Published for SISSA by Springer

Received: February 27, 2017
Accepted: March 13, 2017
Published: March 28, 2017

Dynamical symmetry enhancement near $\mathcal{N} = 2$, $D = 4$ gauged supergravity horizons

J. Gutowski,[a] T. Mohaupt[b] and G. Papadopoulos[c]

[a] *Department of Mathematics, University of Surrey, Guildford, GU2 7XH, U.K.*
[b] *Department of Mathematical Sciences, University of Liverpool, Peach Street, Liverpool L69 7ZL, U.K.*
[c] *Department of Mathematics, King's College London, Strand, London WC2R 2LS, U.K.*

E-mail: j.gutowski@surrey.ac.uk, Thomas.Mohaupt@liverpool.ac.uk, george.papadopoulos@kcl.ac.uk

Abstract: We show that all smooth Killing horizons with compact horizon sections of 4-dimensional gauged $\mathcal{N} = 2$ supergravity coupled to any number of vector multiplets preserve $2c_1(\mathcal{K}) + 4\ell$ supersymmetries, where $\mathcal{K}$ is a pull-back of the Hodge bundle of the special Kähler manifold on the horizon spatial section. We also demonstrate that all such horizons with $c_1(\mathcal{K}) = 0$ exhibit an $\mathfrak{sl}(2, \mathbb{R})$ symmetry and preserve either 4 or 8 supersymmetries. If the orbits of the $\mathfrak{sl}(2, \mathbb{R})$ symmetry are 2-dimensional, the horizons are warped products of AdS_2 with the horizon spatial section. Otherwise, the horizon section admits an isometry which preserves all the fields. The proof of these results is centered on the use of index theorem in conjunction with an appropriate generalization of the Lichnerowicz theorem for horizons that preserve at least one supersymmetry. In all $c_1(\mathcal{K}) = 0$ cases, we specify the local geometry of spatial horizon sections and demonstrate that the solutions are determined by first order non-linear ordinary differential equations on some of the fields.

Keywords: Black Holes, Black Holes in String Theory, Supergravity Models

ArXiv ePrint: 1607.02877

doi:10.1007/JHEP03(2017)150

Contents

1 Introduction

It has been known for some time that there is (super)symmetry enhancement near extreme black hole and brane horizons [1–3]. This observation has been made on a case by case basis and it has been instrumental in the formulation of AdS/CFT correspondence [4].

In the last three years it has been realized that (super)symmetry enhancement is a generic phenomenon for all smooth supergravity Killing horizons with compact spatial sections that preserve at least one supersymmetry. The essential features of this (super)symmetry enhancement mechanism have been described in [5] in the form of the "*horizon conjecture*" following earlier related work in [7, 8]. The horizon conjecture has two parts. One part involves a formula for the number of supersymmetries preserved by such horizons. In the second part, this is used to show that some of the horizons with non-trivial fluxes admit an $\mathfrak{sl}(2,\mathbb{R})$ symmetry subalgebra. So far, the horizon conjecture has been proven for all 10- and 11-dimensional supergravities [5, 6, 8–10] and minimal 5-dimensional gauged supergravity [7].

In this paper, we shall demonstrate the validity of the horizon conjecture [5] for all 4-dimensional gauged $\mathcal{N}=2$ supergravities coupled to any number of abelian vector multiplets, see for example [11]. The supersymmetric black hole solutions of such theories, and hence their near horizon geometries, have been extensively investigated in the context of entropy counting and attractor mechanism, starting from [13–16].

The assumptions which are made for the proof of the horizon conjecture are as follows:

- The near horizon geometry as well as the rest of the fields are smooth,
- the near horizon spatial section is compact without boundary,
- the matrix of gauge couplings $\mathrm{Im}\,\mathcal{N}$ is negative definite and hence invertible,[1]
- the scalar potential V is negative semi-definite, $V \leq 0$.

[1] In turn this implies that the scalar manifold admits a (positive definite) Kähler metric.

The first two assumptions may be replaced by the requirement that the data are such that the Hopf maximum principle applies [12], and that a certain surface term integral over the horizon spatial section vanishes.

A consequence of the proof of the conjecture is that all Killing horizons that satisfy these assumptions:

(i) preserve

$$N = 2c_1(\mathcal{K}) + 4\ell\,, \tag{1.1}$$

supersymmetries, where $N \leq 8$, $\ell = 1, 2$ and $\mathcal{K}$ is the pull-back of the Hodge bundle of the special Kähler geometry on the spatial horizon section $\mathcal{S}$,

(ii) and those with $\ell \neq 0$, or equivalently $c_1(\mathcal{K}) = 0$, admit an $\mathfrak{sl}(2,\mathbb{R})$ symmetry.[2]

Note that if $c_1(\mathcal{K}) = 0$, which as we shall show is the case for all the horizons with $\ell \neq 0$, the number of supersymmetries preserved are either 4 or 8. This Chern class corresponds to the index of a certain Dirac operator defined on $\mathcal{S}$.

We further proceed to investigate the geometry of the horizons with $c_1(\mathcal{K}) = 0$. There are two cases to consider depending on whether the orbits of $\mathfrak{sl}(2,\mathbb{R})$ are 2- or 3-dimensional. In the former case, the horizons are warped products of AdS_2 with the horizon spatial section $\mathcal{S}$, $\mathrm{AdS}_2 \times_w \mathcal{S}$. Furthermore, if the warp factor is trivial, $\mathcal{S}$ is a sphere S^2, a torus T^2 or a (quotient of) hyperbolic space H^2 equipped with the Einstein metric depending on the sign of the right-hand-side term in (6.5) and the rest of the fields either vanish or they are constant. If the warp factor is non-trivial, $\mathcal{S}$ admits an isometry which leaves the rest of the fields invariant. We give the local form of the metric on $\mathcal{S}$ and show that it depends on the scalars of the gauge multiplet. Moreover, we show that all the remaining fields are specified by first order ordinary differential equations. In particular, the scalars flow on the horizon.

If $\mathfrak{sl}(2,\mathbb{R})$ has a 3-dimensional orbit on the spacetime, then $\mathcal{S}$ admits an isometry which leaves all the remaining fields invariant. There are several cases that one can consider. In all cases, we give the local form of the spacetime metric and demonstrate that the remaining fields are determined by first order ordinary differential equations. In most cases, the scalars flow on the horizon. Furthermore as the scalars depend on at most one coordinate, the first Chern class of $\mathcal{K}$ vanishes and so all such horizons preserve either 4 or 8 supersymmetries.

We also present an application of the horizon conjecture. In particular, we show that it is a consequence of the horizon conjecture that all horizons with fluxes and $N_- \neq 0$, see [5] and (4.13), for which the spatial horizon section is a marginally trapped surface contain untrapped surfaces both just inside and outside the horizon. This is a characteristic behavior of extreme black hole horizons. As a result such supersymmetric horizons meet the necessary conditions of [24], see also [25, 26], to be extended to full extreme black hole solutions.

The proof of the horizons conjecture utilizes in a essential way that near a smooth Killing horizon one can adapt a null gaussian coordinate system. Then the Killing spinor equations (KSEs) of $\mathcal{N} = 2$ supergravity are integrated along the lightcone directions to

[2]In 11-dimensional and type II horizons the presence of $\mathfrak{sl}(2,\mathbb{R})$ requires that the horizons must have non-trivial fluxes. This is not necessary here as this assumption is implied by our restrictions on the couplings of $\mathcal{N} = 2$ gauged supergravity.

express the Killing spinors in terms of spinors that depend only on the coordinates of $\mathcal{S}$. The remaining equations involve the reduction of the gravitino and gaugini KSEs on $\mathcal{S}$ as well as a large number of integrability conditions. The latter are shown to be implied by the reduced gravitino and gaugini KSEs on $\mathcal{S}$ as well as the field equations. Unlike similar calculations for $D = 11$ and type II supergravities, the assumption that the horizons admit one supersymmetry is used in an essential way. Then the number of solutions of the reduced gravitino and gaugini KSEs on $\mathcal{S}$ are counted by first making use of Lichnerowicz type theorems to turn the problem into one of counting zero modes of Dirac-like operators on $\mathcal{S}$, and then using the index theorem [30]. After taking into account that the KSEs of the $\mathcal{N} = 2$ theory are linear over the complex numbers, the formula for the number of supersymmetries N is produced (1.1), where the number of supersymmetries N is counted over the reals.

The proof of the second part of the horizon conjecture proceeds after first observing that if $c_1(\mathcal{K}) = 0$ then one can always construct pairs of Killing spinors over the spacetime which in turn give rise to three linearly independent vector bilinears. Then the commutators of these vector fields are calculated and it is found that they satisfy a $\mathfrak{sl}(2, \mathbb{R})$ algebra. The geometry of these horizons is also investigated. For this, appropriate coordinates are adapted on the horizon, and local expressions for the metric and other fields are obtained in all cases.

The paper is organized as follows. In section 2, after a brief description of gauged $\mathcal{N} = 2$ supergravity, we describe the near horizon geometries and evaluate the field equations of the theory on the near horizon fields. In section 3, we solve the KSEs of $\mathcal{N} = 2$ supergravity along the lightcone directions of near horizon geometries and state the remaining independent KSEs. In section 4, we establish that near horizon geometries either preserve 4 or 8 supersymmetries. In section 5, we slow that the near horizon geometries exhibit an $\mathfrak{sl}(2, \mathbb{R})$ symmetry. In section 6, we describe the local geometries of all near horizon geometries of $\mathcal{N} = 2$ gauged supergravity. In section 7, we present an application of the horizon conjecture on trapped surfaces. In appendix A, we give our conventions. In appendix B, we summarize the properties of special Kähler geometry which are essential in all our derivations. In appendix C, we determine the independent KSEs of the near horizon backgrounds. In appendix D, we present the derivation of Lichnerowicz type theorems essential for counting the supersymmetries. In appendix E, we examine some symmetry properties of the near horizon fields. In appendix F, we derive the near horizon data of a class of solutions found previously in [20]. In appendix G, we present the details of the derivation of the local expressions for geometries of all near horizon configurations, and in appendix H we verify some of the field equations.

2 Near-horizon geometry and field equations

2.1 $\mathcal{N} = 2$ gauged supergravity with vector multiplets

The bosonic field content of the gravitational multiplet of $\mathcal{N} = 2$ supergravity is a metric and a U(1) field. The theory can also couple to k vector abelian multiplets in which case contains k additional U(1) fields and $2k$ real scalars. In the coupled theory, all the fields interact and the U(1) field of the gravitational multiplet mixes with the rest. The scalars take values on a sigma model manifold which exhibits a special Kähler structure. The two (real) gravitini of the theory can be described together as a Dirac $\mathfrak{so}(3,1)$ spinor 1-form. The gaugini can also be described as Dirac spinors. The supersymmetry parameter is then a Dirac spinor which is taken in what follows to be commuting.

The action of $\mathcal{N}=2$, 4-dimensional, U(1) gauged supergravity with no gauging of special Kähler isometries [11] in the conventions of [21] is given by

$$
\begin{aligned}
e^{-1}\mathcal{L} = &\frac{1}{2}R + \frac{1}{4}\big(\mathrm{Im}\mathcal{N}\big)_{IJ}F^{I}_{\mu\nu}F^{J\mu\nu} - \frac{1}{8}\big(\mathrm{Re}\mathcal{N}\big)_{IJ}e^{-1}\epsilon^{\mu\nu\rho\sigma}F^{I}_{\mu\nu}F^{J}_{\rho\sigma} \\
&- g_{\alpha\bar\beta}\nabla_\mu z^\alpha \nabla^\mu z^{\bar\beta} - V\,,
\end{aligned}
\tag{2.1}
$$

where R is the Ricci scalar of spacetime, $F^I = dA^I$ are the field strengths of U(1) fields and so $I=1,\dots,k+1$, z are k complex scalars, and V is the scalar potential, for a review see also [17]. We have suppressed all terms in the action that depend on the fermions. The scalar manifold M exhibits special Kähler geometry with metric $g_{\alpha\bar\beta}$; see appendix B for the definition and a summary of some key properties. The rest of the couplings include the gauge couplings matrix $\mathrm{Im}\mathcal{N}$ and the theta angles $\mathrm{Re}\mathcal{N}$ which can depend on the scalars. These couplings are also determined in terms of the special Kähler geometry. Furthermore, the scalar potential is given by

$$
V = 4g^2\Big(U^{IJ} - 3\bar X^I X^J\Big)\xi_I\xi_J = -2g^2\Big(\big(\mathrm{Im}\mathcal{N}\big)^{-1\,IJ} + 8\bar X^I X^J\Big)\xi_I\xi_J\,,
\tag{2.2}
$$

where g is a non-zero constant, and the constants ξ_I are obtained from the U(1) Fayet-Iliopoulos terms. Moreover X^I, $I=1,\dots,k+1$, depend only the scalar fields $z,\bar z$ and are defined in the context of special Kähler geometry, see appendix B. To establish the second identity we have used the expression for U^{IJ} in appendix B.

As we have already mentioned in the introduction, apart from the smoothness of the near horizon data, we shall make two assumptions on the couplings of the theory. These are that the matrix of gauge couplings $\mathrm{Im}\mathcal{N}$ is negative definite, and that $V\le 0$. A consequence of our two assumptions is that $\xi_I X^I$ never vanishes,

$$
\xi_I X^I \neq 0\,.
\tag{2.3}
$$

This is because if $\xi_I X^I = 0$ at any point, then at such a point $V = -2g^2(\mathrm{Im}\mathcal{N})^{-1IJ}\xi_I\xi_J > 0$, in contradiction to our assumption that $V\le 0$.

The Einstein, gauge and scalar field equations of the theory are

$$
R_{\mu\nu} = -2\mathrm{Im}(\mathcal{N})_{IJ}(F^+)^I{}_{\rho\mu}(F^-)^{J\rho}{}_\nu + 2g_{\alpha\bar\beta}\nabla_{(\mu}z^\alpha\nabla_{\nu)}\bar z^{\bar\beta} + g_{\mu\nu}V\,,
\tag{2.4}
$$

$$
-2\nabla_\mu\Big(\mathrm{Im}(\mathcal{N})_{IJ}(F^-)^{J\mu\nu}\Big) + i(\nabla_\mu\mathcal{N}_{IJ})\tilde F^{J\mu\nu} = 0\,,
\tag{2.5}
$$

$$
\begin{aligned}
&\nabla_\mu\nabla^\mu z^\alpha + \frac{1}{4i}(F^+)^I{}_{\mu\nu}(F^+)^{J\mu\nu}g^{\alpha\bar\gamma}\frac{\partial}{\partial\bar z^{\bar\gamma}}\mathcal{N}_{IJ} \\
&\qquad - \frac{1}{4i}(F^-)^I{}_{\mu\nu}(F^-)^{J\mu\nu}g^{\alpha\bar\gamma}\frac{\partial}{\partial\bar z^{\bar\gamma}}\bar{\mathcal{N}}_{IJ} - g^{\alpha\bar\gamma}\frac{\partial}{\partial\bar z^{\bar\gamma}}V = 0\,,
\end{aligned}
\tag{2.6}
$$

respectively, where the definition of $(F^\pm)^I$ is given in (2.11). It should be noted that (2.4) and (2.6) correct typographical errors found in [21].

2.2 Horizon fields and field equations

The black hole horizons that we shall be investigating are extremal Killing horizons with regular spatial horizon sections $\mathcal{S}$. For such horizons, one can adapt a Gaussian Null coordinate system [18, 19] such that the spacetime metric ds^2 and 2-form field strengths F^I can be written as

$$\begin{aligned} ds^2 &= 2\mathbf{e}^+\mathbf{e}^- + \delta_{ij}\mathbf{e}^i\mathbf{e}^j\,, \\ F^I &= \Phi^I\,\mathbf{e}^+\wedge\mathbf{e}^- + r\mathbf{e}^+\wedge d_h\Phi^I + \frac{1}{2}Q^I\epsilon_{ij}\,\mathbf{e}^i\wedge\mathbf{e}^j\,, \end{aligned} \tag{2.7}$$

where u, r are the lightcone coordinates and y^I, $I = 1, 2$, are the remaining coordinates of the spacetime, $d_h\Phi^I = d\Phi^I - h\Phi^I$, and the spatial horizon section $\mathcal{S}$ is given by $u = r = 0$ with induced metric and volume form

$$ds^2_{\mathcal{S}} = \delta_{ij}\mathbf{e}^i\mathbf{e}^j\,,\quad \mathrm{dvol}(\mathcal{S}) = \frac{1}{2}\epsilon_{ij}\mathbf{e}^i\wedge\mathbf{e}^j\,, \tag{2.8}$$

respectively. Furthermore, we have used the frame

$$\mathbf{e}^+ = du\,,\quad \mathbf{e}^- = dr + rh - \frac{1}{2}r^2\Delta du\,,\quad \mathbf{e}^i = e^i{}_J dy^J\,,\quad i,j = 1,2\,. \tag{2.9}$$

The components of fields h, Δ, Φ^I, Q^I and $\mathbf{e}^i$ depend only on y^I. The black hole stationary Killing vector field is identified with ∂_u and becomes null on the hypersurface $r = 0$. The 1-form gauge potential associated to F^I is

$$A^I = -r\Phi^I du + B^I,\qquad dB^I = Q^I\mathrm{dvol}(\mathcal{S})\,. \tag{2.10}$$

Our smoothness assumption asserts that Δ, Φ^I, Q^I are globally defined smooth scalars, and h is a globally defined smooth 1-form on the horizon section $\mathcal{S}$, respectively. In addition, the induced metric on $\mathcal{S}$, $ds^2_{\mathcal{S}}$, is smooth, and $\mathcal{S}$ is compact, connected without boundary. We denote the Levi-Civita connection of $\mathcal{S}$ by $\hat{\nabla}$.

In what follows, it is convenient to define

$$(F^\pm)^I{}_{\mu\nu} = \frac{1}{2}(F^I \pm \tilde{F}^I)_{\mu\nu}\,,\qquad \tilde{F}^I{}_{\mu\nu} = -\frac{i}{2}\epsilon_{\mu\nu}{}^{\rho\sigma}F^I_{\rho\sigma}\,. \tag{2.11}$$

We note that the components of $(F^\pm)^I$ are given by

$$\begin{aligned} (F^\pm)^I_{+-} &= \frac{1}{2}(\Phi^I \mp iQ^I)\,, & (F^\pm)^I_{-j} &= 0\,, \\ (F^\pm)^I_{+i} &= \frac{r}{2}\Big(d_h\Phi_i \pm i\epsilon_i{}^j d_h\Phi_j\Big)\,, & (F^\pm)^I_{ij} &= \pm\frac{i}{2}(\Phi^I \mp iQ^I)\epsilon_{ij}\,. \end{aligned} \tag{2.12}$$

Before proceeding with the analysis of the supersymmetry, we decompose the field equations of the bosonic fields along the lightcone and $\mathcal{S}$ directions. In particular, $\nu = -$ component of field equations of the U(1) gauge fields (2.5) is

$$\begin{aligned} &\hat{\nabla}^j\Big(\mathrm{Im}(\mathcal{N}_{IJ})d_h\Phi^J_j\Big) - \mathrm{Im}(\mathcal{N}_{IJ})h^j d_h\Phi^J_j + \frac{1}{2}\mathrm{Im}(\mathcal{N}_{IJ})(dh)_{ij}\epsilon^{ij}Q^J \\ &+\Big(\hat{\nabla}_j\mathrm{Re}(\mathcal{N}_{IJ})\Big)\epsilon^{jk}d_h\Phi^J_k = 0\,, \end{aligned} \tag{2.13}$$

and the $\nu = j$ component of (2.5) is equivalent to

$$\mathrm{Im}(\mathcal{N}_{IJ}) d_h \Phi^J_j = -d_h \big(\mathrm{Im}(\mathcal{N}_{IJ}) Q^J\big)_k \epsilon^k{}_j - \Big(\hat{\nabla}_k \mathrm{Re}(\mathcal{N}_{IJ})\Big) \Phi^J \epsilon^k{}_j \,. \tag{2.14}$$

The scalar field equation[3] (2.6) can be expressed as

$$\hat{\nabla}_i \hat{\nabla}^i z^\alpha - h^i \hat{\nabla}_i z^\alpha + g^{\alpha\bar{\gamma}} \partial_\lambda g_{\sigma\bar{\gamma}} \hat{\nabla}_i z^\lambda \hat{\nabla}^i z^\sigma - g^{\alpha\bar{\gamma}} \partial_{\bar{\gamma}} V + \frac{1}{2}\Big(Q^I \Phi^J + Q^J \Phi^I\Big) g^{\alpha\bar{\gamma}} \partial_{\bar{\gamma}} \mathrm{Re}(\mathcal{N}_{IJ})$$
$$+ \frac{1}{2}\Big(Q^I Q^J - \Phi^I \Phi^J\Big) g^{\alpha\bar{\gamma}} \partial_{\bar{\gamma}} \mathrm{Im}(\mathcal{N}_{IJ}) = 0\,, \tag{2.15}$$

where the Kähler connection of the scalar manifold involving partial derivatives of $g_{\alpha\bar{\beta}}$ has been given explicitly.

The $+-$ component of the Einstein equations (2.4) is

$$\frac{1}{2}\hat{\nabla}^i h_i - \Delta - \frac{1}{2}h^2 - \frac{1}{2}\mathrm{Im}(\mathcal{N}_{IJ})\Big(\Phi^I \Phi^J + Q^I Q^J\Big) - V = 0\,, \tag{2.16}$$

while $++$ component of the Einstein equations is

$$\hat{\nabla}^i \hat{\nabla}_i \Delta - 3h^i \hat{\nabla}_i \Delta - \Delta \hat{\nabla}^i h_i + 2\Delta h^2 + \frac{1}{2}(dh)_{ij}(dh)^{ij} + 2\mathrm{Im}(\mathcal{N}_{IJ})\delta^{ij} d_h \Phi^I_i d_h \Phi^J_j = 0. \tag{2.17}$$

Next the $+i$ component of the Einstein equations is

$$\frac{1}{2}\hat{\nabla}^j (dh)_{ij} - (dh)_{ij} h^j - \hat{\nabla}_i \Delta + \Delta h_i - \mathrm{Im}(\mathcal{N}_{IJ})\Big(\Phi^I d_h \Phi^I_i - Q^I \epsilon_i{}^j d_h \Phi^I_j\Big) = 0\,, \tag{2.18}$$

and finally ij component of the Einstein equations is

$$\frac{1}{2}\hat{R}\delta_{ij} + \hat{\nabla}_{(i} h_{j)} - \frac{1}{2}h_i h_j + \frac{1}{2}\mathrm{Im}(\mathcal{N}_{IJ})\Big(\Phi^I \Phi^J + Q^I Q^J\Big)\delta_{ij}$$
$$- 2 g_{\alpha\bar{\beta}} \hat{\nabla}_{(i} z^\alpha \hat{\nabla}_{j)} \bar{z}^{\bar{\beta}} - V \delta_{ij} = 0\,, \tag{2.19}$$

where $\hat{R}$ is the Ricci scalar of the spatial horizon section $\mathcal{S}$.

Not all of these field equations are independent. In particular, (2.13) is obtained by taking the divergence of (2.14). (2.17) is obtained from taking the divergence of (2.18), together with (2.14) and (2.13). Equation (2.18) is obtained by taking the divergence of the traceless part of (2.19), together with (2.16), (2.15) and (2.14). So the independent bosonic field equations are (2.14), (2.15), (2.16) and (2.19).

3 Supersymmetric near-horizon geometries

3.1 Killing spinor equations

The KSEs of supergravity theories are the vanishing conditions of the supersymmetry variations of the fermionic fields of these theories evaluated at the locus where all the fermionic

[3] We shall use $\partial_\alpha = \frac{\partial}{\partial z^\alpha}$, and $\partial_{\bar{\alpha}} = \frac{\partial}{\partial \bar{z}^{\bar{\alpha}}}$ to denote differentiation w.r.t. the scalars z^α, $\bar{z}^{\bar{\alpha}}$.

JHEP03(2017)150

fields vanish. The fermionic fields of 4-dimensional $\mathcal{N}=2$, gauged supergravity coupled to U(1) multiplets are the gravitino and the gaugini. In particular, the gravitino KSE is

$$\nabla_\mu\epsilon+\left(\frac{i}{2}A_\mu\Gamma_5+ig\xi_I(A^I)_\mu+g\Gamma_\mu\xi_I\Big(\mathrm{Im}X^I+i\Gamma_5\mathrm{Re}X^I\Big)\right.$$
$$\left.+\frac{i}{4}\Gamma^{\rho\sigma}\Big(\mathrm{Im}\big((F^-)^I_{\rho\sigma}X^J\big)-i\Gamma_5\mathrm{Re}\big((F^-)^I_{\rho\sigma}X^J\big)\Big)\mathrm{Im}\mathcal{N}_{IJ}\Gamma_\mu\right)\epsilon=0\,, \tag{3.1}$$

and the gaugini KSEs are

$$\frac{i}{2}\mathrm{Im}\mathcal{N}_{IJ}\Gamma^{\rho\sigma}\Big(\mathrm{Im}\big((F^-)^J_{\rho\sigma}\mathcal{D}_{\bar\beta}\bar X^I g^{\alpha\bar\beta}\big)-i\Gamma_5\mathrm{Re}\big((F^-)^J_{\rho\sigma}\mathcal{D}_{\bar\beta}\bar X^I g^{\alpha\bar\beta}\big)\Big)\epsilon$$
$$+\Gamma^\mu\nabla_\mu\Big(\mathrm{Re}z^\alpha-i\Gamma_5\mathrm{Im}z^\alpha\Big)\epsilon+2g\xi_I\Big(\mathrm{Im}\big(\mathcal{D}_{\bar\beta}\bar X^I g^{\alpha\bar\beta}\big)-i\Gamma_5\mathrm{Re}\big(\mathcal{D}_{\bar\beta}\bar X^I g^{\alpha\bar\beta}\big)\Big)\epsilon=0\,, \tag{3.2}$$

where ϵ is the supersymmetry parameter that is taken to be Dirac commuting spinor,

$$\nabla_\mu\epsilon=\partial_\mu\epsilon+\frac{1}{4}\Omega_{\mu,\rho\sigma}\Gamma^{\rho\sigma}\epsilon\,,\quad A_\mu=-\frac{i}{2}\Big(\partial_\alpha K\nabla_\mu z^\alpha-\partial_{\bar\alpha}K\nabla_\mu z^{\bar\alpha}\Big)\,, \tag{3.3}$$

and Ω is the frame connection of the spacetime metric. The gravitino KSE is a parallel transport equation for the spinor ϵ, while the gaugini KSEs do not involve derivatives of ϵ and so are algebraic. Our spinor conventions including those for the gamma matrices Γ^μ as well as the realization of Cliff$(3,1)$ used are specified in appendix A. Observe that the KSEs is linear over the complex numbers. So the supersymmetric configurations always admit an even number of supersymmetries as counted over the real numbers. The classification of supersymmetric solutions of gauged $\mathcal{N}=2$ supergravity coupled to any number of vector multiplets has been investigated in [21–23].

3.2 Integrability along the lightcone and independent KSEs

For the near horizon geometries that we are investigating, the KSEs of the 4-dimensional supergravity theory can be explicitly integrated along the lightcone directions. This determines the dependence of the Killing spinors in terms of the u,r coordinates. Then we substitute back the resulting expressions for the Killing spinors into the KSEs to find remaining conditions on the Killing spinors. The remaining conditions include those that one expects by the naive restriction of both the gravitino and gaugini KSEs on the spatial horizon section $\mathcal{S}$ as well as large number of integrability conditions.

To determine all the conditions on the Killing spinors, we first solve the $\mu=-$ component of the gravitino KSE (3.1) to find

$$\epsilon_+=\phi_+\,,\qquad\quad \epsilon_-=\phi_-+r\Gamma_-\Theta_+\phi_+\,, \tag{3.4}$$

where $\partial_r\phi_\pm=0$, $\Gamma_\pm\epsilon_\pm=\Gamma_\pm\phi_\pm=0$, and we have defined

$$\Theta_\pm=\frac{1}{4}h_i\Gamma^i-g\xi_I\big(\mathrm{Im}X^I+i\Gamma_5\mathrm{Re}X^I\big)$$
$$\mp\frac{i}{2}\Big(\mathrm{Im}((\Phi^I+iQ^I)X^J)+i\Gamma_5\mathrm{Re}((\Phi^I+iQ^I)X^J)\Big)\mathrm{Im}\mathcal{N}_{IJ}\,. \tag{3.5}$$

Next,we solve the $\mu = +$ component of the gravitino KSE (3.1) to find that

$$\phi_+ = \eta_+ + u\Gamma_+\Theta_-\eta_-, \qquad \phi_- = \eta_- , \tag{3.6}$$

where $\partial_r\eta_\pm = \partial_u\eta_\pm = 0$, $\Gamma_\pm\eta_\pm = 0$, and so $\eta_\pm$ depend only on the coordinates of $\mathcal{S}$. Thus after solving the gravitino KSE along the lightcone directions the Killing spinor can be written as

$$\epsilon = \epsilon_+ + \epsilon_- , \quad \epsilon_+ = \eta_+ + u\Gamma_+\Theta_-\eta_- , \quad \epsilon_- = \eta_- + r\Gamma_-\Theta_+\big(\eta_+ + u\Gamma_+\Theta_-\eta_-\big) . \tag{3.7}$$

Substituting ϵ back into all the KSEs, one obtains a large number of conditions (C.1)–(C.10) described in appendix C.

Not all these conditions are independent. Using in an essential way that the horizons preserve at least one supersymmetry,[4] and in particular the relations between the fields (C.24), (C.25), (C.29), (C.30), (C.31), (C.32), (C.35) and (C.36) that are implied by such an assumption, and after utilizing the field equations, one finds that the remaining independent conditions implied by the gravitino KSE on the Killing spinors are

$$\nabla^{(\pm)}_i\eta_\pm = 0 , \tag{3.8}$$

where

$$\nabla^{(\pm)}_i \equiv \hat{\nabla}_i + \frac{i}{2}A_i\Gamma_5 + ig\xi_I B^I_i - \Gamma_i\hat{\Theta}_\mp \mp \frac{1}{4}h_i , \tag{3.9}$$

and

$$\Theta_\mp = \frac{1}{4}\Gamma^i h_i + \hat{\Theta}_\mp , \quad A_i = -\frac{i}{2}\big(\partial_\alpha K\partial_i z^\alpha - \partial_{\bar{\alpha}}K\partial_i z^{\bar{\alpha}}\big) . \tag{3.10}$$

Similarly, the gaugini KSEs (3.2) give

$$\mathcal{A}^\alpha_{(\pm)}\eta_\pm = 0 , \tag{3.11}$$

where

$$\begin{aligned}\mathcal{A}^\alpha_{(\pm)} = &\mp i\mathrm{Im}\mathcal{N}_{IJ}\Big[\mathrm{Im}\big((\Phi^J + iQ^J)\mathcal{D}_{\bar{\beta}}\bar{X}^I g^{\alpha\bar{\beta}}\big) - i\Gamma_5\mathrm{Re}\big((\Phi^J + iQ^J)\mathcal{D}_{\bar{\beta}}\bar{X}^I g^{\alpha\bar{\beta}}\big)\Big] \\ &+ \Gamma^i\hat{\nabla}_i\Big[\mathrm{Re}z^\alpha - i\Gamma_5\mathrm{Im}z^\alpha\Big] + 2g\xi_I\Big[\mathrm{Im}\big(\mathcal{D}_{\bar{\beta}}\bar{X}^I g^{\alpha\bar{\beta}}\big) - i\Gamma_5\mathrm{Re}\big(\mathcal{D}_{\bar{\beta}}\bar{X}^I g^{\alpha\bar{\beta}}\big)\Big] .\end{aligned} \tag{3.12}$$

The KSEs (3.8) and (3.12) can be thought of as the naive restriction of the gravitino and gaugini KSEs on the spatial horizon section $\mathcal{S}$.

Furthermore, one also establishes from the analysis of the integrability conditions that if η_- satisfies the above KSEs, then

$$\eta_+ = \Gamma_+\Theta_-\eta_- , \tag{3.13}$$

also is a Killing spinor. To see whether $\eta_+ \neq 0$, one has to show that $\mathrm{Ker}\,\Theta_- = \{0\}$ which is demonstrated below.

[4] Such an assumption is not necessary for the proof of a similar result in 10- and 11-dimensional supergravities [5, 8–10] but it has been used before in the context of minimal 5-dimensional supergravity [7].

3.3 $\mathrm{Ker}\,\Theta_- = \{0\}$

To show this, we shall use contradiction. Suppose that there is exists $\eta_- \neq 0$ such that $\Theta_- \eta_- = 0$. It follows that

$$\begin{aligned}\Theta_+ \eta_- &= (\Theta_+ - \Theta_-)\eta_- \\ &= \mathrm{Im}\mathcal{N}_{IJ}\Big(\Gamma_5 \mathrm{Re}\big((\Phi^I + iQ^I)X^J\big) - i\mathrm{Im}\big((\Phi^I + iQ^I)X^J\big)\Big)\eta_- \,.\end{aligned} \tag{3.14}$$

It then follows from (3.8) that

$$\hat{\nabla}_i \parallel \eta_- \parallel^2 = -h_i \parallel \eta_- \parallel^2 , \tag{3.15}$$

and so $dh = 0$ as η_- is a parallel spinor and so is nowhere vanishing. The integrability condition (C.2) further implies that $\Delta = 0$. On taking the divergence of (3.15), one then obtains

$$\hat{\nabla}^i \hat{\nabla}_i \parallel \eta_- \parallel^2 = \Big(-\mathrm{Im}\mathcal{N}_{IJ}(\Phi^I \Phi^J + Q^I Q^J) - 2V \Big) \parallel \eta_- \parallel^2 , \tag{3.16}$$

As we have assumed that $\mathrm{Im}\mathcal{N}_{IJ}$ is negative definite, and also $V \leq 0$, an application of the maximum principle reveals that

$$\Phi^I = Q^I = 0 \,, \qquad V = 0 \,, \tag{3.17}$$

and also $\parallel \eta_- \parallel^2 =$ const. Substituting the constant norm condition into (3.15), we obtain $h = 0$.

Substituting all of these conditions back into the condition $\Theta_- \eta_- = 0$, one obtains

$$\xi_I \Big(\mathrm{Im}X^I + i\Gamma_5 \mathrm{Re}X^I \Big)\eta_- = 0 \,, \tag{3.18}$$

which implies $\xi_I X^I = 0$ which contradicts our assumptions on the couplings. Thus, we establish that $\mathrm{Ker}\,\Theta_- = \{0\}$.

One consequence of the above result is that for all horizons $\phi_+ \neq 0$. To see this, since our backgrounds are supersymmetric either η_+ or η_- must not vanish. If $\eta_+ \neq 0$, then $\phi_+ \neq 0$. On the other hand if $\eta_- \neq 0$, then also $\phi_+ \neq 0$ as can be seen from (3.6) and $\mathrm{Ker}\,\Theta_- = \{0\}$. In particular, this means that all supersymmetric near-horizon geometries must admit a non-zero spinor ϕ_+ satisfying (C.5), (C.1), (C.3), (C.7), (C.8) and (C.10).

4 Counting the supersymmetries of horizons

In this section, we shall demonstrate the first consequence of the horizons conjecture which is the counting of supersymmetries of $\mathcal{N} = 2$ supergravity horizons as stated in the introduction. For this, we shall establish two Lichnerowicz type theorems and then we shall use index theory to count the number of supersymmetries preserved by the near horizon geometries.

JHEP03(2017)150

4.1 Lichnerowicz type theorem for ϕ_+

The Killing spinor equations on η_+ have been reduced to the naive restriction of the gravitino and gaugini KSEs on $\mathcal{S}$ (3.8) and (3.12), respectively. Let us define the horizon Dirac operators

$$\mathcal{D}^{(\pm)} \equiv \Gamma^i \hat{\nabla}_i^{(\pm)} = \Gamma^i \hat{\nabla}_i + \frac{i}{2}\Gamma^i A_i \Gamma_5 + ig\Gamma^i \xi_I B_i^I - 2\hat{\Theta}_\mp \mp \frac{1}{4}\Gamma^i h_i \,. \tag{4.1}$$

Here we shall establish the Lichnerowicz type theorem

$$\hat{\nabla}_i^{(+)}\phi_+ = 0 \qquad \text{and} \qquad \mathcal{A}^\alpha_{(+)}\phi_+ = 0 \quad \Longleftrightarrow \quad \mathcal{D}^{(+)}\phi_+ = 0\,. \tag{4.2}$$

The proof of the Lichnerowicz type theorem for η_+ spinor is similar. It is clear that if ϕ_+ is Killing, then it is a zero mode of the $\mathcal{D}^{(+)}$ and so one direction is straightforward. To prove the converse, we shall assume that the near horizon geometries preserve one supersymmetry[5] and that the maximum principle applies.

The assumption of the existence of one supersymmetry requires some explanation. We have shown in appendix C that the fields of the near horizon geometries that preserve one supersymmetry satisfy certain at most first order differential conditions which depend on the choice of the Killing spinor via a function κ. These conditions are necessary to establish the Lichnerowicz type theorems. However although the conditions (C.24), (C.25), (C.29), (C.30), (C.31), (C.32), (C.34), (C.35) and (C.36) are used, κ is not required to be related to the spinor under investigation in the Lichnerowicz type theorem. In other words, we use the at most first order differential conditions on the fields that are derived from the requirement of one supersymmetry but the Lichnerowicz type theorems are valid for every zero mode of the horizon Dirac operators irrespectively on whether this zero mode is associated to the Killing spinor used to establish the differential relations.

To proceed one can show utilizing (C.29) that the gaugini algebraic condition can be rewritten as

$$\mathcal{A}^\alpha_{(+)}\phi_+ = 0\,, \tag{4.3}$$

where now

$$\begin{aligned}\mathcal{A}^\alpha_{(+)} = \Gamma^i\hat{\nabla}_i \mathrm{Re}z^\alpha + i\Gamma_5\Gamma^i\hat{\nabla}_i\mathrm{Im}z^\alpha + 2g(1-\kappa\Gamma_5)\Big(&\xi_I \mathrm{Im}\big(\mathcal{D}_{\bar\beta}\bar{X}^I g^{\alpha\bar\beta}\big)\\ &- i\Gamma_5\xi_I\mathrm{Re}\big(\mathcal{D}_{\bar\beta}\bar{X}^I g^{\alpha\bar\beta}\big)\Big)\,.\end{aligned} \tag{4.4}$$

Next assume that ϕ_+ is a zero mode of the horizon Dirac operator, $\mathcal{D}^{(+)}\phi_+ = 0$, then after some computation which is described in appendix D, one can show that

$$\begin{aligned}\hat{\nabla}_i\hat{\nabla}^i \parallel \phi_+ \parallel^2 - h^i\hat{\nabla}_i \parallel \phi_+ \parallel^2 = {}& 2\langle \hat{\nabla}^{(+)i}\phi_+, \hat{\nabla}^{(+)}_i\phi_+\rangle\\ &+ \langle \mathcal{A}^\beta_{(+)}\phi_+, \big(\mathrm{Re}(g_{\alpha\bar\beta}) + i\Gamma_5\mathrm{Im}(g_{\alpha\bar\beta})\big)\mathcal{A}^\alpha_{(+)}\phi_+\rangle\,.\end{aligned} \tag{4.5}$$

[5] This assumption is not necessary for 11-dimensional and type II horizons in [5, 8–10] but this assumption has been used before for the 5-dimensional horizons [7].

The right-hand-side of this expression is a sum of positive definite terms. The maximum principle then implies that η_+ is a Killing spinor and that

$$\partial_i \parallel \phi_+ \parallel = 0\,. \tag{4.6}$$

We conclude by stating the Lichnerowicz type theorem for η_+ spinors. In particular, we have that

$$\hat{\nabla}_i^{(+)}\eta_+ = 0 \qquad \text{and} \qquad \mathcal{A}^\alpha_{(+)}\eta_+ = 0\,, \quad \Longleftrightarrow \quad \mathcal{D}^{(+)}\eta_+ = 0\,, \tag{4.7}$$

and

$$\parallel \eta_+ \parallel = \text{const}\,, \tag{4.8}$$

where $\hat{\nabla}^{(+)}$, $\mathcal{A}^\alpha_{(+)}$ and $\mathcal{D}^{(+)}$ are defined by (3.9), (4.4) and (4.1), respectively.

4.2 Lichnerowicz type theorem for η_- spinors

There is an analogous Lichnerowicz type theorem for η_- spinors. In particular, one can show that

$$\hat{\nabla}_i^{(-)}\eta_- = 0 \qquad \text{and} \qquad \mathcal{A}^\alpha_{(-)}\eta_- = 0 \quad \Longleftrightarrow \quad \mathcal{D}^{(-)}\eta_- = 0\,, \tag{4.9}$$

where the operator $\hat{\nabla}^{(-)}$ is defined in (3.9), $\mathcal{A}_{(-)}$ is given in (3.12) and upon using (C.29) can be expressed as

$$\begin{aligned}\mathcal{A}^\alpha_{(-)} \equiv{}& \Gamma^i\hat{\nabla}_i\text{Re}z^\alpha + i\Gamma_5\Gamma^i\hat{\nabla}_i\text{Im}z^\alpha \\ &+ 2g(1+\kappa\Gamma_5)\Big(\xi_I\text{Im}\big(\mathcal{D}_{\bar{\beta}}\bar{X}^I g^{\alpha\bar{\beta}}\big) - i\Gamma_5\text{Re}\big(\mathcal{D}_{\bar{\beta}}\bar{X}^I g^{\alpha\bar{\beta}}\big)\Big)\,,\end{aligned} \tag{4.10}$$

and the horizon Dirac operator is

$$\mathcal{D}^{(-)} \equiv \Gamma^i\hat{\nabla}_i^{(-)}\,. \tag{4.11}$$

We have again assumed that the near horizon geometries preserve one supersymmetry and we shall use this in a way that has been explained for ϕ_+ spinors in the previous section. It is clear that if η_- is a Killing spinor, then it is also a zero mode of the horizon Dirac operator. To establish the converse, take η_- to be a zero mode of the horizon Dirac operator $\mathcal{D}^{(-)}$, $\mathcal{D}^{(-)}\eta_- = 0$, and after some computation that is described in appendix D, one can establish the identity

$$\begin{aligned}\hat{\nabla}_i\hat{\nabla}^i \parallel \eta_- \parallel^2 + \hat{\nabla}^i\Big(h_i \parallel \eta_- \parallel^2\Big) ={}& 2\langle\hat{\nabla}^{(-)i}\eta_-, \hat{\nabla}_i^{(-)}\eta_-\rangle \\ &+ \langle\mathcal{A}^\beta_{(-)}\eta_-, \big(\text{Re}(g_{\alpha\bar{\beta}}) + i\Gamma_5\text{Im}(g_{\alpha\bar{\beta}})\big)\mathcal{A}^\alpha_{(-)}\eta_-\rangle.\end{aligned} \tag{4.12}$$

The right-hand-side of this expression is a sum of positive definite terms. On integrating both sides of this expression over $\mathcal{S}$, which is taken to be compact without boundary, the contribution from the left-hand-side vanishes. So the integral of the right-hand-side vanishes as well. As it is the sum of positive terms, this implies that η_- is Killing spinor as required.

JHEP03(2017)150

4.3 Counting supersymmetries

The number of supersymmetries of near horizon geometries is $N = N_+ + N_-$ where $N_\pm$ is the number of linearly independent $\eta_\pm$ Killing spinors. On the other hand, the two Lichnerowicz type theorems (4.7) and (4.9) we have established for both the $\eta_\pm$ spinor imply that

$$N_\pm = \dim \text{ Ker } \mathcal{D}^{(\pm)} . \tag{4.13}$$

Moreover one can easily show that

$$\Gamma_- (\mathcal{D}^{(+)})^\dagger = \mathcal{D}^{(-)} \Gamma_- , \tag{4.14}$$

which implies that

$$\dim\big(\text{Ker}(\mathcal{D}^{(+)\dagger})\big) = \dim\big(\text{Ker}(\mathcal{D}^{(-)})\big) . \tag{4.15}$$

On the other hand [30]

$$\text{Index } \big(\mathcal{D}^{(+)}\big) \equiv \dim \text{ Ker } \mathcal{D}^{(+)} - \dim \text{ Ker } (\mathcal{D}^{(+)})^\dagger = N_+ - N_- . \tag{4.16}$$

Therefore, the number of supersymmetries preserved by the near horizon geometries can be expressed as

$$N = \text{index}\big(\mathcal{D}^{(+)}\big) + 2N_- . \tag{4.17}$$

It remains to calculate index $(\mathcal{D}^{(+)})$. For this observe that from (4.1) and using the conventions in appendix A that one can write

$$\begin{aligned} \mathcal{D}^{(+)} &= \Gamma^i \hat{\nabla}_i + \frac{i}{2} \Gamma^i A_i \Gamma_5 + ig \Gamma^i \xi_I B_i^I - 2\hat{\Theta}_- - \frac{1}{4} \Gamma^i h_i \\ &= \sigma^i \otimes \sigma_3 \hat{\nabla}_i - \frac{i}{2} \sigma^i \sigma_3 \otimes \sigma_3 A_i + ig \sigma^i \otimes \sigma_3 \xi_I B_i^I - 2\hat{\Theta}_- - \frac{1}{4} \sigma^i \otimes \sigma_3 h_i \\ &= \sigma^i \hat{\nabla}_i - \frac{i}{2} \sigma^i \sigma_3 A_i + ig \sigma^i \xi_I B_i^I - 2\hat{\Theta}_- - \frac{1}{4} \sigma^i h_i , \end{aligned} \tag{4.18}$$

where in the last equality we have used $\Gamma_+ \eta_+ = 0$, or equivalently $\mathbb{I}_2 \otimes \sigma_3 \eta_+ = \eta_+$, and identified $\sigma^i \otimes 1 = \sigma^i$ and $\sigma^i \sigma_3 \otimes 1 = \sigma^i \sigma_3$. Using the chirality operator σ_3 on $\mathcal{S}$ the above operator further decomposes into two other operators as

$$\mathcal{D}^{(+)} = \mathcal{D}_+^{(+)} \oplus \mathcal{D}_-^{(+)} , \tag{4.19}$$

where

$$\mathcal{D}_\pm^{(+)} = \sigma^i \hat{\nabla}_i \mp \frac{i}{2} \sigma^i A_i + ig \sigma^i \xi_I B_i^I - 2\hat{\Theta}_- - \frac{1}{4} \sigma^i h_i . \tag{4.20}$$

To continue observe that

$$\begin{aligned} \mathcal{D}_+^{(+)} : &\quad \Gamma(S_+ \otimes \mathcal{K}^{\frac{1}{2}} \otimes \mathcal{L}) \to \Gamma(S_- \otimes \mathcal{K}^{\frac{1}{2}} \otimes \mathcal{L}) , \\ \mathcal{D}_-^{(+)} : &\quad \Gamma(S_- \otimes \bar{\mathcal{K}}^{\frac{1}{2}} \otimes \mathcal{L}) \to \Gamma(S_+ \otimes \bar{\mathcal{K}}^{\frac{1}{2}} \otimes \mathcal{L}) , \end{aligned} \tag{4.21}$$

where $S_\pm$ are the bundles of chiral/antichiral spinors on $\mathcal{S}$, respectively, $\mathcal{K}$ is the pull-back of the Hodge bundle on $\mathcal{S}$, $\mathcal{L}$ is the line bundle with connection $\xi_I B^I$ and $\Gamma(E)$ denotes the smooth sections of the vector bundle E.

The index of $\mathcal{D}^{(+)}$ can be calculated as follows.

$$\begin{aligned}\text{index}\big(\mathcal{D}^{(+)}\big) &= \text{index}\big(\mathcal{D}^{(+)}_+\big) + \text{index}\big(\mathcal{D}^{(+)}_-\big) = \text{index}\big(\mathcal{D}^{(+)}_+\big) - \text{index}\big((\mathcal{D}^{(+)}_-)^\dagger\big) \\ &= \left(\frac{1}{2}c_1(\mathcal{K}) + c_1(\mathcal{L})\right) - \left(\frac{1}{2}c_1(\bar{\mathcal{K}}) + c_1(\mathcal{L})\right) = c_1(\mathcal{K})\,,\end{aligned} \tag{4.22}$$

where we have used that $\mathcal{D}^{(+)}_+$ and $(\mathcal{D}^{(+)}_-)^\dagger$ have the same principal symbol as that of twisted Dirac operators with the bundles $\mathcal{K}^{\frac{1}{2}} \otimes \mathcal{L}$ and $\bar{\mathcal{K}}^{\frac{1}{2}} \otimes \mathcal{L}$, respectively, and so the same index.

Therefore, we have found that

$$N = 2c_1(\mathcal{K}) + 2N_- = 2c_1(\mathcal{K}) + 4\ell\,, \tag{4.23}$$

as N_- is an even number because the $\mathcal{D}^{(-)}$ is linear over the complex numbers. The additional factor of 2 in front of $c_1(\mathcal{K})$ appears because the index is computed over the complex numbers while our counting of supersymmetries is over the real numbers.

In many cases of interest $c_1(\mathcal{K})$ vanishes. In particular, we shall see that if $N_- \neq 0$, or equivalently $\ell \neq 0$, then $c_1(\mathcal{K}) = 0$. This is because the pull-back of the Hodge bundle on $\mathcal{S}$ in all these cases is trivial. This will be proven after a detailed analysis of the geometries of the horizons in section 6. Conversely, if $c_1(\mathcal{K}) = 0$ then $N = 4\ell$, so all supersymmetric solutions with $c_1(\mathcal{K}) = 0$ must have $\ell \neq 0$.

5 $\mathfrak{sl}(2,\mathbb{R})$ symmetry

We shall demonstrate that all supersymmetric horizons with $N_- \neq 0$, or equivalently $c_1(\mathcal{K}) = 0$ in (1.1), of $\mathcal{N} = 2$ gauged supergravity exhibit an $\mathfrak{sl}(2,\mathbb{R})$ symmetry which is the second part of the horizons conjecture as stated in the introduction. To prove this, first observe that if ϵ_1 and ϵ_2 are Killing spinors, then the 1-form bilinear

$$K(\epsilon_1, \epsilon_2) = \text{Re}\langle(\Gamma_+ - \Gamma_-)\epsilon_1, \Gamma_\mu \epsilon_2\rangle\, e^\mu, \tag{5.1}$$

is associated with a Killing vector which in addition leaves all other fields invariant, see [21–23] and also appendix E. The former property is a consequence of the gravitino KSE. Suppose now that $N_- \neq 0$. We have also shown that if η_- is Killing spinor, then $\eta_+ = \Gamma_+ \Theta_- \eta_-$ is also a Killing spinor (3.13). Using these, we can construct two linearly independent Killing spinors over the whole spacetime associated with the pairs $(\eta_-, 0)$ and (η_-, η_+) which after a rearrangement can be written as

$$\epsilon_1 = \eta_- + u\eta_+ + ru\Gamma_-\Theta_+\eta_+\,, \quad \epsilon_2 = \eta_+ + r\Gamma_-\Theta_+\eta_+\,; \quad \eta_+ = \Gamma_+\Theta_-\eta_-\,. \tag{5.2}$$

JHEP03(2017)150

These give rise to three 1-form bi-linears as

$$\begin{aligned}
K_1 &= \mathrm{Re}\langle(\Gamma_+ - \Gamma_-)\epsilon_1, \Gamma_\mu \epsilon_2\rangle\, e^\mu = (2r\,\mathrm{Re}\langle\Gamma_+\eta_-, \Theta_+\eta_+\rangle + 4ur^2 \parallel \Theta_+\eta_+ \parallel^2)\,\mathbf{e}^+ \\
&\qquad - 2u \parallel \eta_+ \parallel^2\, \mathbf{e}^- + W_i \mathbf{e}^i\,, \\
K_2 &= \mathrm{Re}\langle(\Gamma_+ - \Gamma_-)\epsilon_2, \Gamma_\mu \epsilon_2\rangle\, e^\mu = 4r^2 \parallel \Theta_+\eta_+ \parallel^2 \mathbf{e}^+ - 2 \parallel \eta_+ \parallel^2 \mathbf{e}^-\,, \\
K_3 &= \mathrm{Re}\langle(\Gamma_+ - \Gamma_-)\epsilon_1, \Gamma_\mu \epsilon_1\rangle\, e^\mu = (2 \parallel \eta_- \parallel^2 + 4ru\mathrm{Re}\langle\Gamma_+\eta_-, \Theta_+\eta_+\rangle + 4r^2u^2 \parallel \Theta_+\eta_+ \parallel^2)\mathbf{e}^+ \\
&\qquad - 2u^2 \parallel \eta_+ \parallel^2 \mathbf{e}^- + 2uW_i\mathbf{e}^i\,,
\end{aligned} \tag{5.3}$$

where to simplify the expressions for K_1, K_2 and K_3 somewhat we have used the fact that $\parallel \eta_+ \parallel$ is constant (4.8), and have set

$$W_i = \mathrm{Re}\,\langle\Gamma_+\eta_-, \Gamma_i\eta_+\rangle\,, \tag{5.4}$$

and also have used

$$\mathrm{Re}\langle\eta_+, \Gamma_i\Theta_+\eta_+\rangle = \mathrm{Re}\langle\eta_+, \Gamma_i\Theta_-\eta_+\rangle = 0\,, \tag{5.5}$$

which follows from a direct computation utilizing the expressions for $\Theta_\pm$.

Furthermore, the requirement that all the above three 1-forms give rise to Killing vector fields implies the conditions, see also appendix E,

$$\begin{aligned}
&\hat{\nabla}_{(i}W_{j)} = 0\,, \quad \hat{\mathcal{L}}_W h = 0\,, \qquad \hat{\mathcal{L}}_W \Delta = 0\,, \quad 4 \parallel \Theta_+\eta_+ \parallel^2 = \Delta \parallel \eta_+ \parallel^2\,, \\
&-2 \parallel \eta_+ \parallel^2 - h_i W^i + 2\mathrm{Re}\,\langle\Gamma_+\eta_-, \Theta_+\eta_+\rangle = 0\,, \quad i_W(dh) + 2d\mathrm{Re}\,\langle\Gamma_+\eta_-, \Theta_+\eta_+\rangle = 0\,, \\
&2\mathrm{Re}\,\langle\Gamma_+\eta_-, \Theta_+\eta_+\rangle - \Delta \parallel \eta_- \parallel^2 = 0\,, \qquad W + \parallel \eta_- \parallel^2 h + d \parallel \eta_- \parallel^2 = 0\,.
\end{aligned} \tag{5.6}$$

Using the above expressions, observe that K_1, K_2 and K_3 can be simplified further and also one can show that

$$\mathcal{L}_W \parallel \eta_- \parallel^2 = 0\,. \tag{5.7}$$

In addition to the Killing vectors associated with K_1, K_2 and K_3, the geometry of spacetime is further restricted by the KSEs and field equations of the theory. An exhaustive description of the geometry of the horizons will be given in the next section.

To demonstrate that the horizons exhibit an $\mathfrak{sl}(2,\mathbb{R})$ symmetry, we use the various identities derived above (5.6) to write the vector fields associated to the 1-forms K_1, K_2 and K_3 (5.3) as

$$\begin{aligned}
K_1 &= -2u \parallel \eta_+ \parallel^2 \partial_u + 2r \parallel \eta_+ \parallel^2 \partial_r + W^i \hat{\partial}_i\,, \\
K_2 &= -2 \parallel \eta_+ \parallel^2 \partial_u\,, \\
K_3 &= -2u^2 \parallel \eta_+ \parallel^2 \partial_u + (2 \parallel \eta_- \parallel^2 + 4ru \parallel \eta_+ \parallel^2)\partial_r + 2uW^i\hat{\partial}_i\,,
\end{aligned} \tag{5.8}$$

where we have used the same symbol for the 1-forms and the associated vector fields. A direct computation then reveals using (5.7) that

$$[K_1, K_2] = 2 \parallel \eta_+ \parallel^2 K_2\,, \ [K_2, K_3] = -4 \parallel \eta_+ \parallel^2 K_1\,, \ [K_3, K_1] = 2 \parallel \eta_+ \parallel^2 K_3\,. \tag{5.9}$$

Therefore all such horizons with non-trivial fluxes admit an $\mathfrak{sl}(2,\mathbb{R})$ symmetry subalgebra.

The orbits of the $\mathfrak{sl}(2,\mathbb{R})$ symmetry are either two or three dimensional depending on whether W vanishes or not. In the former case, the spacetime is a warped product of AdS_2 with $\mathcal{S}$.

6 Geometry of the near-horizon solutions

In this section, we shall summarize the local forms of all near-horizon geometries of $\mathcal{N} = 2$ gauged supergravity with $c_1(\mathcal{K}) = 0$, which implies that $N_- \neq 0$. In fact, as a consequence of the following analysis, it can also be easily seen that the converse holds, i.e. $N_- \neq 0$ implies that $c_1(\mathcal{K}) = 0$. This is because if $N_- \neq 0$ then the scalars locally depend on at most one coordinate or they are constant. As a result the first Chern class of the pull-back of the Hodge bundle on $\mathcal{S}$ vanishes. Hence $c_1(\mathcal{K}) = 0$ if, and only if, $N_- \neq 0$. All such near-horizon geometries preserve either 4 or 8 supersymmetries. As $c_1(\mathcal{K}) = 0$ implies that $N_- = N_+$, the global argument given previously implies that the KSEs (3.8) and (3.11) admit the same number of η_- and η_+ spinor solutions.

The function

$$\kappa = \parallel \phi_+ \parallel^{-2} \langle \phi_+, \Gamma_5 \phi_+ \rangle \,, \tag{6.1}$$

plays a particularly important role in the analysis, because the metric and other fields depend on it, see also appendices C and G. Observe that $|\kappa| \leq 1$ as a consequence of the Cauchy-Schwarz inequality and $\kappa = \pm 1$ iff ϕ_+ is an eigenspinor of Γ_5 with eigenvalue ± 1.

To examine the geometry of near horizon backgrounds, we are mostly concerned with solving the conditions (C.24), (C.25), (C.29), (C.30), (C.31), (C.32), (C.35) and (C.36) on the fields which arise from the KSEs on ϕ_+ and for this we also make use of some of the Einstein equations (2.16)–(2.19) according to need. Note that the independent field equations are (2.14) for the vector fields, (2.15) for the scalars, and the Einstein equations (2.16) and (2.19). Moreover, we have verified that for all supersymmetric near horizon backgrounds (2.14) is automatically satisfied in appendix H. The scalar field equations (2.15) have not at any point been used in the analysis of any of the KSE. Furthermore, it has been shown in [21] that the scalar field equations are implied by supersymmetry and the remaining field equations.

In many of the cases we consider, it turns out that the scalar fields z^α have a non-trivial dependence on the co-ordinate κ. Such near-horizon solutions have also been considered in the context of the entropy function formalism [31], in which a $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{U}(1)$ symmetry was assumed, together with spherical topology of the horizon spatial cross-section $\mathcal{S}$.

The solution of the (C.24), (C.25), (C.29), (C.30), (C.31), (C.32), (C.35) and (C.36) equations and field equations is arranged so that all the other fields are determined in terms of the scalar fields of the vector multiplets and κ. These in turn obey non-linear first order differential equations. In what follows, we shall not give details of the proof. Instead, we shall simply state the results with some minimal explanation. A more detailed derivation can be found in appendix G.

6.1 Warped AdS$_2$ horizons; $W \equiv 0$

It can be shown using the maximum principle that for all these backgrounds

$$\Delta > 0 \,, \quad \xi_I \Phi^I = 0 \,. \tag{6.2}$$

Furthermore, the associated vector field to

$$\tau = \star_{\mathcal{S}} h\,, \tag{6.3}$$

leaves the field $z^\alpha, \Delta, h, \Phi^I, Q^I$ invariant. As a result, there are two cases to consider depending on whether or not $h = 0$.

6.1.1 Solutions with $W = h \equiv 0$

The conditions from supersymmetry and the field equations imply that the fields z^α, κ, Δ, Φ^I and Q^I are all constant, with $\Delta > 0$. The spacetime metric is then given by

$$ds^2 = 2du\left(dr - \frac{1}{2}r^2\Delta du\right) + ds^2_{\mathcal{S}}\,, \tag{6.4}$$

where the Ricci scalar of $\mathcal{S}$ is given by

$$\hat{R} = 2\Delta + 4V\,, \tag{6.5}$$

where

$$\Delta = -\frac{1}{2}\mathrm{Im}\mathcal{N}_{IJ}\big(\Phi^I\Phi^J + Q^I Q^J\big) - V\,. \tag{6.6}$$

So the spacetime is $\mathrm{AdS}_2 \times \mathcal{S}$ where $\mathcal{S}$ is T^2, S^2 or H^2 according to whether $2\Delta + 4V > 0$, $2\Delta + 4V = 0$ or $2\Delta + 4V < 0$, respectively.

The constant fields z^α, κ, Δ, Φ^I and Q^I are not arbitrary. In particular, as $h=0$, (C.24) implies that $\kappa^2 = 1$. Also, (C.29) must be imposed, which relates the electric and magnetic parts of the U(1) fields in terms of the scalars.

6.1.2 Solutions with $W \equiv 0$ and $h \neq 0$

Such solutions are warped products $\mathrm{AdS}_2 \times_w \mathcal{S}$. Adapting suitable coordinates along h and τ and after a further coordinate transformation, one finds that the near-horizon data is then given by

$$\begin{aligned} \Delta &= 4\nu^2 e^x, \qquad h = dx, \\ ds^2_{\mathcal{S}} &= \frac{1}{16g^2|\xi_I X^I|^2(1-\kappa^2)}dx^2 + 16g^2L^2(1-\kappa^2)e^{-x}d\psi^2\,, \end{aligned} \tag{6.7}$$

together with

$$\Phi^I + iQ^I = -4i\nu\frac{|\xi_J X^J|}{\xi_T\bar{X}^T}e^{\frac{x}{2}}\bar{X}^I - 2ig\kappa\xi_J\mathrm{Im}\mathcal{N}^{-1IJ}\,, \tag{6.8}$$

where ν, L are a real constants. The scalars z^α and κ depend only on x and satisfy

$$\frac{dz^\alpha}{dx} = \frac{1}{2\xi_J\bar{X}^J}\xi_I\mathcal{D}_{\bar{\beta}}\bar{X}^I g^{\alpha\bar{\beta}}, \qquad \frac{d\kappa}{dx} = \kappa - \frac{\nu}{2g|\xi_I X^I|}e^{\frac{x}{2}}\,. \tag{6.9}$$

On setting $r = e^{-x}\rho$, the spacetime metric is

$$ds^2 = 2e^{-x}du\big(d\rho - 2\nu^2\rho^2 du\big) + \frac{1}{16g^2|\xi_I X^I|^2(1-\kappa^2)}dx^2 + 16g^2L^2(1-\kappa^2)e^{-x}d\psi^2\,, \tag{6.10}$$

which is a warped product $\mathrm{AdS}_2 \times_w \mathcal{S}$ with warp factor e^{-x}.

In this case, we have solved all the (C.24), (C.25), (C.29), (C.30), (C.31), (C.32), (C.35) and (C.36) equations.

6.2 Solutions with $W \not\equiv 0$

The spacetime metric as well as all the other fields are invariant under the action of $W \neq 0$. W also leaves invariant the metric on $\mathcal{S}$ as well as the other near horizon data h, z^α, Φ^I, Q^I, and Δ. Furthermore, the Lie derivatives of κ, and $\parallel \eta_- \parallel^2$ with respect to W also vanish. We present the proof of these in appendix G. There are several cases to consider and we summarize the local form of the fields below.

6.2.1 Solutions with $W \not\equiv 0$ and $\kappa = \text{const}$

For all these solutions $h \neq 0$, $dh = 0$, and

$$\Phi^I + iQ^I = -2ig\kappa\Big(\xi_J \text{Im}\mathcal{N}^{-1IJ} + 4\xi_J X^J \bar{X}^I\Big)\,, \tag{6.11}$$

and

$$\Delta = 16g^2\kappa^2|\xi_I X^I|^2\,, \tag{6.12}$$

see appendix G. Furthermore because of (G.47), $\hat{R} = (1+\kappa^2)(1-\kappa^2)^{-1}\hat{\nabla}^i h_i$, the Euler number of $\mathcal{S}$ vanishes and so $\mathcal{S}$ is a topological 2-torus. There are two different subcases to consider, corresponding as to whether $\parallel \eta_- \parallel^2$ is constant or not.

$\parallel \eta_- \parallel^2$ constant. If $\parallel \eta_- \parallel^2$ is constant, then one finds that (G.1) implies that

$$\parallel \eta_- \parallel^2 h + W = 0\,, \quad \hat{R} = 0\,, \quad |\xi_I X^I|^2 = \text{const.} \tag{6.13}$$

Thus h is covariantly constant and $\mathcal{S}$ is a torus. Then one can introduce local co-ordinates x, y on $\mathcal{S}$ such that

$$h = dx, \qquad \star_{\mathcal{S}} h = dy\,, \tag{6.14}$$

so that the z^α, Φ^I and Q^I depend only on y. In these co-ordinates, the metric is

$$ds^2 = 2du\Big(dr + rdx - 8g^2\kappa^2|\xi_I X^I|^2 r^2 du\Big) + \frac{1}{16g^2(1-\kappa^2)|\xi_I X^I|^2}\Big(dx^2 + dy^2\Big)\,, \tag{6.15}$$

and the scalars z^α satisfy

$$\frac{dz^\alpha}{dy} = \frac{i}{2\xi_J \bar{X}^J}\xi_I \mathcal{D}_{\bar\beta}\bar{X}^I g^{\alpha\bar\beta}\,. \tag{6.16}$$

The Φ^I and Q^I are given by (6.11) and Δ is constant given by (6.12); the scalars also must satisfy

$$g^{\alpha\bar\beta}\xi_I\mathcal{D}_\alpha X^I \xi_J \mathcal{D}_{\bar\beta}\bar{X}^J = |\xi_I X^I|^2, \quad \text{and} \quad \text{Im}\mathcal{N}^{-1IJ}\xi_I\xi_J = -4|\xi_I X^I|^2\,. \tag{6.17}$$

$\parallel \eta_- \parallel^2$ non-constant. For this class of solutions $i_W h$ is a negative constant. So we set

$$i_W h = -\mu^2\,, \tag{6.18}$$

and introduce coordinates x, ψ on $\mathcal{S}$ as

$$W = \mu^2 \frac{\partial}{\partial\psi}\,, \qquad \mu^2 x = \parallel \eta_- \parallel^2\;. \tag{6.19}$$

Then after some extensive analysis which utilizes the maximum principle and is presented in appendix G, one can show that

$$\Delta = \kappa = 0\,, \qquad \Phi^I = Q^I = 0\,, \qquad h = -d\psi\,, \tag{6.20}$$

$$ds^2_{\mathcal{S}} = \frac{1}{x}\left(\left(x d\psi - dx\right)^2 + \frac{1}{16g^2|\xi_I X^I|^2 x - 1} dx^2\right), \tag{6.21}$$

and

$$\frac{dz^\alpha}{dx} = -\frac{i}{2x}\left(\frac{1}{\sqrt{16g^2|\xi_L X^L|^2 - 1}} - i\right)\frac{1}{\xi_J \bar{X}^J}\xi_I \mathcal{D}_{\bar\beta}\bar{X}^I g^{\alpha\bar\beta}\,. \tag{6.22}$$

It follows that the spacetime metric is given by

$$ds^2 = 2du(dr - r d\psi) + \frac{1}{x}\left(\left(x d\psi - dx\right)^2 + \frac{1}{16g^2|\xi_I X^I|^2 x - 1} dx^2\right), \tag{6.23}$$

which concludes the analysis.

6.2.2 Solutions with $W \not\equiv 0$ and $\kappa \neq const$

The local coordinates in this case are chosen to be

$$\kappa\,, \quad \phi\,; \quad W = \frac{\partial}{\partial\psi}\,. \tag{6.24}$$

The relation between ϕ and ψ can be found in appendix G. Furthermore, one sets

$$A + iB = \kappa \xi_I \bar{X}^I \mathcal{G}\,, \quad \mathcal{G} = -\frac{2ig}{1 - iY}\;; \quad Y \neq 0, -i\,, \tag{6.25}$$

and after some extensive analysis which has been presented in appendix G, one finds that

$$\frac{\bar{Y}}{Y} = \frac{\kappa + ic}{\kappa - ic}\,, \tag{6.26}$$

where c is a real constant, and

$$\frac{d\mathcal{G}}{d\kappa} = \frac{1}{2\kappa(\kappa + ic)}\left(\frac{\kappa\mathcal{G} + ig(\kappa + ic)}{\frac{1}{2}\mathcal{G} + ig}\right)\left(\frac{ig(\kappa - ic)\mathcal{G}}{\kappa\mathcal{G} + ig(\kappa + ic)}\left(1 - \frac{i}{g}\mathcal{G}\right)\right.$$
$$\left. - \frac{1}{|\xi_L X^L|^2}\mathcal{G}\xi_I \mathcal{D}_\alpha X^I \xi_J \mathcal{D}_{\bar\beta}\bar{X}^J g^{\alpha\bar\beta}\right). \tag{6.27}$$

There are two cases to investigate depending on whether $\kappa\mathcal{G} + 2ig(\kappa + ic)$ vanishes or not.

$\kappa\mathcal{G} + 2ig(\kappa + ic) \neq 0$. In this case, after some computation which is explained in appendix G, one finds that

$$\Delta = \frac{16g^2\kappa^2|\xi_I X^I|^2}{|1-iY|^2}\,, \tag{6.28}$$

$$\frac{dz^\alpha}{d\kappa} = \frac{1}{2\kappa\xi_J\bar{X}^J}(1+iY^{-1})\xi_I\mathcal{D}_{\bar{\beta}}\bar{X}^I g^{\alpha\bar{\beta}}\,, \tag{6.29}$$

$$h = \kappa^{-1}\left(1 - \frac{c}{(\kappa+ic)Y}\right)d\kappa - (1-\kappa^2)d\phi\,, \tag{6.30}$$

and

$$\Phi^I + iQ^I = -\frac{8ig\kappa}{1+iY}\xi_J X^J\bar{X}^I - 2ig\kappa\mathrm{Im}\mathcal{N}^{-1IJ}\xi_J\,, \tag{6.31}$$

where $\psi = \frac{p}{16g^2}\phi$ and p is an integration constant which appears at an intermediate step. Moreover, the spacetime metric is

$$ds^2 = 2du\left(dr + r\left[\kappa^{-1}\left(1-\frac{c}{(\kappa+ic)Y}\right)d\kappa - (1-\kappa^2)d\phi\right] - r^2\frac{8g^2\kappa^2|\xi_I X^I|^2}{|1-iY|^2}du\right)$$
$$+\Delta^{-1}\left(\frac{1}{|Y|^2(1-\kappa^2)}d\kappa^2 + (\kappa^2+c^2)(1-\kappa^2)d\phi^2\right). \tag{6.32}$$

From these data after solving the first order non-linear differential equations, one can construct explicit solutions for each of the theories.

$\kappa\mathcal{G} + 2ig(\kappa + ic) = 0$. This special case corresponds to taking $c \neq 0$, with

$$Y = \frac{c}{\kappa + ic}\,. \tag{6.33}$$

Furthermore, $\parallel \eta_- \parallel^2 =$ const, $|\xi_I X^I|^2 =$ const, and (G.89) implies that

$$\xi_I\mathcal{D}_\alpha X^I\xi_J\mathcal{D}_{\bar{\beta}}\bar{X}^J g^{\alpha\bar{\beta}} = |\xi_I X^I|^2\,. \tag{6.34}$$

The remainder of the near-horizon data is given by (6.28)–(6.31) for this choice of Y with $c \neq 0$.

In all the four cases above, we have solved all the (C.24), (C.25), (C.29), (C.30), (C.31), (C.32), (C.35) and (C.36) equations.

7 Degenerate marginally trapped surfaces

The definition of what is a black hole spacetime is a long standing problem in general relativity, see [28] for a review. In particular it is desirable to have a quasi-local definition of what is a black hole horizon. An investigation of this question for extreme black holes has revealed that the degenerate Killing horizons that occur in extreme black holes exhibit a marginally trapped surface which after a suitable deformation becomes untrapped both inside and outside the horizon [24–27]. From the perspective of the Killing horizons, one then turn these conditions into criteria for a near horizon geometry to extend to a full black hole spacetime. In particular, these conditions can be stated as follows [24, 27]. Given the 1-form h on $\mathcal{S}$, there is a unique positive function Γ, and a divergence-free 1-form h' such that

$$h = \Gamma^{-1} h' - d \log \Gamma \,. \tag{7.1}$$

For $\mathcal{S}$ to be a marginally trapped surface, it is required that

$$\int_{\mathcal{S}} \Gamma \gamma^{(1)} > 0 \,, \tag{7.2}$$

where $\gamma^{(1)}$ is a function associated with the deformation of the metric of $\mathcal{S}$. Then the condition to have untrapped surfaces both inside and outside the horizon is that the integral

$$\int_{\mathcal{S}} \gamma^{(1)} (F' - (h')^2) < 0 \,, \tag{7.3}$$

where $F' = -\Gamma^2 \Delta$.

For the supersymmetric horizons of $\mathcal{N} = 2$ supergravity we are considering, as well as the horizons of 11-dimensional and type II supergravities with fluxes [5, 6, 8–10], which satisfy the criteria of the second part of the horizon conjecture and have a marginally trapped surface using (G.1), one finds that

$$h' = -W, \qquad \Gamma = \parallel \eta_- \parallel^2 \,, \tag{7.4}$$

which in turn gives

$$\int_{\mathcal{S}} \Gamma \gamma^{(1)} = \int_{\mathcal{S}} \parallel \eta_- \parallel^2 \gamma^{(1)} > 0 \,. \tag{7.5}$$

Moreover,

$$F' = -\Gamma^2 \Delta = - \parallel \eta_- \parallel^4 \Delta \,. \tag{7.6}$$

For all such supersymmetric near-horizon solutions, the conditions (5.6) imply that

$$W^2 = - \parallel \eta_- \parallel^2 h^i W_i \tag{7.7}$$

and

$$\Delta \parallel \eta_- \parallel^2 - h^i W_i = 2 \parallel \eta_+ \parallel^2 \,, \tag{7.8}$$

and hence

$$F' - (h')^2 = -2 \parallel \eta_- \parallel^2 \parallel \eta_+ \parallel^2 \,. \tag{7.9}$$

So one obtains

$$\int_{\mathcal{S}} \gamma^{(1)} \big(F' - (h')^2\big) = -2 \parallel \eta_+ \parallel^2 \int_{\mathcal{S}} \parallel \eta_- \parallel^2 \gamma^{(1)} < 0 \tag{7.10}$$

as a consequence of (7.5), where we have made use of the condition $\parallel \eta_+ \parallel^2 = \text{const}$.

Hence, (7.3) holds automatically for all supersymmetric near horizon geometries with fluxes and $N_- \neq 0$ satisfying (7.5). Therefore assuming the validity of the horizon conjecture, we have shown the following: *All supersymmetric horizons with fluxes and $N_- \neq 0$ for which the spatial horizon section is a marginally trapped surface contain untrapped surfaces both just inside and outside the horizon.*

8 Concluding remarks

We have confirmed the validity of the *horizon conjecture* for all near horizon geometries of $\mathcal{N} = 2$, $D = 4$, gauged supergravity coupled to any number of vector multiplets under some mild restrictions on the couplings. As a result, we have provided a formula which counts the number of superymmetries of all such backgrounds (1.1) as well as demonstrated that those with $N_- \neq 0$, or equivalently $c_1(\mathcal{K}) = 0$, in (1.1) exhibit a $\mathfrak{sl}(2, \mathbb{R})$ symmetry. We have also provided an exhaustive local description of supersymmetric near horizon geometries.

The horizon conjecture has been confirmed for a large number of theories. It demonstrates that the emergence of conformal symmetry near the horizon of supersymmetric black holes is a consequence of the fluxes of supergravity theories and the smoothness of the horizons. Therefore it is a generic property of these theories and it does not depend on the details of the black hole solution under consideration.

Apart from this, we have demonstrated another application of the horizon conjecture. In particular, we have shown that the horizon conjecture implies that all those horizons for which the horizon section is a marginally trapped surface have untrapped surfaces both just inside and outside the horizon. As a result, it is possible that they may be extended to full extreme black hole solutions. As it is likely that the horizon conjecture holds for all supergravity theories, perhaps under some mild restrictions on the couplings, the above result holds for all such supersymmetric near horizon geometries. As the first obstruction to extend the near horizon geometries to full black hole solutions can be removed, it indicates that many of the supersymmetric horizons could be extended to full black hole solutions. However not all criteria for this are known and so the question of which of the near horizon geometries are extendable and which are not remains an open question.

Other aspects of our results are the plethora of new Lichnerowicz type theorems that have been demonstrated, and the extensive applications that the maximum principle has in the context of horizons. The former results can be adapted to the theory of Clifford bundles and so they can used for applications to geometry. The latter indicate that the maximum principle has a close relationship with supersymmetry. Perhaps this is not too surprising as supersymmetry imposes restrictions on the couplings of various theories which are essential for the validity of the various maximum principle formulae. However the precise relation is not apparent and it would be of interest to investigate it in the future.

Acknowledgments

JG is supported by the STFC grant, ST/1004874/1. TM is partially supported by the STFC consolidated grant ST/L000431/1. GP is partially supported by the STFC consolidated grant ST/J002798/1.

Data management. No additional research data beyond the data presented and cited in this work are needed to validate the research findings in this work.

A Conventions

A.1 Spin connection and curvature

The non-vanishing components of the spin connection of the near horizon geometry (2.7) in the frame basis (2.9) are

$$
\begin{aligned}
&\Omega_{-,+i} = -\frac{1}{2}h_i\,, \qquad \Omega_{+,+-} = -r\Delta, \qquad \Omega_{+,+i} = \frac{1}{2}r^2(\Delta h_i - \partial_i\Delta),\\
&\Omega_{+,-i} = -\frac{1}{2}h_i, \qquad \Omega_{+,ij} = -\frac{1}{2}rdh_{ij}\,, \qquad \Omega_{i,+-} = \frac{1}{2}h_i, \qquad \Omega_{i,+j} = -\frac{1}{2}rdh_{ij},\\
&\Omega_{i,jk} = \hat{\Omega}_{i,jk}\,,
\end{aligned}
\tag{A.1}
$$

where $\hat{\Omega}$ denotes the spin-connection of the spatial horizon cross section $\mathcal{S}$ in with basis $\mathbf{e}^i$. If f is any function of spacetime, then frame derivatives are expressed in terms of co-ordinate derivatives as

$$
\partial_+ f = \partial_u f + \frac{1}{2}r^2\Delta\partial_r f\,, \quad \partial_- f = \partial_r f\,, \quad \partial_i f = \tilde{\partial}_i f - r\partial_r f h_i\,. \tag{A.2}
$$

The non-vanishing components of the Ricci tensor is the basis (2.9) are

$$
\begin{aligned}
R_{+-} &= \frac{1}{2}\hat{\nabla}^i h_i - \Delta - \frac{1}{2}h^2\,, \qquad R_{ij} = \hat{R}_{ij} + \hat{\nabla}_{(i}h_{j)} - \frac{1}{2}h_i h_j\,,\\
R_{++} &= r^2\left(\frac{1}{2}\hat{\nabla}^2\Delta - \frac{3}{2}h^i\hat{\nabla}_i\Delta - \frac{1}{2}\Delta\hat{\nabla}^i h_i + \Delta h^2 + \frac{1}{4}(dh)_{ij}(dh)^{ij}\right),\\
R_{+i} &= r\left(\frac{1}{2}\hat{\nabla}^j(dh)_{ij} - (dh)_{ij}h^j - \hat{\nabla}_i\Delta + \Delta h_i\right),
\end{aligned}
\tag{A.3}
$$

where $\hat{R}$ is the Ricci tensor of the horizon section $\mathcal{S}$ in the $\mathbf{e}^i$ frame.

A.2 Spinor conventions

We first present a matrix representation of Cliff$(3,1)$ adapted to the basis (2.9). The module of Dirac spinors has been identified with $\mathbb{C}^4$ and we have set

$$
\begin{aligned}
&\Gamma_i = \sigma_i \otimes \sigma_3 = \begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}, \; i = 1,2\,; \qquad \Gamma_0 = i\mathbb{I}_2 \otimes \sigma_2\,, \quad \Gamma_3 = \mathbb{I}_2 \otimes \sigma_1\;;\\
&\Gamma_+ = \begin{pmatrix} 0 & \sqrt{2}\,\mathbb{I}_2 \\ 0 & 0 \end{pmatrix}, \qquad \Gamma_- = \begin{pmatrix} 0 & 0 \\ \sqrt{2}\,\mathbb{I}_2 & 0 \end{pmatrix},
\end{aligned}
$$

where σ^i, are the Hermitian Pauli matrices $\sigma^i\sigma^j = \delta^{ij}\mathbb{I}_2 + i\epsilon^{ijk}\sigma^k$. Note that

$$
\Gamma_{+-} = \begin{pmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{pmatrix},
$$

and we define

$$
\Gamma_5 = i\Gamma_{+-12} = -\sigma_3 \otimes \sigma_3\,. \tag{A.4}
$$

It will be convenient to decompose the spinors into positive and negative chiralities with respect to the lightcone directions as

$$\epsilon = \epsilon_+ + \epsilon_- \,, \tag{A.5}$$

where

$$\Gamma_{+-}\epsilon_\pm = \pm\epsilon_\pm, \qquad \text{or equivalently} \qquad \Gamma_\pm\epsilon_\pm = 0\,. \tag{A.6}$$

With these conventions, note that

$$\Gamma_{ij}\epsilon_\pm = \mp i\epsilon_{ij}\Gamma_5\epsilon_\pm\,. \tag{A.7}$$

The inner product, $\langle\cdot,\cdot\rangle$, we use is that for which spacelike gamma matrices are Hermitian while time-like ones are anti-Hermitian. When restricted on Spin(2) is also Spin(2)-invariant. In particular, note that $(\Gamma_{ij})^\dagger = -\Gamma_{ij}$.

B Special Kähler geometry

B.1 Definition

The matter couplings of the $\mathcal{N} = 2$, $d = 4$ supergravity are described by special Kähler geometry data. For this, we shall give a brief summary of special Kähler geometry. For a review of the various approaches to special Kähler geometry, see [32] and references within.

Let M be a Hodge Kähler manifold,[6] $\mathcal{K}$ be the Hodge complex line bundle over M and E be a flat $\mathrm{Sp}(2(k+1),\mathbb{R})$ vector bundle E with typical fibre $\mathbb{C}^{2(k+1)}$ and compatible (symplectic) fibre inner product $\langle\cdot,\cdot\rangle$.

Next, consider $E\otimes\mathcal{K}$ and introduce the connection on the sections $\mathcal{V}$ of $E\otimes\mathcal{K}$

$$\begin{aligned}\mathcal{D}_{\bar\alpha}\mathcal{V} &= D_{\bar\alpha}\mathcal{V} - \frac{1}{2}\partial_{\bar\alpha}K\mathcal{V}\,,\\ \mathcal{D}_{\alpha}\mathcal{V} &= D_{\alpha}\mathcal{V} + \frac{1}{2}\partial_{\alpha}K\mathcal{V}\,,\end{aligned} \tag{B.1}$$

where D is the flat connection of E, $\partial_\alpha = \partial/\partial z^\alpha$ and z^α are homomorphic coordinates of the Kähler manifold. Observe that the curvature of $\mathcal{D}$ is proportional to the Kähler form of M.

Definition: M is a special Kähler manifold provided that $E\otimes\mathcal{K}$ admits a section $\mathcal{V}$ such that it satisfied the following conditions

$$\begin{aligned}\mathcal{D}_{\bar\alpha}\mathcal{V} = 0\,, &\qquad \langle\mathcal{V},\bar{\mathcal{V}}\rangle = i\,,\\ \langle\mathcal{D}_\alpha\mathcal{V},\mathcal{V}\rangle = 0\,, &\quad \langle\mathcal{D}_\alpha\mathcal{V},\mathcal{D}_\beta\mathcal{V}\rangle = 0\,.\end{aligned} \tag{B.2}$$

$\triangle$

[6]A Kähler manifold M is Hodge, if the cohomology class represented by the Kähler form is the Chern class of a line bundle $\mathcal{K}$ on M. We have also denoted with $\mathcal{K}$ the pull back of the Hodge bundle over $\mathcal{S}$. Which line bundle $\mathcal{K}$ refers to is clear from the context.

To investigate the consequences of the above definition, first perform a $\mathrm{GL}(2(k+1),\mathbb{R})$ transformation to bring the symplectic inner product $\langle\cdot,\cdot\rangle$ into canonical form.[7] Then the above conditions can be re-written as

$$\begin{aligned} \mathcal{D}_{\bar\alpha}X^I = \mathcal{D}_{\bar\alpha}F_I = 0\,, \\ X^I\bar F_I - F_I\bar X^I = i\,, \\ \mathcal{D}_\alpha F_I\,X^I - F_I\,\mathcal{D}_\alpha X^I = 0\,, \\ \mathcal{D}_\alpha F_I\mathcal{D}_\beta X^I - \mathcal{D}_\beta F_I\mathcal{D}_\alpha X^I = 0\,, \end{aligned} \tag{B.3}$$

where the section $\mathcal{V}$ has been written in the canonical form as

$$\mathcal{V} = \begin{pmatrix} X^I \\ F_I \end{pmatrix}. \tag{B.4}$$

Observe that the first condition in (B.3) is a covariant holomorphicity condition while the last condition in (B.3) is implied by the third condition.

Taking the covariant derivative of the second condition in (B.3), we find that

$$\mathcal{D}_\alpha X^I\bar F_I - \mathcal{D}_\alpha F_I\bar X^I = 0\,. \tag{B.5}$$

Next taking that $\mathcal{D}_{\bar\beta}$ covariant derivative of the above expression we find that

$$g_{\alpha\bar\beta} \equiv \partial_\alpha\partial_{\bar\beta}K = i[\mathcal{D}_\alpha X^I\mathcal{D}_{\bar\beta}\bar F_I - \mathcal{D}_\alpha F_I\mathcal{D}_{\bar\beta}\bar X^I]\,. \tag{B.6}$$

The gauge couplings $\mathcal{N}$ are then defined as

$$F_I = \mathcal{N}_{IJ}X^I\,, \quad \mathcal{D}_{\bar\alpha}\bar F_I = \mathcal{N}_{IJ}\mathcal{D}_{\bar\alpha}\bar X^J\,. \tag{B.7}$$

The conditions of special Kähler geometry together with the requirement that M is a Kähler manifold imply that $\mathcal{N}$ is a symmetric matrix. In terms of the gauge couplings, the second and third equations in (B.3), and (B.6) can be written as

$$\mathrm{Im}\mathcal{N}_{IJ}X^I\bar X^J = -\frac{1}{2}\,, \tag{B.8}$$

$$\mathrm{Im}\mathcal{N}_{IJ}X^I\mathcal{D}_\alpha X^J = 0\,, \tag{B.9}$$

$$g_{\alpha\bar\beta} = -2\mathrm{Im}\mathcal{N}_{IJ}\mathcal{D}_\alpha X^I\mathcal{D}_{\bar\beta}\bar X^J\,, \tag{B.10}$$

respectively. As the Kähler metric must be positive definite, it is required that $\mathrm{Im}\mathcal{N}$ is negative definite. The fourth equation in (B.3) and (B.5) are automatically implied as $\mathcal{N}$ is a symmetric matrix.

Furthermore from the definition of $\mathcal{N}$, one can establish the identity

$$U^{IJ} \equiv g^{\alpha\bar\beta}\mathcal{D}_\alpha X^I\mathcal{D}_{\bar\beta}\bar X^J = -\frac{1}{2}(\mathrm{Im}\mathcal{N})^{-1\;IJ} - \bar X^I X^J\,. \tag{B.11}$$

This identity is required in the definition of the scalar potential of the supergravity theory.

[7] Of course one then can use a local gauge $\mathrm{Sp}(2(k+1),\mathbb{R})$ transformation to set $D_\alpha = \partial_\alpha$ and $D_{\bar\alpha} = \partial_{\bar\alpha}$ as D is flat. But this is not necessary in what follows.

JHEP03(2017)150

B.2 Prepotential

A special class of solutions for the conditions of special Kähler geometry (B.3) can be expressed in terms of a holomorphic prepotential as follows. It is well-known that the solutions of a covariant holomorphicity condition on sections of a vector bundle with respect to a connection which has (1,1) curvature can be expressed in terms of the holomorphic sections of the associated holomorphic bundle. In this case, write

$$\mathcal{V} = e^{\frac{K}{2}} \mathcal{U} , \tag{B.12}$$

and observe that

$$\mathcal{D}_\alpha \mathcal{V} = e^{\frac{K}{2}} \mathcal{D}_\alpha \mathcal{U} , \quad \mathcal{D}_{\bar{\alpha}} \mathcal{V} = e^{\frac{K}{2}} \mathcal{D}_{\bar{\alpha}} \mathcal{U} , \tag{B.13}$$

with

$$\mathcal{D}_\alpha \mathcal{U} = D_\alpha \mathcal{U} + \partial_\alpha K \mathcal{U} , \quad \mathcal{D}_{\bar{\alpha}} \mathcal{U} = D_{\bar{\alpha}} \mathcal{U} . \tag{B.14}$$

It is clear from this that in the gauge $\mathcal{D}_{\bar{\alpha}} = \partial_{\bar{\alpha}}$, the covariant holomorphicity condition on $\mathcal{V}$ can be solved by setting

$$\mathcal{V} = e^{\frac{K}{2}} \begin{pmatrix} Z^I \\ \frac{\partial}{\partial Z^I} F \end{pmatrix} , \quad \mathcal{U} = \begin{pmatrix} Z^I \\ \frac{\partial}{\partial Z^I} F \end{pmatrix} , \tag{B.15}$$

where $\mathcal{U}$ is a holomorphic section, ie function only of z, and $F(Z)$ is the prepotential which is taken to be a homogeneous function of degree two in Z. The use of the homogeneity condition will become apparent later.

Let us now investigate the remaining conditions of the special Kähler geometry (B.2) or (B.3) in terms of $\mathcal{U}$. The second condition in (B.3) can now be rewritten as

$$e^{-K} = -i(Z^I \bar{\partial}_I \bar{F} - \bar{Z}^I \partial_I F) = -2 Z^I \bar{Z}^J \mathrm{Im}(\partial_I \partial_J F) , \tag{B.16}$$

where we have used the homogeneity of the prepotential. The remaining two conditions in (B.3) are identically satisfied as a consequence of the homogeneity of F. While (B.5) and (B.6) can now be written as

$$\mathrm{Im}(\partial_I \partial_J F) \mathcal{D}_\alpha Z^I \bar{Z}^J = 0 , \quad g_{\alpha\bar{\beta}} = -\partial_\alpha \partial_{\bar{\beta}} \log[Z^I \bar{Z}^J \mathrm{Im}(\partial_I \partial_J F)] . \tag{B.17}$$

Furthermore, the identities involving the gauge couplings in terms of $\mathcal{U}$ can now be written as follows. First the definition of the gauge couplings becomes

$$\partial_I F = \mathcal{N}_{IJ} Z^I , \quad \bar{\partial}_I \bar{\partial}_J \bar{F} \mathcal{D}_{\bar{\alpha}} \bar{Z}^J = \mathcal{N}_{IJ} \mathcal{D}_{\bar{\alpha}} \bar{Z}^J . \tag{B.18}$$

Then the remaining identities can be expressed as

$$e^{-K} = -2 \mathrm{Im} \mathcal{N}_{IJ} Z^I \bar{Z}^J , \tag{B.19}$$

$$\mathrm{Im} \mathcal{N}_{IJ} Z^I \mathcal{D}_\alpha Z^J = 0 , \tag{B.20}$$

$$g_{\alpha\bar{\beta}} = -2 e^K \mathrm{Im} \mathcal{N}_{IJ} \mathcal{D}_\alpha Z^I \mathcal{D}_{\bar{\beta}} \bar{Z}^J . \tag{B.21}$$

Furthermore, U^{IJ} in (B.11) can be easily written in terms of Z. This concludes the description of the geometry.

C Independent KSEs

C.1 KSEs and integrability conditions on $\mathcal{S}$

Substituting the Killing spinor ϵ (3.7) back into all the KSEs, one obtains from the gravitino KSE along the lightcone directions the integrability conditions

$$\left(\frac{1}{2}\Delta + \frac{i}{8}dh_{ij}\epsilon^{ij}\Gamma_5 - ig\xi_I\Phi^I - \Gamma_+\Theta_-\Gamma_-\Theta_+\right)\phi_+ = 0\,, \tag{C.1}$$

and

$$\Big(\Gamma_-\Theta_+\Gamma_+\Theta_- - \frac{1}{2}\Delta - \frac{i}{8}dh_{ij}\epsilon^{ij}\Gamma_5 - ig\xi_I\Phi^I \\ - i\mathrm{Im}\mathcal{N}_{IJ}\Gamma^i\mathrm{Im}\big((d_h\Phi^I_i - i\epsilon_i{}^j d_h\Phi^I_j)X^J\big)\Big)\eta_- = 0\,, \tag{C.2}$$

and

$$\Big(\frac{1}{4}\Gamma^i(\Delta h_i - \hat{\nabla}_i\Delta) + \frac{i}{8}dh_{ij}\epsilon^{ij}\Gamma_5\Theta_+ - ig\xi_I\Phi^I\Theta_+ \\ + i\mathrm{Im}\mathcal{N}_{IJ}\Gamma^i\mathrm{Im}\big((d_h\Phi^I_i - i\epsilon_i{}^j d_h\Phi^I_j)X^J\big)\Theta_+\Big)\phi_+ = 0\,, \tag{C.3}$$

where ϕ_+ is defined in (3.6).

We remark that the conditions (C.2) and (C.3) are obtained by making use of the following identity:

$$\mathrm{Im}\mathcal{N}_{IJ}\Gamma^i\Big(\mathrm{Im}((\hat{\nabla}_i\Phi^I - h_i\Phi^I - i\epsilon_i{}^j(\hat{\nabla}_j\Phi^I - h_j\Phi^I))X^J) \\ \pm i\Gamma_5\mathrm{Re}((\hat{\nabla}_i\Phi^I - h_i\Phi^I - i\epsilon_i{}^j(\hat{\nabla}_j\Phi^I - h_j\Phi^I))X^J)\Big)\xi_\pm = 0. \tag{C.4}$$

Furthermore, substituting ϵ given in (3.7) into the $\mu = i$ component of the gravitino KSE (3.1) gives two parallel transport equations

$$\hat{\nabla}_i\phi_+ + \left(\frac{i}{2}A_i\Gamma_5 + ig\xi_I B^I_i - \Gamma_i\Theta_- - \frac{i}{4}\epsilon_{ij}h^j\Gamma_5\right)\phi_+ = 0\,, \tag{C.5}$$

and

$$\hat{\nabla}_i\eta_- + \left(\frac{i}{2}A_i\Gamma_5 + ig\xi_I B^I_i + \frac{1}{2}h_i - \Gamma_i\Theta_+ + \frac{i}{4}\epsilon_{ij}h^j\Gamma_5\right)\eta_- = 0\,, \tag{C.6}$$

together with an algebraic integrability condition

$$\Big(-\hat{\nabla}_i\Theta_- + \frac{1}{2}\hat{\nabla}_{(i}h_{j)}\Gamma^j - 2g\xi_I\hat{\nabla}_i\big(\mathrm{Im}X^I + i\Gamma_5\mathrm{Re}X^I\big) - \frac{i}{4}\epsilon_{ij}h^j\Theta_+\Gamma_5 \\ - \frac{i}{2}A_i\big(\Gamma_5\Theta_+ + \Theta_+\Gamma_5\big) - 2g\Theta_+\Gamma_i\xi_I\big(\mathrm{Im}X^I + i\Gamma_5\mathrm{Re}X^I\big) - \frac{3}{4}h_i\Theta_+ + \frac{1}{4}\Gamma_i h_j\Gamma^j\Theta_+ \\ + \frac{i}{2}\mathrm{Im}\mathcal{N}_{IJ}\Big[\mathrm{Im}\big((d_h\Phi^I_i - i\epsilon_i{}^j d_h\Phi^I_j)X^J\big) \\ + i\Gamma_5\mathrm{Re}\big((d_h\Phi^I_i - i\epsilon_i{}^j d_h\Phi^I_j)X^J\big)\Big]\Big)\phi_+ = 0\,. \tag{C.7}$$

Next we consider the gaugini KSEs (3.2). Substituting the spinor ϵ (3.7) again, we obtain

$$\begin{aligned}\Big(&-i\mathrm{Im}\mathcal{N}_{IJ}\Big[\mathrm{Im}\big((\Phi^J+iQ^J)\mathcal{D}_{\bar\beta}\bar X^I g^{\alpha\bar\beta}\big)-i\Gamma_5\mathrm{Re}\big((\Phi^J+iQ^J)\mathcal{D}_{\bar\beta}\bar X^I g^{\alpha\bar\beta}\big)\Big]\\&+\Gamma^i\hat\nabla_i\Big[\mathrm{Re}z^\alpha-i\Gamma_5\mathrm{Im}z^\alpha\Big]+2g\xi_I\Big[\mathrm{Im}\big(\mathcal{D}_{\bar\beta}\bar X^I g^{\alpha\bar\beta}\big)-i\Gamma_5\mathrm{Re}\big(\mathcal{D}_{\bar\beta}\bar X^I g^{\alpha\bar\beta}\big)\Big]\Big)\phi_+=0\,,\end{aligned}\tag{C.8}$$

and

$$\begin{aligned}\Big(&i\mathrm{Im}\mathcal{N}_{IJ}\Big[\mathrm{Im}\big((\Phi^J+iQ^J)\mathcal{D}_{\bar\beta}\bar X^I g^{\alpha\bar\beta}\big)-i\Gamma_5\mathrm{Re}\big((\Phi^J+iQ^J)\mathcal{D}_{\bar\beta}\bar X^I g^{\alpha\bar\beta}\big)\Big]\\&+\Gamma^i\hat\nabla_i\Big[\mathrm{Re}z^\alpha-i\Gamma_5\mathrm{Im}z^\alpha\Big]+2g\xi_I\Big[\mathrm{Im}\big(\mathcal{D}_{\bar\beta}\bar X^I g^{\alpha\bar\beta}\big)-i\Gamma_5\mathrm{Re}\big(\mathcal{D}_{\bar\beta}\bar X^I g^{\alpha\bar\beta}\big)\Big]\Big)\eta_-=0\,,\end{aligned}\tag{C.9}$$

and

$$\begin{aligned}\Big(&i\mathrm{Im}\mathcal{N}_{IJ}\Big[\mathrm{Im}\big((\Phi^J+iQ^J)\mathcal{D}_{\bar\beta}\bar X^I g^{\alpha\bar\beta}\big)+i\Gamma_5\mathrm{Re}\big((\Phi^J+iQ^J)\mathcal{D}_{\bar\beta}\bar X^I g^{\alpha\bar\beta}\big)\Big]\Theta_+\\&-\Gamma^i\hat\nabla_i\Big[\mathrm{Re}z^\alpha+i\Gamma_5\mathrm{Im}z^\alpha\Big]\Theta_++2g\xi_I\Big[\mathrm{Im}\big(\mathcal{D}_{\bar\beta}\bar X^I g^{\alpha\bar\beta}\big)+i\Gamma_5\mathrm{Re}\big(\mathcal{D}_{\bar\beta}\bar X^I g^{\alpha\bar\beta}\big)\Big]\Theta_+\\&+i\mathrm{Im}\mathcal{N}_{IJ}\Gamma^i\mathrm{Im}\Big(\big(d_h\Phi^J_i-i\epsilon_i{}^j d_h\Phi^J_j\big)\mathcal{D}_{\bar\beta}\bar X^I g^{\alpha\bar\beta}\Big)\Big)\phi_+=0\,.\end{aligned}\tag{C.10}$$

The KSEs (C.5), (C.6), (C.8) and (C.9) on $\eta_\pm$ can be thought of as the naive reduction of the gravitino and gaugini KSEs on the spatial horizon section $\mathcal{S}$. The remaining conditions should be thought of as integrability conditions. Typically, the integrability conditions are not independent. Rather they are implied by (C.5), (C.6), (C.8) and (C.9) on $\eta_\pm$ and the field equations.

C.2 Conditions on $\parallel \phi_+ \parallel$

Having established that ϕ_+ cannot vanish identically as a consequence of $\mathrm{Ker}\,\Theta_-=\{0\}$ and the assumption that the solutions are supersymmetric, we consider further the conditions on ϕ_+. In particular, we shall establish, via a maximum principle argument, that $\parallel \phi_+ \parallel^2$ does not depend on the co-ordinates of $\mathcal{S}$.

To proceed, note that (C.5) implies that

$$\hat\nabla_i \parallel \phi_+ \parallel^2=\frac{1}{2}h_i \parallel \phi_+ \parallel^2+\langle\phi_+,-2g\Gamma_i\xi_I\big(\mathrm{Im}X^I+i\Gamma_5\mathrm{Re}X^I\big)\phi_+\rangle\,,\tag{C.11}$$

and hence it follows that

$$\begin{aligned}\hat\nabla_i\hat\nabla^i \parallel \phi_+ \parallel^2=&\frac{1}{2}\hat\nabla^i h_i \parallel \phi_+ \parallel^2+\frac{1}{2}h^i\hat\nabla_i \parallel \phi_+ \parallel^2\\&+\langle\phi_+,-gh^i\Gamma_i\xi_I\big(\mathrm{Im}X^I+i\Gamma_5\mathrm{Re}X^I\big)\phi_+\rangle\\&+\langle\phi_+,-4g\xi_I\mathrm{Im}X^I\big(\hat\Theta_-^\dagger+\hat\Theta_-\big)\phi_+\rangle\\&+\langle\phi_+,-4ig\Gamma_5\xi_I\mathrm{Re}X^I\big(\hat\Theta_-^\dagger-\hat\Theta_-\big)\phi_+\rangle\\&+\langle\phi_+,-2g\Gamma^i\hat\nabla_i\big[\xi_I\big(\mathrm{Im}X^I+i\Gamma_5\mathrm{Re}X^I\big)\big]\phi_+\rangle\\&+\mathrm{Re}\Big(\langle\phi_+,-2i\Gamma^iA_i\hat\Theta_+\Gamma_5\phi_+\rangle\Big)\,,\end{aligned}\tag{C.12}$$

where

$$\hat{\Theta}_{\pm} = -g\xi_I\big(\mathrm{Im}X^I + i\Gamma_5\mathrm{Re}X^I\big) \mp \frac{i}{2}\Big(\mathrm{Im}((\Phi^I + iQ^I)X^J) + i\Gamma_5\mathrm{Re}((\Phi^I + iQ^I)X^J)\Big)\mathrm{Im}\mathcal{N}_{IJ}\,. \tag{C.13}$$

Next, contract (C.7) with Γ^i, to obtain

$$\Big(\frac{1}{4}\hat{\nabla}^i h_i - \frac{1}{8}dh_{ij}\Gamma^{ij} - 2g\xi_I\Gamma^i\hat{\nabla}_i\big(\mathrm{Im}X^I + i\Gamma_5\mathrm{Re}X^I\big) - \frac{1}{8}h_ih^i - \Gamma^i\hat{\nabla}_i\hat{\Theta}_- - 2g\Gamma^i\hat{\Theta}_+\Gamma_i\xi_I\big(\mathrm{Im}X^I + i\Gamma_5\mathrm{Re}X^I\big) - i\Gamma^iA_i\Gamma_5\hat{\Theta}_+\Big)\phi_+ = 0\,. \tag{C.14}$$

This expression implies

$$\Big(\frac{1}{2}\hat{\nabla}^i h_i - \frac{1}{4}h_ih^i\Big) \parallel \phi_+ \parallel^2 + \langle\phi_+, -2g\xi_I\Gamma^i\hat{\nabla}_i\big(\mathrm{Im}X^I + i\Gamma_5\mathrm{Re}X^I\big)\phi_+\rangle + \mathrm{Re}\Big(\langle\phi_+, -4g\Gamma^i\hat{\Theta}_+\Gamma_i\xi_I\big(\mathrm{Im}X^I + i\Gamma_5\mathrm{Re}X^I\big)\phi_+\rangle\Big) - \mathrm{Re}\Big(\langle\phi_+, 2i\Gamma^iA_i\hat{\Theta}_+\Gamma_5\phi_+\rangle\Big) = 0\,. \tag{C.15}$$

On substituting (C.15) into (C.12) to eliminate the $\xi_I\Gamma^i\hat{\nabla}_i\big(\mathrm{Im}X^I + i\Gamma_5\mathrm{Re}X^I\big)$ term, and making use of (C.11), we obtain

$$\hat{\nabla}^i\hat{\nabla}_i \parallel \phi_+ \parallel^2 - h^i\hat{\nabla}_i \parallel \phi_+ \parallel^2 = 0\,. \tag{C.16}$$

On applying the maximum principle[8] we find that

$$\hat{\nabla}_i \parallel \phi_+ \parallel^2 = 0\,, \tag{C.17}$$

and hence

$$\frac{1}{2}h_i \parallel \phi_+ \parallel^2 + \langle\phi_+, -2g\Gamma_i\xi_I\big(\mathrm{Im}X^I + i\Gamma_5\mathrm{Re}X^I\big)\phi_+\rangle = 0\,, \tag{C.18}$$

or equivalently

$$\mathrm{Re}\Big(\langle\phi_+, \Gamma_i\Theta_-\phi_+\rangle\Big) = 0\,, \tag{C.19}$$

or, again, equivalently

$$h_i\Gamma^i\phi_+ = 4g\Big(\xi_I\mathrm{Im}X^I + i\Gamma_5\xi_I\mathrm{Re}X^I\Big)\Big(1 - \frac{\langle\phi_+, \Gamma_5\phi_+\rangle}{\parallel \phi_+ \parallel^2}\Gamma_5\Big)\phi_+\,. \tag{C.20}$$

These conditions imply that

$$h^2 = 16g^2|\xi_IX^I|^2\Big(1 - \frac{\langle\phi_+, \Gamma_5\phi_+\rangle^2}{\langle\phi_+, \phi_+\rangle^2}\Big)\,. \tag{C.21}$$

As a consequence of the last equation we conclude that $\langle\phi_+, \Gamma_5\phi_+\rangle \parallel \phi_+ \parallel^{-2}$ does not depend on the coordinate u.

[8]See e.g. [12].

C.3 Independent KSEs on ϕ_+

In this appendix, we shall first prove that, given the gravitino KSE (C.5) on ϕ_+ and (C.22) defined below, the algebraic KSEs which arise as integrability conditions (C.1), (C.3), (C.7), (C.8), (C.10) can be reduced to conditions involving only the bosonic fields and a function κ on $\mathcal{S}$. Then we shall show that, given these bosonic conditions together with the bosonic field equations, the KSEs involving ϕ_+ are equivalent to the naive restriction of the gravitino (C.5) and gaugini (C.8) KSEs on ϕ_+.

To continue, consider

$$h_i\Gamma^i\phi_+ = 4g\big(\xi_I\mathrm{Im}X^I + i\Gamma_5\xi_I\mathrm{Re}X^I\big)\big(1-\kappa\Gamma_5\big)\phi_+\,, \tag{C.22}$$

an additional condition, where κ is a real function. As (C.22) implies that

$$\kappa = \frac{\langle\phi_+,\Gamma_5\phi_+\rangle}{\parallel\phi_+\parallel^2}\,, \tag{C.23}$$

(C.22) is a rewriting of (C.20) but without $\parallel\phi_+\parallel$ being constant. So (C.22) is equivalent to (C.18). Furthermore, (C.22) also implies that

$$\kappa^2 = 1 - \frac{h^2}{16g^2|\xi_I X^I|^2}\,. \tag{C.24}$$

Then (C.5) implies that κ satisfies

$$\hat{\nabla}_i\kappa = \kappa h_i - \mathrm{Im}\left(\frac{(A-iB)}{2g\xi_I X^I}\right)h_i + \mathrm{Re}\left(\frac{(A-iB)}{2g\xi_I X^I}\right)\epsilon_i{}^j h_j\,, \tag{C.25}$$

where we define the scalars A and B via

$$\begin{aligned}
A &= -g\kappa\xi_I\mathrm{Im}X^I + \frac{1}{2}\mathrm{Im}\mathcal{N}_{IJ}\mathrm{Re}\big((\Phi^I + iQ^I)X^J\big)\,,\\
B &= -g\kappa\xi_I\mathrm{Re}X^I - \frac{1}{2}\mathrm{Im}\mathcal{N}_{IJ}\mathrm{Im}\big((\Phi^I + iQ^I)X^J\big)\,.
\end{aligned} \tag{C.26}$$

In the analysis which will follow, we shall also make use of the integrability condition of (C.5), which is

$$\Gamma^j\big(\hat{\nabla}_j\hat{\nabla}_i - \hat{\nabla}_i\hat{\nabla}_j\big)\phi_+ = \frac{1}{2}\Gamma^j\hat{R}_{ij}\phi_+\,, \tag{C.27}$$

where the l.h.s. is evaluated using (C.5). This condition is equivalent to

$$\begin{aligned}
&\bigg(-g_{\alpha\bar\beta}\hat{\nabla}_i z^{\bar\beta}\hat{\nabla}^i z^\alpha - ig_{\alpha\bar\beta}\hat{\nabla}_i z^{\bar\beta}\hat{\nabla}_j z^\alpha\epsilon^{ij} + \frac{1}{4}h_ih^i + \Delta + \mathrm{Im}\mathcal{N}_{IJ}(\Phi^I\Phi^J + Q^IQ^J)\\
&\quad -\frac{1}{4}dh_{ij}\Gamma^{ij} + 2g\Gamma_5\xi_IQ^I - 2\Gamma^i\hat{\nabla}_i\hat{\Theta}_- - 2\Gamma^j\hat{\Theta}_-\Gamma_j\hat{\Theta}_- + 2i\Gamma^iA_i\Gamma_5\hat{\Theta}_-\bigg)\phi_+ = 0\,.
\end{aligned} \tag{C.28}$$

Now we are ready to determine the conditions on the fields implied by the remaining KSEs on ϕ_+. We begin with the condition (C.8). This condition is equivalent to the following two conditions:

$$\begin{aligned}
\Phi^I + iQ^I &= -2\mathrm{Im}\mathcal{N}_{JN}X^J(\Phi^N + iQ^N)\bar{X}^I\\
&\quad - 2ig\kappa\Big(\xi_J\mathrm{Im}\mathcal{N}^{-1IJ} + 2\xi_JX^J\bar{X}^I\Big)\,,
\end{aligned} \tag{C.29}$$

JHEP03(2017)150

and

$$\hat{\nabla}_i \mathrm{Re} z^\alpha - \epsilon_i{}^j \hat{\nabla}_j \mathrm{Im} z^\alpha = \frac{1}{2}\mathrm{Re}\bigg(\frac{1}{\xi_J \bar{X}^J}\xi_I \mathcal{D}_{\bar{\beta}}\bar{X}^I g^{\alpha\bar{\beta}}\bigg)h_i - \frac{1}{2}\mathrm{Im}\bigg(\frac{1}{\xi_J \bar{X}^J}\xi_I \mathcal{D}_{\bar{\beta}}\bar{X}^I g^{\alpha\bar{\beta}}\bigg)\epsilon_i{}^j h_j\,, \tag{C.30}$$

where (C.22) has been used in order to obtain (C.30).

Next we shall consider (C.1); this is equivalent to the following conditions:

$$\Delta = 4(A^2+B^2)\,, \tag{C.31}$$

and

$$\frac{1}{8}dh_{ij}\epsilon^{ij} + 2g\kappa \mathrm{Im}\mathcal{N}_{IJ}\mathrm{Re}\big(\xi_N \bar{X}^N(\Phi^I + iQ^I)X^J\big) = 0\,. \tag{C.32}$$

In particular, (C.31) implies that $\Delta \geq 0$.

Next, we consider (C.10). We remark that with the definition of the scalars A, B in (C.26), together with (C.22), one has

$$\Theta_+\phi_+ = (A\Gamma_5 + iB)\phi_+\,. \tag{C.33}$$

This expression can be used, together with (C.29), to simplify (C.10) considerably. After some computation, we find that (C.10) is equivalent to:

$$\bigg(-\frac{(A-iB)}{2\xi_J X^J}\xi_I \mathcal{D}_{\bar{\beta}}\bar{X}^I g^{\alpha\bar{\beta}} + 2(A+iB)\mathrm{Im}\mathcal{N}_{IJ}X^J\mathcal{D}_{\bar{\beta}}\bar{X}^I g^{\alpha\bar{\beta}}\bigg)\Big(h_i - i\epsilon_i{}^j h_j\Big) - (A+iB)\Big(\hat{\nabla}_i z^\alpha - i\epsilon_i{}^j\hat{\nabla}_j z^\alpha\Big) + \mathrm{Im}\mathcal{N}_{IJ}\mathcal{D}_{\bar{\beta}}\bar{X}^I g^{\alpha\bar{\beta}}\Big(\hat{\nabla}_i\Phi^J - i\epsilon_i{}^j\hat{\nabla}_j\Phi^J\Big) = 0\,. \tag{C.34}$$

Next we consider (C.3). This algebraic KSE is equivalent to

$$\bigg(\frac{1}{4}\Delta + \frac{1}{\xi_J X^J}(A\xi_I\mathrm{Re}X^I - B\xi_I\mathrm{Im}X^I)(A-iB) - (A+iB)\mathrm{Im}\mathcal{N}_{IJ}\Phi^I X^J\bigg)\Big(h_i - i\epsilon_i{}^j h_j\Big) - \frac{1}{4}\Big(\hat{\nabla}_i\Delta - i\epsilon_i{}^j\hat{\nabla}_j\Delta\Big) + (A+iB)\mathrm{Im}\mathcal{N}_{IJ}X^J\Big(\hat{\nabla}_i\Phi^I - i\epsilon_i{}^j\hat{\nabla}_j\Phi^I\Big) = 0\,. \tag{C.35}$$

Finally, we consider the algebraic KSE (C.7). On making use of (C.5), after some further involved computation, one finds

$$\hat{\nabla}_i(A+iB) - \frac{1}{2}(A+iB)h_i - i(A+iB)A_i - \frac{1}{2}\mathrm{Im}\mathcal{N}_{IJ}\bar{X}^J\Big(d_h\Phi^I_i + i\epsilon_i{}^j d_h\Phi^I_j\Big) + \frac{\xi_J\Phi^J}{8\xi_I X^I}\Big(h_i - i\epsilon_i{}^j h_j\Big) = 0\,. \tag{C.36}$$

Having rewritten the algebraic conditions in this fashion, we shall now reconsider the condition (C.22). This was obtained via a global analysis in the previous section. However, no such analogous condition exists for η_-. Hence, we wish to exchange the condition (C.22) for another algebraic condition, (C.8), for which there does exist an analogous condition for

η_-, which is (C.9). First, note that if one assumes (C.22), together with (C.29) and (C.30), then one directly obtains (C.8). Conversely, if one assumes (C.8), together with (C.29) and (C.30), then one obtains the condition

$$\begin{aligned}&\Big(\mathrm{Im}\big(\xi_I \mathcal{D}_{\bar\beta}\bar X^I g^{\alpha\bar\beta}\big) - i\Gamma_5 \mathrm{Re}\big(\xi_I \mathcal{D}_{\bar\beta}\bar X^I g^{\alpha\bar\beta}\big)\Big)\\ &\qquad\Big(h_i\Gamma^i - 4g\big(\xi_J \mathrm{Im}X^J + i\Gamma_5\xi_J\mathrm{Re}X^J\big)(1-\kappa\Gamma_5)\Big)\phi_+ = 0\,.\end{aligned} \tag{C.37}$$

Hence, either $\xi_I\mathcal{D}_\alpha X^I = 0$, or (C.37) implies (C.22).

We remark that in the special case for which $\xi_I\mathcal{D}_\alpha X^I = 0$ then the equations (C.30), (C.29) and (B.11) imply that the scalars z^α are constant, and also

$$\Phi^I + iQ^I = -2\mathrm{Im}\mathcal{N}_{JN}X^J(\Phi^N + iQ^N)\bar X^I\,. \tag{C.38}$$

In this special case, it is then straightforward to show that one can obtain the condition (C.22) directly from the KSE (C.5) and the bosonic conditions listed above. To see this, note that (C.11) holds as a consequence of (C.5), and as the scalars are constant one finds that (C.12) can be simplified to give

$$\begin{aligned}\hat\nabla_i\hat\nabla^i \parallel \phi_+\parallel^2 = {}&\frac{1}{2}\hat\nabla^i h_i \parallel\phi_+\parallel^2 + \frac{1}{2}h^i\hat\nabla_i\parallel\phi_+\parallel^2\\ &+\langle\phi_+, -gh^i\Gamma_i\xi_I\big(\mathrm{Im}X^I + i\Gamma_5\mathrm{Re}X^I\big)\phi_+\rangle\\ &+\langle\phi_+, -4g\xi_I\mathrm{Im}X^I\big(\hat\Theta_-^\dagger + \hat\Theta_-\big)\phi_+\rangle\\ &+\langle\phi_+, -4ig\Gamma_5\xi_I\mathrm{Re}X^I\big(\hat\Theta_-^\dagger - \hat\Theta_-\big)\phi_+\rangle\,,\end{aligned} \tag{C.39}$$

which can then be further rewritten as

$$\begin{aligned}\hat\nabla_i\hat\nabla^i\parallel\phi_+\parallel^2 - h^i\hat\nabla_i\parallel\phi_+\parallel^2 = {}&\Big(\Delta + \frac{1}{4}h_ih^i - 4g^2|\xi_IX^I|^2 - |\mathrm{Im}\mathcal{N}_{IJ}(\Phi^I+iQ^I)X^J|^2\\ &- 4g\frac{\langle\phi_+,\Gamma_5\phi_+\rangle}{\parallel\phi_+\parallel^2}\mathrm{Im}\big(\xi_L\bar X^L\mathrm{Im}\mathcal{N}_{IJ}(\Phi^I+iQ^I)X^J\big)\Big)\parallel\phi_+\parallel^2\,,\end{aligned} \tag{C.40}$$

where we have used (2.16) to eliminate the divergence in h term, together with (C.38). However, on taking the inner product of (C.28) with ϕ_+ and expanding out the terms, one finds that the r.h.s. of (C.40) vanishes as a consequence of (C.5) and the Einstein field equations. Hence, we have

$$\hat\nabla^i\hat\nabla_i\parallel\phi_+\parallel^2 - h^i\hat\nabla_i\parallel\phi_+\parallel^2 = 0\,, \tag{C.41}$$

which, via an application of the maximum principle, we get $\parallel\phi_+\parallel = \mathrm{const}$ on $\mathcal{S}$. Then (C.11), which follows from (C.5), implies (C.22) as claimed.

C.4 Independent KSEs on η_-

In this section, we analyse the various KSEs involving η_-. The conditions involving η_- are the u-dependent parts of the conditions on ϕ_+ together with (C.6), (C.2) and (C.9). We shall assume all of the conditions on the bosonic fields (C.29), (C.30), (C.31), (C.32), (C.34), (C.35) and (C.36), together with (C.24) and (C.25); which we have previously obtained.

A consequence of our assumptions is that all the KSEs involving η_- which come from the u-dependent parts of ϕ_+, apart from that of (C.5), are automatically satisfied. In the case of the u-dependent part of the gaugino equation, (C.8), we remark that this is implied from (C.5), by making use of the Lichnerowicz theorem analysis as set out in section 4.1.

We shall show that the conditions on η_- corresponding to the u-dependent part of (C.5), as well as (C.2), are implied by (C.6) and (C.9) together with the bosonic conditions. We begin with the u-dependent part of (C.5).

C.4.1 The u-dependent part of (C.5)

The u-dependent part of (C.5) can be rewritten as

$$\begin{aligned}
&\bigg(\hat{\nabla}_i\hat{\Theta}_- + \frac{1}{8}dh_{ij}\Gamma^j + \frac{1}{4}\Big(\hat{\nabla}_{(i}h_{j)} - \frac{1}{2}h_ih_j\Big)\Gamma^j - \Big(\frac{i}{4}h_i + \frac{1}{4}\epsilon_{ij}h^j\Gamma_5\Big)\mathrm{Im}\mathcal{N}_{IJ}\big(\mathrm{Im}((\Phi^I + iQ^I)X^J) \\
&\quad + i\Gamma_5\mathrm{Re}((\Phi^I + iQ^I)X^J)\big) - iA_i\Gamma_5\hat{\Theta}_- \\
&\quad - 2g\hat{\Theta}_-\Gamma_i\xi_I\big(\mathrm{Im}X^I + i\Gamma_5\mathrm{Re}X^I\big)\bigg)\eta_- = 0\,. \qquad \text{(C.42)}
\end{aligned}$$

To proceed, note that the integrability condition of (C.6) can be written as

$$\begin{aligned}
&\bigg(\frac{1}{4}\Gamma^j\Big(\hat{\nabla}_{(i}h_{j)} - \frac{1}{2}h_ih_j\Big) + \frac{1}{8}dh_{ij}\Gamma^j - \frac{1}{2}\Gamma^j g_{\alpha\bar{\beta}}\hat{\nabla}_{(i}z^\alpha\hat{\nabla}_{j)}z^{\bar{\beta}} + \Gamma_i\Big(\frac{1}{8}\mathrm{Im}\mathcal{N}_{IJ}(\Phi^I\Phi^J + Q^IQ^J) \\
&\quad - \frac{1}{4}V + \frac{i}{4}g_{\alpha\bar{\beta}}\epsilon^{mn}\hat{\nabla}_m z^{\bar{\beta}}\hat{\nabla}_n z^\alpha - \frac{1}{2}g\xi_IQ^I\Gamma_5 - \frac{1}{2}\Gamma^j\hat{\nabla}_j\hat{\Theta}_+ \\
&\quad + \frac{i}{2}A_j\Gamma^j\hat{\Theta}_+\Gamma_5 - \frac{1}{2}\Gamma^j\hat{\Theta}_+\Gamma_j\hat{\Theta}_+\Big)\bigg)\eta_- = 0\,. \qquad \text{(C.43)}
\end{aligned}$$

On computing the difference of (C.42) from (C.43), one then obtains, after making use of (C.29)

$$\begin{aligned}
&\bigg(\frac{1}{2}\Gamma^j g_{\alpha\bar{\beta}}\hat{\nabla}_{(i}z^\alpha\hat{\nabla}_{j)}z^{\bar{\beta}} + \Gamma_i\Big(g^2g^{\alpha\bar{\beta}}\xi_I\mathcal{D}_\alpha X^I\xi_J\mathcal{D}_{\bar{\beta}}\bar{X}^J(1+\kappa\Gamma_5)^2 - \frac{i}{4}g_{\alpha\bar{\beta}}\epsilon^{mn}\hat{\nabla}_m z^{\bar{\beta}}\hat{\nabla}_n z^\alpha\Big) \\
&\quad - \frac{3}{2}g\mathrm{Im}\big(\xi_I\mathcal{D}_\alpha X^I\hat{\nabla}_i z^\alpha\big) - \frac{3i}{2}g\Gamma_5\mathrm{Re}\big(\xi_I\mathcal{D}_\alpha X^I\hat{\nabla}_i z^\alpha\big) \\
&\quad + \frac{1}{2}g\epsilon_i{}^j\mathrm{Re}\big(\xi_I\mathcal{D}_\alpha X^I\hat{\nabla}_j z^\alpha\big) - \frac{i}{2}g\Gamma_5\epsilon_i{}^j\mathrm{Im}\big(\xi_I\mathcal{D}_\alpha X^I\hat{\nabla}_j z^\alpha\big) \\
&\quad - \Big(\frac{i}{4}h_i + \frac{1}{4}\epsilon_{ij}h^j\Gamma_5\Big)\mathrm{Im}\mathcal{N}_{IJ}\big(\mathrm{Im}((\Phi^I + iQ^I)X^J) + i\Gamma_5\mathrm{Re}((\Phi^I + iQ^I)X^J)\big) \\
&\quad + \frac{i}{4}\hat{\nabla}_i\Big(\mathrm{Im}\mathcal{N}_{IJ}\big(\mathrm{Im}((\Phi^I + iQ^I)X^J) + i\Gamma_5\mathrm{Re}((\Phi^I + iQ^I)X^J)\big)\Big) \\
&\quad + \frac{1}{4}\epsilon_i{}^j\Gamma_5\hat{\nabla}_j\Big(\mathrm{Im}\mathcal{N}_{IJ}\big(\mathrm{Im}((\Phi^I + iQ^I)X^J) + i\Gamma_5\mathrm{Re}((\Phi^I + iQ^I)X^J)\big)\Big) \\
&\quad + \frac{1}{4}A_i\Big(\mathrm{Im}\mathcal{N}_{IJ}\big(i\mathrm{Re}((\Phi^I + iQ^I)X^J) + \Gamma_5\mathrm{Im}((\Phi^I + iQ^I)X^J)\big)\Big) \\
&\quad - \frac{i}{4}\epsilon_i{}^jA_j\Big(\mathrm{Im}\mathcal{N}_{IJ}\big(\mathrm{Im}((\Phi^I + iQ^I)X^J) + i\Gamma_5\mathrm{Re}((\Phi^I + iQ^I)X^J)\big)\Big)\bigg)\eta_- = 0\,. \qquad \text{(C.44)}
\end{aligned}$$

To simplify this expression further, we make use of the algebraic condition (C.9), which can be rewritten, using (C.29), as

$$\begin{aligned}\Big(&\Gamma^i\hat{\nabla}_i\big(\mathrm{Re}z^\alpha - i\Gamma_5\mathrm{Im}z^\alpha\big) + 2g(1+\kappa\Gamma_5)\xi_I\big(\mathrm{Im}(\mathcal{D}_{\bar\beta}\bar{X}^I g^{\alpha\bar\beta})\\ &- i\Gamma_5\mathrm{Re}(\mathcal{D}_{\bar\beta}\bar{X}^I g^{\alpha\bar\beta})\big)\Big)\eta_- = 0\,. \end{aligned}\tag{C.45}$$

Acting on the left-hand-side of this expression with $\mathrm{Im}(\xi_J\mathcal{D}_\alpha X^J) - i\Gamma_5\mathrm{Re}(\xi_J\mathcal{D}_\alpha X^J)$ gives the condition

$$\begin{aligned}\Big(&\Gamma^j\hat{\nabla}_j\big(\mathrm{Re}z^\alpha - i\Gamma_5\mathrm{Im}z^\alpha\big)\big(\mathrm{Im}(\xi_J\mathcal{D}_\alpha X^J) + i\Gamma_5\mathrm{Re}(\xi_J\mathcal{D}_\alpha X^J)\big)\\ &- 2g(1+\kappa\Gamma_5)\xi_I\mathcal{D}_\alpha X^I\xi_J\mathcal{D}_{\bar\beta}\bar{X}^J g^{\alpha\bar\beta}\Big)\eta_- = 0\,. \end{aligned}\tag{C.46}$$

which is used to eliminate the g^2 term from (C.44). Also, a further useful identity is obtained by acting on the left-hand-side of (C.45) with $\mathrm{Im}g_{\alpha\bar\lambda} - i\Gamma_5\mathrm{Re}g_{\alpha\bar\lambda}$, to obtain

$$\begin{aligned}\Big(&g(1+\kappa\Gamma_5)\big(\mathrm{Im}(\xi_J\mathcal{D}_\alpha X^J) + i\Gamma_5\mathrm{Re}(\xi_J\mathcal{D}_\alpha X^J)\big)\\ &+\frac{i}{2}\Gamma_5\big(\mathrm{Im}g_{\alpha\bar\beta} + i\Gamma_5\mathrm{Re}g_{\alpha\bar\beta}\beta)\Gamma^i\hat{\nabla}_i(\mathrm{Re}z^{\bar\beta} + i\Gamma_5\mathrm{Im}z^{\bar\beta})\Big)\eta_- = 0\,. \end{aligned}\tag{C.47}$$

Using this expression, (C.44) can be rewritten as

$$\Big(S_i + \Gamma_5 T_i\Big)\eta_- = 0\,,\tag{C.48}$$

where

$$\begin{aligned}S_i = &-\frac{i}{4}h_i\mathrm{Im}\mathcal{N}_{IJ}\mathrm{Im}((\Phi^I+iQ^I)X^J) - \frac{i}{4}\epsilon_i{}^j h_j\mathrm{Im}\mathcal{N}_{IJ}\mathrm{Re}((\Phi^I+iQ^I)X^J)\\ &+\frac{i}{2}\kappa g\mathrm{Re}(\xi_I\mathcal{D}_\alpha X^I\hat{\nabla}_i z^\alpha) - \frac{i}{2}\kappa g\epsilon_i{}^j\mathrm{Im}(\xi_I\mathcal{D}_\alpha X^I\hat{\nabla}_j z^\alpha)\\ &+\frac{i}{4}\hat{\nabla}_i\Big(\mathrm{Im}\mathcal{N}_{IJ}\mathrm{Im}((\Phi^I+iQ^I)X^J)\Big) + \frac{i}{4}\epsilon_i{}^j\hat{\nabla}_j\Big(\mathrm{Im}\mathcal{N}_{IJ}\mathrm{Re}((\Phi^I+iQ^I)X^J)\Big)\\ &+\frac{i}{4}A_i\mathrm{Im}\mathcal{N}_{IJ}\mathrm{Re}((\Phi^I+iQ^I)X^J) - \frac{i}{4}\epsilon_i{}^j A_j\mathrm{Im}\mathcal{N}_{IJ}\mathrm{Im}((\Phi^I+iQ^I)X^J)\,, \end{aligned}\tag{C.49}$$

and

$$\begin{aligned}T_i = &\frac{1}{4}h_i\mathrm{Im}\mathcal{N}_{IJ}\mathrm{Re}((\Phi^I+iQ^I)X^J) - \frac{1}{4}\epsilon_i{}^j h_j\mathrm{Im}\mathcal{N}_{IJ}\mathrm{Im}((\Phi^I+iQ^I)X^J)\\ &+\frac{1}{2}\kappa g\mathrm{Im}(\xi_I\mathcal{D}_\alpha X^I\hat{\nabla}_i z^\alpha) + \frac{1}{2}\kappa g\epsilon_i{}^j\mathrm{Re}(\xi_I\mathcal{D}_\alpha X^I\hat{\nabla}_j z^\alpha)\\ &-\frac{1}{4}\hat{\nabla}_i\Big(\mathrm{Im}\mathcal{N}_{IJ}\mathrm{Re}((\Phi^I+iQ^I)X^J)\Big) + \frac{1}{4}\epsilon_i{}^j\hat{\nabla}_j\Big(\mathrm{Im}\mathcal{N}_{IJ}\mathrm{Im}((\Phi^I+iQ^I)X^J)\Big)\\ &+\frac{1}{4}A_i\mathrm{Im}\mathcal{N}_{IJ}\mathrm{Im}((\Phi^I+iQ^I)X^J) + \frac{1}{4}\epsilon_i{}^j A_j\mathrm{Im}\mathcal{N}_{IJ}\mathrm{Re}((\Phi^I+iQ^I)X^J)\,. \end{aligned}\tag{C.50}$$

JHEP03(2017)150

As S_i is imaginary and T_i is real, the condition (C.48) is equivalent to

$$\Big(T_i - S_i\Big)\eta_- = 0\,, \tag{C.51}$$

where

$$\begin{aligned} T_i - S_i &= \frac{1}{4}(h_i + i\epsilon_i{}^j)\mathrm{Im}\mathcal{N}_{IJ}(\Phi^I + iQ^I)X^J - \frac{i}{2}\kappa g\xi_I\mathcal{D}_\alpha X^I\hat{\nabla}_i z^\alpha + \frac{1}{2}\kappa g\epsilon_i{}^j\xi_I\mathcal{D}_\alpha X^I\hat{\nabla}_j z^\alpha \\ &\quad - \frac{1}{4}\hat{\nabla}_i\Big(\mathrm{Im}\mathcal{N}_{IJ}(\Phi^I + iQ^I)X^J\Big) - \frac{i}{4}\epsilon_i{}^j\hat{\nabla}_j\Big(\mathrm{Im}\mathcal{N}_{IJ}(\Phi^I + iQ^I)X^J\Big) \\ &\quad - \frac{i}{4}A_i\mathrm{Im}\mathcal{N}_{IJ}(\Phi^I + iQ^I)X^J + \frac{1}{4}\epsilon_i{}^j A_j\mathrm{Im}\mathcal{N}_{IJ}(\Phi^I + iQ^I)X^J\,. \end{aligned} \tag{C.52}$$

However, the conditions we have found on the fields in the previous section imply that $T_i - S_i = 0$. In particular, this can be seen by writing

$$\mathrm{Im}\mathcal{N}_{IJ}(\Phi^I + iQ^I)X^J = 2(A - iB) - 2ig\kappa\xi_I X^I\,, \tag{C.53}$$

and then by making use of (C.36), (C.25), and (C.29). After some manipulation, one obtains $T_i - S_i = 0$.

Hence, it follows that the u-dependent part of (C.5) is implied by (C.6), (C.9) and the bosonic conditions.

C.4.2 The (C.2) KSE

To analyse (C.2) we begin by contracting (C.43) with Γ^i to obtain

$$\begin{aligned} &\Big(\frac{1}{2}\Delta + \frac{1}{8}h_ih^i + \frac{i}{8}\epsilon^{ij}dh_{ij}\Gamma_5 - \frac{1}{2}g_{\alpha\bar{\beta}}\hat{\nabla}_i z^\alpha\hat{\nabla}^i z^{\bar{\beta}} + \frac{1}{2}\mathrm{Im}\mathcal{N}_{IJ}(\Phi^I\Phi^J + Q^IQ^J) \\ &\quad + \frac{i}{2}g_{\alpha\bar{\beta}}\epsilon^{mn}\hat{\nabla}_m z^{\bar{\beta}}\hat{\nabla}_n z^\alpha - g\xi_I Q^I\Gamma_5 - \Gamma^j\hat{\nabla}_j\hat{\Theta}_+ \\ &\quad + i\Gamma^j A_j\hat{\Theta}_+\Gamma_5 - \Gamma^j\hat{\Theta}_+\Gamma_j\hat{\Theta}_+\Big)\eta_- = 0\,. \end{aligned} \tag{C.54}$$

The $\Gamma^j\hat{\nabla}_j\hat{\Theta}_+$ term is evaluated by making use of (C.36) together with (C.25), and the terms quadratic in Φ^I and Q^I are rewritten using (C.29). Then (C.54) is equivalent to

$$\begin{aligned} &\Big(\frac{1}{2}\Delta + \frac{1}{8}h_ih^i + \frac{i}{8}\epsilon^{ij}dh_{ij}\Gamma_5 - \frac{1}{2}g_{\alpha\bar{\beta}}\hat{\nabla}_i z^\alpha\hat{\nabla}^i z^{\bar{\beta}} + \frac{i}{2}g_{\alpha\bar{\beta}}\epsilon^{mn}\hat{\nabla}_m z^{\bar{\beta}}\hat{\nabla}_n z^\alpha \\ &- 4g^2\kappa^2 g^{\alpha\bar{\beta}}\xi_I\mathcal{D}_\alpha X^I\xi_J\mathcal{D}_{\bar{\beta}}\bar{X}^J - g\xi_I Q^I\Gamma_5 - 2g^2|\xi_I X^I|^2 - \frac{1}{2}|\mathrm{Im}\mathcal{N}_{IJ}(\Phi^I + iQ^I)X^J|^2 + ig\xi_i\Phi^I \\ &- i\Gamma^j\Big(- \frac{1}{2}\mathrm{Im}\mathcal{N}_{IJ}\mathrm{Im}((\Phi^I + iQ^I)X^J)h_j + \frac{1}{2}\mathrm{Im}\mathcal{N}_{IJ}\mathrm{Re}((\Phi^I + iQ^I)X^J)\epsilon_j{}^k h_k\Big) \\ &- i\Gamma^j\big(\mathrm{Im}\mathcal{N}_{IJ}\mathrm{Re}X^J\epsilon_j{}^k d_h\Phi^I_k - \mathrm{Im}\mathcal{N}_{IJ}\mathrm{Im}X^J d_h\Phi^I_j\big) \\ &- g\Gamma^j\big(\mathrm{Im}(\xi_I\mathcal{D}_\alpha X^I\hat{\nabla}_j z^\alpha) + i\Gamma_5\mathrm{Re}(\xi_I\mathcal{D}_\alpha X^I\hat{\nabla}_j z^\alpha)\big)\big(- 1 + \kappa\Gamma_5\big)\Big)\eta_- = 0\,. \end{aligned} \tag{C.55}$$

JHEP03(2017)150

This expression can be further simplified in several ways. Firstly, using (C.9), the final line can be written as

$$-g\Gamma^j\big(\mathrm{Im}(\xi_I\mathcal{D}_\alpha X^I\hat{\nabla}_j z^\alpha)+i\Gamma_5\mathrm{Re}(\xi_I\mathcal{D}_\alpha X^I\hat{\nabla}_j z^\alpha)\big)\big(-1+\kappa\Gamma_5\big)\eta_- \\ =2g^2(1+\kappa\Gamma_5)^2 g^{\alpha\bar{\beta}}\xi_I\mathcal{D}_\alpha X^I\xi_J\mathcal{D}_{\bar{\beta}}\bar{X}^J\eta_-\,. \tag{C.56}$$

Also, using (C.29) we have

$$-g\xi_I Q^I\Gamma_5+4\kappa g^2 g^{\alpha\bar{\beta}}\xi_I\mathcal{D}_\alpha X^I\xi_J\mathcal{D}_{\bar{\beta}}\bar{X}^J\Gamma_5=2g\mathrm{Im}\Big(\xi_L\bar{X}^L\mathrm{Im}\mathcal{N}_{IJ}\big(\Phi^I+iQ^I\big)X^J\Big)\Gamma_5. \tag{C.57}$$

Furthermore, it is useful to note the following identity:

$$\bigg(\frac{1}{2}g_{\alpha\bar{\beta}}\hat{\nabla}_i z^\alpha\hat{\nabla}^i z^{\bar{\beta}}-\frac{i}{2}g_{\alpha\bar{\beta}}\epsilon^{mn}\hat{\nabla}_m z^{\bar{\beta}}\hat{\nabla}_n z^\alpha-\frac{i}{2}\Gamma^i\Gamma^j\big(\mathrm{Re}\hat{\nabla}_i z^\alpha+i\Gamma_5\mathrm{Im}\hat{\nabla}_i z^\alpha\big) \\ \Gamma_5\big(\mathrm{Im}g_{\alpha\bar{\beta}}-i\Gamma_5\mathrm{Re}g_{\alpha\bar{\beta}}\big)\big(\mathrm{Re}\hat{\nabla}_j z^{\bar{\beta}}+i\Gamma_5\mathrm{Im}\hat{\nabla}_j z^{\bar{\beta}}\big)\bigg)\eta_-=0\,, \tag{C.58}$$

and on making repeated use of (C.9) this expression implies that

$$\bigg(\frac{1}{2}g_{\alpha\bar{\beta}}\hat{\nabla}_i z^\alpha\hat{\nabla}^i z^{\bar{\beta}}-\frac{i}{2}g_{\alpha\bar{\beta}}\epsilon^{mn}\hat{\nabla}_m z^{\bar{\beta}}\hat{\nabla}_n z^\alpha \\ -2g^2(1-\kappa^2)g^{\alpha\bar{\beta}}\xi_I\mathcal{D}_\alpha X^I\xi_J\mathcal{D}_{\bar{\beta}}\bar{X}^J\bigg)\eta_-=0\,. \tag{C.59}$$

On substituting (C.56), (C.57) and (C.59) info (C.55) in order to rewrite the final line of (C.55), and then eliminate the $\xi_I Q^I$ term and the terms quadratic in $\hat{\nabla}z$, we find that (C.55) is equivalent to

$$\bigg(\frac{1}{2}\Delta+\frac{1}{8}h_i h^i+\frac{i}{8}\epsilon^{ij}dh_{ij}\Gamma_5+2g\mathrm{Im}\big(\xi_L\bar{X}^L\mathrm{Im}\mathcal{N}_{IJ}\big(\Phi^I+iQ^I\big)X^J\big)-2g^2|\xi_I X^I|^2 \\ -\frac{1}{2}|\mathrm{Im}\mathcal{N}_{IJ}(\Phi^I+iQ^I)X^J|^2+ig\xi_I\Phi^I+h_j\Gamma^j\bigg(\frac{i}{2}\mathrm{Im}\mathcal{N}_{IJ}\mathrm{Im}((\Phi^I+iQ^I)X^J) \\ -\frac{1}{2}\Gamma_5\mathrm{Im}\mathcal{N}_{IJ}\mathrm{Re}((\Phi^I+iQ^I)X^J)\bigg) \\ +i\mathrm{Im}\mathcal{N}_{IJ}\Gamma^i\mathrm{Im}\big(d_h\Phi^I_i-i\epsilon_i{}^j d_h\Phi^I_j)X^J\big)\bigg)\eta_-=0\,. \tag{C.60}$$

After some straightforward rearrangement of terms, we find that (C.60) is equivalent to (C.2).

D Lichnerowicz type theorems for $\phi_\pm$

In this appendix, we provide a more detailed description of the proof of the Lichnerowicz type theorems for $\phi_\pm$ spinors. Note that $\phi_-=\eta_-$ and that the Lichnerowicz type theorem on η_+ is implied from that on ϕ_+.

To begin, the covariant derivatives associated with the gravitino KSE (3.8) have been defined in (3.9). Next upon using the condition (C.29), the algebraic operators (3.12) which define the gaugini KSEs (3.11) can be rewritten as

$$\begin{aligned}\mathcal{A}^{\alpha}_{(\pm)} &\equiv \Gamma^i\hat{\nabla}_i\mathrm{Re}z^{\alpha} + i\Gamma_5\Gamma^i\hat{\nabla}_i\mathrm{Im}z^{\alpha}\\ &\quad + 2(1\mp\kappa\Gamma_5)\Big(g\xi_I\mathrm{Im}\big(\mathcal{D}_{\bar{\beta}}\bar{X}^I g^{\alpha\bar{\beta}}\big) - ig\Gamma_5\mathrm{Re}\big(\mathcal{D}_{\bar{\beta}}\bar{X}^I g^{\alpha\bar{\beta}}\big)\Big)\,. \end{aligned}\tag{D.1}$$

The horizon Dirac operators $\mathcal{D}^{(\pm)}$ are

$$\mathcal{D}^{(\pm)} \equiv \Gamma^i\hat{\nabla}^{(\pm)}_i \tag{D.2}$$

and we assume that $\phi_\pm$ are zero modes, ie they satisfy

$$\mathcal{D}^{(\pm)}\phi_\pm = 0\,. \tag{D.3}$$

We also assume all of the conditions on the fields (C.24), (C.25), (C.29), (C.30), (C.31), (C.32), (C.34), (C.35), (C.36) obtained in appendix C. However, we do not assume that the function κ is related to the zero mode ϕ_+. We then have

$$\hat{\nabla}_i\hat{\nabla}^i \parallel \phi_\pm \parallel^2 = 2\mathrm{Re}\langle\phi_\pm, \hat{\nabla}_i\hat{\nabla}^i\phi_\pm\rangle + 2\langle\hat{\nabla}^i\phi_\pm, \hat{\nabla}_i\phi_\pm\rangle \tag{D.4}$$

where after making use of $\mathcal{D}^{(\pm)}\phi_\pm = 0$

$$\begin{aligned}2\mathrm{Re}\langle\phi_\pm, \hat{\nabla}_i\hat{\nabla}^i\phi_\pm\rangle &= \frac{1}{2}\hat{R}\parallel\phi_\pm\parallel^2\\ &\quad + 2\mathrm{Re}\langle\phi_\pm, \Gamma^i\hat{\nabla}_i\Big(\Big(-\frac{i}{2}A_j\Gamma^j\Gamma_5 - ig\xi_I B^I_j\Gamma^j + 2\hat{\Theta}_\mp \pm \frac{1}{4}h_j\Gamma^j\Big)\phi_\pm\Big)\rangle\,. \end{aligned}\tag{D.5}$$

It follows that we can write

$$\begin{aligned}\hat{\nabla}_i\hat{\nabla}^i\parallel\phi_\pm\parallel^2 &= \Big(\frac{1}{2}\hat{R} \pm \frac{1}{2}\hat{\nabla}^i h_i\Big)\parallel\phi_\pm\parallel^2 + 2\langle\hat{\nabla}^{(\pm)i}\phi_\pm, \hat{\nabla}^{(\pm)}_i\phi_\pm\rangle\\ &\quad + 2\mathrm{Re}\langle\phi_\pm, \Gamma^i\Big(-\frac{i}{2}\Gamma^j A_j\Gamma_5 - ig\xi_I B^I_j\Gamma^j + 2\hat{\Theta}_\mp \pm \frac{1}{4}h_j\Gamma^j\Big)\hat{\nabla}_i\phi_\pm\rangle\\ &\quad - 4\mathrm{Re}\langle\phi_\pm, \Big(-\frac{i}{2}A^i\Gamma_5 - ig\xi_I B^{Ii} - \hat{\Theta}^\dagger_\mp\Gamma^i \mp \frac{1}{4}h^i\Big)\hat{\nabla}_i\phi_\pm\rangle\\ &\quad + \mathrm{Re}\langle\phi_\pm, \Big(-\frac{i}{2}\Gamma^{ij}(dA)_{ij}\Gamma_5 - ig\xi_I dB^I_{ij}\Gamma^{ij} + 4\Gamma^i\hat{\nabla}_i\hat{\Theta}_\mp\Big)\phi_\pm\rangle\\ &\quad - 2\langle\phi_\pm, \Big(-\frac{i}{2}A^i\Gamma_5 - ig\xi_I B^{Ii} - \hat{\Theta}^\dagger_\mp\Gamma^i \mp \frac{1}{4}h^i\Big)\\ &\quad \Big(\frac{i}{2}A_i\Gamma_5 + ig\xi_J B^J_i - \Gamma_i\hat{\Theta}_\mp \mp \frac{1}{4}h_i\Big)\phi_\pm\rangle\,. \end{aligned}\tag{D.6}$$

The terms in the second and third lines of the above expression which are linear in $\hat{\nabla}_i\phi_\pm$ can then be rewritten using $\mathcal{D}^{(\pm)}\phi_\pm = 0$ as

$$\begin{aligned}&\pm h^i\hat{\nabla}_i\parallel\phi_\pm\parallel^2 + \mathrm{Re}\langle\phi_\pm, \Big(-i\Gamma^j A_j\Gamma_5 + 2ig\xi_I B^I_j\Gamma^j \mp \frac{1}{2}h_j\Gamma^j - 8g\xi_I(\mathrm{Im}X^I - i\Gamma_5\mathrm{Re}X^I)\Big)\\ &\qquad\qquad \times\Big(-\frac{i}{2}\Gamma^i A_i\Gamma_5 - ig\xi_J B^J_i\Gamma^i \pm \frac{1}{4}h_i\Gamma^i + 2\hat{\Theta}_\mp\Big)\phi_\pm\rangle\,. \end{aligned}\tag{D.7}$$

Furthermore, we also have

$$(dA)_{ij} = -2ig_{\alpha\bar{\beta}}\hat{\nabla}_{[i}z^{\bar{\beta}}\hat{\nabla}_{j]}z^{\alpha}\,. \tag{D.8}$$

On substituting these expressions into (D.6), we find for ϕ_+:

$$\begin{aligned}\hat{\nabla}_i\hat{\nabla}^i \parallel \phi_+ \parallel^2 - h^i\hat{\nabla}_i \parallel \phi_+ \parallel^2 &= 2\langle\hat{\nabla}^{(+)i}\phi_+, \hat{\nabla}^{(+)}_i\phi_+\rangle \\ &\quad + \Big(4g^2(1+\kappa^2)g^{\alpha\bar{\beta}}\xi_I\mathcal{D}_\alpha X^I\xi_J\mathcal{D}_{\bar{\beta}}\bar{X}^J + g_{\alpha\bar{\beta}}\hat{\nabla}_i z^\alpha\hat{\nabla}^i z^{\bar{\beta}}\Big) \parallel \phi_+ \parallel^2 \\ &\quad + \mathrm{Re}\langle\phi_+, \Big(ig_{\alpha\bar{\beta}}\epsilon^{ij}\hat{\nabla}_i z^{\bar{\beta}}\hat{\nabla}_j z^\alpha - 8\kappa g^2 g^{\alpha\bar{\beta}}\xi_I\mathcal{D}_\alpha X^I\xi_J\mathcal{D}_{\bar{\beta}}\bar{X}^J\Gamma_5 \\ &\quad - 4g\Gamma^i\mathrm{Im}(\xi_I\mathcal{D}_\alpha X^I\hat{\nabla}_i z^\alpha) - 4ig\Gamma^i\Gamma_5\mathrm{Re}(\xi_I\mathcal{D}_\alpha X^I\hat{\nabla}_i z^\alpha)\Big)\phi_+\rangle\,,\end{aligned} \tag{D.9}$$

and for ϕ_- we find

$$\begin{aligned}\hat{\nabla}_i\hat{\nabla}^i \parallel \phi_- \parallel^2 + \hat{\nabla}^i\Big(h_i \parallel \phi_- \parallel^2\Big) &= 2\langle\hat{\nabla}^{(-)i}\phi_-, \hat{\nabla}^{(-)}_i\phi_-\rangle \\ &\quad + \Big(4g^2(1+\kappa^2)g^{\alpha\bar{\beta}}\xi_I\mathcal{D}_\alpha X^I\xi_J\mathcal{D}_{\bar{\beta}}\bar{X}^J + g_{\alpha\bar{\beta}}\hat{\nabla}_i z^\alpha\hat{\nabla}^i z^{\bar{\beta}}\Big) \parallel \phi_- \parallel^2 \\ &\quad + \mathrm{Re}\langle\phi_-, \Big(-ig_{\alpha\bar{\beta}}\epsilon^{ij}\hat{\nabla}_i z^{\bar{\beta}}\hat{\nabla}_j z^\alpha + 8\kappa g^2 g^{\alpha\bar{\beta}}\xi_I\mathcal{D}_\alpha X^I\xi_J\mathcal{D}_{\bar{\beta}}\bar{X}^J\Gamma_5 \\ &\quad - 4g\Gamma^i\mathrm{Im}(\xi_I\mathcal{D}_\alpha X^I\hat{\nabla}_i z^\alpha) - 4ig\Gamma^i\Gamma_5\mathrm{Re}(\xi_I\mathcal{D}_\alpha X^I\hat{\nabla}_i z^\alpha)\Big)\phi_-\rangle\,,\end{aligned} \tag{D.10}$$

where we have made use of the Einstein equation

$$\hat{R} = -\hat{\nabla}^i h_i + \frac{1}{2}h_i h^i + 2g_{\alpha\bar{\beta}}\hat{\nabla}_i z^\alpha\hat{\nabla}^i z^{\bar{\beta}} + 2V - \mathrm{Im}\mathcal{N}_{IJ}\big(\Phi^I\Phi^J + Q^I Q^J\big)\,, \tag{D.11}$$

obtained from taking the trace of (2.19), as well as (C.29).

To complete the proof after some computation one can show that

$$\begin{aligned}&\Big(4g^2(1+\kappa^2)g^{\alpha\bar{\beta}}\xi_I\mathcal{D}_\alpha X^I\xi_J\mathcal{D}_{\bar{\beta}}\bar{X}^J + g_{\alpha\bar{\beta}}\hat{\nabla}_i z^\alpha\hat{\nabla}^i z^{\bar{\beta}}\Big) \parallel \phi_\pm \parallel^2 \\ &\quad + \mathrm{Re}\langle\phi_\pm, \Big(\pm ig_{\alpha\bar{\beta}}\epsilon^{ij}\hat{\nabla}_i z^{\bar{\beta}}\hat{\nabla}_j z^\alpha \mp 8\kappa g^2 g^{\alpha\bar{\beta}}\xi_I\mathcal{D}_\alpha X^I\xi_J\mathcal{D}_{\bar{\beta}}\bar{X}^J\Gamma_5 \\ &\quad - 4g\Gamma^i\mathrm{Im}(\xi_I\mathcal{D}_\alpha X^I\hat{\nabla}_i z^\alpha) - 4ig\Gamma^i\Gamma_5\mathrm{Re}(\xi_I\mathcal{D}_\alpha X^I\hat{\nabla}_i z^\alpha)\Big)\phi_\pm\rangle \\ &= \langle\mathcal{A}^\beta_{(\pm)}\phi_\pm, \big(\mathrm{Re}(g_{\alpha\bar{\beta}}) + i\Gamma_5\mathrm{Im}(g_{\alpha\bar{\beta}})\big)\mathcal{A}^\alpha_{(\pm)}\phi_\pm\rangle\,.\end{aligned} \tag{D.12}$$

The positive definiteness of this term follows from positive definiteness of the Kähler metric on the scalar manifold and after further decomposing $\mathcal{A}^\alpha_{(\pm)}\phi_\pm$ into positive and negative chiralities with respect to Γ_5.

E Properties of the isometry W

In this appendix, we shall consider the case for which the vector field W given in (5.4) does not vanish, $W \not\equiv 0$, and we shall prove that it is a symmetry of the full solution.

First W is an isometry of the metric on $\mathcal{S}$. This can be seen from either (5.6) or verified directly using (C.5) and (C.6) which imply that

$$\hat{\nabla}_j W_i = \mathrm{Re}\Big(-2i\langle \Gamma_+\eta_-, \Gamma_5 \hat{\Theta}_- \eta_+\rangle \Big)\epsilon_{ij}\,, \quad \eta_+ = \Gamma_+\Theta_-\eta_-\,, \tag{E.1}$$

and hence

$$\hat{\nabla}_{(i}W_{j)} = 0\,. \tag{E.2}$$

To proceed, consider the algebraic conditions

$$\mathcal{A}^\alpha_{(\pm)}\eta_\pm = 0\,. \tag{E.3}$$

In particular, on comparing the conditions

$$\langle \Gamma_+\eta_-, \mathcal{A}^\alpha_{(+)}\eta_+\rangle = 0, \qquad \text{and} \qquad \langle \mathcal{A}^\alpha_{(-)}\eta_-, \Gamma_-\eta_+\rangle = 0\,, \tag{E.4}$$

one obtains the condition

$$\mathcal{L}_W \mathrm{Re} z^\alpha = 0\,, \tag{E.5}$$

and on comparing the conditions

$$\langle \Gamma_+\eta_-, i\Gamma_5\mathcal{A}^\alpha_{(+)}\eta_+\rangle = 0, \qquad \text{and} \qquad \langle i\Gamma_5\mathcal{A}^\alpha_{(-)}\eta_-, \Gamma_-\eta_+\rangle = 0\,, \tag{E.6}$$

one finds that

$$\mathcal{L}_W \mathrm{Im} z^\alpha = 0\,, \tag{E.7}$$

and hence

$$\mathcal{L}_W z^\alpha = 0\,. \tag{E.8}$$

The components of W can be rewritten as

$$W_i = -\frac{1}{2} \parallel \eta_- \parallel^2 h_i + 2g\xi_I \mathrm{Im} X^I \tau_i + 2g\xi_I \mathrm{Re} X^I \epsilon_{ij}\tau^j\,, \tag{E.9}$$

where

$$\tau_i = \langle \eta_-, \Gamma_i \eta_-\rangle\,, \tag{E.10}$$

and τ satisfies

$$\tau_i\tau^i = (\parallel \eta_- \parallel^2)^2 - \langle \eta_-, \Gamma_5\eta_-\rangle^2\,. \tag{E.11}$$

JHEP03(2017)150

Then (G.1) implies

$$\mathcal{L}_W \parallel \eta_- \parallel^2 = 4g^2|\xi_I X^I|^2 \Big(\langle \eta_-, \Gamma_5 \eta_- \rangle^2 - (\parallel \eta_- \parallel^2)^2 \kappa^2 \Big) . \tag{E.12}$$

The condition (E.1) implies, on expanding out the expression for $\mathrm{Im}\langle \Gamma_+ \eta_-, \Gamma_5 \hat{\Theta}_- \eta_+ \rangle$, that

$$\begin{aligned} dW_{ij}\epsilon^{ij} = 4\Big(& - g\xi_I \mathrm{Im} X^I h^i \epsilon_{ij} \tau^j + g\xi_I \mathrm{Re} X^I h^i \tau_i \\ & - 4g\mathrm{Re}\big(\xi_L \bar{X}^L \mathrm{Im}\mathcal{N}_{IJ}(\Phi^I + iQ^I) X^J\big) \langle \eta_-, \Gamma_5 \eta_- \rangle \Big) . \end{aligned} \tag{E.13}$$

However, on taking the exterior derivative of (G.1), one finds

$$dW = W \wedge h - \parallel \eta_- \parallel^2 dh . \tag{E.14}$$

On comparing the components of dW between (E.13) and (E.14), making use of (C.32), one finds that if $\xi_I \Phi^I \neq 0$, then the r.h.s. of (E.12) vanishes. So, if $\xi_I \Phi^I \neq 0$ then

$$\mathcal{L}_W \parallel \eta_- \parallel^2 = 0 . \tag{E.15}$$

Then, taking the Lie derivative of (G.1) with respect to W implies that

$$\mathcal{L}_W h = 0 , \tag{E.16}$$

and taking the Lie derivative of the trace of the Einstein (2.19) with respect to W gives

$$\mathcal{L}_W \Big(\mathrm{Im}\mathcal{N}_{IJ}(\Phi^I \Phi^J + Q^I Q^J) \Big) = 0 , \tag{E.17}$$

and taking the Lie derivative of the Einstein equation (2.16) with respect to W implies that

$$\mathcal{L}_W \Delta = 0 . \tag{E.18}$$

The condition (C.24) implies also that

$$\mathcal{L}_W \kappa = 0 . \tag{E.19}$$

Next, on taking the Lie derivative of (C.25) with respect to W gives

$$-\mathrm{Im}\Big(\frac{1}{\xi_I X^I} \mathcal{L}_W (A - iB) \Big) h_i + \mathrm{Re}\Big(\frac{1}{\xi_I X^I} \mathcal{L}_W (A - iB) \Big) \epsilon_i{}^j h_j = 0 . \tag{E.20}$$

We remark that it is not consistent to have $h \equiv 0$, because if $h \equiv 0$ then (C.24) implies that $\kappa^2 = 1$, and then the condition (C.32) is inconsistent with our assumption that $\xi_I \Phi^I \neq 0$. Hence, we must have

$$\mathcal{L}_W A = \mathcal{L}_W B = 0 , \tag{E.21}$$

which further implies that

$$\mathcal{L}_W \Phi^I = \mathcal{L}_W Q^I = 0\,, \tag{E.22}$$

as a consequence of (C.29). Hence, if $\xi_I \Phi^I \neq 0$, then W is a symmetry of the full solution.

Next, we consider the case for which $\xi_I \Phi^I \equiv 0$. On taking the Lie derivative of (C.30) with respect to W, it follows that as $\mathcal{L}_W z^\alpha = 0$, one must have either $\xi_I \mathcal{D}_\alpha X^I = 0$, or $\mathcal{L}_W h = 0$. Suppose then that $\xi_I \Phi^I \equiv 0$, but $\xi_I \mathcal{D}_\alpha X^I \neq 0$. Then

$$\mathcal{L}_W h = 0\,. \tag{E.23}$$

As before, the trace of (2.19), (2.16) and (C.24) imply that

$$\mathcal{L}_W \Big(\mathrm{Im}\mathcal{N}_{IJ}(\Phi^I \Phi^J + Q^I Q^J) \Big) = 0, \qquad \mathcal{L}_W \Delta = 0, \qquad \mathcal{L}_W \kappa = 0\,, \tag{E.24}$$

and taking the Lie derivative of (C.25) with respect to W gives

$$\mathcal{L}_W A = \mathcal{L}_W B = 0, \qquad \text{or} \qquad h = 0\,. \tag{E.25}$$

Suppose that $\mathcal{L}_W A = \mathcal{L}_W B = 0$. Then (C.29) implies that

$$\mathcal{L}_W \Phi^I = \mathcal{L}_W Q^I = 0\,, \tag{E.26}$$

and hence W is a symmetry of the full solution.

Alternatively, if $h \equiv 0$, then the Einstein equation (2.17) implies that $\Delta = \text{const}$, $\Phi^I = \text{const}$ and (C.30) implies that $z^\alpha = \text{const}$. The gauge field equation (2.14) then implies that $Q^I = \text{const}$ as well. So if $h \equiv 0$, it follows again that W must be an symmetry of the full solution. Hence, if $\xi_I \Phi^I = 0$ but $\xi_I \mathcal{D}_\alpha X^I \neq 0$, then W is a symmetry of the full solution.

It remains to consider the case for which $\xi_I \Phi^I = 0$ and $\xi_I \mathcal{D}_\alpha X^I = 0$. For such solutions $dh = 0$, and $z^\alpha = \text{const}$. To proceed in this case, consider the gravinito integrability conditions (C.28) and (C.54), which imply

$$\Big(2g(\Gamma_5 - \kappa)\xi_I Q^I - 2\Gamma^i \hat{\nabla}_i \hat{\Theta}_- \Big) \eta_+ = 0\,, \tag{E.27}$$

and

$$\Big(2g(-\Gamma_5 - \kappa)\xi_I A^I + 2\Gamma^i \hat{\nabla}_i \hat{\Theta}_- \Big) \eta_- = 0\,. \tag{E.28}$$

On taking the inner product of (E.27) with $\Gamma_+ \eta_-$, and comparing this with the (complex conjugate of) the inner product of (E.28) with $\Gamma_- \eta_+$, we obtain

$$\mathcal{L}_W \Big(\mathrm{Im}\mathcal{N}_{IJ} \mathrm{Im}\big((\Phi^I + iQ^I) X^J \big) \Big) = 0\,, \tag{E.29}$$

and on taking the inner product of $i\Gamma_5$(E.27) with $\Gamma_+\eta_-$, and comparing with the (complex conjugate of) the inner product of $i\Gamma_5$(E.28) with $\Gamma_-\eta_+$, we find

$$\mathcal{L}_W\Big(\mathrm{Im}\mathcal{N}_{IJ}\mathrm{Re}\big((\Phi^I + iQ^I)X^J\big)\Big) = 0\,. \tag{E.30}$$

So, we have

$$\mathcal{L}_W\Big(\mathrm{Im}\mathcal{N}_{IJ}\big((\Phi^I + iQ^I)X^J\big)\Big) = 0\,. \tag{E.31}$$

The condition (C.29) then implies that

$$\mathcal{L}_W\Phi^I = \mathcal{L}_W Q^I = 0\,. \tag{E.32}$$

On taking the Lie derivative with respect to W of the gauge equation (2.14), we find

$$\mathrm{Im}\mathcal{N}_{IJ}\Phi^J(\mathcal{L}_W h)_j = \mathrm{Im}\mathcal{N}_{IJ}Q^J\epsilon_j{}^k(\mathcal{L}_W h)_k\,, \tag{E.33}$$

which implies that either $\Phi^I = Q^I = 0$, or $\mathcal{L}_W h = 0$. If $\mathcal{L}_W h = 0$, then on taking the Lie derivative of (2.16) with respect to W gives

$$\mathcal{L}_W\Delta = 0\,, \tag{E.34}$$

and hence W is a symmetry of the full solution.

It remains to consider the case for which $\Phi^I = 0$, $Q^I = 0$ and $\xi_I\mathcal{D}_\alpha X^I = 0$. In this case, the Einstein equation (2.16) can be rewritten as

$$\hat\nabla^i h_i + 8g^2(1+\kappa^2)|\xi_I X^I|^2 = 0\,. \tag{E.35}$$

On integrating this expression over $\mathcal{S}$, we see that it admits no solution. Hence, the case for which $\Phi^I = 0$, $Q^I = 0$ and $\xi_I\mathcal{D}_\alpha X^I = 0$ is excluded.[9]

Hence, in all of the above cases, we have shown that the Lie derivative of all near-horizon data (i.e. the metric on $\mathcal{S}$, h, z^α, Φ^I, Q^I, and Δ) with respect to W vanishes. We remark that these conditions, together with (C.24) imply that in all cases $\mathcal{L}_W\kappa = 0$ as well. Furthermore, one also has $\mathcal{L}_W \parallel \eta_- \parallel^2 = 0$ in all cases as well. To see this, take the Lie derivative of (G.1) with respect to W to obtain

$$d\big(\mathcal{L}_W \parallel \eta_- \parallel^2\big) = -\big(\mathcal{L}_W \parallel \eta_- \parallel^2\big)h\,. \tag{E.36}$$

As $\mathcal{L}_W \parallel \eta_- \parallel^2$ must vanish at some point in $\mathcal{S}$, this condition implies that $\mathcal{L}_W \parallel \eta_- \parallel^2 = 0$ everywhere on $\mathcal{S}$.

[9] We remark that this excludes the solution AdS_4 with constant z^α, and $F^I = 0$.

F 1/2 BPS near-horizon geometries

It is instructive to describe the half-supersymmetric near-horizon geometries constructed in [23] in terms of Gaussian null co-ordinates, and extract all the near-horizon data associated with the solutions. This will incorporate these solutions into our classification scheme and so there will be a unified description of all near horizon geometries of $\mathcal{N}=2$ gauged supergravity coupled to any number of multiplets.

In the spacetime coordinates (t,z,x,v) the metric of the solutions given in [23] is

$$ds^2 = -z^2 e^v \Big(dt + 4(e^{-2v} - L)z^{-1}dx\Big)^2 + 4e^{-v}z^{-2}dz^2 \\ + 16e^{-v}(e^{-2v} - L)dx^2 + \frac{4e^{-2v}}{Y^2(e^{-v} - Le^v)}dv^2\,, \tag{F.1}$$

where $L > 0$ is constant, and

$$Y^2 = 64g^2 e^{-v}|\xi_I X^I|^2 - 1\,. \tag{F.2}$$

The scalars depend only on v, and satisfy

$$\frac{dz^\alpha}{dv} = \frac{i}{2\xi_I \bar{X}^I Y}(1 - iY)g^{\alpha\bar{\beta}}\mathcal{D}_{\bar{\beta}}\Big(\xi_J \bar{X}^J\Big)\,. \tag{F.3}$$

Hence the scalars are constant if and only if

$$\mathcal{D}_\alpha(\xi_I X^I) = 0 \tag{F.4}$$

Note in particular that (F.3) implies that

$$\frac{d}{dv}(|\xi_I X^I|^2) = g^{\alpha\bar{\beta}}\mathcal{D}_\alpha(\xi_I X^I)\mathcal{D}_{\bar{\beta}}(\xi_J \bar{X}^J)\,. \tag{F.5}$$

The gauge field strengths are given by

$$F^I = 8ig\left(\frac{\xi_J \bar{X}^J}{1 - iY}X^I - \frac{\xi_J X^J}{1 + iY}\bar{X}^I\right)dt \wedge dz \\ + \frac{4}{Y}\left(\frac{2\xi_J \bar{X}^J}{1 - iY}X^I + \frac{2\xi_J X^J}{1 + iY}\bar{X}^I + \mathrm{Im}\mathcal{N}^{-1IJ}\xi_J\right)(zdt - 4Ldx) \wedge dv\,. \tag{F.6}$$

In order to rewrite the metric (F.1) in Gaussian null co-ordinates, we set

$$w = e^v, \quad t = u + \frac{4}{wr}, \quad x = \frac{1}{2\sqrt{L}}(\psi + \log(wr)), \quad z = -\frac{\sqrt{L}}{2}wr\,. \tag{F.7}$$

Then in the co-ordinates (u,r,ψ,w) the metric is

$$ds^2 = -\frac{1}{4}Lw^3r^2du^2 + 2dudr + 2rdu\big((1 - Lw^2)d\psi + w^{-1}dw\big) \\ + 4(w^{-1} - Lw)d\psi^2 + \frac{4w^{-4}}{Y^2(w^{-1} - Lw)}dw^2\,. \tag{F.8}$$

It follows that the near-horizon data are given by

$$\Delta = \frac{L}{4} w^3, \quad h = (1 - Lw^2) d\psi + w^{-1} dw \,, \tag{F.9}$$

and

$$ds^2_{\mathcal{S}} = 4(w^{-1} - Lw) d\psi^2 + \frac{4w^{-4}}{Y^2 (w^{-1} - Lw)} dw^2 \,. \tag{F.10}$$

We choose the volume form on $\mathcal{S}$ to be

$$\mathrm{dvol}_{\mathcal{S}} = -4w^{-2} Y^{-1} d\psi \wedge dw \,, \tag{F.11}$$

and with this convention, it is straightforward to prove that the scalars in (F.3) satisfy (C.30).

It is also straightforward to compute Φ^I and Q^I from (F.6); one finds

$$\Phi^I + iQ^I = 4i\sqrt{L} g w \left(\frac{1 - iY}{1 + iY} \right) \xi_J X^J \bar{X}^I + 2i\sqrt{L} g w \left(\xi_J \mathrm{Im}\mathcal{N}^{-1IJ} + 2\xi_J X^J \bar{X}^I \right) . \tag{F.12}$$

In particular, this expression implies that

$$\mathrm{Im}\mathcal{N}_{IJ} X^I (\Phi^J + iQ^J) = -2i\sqrt{L} g w \xi_J X^J \left(\frac{1 - iY}{1 + iY} \right) , \tag{F.13}$$

and hence

$$\Phi^I + iQ^I = -2\mathrm{Im}\mathcal{N}_{JN} X^J (\Phi^N + iQ^N) \bar{X}^I + 2i\sqrt{L} g w \left(\xi_J \mathrm{Im}\mathcal{N}^{-1IJ} + 2\xi_J X^J \bar{X}^I \right) , \tag{F.14}$$

which is consistent with (C.29) on setting

$$\frac{\langle \phi_+, \Gamma_5 \phi_+ \rangle}{\parallel \phi_+ \parallel^2} = -\sqrt{L} w \,. \tag{F.15}$$

For convenience, we shall also list here a number of useful identities associated with this class of solutions:

$$\xi_I \Phi^I = 8\sqrt{L} g w \frac{|\xi_I X^I|^2 Y}{1 + Y^2} \,, \tag{F.16}$$

$$\frac{dz^\alpha}{dw} = \frac{i}{2w \xi_I \bar{X}^I Y} (1 - iY) \xi_J \mathcal{D}_{\bar{\beta}} \bar{X}^J g^{\alpha\bar{\beta}} \,, \tag{F.17}$$

and

$$A + iB = \frac{2\sqrt{L} i w g}{1 - iY} \xi_I \bar{X}^I \,, \tag{F.18}$$

where A and B are defined in (C.26). Furthermore, one can establish

$$\frac{dY}{dw} = 32 g^2 w^{-2} Y^{-1} \left(-\frac{1}{2} \mathrm{Im}\mathcal{N}^{-1IJ} \xi_I \xi_J - 2|\xi_I X^I|^2 \right) , \tag{F.19}$$

and

$$\frac{d\Phi^I}{dw} = -4\sqrt{L} g Y^{-1} \left(\frac{1}{2} \mathrm{Im}\mathcal{N}^{-1IJ} \xi_J + \left(\frac{1 + iY}{1 - iY} \right) \xi_J \bar{X}^J X^I + \left(\frac{1 - iY}{1 + iY} \right) \xi_J X^J \bar{X}^I \right) . \tag{F.20}$$

These formulae provide a useful check on our computations.

G Geometry of the near-horizon solutions

The description of the local geometry of horizons depends on whether the vector field W associated with (5.4) vanishes or not. As it has been presented in detail in appendix E, W is a symmetry of the full solution. In what follows it is useful to consider the identity

$$d \parallel \eta_- \parallel^2 = - \parallel \eta_- \parallel^2 h - W\,. \tag{G.1}$$

This is one of the identities presented in (5.6). It can also been proven directly using (C.6). We shall first consider the special case when $W \equiv 0$.

G.1 Solutions with $W \equiv 0$

All these solutions are warped products $\mathrm{AdS}_2 \times_w \mathcal{S}$. In this case, (G.1) implies that

$$d \parallel \eta_- \parallel^2 = - \parallel \eta_- \parallel^2 h\,, \tag{G.2}$$

and as $\parallel \eta_- \parallel^2$ is nowhere vanishing, one concludes that $dh = 0$. We remark that these solutions are distinct from the class of half-supersymmetric BPS near-horizon solutions in [23], because for those solutions $dh \neq 0$.

Next (2.17) can be rewritten as

$$\begin{aligned}\hat{\nabla}^i\hat{\nabla}_i(\Delta \parallel \eta_- \parallel^2) &+ \frac{1}{\parallel \eta_- \parallel^2}\hat{\nabla}^i(\parallel \eta_- \parallel^2)\hat{\nabla}_i(\Delta \parallel \eta_- \parallel^2)\\ &= -2 \parallel \eta_- \parallel^2 \,\mathrm{Im}\delta^{ij}\mathcal{N}_{IJ} d_h\Phi^I_i d_h\Phi^J_j\,.\end{aligned} \tag{G.3}$$

As $\mathrm{Im}\mathcal{N}_{IJ}$ is negative definite, an application of the maximum principle gives the conditions

$$\Delta \parallel \eta_- \parallel^2 = \mathrm{const}\,, \tag{G.4}$$

and

$$d\Phi^I - \Phi^I h = 0\,. \tag{G.5}$$

Also, (2.18) implies that

$$d\Delta - \Delta h = 0\,. \tag{G.6}$$

This condition implies that either $\Delta = 0$ everywhere, or together with (C.31) $\Delta > 0$ everywhere. Also, (C.3) implies that

$$\xi_I \Phi^I \Theta_+ \phi_+ = 0\,. \tag{G.7}$$

It follows using (5.6) that either $\Delta = 0$ or $\xi_I\Phi^I = 0$.

There are no solutions with $\Delta = 0$ and $\xi_I \Phi^I \neq 0$. To see this observe that (2.16) can be rewritten as

$$\hat{\nabla}^i\hat{\nabla}_i \parallel \eta_- \parallel^2 = -2 \parallel \eta_- \parallel^2 \left(\frac{1}{2}\mathrm{Im}\mathcal{N}_{IJ}(\Phi^I\Phi^J + Q^I Q^J) + V\right). \tag{G.8}$$

As the right-hand-side of this expression is non-negative, an application of the maximum principle implies that $\parallel \eta_- \parallel^2 = const$ and that $\Phi^I = 0$. However, this is in contradiction to the assumption that $\xi_I \Phi^I \neq 0$.

Furthermore there are no solutions with $\Delta = \xi_I \Phi^I = 0$. If $\Delta = 0$ then (G.8) again holds, which implies

$$\Phi^I = Q^I = 0, \qquad V = 0\,, \tag{G.9}$$

and $\parallel \eta_- \parallel^2 = const$. The latter condition implies that $h = 0$ as a consequence of (G.2). In addition, $\Delta = 0$ implies that $\Theta_+ \eta_+ = 0$ as a consequence of (C.31) and (C.33). This, together with the previous conditions, implies

$$\xi_I (\mathrm{Im} X^I + i\Gamma_5 \mathrm{Re} X^I)\eta_+ = 0\,, \tag{G.10}$$

and hence

$$\xi_I X^I = 0\,. \tag{G.11}$$

However, the conditions $\xi_I X^I = 0$ and $V = 0$ then lead to a contradiction. So we must have $\Delta > 0$ everywhere and $\xi_I \Phi^I = 0$.

The condition $\xi_I \Phi^I = 0$ implies that

$$\mathrm{Re}\left(\frac{A - iB}{\xi_I X^I}\right) = 0\,. \tag{G.12}$$

Also, as $\Delta > 0$ everywhere, $A + iB \neq 0$. Then (C.34) and (C.30) imply that

$$\hat{\nabla}_i z^\alpha = \frac{1}{2\xi_J \bar{X}^J} \xi_I \mathcal{D}_{\bar{\beta}} \bar{X}^I g^{\alpha\bar{\beta}} h_i\,. \tag{G.13}$$

It will be convenient to define

$$\tau = \star_{\mathcal{S}} h\,. \tag{G.14}$$

Then (G.13) implies that

$$\mathcal{L}_\tau z^\alpha = 0\,. \tag{G.15}$$

In turn, using (C.36) and (C.31), respectively, one has that

$$\mathcal{L}_\tau A = \mathcal{L}_\tau B = 0\,, \tag{G.16}$$

and

$$\mathcal{L}_\tau \Delta = 0\,. \tag{G.17}$$

Also, as $dh = 0$ and $i_\tau h = 0$, we also have

$$\mathcal{L}_\tau h = 0\,, \tag{G.18}$$

and (C.25) implies

$$\mathcal{L}_\tau \kappa = 0\,, \tag{G.19}$$

as well. These conditions, together with (C.26) imply

$$\mathcal{L}_\tau \Big(\mathrm{Im}\mathcal{N}_{IJ}(\Phi^I + iQ^I)X^J \Big) = 0\,, \tag{G.20}$$

and it therefore follows from (C.29) that

$$\mathcal{L}_\tau \Phi^I = \mathcal{L}_\tau Q^I = 0\,. \tag{G.21}$$

In addition, $\mathcal{L}_\tau \kappa = 0$ and $\mathcal{L}_\tau X^I = 0$ imply, together with (C.24) that

$$\mathcal{L}_\tau h^2 = 0\,. \tag{G.22}$$

It then follows from (2.16) that

$$\mathcal{L}_\tau \big(\hat{\nabla}^i h_i\big) = 0\,. \tag{G.23}$$

We shall consider two subcases, corresponding to $h \equiv 0$ and $h \not\equiv 0$.

G.1.1 Solutions with $W \equiv 0$ and $h \equiv 0$

For solutions with $W \equiv 0$ and $h = 0$, the previously obtained conditions on the bosonic fields imply that z^α, κ, A, B, Δ, Φ^I and Q^I are all constant, with $\Delta > 0$. The spacetime geometry is a product $\mathrm{AdS}_2 \times \mathcal{S}$ described in section 6.1.1.

G.1.2 Solutions with $W \equiv 0$ and $h \not\equiv 0$

For solutions with $W \equiv 0$ and $h \not\equiv 0$, it is convenient to introduce local co-ordinates ψ and x on $\mathcal{S}$ so that

$$\tau = \frac{\partial}{\partial \psi}, \qquad h = dx\,. \tag{G.24}$$

A local basis for $\mathcal{S}$ is then given by

$$\mathbf{e}^1 = \frac{1}{\sqrt{h^2}} dx \qquad \mathbf{e}^2 = \sqrt{h^2}\big(d\psi + q(x,\psi)dx\big)\,, \tag{G.25}$$

where $h^2 = h^2(x)$. The condition $\mathcal{L}_\tau(\hat{\nabla}^i h_i) = 0$ then implies that

$$\frac{\partial^2 q}{\partial \psi^2} = 0\,, \tag{G.26}$$

and so we have

$$q = q_0(x) + \psi q_1(x)\,. \tag{G.27}$$

A co-ordinate transformation of the form

$$\psi = f_1(x)\psi' + f_2(x)\,, \tag{G.28}$$

for appropriately chosen functions f_1, f_2 can be used to further simplify the basis for $\mathcal{S}$:

$$\mathbf{e}^1 = \frac{1}{\sqrt{h^2}}dx, \qquad \mathbf{e}^2 = \sqrt{h^2}Pd\psi'\,, \tag{G.29}$$

with $\tau = h^2 Pd\psi'$, where $P = P(x)$. We shall now drop the prime on ψ'. The scalars z^α, together with κ, Δ, h^2, P, Φ^I and Q^I are independent of the co-ordinate ψ, as are all components of the metric.

After some calculation, the Einstein equations (2.16) and (2.19) imply that

$$\begin{aligned}\Big(16g^2\kappa^2|\xi_I X^I|^2 - 8g\kappa\mathrm{Im}\Big(\xi_L\bar{X}^L\mathrm{Im}\mathcal{N}_{IJ}(\Phi^I + iQ^I)X^J\Big)&\\ - \frac{1}{|\xi_L X^L|^2}g^{\alpha\bar{\beta}}\xi_I\mathcal{D}_\alpha X^I\xi_J\mathcal{D}_{\bar{\beta}}\bar{X}^J h^2\Big)h_i + \hat{\nabla}_i h^2 = 0\,,&\end{aligned} \tag{G.30}$$

and (G.13) implies that

$$\hat{\nabla}_i|\xi_I X^I|^2 = g^{\alpha\bar{\beta}}\xi_I\mathcal{D}_\alpha X^I\xi_J\mathcal{D}_{\bar{\beta}}\bar{X}^J h_i\,. \tag{G.31}$$

Furthermore, note that

$$\hat{\nabla}^i h_i = \frac{dh^2}{dx} + \frac{h^2}{P}\frac{dP}{dx}\,. \tag{G.32}$$

On making use of (2.16) and (G.30), together with (C.29) and (C.31), we find that

$$\hat{\nabla}^i h_i - \frac{dh^2}{dx} = -\frac{1}{2}h^2\left(1 + \frac{1}{|\xi_L X^L|^2}\xi_I\mathcal{D}_\alpha X^I\xi_J\mathcal{D}_{\bar{\beta}}\bar{X}^J g^{\alpha\bar{\beta}}\right)\,. \tag{G.33}$$

It follows that (G.32) implies

$$P^{-1}\frac{dP}{dx} = -\frac{1}{2}\left(1 + \frac{d}{dx}\log|\xi_I X^I|^2\right)\,, \tag{G.34}$$

and so

$$P = \frac{Le^{-\frac{x}{2}}}{|\xi_I X^I|}\,, \tag{G.35}$$

for constant L.

Next note that (C.36) implies that

$$\frac{d}{dx}\log\left(\left(\frac{A - iB}{\xi_I X^I}\right)^2|\xi_J X^J|^2\right) = 1\,, \tag{G.36}$$

so that

$$\frac{A - iB}{\xi_I X^I} = \frac{i\nu}{|\xi_J X^J|}e^{\frac{x}{2}}\,, \tag{G.37}$$

JHEP03(2017)150

for $\nu \in \mathbb{R}$ constant, and (C.31) then implies

$$\Delta = 4\nu^2 e^x \,. \tag{G.38}$$

As we require that $\Delta \neq 0$, we must take $\nu \neq 0$. The scalar κ then satisfies

$$\frac{d\kappa}{dx} = \kappa - \frac{\nu}{2g|\xi_I X^I|} e^{\frac{x}{2}} \,, \tag{G.39}$$

as a consequence of (C.25), and h^2 is then given by (C.24) as

$$h^2 = 16g^2|\xi_I X^I|^2(1-\kappa^2) \,. \tag{G.40}$$

The near-horizon data for this class of solutions have been collected in (6.7). The dependence of the fields in terms of x is determined by the equations (6.8) and (6.9).

G.2 Solutions with $W \not\equiv 0$

As we have already mentioned W leaves all the fields invariant. In addition, the Lie derivatives of κ, and $\parallel \eta_- \parallel^2$ with respect to W also vanish. We present the proof of these in appendix E.

G.2.1 Solutions with $W \not\equiv 0$, and $\kappa = \text{const}$ with $|\kappa| \neq 1$

First we consider the special case for which $\kappa = \text{const}$. Then (G.1) implies that if $h \equiv 0$, then $W \equiv 0$. So it follows that $h \not\equiv 0$, and hence (C.25) implies

$$A - iB = 2ig\kappa\xi_I X^I \,. \tag{G.41}$$

Then (C.29) gives that

$$\Phi^I + iQ^I = -2ig\kappa\Big(\xi_J \mathrm{Im}\mathcal{N}^{-1IJ} + 4\xi_J X^J \bar{X}^I\Big) \,, \tag{G.42}$$

and (C.31) implies that

$$\Delta = 16g^2\kappa^2|\xi_I X^I|^2 \,. \tag{G.43}$$

In particular, (G.42) implies that $\xi_I \Phi^I = 0$, and hence (C.32) implies that

$$dh = 0 \,. \tag{G.44}$$

The Einstein equation (2.16) implies that

$$\hat{\nabla}^i h_i = 2(1-\kappa^2)\Big(4g^2 g^{\alpha\bar{\beta}}\xi_I \mathcal{D}_\alpha X^I \xi_J \mathcal{D}_{\bar{\beta}}\bar{X}^J - 4g^2|\xi_I X^I|^2\Big) \,, \tag{G.45}$$

and (C.30) implies that

$$g_{\alpha\bar{\beta}}\hat{\nabla}_i z^\alpha \hat{\nabla}^i z^{\bar{\beta}} = 4g^2(1-\kappa^2)g^{\alpha\bar{\beta}}\xi_I \mathcal{D}_\alpha X^I \xi_J \mathcal{D}_{\bar{\beta}}\bar{X}^J \,. \tag{G.46}$$

So, on taking the trace of the Einstein equation (2.19) we find

$$\hat{R} = 8g^2(1+\kappa^2)\Big(g^{\alpha\bar{\beta}}\xi_I\mathcal{D}_\alpha X^I \xi_J \mathcal{D}_{\bar{\beta}}\bar{X}^J - |\xi_I X^I|^2\Big)\,, \tag{G.47}$$

and hence

$$\hat{R} = \frac{(1+\kappa^2)}{(1-\kappa^2)}\hat{\nabla}^i h_i\,. \tag{G.48}$$

Thus $\mathcal{S}$ is topologically T^2.

There are two different cases to consider, corresponding as to whether $\parallel \eta_- \parallel^2$ is constant, or not constant.

If $\parallel \eta_- \parallel^2$ is constant, then (G.1) implies that

$$\parallel \eta_- \parallel^2 h + W = 0\,. \tag{G.49}$$

As $dh = 0$ this implies that $dW = 0$. Hence, it follows that both h and W are covariantly constant on $\mathcal{S}$. Therefore $\mathcal{S} = T^2$, and $\hat{R} = 0$ implies that

$$g^{\alpha\bar{\beta}}\xi_I\mathcal{D}_\alpha X^I \xi_J \mathcal{D}_{\bar{\beta}}\bar{X}^J = |\xi_I X^I|^2, \quad \text{and} \quad \mathrm{Im}\mathcal{N}^{-1IJ}\xi_I\xi_J = -4|\xi_I X^I|^2\,. \tag{G.50}$$

As h^2 is constant, it follows from (C.24) that $|\xi_I X^I|^2$ is constant, and also Δ is constant. Furthermore, (C.30) implies that

$$\hat{\nabla}_i z^\alpha = \frac{i}{2\xi_J\bar{X}^J}\xi_I\mathcal{D}_{\bar{\beta}}\bar{X}^I g^{\alpha\bar{\beta}}\epsilon_i{}^j h_j\,. \tag{G.51}$$

It is straightforward to obtain local co-ordinates for the metric; as h is covariantly constant, we can introduce local co-ordinates x, y on $\mathcal{S}$ such that

$$h = dx, \qquad \star_{\mathcal{S}} h = dy\,, \tag{G.52}$$

so that the z^α, Φ^I and Q^I depend only on y. The metric and equations that determine the dependence of the remaining fields on x are summarized in section 6.2.1.

Next, consider the case for which $\parallel \eta_- \parallel^2$ is not constant. As $\mathcal{L}_W h = 0$ and $dh = 0$ it follows that $i_W h = const$. Furthermore, from (G.1), together with $\mathcal{L}_W \parallel \eta_- \parallel^2 = 0$, it follows that

$$\parallel \eta_- \parallel^2 i_W h + W^2 = 0\,, \tag{G.53}$$

and hence $i_W h < 0$. We shall set $i_W h = -\mu^2$, and we shall furthermore introduce local co-ordinates x and ψ on $\mathcal{S}$ such that

$$W = \frac{\partial}{\partial\psi}, \qquad x = \parallel \eta_- \parallel^2\,, \tag{G.54}$$

with

$$h = -\mu^2 d\psi\,. \tag{G.55}$$

Then (G.1) implies that

$$\hat{\nabla}_i x \hat{\nabla}^i x = h^2 x^2 - \mu^2 x \,, \tag{G.56}$$

and moreover

$$W^2 = \mu^2 x \,. \tag{G.57}$$

As $i_W dx = 0$, it follows that

$$dx = \beta \star_{\mathcal{S}} W \,, \tag{G.58}$$

for some function β, and on taking the norm of both sides of this expression, using (G.56) and (G.57) one finds that

$$dx = \mu^{-1} \sqrt{h^2 x - \mu^2} \star_{\mathcal{S}} W \,. \tag{G.59}$$

On substituting this expression back into (G.1) it follows that

$$h = -x^{-1} W - \mu^{-1} x^{-1} \sqrt{h^2 x - \mu^2} \star_{\mathcal{S}} W \,. \tag{G.60}$$

Next,substituting this expression into (C.30) and using the fact that dz^α must be proportional to $\star_{\mathcal{S}} W$, we get that

$$\hat{\nabla}_i z^\alpha = -\frac{i}{2x}\left(1 - \frac{i}{\mu}\sqrt{h^2 x - \mu^2}\right)\frac{1}{\xi_J \bar{X}^J} \xi_I \mathcal{D}_{\bar{\beta}} \bar{X}^I g^{\alpha\bar{\beta}} \epsilon_{ij} W^j \,, \tag{G.61}$$

or equivalently

$$\frac{dz^\alpha}{dx} = -\frac{i}{2x}\left(\frac{\mu}{\sqrt{h^2 x - \mu^2}} - i\right)\frac{1}{\xi_J \bar{X}^J} \xi_I \mathcal{D}_{\bar{\beta}} \bar{X}^I g^{\alpha\bar{\beta}} \,. \tag{G.62}$$

Next we shall consider the conditions (C.34), (C.35) and (C.36). In evaluating these expressions, we make use of (G.61), together with

$$\begin{aligned} \hat{\nabla}_i \Phi^I = 2g\kappa x^{-1}\Bigg(&-\left(1 + i\mu^{-1}\sqrt{h^2 x - \mu^2}\right)\frac{1}{\xi_J X^J} \xi_L \mathcal{D}_{\bar{\alpha}} \bar{X}^L \xi_N \mathcal{D}_\beta X^N g^{\bar{\alpha}\beta} X^I \\ &+ \left(1 - i\mu^{-1}\sqrt{h^2 x - \mu^2}\right)\mathcal{D}_\alpha X^I \xi_N \mathcal{D}_{\bar{\beta}} \bar{X}^N g^{\alpha\bar{\beta}} \\ &- \left(1 - i\mu^{-1}\sqrt{h^2 x - \mu^2}\right)\frac{1}{\xi_J \bar{X}^J} \xi_L \mathcal{D}_\alpha X^L \xi_N \mathcal{D}_{\bar{\beta}} \bar{X}^N g^{\alpha\bar{\beta}} \bar{X}^I \\ &+ \left(1 + i\mu^{-1}\sqrt{h^2 x - \mu^2}\right)\mathcal{D}_{\bar{\alpha}} \bar{X}^I \xi_N \mathcal{D}_\beta X^N g^{\bar{\alpha}\beta}\Bigg)\epsilon_{ij} W^j \,, \end{aligned} \tag{G.63}$$

and

$$h_i - i\epsilon_{ij} h^j = -x^{-1}\left(1 + i\mu^{-1}\sqrt{h^2 x - \mu^2}\right)\left(W_i - i\epsilon_{ij} W^j\right) , \tag{G.64}$$

and

$$\hat{\nabla}_i |\xi_I X^I|^2 = -x^{-1}\mu^{-1}\sqrt{h^2 x - \mu^2}\xi_I \mathcal{D}_\alpha X^I \xi_J \mathcal{D}_{\bar{\beta}} \bar{X}^J g^{\alpha\bar{\beta}} \epsilon_{ij} W^j \,. \tag{G.65}$$

Then on decomposing (C.36) into directions parallel and orthogonal to W, we find the condition

$$\kappa\Big(\mathrm{Im}\mathcal{N}^{-1IJ}\xi_I\xi_J+4|\xi_IX^I|^2\Big)=0\,. \tag{G.66}$$

This condition is sufficient to ensure that (C.34), (C.35) and (C.36) are satisfied.

Suppose that $\kappa\neq 0$. Then the condition (G.66), together with (G.45) implies that

$$\hat{\nabla}^i h_i=0\,, \tag{G.67}$$

and it follows on taking the divergence of (G.1) that

$$\hat{\nabla}^i\hat{\nabla}_i \parallel \eta_- \parallel^2 + h^i\hat{\nabla}_i \parallel \eta_- \parallel^2 = 0\,. \tag{G.68}$$

An application of the maximum principle then implies that $\parallel \eta_- \parallel^2=$ const, but this is in contradiction to our assumption that $\parallel \eta_- \parallel^2$ is not constant. So, for this class of solutions, we must have $\kappa=0$, which in turn implies that

$$\Delta=0,\qquad \Phi^I=Q^I=0,\qquad h^2=16g^2|\xi_IX^I|^2\,. \tag{G.69}$$

It remains to choose a local basis for $\mathcal{S}$; we take

$$\mathbf{e}^1=\mu^{-1}x^{-\frac{1}{2}}W=\mu^{-1}x^{-\frac{1}{2}}(\mu^2xd\psi-dx)\,, \tag{G.70}$$

and

$$\mathbf{e}^2=\mu^{-1}x^{-\frac{1}{2}}\star_{\mathcal{S}}W=\frac{x^{-\frac{1}{2}}}{\sqrt{h^2x-\mu^2}}dx\,, \tag{G.71}$$

so that

$$ds^2_{\mathcal{S}}=\frac{1}{x}\Big(\mu^{-2}(\mu^2xd\psi-dx)^2+\frac{1}{h^2x-\mu^2}dx^2\Big)\,. \tag{G.72}$$

This metric can be simplified further by changing co-ordinates as

$$x=\mu^2x',\qquad \psi=\mu^{-2}\psi'\,, \tag{G.73}$$

to obtain (on dropping primes)

$$ds^2_{\mathcal{S}}=\frac{1}{x}\Big(\big(xd\psi-dx\big)^2+\frac{1}{16g^2|\xi_IX^I|^2x-1}dx^2\Big)\,, \tag{G.74}$$

with

$$h=-d\psi\,. \tag{G.75}$$

The results have been summarized in section 6.2.1. The spacetime metric and the equations that determine the dependence of the scalars on x are given in (6.23) and (6.22), respectively.

G.2.2 Solutions with $W \not\equiv 0$ and $\kappa \neq \text{const}$

To proceed with the analysis, we first make use of (C.25) in order to write h in terms of $d\kappa$ and $\star_{\mathcal{S}} d\kappa$. We find

$$h = \frac{1}{\chi}\left(\left(\kappa - \text{Im}\left(\frac{A-iB}{2g\xi_I X^I}\right)\right)d\kappa - \text{Re}\left(\frac{A-iB}{2g\xi_I X^I}\right)\star_{\mathcal{S}} d\kappa\right), \tag{G.76}$$

where

$$\chi = \left(\kappa - \text{Im}\left(\frac{A-iB}{2g\xi_I X^I}\right)\right)^2 + \left(\text{Re}\left(\frac{A-iB}{2g\xi_I X^I}\right)\right)^2. \tag{G.77}$$

As dz^α must be proportional to $d\kappa$, (C.30) implies that

$$\hat{\nabla}_i z^\alpha = \frac{1}{2\chi\xi_J\bar{X}^J}\left(\kappa + \frac{i(A-iB)}{2g\xi_I X^I}\right)\xi_L \mathcal{D}_{\bar{\beta}}\bar{X}^L g^{\alpha\bar{\beta}}\hat{\nabla}_i\kappa\,. \tag{G.78}$$

Next, we consider (C.36), and decompose the resulting expression into terms parallel and orthogonal to $\hat{\nabla}_i\kappa$, by noting that

$$h + i\star_{\mathcal{S}} h = \frac{1}{\chi}\left(\kappa + \frac{i(A-iB)}{2g\xi_I X^I}\right)\left(d\kappa + i\star_{\mathcal{S}} d\kappa\right). \tag{G.79}$$

On eliminating the terms involving $\hat{\nabla}\Phi^I$ from the two expressions obtained in this fashion, we find

$$\begin{aligned}\hat{\nabla}_i(A+iB) = {}& \frac{1}{\chi}\left(\kappa - \frac{i(A+iB)}{2g\xi_L\bar{X}^L}\right)\left(\frac{1}{2}(A+iB) - \frac{\xi_J\Phi^J}{4\xi_I X^I}\right)\hat{\nabla}_i\kappa \\ &+ i(A+iB)A_i\,.\end{aligned} \tag{G.80}$$

In fact, the remaining parts of (C.34), (C.35) and (C.36) also hold automatically. This makes use of (C.31) and (G.78). Furthermore, using (C.29) together with (G.80) and (G.78), we find that

$$\Phi^I = -2(A-iB)\bar{X}^I - 2(A+iB)X^I\,. \tag{G.81}$$

One then finds

$$\hat{\nabla}_i\Phi^I = -\frac{1}{\chi}\text{Re}\left(4(A+iB)\left(\kappa - \frac{i(A+iB)}{2g\xi_N\bar{X}^N}\right)X^I - \frac{\kappa(A+iB)}{\xi_L\bar{X}^L}\text{Im}\mathcal{N}^{-1IJ}\xi_J\right)\hat{\nabla}_i\kappa\,. \tag{G.82}$$

Using these expressions, the remaining content of (C.34), (C.35) and (C.36) holds automatically.

To proceed, we return to the condition (G.80). Motivated by the expression for $A+iB$ in (F.18) for the example in appendix F, we set

$$A + iB = \kappa\xi_I\bar{X}^I\mathcal{G}\,. \tag{G.83}$$

Then (G.80) can be rewritten as

$$\frac{d\mathcal{G}}{d\kappa} = \frac{\kappa^{-1}}{1+\frac{i}{2g}\bar{\mathcal{G}}}\left(\frac{1}{2}\bar{\mathcal{G}}\left(1 - \frac{i}{g}\mathcal{G}\right) - \frac{1}{2|\xi_L X^L|^2}\mathcal{G}\xi_I\mathcal{D}_\alpha X^I\xi_J\mathcal{D}_{\bar{\beta}}\bar{X}^J g^{\alpha\bar{\beta}}\right). \tag{G.84}$$

On taking the complex conjugate of (G.84), one obtains the following condition

$$\frac{d}{d\kappa}\log\left(\frac{\frac{i}{2g}+\frac{1}{\bar{\mathcal{G}}}}{-\frac{i}{2g}+\frac{1}{\mathcal{G}}}\right)=\frac{\kappa^{-1}}{2|\mathcal{G}|^2|1-\frac{i}{2g}\mathcal{G}|^2}(\mathcal{G}+\bar{\mathcal{G}})\left(\bar{\mathcal{G}}-\mathcal{G}-\frac{i}{g}|\mathcal{G}|^2\right). \tag{G.85}$$

To proceed further, we shall set, see appendix F,

$$\mathcal{G}=-\frac{2ig}{1-iY}, \tag{G.86}$$

for Y a complex function, where $Y\not\equiv 0$, and $Y\not\equiv -i$. Then (G.85) is equivalent to

$$\frac{d}{d\kappa}\left(\frac{\bar{Y}}{Y}\right)=\frac{1}{2}\kappa^{-1}\left(1-\left(\frac{\bar{Y}}{Y}\right)^2\right), \tag{G.87}$$

which has the general solution

$$\frac{\bar{Y}}{Y}=\frac{\kappa+ic}{\kappa-ic}, \tag{G.88}$$

for constant $c\in\mathbb{R}$. Using this expression, we can eliminate $\bar{\mathcal{G}}$ in favour of $\mathcal{G}$ in (G.84) to find

$$\frac{d\mathcal{G}}{d\kappa}=\frac{1}{2\kappa(\kappa+ic)}\left(\frac{\kappa\mathcal{G}+ig(\kappa+ic)}{\frac{1}{2}\mathcal{G}+ig}\right)\left(\frac{ig(\kappa-ic)\mathcal{G}}{\kappa\mathcal{G}+ig(\kappa+ic)}\left(1-\frac{i}{g}\mathcal{G}\right)\right.$$
$$\left.-\frac{1}{|\xi_L X^L|^2}\mathcal{G}\xi_I\mathcal{D}_\alpha X^I\xi_J\mathcal{D}_{\bar{\beta}}\bar{X}^J g^{\alpha\bar{\beta}}\right), \tag{G.89}$$

and moreover, on using (G.78) we also have

$$\frac{d}{d\kappa}|\xi_I X^I|^2=\frac{1}{2\kappa(\kappa+ic)}\left(\frac{\kappa\mathcal{G}+2ig(\kappa+ic)}{\frac{1}{2}\mathcal{G}+ig}\right)\xi_I\mathcal{D}_\alpha X^I\xi_J\mathcal{D}_{\bar{\beta}}\bar{X}^J g^{\alpha\bar{\beta}}. \tag{G.90}$$

We shall consider the cases for which $\kappa\mathcal{G}+2ig(\kappa+ic)$ vanishes identically, and is non-zero, separately.

Suppose first that $\kappa\mathcal{G}+2ig(\kappa+ic)\not\equiv 0$. Then the conditions (G.89) and (G.90) can be combined to give

$$\frac{d}{d\kappa}\log\left(\frac{1}{|\xi_I X^I|^2}(1-iY)(i\kappa(1+iY)+c(1-iY))\right)=\frac{c}{\kappa(\kappa+ic)}(Y^{-1}+i). \tag{G.91}$$

Furthermore, we recall that $W=\beta\star_{\mathcal{S}}d\kappa$ for some function $\beta=\beta(\kappa)$. On substituting this into the condition (G.1), one obtains

$$\beta=\frac{\parallel\eta_-\parallel^2}{2g\kappa}\frac{\mathrm{Re}\mathcal{G}}{(1+\frac{i}{2g}\bar{\mathcal{G}})(1-\frac{i}{2g}\mathcal{G})}, \tag{G.92}$$

and

$$\frac{d\parallel\eta_-\parallel^2}{d\kappa}=-\frac{\parallel\eta_-\parallel^2}{2\kappa}\left(\frac{1}{1-\frac{i}{2g}\mathcal{G}}+\frac{1}{1+\frac{i}{2g}\bar{\mathcal{G}}}\right). \tag{G.93}$$

Then (G.93) can be rewritten in terms of Y as

$$\frac{d}{d\kappa}\log \parallel \eta_- \parallel^2 = \frac{c\kappa^{-1}Y^{-1}}{\kappa+ic} - \kappa^{-1} \,. \tag{G.94}$$

Next on combining (G.94) and (G.91), we find that the resulting condition can be integrated up to give

$$\frac{(\kappa+ic)(1-iY)(i\kappa(1+iY)+c(1-iY))}{\kappa^2 \parallel \eta_- \parallel^2 |\xi_I X^I|^2} = ip \,, \tag{G.95}$$

for $p \in \mathbb{R}$ constant, $p \neq 0$. To see that $p \neq 0$, we rewrite (G.95) using (G.88) as

$$\frac{(1+c^2\kappa^{-2})|1-iY|^2}{\parallel \eta_- \parallel^2 |\xi_I X^I|^2} = p \,. \tag{G.96}$$

To obtain local expressions for all the near-horizon data, we take local co-ordinates κ, ψ with $W = \frac{\partial}{\partial \psi}$ and take, without loss of generality

$$W = S d\psi \,, \tag{G.97}$$

for $S = S(\kappa)$.[10] Then

$$S = W^2 = \beta^2 \hat{\nabla}_i \kappa \hat{\nabla}^i \kappa = \frac{4(\parallel \eta_- \parallel^2)^2(1-\kappa^2)|\xi_I X^I|^2(\mathrm{Re}\mathcal{G})^2}{(1+\frac{i}{2g}\bar{\mathcal{G}})(1-\frac{i}{2g}\mathcal{G})} \,, \tag{G.98}$$

where we have used (G.92) together with (C.24) and (C.25). This implies that

$$\star_{\mathcal{S}} d\kappa = 8g\kappa \parallel \eta_- \parallel^2 (1-\kappa^2)|\xi_I X^I|^2(\mathrm{Re}\mathcal{G})d\psi \,. \tag{G.99}$$

In addition, Δ is given by (C.31) as

$$\Delta = \frac{16g^2\kappa^2|\xi_I X^I|^2}{|1-iY|^2} \,, \tag{G.100}$$

and (G.78) implies that

$$\frac{dz^\alpha}{d\kappa} = \frac{1}{2\kappa \xi_J \bar{X}^J}(1+iY^{-1})\xi_I \mathcal{D}_{\bar{\beta}} \bar{X}^I g^{\alpha\bar{\beta}} \,. \tag{G.101}$$

It is convenient to set $\psi = \frac{p}{16g^2}\phi$, then the metric on $\mathcal{S}$ can be written as

$$ds^2_{\mathcal{S}} = \Delta^{-1}\left(\frac{1}{|Y|^2(1-\kappa^2)}d\kappa^2 + (\kappa^2+c^2)(1-\kappa^2)d\phi^2\right) . \tag{G.102}$$

The expression for h is obtained by using (G.1), together with (G.97) and (G.98) and (G.94), to find

$$h = \kappa^{-1}\left(1 - \frac{c}{(\kappa+ic)Y}\right)d\kappa - (1-\kappa^2)d\phi \,. \tag{G.103}$$

Furthermore, (C.29) implies

$$\Phi^I + iQ^I = -\frac{8ig\kappa}{1+i\bar{Y}}\xi_J X^J \bar{X}^I - 2ig\kappa \mathrm{Im}\mathcal{N}^{-1IJ}\xi_J \,. \tag{G.104}$$

The spacetime metric and the equations that determine the near horizon fields are summarized in section 6.2.2. The special case for which $\kappa\mathcal{G} + 2ig(\kappa+ic) = 0$ is summarized in section 6.2.2.

[10] This can always be done by making use of a co-ordinate transformation of the form $\psi = \psi' + H(\kappa)$.

H Gauge field equations

Here, we list the non-trivial content of the gauge field equations (2.14). In a number of cases, these hold automatically. In the remaining cases, only one non-trivial component of (2.14) needs to be checked as the others can be shown to hold automatically.

The cases to be considered are

(1) The class of solution in section 6.1.2. The gauge field equation is

$$\frac{d}{dx}\Big(\mathrm{Im}\mathcal{N}_{IJ}Q^J\Big)+\frac{d}{dx}\Big(\mathrm{Re}\mathcal{N}_{IJ}\Big)\Phi^J-\mathrm{Im}\mathcal{N}_{IJ}Q^J=0\,. \tag{H.1}$$

(2) The first class of solutions in section 6.2.1 — for which $\parallel \eta_- \parallel^2=$ const, i.e. up to equation (6.16). For this case, the gauge field equation content is:

$$\frac{d}{dy}\Big(\mathrm{Im}\mathcal{N}_{IJ}Q^J\Big)+\frac{d}{dy}\Big(\mathrm{Re}\mathcal{N}_{IJ}\Big)\Phi^J-\mathrm{Im}\mathcal{N}_{IJ}\Phi^J=0\,. \tag{H.2}$$

(3) The solution of section 6.2.2. For this case, the gauge field equation content is:

$$\begin{aligned}\frac{d}{d\kappa}\Big(\mathrm{Im}\mathcal{N}_{IJ}Q^J\Big)+\frac{d}{d\kappa}\Big(\mathrm{Re}\mathcal{N}_{IJ}\Big)\Phi^J-\mathrm{Im}\mathcal{N}_{IJ}\Big(&\frac{1}{(\kappa+ic)Y}\Phi^J\\ -\frac{c-(\kappa+ic)Y}{\kappa(\kappa+ic)Y}Q^J\Big)&=0\,.\end{aligned} \tag{H.3}$$

To evaluate these equations it is useful to first note that (C.29) implies that

$$Q^J=\hat{Q}^J-2g\kappa\mathrm{Im}\mathcal{N}^{-1JL}\xi_L \tag{H.4}$$

where

$$\Phi^J-i\hat{Q}^J=WX^J \tag{H.5}$$

for some complex function W whose precise form depends on the case under consideration. For all of the gauge field equations, we must evaluate a term of the type

$$d(\mathrm{Im}\mathcal{N}_{IJ}Q^J)+\Phi^J d\Big(\mathrm{Re}\mathcal{N}_{IJ}\Big)=-2g\xi_I d\kappa+\mathrm{Im}\mathcal{N}_{IJ}d\hat{Q}^J+\mathrm{Re}\Big(WX^J d\mathcal{N}_{IJ}\Big)\,. \tag{H.6}$$

The final term in the above expression can be rewritten using the conditions of special geometry. In particular we have

$$X^J d\mathcal{N}_{IJ}=-2i\mathrm{Im}\mathcal{N}_{IJ}\mathcal{D}_\alpha X^J dz^\alpha\,, \tag{H.7}$$

where we have made use of the special Kähler geometry identities (B.7) in appendix B. On using these identities one obtains

$$\begin{aligned}d(\mathrm{Im}\mathcal{N}_{IJ}Q^J)+\Phi^J d\Big(\mathrm{Re}\mathcal{N}_{IJ}\Big)&=-2g\xi_I d\kappa+\mathrm{Im}\mathcal{N}_{IJ}d\hat{Q}^J\\ &\quad+2\mathrm{Im}\mathcal{N}_{IJ}\mathrm{Im}\Big(W\mathcal{D}_\alpha X^J dz^\alpha\Big)\,.\end{aligned} \tag{H.8}$$

All of the terms in this expression can then be directly calculated using the conditions we have found on the solutions. In particular, the dependence of κ is known, the $d\hat{Q}^I$ term can be calculated directly, as can W and dz^α.

References

[1] B. Carter, *Black Holes*, C. de Witt and B.S. de Witt eds., Gordon and Breach, New York (1973).

[2] G.W. Gibbons, *Supersymmetry, Supergravity and Related Topics*, F. del Aguila, J.A. de Azcarraga and L.E. Ibanez eds., World Scientific (1985).

[3] G.W. Gibbons and P.K. Townsend, *Vacuum interpolation in supergravity via super p-branes*, *Phys. Rev. Lett.* **71** (1993) 3754 [hep-th/9307049] [INSPIRE].

[4] O. Aharony, S.S. Gubser, J.M. Maldacena, H. Ooguri and Y. Oz, *Large-N field theories, string theory and gravity*, *Phys. Rept.* **323** (2000) 183 [hep-th/9905111] [INSPIRE].

[5] U. Gran, J. Gutowski and G. Papadopoulos, *Index theory and dynamical symmetry enhancement near IIB horizons*, *JHEP* **11** (2013) 104 [arXiv:1306.5765] [INSPIRE].

[6] J. Gutowski and G. Papadopoulos, *Heterotic Black Horizons*, *JHEP* **07** (2010) 011 [arXiv:0912.3472] [INSPIRE].

[7] J. Grover, J.B. Gutowski, G. Papadopoulos and W.A. Sabra, *Index Theory and Supersymmetry of* 5*D Horizons*, *JHEP* **06** (2014) 020 [arXiv:1303.0853] [INSPIRE].

[8] J. Gutowski and G. Papadopoulos, *Index theory and dynamical symmetry enhancement of M-horizons*, *JHEP* **05** (2013) 088 [arXiv:1303.0869] [INSPIRE].

[9] U. Gran, J. Gutowski, U. Kayani and G. Papadopoulos, *Dynamical symmetry enhancement near IIA horizons*, *JHEP* **06** (2015) 139 [arXiv:1409.6303] [INSPIRE].

[10] U. Gran, J. Gutowski, U. Kayani and G. Papadopoulos, *Dynamical symmetry enhancement near massive IIA horizons*, *Class. Quant. Grav.* **32** (2015) 235004 [arXiv:1411.5286] [INSPIRE].

[11] L. Andrianopoli et al., $N = 2$ *supergravity and* $N = 2$ *super Yang-Mills theory on general scalar manifolds: Symplectic covariance, gaugings and the momentum map*, *J. Geom. Phys.* **23** (1997) 111 [hep-th/9605032] [INSPIRE].

[12] M. Protter and H. Weinberger, *Maximum principles in differential equations*, Prentice-Hall Inc. (1967).

[13] S. Ferrara, R. Kallosh and A. Strominger, $N = 2$ *extremal black holes*, *Phys. Rev.* **D 52** (1995) R5412 [hep-th/9508072] [INSPIRE].

[14] A. Strominger, *Macroscopic entropy of* $N = 2$ *extremal black holes*, *Phys. Lett.* **B 383** (1996) 39 [hep-th/9602111] [INSPIRE].

[15] S. Ferrara and R. Kallosh, *Supersymmetry and attractors*, *Phys. Rev.* **D 54** (1996) 1514 [hep-th/9602136] [INSPIRE].

[16] S. Ferrara, G.W. Gibbons and R. Kallosh, *Black holes and critical points in moduli space*, *Nucl. Phys.* **B 500** (1997) 75 [hep-th/9702103] [INSPIRE].

JHEP03(2017)150

[17] A. Van Proeyen, $N = 2$ *supergravity in* $d = 4, 5, 6$ *and its matter couplings*, extended version of lectures given during the semester *Supergravity, superstrings and M-theory*, Institut Henri Poincaré, Paris, November 2000, http://itf.fys.kuleuven.be/∼toine/LectParis.pdf.

[18] V. Moncrief and J. Isenberg, *Symmetries of cosmological Cauchy horizons*, *Commun. Math. Phys.* **89** (1983) 387 [INSPIRE].

[19] H. Friedrich, I. Racz and R.M. Wald, *On the rigidity theorem for space-times with a stationary event horizon or a compact Cauchy horizon*, *Commun. Math. Phys.* **204** (1999) 691 [gr-qc/9811021] [INSPIRE].

[20] A. Gnecchi, K. Hristov, D. Klemm, C. Toldo and O. Vaughan, *Rotating black holes in* 4d *gauged supergravity*, *JHEP* **01** (2014) 127 [arXiv:1311.1795] [INSPIRE].

[21] S.L. Cacciatori, D. Klemm, D.S. Mansi and E. Zorzan, *All timelike supersymmetric solutions of* $N = 2$, $D = 4$ *gauged supergravity coupled to abelian vector multiplets*, *JHEP* **05** (2008) 097 [arXiv:0804.0009] [INSPIRE].

[22] D. Klemm and E. Zorzan, *All null supersymmetric backgrounds of* $N = 2$, $D = 4$ *gauged supergravity coupled to abelian vector multiplets*, *Class. Quant. Grav.* **26** (2009) 145018 [arXiv:0902.4186] [INSPIRE].

[23] D. Klemm and E. Zorzan, *The timelike half-supersymmetric backgrounds of* $N = 2$, $D = 4$ *supergravity with Fayet-Iliopoulos gauging*, *Phys. Rev.* **D 82** (2010) 045012 [arXiv:1003.2974] [INSPIRE].

[24] C. Li and J. Lucietti, *Transverse deformations of extreme horizons*, *Class. Quant. Grav.* **33** (2016) 075015 [arXiv:1509.03469] [INSPIRE].

[25] I. Booth and S. Fairhurst, *Extremality conditions for isolated and dynamical horizons*, *Phys. Rev.* **D 77** (2008) 084005 [arXiv:0708.2209] [INSPIRE].

[26] M. Mars, *Stability of MOTS in totally geodesic null horizons*, *Class. Quant. Grav.* **29** (2012) 145019 [arXiv:1205.1724] [INSPIRE].

[27] J. Lucietti and H.S. Reall, *Gravitational instability of an extreme Kerr black hole*, *Phys. Rev.* **D 86** (2012) 104030 [arXiv:1208.1437] [INSPIRE].

[28] I. Booth, *Black hole boundaries*, *Can. J. Phys.* **83** (2005) 1073 [gr-qc/0508107] [INSPIRE].

[29] S.A. Hayward, *General laws of black hole dynamics*, *Phys. Rev.* **D 49** (1994) 6467 [INSPIRE].

[30] M.F. Atiyah and I.M. Singer, *The index of elliptic operators. 1*, *Annals Math.* **87** (1968) 484 [INSPIRE].

[31] D. Astefanesei, K. Goldstein, R.P. Jena, A. Sen and S.P. Trivedi, *Rotating attractors*, *JHEP* **10** (2006) 058 [hep-th/0606244] [INSPIRE].

[32] B. Craps, F. Roose, W. Troost and A. Van Proeyen, *What is special Kähler geometry?*, *Nucl. Phys.* **B 503** (1997) 565 [hep-th/9703082] [INSPIRE].