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# Stability of the travelling front of a decaying brane

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Abstract: The dynamics (in light-cone time) of the tachyon on an unstable brane in the background of a dilaton linear along a null coordinate is a non-local reaction-diffusion type equation, which admits a travelling front solution. We analyze the (in-)stability of this solution using linearized perturbation theory. We find that the front solution obtained in singular perturbation method is stable. However, these inhomogenous solutions (unlike the homogenous solution) also have Lyapunov exponents corresponding to unstable modes around the (meta-)stable vacuum.

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## Contents



## 1 Introduction

It is interesting and important to study the dynamics of instabilities in string theory. While this general question is too broad in its scope, the question of tachyonic instabilities in configurations of D-branes in string theory [1] is more specific and tractable [2–8]. In particular, using light-cone coordinates and putting the open strings in a dilaton background, which is linear along a null direction, the authors of ref. [3] studied the homogeneous[1] decay process in the effective field theory of the tachyon and extended this to a complete set of equations of motion of the open string field theory. If we consider inhomogeneous decay in this framework in which the tachyon field depends on the (light-cone) time and one other (spatial) coordinate along the brane, the equation of motion of the tachyon turns out to resemble a reaction-diffusion type equation that was pioneered in refs. [9–11] and appeared ubiquitously since. There are some additional elements, however. Specifically, the non-linear reaction term of what we call the *Fisher equation for the tachyon on a decaying brane*, eq. (2.1), involves a time delay and spatial averaging with a Gaussian kernel, hence it is non-local [8]. Even though non-locality in reaction-diffusion systems has been considered in the literature, in Mathematical Biology for instance (see [12–14] for example), the combination of delay and (the specific form of) non-local interactions that are inherent in open string field theory is quite characteristic. It also makes the resulting equations more interesting and difficult to analyze.

As is the case for these type of equations, the Fisher equation for the tachyon also admits a *travelling front* solution. This front, which can be found using a singular perturbation analysis [15–17], separates the brane from the (closed string) vacuum, while moving with a constant speed that is attained asymptotically. We have also extended the traveling

[1]Following standard terminology, by *homogeneous decay* we mean the dynamical evolution of the tachyon dependent on (light-cone) time only.

front to a solution of the equations of motion of open string field theory to the first non-trivial order [18]. In terms of the boundary conformal field theory on the worldsheet of the string, which provides the background for the open string field theory, this corresponds to a deformation by a marginal operator which remains marginal when the first stringy corrections are included. The disc one-point functions of the closed string tachyon and graviton vertex operators, in the presence of this marginal deformation, were also studied in ref. [18].

It is worth noting that the inhomogeneous decay described by the travelling front is closer to a natural decay process. One would expect tachyon condensation to start, perhaps due to a fluctuation, in a small region of space. This nucleus, just like the condensation of a droplet in a supercooled gas, would grow in size. In one dimension this would give rise to two fronts travelling in opposite directions. In higher dimensions, the Laplacian $\nabla^2$ would appear in place of $\partial_x^2$ in eq. (2.1) and the resulting equation is not quite a Fisher-type equation. However, for spherically symmetric decay, $\nabla_d^2 = \frac{\partial^2}{\partial r^2} + \frac{d-1}{r}\frac{\partial}{\partial r}$ is approximated by $\frac{\partial^2}{\partial r^2}$ for large $r$, leading to a Fisher-type equation in the asymptotic limit.

In this paper, we shall consider the stability of the traveling front. This analysis will be in the context of the effective field theory of the tachyon. We shall study the behaviour of small fluctuations around the front solution using linearized perturbation theory and argue that it is stable. We do, however, find a potential instability around the stable vacuum, reminiscent of the oscillations in ref. [2]. This does not destabilize the front solution, obtained using a singular perturbation method starting with the solution corresponding to the homogeneous decay.

## 2 Tachyon Fisher equation and the travelling front

We recall that the dynamics of the open string modes are given by the cubic open string field theory. In a given background, the string field can be expanded in terms of the states in the Hilbert space of the underlying boundary conformal field theory on the worldsheet with coefficients that are the 'wavefunctions'. The leading contribution is the tachyon field $\phi(x^\mu)$ on the unstable brane. This is a Klein-Gordon equation with negative mass-square augmented by *non-local* cubic self-interactions. The solutions of this equations are untamed oscillations [2] which may be attributed to the fact that the energy in the D-brane cannot be dissipated to the closed string modes in the absence of any coupling between the open and closed string modes.

A simple and elegant approach to this problem that avoids the complexities of an open-closed string field theory was proposed in ref. [3] and explored further by us [8, 18]. The idea is to consider one of the light-cone coordinates (say $x^+$) as time, and at the same time consider a dilaton background that in linear along the other light-cone direction $x^-$. This changes the essential character of the dynamical equations, while retaining the solvability of the underlying conformal field theory. In particular, the equation of motion of the tachyonic scalar field is

$$b\partial_t\phi - \partial_x^2\phi - m^2\phi + K^3 e^{-2\alpha b\partial_t + \alpha\partial_x^2}\left(e^{\alpha\partial_x^2}\phi\right)^2 = 0, \tag{2.1}$$

where $-m^2 = -1$ is the mass-square of the tachyon, $b$ is the slope of the linear dilaton and $\alpha = \ln K = \ln\left(3\sqrt{3}/4\right)$ is a number originating in the conformal maps that define the string field theory. As mentioned above, $t \equiv x^+$ denotes light-cone time, and for simplicity, we have taken $\phi$ to depend only on one spatial coordinate $x$. This is a reaction-diffusion equation with time *delay* and spatial *non-locality.* We refer to it as the *Fisher equation for the tachyon on a decaying brane.*

Like all equations of this type, of which there are innumerable examples in the literature, the above admits travelling front solutions. To see this, let us change variables to the comoving coordinate[2] and time

$$\xi = x + vt, \qquad \tau = t,$$

in terms of which the equation reads as follows:

$$b\frac{\partial\phi}{\partial\tau} + bv\frac{\partial\phi}{\partial\xi} - \frac{\partial^2\phi}{\partial\xi^2} - \phi + K^3 e^{-2\alpha b\partial_\tau - 2\alpha bv\partial_\xi + \alpha\partial_\xi^2}\left(e^{\alpha\partial_\xi^2}\phi\right)^2 = 0. \tag{2.2}$$

The travelling front does not have an explicit dependence on $t$ and is a function $\xi$ alone. Therefore it satisfies

$$bv\frac{\partial\Phi_v}{\partial\xi} - \frac{\partial^2\Phi_v}{\partial\xi^2} - \Phi_v + K^3 e^{-2\alpha bv\partial_\xi + \alpha\partial_\xi^2}\left(e^{\alpha\partial_\xi^2}\Phi_v\right)^2 = 0. \tag{2.3}$$

The nonlocalities in the equations above can alternatively be written using the Gaussian kernel

$$e^{\alpha\partial_\xi^2} f(\xi) = \frac{1}{2\sqrt{\alpha\pi}}\int_{-\infty}^{\infty} d\xi'\, e^{-\frac{1}{4\alpha}(\xi'-\xi)^2} f(\xi') \equiv \mathfrak{G}_\alpha[f(\xi)], \tag{2.4}$$

and the fact that $e^{-a\partial_x} f(x) = f(x-a)$, (for which $e^{-a\partial_x}\left(f(x)g(x)\right) = f(x-a)g(x-a)$ holds):

$$\left(b\frac{\partial}{\partial\tau} + bv\frac{\partial}{\partial\xi} - \frac{\partial^2}{\partial\xi^2} - 1\right)\phi(\xi,\tau) + K^3\,\mathfrak{G}_\alpha\left[\left(\mathfrak{G}_\alpha\left[\phi(\xi - 2\alpha bv, \tau - 2\alpha b)\right]\right)^2\right] = 0,$$
$$bv\frac{\partial\Phi_v(\xi)}{\partial\xi} - \frac{\partial^2\Phi_v(\xi)}{\partial\xi^2} - \Phi_v(\xi) + K^3\mathfrak{G}_\alpha\left[\left(\mathfrak{G}_\alpha\left[\Phi_v(\xi - 2\alpha bv)\right]\right)^2\right] = 0.$$

The travelling front solution to these equations [8] can be obtained in singular perturbation theory.

### 2.1 Convergence and (in-)stability around the fixed points

The differential equation above for inhomogeneous decay to leading order is of order two. However, as in the case of the homogenous decay studied in ref. [3], it has two fixed points, and the travelling front interpolates between the unstable fixed point $\phi_{\rm U} = 0$ to the stable one at $\phi_{\rm S} = K^{-3} \simeq 0.456$. It goes away from $\phi_{\rm U} = 0$ exponentially, the exponent being determined by the negative mass-square of the tachyon. Around $\phi_{\rm S}$, however, due to the

[2]This corresponds to a front moving to the left. The front moving to the right is obviously also a solution.

presence of the delay and non-locality, convergence is oscillatory. These can be deduced from a linearized perturbation analysis around the fixed points.

First, consider the unstable fixed point $\phi_{\rm U} = 0$. We can ignore the non-linear term in eq. (2.3) and substitute $\phi = e^{\mu\xi}$. This gives

$$\mu = \frac{1}{2}\left(bv \pm \sqrt{b^2v^2 - 4}\right)$$

which gives the minimum speed of the front as $bv = 2$. The nonlocalities in the interaction term does not affect this behaviour, thus it is the same as in the standard Fisher equation.

Indeed, this is true not only of the asymptotic speed, but also the way it is approached. Given a profile $\phi(x, 0)$ at $\tau = 0$, the solution to the equation linearized around $\phi_{\rm U} = 0$, namely $b\partial_\tau\phi = \partial_x^2\phi + \phi$, is given by[3]

$$\begin{aligned}
\phi(x,\tau) &= \int_{-\infty}^{\infty} dy\, \phi(y,0)\, \frac{1}{\sqrt{4\pi\tau/b}} e^{\frac{b}{4\tau}\left(-(x-y)^2+\left(\frac{2\tau}{b}\right)^2\right)} \\
&\propto \frac{1}{\sqrt{4\pi\tau/b}}\, e^{-\frac{b}{4\tau}\xi^2+\xi+\mathcal{O}(y)},
\end{aligned} \tag{2.5}$$

where, we have rewritten the argument of the exponential in terms of the comoving coordinate with the asymptotic velocity $\xi = x + \frac{2\tau}{b}$. (Note that the expression above is valid for $\xi \gtrsim -\infty$, near the unstable fixed point.) Now let $(\xi_{\phi_0}, \tau)$ be the coordinates at which the tachyon profile has reached a specific constant value $\phi_0$. Solving the equation above for $\xi_{\phi_0}(\tau)$, we obtain $\xi_{\phi_0}(\tau) \simeq \frac{1}{2}\ln\left(\frac{\tau}{b}\right)$. Therefore, the asymptotic velocity is reached as $v(\tau) = v_{\rm asym} - \dot{\xi}_{\phi_0} \simeq \frac{2}{b} - \frac{1}{2\tau} + \mathcal{O}(\tau^{-2})$. While this is indeed the qualitative nature of the asymptotics, the coefficient of the $1/\tau$ term is not quite correct. This is because a derivative of the kernel of the diffusion equation also gives a solution, and in particular, taking the correction from the first derivative into account, we find

$$\begin{aligned}
\phi(x,\tau) &\propto \left(x + \frac{2\tau}{b}\right)\, \exp\left[-\frac{b}{4\tau}\left(x+\frac{2\tau}{b}\right)^2 + \left(x + \frac{2\tau}{b} + \frac{3}{2}\ln\left(\frac{\tau}{b}\right)\right)\right] \\
v(\tau) &= v_{\rm asym} - \dot{\xi}_{\phi_0} \simeq \frac{2}{b} - \frac{3}{2\tau} + \cdots
\end{aligned} \tag{2.6}$$

We would like to reiterate that this analysis is exactly as in the case of the standard Fisher equation (see, for example, the review [19]) and is not affected by the non-local interactions.

On the other hand, the linearized equation for $\psi = \phi - \phi_{\rm S} = \phi - K^{-3}$ around the stable fixed point differs from the standard case. The substitution $\psi = e^{\lambda\xi}$ in the linearized equation leads to

$$bv\lambda - \lambda^2 - 1 + 2e^{2\alpha\lambda^2 - 2\alpha bv\lambda} = 0. \tag{2.7}$$

This is a transcendental equation which does not have any real solution, however, it admits an infinite number of complex solutions,[4] for example, the leading behaviour is deter-

[3] A transformation $\phi \to e^{\tau/b}\phi$ brings it to the standard form of the diffusion equation.

[4] The corresponding equation for the homogeneous case, $b\lambda - 1 + e^{-2\alpha b\lambda} = 0$, is also a transcendental equation [3, 8]. Its leading solutions are $-0.249613 \pm i\,1.90371$, however, $-3.91104 \pm i\,14.4748$, $-5.03573 \pm i\,26.7603$, $-5.73776 \pm i\,38.9404$, etc., which also satisfy the equation, are some of the non-leading solutions.

mined by

$$\lambda = -0.327933 \pm i\,0.716793 \tag{2.8}$$

which differs slightly from the homogeneous case. (Some other solutions are $-2.17775 \pm i\,2.27752$, $-3.8092 \pm i\,4.04854$, $-4.42422 \pm i\,4.70868$ etc.) The exponent (2.8) is also not very different from the standard Fisher case, to which our equation reduces when $\alpha = 0$: $(\lambda-1)^2 = 2$, a solution of which is $1-\sqrt{2} \simeq -0.4142$.

One should note, however, $1+\sqrt{2} \simeq +2.4142$ is also a solution of this quadratic equation — the positive real part of $\lambda$ suggests that this corresponds to moving away from the stable fixed point $\phi_{\mathrm{S}}$. However, in the standard analysis [19], this positive exponent is eliminated by fixing the asymptotic conditions at $\xi \to \pm\infty$ determined by the front.

This potential instability is also present in the case of the tachyon. The equation for the exponent in eq. (2.7) has a symmetry around $\lambda = 1$ (for $bv = 2$), and hence admits a solution $2.32793 \pm i\,0.716793$ with a positive real part (and similarly for the other roots). The singular perturbation analysis that starts with the solution of the homogeneous equation as the seed, and thus fixes the asymptotic conditions at $\xi \to \pm\infty$, is not affected by this instability and yields a travelling front solution that converges. Nevertheless this instability could potentially cause the inhomogeneously decaying tachyon to oscillate around $\phi_{\mathrm{S}}$ with increasing amplitude, the behaviour that was seen in the analysis of ref. [2]. In particular, as in ref. [3], one may attempt to solve eq. (2.3) by converting it into a recursion relation:

$$a_n = \frac{e^{\alpha(n^2-4n+3)}}{(n-1)^2} \sum_{m=1}^{n-1} \left(a_m e^{\alpha m^2}\right) \left(a_{n-m} e^{\alpha(n-m)^2}\right) \tag{2.9}$$

for $a_n$ in $\Phi_{bv=2} = \sum a_n e^{n\xi}$. The coefficients increase rapidly, resulting in a divergent series.

## 3 Perturbation of the non-local Fisher equation of the tachyon

In this section, we shall analyze small fluctuations around the travelling front. To this end, let us separate the leading order front solution $\Phi_v$, that depends only on $\xi$, from the (small) perturbations around it

$$\phi(\xi,\tau) = \Phi_v(\xi) + \eta(\xi,\tau), \qquad |\eta| \ll |\Phi_v|.$$

Thanks to eq. (2.3) satisfied by the leading order solution $\Phi_v$ ('classical solution'), the perturbations satisfy the linearized equation

$$b\frac{\partial\eta}{\partial\tau} + bv\frac{\partial\eta}{\partial\xi} - \frac{\partial^2\eta}{\partial\xi^2} - \eta + 2K^3 e^{-2\alpha b\partial_\tau - 2\alpha bv\partial_\xi + \alpha\partial_\xi^2} \left(e^{\alpha\partial_\xi^2}\Phi_v\right)\left(e^{\alpha\partial_\xi^2}\eta\right) = 0 \tag{3.1}$$

where we have neglected terms of $\mathcal{O}(\eta^2)$. As expected, the translation zero-mode $\eta(\xi,\tau) = \partial_\xi\Phi_v(\xi)$ is a solution to this.

Let us expand the perturbation $\eta(\tau,\xi)$ in terms of its (Fourier-Laplace) modes

$$\eta(\tau,\xi) = \int_0^\infty dE \int_{-\infty}^{+\infty} \frac{dp}{2\pi}\, e^{-E\tau + ip\xi}\, \widetilde{\eta}_E(p),$$

but leave the front $\Phi_v$ as it is for the moment. If we make the plausible assumption that the operator $e^{-2\alpha b\partial_\tau - 2\alpha bv\partial_\xi + \alpha\partial_\xi^2}$ in eq. (3.1) is invertible, we arrive at the equation:

$$\begin{aligned} \left(-bE + p^2 - 1 + ibvp\right) e^{-2\alpha bE + \alpha p^2 + i2\alpha bvp}\, \widetilde{\eta}_E(p) &= 2K^3 \mathfrak{G}_\alpha[\Phi_v]\, e^{-\alpha p^2} \widetilde{\eta}_E(p)\,, \\ \left(\frac{1}{2}\frac{\partial}{\partial\alpha}\mathcal{L}_\alpha - 1\right) \widetilde{\eta}_E(p) &= 2K^3 \mathfrak{G}_\alpha[\Phi_v]\, \widetilde{\eta}_E(p)\,. \end{aligned} \tag{3.2}$$

In the above, we have rewritten the equation in terms of a formal derivative of the operator

$$\mathcal{L}_\alpha = e^{-2\alpha bE + 2\alpha p^2 + i2\alpha bvp} \sim e^{2\alpha b\partial_\tau + 2\alpha bv\partial_\xi - \alpha\partial_\xi^2}$$

with respect to $\alpha$ by an abuse of notation. (Recall that $\alpha = \ln\left(3\sqrt{3}/4\right)$ is a fixed number in OSFT.)

Following [20], let us consider the conditions for stability at asymptotic values of the front profile $\xi \to \pm\infty$. The Gaussian convolution $\mathfrak{G}_\alpha[\Phi_v]$ of the front profile $\Phi_v$ softens the oscillations around the stable fixed point.

As $\xi \to -\infty$, the tachyon profile $\Phi_v$ as well as its Gaussian transform $\mathfrak{G}_\alpha[\Phi] \to 0$. In this region, we have the operator equation $\partial_\alpha(\ln \mathcal{L}_\alpha) = 2$:

$$bE = (p^2 - 1) + ibvp. \tag{3.3}$$

This condition is exactly the same as in the case of the standard Fisher equation without any non-locality. This is not unexpected, as the behaviour of the two equations and their travelling front solutions are the same in this region. The travelling solution is said to be linearly stable if the perturbation decays exponentially in time, i.e., if $\mathrm{Re}(E) \geq 0$ (where $E = 0$ corresponds to the translation zero-mode). Thus eq. (3.3) may seem to indicate an instability at first sight because $\mathrm{Re}(bE) = p^2 - 1$, hence it is negative for $|p| < 1$. However, this is just the tachyonic instability at the maximum of the potential — the region $\xi \to -\infty$ still has the unstable D-brane.

Before we analyze the stability conditions in the asymptotic region $\xi \to \infty$ of the travelling front of the tachyon, let us review the situation for the usual Fisher equation, i.e., the case $\alpha \to 0$. Eq. (3.2) reduces to a simple form:

$$bE = \left(p^2 - 1 + 2K^3\Phi_v\right) + ibvp. \tag{3.4}$$

Recall that at the true vacuum, $\Phi_v$ approaches the value $K^{-3}$. This means the solution is stable $\mathrm{Re}(bE) \approx p^2 + 1$ at the non-perturbative vacuum. (As mentioned above, in the region $\xi \to -\infty$ corresponding to the perturbative vacuum $\Phi_v \approx 0$, the stability condition is exactly the same with or without non-locality.)

Getting back to general case with non-locality ($\alpha \neq 0$), the analytic form of the eigenvalue $E$ can be found by integrating the formal first order differential equation (3.2) for the operator $\mathcal{L}_\alpha$ with an integrating factor. By a straightforward integration of

$$\frac{\partial}{\partial\alpha}\left(e^{-2\alpha}\mathcal{L}_\alpha\right) = 4e^{-2\alpha} K^3 \mathfrak{G}_\alpha[\Phi_v]\, \widetilde{\eta}_E(p)$$

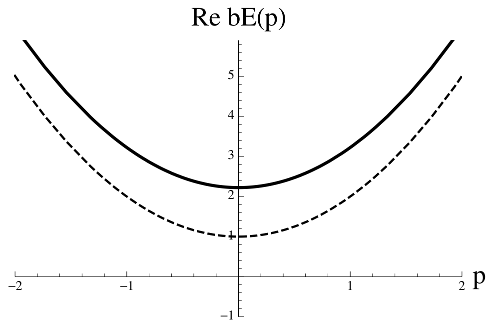


**Figure 1**. Real parts of $bE(p)$, from eqs. (3.5) and (3.4), for the travelling front of the tachyon Fisher equation and the standard Fisher equation around the non-perturbative vacuum (region where tachyon condensation has taken place) in dashed and solid lines, respectively. Both spectra are clearly non-negative, however, the tachyon front is even more stable than the travelling front of the standard Fisher equation.

we obtain

$$bE = p^2 - 1 + ibvp - \frac{1}{2\alpha} \ln\left(1 - 4K^3 \int_0^\alpha \mathfrak{G}_{\alpha'}[\Phi_v]\, e^{-2\alpha'} d\alpha'\right). \tag{3.5}$$

This is valid for any value of $\xi$, and, in particular, the results for the region $\xi \to -\infty$ corresponding to the perturbative (unstable) extremum can be recovered. On the other hand, in the region $\xi \to \infty$, the front has settled to the stable (local) minimum where $\mathfrak{G}_\alpha[\Phi_v] \approx K^{-3}$. Therefore, the argument of the logarithm can be approximated as $2e^{-2\alpha} - 1$ which gives the real part of $\mathrm{Re}[bE] \approx p^2 + 2.22$. As a consequence, the travelling front of the tachyon is even more stable than the standard Fisher equation. The plots of $\mathrm{Re}\,(E\,(p))$ for both cases are shown in figure 1.

### 3.1 Euclidean Schrödinger equation

We can isolate the leading tachyonic instability around the perturbative vaccum from the effect of fluctuations around the travelling front by the substitution

$$\eta(\tau, \xi) = e^{bv\xi/2} \psi(\tau, \xi),$$

which gets rid of the $\partial_\xi \eta$ term in eq. (3.1) and brings the above to the form of an Euclidean Schrödinger equation. However, one should be careful due to subtelties that arise from the fact that $\psi$ does not belong to the Hilbert space of $L^2$-functions (because of the presence of the $\xi$-dependent prefactor). This is true of the standard Fisher case as well [19].

The equation satisfied by $\psi$ is

$$\begin{aligned} b\frac{\partial \psi}{\partial \tau} &= \frac{\partial^2 \psi}{\partial \xi^2} + \left(1 - \frac{1}{4} b^2 v^2\right) \psi \\ &\quad - 2K^3 e^{-\frac{1}{2} bv\xi} \left[ e^{-2\alpha bv \partial_\xi + \alpha \partial_\xi^2} \left(e^{\alpha \partial_\xi^2} \Phi_v\right) \left(e^{\alpha \partial_\xi^2 - 2\alpha b \partial_\tau} e^{+\frac{1}{2} bv\xi} \psi\right)\right]. \end{aligned} \tag{3.6}$$

In order to simplify this further, we use the Campbell-Baker-Hausdorff formulas to write

$$\begin{aligned} e^{\alpha \partial_\xi^2}\, e^{bv\xi/2} &= e^{\alpha b^2 v^2/4}\, e^{bv\xi/2}\, e^{\alpha bv \partial_\xi + \alpha \partial_\xi^2} \\ e^{-2\alpha bv \partial_\xi}\, e^{bv\xi/2} &= e^{-\alpha b^2 v^2}\, e^{bv\xi/2}\, e^{-2\alpha bv \partial_\xi}. \end{aligned}$$

This gives us the Euclidean Schrödinger equation for the perturbation function $\psi(\xi,\tau)$ as

$$\begin{aligned} b\frac{\partial\psi}{\partial\tau} &= \frac{\partial^2\psi}{\partial\xi^2} + \left(1-\frac{1}{4}b^2v^2\right)\psi \\ &\quad -2K^3 e^{-\frac{1}{2}\alpha b^2 v^2}\left[e^{-\alpha bv\partial_\xi+\alpha\partial_\xi^2}\left(e^{\alpha\partial_\xi^2}\Phi_v\right)\left(e^{\alpha\partial_\xi^2+\alpha bv\partial_\xi-2\alpha b\partial_\tau}\psi\right)\right] \\ &= \frac{\partial^2\psi(\xi,\tau)}{\partial\xi^2} + \left(1-\frac{1}{4}b^2v^2\right)\psi(\xi,\tau) \\ &\quad -2K^3 e^{-\frac{1}{2}\alpha b^2v^2}\,\mathfrak{G}_\alpha\left[\mathfrak{G}_\alpha[\Phi_v(\xi-\alpha bv)]\star\mathfrak{G}_\alpha[\psi(\xi,\tau-2\alpha b)]\right]. \end{aligned} \tag{3.7}$$

(Notice that the argument of $\psi$ does not have a shift in $\xi$, though it has one in $\tau$.) In the case of the standard Fisher equation without any non-locality, ($\alpha\to 0$) the above is a usual Schrödinger equation:

$$b\partial_\tau\psi = \partial_\xi^2\psi + \left(1-\frac{1}{4}b^2v^2\right)\psi - 2K^3\Phi_v\psi \tag{3.8}$$

with the 'potential' determined by the 'classical' front solution $\Phi_v(\xi)$.

Let us point out some features of eq. (3.7). The interaction with the 'potential' $\Phi_v$ is non-local and in terms of a convolution product. Moreover, there is a delay in the argument of $\psi$ on the r.h.s. of the above. Due to the delay, we do not get the conventional eigenvalue equation; rather writing $\psi(\xi,\tau) = e^{-E\tau}\,\Psi_E(\xi)$, the 'time-independent' Schrödinger equation

$$\begin{aligned} bE\Psi_E(\xi) &= -\frac{\partial^2\Psi_E(\xi)}{\partial\xi^2} - \left(1-\frac{1}{4}b^2v^2\right)\Psi_E(\xi) \\ &\quad + 2e^{2\alpha bE}\,K^3 e^{-\frac{1}{2}\alpha b^2v^2}\,\mathfrak{G}_\alpha\left[\mathfrak{G}_\alpha[\Phi_v(\xi-\alpha bv)]\star\mathfrak{G}_\alpha[\Psi_E(\xi)]\right] \end{aligned} \tag{3.9}$$

is a transcendental equation for $E$. In order to show that the solution is stable, we need to prove that all the solutions to (3.9) have $E\geq 0$. (Recall that $E=0$ is a solution that corresponds to translating the leading order solution.)

In terms of the (Laplace-Fourier) modes

$$\psi(\tau,\xi) = \int_0^\infty dE\int_{-\infty}^{+\infty}\frac{dp}{2\pi}e^{-E\tau+ip\xi}\;\widetilde{\psi}_E(p), \tag{3.10}$$

eq. (3.8) gives

$$bE = p^2 + \left(\frac{1}{4}b^2v^2 - 1\right) + 2K^3\Phi_v \tag{3.11}$$

in the standard Fisher case. It is obvious from eq. (3.11) that $E(p)\geq 0$ for all values of $p$. This is due to the factor of $\frac{1}{4}b^2v^2$ on the right-hand side, and is expected from the form of the 'potential' $\Phi_v$. In the non-local case of $\alpha\neq 0$, we follow the same steps as in the analysis of $\eta$ to obtain:

$$bE = p^2 + \left(\frac{1}{4}b^2v^2-1\right) - \frac{1}{2\alpha}\ln\left[1-4K^3\int_0^\alpha d\alpha'\,\mathfrak{G}_{\alpha'}[\Phi_v]\,e^{-2\alpha'(1+ibvp)}\right]. \tag{3.12}$$

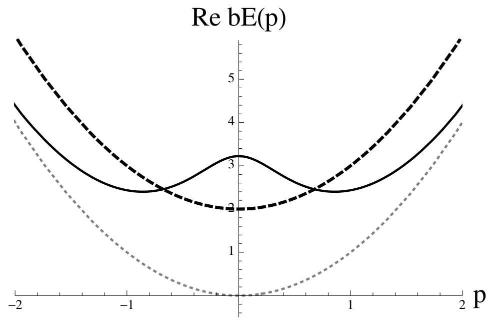


**Figure 2**. Real parts of $bE(p)$, from eqs. (3.11) and (3.13), for the travelling tachyon and standard Fisher cases around the perturbative (lower dotted curve — exactly identical in both cases) and the non-perturbative vacua (upper curves: the solid one for the tachyon and the dashed one for standard Fisher equation). For the travelling tachyon the minimum of the energy spectrum is at $p \approx \pm 0.86$.

Similar to the conclusion for $\eta$, it turns out that at perturbative vacuum the spectrum is in the same form as Fisher case $bE = p^2 + \left(\frac{1}{4}b^2v^2 - 1\right)$, and it is always non-negative. For non-perturbative vacuum, it may seem that non-negativity of Re$[bE]$ is not guaranteed because of the oscillations from $e^{ibvp}$. However, at the non-perturbative vacuum $\mathfrak{G}_\alpha[\Phi_v] \approx K^{-3}$, whence eq. (3.12) reduces to

$$bE = p^2 + \left(\frac{1}{4}b^2v^2 - 1\right) - \frac{1}{2\alpha}\ln\left[1 - 2\left(\frac{1 - e^{-2\alpha(1+ibvp)}}{1 + ibvp}\right)\right]. \tag{3.13}$$

From the equation above, we find that the spectrum is symmetric, but not convex. The minimum of energy is not at zero: Re$[bE] \approx 2.40$ at $p \approx \pm 0.86$. The oscillatory profile of $\Phi_v$ produces small potential wells, however, the positive contribution to the spectrum from the excitations can overcome the negative part from logarithmic term (of non-local effect). Numerical plots of real parts of spectrum Re$[bE(p)]$ for both the standard Fisher and the travelling tachyon cases around the perturbative and non-perturbative vacua are shown in figure 2.

In the above, we have taken the travelling front profile $\Phi_v$ to the leading order in the singular perturbation expansion (that is, we have worked with $\Phi_v^{(0)}$). However, it is straightforward to work with the profile including the effects at higher order. The qualitative behaviour is not expected to change. Plots for both the real and the imaginary parts of the Fourier transform of the tachyon front computed numerically are shown in figure 3. For this, we have put the system in a finite-size box (IR regulator). In spite of the oscillations around the stable vacuum, the difference from the standard Fisher case is small, and restricted to a finite region in momentum space.

One may also attempt to solve the non-local Schrödinger equation by reducing it in to an integral eigenvalue problem. In terms of the modes $\widetilde{\Psi}_E(k)$:

$$\begin{aligned}&\left(bE - k^2 + 1 - \frac{1}{4}b^2v^2\right)\widetilde{\Psi}_E(k)\\ &\qquad = 2K^3\, e^{2\alpha bE - 2\alpha k^2 - i\alpha bvk - \frac{1}{2}\alpha b^2v^2}\int \frac{d\ell}{2\pi}\, e^{2\alpha\ell(k-\ell) + i\alpha b\ell}\,\widetilde{\Phi}_v(k-\ell)\widetilde{\Psi}_E(\ell).\end{aligned} \tag{3.14}$$

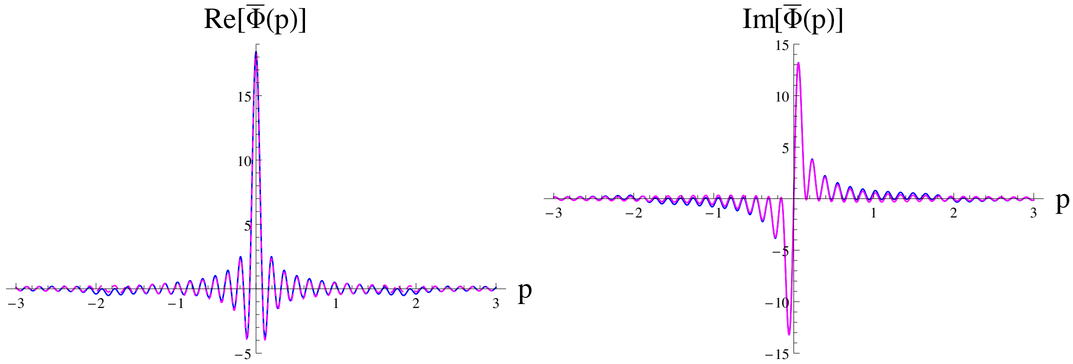


**Figure 3**. Left: real parts of Fourier transform of $\Phi_v^{(0)}$ for the tachyon Fisher and standard Fisher case in are shown in blue and magenta, respectively. The very small deviations between two plots occur in the region $1.4 \leq |p| \leq 2.2$. Right: the corresponding imaginary parts. The very small deviations between the two plots are seen in the region $0.6 \leq |p| \leq 1.4$.

Setting $\alpha = 0$ in (3.14) recovers the standard Fisher equation (no delay or nonlocality) and its perturbation that satisfies

$$\left(bE - k^2 + 1 - \frac{1}{4}b^2v^2\right)\widetilde{\Psi}_E(k) \;=\; 2K^3 \int \frac{d\ell}{2\pi}\,\tilde{\Phi}_v(k-\ell)\widetilde{\Psi}_E(\ell). \tag{3.15}$$

The equations above are Fredholm integral equation of the second kind.

As before, we may introduce

$$\mathcal{U}_\alpha = \exp\left(-2\alpha\left(bE - k^2 + 1 - \frac{1}{4}b^2v^2\right)\right) \;\sim\; \exp\left(2\alpha\left(b\partial_\tau + \partial_\xi^2 + 1 - \frac{1}{4}b^2v^2\right)\right)$$

to write eq. (3.14) compactly as

$$\begin{aligned} -\frac{\partial}{\partial\alpha}\mathcal{U}_\alpha\widetilde{\Psi}_E(k) &= 4K\,e^{-i\alpha bvk}\int\frac{d\ell}{2\pi}\,e^{2\alpha\ell(k-\ell)+i\alpha b\ell}\,\widetilde{\Phi}_v(k-\ell)\widetilde{\Psi}_E(\ell)\\ (\mathbf{1}-\mathcal{U}_\alpha)\,\widetilde{\Psi}_E(k) &= 4K\int\frac{d\ell}{2\pi}\left[\frac{e^{\alpha(2\ell(k-\ell)+ib(\ell-vk))}-1}{2\ell(k-\ell)+ib\,(\ell-vk)}\right]\widetilde{\Phi}_v(k-\ell)\widetilde{\Psi}_E(\ell), \end{aligned}$$

where the last line is the result of integrating over the non-locality parameter from 0 to $\alpha$, the required value.

Before we close this section, since we have not come across it in the literature, it may not be entirely out of place to mention that the (Euclidean) Schrödinger equation for the perturbation of the standard Fisher equation, namely eq. (3.8), can be solved exactly to the lowest order in singular perturbation theory where $\Phi_v^{(0)}(\xi) = K^{-3}y(\xi) = K^{-3}/\big(1+e^{-\xi/bv}\big)$. If we change variable from $\xi$ to $y$ and write

$$\psi(\tau, y(\xi)) = e^{-E\tau}y^\mu(1-y)^\nu F(y),$$

then $F(y)$ satisfies a hypergeometric differential equation with parameters $a = (\mu + \nu)$, $b = (\mu+\nu+1)$ and $c = (2\mu+1)$, where $\mu^2 = -b^2v^2\left(bE + 1 - \frac{b^2v^2}{4}\right)$ and $\nu^2 =$

$-b^2v^2\left(bE-1-\frac{b^2v^2}{4}\right)$. We note in passing that the the change of variable used above cannot be made in case of the tachyon Fisher equation, as the leading front is not monotonic due to non-local effects.

## 4 Stability analysis in singular perturbation theory

The travelling front solution of the tachyon Fisher equation (2.3), was solved by using a singular perturbation analysis [8], in which $\varepsilon = 1/(bv)^2$ was used as a small parameter. It is, therefore, natural to analyze the question of stability in this approach. In terms of the rescaled variable $\zeta = \sqrt{\varepsilon}\xi = bv\xi$ (and the derivative $bv\partial_\xi = \partial_\zeta$), used in the singular perturbation theory, the equation for the perturbation (3.1) takes the form

$$b\frac{\partial\eta}{\partial\tau} + \frac{\partial\eta}{\partial\zeta} - \varepsilon\frac{\partial^2\eta}{\partial\zeta^2} - \eta + 2K^3 e^{-2\alpha b\partial_\tau - 2\alpha\partial_\zeta + \varepsilon\alpha\partial_\zeta^2}\left(e^{\varepsilon\alpha\partial_\zeta^2}\Phi_v\right)\left(e^{\varepsilon\alpha\partial_\zeta^2}\eta\right) = 0. \tag{4.1}$$

Following the expansion of the leading order solution $\Phi_v(\xi) = \Phi_v^{(0)}(\xi) + \varepsilon\Phi_v^{(1)}(\xi) + \cdots$, we now expand the perturbation as well:

$$\eta(\xi,\tau) = \eta^{(0)}(\xi,\tau) + \varepsilon\,\eta^{(1)}(\xi,\tau) + \varepsilon^2\,\eta^{(2)}(\xi,\tau) + \cdots$$

Moreover, since the Gaussian kernel is identity for $\alpha = 0$, it can be divided as [8]

$$\mathfrak{G}_{\varepsilon\alpha}[F(\zeta)] = F(\zeta) + \mathfrak{d}\mathfrak{G}_{\varepsilon\alpha}[F(\zeta)], \tag{4.2}$$

in which we treat $\mathfrak{dg} \sim \mathcal{O}(\varepsilon)$.

This leads to the following equations:

$$\begin{aligned}
\mathcal{O}(1): \quad & b\frac{\partial\eta^{(0)}}{\partial\tau} + \frac{\partial\eta^{(0)}}{\partial\zeta} - \eta^{(0)} + 2K^3 e^{-2\alpha b\partial_\tau - 2\alpha\partial_\zeta}\left(\Phi_v^{(0)}\,\eta^{(0)}\right) = 0,\\
\mathcal{O}(\varepsilon): \quad & b\frac{\partial\eta^{(1)}}{\partial\tau} + \frac{\partial\eta^{(1)}}{\partial\zeta} - \eta^{(1)} + 2K^3 e^{-2\alpha b\partial_\tau - 2\alpha\partial_\zeta}\left(\Phi_v^{(0)}\,\eta^{(1)}\right)\\
&= \frac{\partial^2\eta^{(0)}}{\partial\zeta^2} - 2K^3 e^{-2\alpha b\partial_\tau - 2\alpha\partial_\zeta}\Bigg(\mathfrak{dg}_{\varepsilon\alpha}\left[\Phi_v^{(0)}\eta^{(0)}\right] + \Phi_v^{(0)}\left(\mathfrak{dg}_{\varepsilon\alpha}\left[\eta^{(0)}\right]\right)\\
&\quad + \left(\left(\mathfrak{dg}_{\varepsilon\alpha}\left[\Phi_v^{(0)}\right] + \Phi_v^{(1)}\right)\eta^{(0)}\right)\Bigg)
\end{aligned} \tag{4.3}$$

plus equations for higher order terms. The first equation above for the leading term of the perturbation is not a Schrödinger-type equation being first order in time as well as the space derivatives. However, it is homogeneous, while the equations at second (and higher) order are inhomogeneous, with the sources determined from those at lower order.

Consider the Fourier transformed functions

$$\Phi_v(\zeta) = \int\frac{dk}{2\pi}\,e^{ik\zeta}\,\widetilde{\Phi}_v(k), \qquad \eta(\tau,\zeta) = \int dE\int\frac{dk}{2\pi}\,e^{-E\tau+ik\zeta}\,\widetilde{\eta}_E(k),$$

which are valid at every order in perturbation. The equation at lowest order is

$$(-bE+ik-1)\,e^{-2\alpha bE+2i\alpha k}\,\tilde{\eta}_E^{(0)}(k) = 2K^3\int\frac{d\ell}{2\pi}\,\tilde{\Phi}_v^{(0)}(k-\ell)\,\tilde{\eta}_E^{(0)}(\ell), \tag{4.4}$$

which is again a Fredholm integral equation of second kind. The equations at higher order are Fredholm equation of first kind, consequently these may be solved by iterative technique. For the standard Fisher equation ($\alpha = 0$) one can once again change variable to $y = 1/\big(1 + e^{-\zeta}\big)$, which leads to a simple quadrature

$$\frac{d\eta^{(0)}}{dy} = \frac{1 + bE - 2y}{y(1-y)},$$

integrating which we get $\eta^{(0)}(y) = K^{-3} y^{1+bE}(1-y)^{1-bE}$. We see that for $bE = 0$, $\eta^{(0)}$ is a translation of the 'classical' front $\Phi^{(0)}_{(v)}$

$$\eta^{(0)}\,(y(\zeta); E = 0) \;=\; K^{-3}\, y(1-y) \;=\; \frac{K^{-3}\, e^{-\zeta}}{(1+e^{-\zeta})^2} \;=\; \frac{d}{d\zeta}\Phi^{(0)}_v(\zeta)$$

as expected.

## 5 Conclusions

The dynamical equation of the tachyon on an unstable D-brane does not have a solution that interpolate between the extrema of the potential [2]. However, in the background of a dilaton that is linear along a light-like coordinate $x^-$, the equation of motion (in light-cone time $x^+$) is first order. This admits an interpolating solution that has an oscillatory convergence to the (closed-string) vacuum [3]. This equation is actually a variant of a reaction-diffusion equation, which has nonlocal interactions, including a delay. Therefore, in the case of an inhomogeneous decay, there is a travelling front solution that moves with an asymptotic velocity converting regions of space from the unstable brane to the vacuum in its wake [8]. In this paper, we have carried out a stability analysis of the front solution using linearized perturbation theory. The equations for the perturbation is a nonlocal Euclidean Schrödinger equation, with the front profile acting as a potential. Thanks to the nonlocality, however, the potential and the 'wavefunction' are in a convolution product. We find that the front solution found from a singular perturbation analysis is stable. We have also analyzed (linear) stability around the closed string vacuum. The Lyapunov exponents are determined by transcendental equations, which are different for the case of homogeneous and inhomogeneos decay. For the latter, there are positive solutions that corresponds to (oscillatory) divergence. Even though these modes do not destabilize the travelling front obtained in the singular perturbation theory, their existence suggests that there could be space-time dependent solutions of the equation of motion of the tachyon that exhibit untamed oscillation with increasing magnitude around the (closed-string) vacuum, similar to those of homogenous decay in usual time [2]. The inclusion of the higher string modes may change the dynamics — we know that the tachyon perturbation corresponding to the front solution can be extended to the equations of string field theory to the next order [18].



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