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Special geometry on the 101 dimesional moduli space of the quintic threefold

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Abstract: A new method for explicit computation of the CY moduli space metric was proposed by the authors recently. The method makes use of the connection of the moduli space with a certain Frobenius algebra. Here we clarify this approach and demonstrate its efficiency by computing the Special geometry of the 101-dimensional moduli space of the quintic threefold around the orbifold point.

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Contents

1 Introduction

When compactifying the IIB superstring theory on a Calabi-Yau (CY) threefold X, one can write the low-energy effective theory in terms of the geometry of the CY moduli space [8]. More precisely, the effective Lagrangian of the vector multiplets in the superspace contains $h^{2,1}$ supermultiplets. Scalars from these multiplets take value in the target space $\mathcal{M}$, which is a moduli space of complex structures on a CY manifold and is a special Kähler manifold itself [9, 13, 24]. Metric $G_{a\bar{b}}$ and Yukawa couplings κ_{abc} on this space are given by the following formulae:

$$
\begin{aligned}
G_{a\bar{b}} &= \partial_a \partial_{\bar{b}} K, \qquad e^{-K} = -i \int_X \Omega \wedge \bar{\Omega}, \\
\kappa_{abc} &= \int_X \Omega \wedge \partial_a \partial_b \partial_c \Omega = \frac{\partial^3 F}{\partial z^a \partial z^b \partial z^c},
\end{aligned}
\tag{1.1}
$$

where

$$
z^a = \int_{A_a} \Omega, \; \frac{\partial F}{\partial z^a} = \int_{B^a} \Omega \tag{1.2}
$$

are the period integrals of the holomorphic volume form Ω on X. Here A_a and B^a form the symplectic basis in $H_3(X, \mathbb{Z})$. We can rewrite the expression (1.1) for the Kähler potential using the periods as

$$
e^{-K} = -i\Pi\Sigma\Pi^{\dagger}, \; \Pi = (\partial F, \; z), \tag{1.3}
$$

where matrix $(\Sigma)^{-1}$ is an intersection matrix of cycles A_a, B^a equal to the symplectic unit. In practice, computation of periods in the symplectic basis is a very complicated problem and was done explicitly only in few examples [10–12, 20]. It is due to the fact, that it requires a case by case analysis and geometric description of the symplectic basis of cycles. Recently we proposed a method [1, 2] to easily compute the Kähler metric (and the symplectic basis) for a large class of CY manifolds which can be represented by specific hypersurfaces in weighted projective spaces [6]. Our method does not require the knowledge of symplectic cycles, but instead uses a structure of a Frobenius algebra associated with a CY of this class and its Hodge structure.[1]

Namely, let a CY manifold X be given as a solution of an equation

$$W(x,\phi) = W_0(x) + \sum_{s=1}^{h^{2,1}} \phi_s e_s(x) = 0 \tag{1.4}$$

in some weighted projective space, where $W_0(x)$ is a quasihomogeneous function in $\mathbb{C}^5$ of weight d that defines an isolated singularity at $x = 0$,(see [3]) which is tightly related with the underlying $N = 2$ superconformal theory [18, 21, 22]. The monomials $e_s(x)$ also have weight d and correspond to deformations of the complex structure of X.

Polynomial $W_0(x)$ defines a Milnor ring R_0. Inside R_0 there exists a subring R_0^Q which is invariant w.r.t. the action of the so-called quantum symmetry group Q. This group acts on $\mathbb{C}^5$ diagonally, and preserves $W(x,\phi)$. In many cases $\dim R_0^Q = \dim H^3(X)$ and the ring itself has a Hodge structure $R_0^Q = (R_0^Q)^0 \oplus (R_0^Q)^1 \oplus (R_0^Q)^2 \oplus (R_0^Q)^3$ in correspondence with degrees of the elements $0, d, 2d, 3d$. One can introduce an invariant pairing η on R_0^Q. The pairing turns the ring to a Frobenius algebra [17] and plays an important role in the construction of our formula for e^{-K}.

Using the invariant ring R_0^Q and differentials $D_\pm = \mathrm{d} \pm \mathrm{d}W_0\wedge$ we construct two groups of $Q-$invariant cohomology $H^5_{D_\pm}(\mathbb{C}^5)_{\mathrm{inv}}$. These groups inherit the Hodge structure from R_0^Q. If we denote by $\{e_\mu(x)\}$ some basis of R_0^Q, then $\{e_\mu(x)\,\mathrm{d}^5x\}$ will be a basis of $H^5_{D_\pm}(\mathbb{C}^5)_{\mathrm{inv}}$. As shown by Candelas [7], elements of these cohomology groups are in correspondence with harmonic forms of $H^3(X)$. This isomorphism sends components $H^{3-q,q}(X)$ to the Hodge decomposition components of $H^5_\pm(\mathbb{C}^5)_{\mathrm{inv}}$ spanned by $e_\mu(x)\,\mathrm{d}^5x$ with $e_\mu(x) \in (R_0^Q)^q$ and sends the pairing on the differential forms on X to the invariant Frobenius algebra pairing η. Also the same isomorphism allows to define a complex conjugation (we denote this operation $*$) on the invariant cohomology.

It turns out, that in the basis $\{e_\mu(x)\}$ the operation $*$ reads

$$* \, e_\mu(x)\,\mathrm{d}^5x = M_\mu^\nu e_\nu(x)\,\mathrm{d}^5x\,. \tag{1.5}$$

If we pick a basis $\{e_\mu(x)\}$ such, that the Frobenius pairing η is antidiagonal, the matrix M is antidiagonal as well:

$$* \, e_\mu(x)\,\mathrm{d}^5x = A^\mu \, e_{\mu'}(x)\,\mathrm{d}^5x, \tag{1.6}$$

where $e_{\mu'}(x)$ is the unique element of the basis such, that $\eta(e_\mu(x), e_{\mu'}(x)) = 1$, that is $e_\mu(x)\cdot e_{\mu'}(x) = e_\rho(x)$, which is a unique up to a constant element of degree 3d. Coefficients A_μ are the coefficients of the matrix M in this basis. In particular, we have a useful relation: $A_\mu \overline{A_{\mu'}} = 1$, because M is an anti-involution. We compute A_μ in the section 5 for the quintic threefold, see also section 6.

[1] Actually, moduli space of a CY manifold is closely related with a Frobenius manifold [17], and the Frobenius algebra, we use, is a tangent space to this manifold at one point.

Having $H^5_{D_\pm}(\mathbb{C}^5)_{\text{inv}}$ we define the relative invariant homology groups $\mathcal{H}_5^{\pm,\text{inv}} := H_5(\mathbb{C}^5, W_0 = L,\ \text{Re}L \to \pm\infty)_{\text{inv}}$ inside a relative homology group $H_5(\mathbb{C}^5, W_0 = L,\ \text{Re}L \to \pm\infty)$. For this purpose we use oscillatory integrals. Using the oscillatory integral pairing we define a cycle $\Gamma^\pm_\mu$ in the basis of relative invariant homology to be dual to $e_\mu(x)\,\mathrm{d}^5x$.

At last we define periods $\sigma^\pm_\mu(\phi)$ to be oscillatory integrals over the basis of cycles $\Gamma^\pm_\mu$, which can be effectively computed using the techinque described in [1, 2, 4] that we remind in the section 4. The periods $\sigma^\pm_\mu(\phi)$ are equal to periods of the holomorphic volume form Ω on X in a special basis of cycles $H_3(X,\mathbb{C})$ with complex coefficients.

As shown in [1], Kähler potential for the metric is given by the followng formula

$$e^{-K(\phi)} = \sum_{\mu,\nu,\lambda} \sigma^+_\mu(\phi)\eta^{\mu\lambda}M^\nu_\lambda\overline{\sigma^-_\nu(\phi)}, \tag{1.7}$$

where the real structure matrix M is the same as the one in (1.5). Matrix M can be also represented as $M = T^{-1}\bar{T}$, where T is a transition matrix from periods in arbitrary real basis of cycles $Q^\pm_\mu$ to periods $\sigma^\pm_\mu(\phi)$. In our basis matrix η is antidiagonal, and it follows, that

$$e^{-K(\phi)} = \sum_\mu (-1)^{|\nu|}\sigma^+_\mu(\phi)A^\mu\overline{\sigma^-_\mu(\phi)}. \tag{1.8}$$

Using this we are able to explicitly compute the diagonal matrix elements A^μ and to obtain the explicit expression for the whole e^{-K}.

In [1, 2], to find the real structure, we used the knowledge of periods in some integral basis of homology cycles (e.g. from [5]). However this basis is not always known.

In this paper we propose another method to compute the real structure matrix M and apply it to the 101-dimensional moduli space of the quintic threefold complex structures around the orbifold point to get an explicit exact result for the moduli space Kähler metric. Together with the knowledge of the geometry of the 1-dimensional moduli space of the quintic Kähler structures computed via the mirror symmetry in [11] it presumably gives the geometry of the full moduli space of Calabi-Yau quintic threefold.

In what follows we apply our method for the quintic threefold, where the huge symmetry group $S_5 \ltimes (\mathbb{Z}_5)^5$ simplifies the computations. $(\mathbb{Z}_5)^5$ is called a group of phase symmetries, it acts diagonally on $\mathbb{C}^5$ and preserves $W_0(x)$. It acts naturally on the invariant ring R_0^Q, and this action respects the Hodge decomposition of R_0^Q. This allows to pick a basis $e_\mu(x)$ in each of the Hodge decomposition components of R_0^Q, which consists of eigenvectors of the phase symmetry group action, which simplifies our computations. The S_5, which acts by permutations of x_i among themselves, allows to reduce the amount of computations even further.

If we consider other hypersurfaces in weighted projective spaces, they have less symmetry then the quintic threefold. However, most of the considerations are true and allow to perform explicit computations in the more general case, as we briefly discuss in the conclusion 7.

2 Hodge structure on the middle cohomology of the quintic

First of all we notice, that the formula (1.3) may be written in the arbitrary basis of cycles $q_\mu \in H_3(X,\mathbb{Z})$:

$$\begin{aligned} e^{-K(\phi)} &= \omega_\mu(\phi) C^{\mu\nu} \overline{\omega_\nu(\phi)}, \\ \omega_\mu(\phi) &= \int_{q_\mu} \Omega \end{aligned} \tag{2.1}$$

and $(C^{-1})_{\mu\nu} = q_\mu \cap q_\nu$.

Now let us specialize to the case where X is a quintic threefold:

$$X = \{(x_1 : \cdots : x_5) \in \mathbb{P}^4 \mid W(x,\phi) = 0\}, \tag{2.2}$$

where

$$W(x,\phi) = W_0(x) + \sum_{t=0}^{100} \phi_t e_t(x), \; W_0(x) = x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5\,, \tag{2.3}$$

and $e_t(x)$ are the degree 5 monomials such that each variable has the power that is a non-negative integer less then four. Let us denote monomials $e_t(x) = x_1^{t_1} x_2^{t_2} x_3^{t_3} x_4^{t_4} x_5^{t_5}$ by its degree vector $t = (t_1, \cdots, t_5)$. Then there are precisely 101 of such monomials, which can be divided into 5 sets in respect to the permutation group S_5: $(1,1,1,1,1)$, $(2,1,1,1,0)$, $(2,2,1,0,0)$, $(3,1,1,0,0)$, $(3,2,0,0,0)$. In these groups there are correspondingly 1, 20, 30, 30, 20 different monomials. We denote $e_0(x) := e_{(1,1,1,1,1)}(x) = x_1x_2x_3x_4x_5$ to be the so-called fundamental monomial, which will be somewhat distinguished in our picture.

For this CY $\dim H_3(X) = 204$ and period integrals have the form

$$\omega_\mu(x) = \int_{q_\mu} \frac{x_5\, \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3}{\partial W(x,\phi)/\partial x_4} = \int_{Q_\mu} \frac{\mathrm{d}x_1 \cdots \mathrm{d}x_5}{W(x,\phi)}, \tag{2.4}$$

where $q_\mu \in H_3(X,\mathbb{Z})$ and $Q_\mu \in H_5(\mathbb{C}^5 \backslash (W(x,\phi) = 0), \mathbb{Z})$ are the corresponding cycles. Cohomology groups of a Kähler manifold possess a Hodge structure $H^3(X) = H^{3,0}(X) \oplus H^{2,1}(X) \oplus H^{1,2}(X) \oplus H^{0,3}(X)$. Period integrals measure variation of the Hodge structure on $H^3(X)$ as the complex structure on X varies with ϕ. This Hodge structure variation is equivalent to the one on a certain ring which we will now describe.

3 Hodge structure on the invariant Milnor ring

We can consider $W_0(x)$ as a singularity in $\mathbb{C}^5$. Then there is an associated Milnor (also Jacobi) ring

$$R_0 = \frac{\mathbb{C}[x_1, \cdots, x_5]}{\langle \partial_i W \rangle}. \tag{3.1}$$

We will identify its elements with unique smallest degree polynomial representatives. For the quintic threefold X its Milnor ring R_0 is generated as a vector space by monomials where each variable has degree less than four, and $\dim R_0 = 1024$. Polynomial $W_0(x)$ is homogeneous and, in particular, $W_0(\alpha x_1, \ldots, \alpha x_5) = W_0(x_1, \ldots, x_5)$ for $\alpha^5 = 1$. This action preserves $W_0(x)$ and is trivial in the corresponding projective space and on X. Such a group with this action is called a *quantum symmetry* Q, in our case $Q \simeq \mathbb{Z}_5$. Q obviously acts on the Milnor ring R_0.

Now we define a subring R_0^Q in the Milnor ring R_0,

$$R_0^Q := \{e_\mu(x) \in R_0 \mid e_\mu(\alpha x) = e_\mu(x)\}, \ \alpha^5 = 1, \tag{3.2}$$

to be a Q-invariant part of the Milnor ring.

It is multiplicatively generated by 101 fifth-degree monomials $e_t(x)$ from (2.3). More precisely, R_0^Q consists of elements of degree $0, 5, 10$ and 15, dimensions of the corresponding subspaces are $1, 101, 101$ and 1. This degree filtration defines a Hodge structure on R_0^Q. Basically R_0^Q is isomorphic to $H^3(X)$ and the isomorphism sends the degree filtration to the Hodge filtration on $H^3(X)$ [7]. Let us denote $\chi_{\bar{j}}^i = g^{i\bar{k}}\chi_{\bar{k}\bar{j}}$ as an extrinsic curvature tensor for the hypersurface $W(x,\phi) = 0$ in $\mathbb{P}^4$. Then the isomorphism above can be written as a map from R_0^Q to closed differential forms in $H^3(X)$:

$$\begin{aligned}
1 &\to \Omega_{ijk} \in H^{3,0}(X), \\
e_\mu(x) &\to e_\mu(x(y))\, \chi_{\bar{i}}^l\, \Omega_{ljk} \in H^{2,1}(X) \text{ if } |\mu| = 5, \\
e_\mu(x) &\to e_\mu(x(y))\, \chi_{\bar{i}}^l\, \chi_{\bar{j}}^m\, \Omega_{lmk} \in H^{1,2}(X) \text{ if } |\mu| = 10, \\
e_\rho(x) = x_1^3x_2^3x_3^3x_4^3x_5^3 &\to \chi_{\bar{i}}^l\, \chi_{\bar{j}}^m\, \chi_{\bar{k}}^p\, \Omega_{lmp} = \kappa\bar{\Omega} \in H^{0,3}(X)\,.
\end{aligned} \tag{3.3}$$

The details on this map can be found in [7, 14]. We also introduce the notation $e_\mu(x)$ for elements of the monomial basis of R_0^Q, where $\mu = (\mu_1, \cdots, \mu_5)$, $\mu_i \in \mathbb{Z}_+^5$, $e_\mu(x) = \prod_i x_i^{\mu_i}$ and $|\mu| = \sum \mu_i$ is the degree of $e_\mu(x)$. In particular, $\rho = (3,3,3,3,3)$, that is $e_\rho(x)$ is a unique degree 15 element of R_0^Q.

There is a $\mathbb{Z}_5^5$ phase symmetry group acting diagonally on $\mathbb{C}^5$: $\alpha \cdot (x_1, \cdots, x_5) = (\alpha_1 x_1, \cdots, \alpha_5 x_5)$, $\alpha_i^5 = 1$. This action preserves $W_0 = \sum_i x_i^5$. The mentioned above quantum symmetry Q is a diagonal subgroup of the phase symmetries. Basis $\{e_\mu(x)\}$ is an eigenbasis of the phase symmetry and each $e_\mu(x)$ has a unique weight. Note that phase symmetry preserves the Hodge decomposition.

One additional important fact is that on the invariant ring R_0^Q there exists a natural invariant pairing turning it into a Frobenius algebra [1, 17]:

$$\eta_{\mu\nu} = \mathrm{Res} \frac{e_\mu(x)\, e_\nu(x)}{\prod_i \partial_i W_0(x)}. \tag{3.4}$$

Up to an irrelevant constant for the monomial basis it is $\eta_{\mu\nu} = \delta_{\mu+\nu,\rho}$. This pairing plays a crucial role in our construction.

Let us introduce a couple of differentials [23] on differential forms on $\mathbb{C}^5$: $D_\pm = \mathrm{d} \pm \mathrm{d}W_0(x)\wedge$. They define the cohomology groups $H^*_{D_\pm}(\mathbb{C}^5)$. The cohomologies are only nontrivial in the top dimension $H^5_{D_\pm}(\mathbb{C}^5) \overset{J}{\simeq} R_0$. The isomorphism J has an explicit description

$$J(e_\mu(x)) = e_\mu(x)\, \mathrm{d}^5 x, \ e_\mu(x) \in R_0. \tag{3.5}$$

We see, that $Q = \mathbb{Z}_5$ naturally acts on $H^5_{D_\pm}(\mathbb{C}^5)$ and J sends the Q-invariant part R_0^Q to Q-invariant subspace $H^5_{D_\pm}(\mathbb{C}^5)_{\mathrm{inv}}$. Therefore, the latter space obtains the Hodge structure as well. Actually, this Hodge structure naturally corresponds to the Hodge structure on $H^3(X)$.

The complex conjugation acts on $H^3(X)$ so that $\overline{H^{p,q}(X)} = H^{q,p}(X)$, in particular $\overline{H^{2,1}(X)} = H^{1,2}(X)$. Through the isomorphism between R_0^Q and $H^3(X)$ the complex conjugation acts also on the elements of the ring R_0^Q as $*e_\mu(x) = p_\mu e_{\rho-\mu}(x)$, where p_μ is a constant to be determined. In particular, differential form built from $e_\mu(x) + p_\mu e_{\rho-\mu}(x) \in H^3(X,\mathbb{R})$ is real and $p_\mu p_{\rho-\mu} = 1$.

4 Oscillatory representation and computation of $\boldsymbol{\sigma_\mu(\phi)}$

Relative homology groups $H_5(\mathbb{C}^5, W_0 = L,\ \mathrm{Re}L \to \pm\infty)$ have a natural pairing with Q-invariant cohomology groups $H^5_{D_\pm}(\mathbb{C}^5)_{\mathrm{inv}}$:

$$\langle e_\mu(x)\mathrm{d}^5x, \Gamma^\pm\rangle = \int_{\Gamma^\pm} e_\mu(x) e^{\mp W_0(x)} \mathrm{d}^5x,\ H_5(\mathbb{C}^5, W_0 = L,\ \mathrm{Re}L \to \pm\infty). \tag{4.1}$$

Using this we define two invariant homology groups[2] $\mathcal{H}_5^{\pm,\mathrm{inv}}$ as quotient of $H_5(\mathbb{C}^5, W_0 = L,$ $\mathrm{Re}L \to \pm\infty)$ with respect to the subgroups orthogonal to $H^5_{D_\pm}(\mathbb{C}^5)_{\mathrm{inv}}$. Now we introduce basises $\Gamma_\mu^\pm$ in the homology groups $\mathcal{H}_5^{\pm,\mathrm{inv}}$ using the duality with the basises in $H^5_{D_\pm}(\mathbb{C}^5)_{\mathrm{inv}}$:

$$\int_{\Gamma_\mu^\pm} e_\nu(x) e^{\mp W_0(x)} \mathrm{d}^5x = \delta_{\mu\nu} \tag{4.2}$$

and the corresponding periods

$$\begin{aligned} \sigma^\pm_{\alpha\mu}(\phi) &:= \int_{\Gamma_\mu^\pm} e_\alpha(x) e^{\mp W(x,\phi)} \mathrm{d}^5x, \\ \sigma^\pm_\mu(\phi) &:= \sigma^\pm_{0\mu}(\phi) \end{aligned} \tag{4.3}$$

which are understood as series expansions in ϕ around zero.

Periods $\sigma^\pm_\mu(\phi)$ satisfy the same differential equation as periods $\omega_\mu(\phi)$ of the holomorphic volume form on X. Moreover, these sets of periods span same subspaces as functions of ϕ. It follows, that we can define cycles $Q_\mu^\pm \in \mathcal{H}_5^{\pm,\mathrm{inv}}$ such that

$$\int_{Q_\mu^\pm} e^{\mp W(x,\phi)} \mathrm{d}^5x = \int_{q_\mu} \Omega = \int_{Q_\mu} \frac{\mathrm{d}^5x}{W(x,\phi)}\,, \tag{4.4}$$

and periods $\omega^\pm_{\alpha\mu}(\phi)$ are given by the integrals over these cycles analogous to (4.3).

With these notations the idea of computation of periods [4]

$$\sigma^\pm_\mu(\phi) = \int_{\Gamma_\mu^\pm} e^{\mp W(x,\phi)}\, \mathrm{d}^5x \tag{4.5}$$

can be stated as follows.

To explicitly compute $\sigma^\pm_\mu(\phi)$, first we expand the exponent in the integral (4.5) in ϕ representing $W(x,\phi) = W_0(x) + \sum_s \phi_s e_s(x)$

$$\sigma^\pm_\mu(\phi) = \sum_m \left(\prod_s \frac{(\pm\phi_s)^{m_s}}{m_s!}\right) \int_{\Gamma_\mu^\pm} \prod_r e_r(x)^{m_r}\, e^{\mp W_0(x)}\, \mathrm{d}^5x, \tag{4.6}$$

[2] We are grateful to V. Vasiliev for explaining to us the details about these homology groups and their connection with the middle homology of X.

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where $m := \{m_s\}_s$, $m_s \geq 0$ denotes a multi-index of powers of ψ_s in the expansion above. We note, that $\sigma^-_\mu(\phi) = (-1)^{|\mu|}\sigma^+_\mu(\phi)$, so we focus on $\sigma_\mu(\phi) := \sigma^+_\mu(\phi)$.

For each of the summands in (4.6) the form $\prod_s e_s(x)^{m_s}\, \mathrm{d}^5x$ belongs to $H^5_{D_\pm}(\mathbb{C}^5)_{\mathrm{inv}}$, because it is $Q-$invariant. Therefore, we can expand it in the basis $\{e_\mu(x)\, \mathrm{d}^5x\}_{\mu=1}^{\dim R_0^Q} \subset H^5_{D_\pm}(\mathbb{C}^5)_{\mathrm{inv}}$. Namely we always can find such a polynomial $4-$form U, that

$$\prod_s e_s(x)^{m_s}\, \mathrm{d}^5x = \sum_\nu C_\nu(m)\, e_\nu(x)\, \mathrm{d}^5x + D_+U, \tag{4.7}$$

where $C_\nu(m)$ are uniquely determined as coefficients of the expansion of the l.h.s. in the basis $e_\mu(x)\, \mathrm{d}^5x$. Therefore for the integral in (4.6) we obtain

$$\int_{\Gamma^\pm_\mu} \prod_s e_s(x)^{m_s}\, e^{\mp W_0(x)}\, \mathrm{d}^5x = C_\mu(m). \tag{4.8}$$

Writing (4.6) explicitly we have

$$\sigma_\mu(\phi) = \sum_m \left(\prod_s \frac{\phi_s^{m_s}}{m_s!}\right) \int_{\Gamma^+_\mu} \prod_{s,i} x_i^{m_s s_i}\, e^{-W_0(x)}\, \mathrm{d}^5x. \tag{4.9}$$

Let $m_s s_i = 5n_i + \nu_i$, $\nu_i < 5$. Therefore we want to expand

$$\prod_i x_i^{5n_i+\nu_i}\, \mathrm{d}^5x = \sum_\nu C_\nu(m)\, e_\nu(x)\, \mathrm{d}^5x + D_+U. \tag{4.10}$$

Note that

$$\begin{aligned} D_+\left(\frac{1}{5}x_1^{5n+k-4}\, f(x_2,\cdots,x_5)\, \mathrm{d}x_2\wedge\cdots\wedge \mathrm{d}x_5\right) &= \\ = \left[x_1^{5n+k} + \left(n+\frac{k-4}{5}\right)x_1^{5(n-1)+k}\right] f(x_2,\cdots,x_5)\, \mathrm{d}^5x\,. \end{aligned} \tag{4.11}$$

Therefore in D_+ cohomology we have

$$\prod_i x_i^{5n_i+\nu_i}\, \mathrm{d}^5x = -\left(n_1 + \frac{\nu_1-4}{5}\right) x_1^{5(n_1-1)+\nu_1} \prod_{i=2}^5 x_i^{5n_i+\nu_i}\, \mathrm{d}^5x,\ \nu_i < 5. \tag{4.12}$$

By induction we obtain

$$\prod_i x_i^{5n_i+\nu_i}\, \mathrm{d}^5x = (-1)^{\sum_i n_i} \prod_i \left(\frac{\nu_i+1}{5}\right)_{n_i} \prod_i x_i^{\nu_i}\, \mathrm{d}^5x,\ \nu_i < 5. \tag{4.13}$$

where $(a)_n = \Gamma(a+n)/\Gamma(a)$.

Using (4.11) once again, we see that if any $\nu_i = 4$ then the differential form is trivial and the integral is zero. Hence, r.h.s. of (4.13) is proportional to $e_\nu(x)$ and gives the desired expression. Plugging (4.13) into (4.9) and integrating over Γ^+_μ gives the answer

$$\sigma_\mu(\phi) = \sigma^+_\mu(\phi) = \sum_{n_i\geq 0} \prod_i \left(\frac{\mu_i+1}{5}\right)_{n_i} \sum_{m\in\Sigma_n} \prod_s \frac{\phi_s^{m_s}}{m_s!}, \tag{4.14}$$

where

$$\Sigma_n = \left\{ m \mid \sum_s m_s s_i = 5n_i + \mu_i \right\}. \tag{4.15}$$

Further we will also use the periods with slightly different normalization, which turn out to be convenient

$$\hat{\sigma}_\mu(\phi) = \prod_i \Gamma\left(\frac{\mu_i+1}{5}\right) \sigma_\mu(\phi) = \sum_{n_i \geq 0} \prod_i \Gamma\left(n_i + \frac{\mu_i+1}{5}\right) \sum_{m \in \Sigma_n} \prod_s \frac{\phi_s^{m_s}}{m_s!}. \tag{4.16}$$

5 Computation of the Kähler potential

Pick any basis $Q_\mu^\pm$ of cycles with integer or real coefficients as in (4.4). Then for the Kähler potential we have the formula

$$e^{-K} = \omega_\mu^+(\phi) C^{\mu\nu} \overline{\omega_\nu^-(\phi)} \tag{5.1}$$

in which the matrix $C^{\mu\nu}$ is related with the Frobenius pairing η as

$$\eta_{\alpha\beta} = \omega_{\alpha\mu}^+(0) C^{\mu\nu} \omega_{\beta\nu}^-(0). \tag{5.2}$$

The last expression is due to [15, 16]. Let also $T^\pm$ be a coordinate change matrix $Q_\mu^\pm = (T^\pm)_\mu^\nu \Gamma_\nu^\pm$. Then $M = (T^-)^{-1}\overline{T^-}$ is a real structure matrix, that is $M\bar{M} = 1$ and by construction M doesn't depend on the choice of basis $Q_\mu^\pm$. M is only defined by our choice of $\Gamma_\mu^\pm$.

In [1] we deduced from (5.1) and (5.2) the formula

$$e^{-K(\phi)} = \sigma_\mu^+(\phi) \eta^{\mu\lambda} M_\lambda^\nu \overline{\sigma_\nu^-(\phi)} = \sigma_\mu A^{\mu\nu} \overline{\sigma_\nu}, \tag{5.3}$$

where $\eta^{\mu\nu} = \eta_{\mu\nu} = \delta_{\mu,\rho-\nu}$. In that papers our method to compute the real structure matrix M used the knowledge of the periods in some basis q_μ computed using the residue formula and monodromy considerations. However, this method gives only 4 out of 204 linearly independent periods for the quintic threefold X.

Therefore we propose here a different method to find M.

Lemma 5.1. *Inverse intersection matrix $A^{\mu\nu}$ in (5.3) is diagonal.*

Proof. We may extend the action of the phase symmetry group to the action $\mathfrak{A}$ on the parameter space $\{\phi_s\}$ such that $W = W_0 + \sum_s \phi_s e_s(x)$ is invariant under this new action. Each $e_s(x)$ has a unique weight under this group action.

Action $\mathfrak{A}$ can be compensated using the coordinate tranformation and therefore is trivial on the moduli space of the quintic (implying that point $W = W_0$ is an orbifold point of the moduli space). In particular, $e^{-K} = \int_X \Omega \wedge \bar{\Omega}$ is $\mathfrak{A}$ invariant. Consider

$$e^{-K} = \sigma_\mu A^{\mu\nu} \overline{\sigma_\nu} \tag{5.4}$$

as a series in ϕ_s, $\overline{\phi_t}$ Each monomial has a certain weight under $\mathfrak{A}$. For the series to be invariant, each monomial must have weight 0. But weight of $\sigma_\mu \overline{\sigma_\nu}$ equals to $\mu - \nu$ and due to non-degeneracy of weights of σ_μ only the ones with $\mu = \nu$ have weight zero. □

Thus, (5.3) becomes

$$e^{-K} = \sum_\mu A^\mu |\sigma_\mu(\phi)|^2. \tag{5.5}$$

Moreover, the matrix A should be real and, because $A = \eta \cdot M$, $M\bar{M} = 1$ and $\eta_{\mu\nu} = \delta_{\mu+\nu,\rho}$, we have

$$A^\mu A^{\rho-\mu} = 1. \tag{5.6}$$

Monodromy considerations. To fix the remaining 102 real numbers A^μ we use monodromy invariance of e^{-K} around $\phi_0 = \infty$. Fix some $t = (t_1, t_2, t_3, t_4, t_5)$, $|t| = 5$ and let $\phi_s|_{s\neq t,0} = 0$, also consider only the first order in ϕ_t. Then the condition that period $\sigma_\mu(\phi)$ contains only non-zero summands of the form $\phi_0^{m_0}\,\phi_t$ implies that $\mu = t + const\cdot(1,1,1,1,1)$ mod 5. For each t from the table below the only such possibilities are $\mu = t$ and $\mu = \rho - t' = (3,3,3,3,3) - t'$, where t' denotes a vector obtained from t by permutation (written explicitly in the table below) of its coordinates.

Therefore, in this setting (5.3) becomes

$$e^{-K} = \sum_{k=0}^{3} a_k |\hat\sigma_{(k,k,k,k,k)}|^2 + a_t|\hat\sigma_t|^2 + a_{\rho-t'}|\hat\sigma_{\rho-t'}|^2 + O(\phi_t^2), \tag{5.7}$$

where we used periods $\hat\sigma$ from (4.16), $a_t = A^t/\prod_i \Gamma((t_i+1)/5)^2$ and a_k, $k = 0,1,2,3$ are already known [11]. This expression should be monodromy invariant. We consider the effect of the transport of ϕ_0 around ∞. From the formula (4.16) we have

$$\begin{aligned} F_1 &= \hat\sigma_k(\phi_t,\phi_0) = g_t\phi_k\, F(a,b;a+b\,|\,(\phi_0/5)^5) + O(\phi_t^6), \\ F_2 &= \hat\sigma_{\rho-t'}(\phi_t,\phi_0) = g_{\rho-t'}\phi_t\,\phi_0^{1-a-b}\,F(1-a,1-b;2-a-b\,|\,(\phi_0/5)^5) + O(\phi_t^6), \end{aligned} \tag{5.8}$$

where g_t, $g_{\rho-t'}$ are some constants. Explicitly for all different labels t

t	$\rho - t'$	(a, b)
(2,1,1,1,0)	(3,2,2,2,1)	(2/5,2/5)
(2,2,1,0,0)	(3,3,2,1,1)	(1/5,3/5)
(3,1,1,0,0)	(0,3,3,2,2)	(1/5,2/5)
(3,2,0,0,0)	(1,0,3,3,3)	(1/5,1/5)

and

$$F(a,b;c|z) := \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)}\,{}_2F_1(a,b;c;z). \tag{5.9}$$

When ϕ_0 goes around infinity

$$\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = B \cdot \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}, \tag{5.10}$$

where (e.g. [19][3])

$$B = \frac{1}{is(a+b)} \begin{pmatrix} c(a-b) - e^{i\pi(a+b)} & 2s(a)s(b) \\ 2e^{2\pi i(a+b)}s(a)s(b) & e^{\pi i(a+b)}[e^{2\pi ia} + e^{2\pi ib} - 2]/2 \end{pmatrix}. \tag{5.11}$$

[3]Translated from the Russian, translation edited and with a preface by Alan Jeffrey and Daniel Zwillinger, with one CD-ROM (Windows, Macintosh and UNIX).

Here $c(x) = \cos(\pi x),\ s(x) = \sin(\pi x)$. It is straightforward to show the following

Proposition 1.

$$a_t|\hat{\sigma}_t|^2 + a_{\rho-t'}|\hat{\sigma}_{\rho-t'}|^2 = a_t \prod_i \Gamma\left(\frac{t_i+1}{5}\right)^2 |\sigma_t|^2 + a_{\rho-t'} \prod_i \Gamma\left(\frac{4-t_i}{5}\right)^2 |\sigma_{\rho-t'}|^2 \tag{5.12}$$

is B-invariant iff $a_t = -a_{\rho-t'}$.

Due to symmetry we have $a_{\rho-t'} = a_{\rho-t}$ in each case. From (5.6) it follows that the product of the coefficients at $|\sigma_\mu|^2$ and $|\sigma_{\rho-\mu}|^2$ in the expression for e^{-K} should be 1:

$$A^{\rho-t'} \cdot A^t = a_{\rho-t'} \cdot a_t \prod_i \Gamma\left(\frac{t_i+1}{5}\right)^2 \Gamma\left(\frac{4-t_i}{5}\right)^2 = 1. \tag{5.13}$$

Due to reflection formula $a_t = \pm \prod_i \sin(\pi(t_i+1)/5)$ up to a common factor of π. The sign turns out to be minus for Kähler metric to be positive definite in the origin. Therefore

$$A^\mu = (-1)^{\deg(\mu)/5} \prod \gamma\left(\frac{\mu_i+1}{5}\right). \tag{5.14}$$

Finally the Kähler potential becomes

$$e^{-K(\phi)} = \sum_{\mu=0}^{203} (-1)^{\deg(\mu)/5} \prod \gamma\left(\frac{\mu_i+1}{5}\right) |\sigma_\mu(\phi)|^2, \tag{5.15}$$

where $\gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)}$.

6 Real structure on the cycles $\Gamma^\pm_\mu$

Let cycles $\gamma_\mu \in H_3(X)$ be the images of cycles Γ^+_μ under the isomorphism $\mathcal{H}_5^{+,\mathrm{inv}} \simeq H_3(X)$.

Complex conjugation sends $(2,1)$-forms to $(1,2)$-forms. Similarly it extends to a mapping on the dual homology cycles γ_μ. In the real basis of cycles a version of the formula (5.3) takes an especially simple form, because the real structure matrix M becomes an identity.

Lemma 6.1. *Conjugation of homology classes has the following form: $*\gamma_\mu = p_\mu \gamma_{\rho-\mu}$, where $\rho = (3,3,3,3,3)$ is a unique maximal degree element in the Milnor ring.*

Proof. We perform a proof for the cohomology classes represented by differential forms. For one-dimensional $H^{3,0}(X)$ and $H^{0,3}(X)$ it is obvious. Let

$$\Omega_{2,1} := e_t(x)\, \chi_{\bar{i}}^{l}\, \Omega_{ljk} \in H^{2,1}(X). \tag{6.1}$$

Any element from $H^{1,2}(X)$ is representable by a degree 10 polynomial $P(x)$ as follows from (3.3) as

$$\overline{\Omega_{2,1}} = \Omega_{1,2} := P(x)\, \chi_{\bar{i}}^{l}\, \chi_{\bar{j}}^{m}\, \Omega_{lmk} \in H^{1,2}(X). \tag{6.2}$$

The group of phase symmetries modulo common factor acts by isomorphisms on X. Therefore, it also acts on the differential forms. Lhs and r.h.s. of the previous equation should have the same weigth under this action, and weight of the l.h.s. is equal $-t$ modulo $(1,1,1,1,1)$. It follows that $P(x) = p_t\, e_{\rho-t}(x)$ with some constant p_t. □

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Using this lemma and applying the complex conjugation of cycles to the formula (5.3) to obtain

$$e^{-K} = \sum_{\mu} A^{\mu} |\sigma_{\mu}|^2 = \sum_{\mu} p_{\mu}^2 A^{\mu} \, |\sigma_{\rho-\mu}|^2, \tag{6.3}$$

it follows that $A^{\mu} = \pm 1/p_{\mu}$. Now formula (5.15) implies

$$p_{\mu} = \prod_i \gamma \left(\frac{4-\mu_i}{5} \right). \tag{6.4}$$

7 Conclusions

The method for computing the Kähler potential on the CY moduli space from [1] modified in this paper does not require knowledge of periods in some real homology basis. Instead, we use some simple monodromy considerations to fix the real structure matrix. Another possible interesting method would be to determine this matrix by direct computation of coefficients (6.4) of the complex conjugation in the basis $e_{\mu}(x)$. In this paper we use our modified method to compute Weil-Peterson metric on the whole 101-dimensional complex structure moduli space of the quintic threefold around the orbifold point (5.15). Together with the computation of the moduli space geometry of the Kähler structures through the mirror map [11] it describes the Special geometry of all Ricci flat deformations of CY metric in the region.

Though we present our result for the quintic threefold, our method should be applicable to a bigger class of models, which are connected with Landau-Ginzburg description, in particular hypersurfaces in toric varieties. At least in the case of the hypersurfaces in weighted projective spaces, we can, in principle, compute the basis $\{e_{\mu}(x)\}$ of R_0^Q such, that the pairing η is antidiagonal, and the periods $\sigma_{\mu}(\phi)$. Indeed, it reduces to Jacobi ideal computations. Using the connection of the pairing $\eta_{\mu\nu}$ with the natural pairing in the cohomology $H^3(X)$ it is possible to prove (1.8) in this generality. Then the whole computation of the Kähler potential is reduced to finding of the coefficients A_{μ}. One way to do it is to restrict the expression to the different one-dimensional subspaces of the moduli space and to require the monodromy invariance of the Kähler potential, as we did in the section 5 for the quintic threefold. In general, monodromy invariance translates to properties of generalized hypergeometric functions in one variable.

The main problem of our method in general is to choose the convenient starting point $W_0(x)$ such, that Jacobi ideal computations are not be too complicated. We plan to address possible generalizations in details in the future publications.

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