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The covariant chiral ring

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Abstract: We construct a covariant generating function for the spectrum of chiral primaries of symmetric orbifold conformal field theories with $N = (4, 4)$ supersymmetry in two dimensions. For seed target spaces $K3$ and T^4, the generating functions capture the SO(21) and SO(5) representation theoretic content of the chiral ring respectively. Via string dualities, we relate the transformation properties of the chiral ring under these isometries of the moduli space to the Lorentz covariance of perturbative string partition functions in flat space.

Keywords: Conformal Field Models in String Theory, AdS-CFT Correspondence, String Duality

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1 Introduction

The AdS/CFT duality [1] between two-dimensional conformal field theories and gravitational theories in three-dimensional anti-de Sitter spacetime has been a powerful tool for studying the microscopic origin of black hole entropy [2, 3]. It is a testing ground for holography that is under good control, at both small and large curvature. For instance, protected three-point functions can be computed exactly as a function of α' over the radius of curvature in the bulk string theory [4, 5], and can be matched to the three-point functions of the boundary theory [6, 7]. Moreover, we can calculate all α' corrections to the central charge, without relying on supersymmetry [8].

The three-point functions provided a dynamical test of the holographic correspondence, albeit in a limited sense. In the case where the boundary theory exhibits an $N = (4, 4)$ supersymmetry, it has been proven that certain three-point functions satisfy a non-renormalisation theorem. They are covariantly constant on the moduli space of the superconformal field theory [9, 10]. Consequently, the vector bundle of chiral primaries over the symmetric space of moduli is homogeneous. The bundle is then determined completely by the representation of the holonomy group constituted by the fibres.

In the example of IIB superstring theory on $\mathrm{AdS}_3 \times S^3 \times M$, where M is either a $K3$ manifold or a four-torus T^4, the three-point functions have been computed on both sides of the duality, and successfully compared [4, 5]. In particular, these three-point functions have been calculated in the bulk at a point of moduli space where the theory is described by $\mathrm{AdS}_3 \times S^3 \times M$ with only a background NSNS flux, corresponding to N_1 fundamental strings and N_5 NS5-branes. On the boundary, the three-point functions have been computed at the symmetric orbifold point corresponding to the conformal field theory on $\mathrm{Sym}_{N_1 N_5}(M)$. The identification of the conformal field theory is inspired by the physics of the S-dual D1-D5 Higgs branch (but the identification remains subtle [11]). The comparison of three-point functions has been successfully carried out at leading order in $1/N$ where $N = N_1 N_5$.

In [9, 12], it was argued that the full covariance of the chiral ring of these models is SO(21) for $M = K3$ and SO(5) for $M = T^4$. Moreover, the $\mathrm{SO}(n)$ covariant representation content of the chiral ring was identified for operators with small R-charges for the case $M = K3$. In this paper, we continue this line of reasoning as follows. We briefly review the proposed covariance in section 2. Motivated by the importance of the representation theoretic content not only for a more refined classification of the spectrum, but also for the protected three-point functions, we wish to identify the SO(21) and SO(5) representations appearing in the chiral ring at all levels. To that end, we will construct a fully covariant generating function in section 3, both for the case of $M = K3$ and $M = T^4$.

2 The chiral ring

In this section, we review salient features of the background of type IIB string theory under study. We consider type IIB string theory compactified on $M = T^4$ or $M = K3$, and a string transverse to M, consisting of D1-strings and D5-branes wrapped on M. We will study the background in the vicinity of the string. The near-string geometry is $\mathrm{AdS}_3 \times S^3 \times M$. The bulk string theory is dual [1] to a two-dimensional $N = (4, 4)$ superconformal field theory on the boundary of level $N = N_1 N_5$. The moduli space of these theories is locally of the form

$$\frac{\mathrm{SO}(4, n)}{\mathrm{SO}(4) \times \mathrm{SO}(n)}, \tag{2.1}$$

where $n = 21$ and $n = 5$ for $K3$ and T^4 respectively [12]. It is generally assumed that there is a point in the moduli space where the conformal field theory can be described as a symmetric product conformal field theory $\mathrm{Sym}_N(M)$ where $N = N_1 N_5$.

The chiral ring of the theory with respect to a $N = (2, 2)$ superconformal subalgebra forms a vector bundle over the moduli space. The symmetry group of the bundle of chiral primaries is $\mathrm{SO}(4) \times \mathrm{SO}(n)$. The three-point functions are covariantly constant on the moduli space [9, 10], and the curvature of the bundle of chiral primaries is covariantly constant as well [9]. Thus the chiral primaries give rise to a homogeneous bundle on a

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symmetric space whose geometry is determined in terms of the representation of the fibre with respect to the structure group $\mathrm{SO}(4) \times \mathrm{SO}(n)$ and the connection of the tangent bundle of the symmetric space. The curvature of the tangent bundle can be computed in conformal field theory, and turns out to be trivial in the $\mathrm{SO}(4)$ factor of the structure group [9]. Thus, it is sufficient to characterize the $\mathrm{SO}(n)$ representations of the chiral primaries to fully characterize the connection of the chiral primary vector bundle on the moduli space [9].

Beyond the elementary observation that we obtain knowledge of the spectrum by classifying it with respect to the largest symmetry group of the problem, we believe that the fact that the enhanced symmetry also provides constraints on the three-point functions of the theory are good motivations for explicitly identifying the full representation theory content of the chiral ring. To that end, we will construct a fully covariant generating function.

3 The covariant generating function

In this section, we firstly review the generating function of the ring of left and right chiral primaries. In a $N = (4, 4)$ supersymmetric theory, the (c, c) ring with respect to a given $N = (2, 2)$ superconformal subalgebra of the $N = (4, 4)$ superconformal algebra is isomorphic to its (a, c), (c, a) and (a, a) anti-chiral counterparts. It is therefore sufficient to consider the (c, c) ring only.

3.1 The counting function

Our starting point is the generating function for the chiral ring of the superconformal field theory with target space M, which in a geometric regime can be identified as the Poincaré polynomial $P_{t,\bar{t}}$ of the manifold M. We can alternatively view it as the trace over the NSNS Hilbert space, with states weighted by their left and right $\mathrm{U}(1)_R$ charges $(2J_0, 2\bar{J}_0)$, or as the generating function for the Hodge numbers $h_{r,s}$:

$$\begin{aligned} P_{t,\bar{t}}(M) &= \mathrm{Tr}_{\mathrm{NSNS}}(t^{2J_0}\bar{t}^{2\bar{J}_0}) \\ &= \sum_{r,s=0}^{c/3} h_{r,s} t^r \bar{t}^s, \end{aligned} \tag{3.1}$$

where c is the central charge of the conformal field theory. Such a polynomial will be represented using the associated Hodge diamond in the following way:

$$\sum_{r,s=0}^{m} h_{r,s} t^r \bar{t}^s = \begin{array}{ccccccccc} & & & & h_{0,0} & & & & \\ & & & h_{1,0} & & h_{0,1} & & & \\ & & h_{2,0} & & h_{1,1} & & h_{0,2} & & \\ & \cdots & & \cdots & & \cdots & & \cdots & \\ h_{m,0} & & \cdots & & \cdots & & \cdots & & h_{0,m} \\ & \cdots & & \cdots & & \cdots & & \cdots & \\ & & h_{m,m-2} & & h_{m-1,m-1} & & h_{m-2,m} & & \\ & & & h_{m,m-1} & & h_{m-1,m} & & & \\ & & & & h_{m,m} & & & & \end{array} \tag{3.2}$$

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where m is the complex dimension of M, and we delete the coefficients $h_{r,s}$ where $r+s$ is odd when these coefficients identically vanish (as for instance in diamond (3.5)). Moreover we have the relations $h_{r,s} = h_{s,r}$ from complex conjugation, $h_{r,s} = h_{m-r,m-s}$ from Poincaré duality and $h_{r,s} = h_{m-r,s}$ if M is hyperkähler. In this case m is even and we then sometimes represent only the upper-left eighth of the diamond:

$$\sum_{r,s=0}^{m} h_{r,s} t^r \bar{t}^s = \begin{array}{ccccc} & & & & h_{0,0} \\ & & & h_{1,0} & \\ & & \cdots & & h_{1,1} \\ & \cdots & & \cdots & \\ h_{\frac{m}{2},0} & & \cdots & & \cdots \\ & \cdots & & \cdots & \\ & & \cdots & & h_{\frac{m}{2}-1,\frac{m}{2}-1} \\ & & & h_{\frac{m}{2},\frac{m}{2}-1} & \\ & & & & h_{\frac{m}{2},\frac{m}{2}} \end{array} \tag{3.3}$$

The generating function Z for the ring of the symmetric product theory can be written down in terms of the data of the underlying theory:

$$\begin{aligned} Z(M^N/S_N) &= \sum_{N\geq 0} q^N P_{t,\bar{t}}(M^N/S_N) \\ &= \prod_{m=1}^{+\infty} \prod_{r,s=0}^{c/3} \left(1+(-1)^{r+s+1} q^m t^{m+r-1} \bar{t}^{m+s-1}\right)^{(-1)^{r+s+1} h_{r,s}}. \end{aligned} \tag{3.4}$$

The formula has a physical interpretation in terms of a second quantized string theory, with twisted long string sectors [13]. The new variable q keeps track of the number of copies of the manifold M that are in play. Note that the R-charge is augmented by $m-1$ both on the left and the right, for each extra oscillator that comes in at order m. To make further progress in the analysis of the covariance of the generating function, we consider the two specific examples of $K3$ and T^4 separately, in sections 3.2 and 3.3.

3.2 Type IIB string theory on $\mathrm{AdS}_3 \times S^3 \times K3$

We first review properties of the chiral ring partition function of $K3$ and its second quantized interpretation. We then present the covariant generating function. Finally, we relate the latter to the Lorentz covariant partition function of the chiral, massive and bosonic half of the heterotic string.

3.2.1 The chiral ring partition function

The only input we need to render the chiral ring counting function Z in equation (3.4) explicit is the Hodge diamond of $M = K3$:

$$P_{t,\bar{t}}(K3) = \begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array} = \begin{array}{cc} & 1 \\ 1 & 20 \;\; 1 \\ & 1 \end{array} \tag{3.5}$$

Since the Hodge numbers are only non-zero for $r+s$ even, the second quantized string theory is bosonic (for one chirality) and the partition function consists of a multiplication of denominators:

$$Z(K3^N/S_N) = \sum_{N\geq 0} q^N P_{t,\bar{t}}(K3^N/S_N) = \prod_{m=1}^{+\infty} \frac{1}{\mathfrak{P}_m(q,t,\bar{t})}\,, \tag{3.6}$$

where the factors are determined by the entries of the Hodge diamond:

$$\mathfrak{P}_m(q,t,\bar{t}) = \begin{array}{ccc} & (1-q^m t^{m-1}\bar{t}^{m-1})\times & \\ (1-q^m t^{m+1}\bar{t}^{m-1}) & (1-q^m t^m \bar{t}^m)^{20} & (1-q^m t^{m-1}\bar{t}^{m+1})\times \\ & (1-q^m t^{m+1}\bar{t}^{m+1})\,. & \end{array}$$

Using this generating function, we can compute the number of chiral primaries as a function of R-charges, for any given N. As an elementary example, we quote the informative eighth of the diamond for the case of $N=8$:

$$\begin{array}{ccccc}
 & & & & 1 \\
 & & & 1 & 21 \\
 & & 1 & 22 & 254 \\
 & 1 & 22 & 276 & 2278 \\
1 & 22 & 277 & 2553 & 16744 \\
 & 277 & 2575 & 19274 & 106284 \\
 & & 19528 & 125006 & 599470 \\
 & & & 702926 & 2983928 \\
 & & & & 11251487
\end{array} \tag{3.7}$$

A given coefficient $h_{r,s}$ can only receive contributions from polynomials $\mathfrak{P}_m(q,t,\bar{t})$ with $m \leq r+s$. Thus, the entries of the table stabilise at $m=r+s$. As we increase the number of copies N of the seed manifold M, the entries will no longer change. In this table, we observe for instance that the Hodge number $h_{1,1}=20$ for one copy of $K3$ has been augmented to $h_{1,1}=21$ for any number of copies of $K3$ larger than one.

3.2.2 A second quantized perspective

A convenient way to represent the operators in the chiral ring is found in terms of the second quantized view on the generating function of the elliptic genus [13]. All the chiral primaries can be represented in terms of the set of operators of the original theory and the twisted sector ground states (labeled by m). More explicitly, we can define operators:

$$\alpha_{-m}^{(r,s),i} \tag{3.8}$$

which live in a conformal field theory corresponding to m copies of the original theory, and has left and right R-charges $(m+r-1, m+s-1)$. The label i represents the multiplicity of the (r,s) entry in the original Hodge diamond. For $M=K3$ we have that $i=1,2,\ldots,20$ for $(r,s)=(1,1)$, and the index can be dropped for other values of (r,s). For $K3$, all oscillators are bosonic. The generating function given above can then be interpreted as

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Name	Number	R-charges
$\alpha_{-n}^{(0,0)}$	1	$(n-1,n-1)$
$\alpha_{-n}^{(2,0)}$	1	$(n+1,n-1)$
$\alpha_{-n}^{(1,1),i}$	20	(n,n)
$\alpha_{-n}^{(0,2)}$	1	$(n-1,n+1)$
$\alpha_{-n}^{(2,2)}$	1	$(n+1,n+1)$

Table 1. The list of excitations of the second quantized chiral string.

R-charge	Number of states	States
$(0,0)$	1	$\lvert N\rangle \equiv (\alpha_{-1}^{(0,0)})^N \lvert 0\rangle$
$(2,0)$	1	$\alpha_{-1}^{(2,0)}\lvert N-1\rangle$
$(1,1)$	21	$\alpha_{-2}^{(0,0)}\lvert N-2\rangle$ and $\alpha_{-1}^{(1,1),i}\lvert N-1\rangle$
$(0,2)$	1	$\alpha_{-1}^{(0,2)}\lvert N-1\rangle$
$(4,0)$	1	$(\alpha_{-1}^{(2,0)})^2\lvert N-2\rangle$
$(3,1)$	22	$\alpha_{-1}^{(2,0)}\alpha_{-1}^{(1,1),i}\lvert N-2\rangle\,,\ \alpha_{-1}^{(2,0)}\alpha_{-2}^{(0,0)}\lvert N-3\rangle\,,\ \alpha_{-2}^{(2,0)}\lvert N-2\rangle$
$(2,2)$	254	$\alpha_{-1}^{(1,1),i}\alpha_{-2}^{(0,0)}\lvert N-3\rangle\,,\ \alpha_{-1}^{(1,1),i}\alpha_{-1}^{(1,1),j}\lvert N-2\rangle\,,$ $\alpha_{-3}^{(0,0)}\lvert N-3\rangle\,,\ \alpha_{-2}^{(1,1),i}\lvert N-2\rangle\,,$ $(\alpha_{-2}^{(0,0)})^2\lvert N-4\rangle\,,\ \alpha_{-1}^{(2,0)}\alpha_{-1}^{(0,2)}\lvert N-2\rangle\,,\ \alpha_{-1}^{(2,2)}\lvert N-1\rangle$

Table 2. The list of states with given (left, right) R-charges.

the partition function of a string theory with the following list of possible excitations in table 1. Let us make more explicit lists of low lying chiral primaries [9]. The number of copies of the original theory has to be equal to the total number of copies N. We take N to be sufficiently large for the Hodge numbers of the orbifold to have stabilized. If necessary, we can always add $\alpha_{-1}^{(0,0)}$ to find the total number N without changing any of the other charges of the operator. In accord with the Hodge diamond entries recorded before, we find the states listed in table 2. Our goal is to derive a generating function that organizes the chiral primaries which manifestly form representations of SO(20), into representations of SO(21). We wish to find a generating function at all levels, thus generalizing the analysis at low levels performed in [9].

3.2.3 The covariant generating function

There is a manifest SO(20) symmetry in the original K3 Hodge diamond (3.5) that rotates the vector of $(1,1)$ harmonic forms, and otherwise acts trivially. This SO(20) action is inherited by the symmetric product theory. To render the SO(21) representation content of the generating function manifest, we follow a slightly roundabout route whose logic will become apparent in due course. Firstly, we write a bosonic chiral string partition function Z_{24} with SO(24) symmetry. The fugacities are y_i where $i = 1,2,\ldots,12$ for the vector representation of SO(24):[1]

$$Z_{24} = \prod_{m=1}^{\infty} \prod_{\lambda \in \Lambda_{\mathbf{24}}} \left(1 - q^m \prod_{i=1}^{12} y_i^{\lambda_i}\right)^{-1} . \tag{3.9}$$

We introduced the notations λ for the weights inside the set of weights $\Lambda_{\mathbf{24}}$ of the 24-dimensional vector representation of SO(24), as well as their components λ_i in a basis of fundamental weights. It is important to realize at this stage that we know that this partition function is, in fact, SO(25) covariant starting at level q^2, since the massive modes of a bosonic open string (or the non-supersymmetric chirality of a heterotic string) exhibit the covariance of the little group SO(25) in $25+1$ dimensions. This can be made manifest by defining the SO(25) covariant version which is obtained after the substitution (see equation (A.11)) [18]:

$$y_{11} = z_{11}/z_{12} \, , \tag{3.10}$$

and renaming the other variables $y_i = z_i$ (for $1 \leq i \leq 12$, $i \neq 11$), where the variables z_i are SO(25) fugacities. As an example, let us illustrate how this substitution reconstitutes the character of the SO(25) vector representation from the vector of SO(24) and a singlet: starting from

$$\chi_{\mathbf{24}} = y_1 + \frac{1}{y_1} + \left[\sum_{i=1}^{9} \frac{y_i}{y_{i+1}} + \frac{y_{i+1}}{y_i}\right] + \frac{y_{10}}{y_{11} y_{12}} + \frac{y_{11} y_{12}}{y_{10}} + \frac{y_{11}}{y_{12}} + \frac{y_{12}}{y_{11}} \tag{3.11}$$

we obtain after substitution of the new variables

$$1 + \chi_{\mathbf{24}} \to 1 + z_1 + \frac{1}{z_1} + \left[\sum_{i=1}^{10} \frac{z_i}{z_{i+1}} + \frac{z_{i+1}}{z_i}\right] + \frac{z_{11}}{z_{12}^2} + \frac{z_{12}^2}{z_{11}} = \chi_{\mathbf{25}} \, . \tag{3.12}$$

The underlying mechanics of these substitution rules is the equivalence of the covariant BRST cohomology (with Lorentz invariant ghosts and Lorentz covariant matter) with the light cone gauge spectrum. This SO(25) covariant partition function (at level 2 and higher) [18] can be fruitfully adapted to the present context. Let us proceed formally at this stage. We first of all assign both left and right R-charges equal to the string mass squared, or level, through the substitution $q \to t\bar{t}q$. We furthermore fine-tune the left and right R-charge assignment using the charges of the states under a $\mathrm{U}(1) \times \mathrm{U}(1) \subset \mathrm{SO}(4)$ subgroup of $\mathrm{SO}(4) \times \mathrm{SO}(21) \subset \mathrm{SO}(25)$. We can implement this through the further substitution:

$$\begin{cases} z_i \longrightarrow \tilde{z}_i & \text{for } i = 1, \dots, 8 \\ z_9 \longrightarrow t\bar{t}\tilde{z}_8 & \\ z_{10} \longrightarrow t^2 \tilde{z}_8 & \\ z_{11} \longrightarrow t^2 \tilde{z}_9 & \\ z_{12} \longrightarrow t\tilde{z}_{10} \, . & \end{cases} \tag{3.13}$$

[1] We refer to appendix A for the relevant representation theory and nomenclature.

Following the scheme outlined above, we find the SO(21) covariant generating function — we drop the tildes on the variables $\tilde{z}_i$ in the final result:

$$\begin{aligned}
&\tilde{P}(q,t,\bar{t},z_1,\ldots,z_{10}) = \\
&\quad \prod_{m=1}^{+\infty} \frac{(1-q^m t^m \bar{t}^m)}{\prod_{\lambda' \in \Lambda_{\mathbf{21}}} \left(1-q^m t^m \bar{t}^m \prod_{i=1}^{10} z_i^{\lambda'_i}\right)} \\
&\quad \times \prod_{m=1}^{+\infty} \frac{1}{(1-q^m t^{m-1}\bar{t}^{m-1})(1-q^m t^{m+1}\bar{t}^{m-1})(1-q^m t^{m-1}\bar{t}^{m+1})(1-q^m t^{m+1}\bar{t}^{m+1})}\,.
\end{aligned} \tag{3.14}$$

In the covariant formalism, we find two towers of ghost oscillators, as well as twenty-six towers of bosonic oscillators. One of the ghost towers is cancelled by the oscillators associated to the time direction. In the resulting expression (3.14), we have one numerator factor left, associated to one ghost, and twenty-five factors in the denominator, that are sufficient to capture the SO(25) covariance of the massive spectrum of the string. The SO(25) covariance has been twisted as described above, leading to the result (3.14).

In the final expression, the coefficient of the monomial $t^r \bar{t}^s$ is the character of the representation of SO(21) in which the chiral primaries of left and right R-charges (r,s) transform. As an example we can study the multiplicity coefficient 254 for $(r,s)=(2,2)$. We have to expand to the fourth power in q to find it, i.e. we have to take the number of copies in the symmetric product conformal field theory to be greater or equal than 4. Using a symbolic manipulation program, we recognize the character of the representation $\mathbf{230}+\mathbf{21}+3\times\mathbf{1}$ of SO(21). In table 2, the corresponding chiral primaries have been organized in three lines according to this decomposition. The coefficient 2278 of the term with $(r,s)=(3,3)$ corresponds to the $\mathbf{1750}+\mathbf{230}+\mathbf{210}+4\times\mathbf{21}+4\times\mathbf{1}$ of SO(21). This example is described in technical detail in appendix A.3. As a final example, we can decompose the coefficient 16744 at $(r,s)=(4,4)$ in representations of SO(21) as

$$\mathbf{10395}+\mathbf{3059}+\mathbf{1750}+5\times\mathbf{230}+\mathbf{210}+8\times\mathbf{21}+12\times\mathbf{1}\,. \tag{3.15}$$

As we mentioned, and as is apparent from the numerator in formula (3.14), we made use of the relation between the chiral ring generating function, and the covariantized generating function of 24 chiral bosons, as embedded in the covariant bosonic or heterotic string (through the addition of light-cone oscillators as well as ghosts [18]). Let us finally show how known dualities link up these ideas, and provide method to this madness.

3.2.4 The duality

If we return to our starting point, which was a D1-D5 system on $\mathbb{R}^{5,1}\times K3$, and we compactify on an extra circle along the string, then we can dualize the configuration to a heterotic string with momentum $n=N_1$ and winding $w=N_5$ on that circle.[2] The elliptic genus of the conformal field theory dual to $\mathrm{AdS}_3\times S^3$ is indeed equal to the counting

[2] An example duality chain would be a chain of first S-duality, then T-duality along the circle, and then IIA/Heterotic duality on $K3/T^4$.

function of half BPS states in heterotic string theory on $S^1 \times T^4$ in asymptotically flat space, with momenta $p_{L,R} = n/R \pm wR/\alpha'$ on the circle S^1 and no momentum on the T^4. See e.g. [14] for a foreshadowing of this known fact. The counting is mostly unaffected by the procedure of taking the near-brane limit. The states that are counted preserve half the supersymmetry, are in one of sixteen right-moving ground states, and obey the equations [15]:

$$\begin{aligned} \frac{\alpha' m^2}{4} &= \frac{\alpha'}{4} p_R^2 \\ &= \frac{\alpha'}{4} p_L^2 + N_L - 1 \,, \end{aligned} \tag{3.16}$$

where the mass m is measured transversely to the circle, and N_L is the left-moving oscillator number. For the left-movers, we allow any of the 24 bosonic oscillator excitations (but no further zero mode excitations). The oscillator level is restricted by level matching to satisfy:

$$N_L - 1 = -nw \,. \tag{3.17}$$

We note that for non-zero quantum numbers (n, w), we are dealing with massive states only. We thus find that only the bosonic half of the heterotic string, and only the oscillator modes intervene in the half BPS state counting [15]. Note that the oscillator states are as in the chiral half of a $25+1$ dimensional bosonic string theory, compactified on T^4. The little group for massive string states in this space is SO(21) — we can safely ignore the compactification on the extra S^1, by realizing that we have fixed the momentum in this direction, and that in regard to state counting, the direction is otherwise equivalent to a non-compact direction in space. Thus, the chiral ring spectrum indeed naturally permits an action of SO(21).

We can thus follow [18] and write down the covariant partition function. To perform this final step, we first discuss how to read off the R-charges of the states from the chiral heterotic string perspective. The R-charge of a state in the dual conformal field theory is given by:

$$(r, s) = (N_L, N_L) + (p^D, p^A) \tag{3.18}$$

where (p^D, p^A) are generators of a diagonal and anti-diagonal SO(2) inside the SO(4) action on the tangent space to the four-torus T^4. The oscillators at given level transform in the **4** of SO(4) and will carry charges $(\pm 1, \pm 1)$ under the SO(2) $\times$ SO(2) subgroup. These charges dictate the fugacity substitution rules we declared above. To complete the proof of SO(21) covariance of the generating function from this dual perspective, we firstly remark that we can compute the covariant BRST cohomology at each fixed oscillator number N_L. And the second and final argument is that SO(21) commutes with the SO(4) acting on the tangent space to T^4.

Finally, we note that our proof is also constructive in providing generators of SO(21). The generators are the right-moving and massive part of the SO(21) Lorentz generators in light-cone gauge. Their explicit expression is therefore known (see e.g. [16]). This provides a hands-on construction of the generators, and thus answers a question raised in [9], in this duality frame, at a particular point in the moduli space.

In summary, through string-string duality, we have related the SO(21) covariance of the chiral ring of the superconformal field theory dual to $\mathrm{AdS}_3 \times S^3 \times K^3$ to Lorentz covariance of the heterotic string on T^4. This relation allowed us to construct the covariant generating function for the chiral ring.

3.3 Type IIB string theory on $\mathbf{AdS_3} \times S^3 \times T^4$

We wish to apply a similar logic to obtain the covariant generating function for the chiral ring of the $N = (4,4)$ conformal field theory dual to Type IIB compactified on the manifold $M = T^4$. In this case, the scalar moduli parametrize the coset $\mathrm{SO}(5,5)/\big(\mathrm{SO}(5) \times \mathrm{SO}(5)\big)$. After considering a $D1/D5$ system, and taking the near-brane limit, one finds that in the $\mathrm{AdS}_3 \times S^3 \times T^4$ geometry, five scalars become massive (namely, the volume of T^4, the three components of the anti-self-dual part of the Neveu-Schwarz two-form, and a linear combination of the Ramond-Ramond scalar and four-form). The twenty remaining scalars parametrize the coset $\mathrm{SO}(4,5)/\big(\mathrm{SO}(4) \times \mathrm{SO}(5)\big)$. This moduli space is again a homogeneous space. The holonomy of the tangent bundle of the moduli space is $\mathrm{SO}(4) \times \mathrm{SO}(5)$. As explained in [9], each of the vector bundles of chiral primaries is characterized by a representation of SO(5).

The seed theory for the dual $N = (4,4)$ superconformal field theory is the theory on the four-torus. The chiral ring of the seed is determined by the Hodge diamond of T^4:

$$P_{t,\bar{t}}(T^4) = \begin{array}{ccccc} & & 1 & & \\ & 2 & & 2 & \\ 1 & & 4 & & 1 \\ & 2 & & 2 & \\ & & 1 & & \end{array}. \tag{3.19}$$

The Hodge diamonds for $(T^4)^N/S_N$ are given by formula (3.4). From this we can write the partition function for the number of chiral primaries at given N, that starts out as

$$Z\big((T^4)^N/S_N\big) = 1 + 16q + 144q^2 + 960q^3 + 5264q^4 + 25056q^5 + \dots\,, \tag{3.20}$$

and we can generate the upper-left eighth of the Hodge diamond for, say, the $N = 9$ symmetric product:

$$\begin{array}{ccccccccccc}
 & & & & & & & & & & 1 \\
 & & & & & & & & & 2 & \\
 & & & & & & & & 2 & & 9 \\
 & & & & & & & 2 & & 18 & \\
 & & & & & & 2 & & 23 & & 61 \\
 & & & & & 2 & & 24 & & 116 & \\
 & & & & 2 & & 24 & & 154 & & 327 \\
 & & & 2 & & 24 & & 170 & & 600 & \\
 & & 2 & & 24 & & 175 & & 817 & & 1514 \\
P_{t,\bar{t}}\big((T^4)^9/S_9\big) = & 2 & & 24 & & 176 & & 932 & & 2690 & \\
 & & 24 & & 176 & & 978 & & 3695 & & 6244 \\
 & & & 176 & & 994 & & 4306 & & 10774 & \\
 & & & & 998 & & 4587 & & 14770 & & 23278 \\
 & & & & & 4664 & & 17224 & & 38444 & \\
 & & & & & & 18010 & & 50637 & & 75320 \\
 & & & & & & & 55226 & & 111912 & \\
 & & & & & & & & 127496 & & 181359 \\
 & & & & & & & & & 213214 & \\
 & & & & & & & & & & 253508
\end{array} \tag{3.21}$$

In the symmetric product conformal field theory, we can factor out the center of mass degree of freedom, which is one copy of T^4. This translates into the fact that the Poincaré polynomial $P_{t,\bar{t}}(T^4)$ divides $P_{t,\bar{t}}\big((T^4)^N/S_N\big)$. Let us define

$$Q_N(t,\bar{t}) = \frac{P_{t,\bar{t}}\big((T^4)^N/S_N\big)}{P_{t,\bar{t}}(T^4)} \in \mathbb{Z}[t,\bar{t}]\,. \tag{3.22}$$

For reference, let us also record the corresponding Hodge diamond, again for $N = 9$:

$$Q_9(t,\bar{t}) = \begin{array}{ccccccccc}
 & & & & & & & & 1 \\
 & & & & & & & 0 & \\
 & & & & & & 1 & & 5 \\
 & & & & & 0 & & 4 & \\
 & & & & 1 & & 6 & & 22 \\
 & & & 0 & & 4 & & 28 & \\
 & & 1 & & 6 & & 34 & & 94 \\
 & 0 & & 4 & & 32 & & 140 & \\
1 & & 6 & & 35 & & 167 & & 376 \\
 & 4 & & 32 & & 172 & & 588 & \\
 & & 35 & & 179 & & 718 & & 1400 \\
 & & & 176 & & 768 & & 2200 & \\
 & & & & 787 & & 2696 & & 4714 \\
 & & & & & 2844 & & 7032 & \\
 & & & & & & 7946 & & 12744 \\
 & & & & & & & 15508 & \\
 & & & & & & & & 19617
\end{array} \tag{3.23}$$

It may be useful to stress the fact that three different SO(4) symmetries are at play. One is the R-symmetry $\mathrm{SO}(4)^R = \mathrm{SU}(2)^R_L \times \mathrm{SU}(2)^R_R \supset \mathrm{U}(1)^R_L \times \mathrm{U}(1)^R_R$, generated by the $3+3$ bosonic currents of the superconformal algebra, which rotates both these currents and the supercurrents. Another is the outer automorphism group $\mathrm{SO}(4)^{outer}$ under which only the 4 supercurrents transform. Finally, there is the transformation group $\mathrm{SO}(4)^{T^4} = \mathrm{SU}(2)^{T^4}_L \times \mathrm{SU}(2)^{T^4}_R$ of the tangent space which rotates the two complexified bosons and fermions. A given chiral primary field of the seed T^4 conformal field theory with $\mathrm{U}(1)^R_L \times \mathrm{U}(1)^R_R$ charge (r,s) is a singlet of $\mathrm{SO}(4)^{outer}$, it is the highest weight state of the spin $r/2$ (respectively $s/2$) representation of $\mathrm{SU}(2)^R_L$ (respectively $\mathrm{SU}(2)^R_R$), and belongs to the following representations of $\mathrm{SU}(2)^{T^4}_L \times \mathrm{SU}(2)^{T^4}_R$:

$$\begin{array}{ccccc}
 & & (0,0) & & \\
 & \left(\frac{1}{2},0\right) & & \left(0,\frac{1}{2}\right) & \\
(0,0) & & \left(\frac{1}{2},\frac{1}{2}\right) & & (0,0) \\
 & \left(\frac{1}{2},0\right) & & \left(0,\frac{1}{2}\right) & \\
 & & (0,0) & &
\end{array} \tag{3.24}$$

It is the latter property that will be most useful in the following, since we will embed the $\mathrm{SO}(4)^{T^4}$ acting on the tangent space of T^4 into the Lorentz group of flat space. Combining these representations (3.24) with the formula (3.4) where the product is to be interpreted as tensor product of representations, we readily obtain the representations of $\mathrm{SU}(2)^{T^4}_L \times \mathrm{SU}(2)^{T^4}_R$ in which the chiral primaries of (3.21) transform. For instance, the 2 on the upper line transforms as a $\left(\frac{1}{2},0\right)$ for odd r and as $2\cdot(0,0)$ for even r, while the 9 decomposes as $2\cdot\left(\frac{1}{2},\frac{1}{2}\right)+(0,0)$.

3.3.1 The covariant generating function

String-string duality now motivates the attempt to implement the SO(5) covariance of the chiral ring generating function by exploiting the SO(9) covariance of the massive superstring spectrum in flat space. This time around, we will use the SO(9) covariance of the left chiral half of the type II superstring. A hands-on way to motivate this is to observe that the Hodge diamond (3.19) indicates the presence of 8 world-sheet bosons and 8 world-sheet fermions in the second quantized symmetric product theory. We will encounter a small subtlety in the naive implementation of this idea.

The SO(9) covariance of the flat space partition function was rendered manifest in the Neveu-Schwarz formalism in [18]. To exploit these results, it turns out to be useful to first translate them into the Green-Schwarz formalism. The latter more directly connects to the generating function of the chiral ring of the symmetric product of the four-torus. The translation involves the use of the generalized Jacobi identity for theta-functions, in turn closely related to triality of SO(8). We briefly review the necessary background.

3.3.2 The SO(8) covariant generating function and triality

Before implementing SO(9) covariance, we manipulate the SO(8) covariant partition function of flat space superstring theory to switch from the Neveu-Schwarz-Ramond to the Green-Schwarz representation. The worldsheet bosonic SO(8) covariant factor in the partition function is:

$$Z_B = \prod_{l=1}^{\infty} \prod_{\lambda \in \mathbf{8_v}} \frac{1}{1 - y^\lambda q^l} = q^{1/3} \eta^4 \prod_{a=1}^{4} \frac{2 \sin(\pi \nu_a)}{\theta_1(\nu_a, \tau)} ,$$

where $\mathbf{8_V}$ is the eight-dimensional vector representation of SO(8) whose character is denoted $[1,0,0,0]_8$, see the notation (A.5). We have defined the SO(8) fugacities:

$$e^{2\pi i \nu_1} = y_1 , \qquad e^{2\pi i \nu_2} = \frac{y_2}{y_1} , \qquad e^{2\pi i \nu_3} = \frac{y_3 y_4}{y_2} , \qquad e^{2\pi i \nu_4} = \frac{y_4}{y_3} . \tag{3.25}$$

For the world sheet fermions we have partition functions that are given by (possibly shifted versions of):

$$Z_F = \prod_{l=0}^{\infty} \prod_{\lambda \in \mathbf{8_v}} (1 + f y^\lambda q^l) .$$

For a given state, the parity of the power of f is equal to the world sheet fermion number. With this in mind, the GSO projection in the NS sector $(-1)^F = 1$ consists in keeping the terms in the partition function that are even powers of f (after taking into account the fermion number of the covariant vacuum). In the Ramond sector, we keep both sectors from the perspective of the oscillators $(-1)^F = \pm 1$, and associate each to different chirality vacuum states (spinor $\mathbf{8_s}$ or $\mathbf{8_c}$). The total partition function for the left-movers, taking bosons and fermions into account, is then

$$\begin{aligned} Z_L = {} & \frac{1}{2}([0,0,1,0]_8 - [0,0,0,1]_8) \\ & + 8 \prod_{a=1}^{4} \frac{\sin(\pi\nu_a)}{\theta_1(\nu_a,\tau)} \left[\prod_{a=1}^{4} \theta_3(\nu_a,\tau) - \prod_{a=1}^{4} \theta_4(\nu_a,\tau) + \frac{1}{16}([0,0,0,1]_8 + [0,0,1,0]_8) \prod_{a=1}^{4} \frac{\theta_2(\nu_a,\tau)}{\cos(\pi\nu_a)} \right] . \end{aligned} \tag{3.26}$$

Using the identities between characters and trigonometric functions

$$[0,0,0,1]_8 = 8\Big(\prod_{a=1}^4 \cos(\pi\nu_a) + \prod_{a=1}^4 \sin(\pi\nu_a)\Big) \tag{3.27}$$

$$[0,0,1,0]_8 = 8\Big(\prod_{a=1}^4 \cos(\pi\nu_a) - \prod_{a=1}^4 \sin(\pi\nu_a)\Big)\,, \tag{3.28}$$

we can write this in a more symmetric form:

$$Z_L = 8\prod_{a=1}^4 \frac{\sin(\pi\nu_a)}{\theta_1(\nu_a,\tau)}\left[\prod_{a=1}^4 \theta_3(\nu_a,\tau) - \prod_{a=1}^4 \theta_4(\nu_a,\tau) + \prod_{a=1}^4 \theta_2(\nu_a,\tau) - \prod_{a=1}^4 \theta_1(\nu_a,\tau)\right]. \tag{3.29}$$

These are the Neveu-Schwarz-Ramond expressions which were the starting point of [18]. The elliptic genus generating function however is more akin to a Green-Schwarz formalism partition function. We therefore wish to translate these SO(8) covariant partition sums into the Green-Schwarz language, in which bosonic oscillators on the world sheet form a vector of SO(8) in light-cone gauge, and fermionic oscillators a spinorial representation [16]. It is well-known that we can implement the change of formalism through triality. We can act with triality on the simple roots of SO(8), e.g. exchanging the first and fourth root. The action induces an action on the fugacities (see also equation (3.25))

$$e^{2\pi i\nu_1'} = y_4 \qquad e^{2\pi i\nu_2'} = \frac{y_2}{y_4} \qquad e^{2\pi i\nu_3'} = \frac{y_1 y_3}{y_2} \qquad e^{2\pi i\nu_4'} = \frac{y_1}{y_3}\,. \tag{3.30}$$

This change of variables lifts to the full oscillator spectrum in the generalized Jacobi identity [19]

$$\begin{aligned}\prod_{a=1}^4 \theta_3(\nu_a,\tau) - \prod_{a=1}^4 \theta_4(\nu_a,\tau) + \prod_{a=1}^4 \theta_2(\nu_a,\tau) - \prod_{a=1}^4 \theta_1(\nu_a,\tau) &= 2\prod_{a=1}^4 \theta_1\Big(\nu_a' + \frac{1}{2},\tau\Big)\\ &= 2\prod_{a=1}^4 \theta_2(\nu_a',\tau)\,.\end{aligned} \tag{3.31}$$

Using this theta function identity in the partition function, we obtain

$$Z_L = 16\prod_{a=1}^4 \frac{\sin(\pi\nu_a)}{\theta_1(\nu_a,\tau)}\theta_2(\nu_a',\tau)\,. \tag{3.32}$$

We can rewrite the θ-functions in terms of products, and use the identities

$$[1,0,0,0]_8 = 8\Big(\prod_{a=1}^4 \cos(\pi\nu_a') + \prod_{a=1}^4 \sin(\pi\nu_a')\Big) \tag{3.33}$$

$$[0,0,1,0]_8 = 8\Big(\prod_{a=1}^4 \cos(\pi\nu_a') - \prod_{a=1}^4 \sin(\pi\nu_a')\Big)\,, \tag{3.34}$$

to find:

$$Z_L = \big([1,0,0,0]_8 + [0,0,1,0]_8\big)\prod_{a=1}^4\prod_{n=1}^\infty \frac{(1+e^{2i\pi\nu_a'}q^n)(1+e^{-2i\pi\nu_a'}q^n)}{(1-e^{2i\pi\nu_a}q^n)(1-e^{-2i\pi\nu_a}q^n)}\,. \tag{3.35}$$

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We recognize this as the Green-Schwarz formalism partition function corresponding to a 8_V vector worth of bosonic oscillators and a 8_S spinor of fermionic worldsheet oscillators. The factorized term $[1,0,0,0]_8 + [0,0,1,0]_8$ is the 16-dimensional multiplet of massless ground states. When we take all fugacities to 1, the partition function can be factorized as

$$Z_L = 16 + 256(q + 9q^2 + 60q^3 + 329q^4 + 1566q^5 + \ldots)\,. \tag{3.36}$$

Finally, we can address the subtlety that we encounter in identifying the flat space partition function with the chiral ring generating function. There are 16 chiral ring states for a single copy $N = 1$ of T^4 (corresponding to the term q^1 in the generating function (3.20)), but there are 256 chiral massive states at the first level in the superstring as seen in (3.36). It should also be noted that the smallest non-trivial representations of SO(5) are of dimensions 4 and 5, while the border of the Hodge diamond (3.21) are representations of dimension 2 that are, in some cases, non-trivial representations of $\mathrm{SO}(4) \subset \mathrm{SO}(5)$, according to the discussion below (3.24).

We must therefore further specify that the arguments of [9] apply to the non-trivially interacting part of the boundary superconformal field theory, which we obtain by factoring the center of mass T^4 from the symmetric product conformal field theory. Correspondingly, in the flat space chiral generating function for the massive modes, we will strip off the massive supermultiplet degrees of freedom. The minimal requirement of the matching of the number of degrees of freedom is then met. This can be seen by factoring out a factor of sixteen from equation (3.20) and comparing to the term between parentheses in equation (3.36). In the following, we demonstrate that this heedful observation also allows us to implement SO(5) covariance in the chiral ring generating function.

3.3.3 The SO(9) and SO(5) covariant generating functions

We recall that in [18] it was demonstrated that the SO(9) massive representation content of the superstring can be recuperated from the formula for the partition function in the following way: one splits off the massless representation and substitutes SO(9) fugacities for SO(8) fugacities using the substitution

$$y_3 = z_3/z_4 \quad \text{and} \quad y_i = z_i \quad \text{for } i = 1, 2, 4\,, \tag{3.37}$$

expands, and decomposes. See [18] for details. It was also shown that one can factor out a massive supermultiplet factor. Applying this procedure to the partition function (3.35) gives

$$Z_L = Z_{\text{massless}} + Z_{\text{massive}} \times Z_{\text{fact}} \tag{3.38}$$

where we expand the factorized partition function as

$$Z_{\text{fact}} = \sum_{N=1}^{\infty} Q_N q^N. \tag{3.39}$$

The first term

$$Z_{\text{massless}} = [1,0,0,0]_8 + [0,0,1,0]_8 \tag{3.40}$$

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in (3.38) accounts for the massless multiplet and is only a SO(8) character, while Z_{massive} is the universal massive multiplet factor, given in term of SO(9) characters by

$$Z_{\text{massive}} = [2,0,0,0]_9 + [1,0,0,1]_9 + [0,0,1,0]_9\,. \tag{3.41}$$

This term corresponds to the factor 256 in (3.36). Our main focus is the remaining factor Z_{fact}. Inspired by string-string duality, we generate the SO(5) covariant chiral ring partition function by breaking $\mathrm{SO}(9) \longrightarrow \mathrm{U}(1)\times\mathrm{U}(1)\times\mathrm{SO}(5)$ in the appropriate way. This is entirely analogous to what we did in subsection 3.2.3. We make the substitutions:

$$q \to t\bar{t}q \qquad z_1 \longrightarrow t\bar{t} \qquad z_2 \longrightarrow t^2 \qquad z_3 \longrightarrow t^2\tilde{z}_1 \qquad z_4 \longrightarrow t\tilde{z}_2\,. \tag{3.42}$$

For the massive supermultiplet, we obtain:

$$(t\bar{t})^2 Z_{\text{massive}} = \begin{array}{ccc} & & 1 \\ & [01] & \\ [10]+1 & & [02]+[10]+1 \\ & [11]+2\cdot[01] & \\ & & [20]+[02]+2\cdot[10]+2 \end{array} \tag{3.43}$$

In the final expression, the coefficient of the monomial $t^r\bar{t}^s$ is the character of the SO(5) representation of the chiral primaries of charge (r,s). To summarize, expand (3.35), using the characters of the vector, spinor and pseudo-spinor representations of SO(8), and take the coefficient of q^N. Multiply this coefficient by $(t\bar{t})^{N-1}$ and make the substitution

$$y_1 \longrightarrow t\bar{t} \qquad y_2 \longrightarrow t^2 \qquad y_3 \longrightarrow t\frac{\tilde{z}_1}{\tilde{z}_2} \qquad y_4 \longrightarrow t\tilde{z}_2\,. \tag{3.44}$$

The decomposition of the Hodge diamond into representations of SO(5) is obtained via this procedure. We use Dynkin label notation for the SO(5) representations, and we arrange the matrices such that the $\mathrm{U}(1)\times\mathrm{U}(1)$ quantum numbers act as row and column labels. As argued above, we strip off this factor, and continue to study the remaining factor, which corresponds to the Hodge diamond of the orbifold conformal field theory where we factor out the center of mass degrees of freedom. The factor Z_{fact} is coded in the diamonds Q_N which at levels $N = 1,\ldots,5$ read, depicting only one eighth of the diamond following the convention (3.3):

$$Q_1 = 1 \tag{3.45}$$

$$Q_2 = \begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & [10] & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array} = \begin{array}{cc} & 1 \\ 0 & \\ & [10] \end{array} \tag{3.46}$$

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$$
Q_3 = \begin{array}{ccc}
 & & 1 \\
 & 0 & \\
1 & & [10] \\
 & [01] & \\
\hdashline
 & & [20]+2
\end{array}
\tag{3.47}
$$

$$
Q_4 = \begin{array}{cccc}
 & & & 1 \\
 & & 0 & \\
 & 1 & & [10] \\
0 & & [01] & \\
 & [10]+1 & & [20]+[10]+3 \\
\hdashline
 & & [11]+2[01] & \\
 & & & [30]+[02]+3[10]+2
\end{array}
\tag{3.48}
$$

$$
Q_5 = \begin{array}{ccccc}
 & & & & 1 \\
 & & & 0 & \\
 & & 1 & & [10] \\
 & 0 & & [01] & \\
1 & & [10]+1 & & [20]+[10]+3 \\
 & [01] & & [11]+3[01] & \\
\hdashline
 & & [20]+3[10]+4 & & [30]+[20]+2[02]+4[10]+4 \\
 & & & [21]+3[11]+6[01] & \\
 & & & & [40]+[12]+3[20]+2[02]+6[10]+8
\end{array}
\tag{3.49}
$$

We dropped the index 5 for the characters — they are all associated to SO(5). We note that the (i, j)th entry of Q_N is independent of N provided $N \geq (i-1) + (j-1)$. Those entries are above the dashed line in the diamonds. When this condition is fulfilled, the dimensions of the representations match those of the reduced Hodge diamond (3.23) of the tensor product theory. For instance, assume we want to know the decomposition of the representation of dimension 167 in (3.23), which corresponds to the entry $(6, 4)$ in the matrices Q_N for any $N \geq 8$. The associated character can be computed as outlined above, and one finds for the $(6, 4)$ coefficient of Q_8:

$$
\chi_{\mathbf{167}} = [30] + 3[20] + 4[02] + 9[10] + 10\,.
\tag{3.50}
$$

These decompositions are obtained by the identification of characters in the covariantized generating function.

3.3.4 Summary

We related the SO(5) covariance of the chiral ring of the superconformal dual to $\mathrm{AdS}_3 \times S^3 \times T^4$ to a broken Lorentz little group for chiral massive type II superstring excitations. Again, the inspiration lies in string-string dualities that relate the chiral primaries to half BPS states [15] in asymptotically flat space.

4 Conclusions

We have provided a generating function that codes all of the SO(21) or SO(5) representation theoretic content of the chiral ring. We have already reviewed in the introduction how this information is sufficient to compute the connection on the homogeneous vector bundle spanned by the chiral states over the moduli space [9]. It would be interesting to exploit this knowledge to analyze the SO(21) covariance of known correlation functions. A much more ambitious goal would be to characterize $1/N$ corrections in terms of SO(21) representation theoretic data.

To exploit the covariance fully, it is necessary to identify the states that fill out the various SO(21) or SO(5) multiplets at a given point in the moduli space. Although we did this in the oscillator formalism, one needs to carefully match these states onto those of the orbifold symmetric product conformal field theory, and in this identification one must allow for operator mixing. Moreover, the operator mixing may involve mixing of single and multi-particle states. Still, it is clear that if one is able to overcome these technical challenges, the SO(21) classification of the spectrum will become powerful in organizing the dynamical information of the theory, and for instance in predicting new correlation functions from old ones, by completing SO(21) or SO(5) multiplets. We leave this interesting task of mixing and separation for future work.

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A Representations and characters

In this appendix, we gather some results on the characters and the representation theory of classical Lie algebras and apply them to the $\mathfrak{so}(n)$ algebras appearing in the body of the paper.

A.1 Characters

Each unitary irreducible representation of a classical Lie algebra is characterized by its highest weight λ. The character associated to the representation with highest weight λ is defined as

$$\chi_\lambda = \sum_{\lambda' \in \Lambda_\lambda} \mathrm{mult}_\Lambda(\lambda') e^{\lambda'} , \tag{A.1}$$

where the sum is over all weights λ' in the weight system Λ_λ of the representation of highest weight λ. A useful basis of the lattice of weights is given by the fundamental weights $(\pi_1, \ldots, \pi_r)$. They have the property that they are orthonormal to a basis of simple co-roots with respect to a natural non-degenerate form on the classical Lie algebra. Their number is equal to the rank r of the Lie algebra. The expansion coefficients of a highest weight in the fundamental weight basis are called Dynkin labels. They can be obtained by computing the inner product of the simple co-roots with the weights. Decomposing the weights

$$\lambda' = \sum_{i=1}^{r} \lambda'_i \pi_i \, , \tag{A.2}$$

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in terms of r integers λ'_i, the formula (A.1) for the characters becomes

$$\chi_\lambda = \sum_{\lambda' \in \Lambda_\lambda} \mathrm{mult}_\Lambda(\lambda') \prod_{i=1}^{r} e^{\lambda'_i \pi_i} \,. \tag{A.3}$$

Let's define fugacities $y_i = e^{\pi_i}$. Then the characters can be written as a sum of monomials:

$$\chi_\lambda = \sum_{\lambda' \in \Lambda_\lambda} \mathrm{mult}_\Lambda(\lambda') \prod_{i=1}^{r} {y_i}^{\lambda'_i} \,. \tag{A.4}$$

We use square brackets around Dynkin labels to denote the character associated to a given weight:

$$[\lambda_1, \dots \lambda_r]_n = \chi_\lambda \qquad \text{where} \qquad \lambda = \sum_{i=1}^{r} \lambda_i \pi_i \,. \tag{A.5}$$

In the text we discuss only characters of algebras of type $\mathfrak{so}$, and we add a subscript n to indicate that we consider the character of an $\mathfrak{so}(n)$ representation. Given the character of a reducible representation, one can decompose it into irreducible representations by identifying the highest weight in the reducible representation, subtracting the character of the corresponding irreducible representation, and continuing recursively. For finite dimensional representations, this is a finite algorithm that can be implemented on a computer. An example is presented in section A.3.

A.2 The $\mathfrak{so}(2r+1)$ and $\mathfrak{so}(2r)$ algebras

We will have particular use for the Lie algebras associated to the orthogonal groups, namely $B_r = \mathfrak{so}(2r+1)$ and $D_r = \mathfrak{so}(2r)$. Take an orthonormal basis $(\epsilon_1, \dots, \epsilon_r)$ in the space of weights. The simple roots and the fundamental weights of B_r can be parameterized as

$$\begin{aligned} \alpha_i &= \epsilon_i - \epsilon_{i+1} \qquad \text{for } \forall\, i < r \\ \alpha_r &= \epsilon_r \,, \end{aligned} \tag{A.6}$$

and

$$\begin{aligned} \pi_i &= \epsilon_1 + \ldots + \epsilon_i \qquad \text{for } i < r \\ \pi_r &= (\epsilon_1 + \ldots + \epsilon_r)/2 \,, \end{aligned} \tag{A.7}$$

while for the algebra D_r we have

$$\begin{aligned} \alpha_i &= \epsilon_i - \epsilon_{i+1} \qquad \text{for } i < r \\ \alpha_r &= \epsilon_{r-1} + \epsilon_r \,, \end{aligned} \tag{A.8}$$

and

$$\begin{aligned} \pi_i &= \epsilon_1 + \ldots + \epsilon_i \qquad \text{for } i < r-1 \\ \pi_{r-1} &= (\epsilon_1 + \ldots + \epsilon_{r-1} - \epsilon_r)/2 \\ \pi_r &= (\epsilon_1 + \ldots + \epsilon_{r-1} + \epsilon_r)/2 \,. \end{aligned} \tag{A.9}$$

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We can relate the fundamental weights of these algebras of equal rank through the relations

$$\begin{aligned} \pi_i^B &= \pi_i^D \qquad \text{for } 1 \leq i \leq r\,,\ i \neq r-1 \\ \pi_{r-1}^B &= \pi_{r-1}^D + \pi_r^D\,. \end{aligned} \tag{A.10}$$

Let us define the fugacities $y_i = \exp(\pi_i^D)$ and $z_i = \exp(\pi_i^B)$. Then the relations between the fundamental weights (A.10) translate into relations between the fugacities

$$\begin{cases} z_i = y_i & \forall\, i \neq r-1 \\ z_{r-1} = y_{r-1} y_r \end{cases} \quad \Leftrightarrow \quad \begin{cases} y_i = z_i & \forall\, i \neq r-1 \\ y_{r-1} = z_{r-1}/z_r \end{cases} \tag{A.11}$$

The important result is then that *if* χ^D is the character of a representation of D_r, which is also a representation of the algebra B_r, *then* the latter can be obtained from the former through the substitution of the fugacities y_i by the fugacities z_j.

A.3 Details of a character decomposition

In this subsection, we give an example how we use the character decomposition to read off the detailed representation theory content of chiral primaries with given R-charge. We study chiral primaries that can be read from (3.7) for $r+s \leq 8$, and decompose them into irreducible representations of SO(21). Let us concentrate on a given term $h_{r,s} t^r \bar{t}^s$ in (3.6), and the corresponding coefficient of $t^r \bar{t}^s$ in (3.14). This coefficient is the character of an $h_{r,s}$-dimensional representation of SO(21) that we want to decompose into irreducible representations. This is a well-known problem, the solution of which consists in finding the highest dominant weight in the character, then subtracting the highest-weight representation character, and iterating the procedure until we reach zero.

As an example, let us illustrate the procedure for the coefficient $h_{3,3} = 2278$. The $t^3 \bar{t}^3$ coefficient in (3.14) belongs to $\mathbb{Z}(z_1, \ldots, z_{10})$, each summand corresponding to a weight with multiplicity. Writing only the terms associated with dominant weights, we are left with

$$z_1^3 + z_1 z_2 + z_3 + 2z_1^2 + 3z_2 + 16z_1 + 38 + \ldots \tag{A.12}$$

We then compute the height of each weight, which gives here $(30, 29, 27, 20, 19, 10, 0)$, select the highest weight z_1^3 and subtract the character of the associated 1750-dimensional irreducible representation generated by this weight. We are then left with (again writing only the dominant weights)

$$z_1^2 + 2z_2 + 6z_1 + 28 + \ldots \tag{A.13}$$

with heights $(20, 19, 10, 0)$, which tell us to subtract the 230-dimensional irreducible representation generated by z_1^2. The characters obtained in the following iterations are $z_2 + 5z_1 + 18 + \ldots$, $4z_1 + 8 + \ldots$ and 4. We then add up all the subtracted representations and obtain $\mathbf{1750} + \mathbf{230} + \mathbf{210} + 4 \times \mathbf{21} + 4 \times \mathbf{1}$ as claimed in the text.

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