bar{2}\bar{1}\bar{1}\bar{2}}^{(3)}, I_{2\bar{2}\bar{2}\bar{1}\bar{2}\bar{2}}^{(3)}, I_{2\bar{2}\bar{2}\bar{2}\bar{2}\bar{2}}^{(3)} \end{pmatrix} + \mathcal{O}(\xi_2^4), \quad (\text{C.5})$$

with

$$\begin{aligned}
 I_{111\bar{1}\bar{1}\bar{1}}^{(3)} &= 840q_1 + 2880q_1q_2 + 1176q_1\bar{q}_1 + 8064q_1q_2\bar{q}_1 + 13824q_1q_2\bar{q}_1\bar{q}_2 + \text{c.c.} \\
 I_{111\bar{1}\bar{1}\bar{2}}^{(3)} &= \overline{I_{112\bar{1}\bar{1}\bar{1}}^{(3)}} = 840q_1 + 2880q_1q_2 + 2880\bar{q}_1\bar{q}_2 + 8064q_1\bar{q}_1\bar{q}_2 + 27648q_1q_2\bar{q}_1\bar{q}_2 \\
 I_{111\bar{1}\bar{2}\bar{2}}^{(3)} &= \overline{I_{122\bar{1}\bar{1}\bar{1}}^{(3)}} = 672q_1 + 2304q_1q_2 + 2880\bar{q}_1\bar{q}_2 + 8064q_1\bar{q}_1\bar{q}_2 + 27648q_1q_2\bar{q}_1\bar{q}_2 \\
 I_{111\bar{2}\bar{2}\bar{2}}^{(3)} &= \overline{I_{222\bar{1}\bar{1}\bar{1}}^{(3)}} = 336q_1 + 1152q_1q_2 + 2880\bar{q}_1\bar{q}_2 + 8064q_1\bar{q}_1\bar{q}_2 + 27648q_1q_2\bar{q}_1\bar{q}_2 \\
 I_{112\bar{1}\bar{1}\bar{2}}^{(3)} &= \overline{I_{112\bar{1}\bar{1}\bar{2}}^{(3)}} = 2880q_1q_2 + 2880\bar{q}_1\bar{q}_2 + 27648q_1q_2\bar{q}_1\bar{q}_2 \\
 I_{112\bar{1}\bar{2}\bar{2}}^{(3)} &= \overline{I_{122\bar{1}\bar{1}\bar{2}}^{(3)}} = 2304q_1q_2 + 2880\bar{q}_1\bar{q}_2 + 27648q_1q_2\bar{q}_1\bar{q}_2 \\
 I_{112\bar{2}\bar{2}\bar{2}}^{(3)} &= \overline{I_{222\bar{1}\bar{1}\bar{2}}^{(3)}} = 1152q_1q_2 + 2880\bar{q}_1\bar{q}_2 + 27648q_1q_2\bar{q}_1\bar{q}_2 \\
 I_{122\bar{1}\bar{2}\bar{2}}^{(3)} &= \overline{I_{122\bar{1}\bar{2}\bar{2}}^{(3)}} = 2304q_1q_2 + 2304\bar{q}_1\bar{q}_2 + 27648q_1q_2\bar{q}_1\bar{q}_2 \\
 I_{122\bar{2}\bar{2}\bar{2}}^{(3)} &= \overline{I_{222\bar{1}\bar{2}\bar{2}}^{(3)}} = 1152q_1q_2 + 2304\bar{q}_1\bar{q}_2 + 27648q_1q_2\bar{q}_1\bar{q}_2 \\
 I_{222\bar{2}\bar{2}\bar{2}}^{(3)} &= 1152q_1q_2 + 1152\bar{q}_1\bar{q}_2 + 27648q_1q_2\bar{q}_1\bar{q}_2.
 \end{aligned} \tag{C.6}$$

Together with the perturbative part of $G^{(3)}$ from eq. (5.36), by use of eq. (4.38), one can obtain the topological correlators in the A-twist theory to the order of one-instanton,

$$\begin{aligned}
 \langle \chi_1^3 \rangle_{S^2}(t_1, t_2) &= 20 + 56q_1 + 192q_1q_2, \\
 \langle \chi_1^2\chi_2 \rangle_{S^2}(t_1, t_2) &= 20 + 192q_1q_2, \\
 \langle \chi_1\chi_2^2 \rangle_{S^2}(t_1, t_2) &= 16 + 192q_1q_2 \\
 \langle \chi_2^3 \rangle_{S^2}(t_1, t_2) &= 8 + 192q_1q_2.
 \end{aligned} \tag{C.7}$$

As a simple consistency check, we can use mirror map to obtain Yukawa couplings in τ -coordinates [37], in a similar fashion of eq. (5.38),

$$\mathcal{C}_{ijk}^{(3)}(\tilde{\tau}) = \left(\frac{dt_l}{d\tilde{\tau}_i} \right) \left(\frac{dt_m}{d\tilde{\tau}_j} \right) \left(\frac{dt_n}{d\tilde{\tau}_k} \right) \mathcal{C}_{lmn}^{(3)}(t(\tilde{\tau})), \quad \text{with } i, j, k, l, m, n = 1, 2. \tag{C.8}$$

Applying eq. (C.8) to eq. (C.7), we arrive at

$$\begin{aligned}
 \langle \chi_1^3 \rangle_{S^2}(\tilde{\tau}_1, \tilde{\tau}_2) &= 20 + 296z_1 - 208z_1z_2, \\
 \langle \chi_1^2\chi_2 \rangle_{S^2}(\tilde{\tau}_1, \tilde{\tau}_2) &= 20 + 128z_1 + 80z_2 + 304z_1z_2, \\
 \langle \chi_1\chi_2^2 \rangle_{S^2}(\tilde{\tau}_1, \tilde{\tau}_2) &= 16 + 32z_1 + 160z_2 - 160z_1z_2 \\
 \langle \chi_2^3 \rangle_{S^2}(\tilde{\tau}_1, \tilde{\tau}_2) &= 8 + 192z_2 - 768z_1z_2,
 \end{aligned} \tag{C.9}$$

with $z_i = e^{2\pi i \tilde{\tau}_i}$. Since the chiral correlators are determined up to a Kähler transformation, one can verify that, accompanied by a transformation

$$\begin{aligned}
 T(\tilde{\tau}_1, \tilde{\tau}_2) &= 1 + 2z_1 - 4z_2 - 28z_1z_2, \\
 \mathcal{C}_{ijk}^{(3)}(\tilde{\tau}_1, \tilde{\tau}_2) &\longrightarrow T(\tilde{\tau}_1, \tilde{\tau}_2) \mathcal{C}_{ijk}^{(3)}(\tilde{\tau}_1, \tilde{\tau}_2)
 \end{aligned} \tag{C.10}$$

reproduce the result of [38] to one-instanton order.

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