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Instanton effects in ABJM theory with general R-charge assignments

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Abstract: We study the large N expansion of the partition function of the quiver superconformal Chern-Simons theories deformed by two continuous parameters which correspond to general R-charge assignment to the matter fields. Though the deformation breaks the conformal symmetry, we find that the partition function shares various structures with the superconformal cases, such as the Airy function expression of the perturbative expansion in $1/N$ with the overall constant $A(k)$ related to the constant map in the ABJM case through a simple rescaling of k. We also identify five kinds of the non-perturbative effects in $1/N$ which correspond to the membrane instantons. The instanton exponents and the singular structure of the coefficients depend on the continuous deformation parameters, in contrast to the superconformal case where all the parameters are integers associated with the orbifold action on the moduli space. This implies that the singularity of the instanton effects would be observable also in the gravity side.

Keywords: Supersymmetric gauge theory, Matrix Models, Nonperturbative Effects, M-Theory

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Contents

1 Introduction

Though the fundamental principles of M2-brane interactions are not clear, a particular class of U(N) superconformal quiver Chern-Simons theories are proposed as the worldvolume theory of the N-stack of interacting M2-branes. One of the supporting evidences is that the large N limit of the free energy computed in these theories exhibit the $N^{3/2}$ scaling. This precisely reproduces the result obtained in the eleven dimensional supergravity on $\mathrm{AdS}_4 \times Y_7$ [1–3] (or their consistent truncations in 4d), where Y_7 is some seven dimensional manifold associated with the theory. Taking this gauge/gravity correspondence inversely, the field theory analysis beyond the large N limit is expected to shed new lights on the M-theory beyond the classical supergravity.

Among the theories of N M2-branes the ABJM theory [4] is the most symmetric one, and hence have been studied with the greatest efforts. The ABJM theory is the $\mathcal{N} = 6$ $\mathrm{U}(N)_k \times \mathrm{U}(N)_{-k}$ quiver superconformal Chern-Simons theory. In the $\mathcal{N} = 2$ notation, each vertex of the quiver is assigned with U(N) Chern-Simons vector multiplet $(A_\mu, \sigma, \lambda, D)$ with Chern-Simons levels $\pm k$ while each edge is assigned with a pair of bifundamental hypermultiplets (ϕ, ψ, F) and $(\widetilde{\phi}, \widetilde{\psi}, \widetilde{F})$ which are charged under $\mathrm{U}(1)_R$ as $(R_{\phi,\widetilde{\phi}^\dagger}, R_{\psi,\widetilde{\psi}^\dagger}, R_{F,\widetilde{F}^\dagger}) = (1/2, -1/2, -3/2)$ [4, 5]. The dual geometry to this theory is $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$. With the help of the localization technique, the partition function of the

ABJM theory can be reduced to a matrix model with $2N$ integration variables [6]. After the determination of the leading $N^{3/2}$ behavior [1], the matrix model was further analyzed in the 't Hooft limit $k, N \to \infty$ with $\lambda = N/k$ fixed [1, 7, 8], with the help of the relation between the 't Hooft expansion of the matrix model and the free energy of the topological string theory on local $\mathbb{P}^1 \times \mathbb{P}^1$ [9].

Later a new expression of the ABJM matrix model was discovered as the canonical partition function of a quantum statistical system of N particle ideal Fermi gas, where the level k is converted into the Planck constant $\hbar = 2\pi k$ in the statistical system [10]. This relation enables us a systematic analysis of the large N expansion of the partition function in the M-theoretical regime $k < \infty$, in terms of the grand potential $J(\mu)$ defined by

$$e^{J(\mu)} = 1 + \sum_{N \geq 1} e^{\mu N} Z(N). \tag{1.1}$$

Here μ is an auxiliary parameter called chemical potential dual to N. The original partition function can be recovered by following inverse transformation

$$Z(N) = \int \frac{d\mu}{2\pi i} e^{J(\mu) - \mu N}. \tag{1.2}$$

For finite value of k, the large N expansion of the partition function corresponds to the large μ expansion of the grand potential.

After various efforts [10–15], finally all the $1/\mu$ corrections were completely determined [16], including both perturbative and non-perturbative effects. The perturbative part of the grand potential is a cubic polynomial in μ

$$J^{\text{pert}}(\mu) = \frac{C}{3}\mu^3 + B\mu + A, \tag{1.3}$$

with C, B and A some constants. In the partition function this turns into the all order perturbative sum expressed as an Airy function (as obtained in [8])

$$Z^{\text{pert}}(N) = e^A C^{-\frac{1}{3}} \, \mathrm{Ai}\left[C^{-\frac{1}{3}}(N - B)\right]. \tag{1.4}$$

There are two kinds of non-perturbative effects in the grand potential: $e^{-4m\mu/k}$ and $e^{-2n\mu}$ $(m, n = 1, 2, \cdots)$. Through the inversion formula, these effects turn to the corrections of $\mathcal{O}(e^{-\sqrt{N/k}})$ or $\mathcal{O}(e^{-\sqrt{kN}})$ in the partition function. In gravity side, the non-perturbative effects are quantitatively interpreted as the effects of fundamental M2-branes winding on Y_7. Indeed the exponents of the non-perturbative effects in the partition function are proportional to R^3_{AdS} and hence can be explained in terms of the excitation energy of winding M2-branes. The first kind of non-perturbative effects $\mathcal{O}(e^{-4\mu/k})$ correspond to the M2-branes winding the $\mathbb{Z}_k$-orbifolded cycle and thus called the worldsheet instanton effects [17], while the second ones $\mathcal{O}(e^{-2\mu})$ correspond to the M2-branes winding in other three directions and called the membrane (D2) instanton effects [7, 18]. Although the Chern-Simons level k is originally integer, in the ABJM matrix model we can generalize k to be an irrational number. This allows the separative analysis of two kinds of non-perturbative effects, respectively by the 't Hooft expansion of the partition function and the semiclassical expansion of the grand potential. For the complete determination of the coefficients in front of these exponentials, however, it was essential to observe following singular structures of them at finite and integral k [13, 14]. For integral k, the exponent of two kind of non-perturbative effects coincide when $m = kn/2$. In this case, the individual coefficients are divergent, while the divergence are completely cancelled between the two coefficients. This structure, called as the HMO cancellation mechanism in [16], was used in the extrapolation of the small k expansion of the coefficient of the second kind of instantons for higher n and conjecture their uniformed expression.

Recently similar structures in the large N expansion was discovered in the more general superconformal quiver Chern-Simons theories. The Airy function expression of the all order perturbative corrections in $1/N$ was already claimed for the general $\mathrm{U}(N)$ $\mathcal{N}=3$ circular quiver superconformal Chern-Simons theories in [10]. The non-perturbative effects were also analyzed in detail for a special class of $\mathcal{N}=4$ superconformal quiver Chern-Simons theory [19–24].[1] Each of these theories are characterized by a integer k and the signs $s_a = \pm 1$ assigned on the edges with which the Chern-Simons level on the a-th vertex is given as [29]

$$k_a = k(s_a - s_{a-1})/2. \tag{1.5}$$

A set of signs $\{s_a\}_{a=1}^M$ is labelled by positive integers m, $\{q_a\}_{a=1}^m$ and $\{p_a\}_{a=1}^m$ as[2]

$$\begin{aligned}\{s_a\}_{a=1}^M = \{&\underbrace{1,1,\cdots,1}_{q_1},\underbrace{-1,-1,\cdots,-1}_{p_1},\underbrace{1,1,\cdots,1}_{q_2},\underbrace{-1,-1,\cdots,-1}_{p_2},\\ &\cdots,\underbrace{1,1,\cdots,1}_{q_m},\underbrace{-1,-1,\cdots,-1}_{p_m}\}\end{aligned} \tag{1.6}$$

which we shall abbreviate as

$$\{s_a\}_{a=1}^M = \{(+1)^{q_1},(-1)^{p_1},(+1)^{q_2},(-1)^{p_2},\cdots,(+1)^{q_m},(-1)^{p_m}\}. \tag{1.7}$$

The dual geometry of this theory is the product of AdS_4 and a radial section of $(\mathbb{C}^2/\mathbb{Z}_q \times \mathbb{C}^2/\mathbb{Z}_p)/\mathbb{Z}_k$, which was determined by analyzing the moduli space or the brane construction [31]. Here q and p are the number of edges with $s_a = \pm 1$

$$q = \sum_{a=1}^m q_a, \quad p = \sum_{a=1}^m p_a. \tag{1.8}$$

The instantons effects in these theories were found to subdivide into four kinds $e^{-2\mu/q}$, $e^{-2\mu/p}$, $e^{-\mu}$ [22] and $e^{-4\mu/(kqp)}$ [24] and have richer divergent structures than in the ABJM case which are controlled by (k,q,p).

So far such detailed analyses, especially of the instanton effects, were successful only in the superconformal quiver Chern-Simons theories. On the other hand, it was known

[1] The Airy function structure and the instanton effects are also revealed for the cases of non-circular quivers [25, 26] or non-unitary gauge groups [27, 28].

[2] For $k=1$, $m=1$ and $(q_1,p_1)=(N_f,1)$ the matrix model is identical with the N_f matrix model studied in [20, 30].

that the leading $N^{3/2}$ scaling behavior of the free energy is satisfied even in some theories without conformal invariance. Such theories are expected to be dual to the geometries which are asymptotically AdS_4 while have non-trivial structure in the bulk and exhibit completely different asymptotics in the opposite limit in the radial direction. Therefore it is non-trivial and would be interesting whether the above structures hold, or how they are generalized, in such non-conformal theories.

In this paper we consider following continuous deformation. Starting from the $\mathcal{N}=4$ circular quiver superconformal Chern-Simons theory with the levels (1.5), we modify the R-charge assignments on the bifundamental hypermultiplets (ϕ_a, ψ_a, F_a) and $(\widetilde{\phi}_a, \widetilde{\psi}_a, \widetilde{F}_a)$ on a-th edge

$$(R_{\phi_a}, R_{\psi_a}, R_{F_a}) = \left(\frac{1+\zeta_a}{2}, \frac{-1+\zeta_a}{2}, \frac{-3+\zeta_a}{2}\right),$$
$$(R_{\widetilde{\phi}_a}, R_{\widetilde{\psi}_a}, R_{\widetilde{F}_a}) = \left(\frac{-1+\zeta_a}{2}, \frac{1+\zeta_a}{2}, \frac{3+\zeta_a}{2}\right), \tag{1.9}$$

with

$$-1 < \zeta_a < 1. \tag{1.10}$$

In the flat space these are just a matter of convention, for which we shall call the $\mathrm{U}(1)_R$ symmetry among the $\mathrm{U}(1)$ global symmetries of the theory. Once we realize the theory on a three sphere, however, the choice is relevant to the curvature couplings and results in a distinctive theory for each choice. The theory is conformal only for the canonical choice of the R-charges $\zeta_a = 0$. The partition function of this theory have been studied in detail in the limit of $N \to \infty$ in the context of the F-theorem [32, 33], and the leading $N^{3/2}$ scaling was obtained with the explicit expression of its coefficient [34].

To analyze the large N expansion of the partition function, we first provide the Fermi gas formalism of this theory, with which we can compute the large N expansion of the partition function systematically through the grand potential $J(\mu)$. Restricting ourselves to the minimal separation of $s_a = \pm 1$

$$\{s_a\}_{a=1}^{M} = \{(+1)^q, (-1)^p\}, \tag{1.11}$$

we find that the perturbative corrections again sum up to an Airy function (1.4), with the three parameters A, B and C given by

$$C = \frac{2qp}{\pi^2 k (q^2-\xi^2)(p^2-\eta^2)}, \quad B = \frac{\pi^2 C}{3} - \frac{qp}{6k}\left(\frac{1}{q^2-\xi^2} + \frac{1}{p^2-\eta^2}\right) + \frac{kqp}{24}, \tag{1.12}$$
$$A = \frac{p^2}{4}\Big(A_{\mathrm{ABJM}}((q+\xi)k) + A_{\mathrm{ABJM}}((q-\xi)k)\Big) + \frac{q^2}{4}\Big(A_{\mathrm{ABJM}}((p+\eta)k) + A_{\mathrm{ABJM}}((p-\eta)k)\Big).$$

Here ξ and η are associated with the total deformation over the edges with $s_a = \pm 1$ respectively as

$$\xi = -\sum_{a=1}^{q} \zeta_a, \quad \eta = \sum_{a=q+1}^{q+p} \zeta_a, \tag{1.13}$$

and $A_{\rm ABJM}(k)$ is the quantity called the constant map in the ABJM theory [35]. These expression for the coefficients are natural generalizations of the results in the conformal case [21].

Analyzing the small k expansion exactly in μ, we also discover five kinds of non-perturbative effects $e^{-2\mu/(q\pm\xi)}$, $e^{-2\mu/(p\pm\eta)}$ and $e^{-\mu}$ which are the generalization of the membrane instantons in the ABJM theory. The instanton exponents (4.13) depends on ξ and η, and the individual coefficient diverges at some special values of ξ and η as in the superconformal theories.

On the other hand, the counterparts of the worldsheet instantons are invisible in the small k expansion, as they are non-perturbative in k. These effects will be accessible from the exact values of the partition functions with various finite (k, N). We generalize the method for the systematic computation of these values known in the superconformal case [12, 23, 36] to the general choice of R-charges (1.9). The result is consistent with the Airy function and strongly support the conjectural expression for A (1.12). On the other hand, the deviations from the Airy function are significantly different from the non-perturbative corrections obtained in the small k expansion and will correspond to the worldsheet instantons.

The remaining part of this paper is organized as follows. In the next section we introduce the Fermi gas formalism for general R-charge assignments (1.9), in slightly more general framework of the $\mathcal{N}=3$ U(N) circular quiver superconformal Chern-Simons theory. In the subsequent sections, we concentrate on the theory of minimal separations (1.11) and compute the exact large N expansion of the partition function using the Fermi gas formalism. In section 3 we compute the perturbative corrections in $1/\mu$ and obtain the Airy function expression (1.4) with the explicit expression of the coefficients B and C in (1.12). In section 4 we analyze the small k expansion of the grand potential in more detail and conjecture the expression of A in (1.12). We also determine the explicit coefficients of the five kinds of membrane instantons and argue the mixing and divergent structures of the instantons. In section 5 we explain the exact computation of the partition function and compare the results with the small k expansion. Finally in section 6 we summarize our results and comment on future directions.

2 Partition function in Fermi Gas formalism

In this section we provide the Fermi gas formalism for general R-charge assignments (1.9). In the derivation we use the difference expression of the Chern-Simons levels (1.5), but not the explicit values of s_a. The Fermi gas formalism hold not only for $s_a=\pm 1$ but also for arbitrary values, which correspond to the general $\mathcal{N}=3$ circular quiver superconformal Chern-Simons theories.

With the help of the localization technique, the partition function of this theory reduces into following matrix model [6, 32, 37, 38]

$$Z(N)=\frac{1}{(N!)^M}\prod_{a=1}^M\prod_{i=1}^N\int D\lambda_{a,i}\prod_{a=1}^M\frac{\prod_{i>j}2\sinh\frac{\lambda_{a,i}-\lambda_{a,j}}{2}\prod_{i>j}2\sinh\frac{\lambda_{a+1,i}-\lambda_{a+1,j}}{2}}{\prod_{i,j}2\cosh\frac{\lambda_{a,i}-\lambda_{a+1,j}-\pi i\zeta_a}{2}}, \quad (2.1)$$

where

$$D\lambda_{a,i} = \frac{d\lambda_{a,i}}{2\pi} \exp\left[\frac{ik_a}{4\pi}\lambda_{a,i}^2\right] \tag{2.2}$$

with k_a the Chern-Simons level on the a-th vertex given by (1.5).[3] Compared with the superconformal case $\zeta_a = 0$, the only difference is the shift in the arguments of the cosine-hyperbolic factors which come from the 1-loop determinant of the bifundamental hypermultiplets. This fact allows the straightforward application of the computational techniques in [9] to derive the Fermi gas formalism.[4] First we rewrite the partition function as

$$Z(N) = \frac{1}{N!}\prod_{i=1}^{N}\int \frac{d\lambda_{1,i}}{2\pi} \det_{i,j} \rho_0(\lambda_{1,i}, \lambda_{1,j})\,, \tag{2.3}$$

where

$$\begin{aligned}\rho_0(v,w) = \prod_{a=2}^{M}\int \frac{dz_a}{2\pi}&\left[e^{\frac{iks_1 v^2}{8\pi}}\frac{1}{2\cosh\frac{v-z_2-i\pi\zeta_1}{2}}e^{-\frac{iks_1 z_2^2}{8\pi}}\right]\left[e^{\frac{iks_2 z_2^2}{8\pi}}\frac{1}{2\cosh\frac{z_2-z_3-i\pi\zeta_2}{2}}e^{-\frac{iks_2 z_3^2}{8\pi}}\right]\\ &\cdots\left[e^{\frac{iks_M z_M^2}{8\pi}}\frac{1}{2\cosh\frac{z_M-w-i\pi\zeta_2}{2}}e^{-\frac{iks_M w^2}{8\pi}}\right].\end{aligned} \tag{2.4}$$

The expression (2.3) can be derived with the help of the Cauchy determinant formula

$$\frac{\prod_{i<j} 2\sinh\frac{x_i-x_j}{2}\prod_{i<j} 2\sinh\frac{y_i-y_j}{2}}{\prod_{i,j} 2\cosh\frac{x_i-y_j-\Delta}{2}} = \det_{i,j}\frac{1}{2\cosh\frac{x_i-y_j-\Delta}{2}} \tag{2.5}$$

and the formula (see appendix A in [43])

$$\frac{1}{N!}\int dz^N \left[\det_{i,j} f(x_i,z_j)\right]\left[\det_{i,j} g(z_i,y_j)\right] = \det_{i,j}\left[\int dz f(x_i,z) g(z,y_j)\right]. \tag{2.6}$$

Using the Fourier transformation formula

$$\frac{1}{2\cosh\frac{z-\pi i\zeta}{2}} = \int \frac{dp}{2\pi} e^{\frac{ipz}{2\pi}}\frac{e^{\frac{\zeta p}{2}}}{2\cosh\frac{p}{2}}\,, \tag{2.7}$$

each factor in the square bracket can be rewritten as

$$e^{\frac{iks_a z_a^2}{8\pi}}\frac{1}{2\cosh\frac{z_a-z_{a+1}-i\pi\zeta_2}{2}}e^{-\frac{iks_a z_{a+1}^2}{8\pi}} = k\cdot\left\langle x\!=\!kz_a\right|e^{\frac{is_a\widehat{x}^2}{8\pi k}}\frac{e^{\frac{\zeta\widehat{p}}{2}}}{2\cosh\frac{\widehat{p}}{2}}e^{-\frac{is_a\widehat{x}^2}{8\pi k}}\left|x\!=\!kz_{a+1}\right\rangle, \tag{2.8}$$

[3]If we take ζ_a to be pure imaginary, this matrix model completely coincide to that with the mass deformations, whose large N limit have been studied for real Chern-Simons level k in [39] and for complex k in [40, 41].

[4]The Fermi gas formalism for mass-deformed $\mathrm{U}(N)\times\mathrm{U}(N)$ theory with fundamental matter multiplets was also constructed in [42].

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where we have introduced the canonical position/momentum operators $(\widehat{x}, \widehat{p})$ and their eigenstates $(|x\rangle, |p\rangle)$ normalized so that

$$[\widehat{x}, \widehat{p}] = i\hbar, \quad (\hbar = 2\pi k)$$
$$\langle x|x'\rangle = 2\pi\delta(x-x'), \quad \langle p|p'\rangle = 2\pi\delta(p-p'), \quad \langle x|p\rangle = \frac{1}{\sqrt{k}} e^{\frac{ipx}{2\pi k}}. \tag{2.9}$$

In the operator formalism, the $(M-1)$ integrals in ρ_0 (2.4) together with the k factored out in (2.8) are interpreted as the insertion of unity

$$1 = \int \frac{dx}{2\pi} |x\rangle\langle x|, \quad (x = kz) \tag{2.10}$$

hence the partition function (2.3) can be written as

$$Z(N) = \frac{1}{N!} \prod_{i=1}^{N} \int \frac{dx_i}{2\pi} \det_{i,j} \langle x_i|\widehat{\rho}|x_j\rangle \tag{2.11}$$

with

$$\widehat{\rho} = \frac{e^{\frac{\zeta_1}{2}(\widehat{p}-\frac{s_1}{2}\widehat{x})}}{2\cosh[\frac{1}{2}(\widehat{p}-\frac{s_1}{2}\widehat{x})]} \frac{e^{\frac{\zeta_2}{2}(\widehat{p}-\frac{s_2}{2}\widehat{x})}}{2\cosh[\frac{1}{2}(\widehat{p}-\frac{s_2}{2}\widehat{x})]} \cdots \frac{e^{\frac{\zeta_M}{2}(\widehat{p}-\frac{s_M}{2}\widehat{x})}}{2\cosh[\frac{1}{2}(\widehat{p}-\frac{s_M}{2}\widehat{x})]}, \tag{2.12}$$

where we have used (2.8) and the formula

$$e^{\frac{i}{2\hbar}\widehat{x}^2} f(\widehat{p}) e^{-\frac{i}{2\hbar}\widehat{x}^2} = f(\widehat{p} - \widehat{x}). \tag{2.13}$$

Using the Fredholm determinant formula, the grand potential (1.1) can be written as

$$J(\mu) = \operatorname{Tr}\log(1 + e^{\mu}\widehat{\rho}). \tag{2.14}$$

This is the same form as the grand potential of a quantum statistical system of ideal Fermi gas.

As in the superconformal case, a special simplification occurs if the original theory have the $\mathcal{N} = 4$ supersymmetry (1.7). Since s_a takes ± 1 in this case there are only two kinds of argument in the density matrix

$$\widehat{Q} = -\widehat{p} + \frac{\widehat{x}}{2}, \quad \widehat{P} = \widehat{p} + \frac{\widehat{x}}{2}, \quad ([\widehat{Q}, \widehat{P}] = i\hbar). \tag{2.15}$$

In the remaining part of this paper, we further focus on the class of minimal separation of $s_a = \pm 1$ (1.11) where the (hermitized) density matrix is

$$\widehat{\rho} = \frac{e^{\frac{\xi\widehat{Q}}{4}}}{\left(2\cosh\frac{\widehat{Q}}{2}\right)^{\frac{q}{2}}} \frac{e^{\frac{\eta\widehat{P}}{2}}}{\left(2\cosh\frac{\widehat{P}}{2}\right)^{p}} \frac{e^{\frac{\xi\widehat{Q}}{4}}}{\left(2\cosh\frac{\widehat{Q}}{2}\right)^{\frac{q}{2}}}. \tag{2.16}$$

with ξ and η given as (1.13). Since ζ_a on each edge is bounded as (1.10), ξ and η are bounded as

$$-q < \xi < q, \quad -p < \eta < p. \tag{2.17}$$

This ensures that the density matrix decays at the infinity of the phase space and thus the trace Tr in (2.14) is well defined.

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3 Perturbative expansion in $1/N$

In this section we show that the large μ expansion of the grand potential $J(\mu)$ takes the form of (1.3), with C, B and A given as (1.12), up to the non-perturbative corrections $\mathcal{O}(e^{-\mu})$. Here C, B and A are μ-independent constants given as (1.12). Plugging these expressions into the inversion formula (1.1), we obtain the all order perturbative expansion of the partition function in $1/N$, which sum up to an Airy function as (1.4).

As argued in [9], the perturbative expansion of $J(\mu)$ (1.3) follows from the large E expansion of the number of states $n(E)$ with energy below E

$$n(E) = \mathrm{Tr}\, \theta(E - \widehat{H}) = CE^2 + B - \frac{\pi^2 C}{3} + \mathcal{O}(e^{-E}), \tag{3.1}$$

where $\widehat{H}$ is the Hamiltonian operator given by the logarithm of the density matrix:

$$e^{-\widehat{H}} = e^{-U(\widehat{Q})/2} e^{-T(\widehat{P})} e^{-U(\widehat{Q})/2} \tag{3.2}$$

with

$$U(\widehat{Q}) = q \log\left[2\cosh\frac{\widehat{Q}}{2}\right] - \frac{\xi\widehat{Q}}{2}, \quad T(\widehat{P}) = p \log\left[2\cosh\frac{\widehat{P}}{2}\right] - \frac{\eta\widehat{P}}{2}. \tag{3.3}$$

Below we shall derive the behavior (3.1) as well as the explicit expressions for C and B. On the other hand, the overall constant A requires a non-perturbative analysis of the grand potential and treated in the next section.

First of all, we introduce the Wigner transformation $(\widehat{X})_W$ of an arbitrary operator $\widehat{X}$

$$(\widehat{X})_W = \int \frac{dQ'}{2\pi} \left\langle Q - \frac{Q'}{2} \middle| \widehat{X} \middle| Q + \frac{Q'}{2} \right\rangle e^{\frac{iQ'P}{\hbar}}. \tag{3.4}$$

Then $n(E)$ is approximately given by the volume inside the region $F = \{(Q,P) \in \mathbb{R}^2 | H_W \leq E\}$ divided by $2\pi\hbar$ as[5]

$$n(E) \approx \int \frac{dQdP}{2\pi\hbar} \theta(E - H_W). \tag{3.5}$$

In the limit of $E \to \infty$ we can approximate the Wigner Hamiltonian H_W with the classical Hamiltonian

$$H_0 = U(Q) + T(P) \tag{3.6}$$

and further approximate the functions $U(Q)$ and $T(P)$ as $U \approx (q|Q| - \xi Q)/2$, $T \approx (p|P| - \eta P)/2$. In this limit the region F approaches a polygon

$$F_{\mathrm{pol}} = \left\{ (Q,P) \in \mathbb{R}^2 \middle| \frac{q|Q| - \xi Q}{2} + \frac{p|P| - \eta P}{2} \leq E \right\} \tag{3.7}$$

and the leading part of $n(E)$ is straightforwardly obtained as

$$n(E) = CE^2 + \delta n \tag{3.8}$$

with C given by (1.12).

[5] The approximation "$\approx$" in (3.5) is due to the fact that $f(\widehat{\mathcal{O}})_W \neq f(\mathcal{O}_W)$ in general. The deviation, however, is irrelevant to the perturbative expansion (3.1) as argued in [10, 21].

To compute the correction δn, we have to take into account two effects which deform the boundary of F from that of the polygon F_{pol}: (i) the deviation of the Wigner Hamiltonian from the classical Hamiltonian H_0 (3.6), and (ii) the deviation of $U(Q)$ and $T(P)$ (3.3) from the linear functions. First consider the deviation (i). The Wigner Hamiltonian can be computed order by order in $\hbar$, by first compute the Hamiltonian operator (3.2) and then perform the Wigner transformation using the formulas

$$f(\widehat{Q})_W = f(Q), \quad f(\widehat{P})_W = f(P), \quad (\widehat{X}\widehat{Y})_W = X_W \star Y_W\,, \tag{3.9}$$

where $\star$ is the non-commutative product

$$\star = \exp\left[\frac{i\hbar}{2}\left(\overleftarrow{\partial}_Q \overrightarrow{\partial}_P - \overleftarrow{\partial}_P \overrightarrow{\partial}_Q\right)\right]. \tag{3.10}$$

Notice that the second derivatives of $U(Q)$ and $T(P)$ are exponentially suppressed for large arguments. Therefore, since at least one of Q and P is of order E on the boundary of the polygon F_{pol}, we can neglect all the terms containing $(\partial_Q^m U)(\partial_P^n T)$ with $m, n \geq 2$ for the purpose to compute the deviation δn perturbatively in $1/E$, and the Wigner Hamiltonian can be approximated with

$$H_W = U + T + \frac{\hbar^2}{24}(U')^2 T^{(2)} - \frac{\hbar^2}{12} U^{(2)}(T')^2 + \sum_{\ell\geq 3}\left(c_U^{(\ell)}(U')^\ell T^{(\ell)} + c_T^{(\ell)}(T')^\ell U^{(\ell)}\right) + \cdots, \tag{3.11}$$

where $c_U^{(\ell)}$ and $c_T^{(\ell)}$ are some constants, while $U^{(\ell)} = \partial_Q^\ell U$ and $T^{(\ell)} = \partial_P^\ell T$. The boundary $H_W(Q,P) = E$ of the region F is displayed in figure 1. The deformation of the surface is negligible except around the four corners of the polygon where the deviation (ii) is relevant. To compute δn we shall decompose it into the contributions around each corner

$$\delta n = -\frac{1}{2\pi\hbar}(\mathrm{vol(I)} + \mathrm{vol(II)} + \mathrm{vol(III)} + \mathrm{vol(IV)}). \tag{3.12}$$

First let us consider the region I. Since $Q \sim E$ in this region, we can replace $U \to (q-\xi)Q/2$ in our calculation without loss of any perturbative corrections. Under this approximation the Fermi surface adjacent to region I is characterized as

$$E = \frac{q-\xi}{2}Q + T + \frac{\hbar^2(q-\xi)^2}{96}T^{(2)} + \sum_{\ell\geq 3} c_U^{(\ell)}\left(\frac{q-\xi}{2}\right)^\ell T^{(\ell)}. \tag{3.13}$$

Denoting the points on the boundary of F as $(Q_F(P), P)$ while those on the boundary of the polygon $(Q_{\mathrm{pol}}(P), P)$, we can compute the volume of region I as

$$\begin{aligned} \mathrm{vol(I)} &= \int_{P_-}^{P_+} dP (Q_{\mathrm{pol}} - Q_F) \\ &= \frac{2}{q-\xi}\int_{P_-}^{P_+}\left(T - \frac{p|P| - \eta P}{2} + \frac{\hbar^2(q-\xi)^2}{96}T^{(2)} + \sum_{\ell\geq 3} c_U^{(\ell)}\left(\frac{q-\xi}{2}\right)^\ell T^{(\ell)}\right). \end{aligned} \tag{3.14}$$

Here $P_\pm$ correspond to some upper/lower bound: the midpoints of the edges of the approaching polygon for instance. Since $P_\pm \sim E$, at the perturbative level we can replace them with $\pm\infty$ as the integrand in (3.14) is exponentially suppressed for large P, to obtain

$$\mathrm{vol(I)} = \frac{\pi^2 p}{3(q-\xi)} + \frac{\pi^2 k^2 p(q-\xi)}{12}. \tag{3.15}$$

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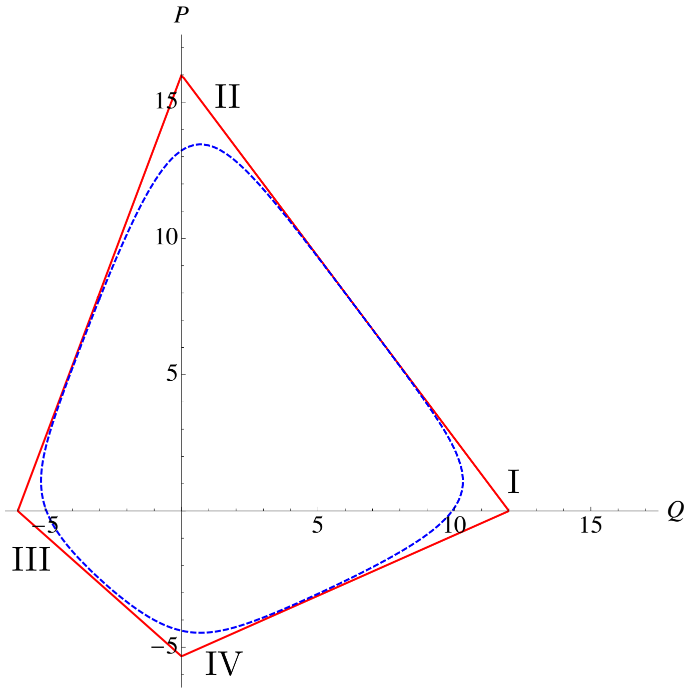

Figure 1. The boundary of the region F with $(q, \xi; p, \eta, E) = (1, 1/3; 1, 1/2, 4)$ (dashed blue line) and that of the polygon F_{pol} (solid red line).

Note that the last terms in (3.14) do not contribute as they give vanishing boundary terms. Similarly we can evaluate the volume of region II, III and IV as

$$\begin{aligned}
&\text{vol(II)} = \frac{\pi^2 q}{3(p-\eta)} - \frac{\pi^2 k^2 q(p-\eta)}{6}, \quad \text{vol(III)} = \frac{\pi^2 p}{3(q+\xi)} + \frac{\pi^2 k^2 p(q+\xi)}{12}, \\
&\text{vol(IV)} = \frac{\pi^2 q}{3(p+\eta)} - \frac{\pi^2 k^2 q(p+\eta)}{6}.
\end{aligned} \tag{3.16}$$

Substituting these results into δn (3.12), we finally obtain the expression of B in (1.12).

4 Non-perturbative effects in grand potential

In this section we study the non-perturbative corrections to the grand potential which we shall call the instantons, as well as the constant A.

To evaluate the large μ expansion of the grand potential systematically, we shall use following Mellin-Barnes expression of the grand potential ($\epsilon > 0$) [44]

$$J(\mu) = -\int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{dt}{2\pi i} \Gamma(t)\Gamma(-t)\mathcal{Z}(t)e^{t\mu} \tag{4.1}$$

with

$$\mathcal{Z}(t) = \operatorname{Tr} e^{-t\widehat{H}}. \tag{4.2}$$

If we assume $\mu < 0$ the r.h.s. of (4.1) can be evaluated by pinching the integration contour surrounding the right-half of the whole complex plane $\mathbb{C}$. Collecting the residues in this region, i.e. the residues at $t = 1, 2, \cdots$ we indeed obtain the small e^{μ} expansion of the original expression (2.14). On the other hand, if $\mu > 0$ we can pinch the contour so that it surrounds the left-half of $\mathbb{C}$. As a result, the grand potential is expressed as the sum of the residues over $\mathrm{Re}(t) \leq 0$. Due to the factor $e^{t\mu}$ in the integrand the residues are typically small or at most polynomial for large μ, which immediately give the large μ expansion of the grand potential.

We can compute $\mathcal{Z}(n)$ order by order in the small $\hbar$ expansion

$$\mathcal{Z}(n) = \sum_{s=0}^{\infty} \hbar^{2s-1} \mathcal{Z}_{2s}(n) \tag{4.3}$$

by the similar calculation as in the case of $\xi = \eta = 0$ performed in [21, 22]. Indeed, the only difference between the density matrix $\widehat{\rho} = e^{-\widehat{H}}$ with $\widehat{H}$ (3.2) and that for $\xi = \eta = 0$ is the definition of unmixed operators $U(\widehat{Q})$ and $T(\widehat{P})$. After the tedious calculation, we have obtained $\mathcal{Z}_0$, $\mathcal{Z}_2$, $\mathcal{Z}_4$ and $\mathcal{Z}_6$. The first few terms are found to have relatively simple expressions

$$\begin{aligned}
\mathcal{Z}_0(n) &= \frac{1}{2\pi} \mathrm{B}\left[\frac{q+\xi}{2}n, \frac{q-\xi}{2}n\right] \mathrm{B}\left[\frac{p+\eta}{2}n, \frac{p-\eta}{2}n\right], \\
\mathcal{Z}_2(n) &= -\frac{n^2(-1+n^2)(q^2-\xi^2)(p^2-\eta^2)}{384(1+qn)(1+pn)} \mathcal{Z}_0(n), \\
\mathcal{Z}_4(n) &= \frac{n^2(1-n^2)(q^2-\xi^2)(p^2-\eta^2)}{92160(1+qn)(1+pn)} \left[\frac{(8q+3n(q^2-\xi^2))(8p+3n(p^2-\eta^2))}{16(3+qn)(3+pn)}(-9+n^2)\right. \\
&\quad \left. + \frac{((2q+n(q^2-\xi^2))(2p+n(p^2-\eta^2))}{(2+qn)(2+pn)}(4-n^2)\right] \mathcal{Z}_0(n),
\end{aligned} \tag{4.4}$$

where B is the Euler beta function

$$\mathrm{B}(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{4.5}$$

4.1 A in the perturbative part

Before going on to the non-perturbative part, let us look the perturbative part of the grand potential again. In the Mellin-Barnes representation (4.1), this comes from the residue at $t = 0$. From the explicit expression of $\mathcal{Z}_{2s}(n)$ in (4.3) ((4.4) for small s), we obtain

$$J^{\text{pert}}(\mu) = \frac{C}{3}\mu^3 + B\mu + A. \tag{4.6}$$

Here C and B are constants which we have already computed in section 1.4, and A is

$$\begin{aligned}
A &= \frac{qp(q^3 - q\xi^2 + p^3 - p\eta^2)\zeta(3)}{\pi^2(q^2-\xi^2)(p^2-\eta^2)k} - \frac{qp(q+p)k}{24} - \frac{\pi^2 qp(qp(q+p) + 3p\xi^2 + 3q\eta^2)k^3}{8640} \\
&\quad + \frac{\pi^4 qp(qp(q^3+p^3) + 5(p\xi^4 + q\eta^4) + 10qp(q\xi^2 + p\eta^2))k^5}{1814400} + \mathcal{O}(k^7).
\end{aligned} \tag{4.7}$$

At first sight the expression looks complicated. With the simple decomposition structure we conjectured in the superconformal case [21] in mind, however, we figure out following decomposition structure again in this case

$$A = \frac{p^2}{4}(f((q+\xi)k) + f((q-\xi)k)) + \frac{q^2}{4}(f((p+\eta)k) + f((p-\eta)k)), \tag{4.8}$$

where $f(k)$ is given in a series expansion as

$$f(k) = \frac{2\zeta(3)}{\pi^2 k} - \frac{k}{12} - \frac{\pi^2 k^3}{4320} + \frac{\pi^4 k^5}{907200} + \mathcal{O}(k^7). \tag{4.9}$$

The series $f(k)$ coincide with the small k expansion of the constant map in the ABJM theory $A_{\rm ABJM}(k)$. Indeed once the structure (4.8) is postulated, we can deduce that $f(k) = A_{\rm ABJM}(k)$ by taking the limit $(q,\xi;p,\eta) \to (1,0;1,0)$ where our theory reduces to the ABJM theory. From these observations we conjecture the exact expression of A for finite k as (1.12). The conjecture is also confirmed from the exact computations of the partition function for $k \in \mathbb{N}$ in section 5.

4.2 Instantons

Due to the factor $e^{t\mu}$ in the Mellin-Barnes representation (4.1), all the residue at poles with $\mathrm{Re}(t) < 0$ are exponentially suppressed in μ. In this section we consider these non-perturbative effects in the grand potential, which we shall call the membrane instantons in an analogy of the ABJM case.

First we observe following universal structure of $\mathcal{Z}_s(n)$

$$\mathcal{Z}_s(n) = f_s(n) \times \mathcal{Z}_0(n), \tag{4.10}$$

where $f_s(n)$ are some rational functions of n. Each $f_s(n)$ have at most a finite number of poles, all of which are cancelled with the zeroes of $\mathcal{Z}_0(n)$ at the same n. This property also hold for $\mathcal{Z}_6(n)$, though we do not display its explicit expression. From these structure it follows that the instanton species are independent of the order of small $\hbar$ expansion. Here we shall display only the $\mathcal{O}(\hbar^{-1})$ part of the non-perturbative part of the grand potential,

$$J^{\rm np}(\mu) = \frac{1}{\hbar} J_0^{\rm np} + \mathcal{O}(\hbar) \tag{4.11}$$

with

$$J_0^{np} = \sum_{n=1}^\infty c_n^{(1)} e^{-\frac{2n\mu}{q+\xi}} + \sum_{n=1}^\infty c_n^{(2)} e^{-\frac{2n\mu}{q-\xi}} + \sum_{n=1}^\infty c_n^{(3)} e^{-\frac{2n\mu}{q+\eta}} + \sum_{n=1}^\infty c_n^{(4)} e^{-\frac{2n\mu}{q-\eta}} + \sum_{n=1}^\infty c_n^{(5)} e^{-n\mu}. \tag{4.12}$$

The instanton coefficients are

$$\begin{aligned}
c_n^{(1)} &= \frac{(-1)^{n-1}}{\pi n!(q+\xi)} \frac{\Gamma\big(-\frac{2n}{q+\xi}\big)\Gamma\big(\frac{2n}{q+\xi}\big)\Gamma\big(-\frac{q-\xi}{q+\xi}n\big)\Gamma\big(-\frac{p+\eta}{q+\xi}n\big)\Gamma\big(-\frac{p-\eta}{q+\xi}n\big)}{\Gamma\big(-\frac{2q}{q+\xi}n\big)\Gamma\big(-\frac{2p}{q+\xi}n\big)}, \\
c_n^{(2)} &= \frac{(-1)^{n-1}}{\pi n!(q-\xi)} \frac{\Gamma\big(-\frac{2n}{q-\xi}\big)\Gamma\big(\frac{2n}{q-\xi}\big)\Gamma\big(-\frac{q+\xi}{q-\xi}n\big)\Gamma\big(-\frac{p+\eta}{q-\xi}n\big)\Gamma\big(-\frac{p-\eta}{q-\xi}n\big)}{\Gamma\big(-\frac{2q}{q-\xi}n\big)\Gamma\big(-\frac{2p}{q-\xi}n\big)}, \\
c_n^{(3)} &= \frac{(-1)^{n-1}}{\pi n!(p+\eta)} \frac{\Gamma\big(-\frac{2n}{p+\eta}\big)\Gamma\big(\frac{2n}{p+\eta}\big)\Gamma\big(-\frac{p-\eta}{p+\eta}n\big)\Gamma\big(-\frac{q+\xi}{p+\eta}n\big)\Gamma\big(-\frac{q-\xi}{q+\eta}n\big)}{\Gamma\big(-\frac{2p}{p+\eta}n\big)\Gamma\big(-\frac{2q}{p+\eta}n\big)}, \\
c_n^{(4)} &= \frac{(-1)^{n-1}}{\pi n!(p-\eta)} \frac{\Gamma\big(-\frac{2n}{p-\eta}\big)\Gamma\big(\frac{2n}{p-\eta}\big)\Gamma\big(-\frac{p+\eta}{p-\eta}n\big)\Gamma\big(-\frac{q+\xi}{p-\eta}n\big)\Gamma\big(-\frac{q-\xi}{q-\eta}n\big)}{\Gamma\big(-\frac{2p}{p-\eta}n\big)\Gamma\big(-\frac{2q}{p-\eta}n\big)}, \\
c_n^{(5)} &= -\frac{(-1)^{n-1}}{2\pi n} \frac{\Gamma\big(-\frac{q+\xi}{2}n\big)\Gamma\big(-\frac{q-\xi}{2}n\big)\Gamma\big(-\frac{p+\eta}{2}n\big)\Gamma\big(-\frac{p-\eta}{2}n\big)}{\Gamma(-qn)\Gamma(-pn)},
\end{aligned} \tag{4.13}$$

where we have used the expression for the Euler beta function $\mathrm{B}(x, y)$ in the Gamma functions (4.5) to clarify the pole structure of the instanton coefficients.

4.3 Divergence and mixing of instantons

Lastly let us study the divergent structure of the instantons. Respecting the original setup of the quiver Chern-Simons theory here we assume $q, p \in \mathbb{N}$.

First we consider the fifth kind of instanton. Since the coefficients of this instanton $c_n^{(5)}$ contain divergent factors $\Gamma(-qn)\Gamma(-pn)$ in the denominator, the coefficients generically vanishes for all $n \geq 1$. The only exception happens if the arguments of the Gamma functions in the denominator are also negative integers so that the divergences in the numerator compensate the divergences in the denominator. In those case the exponent coincides that of some instantons in the other four kinds and the HMO pole cancellation mechanism occurs [13, 22]. In this sense, these fifth kind of instantons never produce distinctive instanton effects, as called the "ghost instantons" in [22].

Next we consider the other four kinds of instantons. In the superconformal limit $\xi = \eta = 0$ they reduce to the two kinds of membrane instantons $(e^{-2\mu/q}, e^{-2\mu/p})$ which would be associated with the $\mathbb{Z}_q$- and $\mathbb{Z}_p$-orbifold in the background geometry. Indeed each instanton coefficient (4.13) have similar structure individually. The rules for divergence and mixing are, however, slightly complicated than that in the superconformal case:

- In the superconformal case, the coefficient always diverges when the instanton exponent coincide with that of another instanton. This divergence is cancelled by the divergence of the other instanton with same exponent. For the mixing among the four instantons in current case, this is not alway the case. When the mixing is between only the first two $(e^{-2\mu/(q+\xi)}, e^{-2\mu/(q-\xi)})$ or the last two $(e^{-2\mu/(p+\eta)}, e^{-2\mu/(p-\eta)})$ the individual coefficients remains finite.

- In the superconformal case, the mixing and pole cancellation are inevitable, since $q, p \in \mathbb{N}$ as obvious from their roles in the orbifold. Due to this restriction the slight modification such as $q \to q + \epsilon$ to disentangle the mixing pair is unphysical. In current case, however, ξ, η are continuous parameters of the original theory. This suggest that, not only the finite part remaining after the cancellation but also the divergence itself would have some gravitational counterpart.

5 Exact partition function for finite (k, N)

In the superconformal case $\xi = \eta = 0$, a particular structure of the density matrix allows the systematic computation of the partition function with finite $k, N \in \mathbb{N}$. As we will see below, the method can be generalized for the case without superconformal symmetry if ξ, η are rational numbers. In this section we concentrate on the deformation of the ABJM theory, $q = p = 1$ and compute the partition function for various $k, N \in \mathbb{N}$ and $\xi, \eta \in \mathbb{Q}$.

5.1 Systematic computation of partition function

In this section we display the algorithm to compute the traces of the density matrix $\mathrm{Tr}\,\widehat{\rho}^n$ of given k, ξ, η recursively in n. The partition functions $Z(N)$ can be read off through the definition of the grand potential (1.1) and (2.14). Here we would like to consider only the case with $q = p = 1$. The way to extend the method for the case with general $q, p \in \mathbb{N}$ is identical to that in the superconformal case [23, 24].

The essence for this method is following schematic structure of the density matrix

$$\rho(Q_1, Q_2) = \frac{1}{2\pi}\langle Q_1|\widehat{\rho}|Q_2\rangle = \frac{E(Q_1)E(Q_2)}{\alpha A(Q_1) + \alpha^{-1}A(Q_2)}. \tag{5.1}$$

The explicit expression of each ingredient is

$$E(Q) = \frac{e^{(\frac{\xi}{4}+\frac{1}{2k})Q}}{\left(2\cosh\frac{Q}{2}\right)^{\frac{1}{2}}}, \quad A(Q) = 2\pi k e^{\frac{Q}{k}}, \quad \alpha = e^{-\frac{\pi i \eta}{2}}. \tag{5.2}$$

From the structure (5.1) the powers of the density matrix are

$$\begin{aligned}\rho^n(Q_1, Q_2) &= \frac{1}{2\pi}\langle Q_1|\widehat{\rho}^n|Q_2\rangle \\ &= \frac{E(Q_1)E(Q_2)}{A(Q_1) - (-1)^n\alpha^{-2n}A(Q_2)}\sum_{m=0}^{n-1}(-1)^m\alpha^{-2m-1}\psi_m(Q_1)\phi_{n-m-1}(Q_2)\end{aligned} \tag{5.3}$$

with

$$\psi_m(Q) = \frac{1}{E(Q)}\int dQ'\rho^m(Q, Q')E(Q'), \quad \phi_m(Q) = \int dQ' E(Q')\rho^m(Q', Q)\frac{1}{E(Q)}. \tag{5.4}$$

Note that ϕ_m are related to ψ_m through the complex conjugation of the replacement $\alpha \to \alpha^{-1}$. Now the task to compute the powers of matrix $\rho^n(Q_1, Q_2)$ is reduced into the computation of vectors $\psi_n(Q)$ and $\phi_n(Q)$. Moreover, these vectors can be computed by simple iterative steps if $\xi \in \mathbb{Q}$, as we shall see below.

We would like to focus on the vector ψ_m which obeys the recursion relation

$$\psi_{m+1}(Q) = \frac{1}{E(Q)}\int dQ'\rho(Q, Q')E(Q')\psi_m(Q'). \tag{5.5}$$

For ξ being a rational number, say $\xi = b/a$ for some $a, b \in \mathbb{N}$ ($a \perp b$), we can introduce new integration variable $u = e^{Q/w}$ with $w = \mathrm{lcm}(2a, k)$ to rewrite the recursion relation (5.5) as

$$\psi_{m+1}(u) = \frac{x\alpha}{2\pi}\int_0^\infty dv \frac{1}{v^x + \alpha^2 u^x}\frac{v^{x+y+\frac{w}{2}-1}}{v^w + 1}\psi_m(v). \tag{5.6}$$

Here x and y are integers given by $x = w/k$ and $y = w\xi/2$. This integration relation is in the same type as that used in the ABJM theory [12] and can be simplified by expanding $\psi_m(u)$ in the series of $\log u$

$$\psi_m(u) = \sum_{j\geq 0}\psi_m^{(j)}(u)(\log u)^j \tag{5.7}$$

with $\psi_m^{(j)}(u)$ rational functions in u, as

$$\psi_m(u) = -\frac{x\alpha}{2\pi}\sum_{j\geq 0}\frac{(2\pi i)^{j+1}}{j+1}\sum_{v_a\in\mathbb{C}\setminus\mathbb{R}^+}\mathrm{Res}\left[\frac{1}{v^x+\alpha^2u^x}\frac{v^{x+y+\frac{w}{2}-1}}{v^w+1}\psi_m^{(j)}(v)B_{j+1}\left[\frac{\log^{(+)}v}{2\pi i}\right], v\to v_a\right]. \tag{5.8}$$

Here $\log^{(+)}$ is the logarithm function with branch cut on $\mathbb{R}^+$, and $B_j(z)$ are the Bernoulli polynomials. In the contribution of the poles associated with the first factor $1/(v^x+\alpha^2u^x)$ we assume $u\in\mathbb{R}^+$.

Once ψ_m and ϕ_m are computed in this manner, the integration in the computation of $\mathrm{Tr}\,\widehat{\rho}^n$ from (5.1) can be manipulated in the same way, and we finally obtain when $(-1)^n\alpha^{-2n}\neq 1$

$$\mathrm{Tr}\widehat{\rho}^n = -\frac{x}{2\pi}\frac{1}{1-(-1)^n\alpha^{-2n}}\sum_{j\geq 0}\frac{(2\pi i)^{j+1}}{j+1}\mathrm{Res}\left[\sum_{v_a\in\mathbb{C}\setminus\mathbb{R}^+}\frac{v^{y+\frac{w}{2}-1}}{v^w+1}f_n^{(j)}(v)B_{j+1}\left[\frac{\log' v}{2\pi i}\right], v\to v_a\right], \tag{5.9}$$

while for $(-1)^n\alpha^{-2n}=1$

$$\mathrm{Tr}\widehat{\rho}^n = -\frac{1}{2\pi}\sum_{j\geq 0}\frac{(2\pi i)^{j+1}}{j+1}\mathrm{Res}\left[\sum_{v_a\in\mathbb{C}\setminus\mathbb{R}^+}\frac{v^{y+\frac{w}{2}}}{v^w+1}g_n^{(j)}(v)B_{j+1}\left[\frac{\log' v}{2\pi i}\right], v\to v_a\right]. \tag{5.10}$$

Here $f_n^{(j)}(u)$ and $g_n^{(j)}(u)$ are the rational functions defined by

$$\begin{aligned}\sum_{j\geq 0}f_n^{(j)}(u)(\log u)^j &= \sum_{m=0}^{n-1}(-1)^m\alpha^{-2m-1}\psi_m(u)\phi_{n-m-1}(u),\\ \sum_{j\geq 0}g_n^{(j)}(u)(\log u)^j &= \sum_{m=0}^{n-1}(-1)^m\alpha^{-2m-1}\partial_u\psi_m(u)\phi_{n-m-1}(u).\end{aligned} \tag{5.11}$$

5.1.1 Poles generated by iterations

In the case $\eta=0$, the poles we need to take into account in the iteration (5.8) are always following two series

$$\begin{aligned} v &= e^{\frac{\pi i(2\ell-1)}{x}}\alpha^{\frac{2}{x}}u, &&(\ell=1,2,\cdots x)\\ v &= e^{\pi i(2\ell-1)/w}, &&(\ell=1,2,\cdots w)\end{aligned} \tag{5.12}$$

where the poles in the first line come from the first factor $1/(v^x+\alpha^2u^x)$ and the second line from the second factor $1/(v^w+1)$.

The situation is different for general values of η, since the third factor $\psi_m^{(j)}(v)$ may have distinct poles. These poles are generated by both the residue at the u-dependent poles and that for the u-independent poles. To clarify the pole contents of ψ_m,[6] first let us study

[6] Here we do not mind the order of each pole.

$\xi\backslash\eta$	1/4	1/3	1/2
0	4	5	8
1/4	3	4	5
1/3	–	2	6
1/2	–	–	5

Table 1. The values of $N_{\max}$ for each $(k=4,\xi,\eta)$ in our computation. We have chosen ξ,η as $\xi\le\eta$ since the matrix model is trivially symmetric under $\xi\leftrightarrow\eta$.

the poles generated in the step $\psi_1\to\psi_2$. From the residues at the poles in the first line of (5.12), we find

$$\begin{aligned}\psi_2(u)&\propto\frac{1}{(\alpha^{\frac{2}{x}}ue^{\frac{\pi i(2j-1)}{x}})^w+1}\\&\propto\frac{1}{u^w+e^{\pi ik}\alpha^{-2k}},\end{aligned}\tag{5.13}$$

where we have used the fact $w/x=k\in\mathbb{N}$. On the other hand the poles in the second line in (5.12) generates

$$\begin{aligned}\psi_2(u)&\propto\frac{1}{(e^{\frac{\pi i(2\ell-1)}{w}})^x+\alpha^2u^x}\\&\propto\frac{1}{u^w+e^{\pi ik}\alpha^{-2k}},\end{aligned}\tag{5.14}$$

where in the second line we have reduced together the fractions with $\ell=1,2,\cdots w$. From these results we conclude that $\psi_2(u)$ have new poles associated with the factor $1/(u^w+e^{\pi ik}\alpha^{-2k})$. In the step $\psi_2\to\psi_3$ the cross substitution of the poles of this factor and those of the first factor in (5.8) again generates the new pole factors $1/(u^w+\alpha^{-4k})$. Repeating these arguments we conclude that ψ_m have following poles

$$\psi_m\propto\prod_{\ell=0}^{m-1}\frac{1}{u^w+e^{\pi ik\ell}\alpha^{-2k\ell}}.\tag{5.15}$$

In the computation of ψ_m (5.8) and $\mathrm{Tr}\widehat{\rho}^n$ (5.9), (5.10), we need to take into account of these poles.

5.2 Comparison with small k expansion

In this section we compare the exact values with the results of the semiclassical analysis in section 3, 4 to confirm the conjectural expression for A (1.12) and observe the non-perturbative effects corresponding to the worldsheet instantons $\mathcal{O}(e^{-\frac{\mu}{k}})$. For this purpose we have computed the exact partition functions for $k=4$ and various pairs of ξ,η, $Z_k^{(\xi;\eta)}(N)$ with $N=1,2,\cdots,N_{\max}$, where the values of $N_{\max}$ are listed in table 1. The exact values are collected in appendix A.

The expression of A in (1.12) can be evaluated for finite k by the integral expression of the constant map in the ABJM theory [20]

$$A_{\mathrm{ABJM}}(k)=\frac{2\zeta(3)}{\pi^2k}\left(1-\frac{k^3}{16}\right)+\frac{k^2}{\pi^2}\int_0^\infty dx\frac{x}{e^{kx}-1}\log(1-e^{-2x}).\tag{5.16}$$

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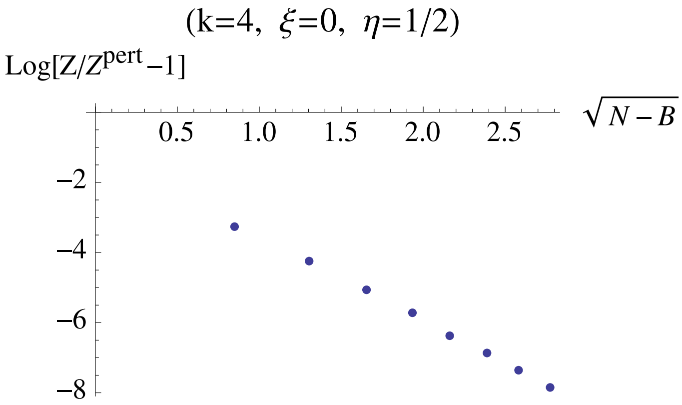

Figure 2. The non-perturbative part of the exact partition function remaining after the subtraction of the perturbative part $Z^{\text{pert}}(N)$ for $(k,\xi,\eta)=(4,0,1/2)$. The combination $Z(N)/Z^{\text{pert}}(N)-1$ behaves like $e^{-\omega\sqrt{\frac{N-B}{C}}}$ even for small N.

Comparing these values and those obtained by fitting the exact values of the partition function (A.1)–(A.9) with the Airy function (1.4) with B and C (1.12), we confirm that our conjecture (1.12) is indeed correct (see table 2 for $k=4$).

Next let us compare the instanton exponent. With the help of the inversion formula (1.2), the leading non-perturbative effect can be directly related to the exact values as [12]

$$J(\mu)-J^{\text{pert}}(\mu)\sim e^{-\omega\mu}\iff\frac{Z(N)}{Z^{\text{pert}}(N)}-1\sim e^{-\sqrt{\frac{N-B}{C}}}\,. \tag{5.17}$$

We observe that this approximation for the exact values holds even for small N (see figure 2) and thus we can estimate the leading instanton exponents ω by fitting as in table 2. Since the results are considerably different from the leading exponent $e^{-2\mu/(1+\eta)}$ obtained from the WKB expansion, we conclude that they correspond to the worldsheet instanton effects $\mathcal{O}(e^{-\frac{\mu}{k}})$.

6 Discussion

In this paper we have studied the partition function of a continuous deformation of the U(N) circular quiver superconformal Chern-Simons theories. The deformation corresponds to general R-charge assignments on the bifundamental matter fields. Formally the partition function of the deformed theory have similar structures as the superconformal case. We can use the Fermi gas formalism to compute the large N expansion of the partition function. Applying this technique to the deformation of $\mathcal{N}=4$ theory with the levels characterized by two integers q and p through (1.5) and (1.11), we have achieved to determine all order perturbative corrections in $1/N$, which sum up to an Airy function. The restriction to $\mathcal{N}=4$ theories with special choice (1.11) allows us the complete determination of the coefficients B and A appearing in the Airy function, as well as the discovery of the five series of non-perturbative effects (membrane instantons). We find a beautiful decomposition structure of A (1.12) which is a natural generalization of the similar structure in the undeformed theories. Though there are no clear explanation so far, the rule of decomposition is strongly

k	ξ	η	A (1.12)	A (fitting)	fitting/(1.12)	ω(fitting)	$\frac{2}{1+\eta}$	$\frac{4}{k(1+\xi)(1+\eta)}$
4	0	1/4	-0.38131	-0.38134	1.00009	0.79	1.6	0.8
4	0	1/3	-0.38442	-0.38443	1.00004	0.74	1.5	0.75
4	0	1/2	-0.39191	-0.39194	1.00007	0.66	1.3	0.67
4	1/4	1/4	-0.3856	-0.3862	1.0015	0.66	1.6	0.64
4	1/4	1/3	-0.3887	-0.3893	1.0016	0.61	1.5	0.6
4	1/4	1/2	-0.3962	-0.3976	1.0036	0.55	1.3	0.53
4	1/3	1/3	-0.3918	-0.3936	1.0045	0.57	1.5	0.56
4	1/3	1/2	-0.3993	-0.4008	1.0037	0.51	1.3	0.5
4	1/2	1/2	-0.4068	-0.4107	1.0096	0.45	1.3	0.44

Table 2. Left: the value of A in (1.12) computed with (5.16) and the results of fitting. Right: the leading instanton exponent ω estimated by fitting, which disagree with the leading exponents expected from WKB expansion $2/(1+\eta)$ but rather agree with the conjectural worldsheet instanton exponent $4/(k(1+\xi)(1+\eta))$.

correlated to the subdivision of the membrane instantons.[7] We wish to provide some interpretation to this structure in future.

We can also determine the instanton coefficients in the limit $k \to 0$ while exactly in the other parameters (q, p, ξ, η). We find the singular structure of the coefficients with respect to these parameters, which are again reminiscent of the superconformal case. The major difference from the superconformal case is the subdivision of the membrane instantons (4.13) by the two *continuous* deformation parameters (ξ, η) (1.13). Correspondingly, the instanton coefficients diverge for special values of ξ and η. The interpretation of these divergences is conceptually different from those appearing in the superconformal case. In the original superconformal theory q and p are associated to the number of vertices and must be positive integers. Especially, since they characterize the orbifold structure of Y_7 [31] in the gravity side, there are no dual geometry corresponding to the non-integral (q, p). Though we can visualize the divergence of the coefficients by continuing these parameters to irrational numbers in the analysis of the matrix model, at the physical values of (k, q, p) there only remain the finite coefficients resulting after the pole cancellation. In contrast, the R-charge assignments ξ and η can be chosen to be arbitrary real number under (2.17), and thus the coefficient of individual instanton can be infinitely large. This implies that the divergences would be meaningful phenomena in the context of the AdS/CFT correspondence.

On the other hand, our deformation drastically modify the dual geometry. Any non-canonical choice of the R-charges break the conformal symmetry, which induces in the gravity side a non-trivial dependence on the radial direction or holographic RG flow [45, 46]. It will be interesting to reveal the dual geometry to our theory, construct instanton solution in that background and reveal what occurs near the special values of (q, p, ξ, η) where the individual instanton coefficients (4.13) diverge in the field theory side.

[7]Similar decomposition structure was observed also in the supercoformal theories with affine D-type quiver [26] and those with $O(N)$ and $\mathrm{USp}(2N)$ gauge groups [28].

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We have also analyzed the non-perturbative effects in $1/N$ for finite k and found disagreement with the membrane instantons. We have concluded them to be the analog of the worldsheet instantons and conjecture the exponents as $e^{-\frac{4\ell\mu}{k(q\pm\xi)(p\pm\eta)}}$ $(\ell=1,2,\cdots)$. Though we have confirmed the exponents for $k=4$ (and also for $k=3,6$ with few undisplayed data), we could not determine the exact values of their coefficients. We wish to determine these effects more quantitatively in future. It would also be interesting to consider similar continuous deformation for the theories with non-circular quiver [25, 26] or non-unitary gauge groups [27, 28, 47]

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A List of exact values $Z_4^{(\xi;\eta)}(N)$

$$
\begin{aligned}
&Z_4^{(0;1/4)}(1)=\frac{\sqrt{2}}{8}\sin\frac{\pi}{8},\quad Z_4^{(0;1/4)}(2)=\frac{-4+3\sqrt{2}}{512},\\
&Z_4^{(0;1/4)}(3)=\frac{16(2+\sqrt{2})+(1-13\sqrt{2})\pi}{2048\pi}\sin\frac{\pi}{8},\quad Z_4^{(0;1/4)}(4)=\frac{-128(2+\sqrt{2})+(9+92\sqrt{2})\pi}{262144\pi},
\end{aligned}
\tag{A.1}
$$

$$
\begin{aligned}
&Z_4^{(0;1/3)}(1)=\frac{\sqrt{3}}{24},\quad Z_4^{(0;1/3)}(2)=\frac{1}{1728},\quad Z_4^{(0;1/3)}(3)=\frac{-216+(216-85\sqrt{3})\pi}{20736\pi},\\
&Z_4^{(0;1/3)}(4)=\frac{4320\sqrt{3}+(-137-1296\sqrt{3})\pi}{5971968\pi},\quad Z_4^{(0;1/3)}(5)=\frac{-10584+(25056-12521\sqrt{3})\pi}{35831808\pi},
\end{aligned}
\tag{A.2}
$$

$$
\begin{aligned}
&Z_4^{(0;1/2)}(1)=\frac{\sqrt{2}}{16},\quad Z_4^{(0;1/2)}(2)=\frac{4-\pi}{256\pi},\quad Z_4^{(0;1/2)}(3)=\frac{-8\sqrt{2}+12\sqrt{2}\pi-3\sqrt{2}\pi^2}{4096\pi^2},\\
&Z_4^{(0;1/2)}(4)=\frac{-304-120\pi+69\pi^2}{393216\pi^2},\\
&Z_4^{(0;1/2)}(5)=\frac{-192\sqrt{2}-960\sqrt{2}\pi-3424\sqrt{2}\pi^2-1832\sqrt{2}\pi^3+963\sqrt{2}\pi^4}{18874368\pi^4},\\
&Z_4^{(0;1/2)}(6)=\frac{-3840-20160\pi+42640\pi^2+36236\pi^3-15165\pi^4}{1509949440\pi^4},\\
&Z_4^{(0;1/2)}(7)=\frac{23040\sqrt{2}-241920\sqrt{2}\pi+3019200\sqrt{2}\pi^2-7891200\sqrt{2}\pi^3-1138088\sqrt{2}\pi^4}{1087163596800\pi^6}\\
&\qquad+\frac{6964164\sqrt{2}\pi^5-1877175\sqrt{2}\pi^6}{1087163596800\pi^6},
\end{aligned}
$$

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$$Z_4^{(0;1/2)}(8) = \frac{-3870720+16441600\pi+85612800\pi^2-107341600\pi^3-115970736\pi^4}{54116587929600\pi^5} + \frac{44873325\pi^5}{54116587929600\pi^5}, \tag{A.3}$$

$$\begin{aligned} Z_4^{(1/4;1/4)}(1) &= \frac{2-\sqrt{2}}{8}, \quad Z_4^{(1/4;1/4)}(2) = \frac{1-2\sqrt{2}+2(1+\sqrt{2})\sin\frac{\pi}{8}}{32}, \\ Z_4^{(1/4;1/4)}(3) &= \frac{-(4+3\sqrt{2})\sin\frac{\pi}{8}+(4-(5+2\sqrt{2})\sin\frac{\pi}{8})\pi}{128\pi}, \end{aligned} \tag{A.4}$$

$$\begin{aligned} Z_4^{(1/4;1/3)}(1) &= \frac{\sqrt{6}}{12}\sin\frac{\pi}{8}, \quad Z_4^{(1/4;1/3)}(2) = \frac{-1-\sqrt{2}+\sqrt{6}}{48}, \\ Z_4^{(1/4;1/3)}(3) &= \frac{72+36\sqrt{2}+(18+81\sqrt{2}-68\sqrt{3}-22\sqrt{6})\pi}{1728\pi}\sin\frac{\pi}{8}, \\ Z_4^{(1/4;1/3)}(4) &= \frac{24\sqrt{6}+\left(228-77\sqrt{2}+84\sqrt{3}-120\sqrt{6}+3\sqrt{6(443-180\sqrt{6})}\right)\pi}{27648\pi}, \end{aligned} \tag{A.5}$$

$$\begin{aligned} Z_4^{(1/4;1/2)}(1) &= \frac{1}{4}\sin\frac{\pi}{8}, \quad Z_4^{(1/4;1/2)}(2) = \frac{3-2\sqrt{2}}{128}, \\ Z_4^{(1/4;1/2)}(3) &= \frac{-32-16\sqrt{2}+(16+\sqrt{2})\pi}{2048\pi}\sin\frac{\pi}{8}, \quad Z_4^{(1/4;1/2)}(4) = \frac{-64+(43-16\sqrt{2})\pi}{131072\pi}, \\ Z_4^{(1/4;1/2)}(5) &= \frac{768+768\sqrt{2}+(3424+3712\sqrt{2})\pi+(-2079-615\sqrt{2})\pi^2}{3145728\pi^2}\sin\frac{\pi}{8}, \end{aligned} \tag{A.6}$$

$$Z_4^{(1/3;1/3)}(1) = \frac{1}{12}, \quad Z_4^{(1/3;1/3)}(2) = \frac{-3+4\sqrt{3}\sin\frac{\pi}{9}+4\sin\frac{\pi}{18}}{72}, \tag{A.7}$$

$$\begin{aligned} Z_4^{(1/3;1/2)}(1) &= \frac{\sqrt{6}}{24}, \quad Z_4^{(1/3;1/2)}(2) = \frac{-3+2\sqrt{3}}{288}, \quad Z_4^{(1/3;1/2)}(3) = \frac{144\sqrt{2}+(-96\sqrt{2}+29\sqrt{6})\pi}{13824\pi}, \\ Z_4^{(1/3;1/2)}(4) &= \frac{-64\sqrt{3}+(63-16\sqrt{3})\pi}{221184\pi}, \\ Z_4^{(1/3;1/2)}(5) &= \frac{-1344\sqrt{2}+384\sqrt{6}+(-1632\sqrt{2}+1067\sqrt{6})\pi}{5308416\pi}, \\ Z_4^{(1/3;1/2)}(6) &= \frac{311040+(1009152+159840\sqrt{3})\pi-1141425\pi^2+404470\sqrt{3}\pi^2}{2866544640\pi^2}, \end{aligned} \tag{A.8}$$

$$\begin{aligned} Z_4^{(1/2;1/2)}(1) &= \frac{1}{8}, \quad Z_4^{(1/2;1/2)}(2) = \frac{-2+\pi}{128\pi}, \quad Z_4^{(1/2;1/2)}(3) = \frac{2-10\pi+3\pi^2}{1024\pi^2}, \\ Z_4^{(1/2;1/2)}(4) &= \frac{70+6\pi-9\pi^2}{49152\pi^2}, \quad Z_4^{(1/2;1/2)}(5) = \frac{3+66\pi+406\pi^2+175\pi^3-99\pi^4}{589824\pi^4}. \end{aligned} \tag{A.9}$$

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