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Probing the string winding sector

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Abstract: We probe a slice of the massive winding sector of bosonic string theory from toroidal compactifications of Double Field Theory (DFT). This string subsector corresponds to states containing one left and one right moving oscillators. We perform a generalized Kaluza Klein compactification of DFT on generic $2n$-dimensional toroidal constant backgrounds and show that, up to third order in fluctuations, the theory coincides with the corresponding effective theory of the bosonic string compactified on n-dimensional toroidal constant backgrounds, obtained from three-point amplitudes. The comparison between both theories is facilitated by noticing that generalized diffeomorphisms in DFT allow to fix generalized harmonic gauge conditions that help in identifying the physical degrees of freedom. These conditions manifest as conformal anomaly cancellation requirements on the string theory side. The explicit expression for the gauge invariant effective action containing the physical massless sector (gravity+antisymmetric+gauge+ scalar fields) coupled to towers of generalized Kaluza Klein massive states (corresponding to compact momentum and winding modes) is found. The action acquires a very compact form when written in terms of fields carrying $O(n,n)$ indices, and is explicitly T-duality invariant. The global algebra associated to the generalized Kaluza Klein compactification is discussed.

Keywords: Flux compactifications, String Duality, Gauge Symmetry

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Contents

1 Introduction

Many amazing properties and symmetries of string theory can be tracked down to the extended nature of the strings. In particular, the presence of an antisymmetric tensor $B_{\hat{\mu}\hat{\nu}}$ in the spectrum is expected because, being one dimensional, the string directly couples to it. Actually, a distinctive feature of all string theories is that, besides the metric $g_{\hat{\mu}\hat{\nu}}$, the gravitational sector also includes the Kalb-Ramond field $B_{\hat{\mu}\hat{\nu}}$ and a scalar dilaton ϕ, with extended

$$S = \frac{1}{2\kappa^2}\int d^Dx\,\sqrt{-g}\,e^{-2\phi}\left(R + 4\partial_{\hat{\mu}}\phi\partial^{\hat{\mu}}\phi - \frac{1}{12}H_{\hat{\mu}\hat{\nu}\hat{\lambda}}H^{\hat{\mu}\hat{\nu}\hat{\lambda}}\right)\,, \tag{1.1}$$

where $H_{\hat{\mu}\hat{\nu}\hat{\lambda}} \equiv \partial_{[\hat{\mu}} B_{\hat{\nu}\hat{\lambda}]}$. The occurrence of this universal gravitational sector is ultimately due to the fact that NS-NS massless fields are constructed from the tensor product of one left and one right moving oscillators, transforming in the fundamental representation of the D-dimensional Lorentz group $\mathrm{SO}(1, D-1)$, and hence accounting for the degrees of freedom of $g_{\hat{\mu}\hat{\nu}}$, $B_{\hat{\mu}\hat{\nu}}$ and ϕ according to the decomposition

$$D^2 = \left(\frac{D(D+1)}{2} - 1\right) \oplus \frac{D(D-1)}{2} \oplus 1\,. \tag{1.2}$$

If the space is compact, the closed string can wind around non-contractible cycles, leading to the so-called winding states. Again, from the world sheet point of view, these states are created by vertex operators involving both coordinates associated with momentum excitations and dual coordinates associated with winding excitations or, equivalently, left and right moving coordinates.

The presence of winding and momentum modes underlies T-duality, a genuine stringy feature, which manifests itself by connecting the physics of strings defined on geometrically very different backgrounds and give rise to enhanced gauge symmetries at specific points of the compact space. Indeed, T-duality implies that n-dimensional toroidal backgrounds of closed string theory related by the non-compact group $O(n, n, \mathbb{Z})$ are physically equivalent. This duality appears as a continuous global $O(n, n, \mathbb{R})$ symmetry in the Kaluza-Klein (KK) toroidal compactification of the corresponding low energy effective gravity theory (1.1), if only the massless modes are kept. Once the massive KK modes are taken into account, the continuous symmetry is broken.

Double Field Theory (DFT) aims at incorporating these stringy features, and in particular information about winding, into a field theory [1–7]. Inspired by string compactification on tori, DFT is formulated on a doubled configuration space, with coordinates $\mathbb{X}^{\mathcal{M}} = (\tilde{x}_{\hat{\mu}},\, x^{\hat{\mu}})$, where new coordinates $\tilde{x}_{\hat{\mu}}$, conjugate to windings, are added to the standard coordinates $x^{\hat{\mu}}$, conjugate to momenta. Here $\mathcal{M} = 0, \ldots, 2D-1$ and $\hat{\mu} = 0, \cdots, D-1$. A manifestly $O(D, D)$ invariant action is then constructed on the doubled space, in which the global $O(D, D)$ symmetry is linearly realized. An interesting feature of DFT is that the metric $g_{\hat{\mu}\hat{\nu}}$ and antisymmetric tensor $B_{\hat{\mu}\hat{\nu}}$ fields can be incorporated into a unique field, the so-called generalized metric, transforming as a tensor of the $O(D, D)$ group.

DFT has local invariances that are well defined only if consistency constraints are satisfied. A solution to these constraints is the so called section condition, which effectively leads to the elimination of half of the coordinates. Under this solution and in the frame in which the fields do not depend on $\tilde{x}_{\hat{\mu}}$, the DFT action reduces to (1.1) and the generalized infinitesimal transformations reduce to the standard diffeomorphisms and gauge transformations of $B_{\hat{\mu}\hat{\nu}}$ that leave (1.1) invariant.

Even if the original motivation is lost when choosing the section condition, DFT still provides an interesting tool for understanding underlying symmetries of string theory. In particular, it shares the basic features of Generalized Complex Geometry [8–11] (both frameworks are based on an ordinary, undoubled, manifold) and the $2D$-dimensional tangent bundle of the doubled space is an extension of the D-dimensional tangent-bundle of ordinary spacetime by its cotangent bundle, with $B_{\hat{\mu}\hat{\nu}}$ parametrizing the structure of the fibration. Actually, some distinctive ingredients of string theory, like α' corrections, have been recently incorporated in these formulations [12–15].

Other solutions to the constraint equations are provided by generalized Scherk-Schwarz compactifications [16–18]. It is worth noticing that Scherk-Schwarz compactifications of DFT give rise to all the gaugings of gauged supergravity theories (not obtainable from compactifications of low energy effective supergravities) allowing for a geometric interpretation of all of them [19], albeit in a double space. In this framework, the doubled coordinates enter in a very particular way through the twist matrix, which gives rise to the constant gaugings.

While winding modes are essential for T-duality, they are not truly present and their role is not evident in these approaches. Clearly, to probe the winding sector requires to relax the section condition. Moreover, in toroidal string compactifications, winding states are massive for generic tori. Therefore, understanding the role of winding modes implies facing the massive sector of the theory and consequently dealing with an infinite number of physical states, with different spins and mass scales. However, at specific points of the compact space, some winding states become massless and an effective theory containing only massless states and enhanced gauge symmetry emerges. This scenario appears particularly suitable to identify the explicit part played by windings and a DFT description of the massless winding sector of bosonic string theory compactified on a circle was suggested in [20].

In the present work we propose a way to probe a slice of the massive winding sector of bosonic string theory in an organized fashion. Namely, we consider compactifications of DFT on generic double tori.[1] The generalized dilaton and metric fields of DFT contain bosonic string states constructed with one left and one right moving oscillators, and therefore we concentrate on this sector of the string spectrum. Even if the bosonic string is ill defined, due to the presence of tachyons, we will use it as a reference since string computations are simpler to deal with. However, for the sector we are interested in here, similar reasoning would apply for the heterotic or Type IIB string theories.

The comparison between DFT and string theory is done by expanding the generalized fields around a generic toroidal background with constant dilaton and two-form field.[2] We then expand the DFT action up to third order in fluctuations around the constant background and contrast the result with the corresponding string theory three point amplitudes.

As a first outcome of the calculations, we find that both the DFT and string spectra containing Kaluza-Klein (KK) momenta and windings coincide as long as a "level matching" constraint (LMC) is imposed on the mode expansion of the DFT fields. Furthermore, we show that the compactified DFT action (up to this order in fluctuations) is invariant under generalized gauge transformations generated by a generalized Lie derivative, if the LMC is imposed. This gauge invariance allows to choose a generalized harmonic gauge which provides a convenient "gauge fixing", as it imposes conditions on massless and massive states that can be easily identified with conformal anomaly cancellation conditions on

[1] See [21] for previous work on this subject.

[2] Expansions around generic backgrounds have been performed in [22].

the vertex operators creating these states in string theory. Using these conditions, we then show that cubic vertices in the DFT action can be reproduced by three point amplitudes in string theory. Actually, DFT appears to provide a straightforward way of organizing these amplitudes in an effective T-duality invariant field theory. We obtain an explicit expression for the gauge invariant effective action containing the physical massless sector (gravity+antisymmetric+gauge+scalar fields) coupled to towers of generalized Kaluza Klein (GKK) massive states (corresponding to compact momentum and winding modes).

The article is organized as follows. In section 2 we present some basic introduction to DFT. We write the DFT action in a generalized Einstein frame and fix the gauge freedom in terms of generalized harmonic coordinates. In section 3 we perform the expansion of the generalized fields in fluctuations around a constant generic background, we discuss the gauge fixing conditions and carry out a GKK decomposition of the fields. In section 4 we consider the mode expansion of the fields on a double torus with constant background fields. We identify massless and massive states and examine the generalized harmonic gauge equations to distinguish physical states and Goldstone like states. The analysis of the cubic interaction terms in the effective action and the identification of unbroken symmetries is also performed. Finally the resulting gauge invariant action in d lower dimensions is presented. Section 5 is devoted to string theory amplitudes on toroidal backgrounds. The equivalence between conformal anomaly cancellation conditions on the string vertex operators and the generalized harmonic gauge conditions on the DFT fields is determined. We compute three point string scattering amplitudes of massless and massive states and show the complete agreement with the expansions in DFT. The comparison involves a huge number of terms and so it is performed with the help of a computer (cadabra program [23]). The simple example of circle compactification is worked out explicitly and the manifestly T-duality invariant effective action is also presented. A discussion on the limitations and possible extensions of this work and a brief outlook are contained in the concluding remarks in section 6.

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2 Double Field Theory basics

In this section we briefly review some of the basic features of DFT that are needed in our discussion.

The theory is defined on a double space with coordinates $\mathbb{X}^{\mathcal{M}} = (\tilde{x}_{\hat{\mu}}, x^{\hat{\mu}})$, defined in the fundamental representation of $O(D,D)$. Here $\mathcal{M} = 0, \cdots, 2D-1$ and $\hat{\mu} = 0, \cdots, D-1$.

The generalized tensors transform under generalized diffeomorphisms as

$$\mathcal{L}_V W^{\mathcal{M}\cdots\mathcal{N}} = V^{\mathcal{P}}\partial_{\mathcal{P}} W^{\mathcal{M}\cdots\mathcal{N}} + (\partial^{\mathcal{M}} V_{\mathcal{P}} - \partial_{\mathcal{P}} V^{\mathcal{M}}) W^{\mathcal{P}\cdots\mathcal{N}} + \cdots + (\partial^{\mathcal{N}} V_{\mathcal{P}} - \partial_{\mathcal{P}} V^{\mathcal{N}}) W^{\mathcal{M}\cdots\mathcal{P}}. \tag{2.1}$$

The natural SO(D,D) metric

$$\eta_{\mathcal{M}\mathcal{N}} = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix} \tag{2.2}$$

is invariant under the above generalized transformations. It can be decomposed into a positive-definite and a negative-definite metric, $\eta|_{C_{\pm}}$, acting on each of the two D-

dimensional orthogonal subspaces of the doubled space $E = C_+ \oplus C_-$, that are generated by the coordinates $\mathbb{X}^{\mathcal{M}}_{\pm} = x^{\hat{\mu}} \pm \tilde{x}_{\hat{\mu}}$. Making use of $\eta|_{C_\pm}$, a positive-definite metric can be defined on E

$$\mathcal{H}_{\mathcal{M}\mathcal{N}} = \left(\eta|_{C_+} - \eta|_{C_-}\right)_{\mathcal{M}\mathcal{N}} = \begin{pmatrix} g^{-1} & -g^{-1}B \\ B\,g^{-1} & g - B\,g^{-1}\,B \end{pmatrix}, \tag{2.3}$$

with

$$\mathcal{H}_{\mathcal{M}\mathcal{P}}\,\eta^{\mathcal{P}\mathcal{Q}}\,\mathcal{H}_{\mathcal{Q}\mathcal{N}} = \eta_{\mathcal{M}\mathcal{N}}\,. \tag{2.4}$$

Under $O(D,D)$ transformations $h_{\mathcal{M}}{}^{\mathcal{P}}$, $\mathbb{X} \to h\mathbb{X}$ and the fields change as

$$\mathcal{H}_{\mathcal{M}\mathcal{N}}(\mathbb{X}) \to h_{\mathcal{M}}{}^{\mathcal{P}} h_{\mathcal{N}}{}^{\mathcal{Q}} \mathcal{H}_{\mathcal{P}\mathcal{Q}}(h\mathbb{X}), \qquad d(\mathbb{X}) \to d(h\mathbb{X}) \tag{2.5}$$

Upper and lower indices are lowered and raised with $\eta_{\mathcal{M}\mathcal{N}}$ and its inverse $\eta^{\mathcal{M}\mathcal{N}}$, respectively.

It is sometimes useful to express the metric $\mathcal{H}_{\mathcal{M}\mathcal{N}}$ in terms of a vielbein

$$\mathcal{H}_{\mathcal{M}\mathcal{N}} = E^{\mathcal{A}}{}_{\mathcal{M}}\, S_{\mathcal{A}\mathcal{B}}\, E^{\mathcal{B}}{}_{\mathcal{N}}\,, \qquad E^{\mathcal{A}}{}_{\mathcal{M}} = \begin{pmatrix} e & e\,B \\ 0 & e^{-1} \end{pmatrix}, \tag{2.6}$$

where $g_{\hat{\mu}\hat{\nu}} = e^{a}{}_{\hat{\mu}}\, s_{ab}\, e^{b}{}_{\hat{\nu}}$,

$$S_{\mathcal{A}\mathcal{B}} = \begin{pmatrix} s^{ab} & 0 \\ 0 & s_{ab} \end{pmatrix} \tag{2.7}$$

and s_{ab} is the D dimensional Minkowski metric. $\mathcal{A}, \mathcal{B}, \cdots$ indices are lowered and raised with the flat SO(D,D) metric defined as

$$\eta_{\mathcal{A}\mathcal{B}} = E_{\mathcal{A}}{}^{\mathcal{M}}\, \eta_{\mathcal{M}\mathcal{N}}\, E_{\mathcal{B}}{}^{\mathcal{N}} \tag{2.8}$$

and its inverse, respectively, which numerically coincide with (2.2).

Since the Minkowski metric is invariant under Lorentz $O(1, D-1)$ transformations, the metric $S_{\mathcal{A}\mathcal{B}}$ is invariant under double transformations $O(1, D-1) \times O(D-1, 1)$ and as a result the generalized metric $\mathcal{H}$ parametrizes the coset

$$\frac{O(D,D)}{O(1,D-1) \times O(D-1,1)}\,. \tag{2.9}$$

From the transformation law (2.1), the generalized metric transforms as

$$\mathcal{L}_V \mathcal{H}_{\mathcal{M}\mathcal{N}} = V^{\mathcal{P}} \partial_{\mathcal{P}} \mathcal{H}_{\mathcal{M}\mathcal{N}} + (\partial_{\mathcal{M}} V^{\mathcal{P}} - \partial^{\mathcal{P}} V_{\mathcal{M}}) \mathcal{H}_{\mathcal{P}\mathcal{N}} + (\partial_{\mathcal{N}} V^{\mathcal{P}} - \partial^{\mathcal{P}} V_{\mathcal{N}}) \mathcal{H}_{\mathcal{M}\mathcal{P}}. \tag{2.10}$$

In terms of $\mathcal{H}_{\mathcal{M}\mathcal{N}}$, and keeping up to two derivatives, the action of DFT in the $2D$-dimensional space E can be expressed as[3]

$$S = \frac{1}{G_{\text{DFT}}} \int d^D x d^D \tilde{x}\, e^{-2d}\, \mathcal{R}(\mathcal{H}, d)\,, \tag{2.11}$$

[3] In the original frame formulation of DFT by Siegel [1, 2] the action includes extra terms that are not contained in (2.11). Up to total derivatives those can be recast as [18]

$$\Delta S = \frac{1}{G_{\text{DFT}}} \int d^D x d^D \tilde{x}\, e^{-2d} \left[\frac{1}{2} (S_{\mathcal{A}\mathcal{B}} - \eta_{\mathcal{A}\mathcal{B}}) \eta^{\mathcal{P}\mathcal{Q}}\, \partial_{\mathcal{M}} E^{\mathcal{A}}{}_{\mathcal{P}} \partial^{\mathcal{M}} E^{\mathcal{B}}{}_{\mathcal{Q}} + 4 \partial_{\mathcal{M}} d \partial^{\mathcal{M}} d - 4 \partial_{\mathcal{M}} \partial^{\mathcal{M}} d \right].$$

In the appendix we show that these terms vanish once the level matching condition described below, is imposed, and therefore we do not consider them in this work.

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where the generalized Ricci scalar is given by [4]

$$\begin{aligned}\mathcal{R} = {}& 4\mathcal{H}^{\mathcal{M}\mathcal{N}}\partial_{\mathcal{M}}\partial_{\mathcal{N}}d - \partial_{\mathcal{M}}\partial_{\mathcal{N}}\mathcal{H}^{\mathcal{M}\mathcal{N}} - 4\mathcal{H}^{\mathcal{M}\mathcal{N}}\partial_{\mathcal{M}}d\partial_{\mathcal{N}}d + 4\partial_{\mathcal{M}}\mathcal{H}^{\mathcal{M}\mathcal{N}}\partial_{\mathcal{N}}d \\ & + \frac{1}{8}\mathcal{H}^{\mathcal{M}\mathcal{N}}\partial_{\mathcal{M}}\mathcal{H}^{\mathcal{K}\mathcal{L}}\partial_{\mathcal{N}}\mathcal{H}_{\mathcal{K}\mathcal{L}} - \frac{1}{2}\mathcal{H}^{\mathcal{M}\mathcal{N}}\partial_{\mathcal{M}}\mathcal{H}^{\mathcal{K}\mathcal{L}}\partial_{\mathcal{K}}\mathcal{H}_{\mathcal{N}\mathcal{L}}\,, \end{aligned} \tag{2.12}$$

the generalized dilaton

$$e^{-2d} = e^{-2\phi}\sqrt{-g} \tag{2.13}$$

is an $O(D,D)$ scalar and G_{DFT} will be defined below.[4]

Comparison with string theory, as we re-discuss in more detail below, requires the *level matching condition* (LMC)

$$\partial_{\mathcal{M}}\partial^{\mathcal{M}}\cdots = (N-\bar{N})\cdots\,, \tag{2.14}$$

where N and $\bar{N}$ are the left and right oscillator numbers of the string and the dots stand for fields or gauge parameters. Given that g, B and ϕ correspond to $N = \bar{N} = 1$, we would expect $N - \bar{N} = 0$. However, in a compact space this difference could be a non-vanishing integer. Even though this is a key ingredient of symmetry enhancing at certain compactification radii (see [20]), we will only consider states satisfying $N - \bar{N} = 0$ in the main body of this article. Introducing the $2D$-dimensional momentum vector $\mathbb{P}^{\mathcal{M}} = (\tilde{p}^{\hat{\mu}}, p_{\hat{\mu}})$, generated by the partial derivatives $-i(\tilde{\partial}^{\hat{\mu}}, \partial_{\hat{\mu}})$ acting on the corresponding field, the constraint reads

$$||\mathbb{P}||^2 \equiv \mathbb{P}^{\mathcal{M}}\mathbb{P}_{\mathcal{M}} = 0 \tag{2.15}$$

for $\mathcal{H}_{\mathcal{M}\mathcal{N}}$ and d. In general, this constraint is not sufficient to ensure consistency. For instance, the product of fields generically does not satisfy it and the generalized transformations (2.1) fail to close.

This failure can be expected from string theory. Namely, many other terms (actually infinite) are expected to complete the effective action, containing higher derivatives but also higher spin fields. Hopefully, in the full action variations could compensate among different terms and the algebra would close. But in the truncated theory involving only massless fields with $N = \bar{N} = 1$ in the non compact case, consistency constraints are necessary. One solution of these constraints is the so-called section condition

$$\partial_{\mathcal{M}}\cdots\partial^{\mathcal{M}}\cdots = 0\,, \tag{2.16}$$

where the dots stand for products of fields or gauge parameters. It implies that half of the coordinates drop from the theory. These coordinates can be chosen to be the dual coordinates $\tilde{x}_{\hat{\mu}}$. This choice is named *gravity frame* since in this case the action (2.11) simply reduces to eq. (1.1) when $\mathcal{H}_{\mathcal{M}\mathcal{N}}$ is parametrized as in (2.3) and $G_{\mathrm{DFT}} \equiv 2\kappa^2 \int d^D\tilde{x}$.

The section condition is sufficient to satisfy the closure constraints, but there are more general solutions [18, 25] when there is a compact sector. It is important to stress that (2.11) describes more physical degrees of freedom than the standard D-dimensional action (1.1) for g, B and ϕ. Indeed, by introducing coordinates $\tilde{x}_{\hat{\mu}}$, and their corresponding partial derivatives $\tilde{\partial}^{\hat{\mu}}$, fields can carry momentum along these directions and the backgrounds can

[4]The overall constant G_{DFT} was introduced in [24].

Φ	$\mathcal{H}_{\mathcal{K}\mathcal{L}}$	$\eta_{\mathcal{K}\mathcal{L}}$	$E^{\mathcal{A}}{}_{\mathcal{M}}$	$S_{\mathcal{A}\mathcal{B}}$	$\eta_{\mathcal{A}\mathcal{B}}$
Δ_Φ	-2	-2	-1	0	0

Table 1. Conformal weights of the various tensors that appear in DFT.

also depend on these coordinates. Such dependence is not an artifact of DFT: backgrounds with non-trivial dependence on the coordinates $\tilde{x}^{\hat{\mu}}$ cannot be described in terms of D-dimensional gravity, but are however expected to be consistent solutions of string theory. In particular, such backgrounds lead upon compactification to fully consistent effective gauged gravities with momenta along the internal coordinates associated to winding excitations. DFT contains more degrees of freedom than D-dimensional gravity, and in particular, it allows to compute observables and describe settings that cannot be accounted for in standard D-dimensional theories.

In what follows we will compactify DFT on generic tori with constant background fields and fluctuations around them. The constraints to be used will be extracted from comparison with string theory results. In our computations, the section condition must be imposed in the spacetime sector but only the LMC constraint is required in the toroidal compact space. This appears to be consistent if fluctuations are considered only up to third order. When going to higher orders, the failure of the gauge algebra to close should be interpreted as an indication that new degrees of freedom must be included. A brief discussion on this issue is offered in the concluding remarks.

2.1 Einstein frame and harmonic coordinates

The generalized metric $\mathcal{H}_{\mathcal{M}\mathcal{N}}$ defined in (2.3) contains the g and B fields, and the generalized dilaton d involves ϕ. We can combine both d and $\mathcal{H}_{\mathcal{M}\mathcal{N}}$ into a single generalized Einstein-frame metric $\tilde{\mathcal{H}}_{\mathcal{M}\mathcal{N}}$ with non-zero determinant. For that aim, we perform a Weyl transformation

$$\tilde{\mathcal{H}}_{\mathcal{M}\mathcal{N}} = e^{2\Omega}\,\mathcal{H}_{\mathcal{M}\mathcal{N}}\,, \tag{2.17}$$

under which a tensor with conformal weight Δ_Φ transforms as

$$\tilde{\Phi} = e^{-\Omega\Delta_\Phi}\,\Phi\,. \tag{2.18}$$

We list the conformal weights of the tensors introduced in the previous section in table 1.

Making use of these transformations, one can easily check that

$$\begin{aligned}\mathcal{R} = e^{2\Omega}\Big[&\tilde{\mathcal{R}} - 2\partial_{\mathcal{M}}\tilde{\mathcal{H}}^{\mathcal{M}\mathcal{N}}\partial_{\mathcal{N}}\Omega - 2\tilde{\mathcal{H}}^{\mathcal{M}\mathcal{N}}\partial_{\mathcal{M}}\partial_{\mathcal{N}}\Omega - (2+D)\tilde{\mathcal{H}}^{\mathcal{M}\mathcal{N}}\partial_{\mathcal{M}}\Omega\partial_{\mathcal{N}}\Omega \\ &+ 8\tilde{\mathcal{H}}^{\mathcal{M}\mathcal{N}}\partial_{\mathcal{M}}\Omega\partial_{\mathcal{N}}d - \frac{1}{2}\tilde{\mathcal{H}}_{\mathcal{K}\mathcal{L}}\tilde{\mathcal{H}}^{\mathcal{M}\mathcal{N}}\partial_{\mathcal{M}}\tilde{\mathcal{H}}^{\mathcal{K}\mathcal{L}}\partial_{\mathcal{N}}\Omega\Big]\,.\end{aligned} \tag{2.19}$$

Taking $\Omega = d$ and integrating by parts, we can express (2.11) in the Einstein frame as

$$S = \frac{1}{G_{\rm DFT}}\int d^Dx d^D\tilde{x}\,\hat{\mathcal{R}}(\tilde{\mathcal{H}}, d)\,, \tag{2.20}$$

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where

$$\hat{\mathcal{R}}(\tilde{\mathcal{H}},d) = (2-D)\tilde{\mathcal{H}}^{\mathcal{MN}}\partial_{\mathcal{M}}d\partial_{\mathcal{N}}d - \frac{1}{2}\tilde{\mathcal{H}}_{\mathcal{KL}}\tilde{\mathcal{H}}^{\mathcal{MN}}\partial_{\mathcal{M}}\tilde{\mathcal{H}}^{\mathcal{KL}}\partial_{\mathcal{N}}d + \partial_M\partial_N\tilde{\mathcal{H}}^{MN} + \frac{1}{8}\tilde{\mathcal{H}}^{\mathcal{MN}}\partial_{\mathcal{M}}\tilde{\mathcal{H}}^{\mathcal{KL}}\partial_{\mathcal{N}}\tilde{\mathcal{H}}_{\mathcal{KL}} - \frac{1}{2}\tilde{\mathcal{H}}^{\mathcal{MN}}\partial_{\mathcal{M}}\tilde{\mathcal{H}}^{\mathcal{KL}}\partial_{\mathcal{K}}\tilde{\mathcal{H}}_{\mathcal{NL}}\,. \tag{2.21}$$

This action (2.20) behaves similarly to the more familiar Einstein-Hilbert action in many aspects. In particular, the equations of motion are greatly simplified by taking a harmonic coordinate condition to fix the gauge freedom under generalized diffeomorphisms. This can be achieved by requiring the coordinates $\mathbb{X}^{\mathcal{R}}$ to be solutions of the Laplacian equation

$$\partial_{\mathcal{M}}\left(\tilde{\mathcal{H}}^{\mathcal{MN}}\partial_{\mathcal{N}}\right)\mathbb{X}^{\mathcal{R}} = 0 \quad \Rightarrow \quad \partial_{\mathcal{M}}\tilde{\mathcal{H}}^{\mathcal{MN}} = 0\,, \tag{2.22}$$

which amounts to the gauge fixing condition[5]

$$\partial_{\mathcal{M}}\mathcal{H}^{\mathcal{MN}} - 2\mathcal{H}^{\mathcal{MN}}\partial_{\mathcal{M}}d = 0\,, \tag{2.23}$$

when written in terms of the metric $\mathcal{H}_{\mathcal{MN}}$ and the scalar d. Alternatively this equation can be expressed as

$$\partial_{\mathcal{M}}d = \frac{1}{2}\mathcal{H}_{\mathcal{MN}}\partial_{\mathcal{R}}\mathcal{H}^{\mathcal{NR}} = -\frac{1}{2}\mathcal{H}^{\mathcal{NR}}\partial_{\mathcal{R}}\mathcal{H}_{\mathcal{MN}}\,. \tag{2.24}$$

Making use of these conditions and integrating by parts, the action (2.11) can be expressed in harmonic coordinates in a particularly compact form

$$S = \frac{1}{G_{\mathrm{DFT}}}\int d^Dx d^D\tilde{x}\, e^{-2d}\left[\frac{1}{8}\mathcal{H}^{\mathcal{MN}}\partial_{\mathcal{M}}\mathcal{H}^{\mathcal{KL}}\partial_{\mathcal{N}}\mathcal{H}_{\mathcal{KL}} - \frac{1}{2}\mathcal{H}^{\mathcal{MN}}\partial_{\mathcal{M}}\mathcal{H}^{\mathcal{KL}}\partial_{\mathcal{K}}\mathcal{H}_{\mathcal{NL}}\right], \tag{2.25}$$

or, in Einstein-like frame,

$$S_{\mathrm{DFT}} = \frac{1}{G_{\mathrm{DFT}}}\int d^Dx d^D\tilde{x}\left(\frac{1}{8}\tilde{\mathcal{H}}^{\mathcal{MN}}\partial_{\mathcal{M}}\tilde{\mathcal{H}}^{\mathcal{KL}}\partial_{\mathcal{N}}\tilde{\mathcal{H}}_{\mathcal{KL}} - \frac{1}{2}\tilde{\mathcal{H}}^{\mathcal{MN}}\partial_{\mathcal{M}}\tilde{\mathcal{H}}^{\mathcal{KL}}\partial_{\mathcal{K}}\tilde{\mathcal{H}}_{\mathcal{LN}} + (2-D)\tilde{\mathcal{H}}^{\mathcal{MN}}\partial_{\mathcal{M}}d\partial_{\mathcal{N}}d\right). \tag{2.26}$$

It is also interesting to express the gauge fixing condition in terms of g, B and ϕ. For standard D-dimensional gravity backgrounds with $\tilde{p}^{\hat{\mu}} = 0$, one may easily check that its components reduce to

$$\partial_{\hat{\nu}}\left(\sqrt{-g}\,g^{\hat{\mu}\hat{\nu}}e^{-2\phi}\right) = 0\,, \qquad g^{\hat{\mu}\hat{\nu}}\partial_{\hat{\nu}}B_{\hat{\lambda}\hat{\mu}} = 0\,. \tag{2.27}$$

In particular, for vanishing dilaton $\phi = 0$, the first equation is the usual harmonic gauge fixing condition of General Relativity. More generally, for generic DFT backgrounds, the gauge fixing conditions for B, g and ϕ read

$$\partial_{\hat{\nu}}\left(\sqrt{-g}\,g^{\hat{\mu}\hat{\nu}}e^{-2\phi}\right) = \tilde{\partial}^{\hat{\lambda}}\left(\sqrt{-g}\,g^{\hat{\mu}\hat{\sigma}}B_{\hat{\sigma}\hat{\lambda}}\,e^{-2\phi}\right) = 0\,, \qquad \tilde{\partial}^{\nu}\left(\sqrt{-g}\,g_{\hat{\mu}\hat{\nu}}e^{-2\phi}\right) = -e^{-2\phi}g^{\hat{\sigma}\hat{\nu}}\left(\partial_{\hat{\nu}} - B_{\hat{\nu}\hat{\lambda}}\tilde{\partial}^{\hat{\lambda}}\right)B_{\hat{\mu}\hat{\sigma}} = 0\,. \tag{2.28}$$

[5] It can be shown that, in terms of the generalized connection of [1, 2, 26], this is equivalent to requiring $\mathcal{H}^{\mathcal{MP}}\Gamma_{\mathcal{MP}}{}^{\mathcal{Q}} = 0$.

3 Perturbative DFT

The physical content of a quantum field theory can be recast in terms of its S-matrix elements, that are usually computed perturbatively. In the particular case of General Relativity, perturbative computations are however specially complex due to the huge number of vertices, rendering most of the brute force computations of scattering amplitudes infeasible. Fortunately, the field theory limit of Kawai-Lewellen-Tye (KLT) relations [27] allows to express gravity amplitudes in terms of two copies of gluon amplitudes, which are much simpler to compute. In particular, starting from gluon amplitudes and using KLT relations, it has been possible to construct a Lagrangian for gravity [28]. The resulting Lagrangian is particularly simple and is related to the usual Einstein-Hilbert action by non-linear redefinitions and gauge fixing similar to those used in [29]. Moreover graviton spacetime indices can be split into two types (left and right), in such a way that contractions do not mix indices of different type.

KLT relations originate from the fact that the integrand of a closed string amplitude involves two open string components, corresponding to left and right movers. It is then natural to expect that this hidden simplification of gravity amplitudes also holds in DFT. Indeed, this is already manifest in the extreme simplicity of the Lagrangian (2.25). To be more specific, let us split $\mathcal{H}_{\mathcal{MN}}$ into background $\overline{\mathcal{H}}_{\mathcal{MN}}$ and quantum fluctuations $\hat{h}_{\mathcal{MN}}$,

$$\mathcal{H}_{\mathcal{MN}} = \overline{\mathcal{H}}_{\mathcal{MN}} + \hat{h}_{\mathcal{MN}}\,, \qquad d = \overline{d} + \hat{d}\,. \tag{3.1}$$

For simplicity, we consider $\overline{\mathcal{H}}_{\mathcal{MN}}$ and $\overline{d}$ to be constant. Due to the presence of two metrics, namely $\mathcal{H}_{\mathcal{MN}}$ and $\eta_{\mathcal{MN}}$, (3.1) can be inverted in two different ways: by making use of $\eta^{\mathcal{MN}}$ or by using the geometric series for matrices. We thus obtain for the inverse

$$\mathcal{H}^{\mathcal{MN}} = \overline{\mathcal{H}}^{\mathcal{MN}} + \hat{h}^{\mathcal{MN}} = \overline{\mathcal{H}}^{\mathcal{MN}} - \hat{h}^{\dot{\mathcal{M}}\dot{\mathcal{N}}} + \hat{h}^{\dot{\mathcal{M}}}{}_{\mathcal{P}}\,\hat{h}^{\dot{\mathcal{P}}\dot{\mathcal{N}}} - \hat{h}^{\dot{\mathcal{M}}}{}_{\mathcal{Q}}\,\hat{h}^{\dot{\mathcal{Q}}\dot{\mathcal{P}}}\,\hat{h}_{\mathcal{P}}{}^{\dot{\mathcal{N}}} + \ \ldots\,, \tag{3.2}$$

where we have introduced the short-hand notation $A^{\dot{\mathcal{M}}} \equiv \overline{\mathcal{H}}^{\mathcal{MN}} A_{\mathcal{N}}\,,\ A_{\dot{\mathcal{M}}} \equiv \overline{\mathcal{H}}_{\mathcal{MN}} A^{\mathcal{N}}$, and it is useful to note that, up to first order,

$$\hat{h}^{\mathcal{MN}} = \eta^{\mathcal{MP}}\eta^{\mathcal{NQ}}\hat{h}_{\mathcal{QP}} = -\overline{\mathcal{H}}^{\mathcal{MP}}\overline{\mathcal{H}}^{\mathcal{NQ}}\hat{h}_{\mathcal{QP}} = -\hat{h}^{\dot{\mathcal{M}}\dot{\mathcal{N}}}\,. \tag{3.3}$$

The single field $\hat{h}^{\mathcal{MN}} \equiv \eta^{\mathcal{MP}}\eta^{\mathcal{MQ}}\hat{h}_{\mathcal{PQ}}$ therefore encodes an infinite set of operators when expressed in terms of the background metric $\overline{\mathcal{H}}_{\mathcal{MN}}$.

Note also that by construction $\overline{\mathcal{H}}^{\mathcal{MN}}\overline{\mathcal{H}}_{\mathcal{NQ}} = \delta^{\mathcal{M}}{}_{\mathcal{Q}}$, however $\hat{h}^{\mathcal{MN}}\hat{h}_{\mathcal{NQ}} \neq \delta^{\mathcal{M}}{}_{\mathcal{Q}}$. Instead, one may easily check the following relation

$$\hat{h}^{\mathcal{MN}}\hat{h}_{\mathcal{NQ}} = -\left(h^{\dot{\mathcal{M}}}{}_{\mathcal{Q}} + h^{\mathcal{M}}{}_{\dot{\mathcal{Q}}}\right)\,. \tag{3.4}$$

For comparison with string theory results, it proves convenient to look at fluctuations in the so-called modified Einstein frame, namely the Einstein frame discussed above with the vacuum value of the generalized dilaton $e^{-2\overline{d}}$ extracted out.[6] Thus, the generalized metric is, up to first order

$$\tilde{\mathcal{H}}_{\mathcal{MN}} = \tilde{\bar{\mathcal{H}}}_{\mathcal{MN}} + \tilde{\hat{h}}_{\mathcal{MN}} = \bar{\mathcal{H}}_{\mathcal{MN}} + (\hat{h}_{\mathcal{MN}} + 2\hat{d}\bar{\mathcal{H}}_{\mathcal{MN}})\,. \tag{3.5}$$

[6] In what follows Einstein frame means modified Einstein frame.

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3.1 Expansion of DFT in fluctuations

Following the discussion above, by using (3.5) we expand the DFT harmonic gauge fixed action (2.26) into background and quantum fluctuations. We get, up to third order in fluctuations,

$$L_{\rm DFT} = \frac{1}{8}\bar{\hat{\mathcal{H}}}^{\mathcal{MN}}\partial_{\mathcal{M}}\hat{\hat{h}}^{\mathcal{KL}}\partial_{\mathcal{N}}\hat{\hat{h}}_{\mathcal{KL}} - (D-2)\bar{\hat{\mathcal{H}}}^{\mathcal{MN}}\partial_{\mathcal{M}}\hat{d}\partial_{\mathcal{N}}\hat{d} \\ -\frac{1}{2}\hat{\hat{h}}^{\mathcal{MN}}\partial_{\mathcal{M}}\hat{\hat{h}}^{\mathcal{KL}}\partial_{\mathcal{K}}\hat{\hat{h}}_{\mathcal{NL}} + \frac{1}{8}\hat{\hat{h}}^{\mathcal{MN}}\partial_{\mathcal{M}}\hat{\hat{h}}^{\mathcal{KL}}\partial_{\mathcal{N}}\hat{\hat{h}}_{\mathcal{KL}} - (D-2)\hat{\hat{h}}^{\mathcal{MN}}\partial_{\mathcal{M}}\hat{d}\partial_{\mathcal{M}}\hat{d}\,. \tag{3.6}$$

Recall that in terms of fields, the fluctuations $\hat{h}_{\mathcal{MN}} = \hat{h}_{(1)\mathcal{MN}} + \hat{h}_{(2)\mathcal{MN}} + \dots$ contain contributions from higher orders. In particular, terms quadratic in $\hat{h}_{\mathcal{MN}}$ could give third order interaction terms. However, this is not the case. Actually, integrating by parts the term $\frac{1}{4}\bar{\hat{\mathcal{H}}}^{\mathcal{MN}}\partial_{\mathcal{M}}\hat{\hat{h}}^{\mathcal{KL}}_{(2)}\partial_{\mathcal{N}}\hat{\hat{h}}_{(1)\mathcal{KL}}$, one gets the equations of motion (see (4.8) below), and so this cubic term vanishes on shell. The same conclusion holds for the second term. Therefore, the third order terms in the action involve only the first order fluctuations of the generalized fields, and we finally have the Lagrangian (3.6) with $\hat{\hat{h}}_{\mathcal{KL}} = \hat{\hat{h}}_{(1)\mathcal{KL}}$.

Before compactification, in a flat background

$$\overline{\mathcal{H}}_{\mathcal{MN}} = \begin{pmatrix} \eta^{\hat{\mu}\hat{\nu}} & 0 \\ 0 & \eta_{\hat{\mu}\hat{\nu}} \end{pmatrix}, \tag{3.7}$$

and to first order in fluctuations, $g_{\hat{\mu}\hat{\nu}} = \eta_{\hat{\mu}\hat{\nu}} + h_{\hat{\mu}\hat{\nu}}$, $B_{\hat{\mu}\hat{\nu}} = b_{\hat{\mu}\hat{\nu}}$, we have

$$\hat{h}_{\mathcal{MN}} = \begin{pmatrix} \hat{h}^{\hat{\mu}\hat{\nu}} & \hat{h}^{\hat{\mu}}{}_{\hat{\nu}} \\ \hat{h}_{\hat{\mu}}{}^{\hat{\nu}} & \hat{h}_{\hat{\mu}\hat{\nu}} \end{pmatrix} = \begin{pmatrix} -h^{\hat{\mu}\hat{\nu}} & -\eta^{\hat{\mu}\hat{\rho}}b_{\hat{\rho}\hat{\nu}} \\ -\eta^{\hat{\nu}\hat{\rho}}b_{\hat{\rho}\hat{\mu}} & h_{\hat{\mu}\hat{\nu}} \end{pmatrix}. \tag{3.8}$$

Then from the second order terms in fluctuations and imposing the strong constraint in the gravity frame (namely, dropping the dependence on the $\tilde{x}_{\hat{\mu}}$ coordinates), we recover the quadratic terms in the action (1.1) in the de Donder gauge [30, 32]. Actually, in the string frame we get

$$S = \frac{1}{2\kappa^2}\int d^Dx\, e^{-2\bar{\phi}}\left[\partial_{\hat{\sigma}}\left(\frac{h_{\hat{\nu}}{}^{\hat{\nu}}}{2} - 2\hat{\phi}\right)\partial^{\hat{\sigma}}\left(\frac{h_{\hat{\rho}}{}^{\hat{\rho}}}{2} - 2\hat{\phi}\right) - \frac{1}{2}\partial_{\hat{\sigma}}(h_{\hat{\nu}\hat{\lambda}} + b_{\hat{\nu}\hat{\lambda}})\partial^{\hat{\sigma}}(h^{\hat{\nu}\hat{\lambda}} + b^{\hat{\nu}\hat{\lambda}})\right]. \tag{3.9}$$

Transforming this action into momentum space, we obtain the propagators for h, b and $\hat{\phi}$

$$D^h_{\hat{\mu}\hat{\nu};\hat{\rho}\hat{\sigma}} = \frac{e^{-2\bar{\phi}}}{4}\frac{\eta_{\hat{\mu}\hat{\rho}}\eta_{\hat{\nu}\hat{\sigma}}}{p^2},$$
$$D^b_{\hat{\mu}\hat{\nu};\hat{\rho}\hat{\sigma}} = \frac{e^{-2\bar{\phi}}}{4}\frac{\eta_{\hat{\mu}\hat{\rho}}\eta_{\hat{\nu}\hat{\sigma}} - \eta_{\hat{\mu}\hat{\sigma}}\eta_{\hat{\nu}\hat{\rho}}}{p^2},$$
$$D^{2\hat{\phi} - \frac{h^{\hat{\nu}}{}_{\hat{\nu}}}{2}} = \frac{4e^{-2\bar{\phi}}}{p^2}.$$

The first lesson to be drawn from this calculation is that the strong constraint must be imposed on the space-time coordinates in order to recover ordinary gravity theories, as expected.

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3.2 Generalized Kaluza-Klein compactification

The generalized Kaluza-Klein (GKK) decomposition of the generalized metric reads

$$\mathcal{H}_{\mathcal{M}\mathcal{N}} = \begin{pmatrix} g^{\mu\nu} & -g^{\mu\rho}c_{\rho\nu} & -g^{\mu\rho}A_{N\rho} \\ -g^{\nu\rho}c_{\rho\mu} & g_{\mu\nu} + A^N{}_\mu \mathcal{M}_{NP}A^P{}_\nu + c_{\rho\mu}g^{\rho\sigma}c_{\sigma\nu} & \mathcal{M}_{NP}A^P{}_\mu + A_{N\rho}g^{\rho\sigma}c_{\sigma\mu} \\ -g^{\nu\rho}A_{M\rho} & \mathcal{M}_{MP}A^P{}_\nu + A_{M\rho}g^{\rho\sigma}c_{\sigma\nu} & \mathcal{M}_{MN} + A_{M\rho}g^{\rho\sigma}A_{N\sigma} \end{pmatrix}, \quad (3.10)$$

where now the $\mathcal{M}, \mathcal{N}$ indices split into spacetime $\mu, \nu, \cdots$ indices taking the values $0, \cdots, d-1$, and internal doubled indices $M, N, \cdots = 1, \cdots, 2(D-d)$. We have introduced the combination $c_{\mu\nu} = b_{\mu\nu} + \frac{1}{2}A^N_\mu A_{N\nu}$, A^N_μ denote the vectors and $\mathcal{M}_{MN}$ is the scalar matrix defined below.

In terms of components, the constant generalized background metric reads now

$$\overline{\mathcal{H}}_{\mathcal{M}\mathcal{N}} = \begin{pmatrix} \eta^{\mu\nu} & 0 & 0 \\ 0 & \eta_{\mu\nu} & 0 \\ 0 & 0 & \overline{\mathcal{M}}_{MN} \end{pmatrix}, \quad (3.11)$$

with

$$\overline{\mathcal{M}}_{MN} = \begin{pmatrix} G^{mn} & -G^{mp}B_{pn} \\ B_{mp}G^{pn} & G_{mn} - B_{mp}G^{pq}B_{qn} \end{pmatrix}, \quad (3.12)$$

where $m, n, \cdots = 1, \cdots, D-d$. The fluctuations up to first order are

$$\hat{h}_{(1)\mathcal{M}\mathcal{N}} = \begin{pmatrix} \hat{h}^{\mu\nu}_{(1)} & \hat{h}^{\mu}_{(1)\nu} & \hat{h}^{\mu}_{(1)N} \\ \hat{h}_{(1)\mu}^{\ \ \nu} & \hat{h}_{(1)\mu\nu} & \hat{h}_{(1)\mu N} \\ \hat{h}_{(1)M}^{\ \ \nu} & \hat{h}_{(1)M\nu} & \hat{h}_{(1)MN} \end{pmatrix} = \begin{pmatrix} -h^{\mu\nu} & -\eta^{\mu\rho}b_{\rho\nu} & -\eta^{\mu\rho}A_{N\rho} \\ -\eta^{\nu\rho}b_{\rho\mu} & h_{\mu\nu} & \overline{\mathcal{M}}_{NP}A^P{}_\mu \\ -\eta^{\nu\rho}A_{M\rho} & \overline{\mathcal{M}}_{MP}A^P{}_\nu & h_{MN} \end{pmatrix}. \quad (3.13)$$

The matrix h_{MN} encoding the scalar field content reads

$$h_{MN} = \begin{pmatrix} -G^{nk}h_{kl}G^{lm} & -G^{nk}b_{km} + G^{nk}h_{ks}G^{sl}B_{lm} \\ -B_{ns}G^{sl}h_{lk}G^{km} + b_{nl}G^{lm} & h_{nm} - b_{nl}G^{lk}B_{km} + B_{nk}G^{ks}h_{sr}G^{rb}B_{bm} - B_{nk}G^{kl}b_{lm} \end{pmatrix} \quad (3.14)$$

where h_{nl}, b_{nl} are the scalar fields derived from the higher dimensional graviton and antisymmetric fields, respectively.

From the definition of the generalized dilaton (2.13) and recalling that $d = \overline{d} + \hat{d}$, we have

$$e^{-2\overline{d}} = e^{-2\phi_0}\sqrt{detG}\,, \quad (3.15)$$

$$\hat{d} = \hat{\phi} - \frac{1}{4}h^\mu{}_\mu\,. \quad (3.16)$$

In Einstein frame, the only fluctuations that are modified are

$$\begin{aligned} \hat{\hat{h}}_{\mu\nu} &= (h_{\mu\nu} + 2\hat{d}\,\eta_{\mu\nu}) \equiv \tilde{h}_{\mu\nu}\,, \\ \hat{\hat{h}}^{\mu\nu} &= (-h^{\mu\nu} + 2\hat{d}\,\eta^{\mu\nu}) \equiv -\tilde{h}^{\mu\nu}\,, \\ \hat{\hat{h}}_{MN} &= (h_{MN} + 2\hat{d}\,\overline{\mathcal{M}}_{MN}) \equiv \tilde{h}_{MN}\,. \end{aligned} \quad (3.17)$$

The harmonic gauge conditions in the Einstein frame ($\partial^{\mathcal{M}}\tilde{\mathcal{H}}_{\mathcal{M}\mathcal{N}} = 0$) become, in terms of fluctuations,

$$\partial^{\mathcal{M}}\hat{\hat{h}}_{\mathcal{M}\mathcal{N}} = \partial_\mu \hat{\hat{h}}^\mu{}_{\mathcal{N}} + \partial^L \hat{\hat{h}}_{L\mathcal{N}} = 0\,, \quad (3.18)$$

where we have used the strong constraint in the spacetime sector. Therefore, when specifying values for the index $\mathcal{N}$, we have

$$\partial_\mu \hat{\tilde{h}}^{\mu\nu} + \partial_N \hat{\tilde{h}}^{N\nu} = 0 \to \partial_\mu \tilde{h}^{\mu\nu} + \partial_N A^{N\nu} = 0\,, \tag{3.19}$$

$$\partial_\mu \hat{\tilde{h}}^{\mu}{}_{\nu} + \partial_N \hat{\tilde{h}}^{N}{}_{\nu} = 0 \to \partial^\mu b_{\mu\nu} - i(\mathbb{P}\overline{M}A)_\nu = 0\,, \tag{3.20}$$

$$\partial_\mu \hat{\tilde{h}}^{\mu N} + \partial_M \hat{\tilde{h}}^{MN} = 0 \to \partial^\mu A^N_\mu - i(\mathbb{P}\overline{M}\tilde{h}\overline{M})^N = 0\,. \tag{3.21}$$

We will discuss the link between this set of equations and the vanishing of conformal anomalies in string theory in section (5) below.

4 Toroidal compactification

We consider the mode expansion of fields on an internal $2n$-dimensional double torus with constant background (metric, dilaton and antisymmetric fields) turned on. It corresponds to a compactification on $2(D-d) = 2n$ circles, which are generically non-orthogonal since the background metric is in general non-diagonal.[7]

The internal coordinates $\mathbb{Y}^M = (\tilde{y}_m, y^m)$ have periodicity

$$\tilde{y}_m \sim \tilde{y}_m + 2\pi \tilde{R}_{(m)}\,, \qquad y^m \sim y^m + 2\pi R^{(m)}\,, \tag{4.1}$$

where $R^{(m)}$ and $\tilde{R}_{(m)} = \alpha' R^{(m)^{-1}}$ denote the radii of the m-th cycle and its dual, respectively. The internal momenta are encoded in the $O(n,n)$ vector $\mathbb{P}_M$ of components

$$\mathbb{P}_M \equiv (\mathbb{P}_m, \mathbb{P}_{n+m}) = (p_m, \tilde{p}^m) = \left(\frac{n_m}{R^{(m)}}, \frac{w^m}{\tilde{R}_{(m)}}\right)\,, \tag{4.2}$$

n_m and w^m being the integer momentum and winding numbers.

On the torus background, the non-trivial identifications (4.1) are only preserved by $O(n,n)$ transformations with integer-valued matrix entries. Thus, the $O(n,n,\mathbb{R})$ symmetry is broken to the discrete $O(n,n,\mathbb{Z})$ group.

The mode expansion of the generalized metric would be $\mathcal{H}(x,\mathbb{Y}) = \bar{\mathcal{H}} + \hat{h}(x,\mathbb{Y})$ with

$$\hat{h}(x,\mathbb{Y}) = \sum_{\mathbb{P}}{}' \hat{h}^{(\mathbb{P})}(x) e^{i\mathbb{P}_M \mathbb{Y}^M}\,, \tag{4.3}$$

where the dependence on the dual space time coordinates $\tilde{x}_\mu$ has been dropped. The expansion of the component fields is

$$g_{\mu\nu}(x,\mathbb{Y}) = \eta_{\mu\nu} + \sum_{\mathbb{P}}{}' h^{(\mathbb{P})}_{\mu\nu}(x) e^{i\mathbb{P}_M \mathbb{Y}^M}\,, \tag{4.4}$$

$$b_{\mu\nu}(x,\mathbb{Y}) = \sum_{\mathbb{P}}{}' b^{(\mathbb{P})}_{\mu\nu}(x) e^{i\mathbb{P}_M \mathbb{Y}^M}\,, \tag{4.5}$$

and similarly for $\hat{d}(x,\mathbb{Y})$, gauge parameters, etc.

The sum over $\mathbb{P}$ involves, in principle, all integer values of momenta and windings (n_m, w^m). Possible constraints are indicated with a prime on the sum. Also, since all the fields we are dealing with are real, we require $\mathcal{H}^{(-\mathbb{P})}(x) = \mathcal{H}^{(\mathbb{P})}(x)^*$.

[7] Here we consider dimensionful internal coordinates whereas the metric is dimensionless. Alternatively, we could absorb the dimensions in the metric just by redefining $\overline{G}_{mn} \to \overline{G}_{mn} R^{(m)} R^{(n)}$.

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Remember that we have dropped the field dependence on dual space-time coordinates, or in DFT words, we have imposed the strong constraint in order to stay in the gravity frame. This means that there will be a $\frac{1}{2\kappa^2}$ overall factor in the action, where κ is now the gravitational constant in $d+2n$ dimensions. In terms of the DFT coupling above it would formally read $\frac{1}{2\kappa^2} = \frac{1}{G_{\rm DFT}} \int d^d\tilde{x}$.

Due to the contributions from both, a circle and its dual, the usual R dependent volume factor of dimensional reduction is not present here, and instead an α' factor is left, namely

$$d^{2n}\mathbb{Y} = \Pi_{i=1}^{n} \frac{dy^i}{2\pi R_i} \frac{d\tilde{y}_i}{2\pi \tilde{R}_i} = \Pi_{i=1}^{n} \frac{1}{(2\pi)^2 \alpha'} dy^i d\tilde{y}_i \,. \tag{4.6}$$

Furthermore, we use that

$$\int d^{2n}\mathbb{Y} e^{i(\mathbb{P}_M + \mathbb{Q}_M)\mathbb{Y}^M} = \delta^{2n}(\mathbb{P}_M + \mathbb{Q}_M)\,, \tag{4.7}$$

since $\int_0^{2\pi R_i} \frac{dy_i}{2\pi R_i} = 1$. We will see below that the dependence on radii shows up when vector fields are redefined in order to have integer U(1) charges. Also a scaling factor appears through the expectation value of the generalized dilaton $e^{-2\bar{d}}$ containing both the determinant of the background metric $\bar{G}$ and dilaton $\bar{\phi}$ fields.

4.1 Quadratic terms and masses

We first concentrate on the quadratic terms in the action. Inserting the GKK expansion in the first line of the Lagrangian (3.6), we obtain

$$\begin{aligned} S^{(2)}_{\rm DFT} = \frac{1}{2\kappa_d^2} \sum_{\mathbb{P}}{}' \int d^d x \Big[& \hat{d}(x)^{(\mathbb{P})} (\partial_\mu \partial^\mu - \mathbb{P}_M \overline{\mathcal{M}}^{MN} \mathbb{P}_N) \hat{d}(x)^{(-\mathbb{P})} \\ & - \frac{1}{8} \hat{\tilde{h}}^{(\mathbb{P})\mathcal{K}\mathcal{L}}(x) (\partial_\mu \partial^\mu - \mathbb{P}_M \overline{\mathcal{M}}^{MN} \mathbb{P}_N) \hat{\tilde{h}}^{(-\mathbb{P})}_{\mathcal{K}\mathcal{L}}(x) \Big] \,, \end{aligned} \tag{4.8}$$

where we have redefined $\hat{d} \to (D-2)^{1/2}\hat{d}$, and by using (3.15),

$$\frac{1}{2\kappa_d^2} = \frac{1}{2\kappa^2} e^{-2\bar{d}}. \tag{4.9}$$

The equations of motion read

$$\left(\partial_\mu \partial^\mu - \mathbb{P}_M \overline{\mathcal{M}}^{MN} \mathbb{P}_N\right) \hat{\tilde{h}}^{(\mathbb{P})}_{\mathcal{K}\mathcal{L}}(x) = 0, \qquad \left(\partial_\mu \partial^\mu - \mathbb{P}_M \overline{\mathcal{M}}^{MN} \mathbb{P}_N\right) \hat{d}^{(\mathbb{P})}(x) = 0. \tag{4.10}$$

Interestingly enough, these expressions not only reproduce the propagators for the gravity multiplet[8] but they also contain the propagators for GKK states. In particular, we can identify the mass squared of the GKK ($\mathbb{P}$) modes[9] as

$$M^2 = -k^2 = \mathbb{P}_M \overline{\mathcal{M}}^{MN} \mathbb{P}_N = \mathbb{P}^{\dot{M}} \mathbb{P}_M \,. \tag{4.11}$$

[8] A careful discussion about physical degrees of freedom is presented in next section.

[9] Here the dot refers to contractions with the internal metric $\overline{\mathcal{M}}$.

This is exactly the mass squared of string states on generic toroidal backgrounds for $N + \bar{N} - 2 = 0$. We expect this condition is satisfied since we started with $N = \bar{N} = 1$. However, the string states also satisfy the LMC, namely

$$\frac{1}{2}\mathbb{P}_M\mathbb{P}^M = N - \bar{N} = 0\,. \tag{4.12}$$

Therefore, it appears that in order to recover the string theory results, we must consider the following constrained GKK expansion

$$\hat{h}(x, \mathbb{Y}) = \sum_{\mathbb{P}} \hat{h}^{(\mathbb{P})}(x) e^{i\mathbb{P}_M \mathbb{Y}^M} \delta(\mathbb{P}^2)\,, \tag{4.13}$$

and similarly for $\hat{d}(x, \mathbb{Y})$.

Let us look at the transformation of the compactified action under the generalized diffemorphisms (2.1). From the discussion above, we know that this variation should be proportional to terms that vanish if the strong constraint $\partial_{\mathcal{P}} \otimes \partial^{\mathcal{P}} = 0$ is imposed. Moreover, since the space-time part already satisfies it, the transformation must be proportional to $\partial_P \otimes \partial^P = 0$, where now P labels the internal compact coordinates. Since the variation is proportional to the gauge parameter, it can be written as $\partial_P \xi_M J^{PM}$, with J^{PM} a product of generalized metric and dilaton fields with a ∂^P derivative acting on one of them. By mode expanding the generalized fields, these derivatives lead to a $\mathbb{Q}^i_P \mathbb{Q}^{jP}$ factor times a $\delta^{2n}(\sum_i \mathbb{Q}^i)$ requiring total momentum conservation. If up to third order terms in fluctuations are kept in the action, momentum conservation and level matching $\mathbb{Q}^{i2} = 0$ for each field (including ξ_M) leads to $\mathbb{Q}^i \cdot \mathbb{Q}^j = 0$ and we conclude that the action, up to this order, is invariant under generalized diffeomorphisms.

4.2 Physical degrees of freedom

The mass formula (4.11) is generic and does not allow us to isolate physical states. For instance, $\hat{\hat{h}}^{(\mathbb{P})\mu}_M(x)$ seems to denote $2(D-d)$ massive vector states. However, we know that some of these vectors must be absorbed by the gravitational and two-form fields to become massive. Actually, the harmonic gauge condition allows to identify the physical degrees of freedom. In order to see this, first recall the expected physical fields in lower dimensions.

A symmetric massless two-tensor in D dimensions has $(D-2)(D-1)/2$ degrees of freedom.[10] With n compact dimensions, we can write

$$\begin{aligned} \frac{1}{2}(D-2)(D-1) &= \frac{1}{2}(D-n-2)(D-n-1) + n(D-n-2) + \frac{1}{2}n(n+1) \quad \text{or} \\ \frac{1}{2}(D-2)(D-1) &= \frac{1}{2}(D-n-1)(D-n) + (n-1)(D-n-1) + \frac{1}{2}n(n-1) \end{aligned}$$

Starting with the metric in D dimensions, decomposing the indices into $D-n$ spacetime and n internal indices, for massless states (corresponding to zero modes in the KK expansion) we would have $\frac{1}{2}(D-n-2)(D-n-1)$ d.o.f. for $g_{\mu\nu}$, n vectors $g_{\mu m}$ leading to $n(D-n-2)$ d.o.f.

[10] We count here the degree of freedom of the trace, associated to the dilaton field. We discuss the splitting of traceless and trace parts below, in order to compare with string theory results.

and $\frac{1}{2}n(n+1)$ scalars g_{mn}, consistent with the first equation. On the other hand, if the states are massive, we must decompose them as in the second equation, corresponding to a massive symmetric two-tensor, $n-1$ massive vectors and $\frac{1}{2}n(n-1)$ scalars. We can understand this combination by interpreting that a scalar is eaten by a massless vector to become massive, leaving $\frac{1}{2}n(n+1)-n = \frac{1}{2}n(n-1)$ scalars and n massive vectors with $(D-n-1)$ degrees of freedom. However, one of these vectors is eaten by the massless graviton to become massive, leaving a massive two-tensor with $\frac{1}{2}(D-n-2)(D-n-1)+(D-n-1) = \frac{1}{2}(D-n-1)(D-n)$ d.o.f., and $n-1$ massive vectors.

A similar computation can be done for the antisymmetric tensor. Namely, a massless two-tensor with $\frac{1}{2}(D-2)(D-3)$ d.o.f. can be decomposed as

$$\begin{aligned}\frac{1}{2}(D-2)(D-3) &= \frac{1}{2}(D-n-2)(D-n-3)+n(D-n-2)+\frac{1}{2}n(n-1) \qquad (4.14)\\ \frac{1}{2}(D-2)(D-3) &= \frac{1}{2}(D-n-1)(D-n-2)+(n-1)(D-n-1)+\frac{1}{2}(n-2)(n-1).\end{aligned}$$

The first equation leads to the familiar KK decomposition in terms of a massless two-tensor $b_{\mu\nu}$, n massless vectors $b_{\mu m}$ and $\frac{1}{2}n(n-1)$ massless scalars b_{mn}. For the massive case, a massless antisymmetric tensor eats a massless vector, leaving a massive antisymmetric tensor with $\frac{1}{2}(D-n-2)(D-n-3)+(D-n-2) = \frac{1}{2}(D-n-1)(D-n-2)$ d.o.f. The $n-1$ massless vectors left eat $n-1$ scalars to become $n-1$ massive vectors, leaving $\frac{1}{2}n(n-1)-(n-1) = \frac{1}{2}(n-2)(n-1)$ massive scalars.

On the whole, a massive GKK level is characterized by the generalized momentum $\mathbb{P}$, with $\mathbb{P}^2 = 0$, and it contains a spin two symmetric tensor (which can be decomposed into a traceless tensor + trace), an antisymmetric tensor, $2(n-1)$ vectors and $n(n-1)$ scalars, all mass degenerate with mass $M^2 = \mathbb{P}\overline{\mathcal{M}}\mathbb{P}$. Note that a non-equivalent level $\mathbb{P}' = h\mathbb{P}$ will have the same mass if h is an $O(n,n)$ transformation, namely h is a duality transformation. Recall that, in the $n=1$ double circle case no extra massive vectors or scalars are present.

In (both spacetime and internal) momentum space, the generalized harmonic gauge conditions (3.21) for the modes $\hat{\tilde{h}}^{(\mathbb{P})}_{\mathcal{M}\mathcal{N}}(k)$ read

$$k^\mu \hat{\tilde{h}}^{(\mathbb{P})}_{\mu\mathcal{N}}(k) + \left(\mathbb{P}\hat{\tilde{h}}^{(\mathbb{P})}\right)_{\mathcal{N}}(k) = k^\mu\left[\hat{\tilde{h}}^{(\mathbb{P})}_{\mu\mathcal{N}}(k) - \frac{1}{M^2}k_\mu\left(\mathbb{P}\hat{\tilde{h}}^{(\mathbb{P})}\right)_{\mathcal{N}}(k)\right] = 0\,, \qquad (4.15)$$

where we have used that $-k^2 = M^2$ is the (squared) mass of the states as given in (4.11).

This is an indication that there is a physical massive field

$$\hat{\tilde{h}}'^{(\mathbb{P})}_{\mu\mathcal{N}}(k) = \hat{\tilde{h}}^{(\mathbb{P})}_{\mu\mathcal{N}}(k) - \frac{1}{M^2}k_\mu\left(\mathbb{P}\hat{\tilde{h}}^{(\mathbb{P})}\right)_{\mathcal{N}}(k) + \ldots\,,$$

(where $\ldots$ indicate possible terms vanishing when contracted with k^μ) or equivalently

$$\hat{\tilde{h}}'^{(\mathbb{P})}_{\mu\mathcal{N}}(x) = \hat{\tilde{h}}^{(\mathbb{P})}_{\mu\mathcal{N}}(x) + i\frac{1}{M^2}\partial_\mu\left(\mathbb{P}\hat{\tilde{h}}^{(\mathbb{P})}\right)_{\mathcal{N}}(x)\,,$$

satisfying $\partial^\mu\hat{\tilde{h}}'^{(\mathbb{P})}_{\mu\mathcal{N}}(x) = 0$. The field combinations $(\mathbb{P}\hat{\tilde{h}}^{(\mathbb{P})})_{\mathcal{N}}$ play the role of eaten Goldstone fields to provide the physical degrees of freedom. Let us analyze them in terms of component

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fields. Using (3.13), (4.15) can be decomposed into graviton, antisymmetric tensor and vector field polarization tensors as

$$
\begin{aligned}
k^{\mu}\left[\tilde{h}^{(\mathbb{P})}_{\mu\nu}(k)-\frac{1}{M^2}k_{\mu}\left(\mathbb{P}\cdot A^{(\mathbb{P})}(k)\right)_{\nu}\right]&=0\,,\\
k^{\mu}\left[b^{(\mathbb{P})}{}_{\mu\nu}(k)+\frac{1}{M^2}k_{\mu}\left(\mathbb{P}\cdot\overline{\mathcal{M}}\cdot A^{(\mathbb{P})}(k)\right)_{\nu}\right]&=0\,,\\
k^{\mu}\left[A^{(\mathbb{P})N}_{\mu}(k)-\frac{1}{M^2}k_{\mu}\left(\mathbb{P}\cdot\overline{\mathcal{M}}\cdot\tilde{h}^{(\mathbb{P})}(k)\cdot\overline{\mathcal{M}}\right)^{N}\right]&=0\,.
\end{aligned}
\tag{4.16}
$$

Gravitons. The first equation in (4.16) can be recast as

$$
k^{\mu}\left\{\tilde{h}^{(\mathbb{P})}_{\mu\nu}-\frac{1}{M^2}\left[k_{\mu}\left(\mathbb{P}\cdot A^{(\mathbb{P})}\right)_{\nu}+k_{\nu}\left(\mathbb{P}\cdot A\right)_{\mu}+k_{\nu}k_{\mu}\frac{1}{M^2}\left(\mathbb{P}\cdot\overline{\mathcal{M}}\cdot\tilde{h}^{(\mathbb{P})}.\overline{\mathcal{M}}\cdot\mathbb{P}\right)\right]\right\}=0\,,
$$

where we have used the third equation in (4.16). Thus, we have an effective symmetric tensor with polarization $\tilde{h}'^{(\mathbb{P})}_{\mu\nu}$ satisfying

$$
k^{\mu}\tilde{h}'^{(\mathbb{P})}_{\mu\nu}(k)=0\,,
\tag{4.17}
$$

where

$$
\tilde{h}'^{(\mathbb{P})}_{\mu\nu}(k)=\tilde{h}^{(\mathbb{P})}_{\mu\nu}-\frac{1}{M^2}\left[k_{\mu}\left(\mathbb{P}\cdot A^{(\mathbb{P})}\right)_{\nu}+k_{\nu}\left(\mathbb{P}\cdot A^{(\mathbb{P})}\right)_{\mu}-k_{\nu}k_{\mu}\frac{1}{M^2}\left(\mathbb{P}\cdot\overline{\mathcal{M}}\cdot\tilde{h}^{(\mathbb{P})}\cdot\overline{\mathcal{M}}\cdot\mathbb{P}\right)\right]
\tag{4.18}
$$

is constructed from the original graviton polarization tensor, one vector field $(\mathbb{P}\cdot A)_{\nu}$ and a scalar field $\mathbb{P}\cdot\overline{\mathcal{M}}\cdot\tilde{h}\cdot\overline{\mathcal{M}}\cdot\mathbb{P}$, as expected from the above counting of degrees of freedom.

Antisymmetric tensor. We can proceed similarly with the antisymmetric field. Namely, the second equation in (4.16) can be rewritten as

$$
k^{\mu}\left\{b^{(\mathbb{P})}{}_{\mu\nu}+\frac{1}{M^2}\left[k_{\mu}\left(\mathbb{P}\cdot\overline{\mathcal{M}}\cdot A^{(\mathbb{P})}\right)_{\nu}-k_{\nu}\left(\mathbb{P}\cdot\overline{\mathcal{M}}\cdot A^{(\mathbb{P})}\right)_{\mu}\right]\right\}+\frac{1}{M^2}k_{\nu}k^{\mu}\left(\mathbb{P}\cdot\overline{\mathcal{M}}\cdot A^{(\mathbb{P})}\right)_{\mu}=0\,,
$$

and using the third equation in (4.16), the last term reads

$$
k^{\mu}\left(\mathbb{P}\cdot\overline{\mathcal{M}}\cdot A^{(\mathbb{P})}\right)_{\mu}=-\mathbb{P}\cdot\overline{\mathcal{M}}\cdot h^{(\mathbb{P})}\cdot\mathbb{P}\,.
\tag{4.19}
$$

However, this term vanishes at first order,[11] and then we are left with an effective antisymmetric polarization

$$
b'^{(\mathbb{P})}_{\mu\nu}=b^{(\mathbb{P})}_{\mu\nu}+\frac{1}{M^2}\left[k_{\mu}\left(\mathbb{P}\cdot\overline{\mathcal{M}}\cdot A^{(\mathbb{P})}\right)_{\nu}-k_{\nu}\left(\mathbb{P}\cdot\overline{\mathcal{M}}\cdot A^{(\mathbb{P})}\right)_{\mu}\right]\,,
\tag{4.21}
$$

where the original polarization $b^{(\mathbb{P})}_{\mu\nu}$ "eats" a vector $(\mathbb{P}\cdot\overline{\mathcal{M}}\cdot A^{(\mathbb{P})})_{\nu}$, in agreement with the discussion above.

[11] In fact, this can be easily seen by rewriting the condition $\mathbb{P}^2=0$. Namely

$$
\begin{aligned}
\mathbb{P}^2&=\mathbb{P}^{M}\mathcal{M}_{MN}\eta^{NK}\mathcal{M}_{KL}\mathbb{P}^{L}=\mathbb{P}^{M}\left(\overline{\mathcal{M}}_{MN}+\tilde{h}_{MN}\right)\eta^{NK}\left(\overline{\mathcal{M}}_{KL}+\tilde{h}_{KL}\right)\mathbb{P}^{L}\\
&=\mathbb{P}^2+2\mathbb{P}\cdot\overline{\mathcal{M}}\cdot\tilde{h}\cdot\mathbb{P}+\mathcal{O}\left(\tilde{h}^2\right)
\end{aligned}
\tag{4.20}
$$

and, therefore $\mathbb{P}\cdot\overline{\mathcal{M}}\cdot\tilde{h}\cdot\mathbb{P}=0$.

Vectors. The third equation (4.16) directly tells us that there are massive vector polarizations

$$A'^{(\mathbb{P})N}_{\nu}(k) = A^{(\mathbb{P})N}_{\nu}(k) + \frac{1}{M^2} k_\nu \left(\mathbb{P}\cdot\overline{\mathcal{M}}\cdot\tilde{h}^{(\mathbb{P})}\cdot\overline{\mathcal{M}}\right)^N + \dots\,, \tag{4.22}$$

satisfying $k^\nu A'^{(\mathbb{P})N}_{\nu} = 0$.

Thus, from the $2n$ original vectors $A^{(\mathbb{P})N}_{\mu}$, the combination $\mathbb{P}\cdot A^{(\mathbb{P})}_{\mu}$ is eaten by the graviton and the combination $\mathbb{P}\cdot\overline{\mathcal{M}}\cdot A^{(\mathbb{P})}_{\mu}$ is eaten by the $b^{(\mathbb{P})}_{\mu\nu}$ field to become massive, and we are left with $2n-2$ vectors. These vectors become massive by absorbing $2n-2$ scalars from the n^2 original $\tilde{h}^{(\mathbb{P})}_{MN}$. One more scalar (the combination $\mathbb{P}\cdot\overline{\mathcal{M}}\cdot\tilde{h}^{(\mathbb{P})}\cdot\overline{\mathcal{M}}\cdot\mathbb{P}$) is eaten by the graviton, so finally we are left with $n^2-(2n-2)-1=(n-1)^2$ scalars.

Notice that the vector eaten by the graviton should be different from the one eaten by $b^{(\mathbb{P})}_{\mu\nu}$. Indeed, this appears to be the case. If $\mathbb{P}\cdot A^{(\mathbb{P})}$ selects some combination, then $\mathbb{P}\cdot\overline{\mathcal{M}}\cdot A^{(\mathbb{P})}$ selects an independent one. Actually, $\overline{\mathcal{M}}$ acts effectively by changing lower to upper indices (see (3.3)).

The physical states found above should be interpreted from the generalized gauge transformations. Starting with generic states, there should be a choice of gauge parameters $\xi^{\mathcal{M}} = (\xi^\mu, \tilde{\xi}_\mu, \Lambda^M)$ such that, by performing a generalized transformation of the form (2.1), unphysical states are gauged away. Let us show that this is indeed the case. The generalized diffeomorphisms (2.10), in terms of component fields and up to first order in fluctuations, read

$$\delta_\xi \tilde{h}_{\mu\nu} = \partial_\mu \xi^\lambda \eta_{\lambda\nu} + \partial_\nu \xi^\lambda \eta_{\lambda\mu}\,, \tag{4.23}$$

$$\delta_\xi b_{\mu\nu} = \partial_\mu \tilde{\xi}_\nu - \partial_\nu \tilde{\xi}_\mu\,, \tag{4.24}$$

$$\delta_\xi A^N_\mu = \partial_\mu \Lambda^N + \eta_{\lambda\mu}\overline{\mathcal{M}}^{N\mathcal{M}}\partial_{\mathcal{M}}\xi^\lambda - \partial^N \tilde{\xi}_\mu\,, \tag{4.25}$$

$$\delta_\xi \tilde{h}_{MN} = \overline{\mathcal{M}}_{MP}\partial_N \Lambda^P + \overline{\mathcal{M}}_{PN}\partial_M \Lambda^P - \overline{\mathcal{M}}_{MP}\partial^P \Lambda_N - \overline{\mathcal{M}}_{PN}\partial^P \Lambda_M\,. \tag{4.26}$$

In terms of GKK modes, the gauge transformed fields will be

$$\begin{aligned}
\tilde{h}'^{(\mathbb{P})}_{\mu\nu} &= \tilde{h}^{(\mathbb{P})}_{\mu\nu} + \delta_\xi \tilde{h}^{(\mathbb{P})}_{\mu\nu} = \tilde{h}^{(\mathbb{P})}_{\mu\nu} + i k_{(\mu}\eta_{\nu)\lambda}\xi^{\lambda(\mathbb{P})}\,, \qquad (4.27)\\
b'^{(\mathbb{P})}_{\mu\nu} &= b^{(\mathbb{P})}_{\mu\nu} + \delta_\xi b^{(\mathbb{P})}_{\mu\nu} = b^{(\mathbb{P})}_{\mu\nu} + i k_{[\mu}\tilde{\xi}^{\mathbb{P})}_{\nu]}\\
A'^{(\mathbb{P})N}_{\mu} &= A^{(\mathbb{P})N}_{\mu} + \delta A^{(\mathbb{P})N}_{\mu} = A^{(\mathbb{P})N}_{\mu} + i k_\mu \Lambda^{(\mathbb{P})N} + i\eta_{\lambda\mu}(\overline{\mathcal{M}}\mathbb{P})^N \xi^{(\mathbb{P})\lambda} - i\mathbb{P}^N \tilde{\xi}^{(\mathbb{P})}_{\mu}\,,\\
\tilde{h}'^{(\mathbb{P})}_{MN} &= \tilde{h}^{(\mathbb{P})}_{MN} + \delta_\xi \tilde{h}^{(\mathbb{P})}_{MN} = i\left(\overline{\mathcal{M}}\Lambda^{(\mathbb{P})}_{M}\right)\mathbb{P}_N + i\left(\overline{\mathcal{M}}\Lambda^{(\mathbb{P})}_{N}\right)\mathbb{P}_M - i\left(\overline{\mathcal{M}}\mathbb{P}\right)_M \Lambda^{(\mathbb{P})}_N - i\left(\overline{\mathcal{M}}\mathbb{P}\right)_N \Lambda^{(\mathbb{P})}_M\,.
\end{aligned}$$

In order to fix the gauge parameters, we first impose the conditions

$$\mathbb{P}\cdot A'^{(\mathbb{P})}_{\mu} = 0\,, \tag{4.28}$$

$$\mathbb{P}\cdot\overline{\mathcal{M}}\cdot A'^{(\mathbb{P})}_{\mu} = 0\,, \tag{4.29}$$

as it should, since the first equation corresponds to the combination eaten by the massive graviton and the second one to the combination eaten by the antisymmetric tensor $b^{(\mathbb{P})}_{\mu\nu}$. These conditions fix the form of $\Lambda^{(\mathbb{P})}_N, \xi^{(\mathbb{P})\lambda}$ up to a coefficient and $\tilde{\xi}^{(\mathbb{P})}_{\nu}$. By also requiring that

$$\mathbb{P}\cdot\overline{\mathcal{M}}\cdot\tilde{h}'^{(\mathbb{P})}\cdot\overline{\mathcal{M}}\cdot\mathbb{P} = 0\,, \tag{4.30}$$

since this is the scalar absorbed by the graviton, this coefficient is fixed. Finally, the gauge transformations required to gauge away the non-physical fields read

$$
\begin{aligned}
\xi^{(\mathbb{P})\lambda} &= i\frac{1}{M^2}\eta^{\lambda\mu}\left[\left(\mathbb{P}\cdot A_\mu^{(\mathbb{P})}\right) - k_\mu\frac{1}{2M^2}\left(\mathbb{P}\cdot\overline{\mathcal{M}}\cdot\tilde{h}^{(\mathbb{P})}\cdot\overline{\mathcal{M}}\cdot\mathbb{P}\right)\right],\\
\tilde{\xi}_\mu^{(\mathbb{P})} &= -i\frac{1}{M^2}\mathbb{P}\cdot\overline{\mathcal{M}}\cdot A_\mu^{(\mathbb{P})},\\
\Lambda^{(\mathbb{P})N} &= i\frac{1}{M^2}\left[\left(\mathbb{P}\cdot\overline{\mathcal{M}}\cdot h^{(\mathbb{P})}\cdot\mathcal{M}\right)^N - \frac{1}{2M^2}\left(\overline{\mathcal{M}}\mathbb{P}\right)^N\mathbb{P}\cdot\overline{\mathcal{M}}\cdot h^{(\mathbb{P})}\cdot\overline{\mathcal{M}}\cdot\mathbb{P}\right].
\end{aligned}
\tag{4.31}
$$

By noticing that $\overline{\mathcal{M}}\cdot\mathbb{P}\cdot\Lambda^{(\mathbb{P})N}=0$ and $\mathbb{P}\cdot\Lambda^{(\mathbb{P})N}=-\frac{1}{2M^2}\mathbb{P}\cdot\overline{\mathcal{M}}\cdot\tilde{h}^{(\mathbb{P})}\cdot\overline{\mathcal{M}}\cdot\mathbb{P}$, and using the LMC ($\mathbb{P}^2=0$), it is easy to check that (4.29) and (4.30) are satisfied. By replacing these gauge parameters in (4.27), we obtain the explicit expressions in terms of the old fields. The resulting physical fields $\tilde{h}'^{(\mathbb{P})}_{\mu\nu}, b'^{(\mathbb{P})}_{\mu\nu}$ are the ones given in (4.18) and (4.21), respectively.

Also,

$$
\begin{aligned}
\tilde{h}'^{(\mathbb{P})}_{MN} = &-\left(\mathbb{P}\cdot\overline{\mathcal{M}}\cdot\tilde{h}^{(\mathbb{P})}\right)_M\mathbb{P}_N\\
&+\frac{1}{M^2}\left(\overline{\mathcal{M}}\mathbb{P}\right)_M\left[\mathbb{P}\cdot\overline{\mathcal{M}}\cdot\tilde{h}^{(\mathbb{P})}\cdot\overline{\mathcal{M}}\right)_N-\frac{1}{2M^2}\left(\overline{\mathcal{M}}\mathbb{P}\right)_N\mathbb{P}\cdot\overline{\mathcal{M}}\cdot\tilde{h}^{(\mathbb{P})}\cdot\overline{\mathcal{M}}\cdot\mathbb{P}]+M\leftrightarrow N
\end{aligned}
$$

and

$$
\begin{aligned}
A'^{(\mathbb{P})N}_\mu = A_\mu^{(\mathbb{P})N} &- k_\mu\frac{1}{M^2}\left(\mathbb{P}\cdot\overline{\mathcal{M}}\cdot\tilde{h}^{(\mathbb{P})}\cdot\overline{\mathcal{M}}\right)^N\\
&-\frac{1}{M^2}\left(\overline{\mathcal{M}}\mathbb{P}\right)^N\left[\left(\mathbb{P}\cdot A_\mu^{(\mathbb{P})}\right)-\frac{1}{M^2}k_\mu\left(\mathbb{P}\cdot\overline{\mathcal{M}}\cdot\tilde{h}^{(\mathbb{P})}\cdot\overline{\mathcal{M}}\cdot\mathbb{P}\right)\right]\\
&-\frac{1}{M^2}\mathbb{P}^N\left(\mathbb{P}\cdot\overline{\mathcal{M}}\cdot A_\mu^{(\mathbb{P})}\right).
\end{aligned}
\tag{4.32}
$$

Interestingly enough, since in the harmonic gauge

$$
k\cdot A^{(\mathbb{P})N} = -\left(\mathbb{P}\cdot\overline{\mathcal{M}}\cdot\tilde{h}^{(\mathbb{P})}\cdot\overline{\mathcal{M}}\right)^N,
\tag{4.33}
$$

then the physical vectors satisfy

$$
k\cdot A'^{(\mathbb{P})N}=0\,.
\tag{4.34}
$$

Moreover, it can also be checked that the physical fields $A'^{(\mathbb{P})N}_\mu, \tilde{h}'^{(\mathbb{P})}_{\mu\nu}, b'^{(\mathbb{P})}_{\mu\nu}, \tilde{h}'^{(\mathbb{P})}_{MN}$ are invariant under generic linearized diffeomorphisms $\delta_{\xi^{(\mathbb{P})}}=(\delta_{\xi^{\mu(\mathbb{P})}},\delta_{\tilde{\xi}_\mu^{(\mathbb{P})}},\delta_{\Lambda^{M(\mathbb{P})}})$ as given in (4.27). This is due to the fact that these combinations correspond to physical fields. The situation is analogous in electromagnetism where the electric and magnetic fields are a gauge invariant combination. Here $A'^{(\mathbb{P})N}_\mu, \tilde{h}'^{(\mathbb{P})}_{\mu\nu}, b'^{(\mathbb{P})}_{\mu\nu}, \tilde{h}'^{(\mathbb{P})}_{MN}$ would be the physical combinations for the internal symmetries (symmetries associated to $\delta_{\xi^{(0)}}$ must still be fixed).

Finally let us discuss the splitting of the symmetric tensor into a traceless part and a trace contribution. Of course the splitting can be performed just by adding and subtracting the trace. Let us consider a trace contribution of the form $\tilde{h}^{\phi(\mathbb{P})}_{\mu\nu}=\tilde{h}'^{\lambda(\mathbb{P})}_\lambda(k)\epsilon^\phi_{\mu\nu}(k)$ with $\epsilon^\phi_{\mu\nu}(k)=f_d(\mathbb{P})(\eta_{\mu\nu}+k_\mu\chi^{(\mathbb{P})}_\nu+k_\nu\chi^{(\mathbb{P})}_\mu)$, where we have used the freedom of including a diffeomorphism parameter χ_ν and f_d is a numerical factor (different for massive and massless states). The parameters χ_ν are chosen such that $k^\mu \epsilon^\phi_{\mu\nu}(k) = 0$. For the massless modes, this leads to the requirement $k_\mu \chi^{(0)\mu} = -1$, whereas for massive modes with $M^2 = \mathbb{P} \cdot \overline{\mathcal{M}} \cdot \mathbb{P}$, we find $\chi^{(\mathbb{P})}_\mu = \frac{1}{2M^2} k_\mu$. Therefore, the polarization tensor for the traceless symmetric graviton is

$$\tilde{h}'^{G(\mathbb{P})}_{\mu\nu}(k) = \tilde{h}'^{(\mathbb{P})}_{\mu\nu}(k) - \tilde{h}'^{\mu(\mathbb{P})}_{\mu}(k)\epsilon^\phi_{\mu\nu}(k)\,, \tag{4.35}$$

with $f_d = \frac{1}{d-2}$ for the massless modes and $f_d = \frac{1}{d-1}$ for the massive ones.

However, we still have the freedom to fix the trace $\tilde{h}'^\lambda{}_\lambda$. A convenient choice is $Tr(\tilde{h}') = \tilde{h}'^\lambda{}_\lambda = 4\phi$, where ϕ is the dilaton field, which amounts to setting $\hat{d} = 0$ (see (3.15)).

Actually, in order to compare with string theory results, it proves useful to redefine the dilaton as $\phi'^{(\mathbb{P})} = \sqrt{f_d}\phi^{(\mathbb{P})}$, and therefore the dilaton polarization becomes

$$\epsilon^{\phi'}_{\mu\nu}(k) = \sqrt{f_d}\,(\mathbb{P})\left(\eta_{\mu\nu} + k_\mu\chi^{(\mathbb{P})}_\nu + k_\nu\chi^{(\mathbb{P})}_\mu\right). \tag{4.36}$$

It is normalized as $\epsilon^{\phi'}(\mathbb{P}) \cdot \epsilon^{\phi'}(\mathbb{P}) = 1$ and also $\epsilon^{\phi'}(\mathbb{P}) \cdot \epsilon^{G}(\mathbb{P}) = 0$, by construction.

We notice that the choice $\hat{d} = 0$ eliminates the last $\hat{d}$ dependent term from the Lagrangian (3.6). However the dilaton part is now included in the previous terms due to the splitting $\tilde{h}'_{\mu\nu} = \tilde{h}'^G_{\mu\nu} + \tilde{h}'^\phi_{\mu\nu}$.

Finally, the cubic order Lagrangian to be considered is

$$L_{\rm DFT} = -\frac{1}{2}\hat{\tilde{h}}'^{\mathcal{MN}}\partial_{\mathcal{M}}\hat{\tilde{h}}'^{\mathcal{KL}}\partial_{\mathcal{K}}\hat{\tilde{h}}'_{\mathcal{NL}} + \frac{1}{8}\hat{\tilde{h}}'^{\mathcal{MN}}\partial_{\mathcal{M}}\hat{\tilde{h}}'^{\mathcal{KL}}\partial_{\mathcal{N}}\hat{\tilde{h}}'_{\mathcal{KL}}\,, \tag{4.37}$$

where only the physical fields identified above must be considered.

Recall that, even if diffeomorphisms have been used in order to fix the physical degrees of freedom, the expression of the action in the harmonic gauge can still be used since these transformations, up to first order in the fields and on shell, do preserve the gauge. More explicitly, $\partial_{\mathcal{M}}\mathcal{H}^{\mathcal{ML}}$ changes as $\partial_{\mathcal{M}}\mathcal{H}^{\mathcal{ML}} \longrightarrow \partial_{\mathcal{M}}\mathcal{H}^{\mathcal{ML}} + \delta_\xi(\partial_{\mathcal{M}}\mathcal{H}^{\mathcal{ML}})$, where from (2.10) we read that

$$\begin{aligned}\delta_\xi(\partial_{\mathcal{M}}\mathcal{H}^{\mathcal{ML}}) = &\,\partial_{\mathcal{P}}\xi_{\mathcal{M}}\partial^{\mathcal{P}}\mathcal{H}^{\mathcal{ML}} - 2\partial_{\mathcal{M}}\partial_{\mathcal{P}}\xi^{\mathcal{M}}\mathcal{H}^{\mathcal{LP}} - 2\partial_{\mathcal{M}}\partial_{\mathcal{P}}\xi^{\mathcal{L}}\mathcal{H}^{\mathcal{MP}}\\ &+ 2\partial_{\mathcal{M}}\partial^{\mathcal{M}}\xi_{\mathcal{P}}\mathcal{H}^{\mathcal{LP}} + 2\partial_{\mathcal{M}}\partial^{\mathcal{L}}\xi_{\mathcal{P}}\mathcal{H}^{\mathcal{MP}}\,.\end{aligned}$$

Since the gauge parameters $\xi^{\mathcal{M}}$ are already first order in the fields, we obtain (using LMC)

$$\delta_\xi\left(\partial_{\mathcal{M}}\mathcal{H}^{\mathcal{ML}}\right) = -2\bar{\mathcal{H}}^{\mathcal{LP}}\partial_{\mathcal{P}}\partial_{\mathcal{M}}\xi^{\mathcal{M}} - 2\bar{\mathcal{H}}^{\mathcal{MP}}\partial_{\mathcal{M}}\partial_{\mathcal{P}}\xi^{\mathcal{L}} + 2\partial^{\mathcal{L}}\left(\bar{\mathcal{H}}^{\mathcal{MP}}\partial_{\mathcal{M}}\xi_{\mathcal{P}}\right). \tag{4.38}$$

The second term vanishes due to the e.o.m (see (4.8)) and it can be easily checked that $\partial_{\mathcal{M}}\xi^{\mathcal{M}}$ and $\bar{\mathcal{H}}^{\mathcal{MN}}\partial_{\mathcal{N}}\xi_{\mathcal{M}}$ are identically zero for the $\xi^{\mathcal{M}}$ parameters found above. Thus, the harmonic gauge does not completely fix the gauge freedom, and we can still use the remaining symmetries to gauge away the Goldstone bosons.

4.3 Unbroken symmetries

The fact that physical fields can be defined by absorbing "Goldstone like fields" is associated to the spontaneous symmetry breaking by the background. This issue has been extensively discussed in the literature about KK compactification (see for instance [33–38]).

Actually, most of the generalized diffeomorphisms are spontaneously broken by the choice of vacuum, namely

$$\begin{aligned} \langle g_{\mu\nu}\rangle &= \eta_{\mu\nu}, \\ \langle A_\mu^M\rangle = \langle b_{\mu\nu}\rangle &= 0\,, \\ \langle \mathcal{H}_{MN}\rangle &= \overline{\mathcal{M}}_{MN}\,. \end{aligned}$$

In fact, only the zero modes $\xi_{\mathcal{M}}^{(0)}(x)$ parametrize the local symmetries, whereas the transformations associated to non-zero modes $\xi_{\mathcal{M}}^{(\mathbb{P})}$ are spontaneously broken.[12] Thus, for instance, the generalized internal diffeomorphism parameter $\Lambda^M(x)$ becomes the $\mathrm{U}(1)_M$ gauge parameter, under which the physical fields transform as

$$\begin{aligned} \delta g^{\mu\nu} &= \Lambda^M \partial_N g^{\mu\nu}\,, \\ \delta A^N &= \Lambda^M \partial_M A^N + d\Lambda^M\,, \\ \delta b &= \Lambda^M \partial_M b + \frac{1}{2} A^M \wedge d\Lambda_M\,, \\ \delta \mathcal{M}_{MN} &= \Lambda^P \partial_P \mathcal{M}_{MN}\,, \end{aligned} \tag{4.39}$$

where these equations must be understood as holding for all GKK modes. Recall that $\partial_M \Lambda^N = 0$, i.e. the gauge parameters do not depend on the internal coordinates, and $d\Lambda_M = 0$ for massive modes. For massless modes $A^{(0)N}$, the usual gauge transformations are obtained. The gauge transformation of the two-form field b is particularly interesting since it involves the vector bosons and, for massless fields, it gives rise to the familiar Chern Simons three-form. Actually, there exist two simple covariant combinations of fields under the above gauge transformations, namely

$$\begin{aligned} H &= \left(d - A^M \wedge \partial_M\right) b + \frac{1}{2} A^M \wedge \left(d - A^N \wedge \partial_N\right) A_M\,, \\ B_M &= \partial_M b + \frac{1}{2} A^N \wedge \partial_M A_N\,, \end{aligned} \tag{4.40}$$

where H and B_M are spacetime three-form and two-form, respectively. We will see that fields in the Lagrangian do group into these combinations.

4.4 Cubic terms and effective action

Once the physical states have been identified, we proceed to consider the third order action[13] (4.37). By splitting the indices of the fluctuations into spacetime and internal com-

[12] The algebra of diffeomorphisms is discussed in the appendix.

[13] A Kaluza-Klein inspired rewritting of the strongly constrained double field theory action was performed in [21].

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ponents, the Lagrangian containing only physical fields, reads

$$
\begin{aligned}
\mathcal{L} = & -\frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho} + \frac{1}{4}D_\mu g_{\nu\rho}D_\sigma g^{\nu\rho}g^{\mu\sigma} - \frac{1}{2}D_\mu g_{\nu\rho}D_\sigma g^{\mu\nu}g^{\rho\sigma} \\
& -\frac{1}{4}\mathcal{M}_{MN}F^M_{\mu\nu}F^{N\mu\nu} + \frac{1}{8}g^{\mu\nu}D_\mu\mathcal{M}_{MN}D^\mu\mathcal{M}^{MN} \\
& +\frac{1}{4}\mathcal{M}^{MN}\partial_M g_{\mu\nu}\partial_N g^{\mu\nu} - \frac{1}{2}\mathcal{M}_{MN}\partial_P A^M_\mu \partial_Q A^N_\nu g^{\mu\nu}\mathcal{M}^{PQ} + \frac{1}{8}\mathcal{M}^{PQ}\partial_P\mathcal{M}_{MN}\partial_Q\mathcal{M}^{MN} \\
& -\frac{1}{4}\mathcal{M}^{MN}B_{M\mu\nu}B_{N\rho\sigma}g^{\mu\rho}g^{\nu\sigma} - \frac{1}{2}\mathcal{M}^{MN}B_{M\mu\nu}F_{N\rho\sigma}g^{\mu\rho}g^{\nu\sigma} \\
& +\frac{1}{2}\mathcal{M}^{MN}\partial_M A^P_\mu D_\nu\mathcal{M}_{NP}g^{\mu\nu} - \frac{1}{2}\mathcal{M}^{MN}\partial^P A_{M\mu}D_\nu\mathcal{M}_{NP}g^{\mu\nu} - \frac{1}{2}\partial^M A^N_\mu \partial_N A_{M\nu}g^{\mu\nu} \\
& -\frac{1}{2}\mathcal{M}^{MN}\partial_M\mathcal{M}^{PQ}\partial_P\mathcal{M}_{NQ} + \frac{1}{2}\mathcal{M}^{MN}\mathcal{M}^{PQ}\partial_M A_{P\mu}\partial_Q A_{N\nu}g^{\mu\nu}\,,
\end{aligned}
\tag{4.41}
$$

where we have included cubic interactions plus some higher order terms required by space-time diffeomorphism and gauge invariances.

Here

$$
\begin{aligned}
F^M_{\mu\nu} &= D_{[\mu}A^M_{\nu]} \equiv \partial_{[\mu}A^M_{\nu]} - A^N_{[\mu}\partial_N A^M_{\nu]} \equiv \partial_\mu A^M_\nu - \partial_\nu A^M_\mu - A^N_\mu\partial_N A^M_\nu + A^N_\nu\partial_N A^M_\mu \\
H_{\mu\nu\rho} &= D_{[\mu}b_{\nu\rho]} - \frac{1}{2}A^M_{[\mu}D_\nu A_{\rho]M} \\
&\equiv D_\mu b_{\nu\rho} + D_\nu b_{\rho\mu} + D_\rho b_{\mu\nu} - \frac{1}{2}(A^M_\mu D_\nu A_{\rho M} + A^M_\nu D_\rho A_{\mu M} + A^M_\rho D_\mu A_{\nu M}) \\
&\quad + \frac{1}{2}\left(A^M_\mu D_\rho A_{\nu M} + A^M_\nu D_\mu A_{\rho M} + A^M_\rho D_\nu A_{\mu M}\right) \\
B_{M\mu\nu} &= \partial_M b_{\mu\nu} + \frac{1}{2}A^N_{[\mu}\partial_M A_{N\nu]} \\
&\equiv \partial_M b_{\mu\nu} + \frac{1}{2}A^N_\mu\partial_M A_{N\nu} - \frac{1}{2}A^N_\nu\partial_M A_{N\mu}\,,
\end{aligned}
\tag{4.42}
$$

and the derivatives are

$$
D_\mu = \partial_\mu - A^M_\mu\partial_M\,. \tag{4.43}
$$

Recall that $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \mathcal{M}_{MN} = \overline{\mathcal{M}}_{MN} + h_{MN}$, etc.

The Lagrangian (4.41) has a rather compact expression due to the explicit $O(n,n)$ invariant setting. The fields here depend on both space time and internal coordinates and must still be mode expanded in generalized momenta, according to (4.13). Modes correspond to physical fields, in terms of which the contributions acquire a more familiar shape. Recall that, when acting on the field mode $(\mathbb{P})$, $-i\partial_M \to \mathbb{P}_M$ is just the charge operator.

The action contains both kinetic and cubic interaction terms of massless and massive fields. Covariant derivatives and Chern-Simons terms in the antisymmetric tensor field strength appear as usual. For instance, the derivative D_μ in (4.43) leads, when mode expanded and acting on a generic field $\Phi^{(\mathbb{P})}(x)$, to the covariant derivative

$$
D_\mu\Phi^{(\mathbb{P})}(x) = (\partial_\mu - iA^{(0)M}_\mu\mathbb{P}_M)\Phi^{(\mathbb{P})}(x)\,, \tag{4.44}
$$

where $\mathbb{P}_M$ is the electric charge with respect to the $\mathrm{U}(1)_M$ gauge field $A^{(0)M}_\mu$.

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We know from (4.2) that fields charged under $(A_\mu^{(0)m}, \tilde{A}^{(0)}_{m\mu})$ carry charge $(\frac{n_m}{R^{(m)}}, \frac{w^m}{\tilde{R}_{(m)}})$ and therefore, in order to have integer charge, field redefinitions

$$A_\mu^{(0)m} \to R^{(m)} A'^{(0)m}_\mu \,, \tag{4.45}$$
$$\tilde{A}^{(0)}_{m\mu} \to \tilde{R}_{(m)} \tilde{A}'^{(0)}_{m\mu} \,, \tag{4.46}$$

must be performed. Therefore, by using the standard definition $-\frac{1}{4g_d^2}$ for the coefficient of the field strength squared term in the d-dimensional Lagrangian, we see that the corresponding gauge and gravitational coupling constants are

$$g'^{(m)2}_d = \frac{2\kappa_d^2}{{R^{(m)}}^2}, \qquad \tilde{g}'^2_{d(m)} = \frac{2\kappa_d^2}{\tilde{R}_{(m)}}, \qquad \kappa_d^2 = \kappa^2 e^{2\bar{d}} \,. \tag{4.47}$$

Recall that, since the generalized dilaton is $O(n,n)$ invariant, κ_d is invariant, as expected.

The massless modes in the first line of (4.41) give rise to the extended Hilbert-Einstein action (1.1), now in d dimensions. The second line contains Abelian field strength kinetic terms $-\frac{1}{4}\overline{\mathcal{M}}_{MN} F^{(0)M}_{\mu\nu} F^{(0)N\mu\nu} = -\frac{1}{4} F^{(0)\dot{M}}_{\mu\nu} F^{(0)\mu\nu}_{M}$ as well as kinetic terms for the scalars. The third line has the massive terms for gravitons, vectors and scalars. For instance, the term for the vector bosons leads to

$$-\frac{1}{2}\overline{\mathcal{M}}_{MN} A^{(\mathbb{P})M}_\mu A^{(-\mathbb{P})N}_\nu g^{\mu\nu} \mathbb{P}\overline{\mathcal{M}}\mathbb{P} = -\frac{1}{2} A^{(\mathbb{P})}_{\mu M} A^{(-\mathbb{P})\dot{M}}_\nu g^{\mu\nu} \mathbb{P}^{\dot{M}} \mathbb{P}_M \,, \tag{4.48}$$

with $M^2 = \mathbb{P}\overline{\mathcal{M}}\mathbb{P} = \mathbb{P}^{\dot{M}}\mathbb{P}_M$ the mass of the vector, etc.

We present the full expanded expression in the case of circle compactification in (5.22) below.

Let us stress that the action (4.41) is an effective gauge invariant action. The massless sector contains gravity+Kalb-Ramond field+ vector bosons + scalars, coupled to the corresponding towers of massive fields associated to KK momenta as well as windings. Propagators, Feynman rules, etc. which are necessary for field amplitudes computations can be explicitly obtained. It provides a generalization of previous constructions (see for instance [39, 40]) where KK compactifications of gravity were considered, in diverse phenomenological proposals.

For comparison with string theory amplitudes we will be interested in the on shell action.

5 String theory amplitudes

In this section we consider string theory with constant toroidal backgrounds G_{pn} and B_{mp} for the metric and antisymmetric tensor, respectively. We analyze the vertex operators creating physical states, discuss the computation of their three-point functions and contrast with the results obtained in the previous sections from DFT. We restrict to states with left and right moving oscillator numbers $N = \bar{N} = 1$. The vertex operators creating these states are analyzed in two different ways:

On the one hand, we show that a combination of different vertex operators (associated to vectors, two-tensors or scalars) is needed in order to cancel conformal anomalies. These combinations can be identified with the expressions determined by the generalized harmonic gauge choice on the DFT side and correspond to a worldsheet manifestation of a built-in string Higgs mechanism.

On the other hand, consistency requirements on the full vertex operator, once the harmonic gauge was chosen, fix the physical polarizations and it is with these operators, corresponding to physical degrees of freedom, that all scattering amplitudes are computed.

5.1 Conformal anomalies and DFT harmonic condition

It is known that the cancellation of conformal anomalies at the string world sheet level manifests as gauge symmetry requirements on the target space fields. This is indeed the case here. The different vertex operators corresponding to two-tensor, vector and scalar fields will generically have anomalous OPEs (Operator Product Expansions) with the world sheet stress energy tensor. For massless fields, the cancellation of anomalous terms leads to the familiar gauge conditions $k^\mu \epsilon^G_{\mu\nu}(k) = 0$, $k^\mu \epsilon^M_\mu(k) = 0$, etc. for the polarization tensors of gravitons, vectors, etc. These correspond to equations (4.16) for zero generalized momentum.

For massive fields, a combination of the different vertex operators must be considered, such that the sum of the different anomalous contributions cancel. This is, indeed, a world sheet manifestation of the Higgs mechanism. The conditions for cancellation of the anomalous terms can be written in an $O(n,n)$ language and can be shown to coincide with the harmonic gauge conditions found in DFT.

The vertex operators we are interested in are, up to normalizations,

$$\begin{aligned}
V_G &= \epsilon^G_{\rho\sigma}(k, k_L, k_R) : \partial X^\rho \bar\partial X^\sigma \, e^{ik\cdot X + ik_L\cdot Y + ik_R\cdot \bar Y} :, \\
V_{A_R} &= \epsilon^a_{R\rho}(k, k_L, k_R) : \partial X^\rho \bar\partial \bar Y^a \, e^{ik\cdot X + ik_L\cdot Y + ik_R\cdot \bar Y} :, \\
V_{A_L} &= \epsilon^a_{L\rho}(k, k_L, k_R) : \partial Y^a \, \bar\partial X^\rho \, e^{ik\cdot X + ik_L\cdot Y + ik_R\cdot \bar Y} :, \\
V_\phi &= \phi_{ab}(k, k_L, k_R) : \partial Y^a \, \bar\partial \bar Y^b \, e^{ik\cdot X + ik_L\cdot Y + ik_R\cdot \bar Y} :
\end{aligned} \tag{5.1}$$

The label G generically denotes a symmetric traceless, antisymmetric or trace polarization, A_L, A_R refer to vectors and ϕ to scalars. Here $\bar\partial = \partial_{\bar z}, \partial = \partial_z$ and $Y = Y(z), \bar Y = \bar Y(\bar z)$ denote left and right moving coordinates. It is convenient to use coordinates $Y^a = e_m{}^a Y^m$ with tangent space indices $a, b, \ldots$, defined in terms of the vielbein $e_m{}^a$ ($\delta^{ab} = e_m{}^a g^{mn} e_n{}^b$) since they have the standard OPEs. Namely, the propagators read

$$\begin{aligned}
\langle X^\mu(z,\bar z) X^\nu(w,\bar w)\rangle &= -\frac{\alpha'}{2}\eta^{\mu\nu} ln|z-w|^2 \,, \\
\langle Y^a(z) Y^b(w)\rangle = -\delta^{ab}\frac{\alpha'}{2} ln(z-w)\,, &\qquad \langle \bar Y^a(\bar z)\bar Y^b(\bar w)\rangle = -\delta^{ab}\frac{\alpha'}{2} ln(\bar z-\bar w)\,.
\end{aligned}$$

The vertex operator momenta are

$$k_{aL} = e_a{}^m p_{mL}\,, \qquad k_{aR} = e_a{}^m p_{mR}\,, \tag{5.2}$$

where

$$p_L^m = \tilde{p}^m + g^{mn}(p_n - B_{nk}\tilde{p}^k)\,, \qquad p_R^m = -\tilde{p}^m + g^{mn}(p_n - B_{nk}\tilde{p}^k)\,.$$

The stress energy tensor is

$$T(z) = -\frac{1}{\alpha'}(\eta_{\mu\nu} : \partial X^\mu(z)\partial X^\nu(z) : +\delta_{ab} : \partial Y^a(z)\partial Y^b(z) :)\,,$$

and similarly for the right moving one. The OPEs are

$$\begin{aligned}
T\left(z_1\right) V_G\left(z_2\right) &= \left[\frac{\alpha'}{4}\left(k^2+k_L^2\right)+1\right]\frac{V_G}{z_{12}^2} - 2i\frac{\alpha'}{4z_{12}^3}\left[: k^\rho \epsilon^G_{\rho\sigma}\partial\bar{X}^\sigma\, e^{ik\cdot X + ik_L\cdot Y + ik_R\cdot\bar{Y}} :\right] + \dots\,,\\
T\left(z_1\right) V_{A_L} &= \left[\frac{\alpha'}{4}\left(k^2+k_L^2\right)+1\right]\frac{V_{A_L}}{z_{12}^2} - 2i\frac{\alpha'}{4z_{12}^3}\left[: k_L^a \epsilon^a_{L\rho}\partial\bar{X}^\rho\, e^{ik\cdot X + ik_L\cdot Y + ik_R\cdot\bar{Y}} :\right] + \dots\,,\\
T\left(z_1\right) V_{A_R} &= \left[\frac{\alpha'}{4}\left(k^2+k_L^2\right)+1\right]\frac{V_{A_R}}{z_{12}^2} - 2i\frac{\alpha'}{4z_{12}^3}\left[: k^\rho \epsilon^a_{R\rho}\partial\bar{Y}^a\, e^{ik\cdot X + k_L Y + k_R\bar{Y}} :\right] + \dots\,,\\
T\left(z_1\right) V_\phi &= \left[\frac{\alpha'}{4}\left(k^2+k_L^2\right)+1\right]\frac{V_\phi}{z_{12}^2} - 2i\frac{\alpha'}{4z_{12}^3}\left[: k_L^a \phi_{ab}\partial\bar{Y}^b\, e^{ik\cdot X + ik_L\cdot Y + ik_R\cdot\bar{Y}} :\right] + \dots\,.
\end{aligned}$$

Since $k_L^2 = -k^2$, the vertex operators have the correct conformal weight $h = 1$ (and similarly $\bar{h} = 1$), however, there are cubic anomalies which suggest that the physical fields should be created from combinations of these operators. Consider then the vertex associated with the massive graviton

$$V = \alpha V_G + \beta V_{A_L} + \gamma V_{A_R} + \delta V_\phi\,, \tag{5.3}$$

with constant $\alpha, \beta, \gamma, \delta$. From the OPE with T and $\bar{T}$, the anomaly cancellation conditions are

$$\begin{aligned}
\alpha k^\rho \epsilon_{\rho\sigma} + \beta k_L^a \epsilon^a_{L\sigma} &= 0\,, \qquad \alpha k^\rho \epsilon_{\rho\sigma} + \gamma k_R^a \epsilon^a_{R\sigma} = 0\,,\\
\delta k_L^a \phi_{ab} + \gamma \epsilon^b_{R\sigma} k^\sigma &= 0\,, \qquad \delta k_R^a \phi_{ba} + \gamma k^\rho \epsilon^b_{L\rho} = 0
\end{aligned} \tag{5.4}$$

Choosing $2\alpha = \gamma = \beta$, the sum of the first two equations leads to

$$k^\rho \epsilon^G_{\rho\sigma} + k_L^a \epsilon_{La} + k_R^a \epsilon_{Ra} = k^\rho \epsilon^G_{\rho\sigma} + \tilde{p}^m \tilde{\epsilon}_{m\sigma} + g^{mn} p_n \epsilon_{m\sigma} = k^\rho \tilde{h}^{(\mathbb{P})}_{\rho\sigma} + \mathbb{P}\cdot A^{(\mathbb{P})}_\sigma = 0\,, \tag{5.5}$$

where we have defined

$$\epsilon_{m\sigma} = \epsilon_{Lm\sigma} + \epsilon_{Rm\sigma}, \qquad \tilde{\epsilon}_{m\sigma} + B_{mn}\epsilon^n_\sigma = \epsilon_{Lm\sigma} - \epsilon_{Rm\sigma}\,, \tag{5.6}$$

and we have made the identifications

$$A^{(\mathbb{P})}_{M\sigma} = \left(A^{(\mathbb{P})}_{m\sigma}, A^{(\mathbb{P})m}_\sigma\right) \equiv (\tilde{\epsilon}_{m\sigma}, \epsilon^m_\sigma), \qquad \epsilon^G_{\rho\sigma} \equiv \tilde{h}^{(\mathbb{P})}_{\rho\sigma}. \tag{5.7}$$

Therefore, (5.5) is nothing but the first harmonic gauge condition in (3.21) in momentum space.

On the other hand, by subtracting the first two equations in (5.4), we obtain

$$k_L\epsilon_L - k_R\epsilon_R = \tilde{p}^m \epsilon_{m\sigma} + g^{mn}(p_n + B_{nk}\tilde{p}^k)(\tilde{\epsilon}_{m\sigma} + B_{mp}\epsilon^p_\sigma) = 0\,,$$

which can be written as

$$\mathbb{P}\cdot\overline{\mathcal{M}}\cdot A_{\sigma}{}^{(\mathbb{P})}=0\,, \tag{5.8}$$

as found in (4.29).

The other two equations involving the scalars lead to

$$\begin{aligned}\delta(k_L^m\phi_{mn}+k_R^n\phi_{mn})+\gamma k.(\epsilon_L^m+\epsilon_R^m)&=2\delta g^{mn}(p_n+B_{nk}\tilde{p}^k)\phi_{mn}+\gamma k\cdot\epsilon_n=0\,,\\ \delta(k_L^m\phi_{mn}-k_R^n\phi_{mn})-\gamma k.(\epsilon_{mL}-\epsilon_{mR})&=2\delta\tilde{p}^m\phi_{mn}-\gamma k\cdot(\tilde{\epsilon}_n+B_{nl}\cdot\epsilon^l)=0\,,\end{aligned}$$

which can be shown to coincide with the third equation of the harmonic gauge conditions in (3.21) when choosing $\delta=\frac{1}{2}\gamma$ and establishing the identification with DFT scalar fields (3.14)

$$\begin{aligned}\phi^{mn}+\phi^{nm}&=\tilde{h}^{mn}\,,\\ \phi^{mn}-\phi^{nm}&=b^{mn}.\end{aligned} \tag{5.9}$$

Thus, the physical vertex operator for the massive graviton is

$$V=\frac{1}{2}\,V_G+V_{A_L}+V_{A_R}+\frac{1}{2}V_{\phi}. \tag{5.10}$$

The effective symmetric polarization tensor can be shown to coincide with (4.18).

Similar steps can be followed for the Kalb-Ramond field and the second equation in (3.21) is obtained.

In the next section we introduce the physical vertex operators used in the computation of scattering amplitudes. The anomaly free conditions on the polarizations coincide with those of the physical fields redefined through the use of the harmonic gauge condition.

5.2 Physical vertex operators on the torus

In the same way that we found the anomaly free combinations of vertex operators (or equivalently, the harmonic gauge conditions), we can impose that each one of the vertex operators (5.1) be anomaly free. This would give the conditions to be satisfied by the physical polarizations, that now we distinguish with a prime. Note that this procedure will give identically zero polarizations for massive vectors and scalars in the case of only one compact dimension, thus confirming that there are no such degrees of freedom on a circle compactification.

The anomaly cancellation conditions for vectors are

$$\begin{aligned}k_L^a\epsilon'^a_{L\rho}&=0\,, &\qquad k_R^a\epsilon'^a_{R\rho}&=0\,,\\ k^\rho\epsilon'^a_{L\rho}&=0\,, &\qquad k^\rho\epsilon'^a_{R\rho}&=0\,.\end{aligned} \tag{5.11}$$

The first two equations can be combined as

$$k_L^a\epsilon'^a_{L\rho}+k_R^a\epsilon'^a_{R\rho}=0\quad\text{or as}\quad k_L^a\epsilon'^a_{L\rho}-k_R^a\epsilon'^a_{R\rho}\;\;=0\,, \tag{5.12}$$

which are equivalent to

$$\begin{aligned}\mathbb{P}\cdot A'_\mu&=0\,,\\ \mathbb{P}\cdot\mathcal{M}\cdot A'_\mu&=0\,,\\ \partial^\mu A'^B_\mu&=0\,.\end{aligned} \tag{5.13}$$

Namely, the conditions found in (4.29) after gauge fixing. In the same way, for scalars we find

$$k_L^a \phi'^{ab} = 0\,, \qquad k_R^b \phi'^{ab} = 0\,, \tag{5.14}$$

which can be expressed in terms of $\tilde{h}'_{mn}$ and b'_{mn} as the following two conditions

$$\begin{aligned} -\tilde{p}^m \tilde{h}'_{mn} + \tilde{p}^m B_{mk} G^{ks} b'_{sn} + p_m G^{mk} b'_{kn} = 0\,, \\ -\tilde{p}^m b'_{mn} + \tilde{p}^m B_{mk} G^{ks} \tilde{h}'_{sn} + p_m G^{mk} \tilde{h}'_{kn} = 0\,. \end{aligned} \tag{5.15}$$

These coincide with the DFT condition (see 4.30)

$$\mathbb{P} \cdot M \cdot \tilde{h}' \cdot M = 0\,, \tag{5.16}$$

which represents the Goldstone boson absorbed by the massive vectors.

For the tensors $\tilde{h}'_{\mu\nu}$ and $b'_{\mu\nu}$ we get the usual transverse gauge conditions

$$\begin{aligned} k^\mu \tilde{h}'_{\mu\nu} = 0\,, \\ k^\mu b'_{\mu\nu} = 0\,. \end{aligned} \tag{5.17}$$

Finally, the dilaton vertex can be written as

$$V_\phi = \phi \epsilon^\phi_{\mu\nu} \partial X^\mu \bar{\partial} X^\nu e^{ik\cdot X}\,, \tag{5.18}$$

with

$$\epsilon^\phi_{\mu\nu} = \sqrt{f_d}\big(\eta_{\mu\nu} + k_\mu \bar{k}_\nu + k_\nu \bar{k}_\mu\big)\,, \tag{5.19}$$

as found in (4.36) by identifying $\bar{k}_\nu \equiv \chi_\nu^{(0)}$ for the massless case and $\bar{k}_\nu \equiv \chi_\nu^{(\mathbb{P})}$ for massive dilatons.

Thus, we have obtained the requirements that physical polarizations must satisfy.

Notice that the two approaches to deal with vertex operators provide different information on the theory: the first one displays a built in Higgs mechanism exhibiting the Goldstone bosons. The second one deals with the physical degrees of freedom once the gauge was chosen. Of course, one can obtain the latter using the former, as was shown in the previous section. We will use physical polarization tensors to compute scattering amplitudes.

5.3 Three-point interaction terms

In this section we consider three point functions of the massless and massive string states created by the vertex operators described above. The resulting amplitudes are then compared with the DFT action (4.37), evaluated on shell. We sketch the computation here and provide some details for the circle case in the appendix.

For the sake of clarity we first concentrate on the circle compactification. This case is particularly simple since neither physical massive vectors nor massive scalars are present. The string S-matrix three-point amplitudes are presented in (B). When mode expanding (4.37) and by using the identifications (5.7) and (5.9) between string polarization tensors and DFT fields polarizations, complete agreement is achieved if we further identify

$$\pi g_c = \frac{1}{2\kappa_d^2}\,, \tag{5.20}$$

where g_c is the closed string coupling.

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The effective $\mathrm{U}(1)\times\mathrm{U}(1)$ gauge invariant action, containing massless as well as massive states with these S-matrix elements, can be written down. By including terms required from gauge invariance and diffeomorphism invariance, this action reads

$$S=\frac{1}{2\kappa_d^2}\int d^{D-1}x\sqrt{-g}\mathcal{L} \tag{5.21}$$

with

$$\begin{aligned}
\mathcal{L}=R&-\frac{1}{12}H^2_{\mu\nu\rho}-\frac{1}{4}\partial_\mu\Phi\partial^\mu\Phi\\
&-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}-\frac{1}{4}\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu}+\frac{1}{2}F_{\mu\nu}F^{\mu\nu}\Phi-\frac{1}{2}\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu}\Phi\\
&-\frac{1}{2}\sum_{n=1}^{\infty}\left(\mathcal{D}_\rho h^{*\,(n)}_{\mu\nu}\mathcal{D}^\rho h^{(n)\,\mu\nu}-2\mathcal{D}_\mu h^{*\,(n)}_{\nu\rho}\mathcal{D}^\nu h^{(n)\,\mu\rho}+m_n^2 h^{*\,(n)}_{\mu\nu}h^{(n)\,\mu\nu}\right)\\
&-\frac{1}{2}\sum_{w=1}^{\infty}\left(\mathcal{D}_\rho \tilde{h}^{*\,(w)}_{\mu\nu}\mathcal{D}^\rho \tilde{h}^{(w)\,\mu\nu}-2\mathcal{D}_\mu \tilde{h}^{*\,(w)}_{\nu\rho}\mathcal{D}^\nu \tilde{h}^{(w)\,\mu\rho}+m_w^2 \tilde{h}^{*\,(w)}_{\mu\nu}\tilde{h}^{(w)\,\mu\nu}\right)\\
&+\sum_{n=1}^{\infty}\left(\frac{1}{6}|H^{(n)}_{\mu\nu\rho}|^2+\frac{m_n^2}{2}|b^{(n)}_{\mu\nu}|^2\right)+\sum_{w=1}^{\infty}\left(\frac{1}{6}|\tilde{H}^{(w)}_{\mu\nu\rho}|^2+\frac{m_w^2}{2}|\tilde{b}^{(w)}_{\mu\nu}|^2\right)\\
&+\sum_{n=1}^{\infty}\frac{1}{2}\frac{n^2}{R^2}\left(|h^{(n)}_{\mu\nu}|^2+|b^{(n)}_{\mu\nu}|^2\right)\Phi-\sum_{w=1}^{\infty}\frac{1}{2}\frac{w^2}{\tilde{R}^2}\left(|\tilde{h}^{(w)}_{\mu\nu}|^2+|\tilde{b}^{(w)}_{\mu\nu}|^2\right)\Phi\\
&-i\sum_{n=1}^{\infty}\frac{n}{R}\left(h^{*(n)}_{\mu\nu}b^{(n)\nu}_{\rho}+h^{(n)}_{\mu\nu}b^{*(n)\nu}_{\rho}\right)\tilde{F}^{\mu\rho}-i\sum_{w=1}^{\infty}\frac{w}{\tilde{R}}\left(\tilde{h}^{*(w)}_{\mu\nu}\tilde{b}^{(w)\nu}_{\rho}+\tilde{h}^{(w)}_{\mu\nu}\tilde{b}^{*(w)\nu}_{\rho}\right)F^{\mu\rho}\\
&+\sum_{n_1+n_2+n_3=0}^{n_i\neq 0}\left(\frac{1}{4}\mathcal{D}_\mu h^{(n_1)}_{\rho\sigma}\mathcal{D}_\nu h^{(n_2)\,\rho\sigma}h^{(n_3)\,\mu\nu}-\frac{1}{2}h^{(n_1)}_{\mu\rho}\mathcal{D}^\mu h^{(n_2)}_{\nu\sigma}\mathcal{D}^\nu h^{(n_3)\,\rho\sigma}\right)\\
&+\sum_{w_1+w_2+w_3=0}^{w_i\neq 0}\left(\frac{1}{4}\mathcal{D}_\mu \tilde{h}^{(w_1)}_{\rho\sigma}\mathcal{D}_\nu \tilde{h}^{(w_2)\,\rho\sigma}\tilde{h}^{(w_3)\,\mu\nu}-\frac{1}{2}\tilde{h}^{(w_1)}_{\mu\rho}\mathcal{D}^\mu \tilde{h}^{(w_2)}_{\nu\sigma}\mathcal{D}^\nu \tilde{h}^{(w_3)\,\rho\sigma}\right)\\
&+\sum_{n_1+n_2+n_3=0}^{n_3\neq 0}\left(\frac{1}{4}\mathcal{D}_\mu b^{(n_1)}_{\rho\sigma}\mathcal{D}_\nu b^{(n_2)\,\rho\sigma}h^{(n_3)\,\mu\nu}-\mathcal{D}_\mu b^{(n_1)\,\sigma\nu}\mathcal{D}_\nu b^{(n_2)}_{\sigma\rho}h^{(n_3)\,\mu\rho}\right.\\
&\qquad\qquad\left.-\frac{1}{2}b^{(n_1)\,\rho\mu}\mathcal{D}_\mu b^{(n_2)\,\sigma\nu}\mathcal{D}_\nu h^{(n_3)}_{\rho\sigma}\right)\\
&+\sum_{w_1+w_2+w_3=0}^{w_3\neq 0}\left(\frac{1}{4}\mathcal{D}_\mu \tilde{b}^{(w_1)}_{\rho\sigma}\mathcal{D}_\nu \tilde{b}^{(w_2)\,\rho\sigma}\tilde{h}^{(w_3)\,\mu\nu}-\mathcal{D}_\mu \tilde{b}^{(w_1)\,\sigma\nu}\mathcal{D}_\nu \tilde{b}^{(w_2)}_{\sigma\rho}\tilde{h}^{(w_3)\,\mu\rho}\right.\\
&\qquad\qquad\left.-\frac{1}{2}\tilde{b}^{(w_1)\,\rho\mu}\mathcal{D}_\mu \tilde{b}^{(w_2)\,\sigma\nu}\mathcal{D}_\nu \tilde{h}^{(w_3)}_{\rho\sigma}\right)
\end{aligned} \tag{5.22}$$

where Φ denotes the massless scalar; $h^{(n)}_{\mu\nu}$ and $\tilde{h}^{(w)}_{\mu\nu}$ the modes of the massive graviton with momentum n and winding w respectively; $b^{(n)}_{\mu\nu}$ and $\tilde{b}^{(w)}_{\mu\nu}$ the modes of the massive antisymmetric tensor with momentum n and winding w respectively.

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We have introduced the following definitions

$$\begin{aligned}
\nabla_\mu f_{\rho\sigma} &= \partial_\mu f_{\rho\sigma} - \Gamma^\lambda_{\rho\mu} f_{\lambda\sigma} - \Gamma^\lambda_{\sigma\mu} f_{\rho d}\,, \\
\Gamma^\lambda_{\mu\nu} &= \frac{1}{2} g^{\lambda\sigma}\left(\partial_\nu g_{\sigma\mu} + \partial_\mu g_{\sigma\nu} - \partial_\sigma g_{\mu\nu}\right)\,, \\
F_{\mu\nu} &= \nabla_\mu A_\nu - \nabla_\nu A_\mu\,, \\
\mathcal{D}_\mu &= \nabla_\mu - iA_\mu \hat{q}_n - i\tilde{A}_\mu \hat{q}_w\,, \\
H_{\mu\nu\rho} &= \mathcal{D}_\mu b_{\nu\rho} + \mathcal{D}_\rho b_{\mu\nu} + \mathcal{D}_\nu b_{\rho\mu}\,.
\end{aligned} \tag{5.23}$$

Here indices are raised with the inverse of the metric tensor $g_{\mu\nu}$, $\hat{q}$ is the charge operator, complex conjugation is denoted with $*$ and, under charge conjugation, the momentum or winding change sign i.e $h^{(n)\,*} = h^{(-n)}$.

The kinetic terms of the symmetric massive states produce the known Fierz Pauli Lagrangian [31, 32], and the T-duality symmetry $R \leftrightarrow \tilde{R}, n \leftrightarrow w$ is manifest. This action coincides with (4.41) when specified for the circle case.

5.4 Strings vs DFT on generic tori

Generalizing the results obtained for the circle to generic tori is formally straightforward. However, the number of terms involved is much bigger. The massless sector contains, besides the graviton, dilaton, antisymmetric and scalar fields, the $2n$ gauge fields associated to $\mathrm{U}(1)^n \times \mathrm{U}(1)^n$. The massive sector includes now, generically, massive vectors and scalars. The comparison of DFT cubic interactions contained in the mode expansion of (4.37) with three point scattering amplitudes computed using the vertex operators (5.1) is now performed with the help of the symbolic algebra computer program XCadabra [23]. Our algorithm compares three point scattering amplitudes of string states and DFT cubic interaction terms by systematic use of momentum conservation and on shell conditions.[14]

As an example of the calculated quantities, we present the result of the scattering amplitude between one antisymmetric tensor $b_{\mu\nu}$ (with momentum $k_{1\,\mu}$, and charges $p_{1\,m}$ and $w^{1\,m}$), one vector $A^m_{L\mu}$ (with momentum $k_{2\,\mu}$, and charges $p_{2\,m}$ and $w^{2\,m}$) and one antisymmetric scalar b_{mn} (with momentum $k_{3\,\mu}$, and charges $p_{3\,m}$ and $w^{3\,m}$).

In the DFT action there is only one place where the interaction vertex can be found, namely

$$-\frac{1}{2\kappa_d^2}\partial_M b_{\mu\nu}\partial_\rho A_{N\sigma} M^{MN} g^{\mu\rho} g^{\nu\sigma}\,. \tag{5.24}$$

Splitting the double internal indices, in order to exhibit the explicit contributions of b_{mn} and h_{mn} scalars, one can collect the required interactions and compute the three point amplitude. The result is

$$\frac{1}{2\kappa_d^2}\epsilon_{\mu\nu}(k_1)\epsilon_{L\mu m}(k_2)G^{nm}b_{nk}(k_3)\left[k_{2\nu}w^{1k} - G^{ks}(k_3)k_{2\nu}p_{1s} + B_{sl}G^{sk}(k_3)k_{2\nu}w^{1l}\right]\,,$$

where $\epsilon_{\mu\nu}$, $\epsilon_{L\mu m}$ and b_{nk} are the polarizations of the two-form, the left vector and the scalar, respectively. The same result is obtained in string theory if we choose $\frac{1}{2\kappa_d^2} = \pi g_c$.

[14]The program is available upon request to the authors.

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6 Conclusions and outlook

Double Field Theory was originally motivated by toroidal compactifications and a double set of coordinates was proposed as conjugate variables of compact momenta and windings. However, a specific realization of momentum and winding modes, which generically requires dealing with massive states, was lacking.

In this work we have dealt with massless and massive states of DFT compactifications on generic double tori (in presence of constant background fields) and compared them with a slice of the massless and massive states of bosonic string theory compactified on a torus. The slice considered corresponds to states with excitation numbers $N = \bar{N} = 1$, namely, a subsector of the bosonic string arising from states containing one left and one right moving oscillators.

We found complete agreement between the spectra of both DFT and string theory when a level matching constraint is imposed on the DFT side. Moreover, by expanding the generalized fields of DFT at first order in fluctuations around the constant background, the resulting third order action agrees with the effective action arising from three-point scattering amplitudes in string theory. For n dimensional tori and d space-time dimensions the obtained action corresponds to a gauge theory with $G_n = \mathrm{U}(1)^n \times \mathrm{U}(1)^n$ Abelian gauge group coupled to gravity. The computations involve both KK and winding modes, named here GKK modes, and therefore the action contains an infinite number of charged massive fields.

It is worth emphasizing that DFT provides a concise and manifestly $O(n,n)$ realization of this effective string theory action. Moreover, on a $2n$-dimensional double torus background, the global $O(n,n,\mathbb{R})$ symmetry of DFT is broken to $O(n,n,\mathbb{Z})$, the discrete T-duality group of the full string theory.

As is well known, physical states in string theory are selected by ensuring cancellation of conformal anomalies in the world sheet. We found that the DFT manifestation of these requirements is the invariance under generalized diffeomorphisms. By using such invariance, we have shown that a generalized harmonic gauge condition can be chosen, and established a correspondence with conditions derived from string theory. Interestingly enough, this gauge choice allows to identify the different Goldstone modes that are absorbed to generate physical fields. Besides the gravity multiplet and massless vectors associated to the compactified gravitational and antisymmetric fields, physical massive fields correspond to massive symmetric and antisymmetric tensors, vectors and scalars charged under the G_n gauge group. The charges, corresponding to momentum and winding numbers, are simply encoded in the generalized DFT momenta $\mathbb{P}$. Generalizing known results in KK compactifications, we found the infinite global symmetry algebra associated to infinite local generalized parameters. In particular, it contains a finite Poincaré $\times \mathrm{SO}(1,2)^n \times \mathrm{SO}(1,2)^n$ subalgebra and massive states should organize in its (infinite dimensional) representations.

Of course the effective action reproducing the three-point amplitudes of these physical massless and massive string states is not a low energy effective action since all possible massive levels are involved. The action provides an organized truncation of string theory. However this truncation is incomplete since it contains states with masses of the order or higher than those of string states with N and/or $\bar{N} \neq 1$ that were not included here. Indeed, we know from string theory that new fields involving higher spins (associated with N and/or $\bar{N} \neq 1$) appear in the spectrum and play a crucial role in higher point-amplitudes. In DFT language, higher order $O(n,n)$ generalized tensors, incorporating these missing string degrees of freedom, are expected.

We also know that a gauge symmetry enhancing, associated to the presence of windings, occurs in string theory at self dual points. This enhancing involves states with $N - \bar{N} \neq 0$ (e.g. $N, \bar{N} = 0, \pm 1$) and for this reason it cannot be seen in our construction. In [20], a DFT description of gauge enhancing in circle compactification at self dual radius R_0 was provided. There, it is shown that enhancing from $\mathrm{U}(1) \times \mathrm{U}(1)$ to $\mathrm{SU}(2) \times \mathrm{SU}(2)$ requires a dependence of the fields on the internal coordinates $y, \tilde{y}$ associated to a double circle, as we indeed have here. But it also requires an extension of the tangent space, leading to an $O(d+1+2, d+1+2)$ structure, that accommodates the extra massless vector fields associated to winding modes. The computation was performed at $R = \tilde{R} = R_0$ by keeping only massless states, and it could be extended to $R - \tilde{R} = R_0 \epsilon$ by keeping small masses. If we tried to generalize in this direction the procedure described in the previous sections, namely by including states with $N, \bar{N} = 0, \pm 1$ and keeping GKK massive modes, we would immediately run into trouble. Since the gauge group is enhanced, now the massive states (massive gravitons, two-forms, vectors and scalars) must transform under $\mathrm{SU}(2) \times \mathrm{SU}(2)$. However, there are not enough states, for a given mass, to fill up these representations. This is again an indication that new fields are needed. Actually, a string theory analysis, for instance by considering the OPE of $\mathrm{SU}(2)$ currents with massive gravitons (with $N = \bar{N} = 1$), shows that for masses $M^2 = 2m\alpha'$, gravitons organize into $(2,2), (3,3), \ldots (m+1, m+1)$ representations. In order to fill up these representations, higher spin fields are required, which are not contained in the present version of DFT. Again, the presence of higher order tensors is claimed for, now from gauge invariance.

Massive particles with spin larger than 2 would also be needed if higher powers of momentum were considered. Actually, the three-point functions presented in the appendix contain higher powers of momentum that we have not included since they go beyond the aim of this paper. However, these higher order terms lead to higher derivative contributions to the effective action which would of course be necessary if quantum corrections were considered. In particular, the inclusion of higher order terms in curvature invariants is known to demand the addition of massive tensors in order to fix the short-distance violations of causality [41], and the Regge behavior required for the resolution of the causality problem [42] also calls for higher order tensors in DFT.

Certainly, the effective theory we have constructed does not work as a fundamental theory. Nevertheless, despite the absence of essential ingredients for full consistency, it might be appealing by itself. It encodes an effective gauge invariant theory with a massless sector containing gravity, antisymmetric tensor plus gauge bosons and scalars coupled to towers of GKK massive modes. It is interesting to notice that, even if a given field has a zero mode, it spreads out into towers of momenta and windings. The simplest case of a non-zero graviton mass is an interesting theoretical possibility since it was not until recently that a consistent non-linear theory of massive gravity could be constructed [43].

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Even in this simple toroidal scenario it could be interesting to look at possible phenomenological consequences and to explore them in more detail. This aspect is beyond the scope of the present work but let us signal some new features that could be worth exploring. Many scenarios including KK excitations have been proposed in the literature for different physical models. These proposals deserve being reconsidered in this GKK scenario including windings as well as other fields. On the one hand new fields, associated to antisymmetric tensor and dilaton, can be present. Also a new energy scale is built in. In fact, even at the circle level two different energy scales $\lambda_{\rm KK} = 1/R$ and $\lambda_{\rm windings} = 1/\tilde{R}$ appear now which can lead to relevant physical consequences.

For instance, the type of models proposed in [39] in the large extra dimensions scenario of [44, 45] appear to be drastically modified. There, toroidal bulk KK gravity modes were coupled to Standard Model fields with radii $\lambda_{\rm KK} \lesssim M_{\rm string} \sim TeV$. However now, besides the fact that other fields are present, the $\lambda_{\rm windings}$ energy scale will also be present. Leaving aside stringy gauge symmetry enhancing, $R = \tilde{R}$ self dual point situations, where both windings and KK modes contribute on the same footing, are also possible.

KK universal scenarios for dark matter [46] have been extensively discussed. The consistent incorporation of massive antisymmetric tensors coupled to Einstein gravity plus other massless and massive fields could be also appealing in this context (see for example [47–49]). More complex situations, that would require generalizations of this simpler toroidal case, provide attractive candidates for dark matter [50, 51]. Phenomenology of massive KK gravitons at the LHC was recently discussed in [52], composite Higgs models associated to bulk KK modes have been considered in [53], etc.

The ideas developed here could in principle be extended to GKK reductions in which the starting theory has non Abelian gauge fields already in higher dimensions (e.g. the heterotic string). These are just plausible roads of research that call for careful study.

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A Extra terms in the DFT action

In the original frame formulation of DFT by Siegel [1, 2]. the action contains extra terms that are not contained in (2.11). Up to total derivatives those can be recast as [18]

$$\Delta S = \int d^{2D}X\, e^{-2d} \left[\frac{1}{2}(S_{\bar{A}\bar{B}} - \eta_{\bar{A}\bar{B}})\eta^{PQ}\, \partial_M E^{\bar{A}}{}_P \partial^M E^{\bar{B}}{}_Q + 4\partial_M d \partial^M d - 4\partial_M \partial^M d\right]. \quad \text{(A.1)}$$

Here we show that these terms vanish once the level matching condition (2.15) is imposed.

To show the vanishing of the term proportional to $S_{\bar{A}\bar{B}}$ we consider the following integral

$$I_1 = \int d^{2D}X\, \partial^M \partial_M \left(e^{-2d}\eta^{PQ}\mathcal{H}_{PQ}\right) = \int d^{2D}X\, \partial^M \partial_M \left(e^{-2d}\eta^{PQ} S_{\bar{A}\bar{B}} E^{\bar{A}}{}_P E^{\bar{B}}{}_Q\right) = 0\,. \tag{A.2}$$

A little of algebra, making use of the property $\mathcal{H}_{PQ}\eta^{PQ} = 0$, shows that

$$I_1 = 2\int d^{2D}X\, e^{-2d}\eta^{PQ} S_{\bar{A}\bar{B}}\left(\partial_M E^{\bar{A}}{}_P \partial^M E^{\bar{B}}{}_Q + E^{\bar{A}}{}_P \partial_M \partial^M E^{\bar{B}}{}_Q\right)\,. \tag{A.3}$$

Similarly, for the term in (A.1) proportional to $\eta_{\bar{A}\bar{B}}$ we consider the integral

$$I_2 = \int d^{2D}X\, \partial^M \partial_M \left(e^{-2d}\eta^{PQ}\eta_{PQ}\right) = \int d^{2D}X\, \partial^M \partial_M \left(e^{-2d}\eta^{PQ}\eta_{\bar{A}\bar{B}} E^{\bar{A}}{}_P E^{\bar{B}}{}_Q\right) = 0\,, \tag{A.4}$$

that can be recast as

$$I_2 = 2\int d^{2D}X\, e^{-2d}\eta^{PQ}\left(-\eta_{PQ}\partial_M\partial^M d + \eta_{\bar{A}\bar{B}}\partial_M E^{\bar{A}}{}_P \partial^M E^{\bar{B}}{}_Q + \eta_{\bar{A}\bar{B}} E^{\bar{A}}{}_P \partial_M \partial^M E^{\bar{B}}{}_Q\right)\,. \tag{A.5}$$

Finally, for the term proportional to $\partial_M d \partial^M d$, we consider the integral

$$I_3 = \int d^{2D}X\, \partial^M \partial_M e^{-2d} = 2\int d^{2D}X\, e^{-2d}\left(2\partial_M d \partial^M d - \partial_M \partial^M d\right) = 0\,. \tag{A.6}$$

From I_1, I_2 and I_3, we can therefore express ΔS as

$$\Delta S = \int d^{2D}X\, e^{-2d}\left[-\frac{1}{2}(S_{\bar{A}\bar{B}} - \eta_{\bar{A}\bar{B}})\eta^{PQ}\, E^{\bar{A}}{}_P \partial_M \partial^M E^{\bar{B}}{}_Q - (2+D)\partial_M\partial^M d\right]\,. \tag{A.7}$$

And therefore, transforming into momentum space and imposing the level-matching condition (2.15), we get $\Delta S = 0$.

B String computations

The results of three-point scattering amplitudes in bosonic string theory are presented here for the case of one compact dimension on a circle of radius R. They are computed with the vertex operators defined in (5.1). We first collect the amplitudes involving only massless states and then the ones containing at least one massive state. We use a shorthand notation with $h, b, \phi, A, \tilde{A}$ denoting graviton, antisymmetric tensor, scalar and vector fields. Recall that no massive vectors or scalars appear in the circle compactification and the massive fields are only h and b. Dots indicate contractions with Minkowski space-time metric $\eta_{\mu\nu}$.

3-point amplitudes for massless states

$$\begin{aligned}
\langle \Phi\Phi h\rangle &= -(\pi g_c)\frac{1}{2}\Phi\Phi(k_1\cdot\epsilon^h\cdot k_2) \qquad\qquad (B.1)\\
\langle hhh\rangle &= -(\pi g_c)\frac{1}{2}\Big((k_2\cdot\epsilon_1^h\cdot\epsilon_3^h\cdot\epsilon_2^h\cdot k_3) + (k_3\cdot\epsilon_1^h\cdot\epsilon_2^h\cdot\epsilon_3^h\cdot k_2) + (k_3\cdot\epsilon_2^h\cdot\epsilon_1^h\cdot\epsilon_3^h\cdot k_1)\\
&\quad - \frac{1}{2}(k_3\cdot\epsilon_1^h\cdot k_2)Tr(\epsilon_2^h\epsilon_3^h) - \frac{1}{2}(k_3\cdot\epsilon_2^h\cdot k_1)Tr(\epsilon_1^h\epsilon_3^h) - \frac{1}{2}(k_1\cdot\epsilon_3\cdot k_2)Tr(\epsilon_1^h\epsilon_2^h)\Big)
\end{aligned}$$

$$
\begin{aligned}
\langle AA\Phi \rangle &= (\pi g_c)\Phi(k_2\cdot\epsilon_1)(k_1\cdot\epsilon_2)\\
\langle \tilde{A}\tilde{A}\Phi \rangle &= -(\pi g_c)\Phi(k_2\cdot\epsilon_1)(k_1\cdot\epsilon_2)\\
\langle AAh \rangle &= (\pi g_c)\left((\epsilon_1\cdot\epsilon_2)(k_1\cdot\epsilon^h\cdot k_1)+(k_1\cdot\epsilon^h\cdot\epsilon_2)(\epsilon_1\cdot k_2)+(k_2\cdot\epsilon^h\cdot\epsilon_1)(\epsilon_2\cdot k_1)\right)\\
\langle A\tilde{A}b \rangle &= (\pi g_c)\left((k_1\cdot\epsilon^b\cdot\epsilon_2)(\epsilon_1\cdot k_2)+(k_2\cdot\epsilon^b\cdot\epsilon_1)(\epsilon_2\cdot k_1)\right)\\
\langle bbh \rangle &= (\pi g_c)\frac{1}{2}\left(\frac{1}{2}Tr(\epsilon_1^b\cdot\epsilon_2^b)(k_1\cdot\epsilon_3^h\cdot k_2)+(k_1\cdot\epsilon_2^b\cdot\epsilon_1^b\cdot\epsilon_3^h\cdot k_1)+(k_2\cdot\epsilon_1^b\cdot\epsilon_3^h\cdot\epsilon_2^b\cdot k_3)\right)
\end{aligned}
$$

3-point amplitudes with at least one massive state

$$
\begin{aligned}
\langle hhA \rangle &= (\pi g_c)\frac{p_1}{R}\left((k_1\cdot\epsilon_3)Tr(\epsilon_1^h\cdot\epsilon_2^h)+(\epsilon_3\cdot\epsilon_2^h\cdot\epsilon_1^h\cdot k_2)-(\epsilon_3\cdot\epsilon_1^h\cdot\epsilon_2^h\cdot k_1)\right)\\
\langle hh\tilde{A} \rangle &= (\pi g_c)\frac{\tilde{p}_1}{\tilde{R}}\left((k_1\cdot\epsilon_3)Tr(\epsilon_1^h\cdot\epsilon_2^h)+(\epsilon_3\cdot\epsilon_2^h\cdot\epsilon_1^h\cdot k_2)-(\epsilon_3\cdot\epsilon_1^h\cdot\epsilon_2^h\cdot k_1)\right)\\
\langle hh\Phi \rangle &= (\pi g_c)\frac{1}{2}\Phi Tr(\epsilon_1^h\cdot\epsilon_2^h)k_{1L}k_{1R}\\
\langle bb\Phi \rangle &= -(\pi g_c)\frac{1}{2}\Phi Tr(\epsilon_1^b\cdot\epsilon_2^b)k_{1L}k_{1R}\\
\langle bbA \rangle &= -(\pi g_c)\frac{p_1}{R}\left((\epsilon_3\cdot k_1)Tr(\epsilon_1^b\cdot\epsilon_2^b)+(k_2\cdot\epsilon_1^b\cdot\epsilon_2^b\cdot\epsilon_3)-(k_1\cdot\epsilon_2^b\cdot\epsilon_1^b\cdot\epsilon_3)\right)\\
\langle bb\tilde{A} \rangle &= -(\pi g_c)\frac{\tilde{p}_1}{\tilde{R}}\left((\epsilon_3\cdot k_1)Tr(\epsilon_1^b\cdot\epsilon_2^b)+(k_2\cdot\epsilon_1^b\cdot\epsilon_2^b\cdot\epsilon_3)-(k_1\cdot\epsilon_2^b\cdot\epsilon_1^b\cdot\epsilon_3)\right)\\
\langle hbA \rangle &= (\pi g_c)\left(\frac{p_1}{R}(k_2\cdot\epsilon_1^h\cdot\epsilon_2^b\cdot\epsilon_3)+\frac{p_2}{R}(\epsilon_3\cdot\epsilon_1^h\cdot\epsilon_2^b\cdot k_1)\right)\\
\langle hb\tilde{A} \rangle &= (\pi g_c)\left(\frac{\tilde{p}_1}{\tilde{R}}(k_2\cdot\epsilon_1^h\cdot\epsilon_2^b\cdot\epsilon_3)+\frac{\tilde{p}_2}{\tilde{R}}(\epsilon_3\cdot\epsilon_1^h\cdot\epsilon_2^b\cdot k_1)\right)
\end{aligned} \tag{B.2}
$$

where $k_L = \frac{p}{R}+\frac{\tilde{p}}{\tilde{R}}$ and $k_R = \frac{p}{R}-\frac{\tilde{p}}{\tilde{R}}$.

C Algebra of diffeomorphisms

Following the discussion in [33–37], we can associate a global infinite parameter algebra to the infinite modes $\xi^{\mathcal{P}(\mathbb{M})}(x)$ of the GKK expansion of the parameters of local transformations, in much the same way as a global Poincaré algebra is associated to general coordinate transformations. From

$$
\xi^{\mathcal{P}}(x,\mathbb{Y}) = \sum_{\mathbb{M}}{}'\xi^{\mathcal{P}(\mathbb{M})}(x)e^{i\mathbb{M}.\mathbb{Y}}\,, \tag{C.1}
$$

with $\mathcal{P}=(\rho,L)$, we restrict to

$$
\xi^{\rho(\mathbb{M})}(x) = a^{\rho(\mathbb{M})} + \omega^{(\mathbb{M})\rho}{}_{\nu}x^{\nu}\,, \tag{C.2}
$$

$$
\xi^{L(\mathbb{M})}(x) = C^{L(\mathbb{M})}\,, \tag{C.3}
$$

where $a^{\rho(\mathbb{M})}, \omega^{(\mathbb{M})\rho}{}_{\nu}, C^{L(\mathbb{M})}$ are constants. The corresponding generators are

$$
\hat{P}_{\rho}^{(\mathbb{M})} = ie^{i\mathbb{M}\cdot\mathbb{Y}}\partial_{\rho}\,, \tag{C.4}
$$

$$
\hat{M}_{\mu\nu}^{(\mathbb{M})} = e^{i\mathbb{M}\cdot\mathbb{Y}}(x_{\mu}\partial_{\nu}-x_{\nu}\partial_{\mu})\,, \tag{C.5}
$$

$$
\hat{Q}_{L}^{(\mathbb{M})} = ie^{i\mathbb{M}\cdot\mathbb{Y}}\partial_{L}\,. \tag{C.6}
$$

It is easy to check that these operators generate an algebra that corresponds to the direct generalization of the algebra found in [33]. Namely,

$$\begin{aligned}
\left[\hat{M}^{(\mathbb{M})}_{\mu\nu}, \hat{M}^{(\mathbb{N})}_{\rho\sigma}\right] &= i\left[\eta_{\nu\rho}\hat{M}^{(\mathbb{M}+\mathbb{N})}_{\mu\sigma} + \eta_{\mu\sigma}\hat{M}^{(\mathbb{M}+\mathbb{N})}_{\nu\rho} - \eta_{\mu\rho}\hat{M}^{(\mathbb{M}+\mathbb{N})}_{\nu\sigma} - \eta_{\nu\sigma}\hat{M}^{(\mathbb{M}+\mathbb{N})}_{\mu\rho}\right] \\
\left[\hat{M}^{(\mathbb{M})}_{\mu\nu}, P^{(\mathbb{N})}_{\lambda}\right] &= i\left[\eta_{\lambda\nu}P^{(\mathbb{M}+\mathbb{N})}_{\mu} - \eta_{\lambda\mu}P^{(\mathbb{M}+\mathbb{N})}_{\nu}\right] \\
\left[P^{(\mathbb{M})}_{\rho}, P^{(\mathbb{N})}_{\mu}\right] &= 0 \\
\left[Q^{(\mathbb{M})}_{L}, \hat{M}^{(\mathbb{N})}_{\mu\nu}\right] &= -\mathbb{N}_L M^{(\mathbb{M}+\mathbb{N})}_{\mu\nu} \\
\left[Q^{(\mathbb{M})}_{L}, P^{(\mathbb{N})}_{\mu}\right] &= -\mathbb{N}_L P^{(\mathbb{M}+\mathbb{N})}_{\mu} \\
\left[Q^{(\mathbb{M})}_{L}, Q^{(\mathbb{N})}_{S}\right] &= -\mathbb{N}_L Q^{(\mathbb{M}+\mathbb{N})}_{S} + \mathbb{M}_S Q^{(\mathbb{M}+\mathbb{N})}_{L}
\end{aligned} \tag{C.7}$$

We see that the zero modes lead to the d dimensional Poincaré algebra. Also, from the last equation we notice that, for $L = S$

$$\left[Q^{(\mathbb{M})}_{L}, Q^{(\mathbb{N})}_{L}\right] = (\mathbb{M}_L - \mathbb{N}_L)Q^{(\mathbb{M}+\mathbb{N})}_{L}\,, \tag{C.8}$$

which is a Virasoro algebra (with no central charge) for each value of $L = 1, \dots 2n$.

For the case of the circle we would have $\mathbb{M} = (m_1, m_2) = (m, \tilde{m})$ with $m = 0$ or $\tilde{m} = 0$ due to LMC.

Notice that if we choose $\mathbb{M} = (m, 0)$ and $\mathbb{N} = (n, 0)$ with $m, n = \pm 1, 0$. Then $\hat{Q}^{(\mathbb{M})}_1 \equiv \hat{Q}^{(\pm 1)}_1, Q^{(0)}_1$ and $\hat{P}^{(0)}_\mu, \hat{M}^{(0)}_{\mu\nu}, \hat{Q}^{(0)}_2$ close a Poincaré$\otimes$SO$(1,2)$ algebra. In the same way, exchanging $1 \leftrightarrow 2$, namely, windings with momenta, another SO$(1,2)$ algebra is obtained. Thus, finally the original Poincaré algebra is enlarged to Poincaré $\otimes$SO$(1,2)^2$. It was shown in [38] that, in the circle case in field theory, the massive KK states organize into an infinite dimensional (non-unitary) R representation of SO$(1,2)$. In DFT on the circle, windings and momenta are decoupled, so massive KK momenta states will fill up the infinite dimensional representation of the first algebra whereas windings will organize in a similar representation of the second one, namely $(R, 1) + (1, R)$.

In the generic case we can proceed in the same way by choosing the GKK momenta at the position L, $\mathbb{M}_L = 0, \pm 1$ with all other components vanishing. In this case we would have Poincaré $\otimes$SO$(1,2)^{2n}$. Since massive states with $M^2 = \mathbb{P}\cdot\mathcal{M}\cdot\mathbb{P}$ mix windings and momenta the analysis of representations is more involved and we will not perform it in the present work.

Even if the above algebra is a symmetry of the original Lagrangian, it is broken to Poincaré $\times$U$(1)^n \times$ U$(1)^n$ by the vacuum (4.3). This can be easily verified by inserting the mode expansions (C.1) to compute the transformations of the fields $g_{\mu\nu}, A^M_\mu, b_{\mu\nu}, \mathcal{H}_{MN}$ and by requiring the vacuum (4.3) to be invariant under these transformations. $\xi^{\mathcal{P}(\mathbb{M})}$, with $\mathbb{M} \neq 0$ correspond to broken generators associated to Goldstone bosons.

References

[1] W. Siegel, *Two vierbein formalism for string inspired axionic gravity*, Phys. Rev. **D 47** (1993) 5453 [hep-th/9302036] [INSPIRE].

[2] W. Siegel, *Superspace duality in low-energy superstrings*, Phys. Rev. **D 48** (1993) 2826 [hep-th/9305073] [INSPIRE].

[3] C. Hull and B. Zwiebach, *Double Field Theory*, JHEP **09** (2009) 099 [arXiv:0904.4664] [INSPIRE].

[4] O. Hohm, C. Hull and B. Zwiebach, *Generalized metric formulation of double field theory*, JHEP **08** (2010) 008 [arXiv:1006.4823] [INSPIRE].

[5] G. Aldazabal, D. Marques and C. Núñez, *Double Field Theory: A Pedagogical Review*, Class. Quant. Grav. **30** (2013) 163001 [arXiv:1305.1907] [INSPIRE].

[6] O. Hohm, D. Lüst and B. Zwiebach, *The Spacetime of Double Field Theory: Review, Remarks and Outlook*, Fortsch. Phys. **61** (2013) 926 [arXiv:1309.2977] [INSPIRE].

[7] D.S. Berman and D.C. Thompson, *Duality Symmetric String and M-theory*, Phys. Rept. **566** (2014) 1 [arXiv:1306.2643] [INSPIRE].

[8] N.J. Hitchin, *The Geometry of Three-Forms in Six Dimensions*, J. Diff. Geom. **55** (2000) 547 [math/0010054] [INSPIRE].

[9] N.J. Hitchin, *Stable forms and special metrics*, in *Global Differential Geometry: The Mathematical Legacy of Alfred Gray*, M.Fernandez and J.A.Wolf eds., Contemporary Mathematics **288**, American Mathematical Society, Providence U.S.A. (2001), math.DG/0107101 [INSPIRE].

[10] N. Hitchin, *Generalized Calabi-Yau manifolds*, Quart. J. Math. **54** (2003) 281 [math/0209099] [INSPIRE].

[11] M. Gualtieri, *Generalized complex geometry*, math/0401221 [INSPIRE].

[12] O. Hohm, W. Siegel and B. Zwiebach, *Doubled α'-geometry*, JHEP **02** (2014) 065 [arXiv:1306.2970] [INSPIRE].

[13] O.A. Bedoya, D. Marques and C. Núñez, *Heterotic α'-corrections in Double Field Theory*, JHEP **12** (2014) 074 [arXiv:1407.0365] [INSPIRE].

[14] D. Marques and C.A. Núñez, *T-duality and α'-corrections*, JHEP **10** (2015) 084 [arXiv:1507.00652] [INSPIRE].

[15] A. Coimbra, R. Minasian, H. Triendl and D. Waldram, *Generalised geometry for string corrections*, JHEP **11** (2014) 160 [arXiv:1407.7542] [INSPIRE].

[16] G. Aldazabal, W. Baron, D. Marques and C. Núñez, *The effective action of Double Field Theory*, JHEP **11** (2011) 052 [*Erratum ibid.* **11** (2011) 109] [arXiv:1109.0290] [INSPIRE].

[17] D. Geissbuhler, *Double Field Theory and $N = 4$ Gauged Supergravity*, JHEP **11** (2011) 116 [arXiv:1109.4280] [INSPIRE].

[18] D. Geissbuhler, D. Marques, C. Núñez and V. Penas, *Exploring Double Field Theory*, JHEP **06** (2013) 101 [arXiv:1304.1472] [INSPIRE].

[19] G. Dibitetto, J.J. Fernandez-Melgarejo, D. Marques and D. Roest, *Duality orbits of non-geometric fluxes*, Fortsch. Phys. **60** (2012) 1123 [arXiv:1203.6562] [INSPIRE].

JHEP03(2017)096

[20] G. Aldazabal, M. Graña, S. Iguri, M. Mayo, C. Nuñez and J.A. Rosabal, *Enhanced gauge symmetry and winding modes in Double Field Theory*, JHEP **03** (2016) 093 [arXiv:1510.07644] [INSPIRE].

[21] O. Hohm and H. Samtleben, *Gauge theory of Kaluza-Klein and winding modes*, Phys. Rev. **D 88** (2013) 085005 [arXiv:1307.0039] [INSPIRE].

[22] O. Hohm and D. Marques, *Perturbative Double Field Theory on General Backgrounds*, Phys. Rev. **D 93** (2016) 025032 [arXiv:1512.02658] [INSPIRE].

[23] K. Peeters, *A field-theory motivated approach to symbolic computer algebra*, Comput. Phys. Commun. **176** (2007) 550 [cs/0608005] [INSPIRE].

[24] A.S. Arvanitakis and C.D.A. Blair, *Black hole thermodynamics, stringy dualities and double field theory*, Class. Quant. Grav. **34** (2017) 055001 [arXiv:1608.04734] [INSPIRE].

[25] M. Graña and D. Marques, *Gauged Double Field Theory*, JHEP **04** (2012) 020 [arXiv:1201.2924] [INSPIRE].

[26] O. Hohm and B. Zwiebach, *On the Riemann Tensor in Double Field Theory*, JHEP **05** (2012) 126 [arXiv:1112.5296] [INSPIRE].

[27] H. Kawai, D.C. Lewellen and S.H.H. Tye, *A Relation Between Tree Amplitudes of Closed and Open Strings*, Nucl. Phys. **B 269** (1986) 1 [INSPIRE].

[28] Z. Bern and A.K. Grant, *Perturbative gravity from QCD amplitudes*, Phys. Lett. **B 457** (1999) 23 [hep-th/9904026] [INSPIRE].

[29] A.E.M. van de Ven, *Two loop quantum gravity*, Nucl. Phys. **B 378** (1992) 309 [INSPIRE].

[30] T. De Donder, *La gravique Einsteinienne*, Gauthier-Villars, Paris, France (1921).

[31] M. Fierz and W. Pauli, *On relativistic wave equations for particles of arbitrary spin in an electromagnetic field*, Proc. Roy. Soc. Lond. **A 173** (1939) 211 [INSPIRE].

[32] T. Ortin, *Gravity and Strings*, *Cambridge Monographs on Mathematical Physics*, Cambridge University Press (2015), ISBN: 9780521768139.

[33] L. Dolan, *Massive Kaluza-Klein Theories and Bound States in Yang-Mills*, RU84/B/96, DOE-ER-40033B-56, DOE/ER/40033B-63, RU83-B-75.

[34] D. Bailin and A. Love, *Kaluza-Klein Theories*, Rept. Prog. Phys. **50** (1987) 1087 [INSPIRE].

[35] L. Dolan, *Symmetries of Massive Fields in Kaluza-Klein Supergravity*, Phys. Rev. **D 30** (1984) 2474 [INSPIRE].

[36] L. Dolan and M.J. Duff, *Kac-Moody Symmetries of Kaluza-Klein Theories*, Phys. Rev. Lett. **52** (1984) 14 [INSPIRE].

[37] Y.M. Cho and S.W. Zoh, *Explicit construction of massive spin two fields in Kaluza-Klein theory*, Phys. Rev. **D 46** (1992) R2290 [INSPIRE].

[38] A. Salam and J.A. Strathdee, *On Kaluza-Klein Theory*, Annals Phys. **141** (1982) 316 [INSPIRE].

[39] T. Han, J.D. Lykken and R.-J. Zhang, *On Kaluza-Klein states from large extra dimensions*, Phys. Rev. **D 59** (1999) 105006 [hep-ph/9811350] [INSPIRE].

[40] G.F. Giudice, R. Rattazzi and J.D. Wells, *Quantum gravity and extra dimensions at high-energy colliders*, Nucl. Phys. **B 544** (1999) 3 [hep-ph/9811291] [INSPIRE].

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[41] X.O. Camanho, J.D. Edelstein, J. Maldacena and A. Zhiboedov, *Causality Constraints on Corrections to the Graviton Three-Point Coupling*, *JHEP* **02** (2016) 020 [arXiv:1407.5597] [INSPIRE].

[42] G. D'Appollonio, P. Di Vecchia, R. Russo and G. Veneziano, *Regge behavior saves String Theory from causality violations*, *JHEP* **05** (2015) 144 [arXiv:1502.01254] [INSPIRE].

[43] C. de Rham, G. Gabadadze and A.J. Tolley, *Ghost free Massive Gravity in the Stúckelberg language*, *Phys. Lett.* **B 711** (2012) 190 [arXiv:1107.3820] [INSPIRE].

[44] I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos and G.R. Dvali, *New dimensions at a millimeter to a Fermi and superstrings at a TeV*, *Phys. Lett.* **B 436** (1998) 257 [hep-ph/9804398] [INSPIRE].

[45] N. Arkani-Hamed, S. Dimopoulos and G.R. Dvali, *The hierarchy problem and new dimensions at a millimeter*, *Phys. Lett.* **B 429** (1998) 263 [hep-ph/9803315] [INSPIRE].

[46] G. Servant and T.M.P. Tait, *Is the lightest Kaluza-Klein particle a viable dark matter candidate?*, *Nucl. Phys.* **B 650** (2003) 391 [hep-ph/0206071] [INSPIRE].

[47] T. Prokopec and W. Valkenburg, *Antisymmetric Metric Field as Dark Matter*, astro-ph/0606315 [INSPIRE].

[48] T. Prokopec and W. Valkenburg, *The cosmology of the nonsymmetric theory of gravitation*, *Phys. Lett.* **B 636** (2006) 1 [astro-ph/0503289] [INSPIRE].

[49] J.W. Moffat, *Gravitational theory, galaxy rotation curves and cosmology without dark matter*, *JCAP* **05** (2005) 003 [astro-ph/0412195] [INSPIRE].

[50] K. Agashe and G. Servant, *Warped unification, proton stability and dark matter*, *Phys. Rev. Lett.* **93** (2004) 231805 [hep-ph/0403143] [INSPIRE].

[51] H.-C. Cheng, J.L. Feng and K.T. Matchev, *Kaluza-Klein dark matter*, *Phys. Rev. Lett.* **89** (2002) 211301 [hep-ph/0207125] [INSPIRE].

[52] E. Alvarez, L. Da Rold, J. Mazzitelli and A. Szynkman, *Graviton resonance phenomenology and a PNGB Higgs at the LHC*, arXiv:1610.08451 [INSPIRE].

[53] E. Alvarez, L. Da Rold and A. Szynkman, *A composite Higgs model analysis of forward-backward asymmetries in the production of tops at Tevatron and bottoms at LEP and SLC*, *JHEP* **05** (2011) 070 [arXiv:1011.6557] [INSPIRE].