On $\mathcal{N}=1$ partition functions without R-symmetry

Gino Knodel, James T. Liu and Leopoldo A. Pando Zayas

Michigan Center for Theoretical Physics, Randall Laboratory of Physics,
The University of Michigan, Ann Arbor, MI 48109-1040, U.S.A.

E-mail: gknodel@umich.edu, jimliu@umich.edu, lpandoz@umich.edu

ABSTRACT: We examine the dependence of four-dimensional Euclidean $\mathcal{N}=1$ partition functions on coupling constants. In particular, we focus on backgrounds without R-symmetry, which arise in the rigid limit of old minimal supergravity. Backgrounds preserving a single supercharge may be classified as having either trivial or SU(2) structure, with the former including S^4. We show that, in the absence of additional symmetries, the partition function depends non-trivially on all couplings in the trivial structure case, and (anti)-holomorphically on couplings in the SU(2) structure case. In both cases, this allows for ambiguities in the form of finite counterterms, which in principle render the partition function unphysical. However, we argue that on dimensional grounds, ambiguities are restricted to finite powers in relevant couplings, and can therefore be kept under control. On the other hand, for backgrounds preserving supercharges of opposite chiralities, the partition function is completely independent of all couplings. In this case, the background admits an R-symmetry, and the partition function is physical, in agreement with the results obtained in the rigid limit of new minimal supergravity. Based on a systematic analysis of supersymmetric invariants, we also demonstrate that $\mathcal{N}=1$ localization is not possible for backgrounds without R-symmetry.

Contents

1   Introduction and summary

Localization of supersymmetric field theories on curved spaces has recently played a central role in elucidating some long standing puzzles. Pestun made use of localization to compute the expectation value of half supersymmetric Wilson loops in $\mathcal{N} = 4$ supersymmetric Yang-Mills on S^4 and to prove that it is given by a Gaussian matrix model [1], a conjecture made more than a decade ago [2, 3]. Kapustin, Willett and Yaakov computed the partition function of supersymmetric field theories on S^3 [4], paving the way to a better understanding of the number of degrees of freedom of such theories, and clarifying various three-dimensional dualities (for a review, see e.g. [5]).

The program of computing supersymmetric observables on curved spaces thus highlights the question of how to systematically construct such field theories. Festuccia and Seiberg initiated a program to answer this question in general, based on the principle of rigid supergravity [6]. According to this principle, one considers the field theory as a matter sector of a supergravity theory, and then proceeds to decouple supergravity. The conditions

for the background to be supersymmetric are obtained by demanding that the gravitino variation vanishes, and all couplings of the matter sector to the background supergravity fields are dictated by the form of the supergravity Lagrangian.

Rigid supergravity provides us with a powerful set of tools for answering questions within a broad family of theories on curved spaces [7–13]. One practically-minded question is whether we can perform localization to calculate the partition function and other observables on various curved backgrounds. Of particular interest are four-dimensional backgrounds that do not possess an R-symmetry, such as the round and squashed S^4, for which exact results for $\mathcal{N} = 1$ theories have so far been elusive. These backgrounds can be naturally studied in the framework of old minimal supergravity, but they can not be found as solutions to new minimal supergravity [6, 9, 10, 14–16].

The standard localization procedure makes use of the fact that there is at least one supersymmetric operator $\mathcal{O}$ such that the partition function for a theory with Lagrangian $\mathcal{L} \supset t\mathcal{O}$ is independent of the corresponding coupling constant t, i.e.

$$\frac{dZ(t)}{dt} = 0. \tag{1.1}$$

If $\mathcal{O}$ has a positive semi-definite part, one can evaluate the partition function at $t \gg 1$, where it is given by a 1-loop determinant around the classical configuration $\mathcal{O}_{cl} = 0$. To understand in which cases localization is in principle possible, one needs to determine under which circumstances (1.1) is satisfied. This condition is equivalent to the statement that there is at least one "flat" direction in the space of coupling constants. The first goal of this paper is therefore to better understand the geometry of the space of couplings of $\mathcal{N} = 1$ theories on four-dimensional curved (Euclidean) backgrounds.

Our first step is to determine the supersymmetric invariants for a given multiplet, which are the building blocks of supersymmetric Lagrangians. In the standard approach, invariants are constructed using the tensor calculus of supergravity [17–20], and one finds the curved space generalizations of the flat-space D-term, as well as a chiral F- and antichiral $\overline{F}$-term. However, the supergravity approach assumes that the background manifold preserves all four complex supercharges of Euclidean $\mathcal{N} = 1$. As was shown in [10, 11, 13], there is a large set of interesting backgrounds with reduced supersymmetry, which preserve fewer than four supercharges. In this case, there are more than just the three standard SUSY-invariants. Furthermore, if the background is R-symmetric, one may also combine superfields using an antisymmetric product $S_1 \wedge S_2$ to construct Lagrangians.

Since both of these subtleties are essentially invisible in the "top-down" approach of supergravity, we instead employ a "bottom-up" approach: we take as our only input the curved space SUSY algebra, derived via rigid supergravity [6, 7, 17, 19–21]. Using the transformation rules, we can then construct the complete set of SUSY invariants, as well as the multiplication rules for combining supermultiplets. For a general Euclidean $\mathcal{N} = 1$ multiplet $S = (C, \psi_L, \psi_R, F, \overline{F}, A_\mu, \lambda_L, \lambda_R, D)$, bosonic SUSY invariants take the form

$$E = \alpha_1 D + \alpha_2 F + \alpha_3 \overline{F} + \alpha_4 C + \beta^\mu A_\mu, \tag{1.2}$$

with background-dependent coefficients α_i, β^μ. Demanding E to be supersymmetric, we derive the differential conditions on the coefficients and give examples of invariants.

A flat direction t_i in the space of couplings is equivalent to the statement that the corresponding invariant E_i is δ-exact. One central result of this paper is that every invariant can be written as a SUSY-exact term, plus extra terms that depend on the geometry of the background. Schematically, we find

$$E_i = \delta V_i + \xi_i^\mu A_\mu + \eta_i C, \tag{1.3}$$

up to a total derivative, where ξ_i^μ and η_i are background-dependent. A flat direction exists only if $\xi_i^\mu = \eta_i = 0$ for some i. We analyze eq. (1.3) for the backgrounds of old minimal supergravity and extract properties of the space of couplings. Our results can be summarized as follows:

1. Backgrounds with non-chiral Killing spinors of the form (ϵ_L, ϵ_R): such manifolds are characterized by a trivial structure group $G = \mathbb{1}$, and S^3-isometry (for example, the round and squashed S^4). They do not admit an R-symmetry. We find that $\xi_i^\mu, \eta_i \neq 0$ for all invariants. This means that SUSY-closed terms are not exact, and the partition function depends nontrivially on all coupling constants.

2. Backgrounds with chiral Killing spinors of the same chirality, i.e. either $(\epsilon_L, 0)$ or $(0, \epsilon_R)$: manifolds of this kind are characterized by $\mathrm{SU}(2)_R$ or $\mathrm{SU}(2)_L$ structure respectively, and possess a $\mathrm{U}(1)_R$ R-symmetry. Focussing on the former case, we find that all but one invariant are exact. The exception is a generalized $\overline{F}$-term, so the partition function only depends on the corresponding coupling[1] $\overline{\lambda}_F$.

3. Backgrounds with chiral Killing spinors of opposite chirality, i.e. at least one pair $(\epsilon_L, 0), (0, \epsilon_R)$: these are torus fibrations $T^2 \times \Sigma$, where Σ is a Riemann surface. Their structure group is reduced to the trivial group, and there is a $\mathrm{U}(1)_R$ R-symmetry. We find that all invariants are exact, so the partition function is completely independent of couplings.

In particular, we use our results to argue that localization of $\mathcal{N} = 1$ theories on S^4 and the related cases in point 1 above is not possible: since there are simply no flat directions available, there is no freedom in tuning the couplings. In the cases 2 and 3, localization proceeds in the usual way. We give an explicit prescription for performing localization on such backgrounds in section 4.3.

In the cases 1 and 2, the obvious question that arises is *how* the partition function depends on the couplings. The second goal of this paper is therefore to analyze this dependence in detail, or in other words, to determine which features of the space of couplings are captured by the partition function. For certain superconformal field theories (SCFTs), it was shown that the partition function computes the Zamolodchikov metric on the space of exactly marginal couplings [22–29]. Inspired by these results, we determine under which circumstances one can extract similar physical quantities from Z. A generic complication

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[1]Considering instead backgrounds with $\mathrm{SU}(2)_L$ structure amounts to a flip of chiralities, so in this case there is a dependence on λ_F.

that arises is the fact that Z itself does not always have an unambiguous physical interpretation [24, 30]: in general, finite counterterms can shift the partition function according to

$$\log Z \to \log Z + \mathcal{F}(\lambda_i), \tag{1.4}$$

where $\mathcal{F}$ is a function of the couplings λ_i. If such ambiguities are present, Z is regularization scheme dependent, and thus unphysical.

To determine the physical content of Z, it is therefore necessary to classify the set of possible finite, supersymmetric counterterms. Focussing on couplings to chiral/antichiral F- and $\overline{F}$-terms, we perform a spurion analysis to construct such counterterms explicitly, and determine whether or not they give rise to ambiguities in the partition function. Let us again highlight some of our results:

1. For backgrounds with trivial structure and S^3-isometry (i.e. backgrounds without R-symmetry), there is an ambiguity of the form

$$\log Z \sim \log Z + F(\lambda, \overline{\lambda}) + G(\lambda) + H(\overline{\lambda}), \tag{1.5}$$

where F, G and H are a priori unconstrained function of all chiral/antichiral couplings $\lambda, \overline{\lambda}$. If we compute Z using different regularization schemes, we will find different answers for its finite part, so the partition function itself is not a sensible physical observable. However, if the theory contains relevant couplings m, simple dimensional analysis reveals that the functions F, G and H are in fact more constrained: they can only contain terms up to cubic order in m. We therefore argue that all ambiguities can be removed by taking a suitable number of derivatives of $\log Z$ with respect to relevant couplings.

2. For backgrounds with $U(1)_R$ R-symmetry and $SU(2)_R$ structure, the only nontrivial coupling is $\overline{\lambda}_F$. We find that the only ambiguity arises at quartic order in relevant couplings $\overline{m}$, and takes the form

$$\log Z \sim \log Z + b(\overline{m}r)^4, \tag{1.6}$$

where b is a background-dependent constant and r is a characteristic length scale.

The rest of this paper is organized as follows. In section 2, we review the framework of rigid supersymmetry as applied to old minimal supergravity, following the particular conventions and notation of [11]. In section 3, we discuss supersymmetric theories on manifolds with trivial structure. We write down the general D-type invariants, as well as the additional F-type invariants for chiral superfields and analyze them in some detail. In particular, we determine for which backgrounds they can be written as SUSY-exact terms, so the partition function is independent of the corresponding couplings. Using these results, we argue that partition functions on manifolds with S^3-isometry depend nontrivially on all couplings, which implies that $\mathcal{N} = 1$ localization is not possible. We proceed to discuss the issue of ambiguities of the partition function by constructing finite counterterms for chiral couplings. In section 4, we analyze manifolds with $SU(2)$ structure. We construct

supersymmetric invariants in an analogous way, and show that with one exception, all SUSY-closed terms are also SUSY-exact, so Z is again independent of couplings. We then present the general philosophy of localization using a simple toy model. We demonstrate that the dependence of Z on antichiral $\overline{F}$-term couplings is ambiguous, and highlight how the presence of a second supercharge of opposite chirality removes the ambiguity completely, hence identifying torus fibrations as the only compact backgrounds without ambiguities. Finally, we comment on explicit R-symmetry breaking and the role of the auxiliary fields of supergravity. We conclude with a discussion in section 5.

2 Rigid supersymmetry

The general approach of rigid supersymmetry [6] is to first start with a matter coupled supergravity theory and then freeze out the gravitational sector, thus leaving a supersymmetric field theory in a non-trivial background. Since we do not wish to impose any gravitational dynamics on the background, it is necessary to work in an off-shell formulation. In four dimensions, there are two off-shell $\mathcal{N} = 1$ supergravities — one with the "old minimal" set of auxiliary fields [31, 32] and one with the "new minimal" set [14, 33] — both of which have been extended to the Euclidean case. Backgrounds preserving an R-symmetry are naturally constructed in new minimal supergravity, while those without R-symmetry only arise in old minimal supergravity.

To avoid confusion about terminology, let us note that the theory we refer to as $\mathcal{N} = 1$ possesses 4 real supercharges in Minkowski space. In Euclidean signature, one a priori has 4 complex supercharges, although certain backgrounds might break some of the supersymmetries. Our analysis does not apply to, for example, the SUSY theories on squashed 4-spheres considered in [34]. Although these backgrounds admit either 2 or 4 supercharges, the SUSY algebra descends from a theory with 8 real supercharges in Minkowski space, i.e. $\mathcal{N} = 2$.

2.1 Supersymmetric backgrounds from old minimal supergravity

The supergravity multiplet for off-shell supergravity with the "old minimal" set of auxiliary fields is given by [20, 31, 32]

$$(g_{\mu\nu}, \psi_{L\mu}, \psi_{R\mu}, b_\mu, M, \overline{M}). \tag{2.1}$$

In Euclidean signature, the chiral spinors $\psi_{L\mu}$ and $\psi_{R\mu}$ are independent, and transform under the left-/right-handed part of $\mathrm{SO}(4) = \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$. The auxiliary fields are a complex vector b_μ, and two independent complex scalars M, $\overline{M}$.

To find supersymmetric backgrounds, we assume a nontrivial background metric $g_{\mu\nu}$, keeping the auxiliary fields arbitrary, but set the gravitino and its variation equal to zero:

$$\delta\psi_{L\mu} = \delta\psi_{R\mu} = 0. \tag{2.2}$$

This condition gives rise to the following Killing spinor equations:

$$\nabla_\mu \epsilon_L = \frac{1}{6} M \gamma_\mu \epsilon_R + \frac{i}{2} b_\mu \epsilon_L - \frac{i}{6} b^\nu \gamma_\mu \gamma_\nu \epsilon_L,$$
$$\nabla_\mu \epsilon_R = \frac{1}{6} \overline{M} \gamma_\mu \epsilon_L - \frac{i}{2} b_\mu \epsilon_R + \frac{i}{6} b^\nu \gamma_\mu \gamma_\nu \epsilon_R. \tag{2.3}$$

A solution $\epsilon \equiv (\epsilon_L, \epsilon_R)$ corresponds to a preserved supercharge. Generically, a background is specified by an arbitrary configuration of the bosonic fields $(g_{\mu\nu}, b_\mu, M, \overline{M})$. However, the condition that the background preserves supersymmetry yields nontrivial constraints on the background fields. For each preserved supercharge ϵ, such a constraint is provided by the integrability condition

$$[\nabla_\mu, \nabla_\nu]\epsilon = \frac{1}{4} R_{\mu\nu\lambda\sigma}\gamma^{\lambda\sigma}\epsilon, \tag{2.4}$$

which relates the auxiliary fields $b_\mu, M, \overline{M}$ to the metric $g_{\mu\nu}$. A complete analysis of integrability conditions in a case-by-case study was performed, for example, in [11, 13]. For our purposes, it is sufficient to note that demanding at least one unbroken supersymmetry gives rise to the conditions

$$\gamma^\mu \nabla_\mu M \epsilon_R = \left(-\frac{1}{2}R + i\nabla_\mu b^\mu - \frac{2}{3}M\overline{M} - \frac{1}{3}b_\mu b^\mu \right)\epsilon_L,$$
$$\gamma^\mu \nabla_\mu \overline{M} \epsilon_L = \left(-\frac{1}{2}R - i\nabla_\mu b^\mu - \frac{2}{3}M\overline{M} - \frac{1}{3}b_\mu b^\mu \right)\epsilon_R. \tag{2.5}$$

We can form a complete set of spinor bilinears that characterize the background manifold:

$$
\begin{aligned}
f_L &= \epsilon_L^\dagger \epsilon_L, & f_R &= \epsilon_R^\dagger \epsilon_R, \\
Q_\mu &= \epsilon_R^\dagger \gamma_\mu \epsilon_L, & K_\mu &= \epsilon_R^c \gamma_\mu \epsilon_L, \\
J_{\mu\nu}^L &= i\epsilon_L^\dagger \gamma_{\mu\nu} \epsilon_L, & J_{\mu\nu}^R &= i\epsilon_R^\dagger \gamma_{\mu\nu} \epsilon_R, \\
\Omega_{\mu\nu}^L &= \epsilon_L^c \gamma_{\mu\nu} \epsilon_L, & \Omega_{\mu\nu}^R &= \epsilon_R^c \gamma_{\mu\nu} \epsilon_R.
\end{aligned}
\tag{2.6}
$$

Throughout this paper, we follow the notation and conventions of [11].

The existence of a nowhere vanishing Killing spinor ϵ imposes additional structure on the supersymmetric backgrounds $\mathcal{M}$ considered here. There are two basic cases [8, 11, 13, 35]:

- If the Killing spinor is of the form (ϵ_L, ϵ_R), with $f_L f_R \neq 0$ (except at isolated points) the four linearly independent vectors $Q_\mu, Q_\mu^\star, K_\mu, K_\mu^\star$ explicitly trivialize the tangent bundle $T\mathcal{M}$. The structure group is broken down from $\mathrm{SO}(4) = \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ to the trivial group $G = \mathbb{1}$. We refer to these manifolds as backgrounds with trivial structure. They are discussed in section 3.

- An interesting feature of eq. (2.3) is that a nowhere vanishing solution ϵ still allows for either ϵ_L or ϵ_R to vanish identically, i.e. $f_L f_R = 0$. Assuming for concreteness that $\epsilon_R = 0$, there are two linearly independent spinors ϵ_L and $C\epsilon_L^\star$ characterizing the background. Both spinors transform as singlets under $\mathrm{SU}(2)_R$, and the remaining structure group is $G = \mathrm{SU}(2)_R$. Backgrounds with SU(2)-structure are discussed in section 4.

2.2 General multiplet and SUSY algebra

We now turn to the matter sector and its coupling to the supergravity background. The general SUSY multiplet is given by [17, 19, 21]

$$S = (C, \psi_L, \psi_R, F, \overline{F}, A_\mu, \lambda_L, \lambda_R, D), \tag{2.7}$$

and has $8 + 8$ components in Minkowski signature. In the Euclidean case, the chiral spinors are taken to be independent, and all bosonic fields are complex holomorphic variables. In particular, note that F and $\overline{F}$ are a priori independent, but will be related to each other later by choosing an appropriate integration contour in the path integral.

The curved space supersymmetry transformations of S are found by taking the rigid limit of the corresponding supergravity variations [19]:

$$\delta C = -\epsilon_L^c \psi_L - \epsilon_R^c \psi_R,$$

$$\delta \psi_L = \frac{1}{2} \gamma^\mu (A_\mu - \nabla_\mu C) \epsilon_R - \epsilon_L F,$$

$$\delta \psi_R = \frac{1}{2} \gamma^\mu (A_\mu + \nabla_\mu C) \epsilon_L + \epsilon_R \overline{F},$$

$$\delta F = \nabla^\mu (\epsilon_R^c \gamma_\mu \psi_L) - \overline{M} \epsilon_L^c \psi_L - \epsilon_R^c \lambda_R,$$

$$\delta \overline{F} = \nabla^\mu (\epsilon_L^c \gamma_\mu \psi_R) - M \epsilon_R^c \psi_R - \epsilon_L^c \lambda_L,$$

$$\delta A_\mu = \epsilon_R^c \gamma_\mu \lambda_L - \epsilon_L^c \gamma_\mu \lambda_R + \nabla_\mu (\epsilon_L^c \psi_L - \epsilon_R^c \psi_R),$$

$$\delta \lambda_L = \frac{1}{2} \gamma^{\mu\nu} \epsilon_L \nabla_\mu A_\nu - \frac{1}{2} \epsilon_L D,$$

$$\delta \lambda_R = \frac{1}{2} \gamma^{\mu\nu} \epsilon_R \nabla_\mu A_\nu + \frac{1}{2} \epsilon_R D,$$

$$\delta D = \nabla^\mu (\epsilon_L^c \gamma_\mu \lambda_R + \epsilon_R^c \gamma_\mu \lambda_L) + \frac{2i}{3} b_\mu (\epsilon_L^c \gamma^\mu \lambda_R - \epsilon_R^c \gamma^\mu \lambda_L)$$
$$- \frac{2}{3} M \epsilon_R^c \lambda_R - \frac{2}{3} \overline{M} \epsilon_L^c \lambda_L. \tag{2.8}$$

Irreducible representations can be embedded into S by making certain identifications [21]. For example, a chiral multiplet is given by

$$\Phi = (\phi, \psi_L, 0, F, 0, -\nabla_\mu \phi, 0, 0, 0). \tag{2.9}$$

Similarly, an antichiral multiplet is embedded via

$$\overline{\Phi} = (\overline{\phi}, 0, \psi_R, 0, \overline{F}, \nabla_\mu \overline{\phi}, 0, 0, 0). \tag{2.10}$$

The rules for multiplying two superfields S_1, S_2 are worked out in the appendix. For SU(2) structure, there is an antisymmetric product $S_1 \wedge S_2$ in addition to the standard symmetric product $S_1 \times S_2$. This gives rise to some interesting features when building supersymmetric Lagrangians (see section 4.3).

Throughout this paper, we take ϵ to be a commuting spinor parameter. The closure relation of the algebra then takes the form

$$\{\delta_1, \delta_2\} = \mathcal{L}_\xi, \tag{2.11}$$

where $\mathcal{L}_\xi$ is the Lie derivative along the vector field

$$\xi^\mu = \epsilon_{1L}^c \gamma^\mu \epsilon_{2R} + \epsilon_{2L}^c \gamma^\mu \epsilon_{1R}. \tag{2.12}$$

Since ϵ_L and ϵ_R transform independently in Euclidean signature, the SUSY variation splits up into the action of left- and right-handed components

$$\delta = \delta_L + \delta_R = \epsilon_L^c Q_L + \epsilon_R^c Q_R, \tag{2.13}$$

corresponding to an anticommuting supercharge of the form $Q = (Q_L, Q_R)$. Given this decomposition, we have

$$\delta_L^2 = \delta_R^2 = 0, \qquad \delta^2 = \{\delta_L, \delta_R\} = -2\mathcal{L}_K. \tag{2.14}$$

While each δ_L and δ_R is nilpotent, the total supercharge squares to a Lie derivative along the Killing vector K^μ. Since K^μ is in general complex, this provides an obstruction to carrying out the usual localization procedure. Deforming the Lagrangian by a SUSY-exact term $\sim \delta V$ generically breaks supersymmetry. While this is an obvious complication for localization, it is not sufficient to show that localization is not possible. One of the goals in the remainder of this paper is to make the obstruction to localization more precise, and provide a no-go theorem for localization on certain manifolds with trivial structure.

One obvious way to avoid the above complication is to consider manifolds with SU(2) structure, where either ϵ_L or ϵ_R (and thus K^μ) vanishes identically. In this case, δ is nilpotent and localization proceeds in the standard way. We analyze this case in some detail in section 4.

3 $\mathcal{N} = 1$ theories on manifolds with trivial structure

We first consider the trivial structure case because it allows us to study manifolds that do not admit an R-symmetry. This includes, in particular, the round and squashed S^4. In this case, the space is spanned by four linearly independent vectors $Q_\mu, Q_\mu^\star, K_\mu, K_\mu^\star$, with $Q_\mu^\star Q^\mu = K_\mu^\star K^\mu = 2f_L f_R$. The two-forms in (2.6) can be expressed in terms of these vectors as

$$J^{L/R} = -\frac{i}{2f_{R/L}} \left(K \wedge K^\star \pm Q \wedge Q^\star \right), \tag{3.1}$$

$$\Omega^L = -\frac{1}{f_R} K \wedge Q, \tag{3.2}$$

$$\Omega^R = \frac{1}{f_L} K \wedge Q^\star. \tag{3.3}$$

One can check that $\nabla_{(\mu} K_{\nu)} = 0$, so that K and $K^\star$ are Killing vectors. An interesting non-trivial feature in Euclidean signature is that since K and $K^\star$ are linearly independent, their commutator may give rise to a third Killing vector

$$L_\mu \equiv [K, K^\star]_\mu = \mu Q_\mu - \mu^\star Q_\mu^\star, \tag{3.4}$$

where

$$\mu = \frac{1}{3}(f_L \overline{M} - f_R M^\star) - \frac{2}{3}\text{Im}(b^\mu)Q_\mu^\star. \tag{3.5}$$

Notice that L is purely imaginary. The backgrounds then fall into two different classes [11, 13]:

1. For $L \neq 0$, the three Killing vectors $\text{Re}K$, $\text{Im}K$ and L satisfy an $\mathfrak{su}(2)$-algebra, which allows us to locally write the metric as a warped product $S^3 \times \mathbb{R}$:

$$ds^2 = d\xi^2 + f(\xi)^2(\sigma_1^2 + \sigma_2^2 + \sigma_3^2). \tag{3.6}$$

Here σ_i are the standard left-invariant one-forms on S^3. Backgrounds of this form have been explicitly constructed [11, 13] and include the round S^4, $\mathbb{R}^4$, $\mathbb{H}^4$, $S^3 \times \mathbb{R}$ and $\mathbb{H}^3 \times \mathbb{R}$, all of which preserve four supercharges. Another interesting case is the squashed S^4, which only preserves two supercharges. Notice that at points where $f(\xi) = 0$, either ϵ_L or ϵ_R vanishes.

2. The case $L = 0$ corresponds to a two-torus fibration over a Riemann surface. This case splits up into two subclasses:

   (a) The background has $M = \overline{M} = 0$ and admits Killing spinors of opposite chirality, namely $(\epsilon_L, 0)$ and $(0, \epsilon_R)$. This is equivalent to having independent left- and right-handed supercharges, both of which are nilpotent.

   (b) The Killing spinor has a chiral form $(\epsilon_L, 0)$ or $(0, \epsilon_R)$. This is the case of SU(2) structure with a chiral supercharge (see section 4).

3.1 General invariants

In order to construct supersymmetric Lagrangians on curved backgrounds, we will need the complete set of supersymmetric invariants, which can be derived from the SUSY algebra. In the flat space case with four supercharges, the bosonic invariants are the usual D-terms and chiral F, $\overline{F}$-terms. Using the tensor calculus for supergravity, these terms can be generalized to curved space. The D-type invariant takes the form [6, 7, 19, 20]

$$e^{-1} \int d^2\Theta(2\epsilon)(\overline{\mathcal{D}\mathcal{D}} - 8R)S = D + \frac{2}{3}(ib_\mu A^\mu - MF - \overline{M}F) - \left(\frac{1}{3}R - \frac{2}{9}M\overline{M} + \frac{2}{9}b_\mu b^\mu\right)C, \tag{3.7}$$

where $2\epsilon = e\left(1 - \Theta^2\overline{M}\right)$ is the chiral density, R is the curvature superfield, and S is a general superfield. In addition, there are generalized chiral F- and $\overline{F}$-terms

$$e^{-1} \int d^2\Theta(2\epsilon)S = F - \overline{M}\phi,$$

$$e^{-1} \int d^2\overline{\Theta}(2\bar{\epsilon})S = \overline{F} - M\overline{\phi}. \tag{3.8}$$

However, the superspace formalism generally assumes that the background preserves the maximum number of supercharges. As we will demonstrate in this section, relaxing the condition on the number or type of preserved supercharges can give rise to additional invariants that are absent in the top-down approach via supergravity. Hence we proceed with a more systematic analysis of SUSY-invariants in curved space.

We consider a general superfield S, and make the following ansatz for bosonic invariants:

$$E = \alpha_1 D + \alpha_2 F + \alpha_3 \overline{F} + \alpha_4 C + \beta^\mu A_\mu. \tag{3.9}$$

We generally expect the coefficients α_i, β^μ to be given in terms of the background fields $(g_{\mu\nu}, b_\mu, M, \overline{M})$. However, as we will show, in some cases this restriction is too strong (see section 4.5), so we treat them as a priori arbitrary functions of x. On a compact manifold,[2] E is invariant if δE is a total derivative. Assuming no special field content (such as chiral/anti-chiral fields), this gives rise to the following conditions:

$$(-\nabla^\mu \alpha_1 + \frac{2i}{3}\alpha_1 b^\mu - \beta^\mu)\gamma_\mu \epsilon_L - (\frac{2}{3}\alpha_1 M + \alpha_2)\epsilon_R = 0,$$
$$(-\nabla^\mu \alpha_1 - \frac{2i}{3}\alpha_1 b^\mu + \beta^\mu)\gamma_\mu \epsilon_R - (\frac{2}{3}\alpha_1 \overline{M} + \alpha_3)\epsilon_L = 0,$$
$$\nabla^\mu \alpha_2 \gamma_\mu \epsilon_R + (\alpha_2 \overline{M} + \nabla^\mu \beta_\mu + \alpha_4)\epsilon_L = 0,$$
$$\nabla^\mu \alpha_3 \gamma_\mu \epsilon_L + (\alpha_3 M - \nabla^\mu \beta_\mu + \alpha_4)\epsilon_R = 0. \tag{3.10}$$

Away from isolated points where one of the chiral Killing spinors might vanish, we can write down a formal solution to this system of equations:

$$\alpha_2 = -(\nabla^\mu \alpha_1 + \beta^\mu - \frac{2i}{3}\alpha_1 b^\mu)\frac{Q_\mu}{f_R} - \frac{2}{3}\alpha_1 M,$$
$$\alpha_3 = (-\nabla^\mu \alpha_1 + \beta^\mu - \frac{2i}{3}\alpha_1 b^\mu)\frac{Q_\mu^\star}{f_L} - \frac{2}{3}\alpha_1 \overline{M},$$
$$\alpha_4 = -\frac{1}{2}\left(\alpha_2 \overline{M} + \alpha_3 M + \nabla^\mu \alpha_2 \frac{Q_\mu^\star}{f_L} + \nabla^\mu \alpha_3 \frac{Q_\mu}{f_R}\right),$$
$$K^\mu \nabla_\mu \alpha_1 = K^\mu \nabla_\mu \alpha_2 = K^\mu \nabla_\mu \alpha_3 = K^\mu \left(\beta_\mu - \frac{2i}{3}\alpha_1 b_\mu\right) = 0,$$
$$\nabla^\mu \beta_\mu = \frac{1}{2}\left(\nabla^\mu \alpha_3 \frac{Q_\mu}{f_R} - \nabla^\mu \alpha_2 \frac{Q_\mu^\star}{f_L} + \alpha_3 M - \alpha_2 \overline{M}\right). \tag{3.11}$$

For given functions α_1 and β^μ, the first three equations in (3.11) determine α_2, α_3 and α_4, respectively. The final two equations can then be viewed as constraints on the form of α_1 and β^μ.

It is in general nontrivial to find solutions to the above system. However, the analysis simplifies in the case of four supercharges. Since, we can construct four linearly independent vectors K_i^μ (and similarly for Q_i^μ), we conclude that $\alpha_1 = $ constant. Hence the only

---

[2]For non-compact manifolds, one may impose suitable fall-off conditions at infinity.

solution is[3]

$$E \equiv D + \frac{2}{3}(ib_\mu A^\mu - MF - \overline{MF} + M\overline{M}C), \tag{3.12}$$

up to a constant rescaling. Using the integrability conditions (2.5), one can check that this is in fact a special case of the standard D-type invariant (3.7) of supergravity. For backgrounds with less than maximal supersymmetry, there may be additional solutions to (3.11), and hence more SUSY invariants. We will not attempt to write down all invariants, but content ourselves with giving some examples of additional invariants that arise in the case of $SU(2)$ structure in section 4.1.

We can nevertheless study the dependence of the partition function on couplings to general invariants, without making use of explicit solutions for α_i, β_μ. An obvious question that arises is whether or not E can be written as a SUSY-exact term. If this were true, the partition function would then be independent of the coupling to such terms. To proceed, we again assume that $\epsilon_{L/R} \neq 0$. This allows us to rewrite the fermionic variations in (2.8) as eight scalar equations by contracting with $\epsilon_{L/R}^\dagger$ and $\epsilon_{L/R}^c$:

$$F = -\delta\left(\frac{\epsilon_L^\dagger \psi_L}{f_L}\right) + \frac{1}{2f_L}Q_\mu^\star(A^\mu - \nabla^\mu C),$$

$$\overline{F} = \delta\left(\frac{\epsilon_R^\dagger \psi_R}{f_R}\right) - \frac{1}{2f_R}Q_\mu\left(A^\mu + \nabla^\mu C\right),$$

$$K^\mu A_\mu = \delta(\epsilon_R^c \psi_R - \epsilon_L^c \psi_L),$$

$$K^\mu \nabla_\mu C = \delta(\epsilon_R^c \psi_R + \epsilon_L^c \psi_L),$$

$$D = \delta\left(\frac{\epsilon_R^\dagger \lambda_R}{f_R} - \frac{\epsilon_L^\dagger \lambda_L}{f_L}\right) + \left(\frac{i}{2f_R}J_{\mu\nu}^R - \frac{i}{2f_L}J_{\mu\nu}^L\right)\nabla^\mu A^\nu,$$

$$\left(\frac{i}{2f_R}J_{\mu\nu}^R + \frac{i}{2f_L}J_{\mu\nu}^L\right)\nabla^\mu A^\nu = -\delta\left(\frac{\epsilon_R^\dagger \lambda_R}{f_R} + \frac{\epsilon_L^\dagger \lambda_L}{f_L}\right),$$

$$\Omega_{\mu\nu}^L \nabla^\mu A^\nu = \delta(2\epsilon_L^c \lambda_L),$$

$$\Omega_{\mu\nu}^R \nabla^\mu A^\nu = \delta(2\epsilon_R^c \lambda_R). \tag{3.13}$$

Using these relations along with (3.11) and the integrability conditions (2.5), we find that the general invariant (3.9) reduces to

$$E = \delta V_E + \xi^\mu A_\mu + \eta C + \nabla(\ldots), \tag{3.14}$$

where

$$V_E = \alpha_1\left(\frac{\epsilon_R^\dagger \lambda_R}{f_R} - \frac{\epsilon_L^\dagger \lambda_L}{f_L}\right) - \alpha_2\left(\frac{\epsilon_L^\dagger \psi_L}{f_L}\right) + \alpha_3\left(\frac{\epsilon_R^\dagger \psi_R}{f_R}\right) + \frac{2}{3f_L f_R}(\text{Im}b \cdot K^\star)(\epsilon_L^c \psi_L - \epsilon_R^c \psi_R), \tag{3.15}$$

---

[3]Note that this is an invariant even in the vicinity of isolated zeroes of $\epsilon_{L/R}$. For a general invariant, one would need to check this explicitly by plugging the solution to (3.11) back into (3.10).

and

$$\xi_\mu = \frac{1}{(2f_L f_R)^2} \alpha_1 (Q_\nu^\star Q_\mu - Q_\nu Q_\mu^\star) L^\nu,$$

$$\eta = \frac{1}{6f_L f_R} \alpha_1 (ML \cdot Q^\star + \overline{M} L \cdot Q) + \frac{1}{2f_L f_R} L^\mu \nabla_\mu \alpha_1 + \xi^\mu \left(\beta_\mu - \frac{2i}{3} \alpha_1 b_\mu \right), \qquad (3.16)$$

and $\nabla(\ldots)$ denotes total derivatives. We see that in general, E cannot be written as a SUSY-exact term: there is an obstruction in the form of additional terms that depend on the geometry.

- Assuming $\alpha_1 \neq 0$, the extra terms vanish if and only if $L = [K, K^\star] = 0$, which is the case of torus fibrations. For $L = 0$, we then have two options:

  - If both ϵ_L and ϵ_R are nowhere vanishing, $E = \delta V_E$ holds everywhere. We conclude that all SUSY invariants are exact, and the partition function does not depend on the corresponding couplings. This result is not surprising: as we noticed earlier, this case corresponds to a pair of nilpotent supercharges $\delta_L^2 = \delta_R^2 = 0$.

  - If, for example, $\epsilon_R = 0$ (which implies $G = \mathrm{SU}(2)_R$), the invariants can be written as a variation with respect to the left-handed supercharge, $E = \delta_L V_E$. We discuss this case in more detail in section 4.1. Here we only note that the partition function will again be independent of the couplings.

- For $L \neq 0$, which includes the interesting case of S^4, equation (3.14) demonstrates that there is no SUSY invariant that is also exact, and hence we expect Z to depend nontrivially on all coupling constants. We analyze this dependence further in section 3.4, where we discuss the issue of finite counterterms.

There is one invariant that needs to be discussed separately. Choosing $\alpha_1 = 0$, eqs. (3.11) imply that $\alpha_2 = \alpha_3 = \alpha_4 = 0$ and $\beta^\mu \sim K^\mu$. This corresponds to

$$K^\mu A_\mu = \delta \left(\epsilon_R^c \psi_R - \epsilon_L^c \psi_L \right), \qquad (3.17)$$

which is SUSY-exact. This invariant generically only conserves a single supercharge. We will further comment on the relevance of this term in section 3.3.

3.2 Chiral invariants

In our analysis so far, we assumed that there are no restrictions on the field content. Of course, any realistic theory will have such restrictions. For example, a theory with chiral and antichiral fields will admit generalized F-type and $\overline{F}$-type invariants, in addition to the general D-type invariants (3.9).

To find these additional chiral/antichiral invariants, we proceed in a similar fashion as before. Chiral and antichiral multiplets are embedded into the general multiplet as in (2.9)

and (2.10). The SUSY variations for a chiral multiplet are

$$
\begin{aligned}
\delta\phi &= -\epsilon_L^c \psi_L, \\
\delta\psi_L &= -\gamma^\mu \epsilon_R \nabla_\mu \phi - \epsilon_L F, \\
\delta F &= \nabla^\mu \left(\epsilon_R^c \gamma_\mu \psi_L \right) - \overline{M} \epsilon_L^c \psi_L,
\end{aligned}
\tag{3.18}
$$

while for an antichiral multiplet, we have

$$
\begin{aligned}
\delta\overline{\phi} &= -\epsilon_R^c \psi_R, \\
\delta\psi_R &= \gamma^\mu \epsilon_L \nabla_\mu \overline{\phi} + \epsilon_R \overline{F}, \\
\delta\overline{F} &= \nabla^\mu \left(\epsilon_L^c \gamma_\mu \psi_R \right) - M \epsilon_R^c \psi_R.
\end{aligned}
\tag{3.19}
$$

The most general bosonic chiral/antichiral invariant may be written as

$$
\begin{aligned}
I &= \beta_1 F + \beta_2 \phi, \\
\overline{I} &= \overline{\beta}_1 \overline{F} + \overline{\beta}_2 \overline{\phi},
\end{aligned}
\tag{3.20}
$$

with functions β_1, β_2, $\overline{\beta}_1$ and $\overline{\beta}_2$ to be determined. Demanding SUSY-invariance of I and $\overline{I}$ yields the conditions

$$
\begin{aligned}
\nabla^\mu \beta_1 \gamma_\mu \epsilon_R + (\beta_2 + \beta_1 \overline{M})\epsilon_L &= 0, \\
\nabla^\mu \overline{\beta}_1 \gamma_\mu \epsilon_L + (\overline{\beta}_2 + \overline{\beta}_1 M)\epsilon_R &= 0,
\end{aligned}
\tag{3.21}
$$

or equivalently

$$
\begin{aligned}
\beta_2 &= -\beta_1 \overline{M} - \nabla^\mu \beta_1 \frac{Q_\mu^\star}{f_L}, \\
\overline{\beta}_2 &= -\overline{\beta}_1 M - \nabla^\mu \overline{\beta}_1 \frac{Q_\mu}{f_R}, \\
K^\mu \nabla_\mu \beta_1 &= K^\mu \nabla_\mu \overline{\beta}_1 = 0.
\end{aligned}
\tag{3.22}
$$

Again, for a background that preserves four supercharges, the only solution is to take β_1, $\overline{\beta}_1$ to be constants, so the invariants are

$$
\begin{aligned}
I &= F - \overline{M}\phi, \\
\overline{I} &= \overline{F} - M\overline{\phi}.
\end{aligned}
\tag{3.23}
$$

These are the curved space generalization of the standard F, $\overline{F}$-terms. The coupling to the background fields can be thought of as originating from the nontrivial chiral density 2ϵ in the superspace formalism:

$$
S\big|_I = e^{-1} \int d^2\Theta(2\epsilon)S, \qquad S\big|_{\overline{I}} = e^{-1} \int d^2\overline{\Theta}(2\overline{\epsilon})\overline{S}.
\tag{3.24}
$$

After setting the gravitino to zero, we find $2\epsilon = e\left(1 - \Theta^2 \overline{M}\right)$, which shifts the F-terms as in (3.23).

For backgrounds that preserve fewer than four supercharges, there may be more solutions to (3.21). We can ask if a general invariant $I, \overline{I}$, with $\beta_1, \overline{\beta}_1$ unspecified, is SUSY-exact. We find that

$$I = \delta \left(-\beta_1 \frac{\epsilon_L^\dagger \psi_L}{f_L} \right) - \frac{1}{2f_L^2 f_R} \beta_1 (L \cdot Q^\star) \phi + \nabla(\ldots),$$

$$\overline{I} = \delta \left(\overline{\beta}_1 \frac{\epsilon_R^\dagger \psi_R}{f_R} \right) - \frac{1}{2f_L f_R^2} \overline{\beta}_1 (L \cdot Q) \overline{\phi} + \nabla(\ldots). \tag{3.25}$$

As before, the obstruction to exactness is related to K^μ not commuting with its complex conjugate.

- For backgrounds with S^3-isometry, where $L \neq 0$, neither I nor $\overline{I}$ are exact, and the partition function depends nontrivially on all chiral/antichiral couplings.

- For torus fibrations, where $L = 0$, I and $\overline{I}$ are in general exact, and the partition function is independent of chiral/antichiral couplings. Notice however that if one of the chiral spinors ϵ_L or ϵ_R vanishes identically, then either the I or $\overline{I}$ equation in (3.25) is no longer valid. We discuss this case separately in section 4.2.

3.3 Lagrangians and localization

With the knowledge of the SUSY invariants, one can construct Lagrangians for an arbitrary field content. As an instructive example, we will discuss the case of a chiral and antichiral multiplet $(\Phi, \overline{\Phi})$. Guided by the "no-miracles" principle, we should write down the most general terms consistent with the symmetries of the theory. We have seen that the invariants are the D-type terms (3.9), and the chiral/antichiral F-type invariants (3.20). Hence the most general Lagrangian is

$$e^{-1}\mathcal{L} = -\frac{1}{2} \sum_E K(\Phi, \overline{\Phi}) \Big|_E - \sum_I W(\Phi) \Big|_I - \sum_{\overline{I}} \overline{W}(\overline{\Phi}) \Big|_{\overline{I}}. \tag{3.26}$$

Here K is a Kähler potential, which can be written as a power series involving the $\times$-multiplication (see appendix A), and W is the holomorphic superpotential. The sums are taken over all possible invariants for a given background. For a maximally supersymmetric space, there are only three invariants, namely the E invariant of (3.7), and the I and $\overline{I}$ invariants of (3.23), so the analysis simplifies somewhat. In this case, evaluating (3.26) yields

$$\begin{aligned}
e^{-1}\mathcal{L} = {} & K \left(\frac{1}{6}R + \frac{1}{9}b^\mu b_\mu - \frac{1}{9}M\overline{M} \right) + K^{(1,1)} \left(\partial^\mu \bar{\phi} \partial_\mu \phi - F\overline{F} \right) \\
& + \frac{i}{3}b^\mu \left(K^{(1,0)} \partial_\mu \phi - K^{(0,1)} \partial_\mu \bar{\phi} \right) + F \left(\frac{1}{3}MK^{(1,0)} - W^{(1)} \right) \\
& + \bar{F} \left(\frac{1}{3}\overline{M}K^{(0,1)} - \overline{W}^{(1)} \right) + W\overline{M} + \overline{W}M \\
& + K^{(1,1)} \psi_R^c \gamma^\mu \widetilde{\nabla}_\mu \psi_L + \frac{1}{2} \left(W^{(2)} + K^{(2,1)}\overline{F} - \frac{1}{3}MK^{(2,0)} \right) \psi_L^c \psi_L \\
& \frac{1}{2} \left(\overline{W}^{(2)} + K^{(1,2)}F - \frac{1}{3}\overline{M}K^{(0,2)} \right) \psi_R^c \psi_R - \frac{1}{4}K^{(2,2)} \psi_L^c \psi_L \psi_R^c \psi_R, \tag{3.27}
\end{aligned}$$

where $K^{(n,m)} \equiv \partial^{n+m} K / \partial \phi^n \partial \overline{\phi}^m$ and $W^{(n)} \equiv \partial^n W / \partial \phi^n$. We have also defined

$$\widetilde{\nabla}_\mu \psi_L = \left(\nabla_\mu + \frac{i}{6} K^{(1,1)} b_\mu + K^{(1,1)} K^{(2,1)} \partial_\mu \phi \right) \psi_L. \tag{3.28}$$

The Lagrangian (3.27) is of course the same result one obtains by taking the rigid limit of the supergravity Lagrangian, analytically continued to Euclidean signature [6, 7, 20].

An interesting question is whether the partition function for (3.26) can be computed via localization. Let us therefore review the general philosophy of localization [1, 4]. Given a supersymmetric background and field content, one should consider a Lagrangian that includes *all* possible terms consistent with symmetries. In our case, the symmetries are $\mathcal{N} = 1$ with a certain number of supercharges, and potentially global symmetries and R-symmetries. Schematically, we have

$$\mathcal{L} = \sum_i \lambda_i \mathcal{L}_i, \tag{3.29}$$

where each $\mathcal{L}_i$ is supersymmetric. A priori, the partition function depends on all the coupling constants λ_i, i.e.

$$Z = Z[\lambda_1, \lambda_2, \ldots]. \tag{3.30}$$

Now, suppose there is a linear combination of couplings, called t, such that

$$\frac{dZ}{dt} = 0. \tag{3.31}$$

We can then evaluate Z for any given value of t, and are guaranteed to get the same result. In particular, we can go to a corner in the space of couplings where $t \gg \lambda_i$ (i.e. formally take $t \to \infty$), and compute Z there. If $\mathcal{L}_t$ has a positive semi-definite bosonic part, the theory localizes around the classical locus $\mathcal{L}_t\big|_{\text{bos.}} = 0$ and the partition function is one-loop exact.

The necessary and sufficient condition for (3.31) to hold is that the corresponding term in the Lagrangian is SUSY-exact:

$$\mathcal{L}_t = \delta V. \tag{3.32}$$

This is why localization is usually thought of as "adding" an exact deformation to the Lagrangian, and consequently taking the coupling to infinity. While there is certainly nothing wrong with this point of view, it seems more useful to think about the localization term δV as already being part of the original Lagrangian. Localization then simply utilizes the fact that there are "flat" directions in the space of coupling constants.

To summarize, there are two basic conditions that need to be satisfied for localization:

1. The Lagrangian contains a term $\mathcal{L}_t$ that is both SUSY-closed and exact.

2. The bosonic part of $\mathcal{L}_t$ is positive semi-definite.

Using our results from the previous section, we can easily check these conditions for a broad class of manifolds.

- For trivial structure with $L = [K, K^\star] = 0$, all SUSY invariants are exact. In principle, there is no obstruction to performing localization. The nontrivial task is to find a positive semi-definite localization term. We will do so for the closely related case of manifolds with SU(2) structure in section 4.3.

- For $L \neq 0$, none of the invariants are exact. We conclude that for manifolds with S^3-isometry, in particular the squashed and round S^4, the partition function does not localize.

Finally, there is one invariant that needs to be discussed separately, namely (3.17), which preserves only one supercharge. Evaluated on a Kähler potential $K(\Phi, \overline{\Phi})$, it reads

$$K^\mu A_\mu = K^\mu \left(K^{(1,0)} \nabla_\mu \overline{\phi} - K^{(0,1)} \nabla_\mu \phi + K^{(1,1)} \psi_L^c \gamma_\mu \psi_R \right). \tag{3.33}$$

This can be regarded as a coupling of the global U(1)-current to the background. It is obvious that its bosonic part cannot be made positive semi-definite, so (3.33) cannot be used for localization.

We should note that our result strictly speaking only holds for a chiral/antichiral field content. Considering other irreducible representations, such as gauge or linear multiplets, might lead to additional invariants, analogous to the chiral/antichiral I and $\overline{I}$ terms. However, our analysis of the general D-type terms was independent of the field content, so it is still true that for $L \neq 0$, there are no D-type invariants that are exact. Since the kinetic terms for fields are generally only found among these D-terms, we conjecture that the possible additional invariants cannot be utilized for localization.

3.4 Counterterms and the physical part of Z

For the case $L \neq 0$, we have established that the partition function is a nontrivial function of all couplings (with one exception, see above). The next question to ask is whether this dependence is non-ambiguous.

It is instructive to review the logic of extracting physical data from partition functions. On compact manifolds, infrared divergences in the partition function are absent, due to the finite volume of the background. However, there might still be ultraviolet divergences that need to be regularized. Very schematically, the partition function may take the form

$$\log Z(\lambda_i) = \sum_j a_j(\lambda_i) \Lambda^j + A(\lambda_i) \log \Lambda + F(\lambda_i). \tag{3.34}$$

The first term captures power law divergences, with Λ being the UV cutoff. The log-divergent term is the analog of the A-type anomaly in CFTs. The last part is the finite contribution F to the free energy. From our analysis above, we expect all terms to be nontrivial functions of the couplings.

A regularization scheme corresponds to choosing a certain set of counterterms, which can be used to tune some of the terms in (3.34) to zero. Only the parts of the partition

function that are unaffected by counterterms are physical observables.[4] Let us now determine the physical content of $\mathcal{N} = 1$ theories on backgrounds with $L^\mu \neq 0$. Instead of considering all possible couplings, we focus on couplings to chiral invariants $I, \overline{I}$. The interactions take the form

$$e^{-1}\mathcal{L}_{\text{int}} = W\big|_I + \overline{W}\big|_{\overline{I}}. \tag{3.35}$$

For a renormalizable theory, the superpotential W contains relevant couplings m_i and marginal couplings λ_i. We can classify the possible counterterms by performing a spurion analysis. For clarity of presentation, we will focus on the case of only a single pair of relevant couplings $(m, \overline{m})$ and marginal couplings $(\lambda, \overline{\lambda})$ each. The generalization to an arbitrary number of couplings should be straightforward. Treating the couplings as the lowest components of spurious chiral/antichiral superfields $(\Sigma_m, \overline{\Sigma}_m, \Sigma_\lambda, \overline{\Sigma}_\lambda)$, we see that renormalizable interactions arise from

$$e^{-1}\mathcal{L}_{\text{int}} = \left[\Sigma_m \Phi^2 + \Sigma_\lambda \Phi^3\right]\big|_I + \left[\overline{\Sigma}_m \overline{\Phi}^2 + \overline{\Sigma}_\lambda \overline{\Phi}^3\right]\big|_{\overline{I}} \tag{3.36}$$

upon taking expectation values.

The possible finite counterterms are local interactions of spurions, consistent with the symmetries of the underlying theory. Let us start by choosing a supersymmetric background with the smallest possible set of symmetries. Those are manifolds with trivial structure and only one conserved supercharge, so the desired counterterms are all local, diffeomorphism invariant terms that preserve one supersymmetry. We have already derived the complete set of such terms, so we can conclude that the counterterms are given by the E-, I-, and $\overline{I}$-terms of sections 3.1 and 3.2. Hence the possible finite counterterms arise from interactions of the form

$$F(\Sigma_\lambda, \overline{\Sigma}_\lambda, \Sigma_m, \overline{\Sigma}_m)\big|_E + G(\Sigma_\lambda, \Sigma_m)\big|_I + H(\overline{\Sigma}_\lambda, \overline{\Sigma}_m)\big|_{\overline{I}}. \tag{3.37}$$

Taking the appropriate expectation values, we find the following counterterm Lagrangian:

$$e^{-1}\mathcal{L}_{ct} = \alpha_4 F(\lambda, \overline{\lambda}, m, \overline{m}) + \beta_2 G(\lambda, m) + \overline{\beta}_2 H(\overline{\lambda}, \overline{m}). \tag{3.38}$$

Here α_4, β_2 and $\overline{\beta}_2$ are solutions to the system (3.11). Instead of attempting to work with the most general solution, let us simply note that the standard choice

$$\alpha_4 = -\frac{1}{3}R + \frac{2}{9}M\overline{M} - \frac{2}{9}b_\mu b^\mu, \tag{3.39}$$

$$\beta_2 = -\overline{M}, \tag{3.40}$$

$$\overline{\beta}_2 = -M, \tag{3.41}$$

is a solution for any number of preserved supercharges and work with the invariants corresponding to this choice.

---

[4]Note that in some cases one might have to take a certain number of derivatives of $\log Z$ with respect to the couplings λ_i to extract the unambiguous physical data. One example is the Zamolodchikov metric on the space of exactly marginal couplings of a CFT [22–29], $g_{i\overline{j}} \sim \partial_i \partial_{\overline{j}} \log Z(\lambda_k, \overline{\lambda}_l)$.

Using dimensional analysis, we can further constrain the form of the counterterms. Assuming that Φ is canonically normalized, we have $[m] = 1$, so the function F in (3.38) needs to be a quadratic function of relevant couplings, while G and H are cubic. Carrying out the volume integral to compute the action will produce a curvature scale $\int \sqrt{g} \alpha_4 \sim r^2$, and similarly for β_2, $\overline{\beta}_2$. Thus the partition function itself exhibits a regularization scheme dependent ambiguity of the form

$$\log Z \sim \log Z + f\left(mr, \overline{m}r, \lambda, \overline{\lambda}\right) + (mr)^3 g(\lambda) + (\overline{m}r)^3 h(\overline{\lambda}), \tag{3.42}$$

where f contains only terms that are quadratic in relevant couplings. To be completely general, we should also consider counterterms that involve curvature multiplets [24, 30]. For example, there are D-type counterterms of the form

$$\mathcal{R}^i \overline{\mathcal{R}}^j \Sigma_m^k \overline{\Sigma}_m^l F(\Sigma_\lambda, \overline{\Sigma}_\lambda)\big|_E, \tag{3.43}$$

where $\mathcal{R}$ is the chiral curvature superfield, with expectation value

$$-6\langle \mathcal{R} \rangle = M + \Theta^2 \left(\frac{1}{2}R + \frac{2}{3}M\overline{M} + \frac{1}{3}b^\mu b_\mu - i\nabla^\mu b_\mu\right). \tag{3.44}$$

Since its lowest component has mass dimension 1, we see that $i + j + k + l = 2$ in (3.43). In addition, we should also consider the more general chiral/antichiral counterterms

$$\mathcal{R}^i \Sigma_m^{3-i} G(\Sigma_\lambda)\big|_I, \quad \overline{\mathcal{R}}^i \overline{\Sigma}_m^{3-i} H(\overline{\Sigma}_\lambda)\big|_{\overline{I}}. \tag{3.45}$$

If we include all such mixed matter-gravity counterterms, the ambiguity becomes

$$\log Z \sim \log Z + F_2\left(mr, \overline{m}r, \lambda, \overline{\lambda}\right) + G_3(mr, \lambda) + H_3(\overline{m}r, \overline{\lambda}), \tag{3.46}$$

where F_2, G_3, H_3 are now general quadratic (cubic) polynomials in the relevant couplings, but arbitrary functions of marginal couplings:

$$\begin{aligned}
F_2\left(mr, \overline{m}r, \lambda, \overline{\lambda}\right) &= \sum_{i+j\leq 2} a_{i,j}(mr)^i(\overline{m}r)^j f_{i,j}(\lambda, \overline{\lambda}), \\
G_3(mr, \lambda) &= \sum_{i\leq 3} b_i(mr)^i g_i(\lambda), \\
H_3(\overline{m}r, \overline{\lambda}) &= \sum_{i\leq 3} c_i(\overline{m}r)^i h_i(\overline{\lambda}).
\end{aligned} \tag{3.47}$$

The coefficients a, b, c are dimensionless, background-dependent constants that arise from integrating curvature invariants.

We conclude that in general, finite counterterms may shift the free energy by regularization scheme dependent terms according to (3.46). If we expand $\log Z(m, \lambda)$ in powers of relevant couplings, all terms up to cubic order are subject to ambiguities, and thus unphysical. However, higher powers of m are free from ambiguities, so we may extract the physical part of the partition function by taking suitable derivatives with respect to coupling constants. Inspecting (3.47), we see that, for example,

$$\frac{\partial^4}{\partial(mr)^4}\log Z, \quad \frac{\partial^3}{\partial(mr)^3}\frac{\partial^2 \log Z}{\partial\lambda\partial\overline{\lambda}}, \tag{3.48}$$

are unambiguous physical observables. The minimum number of derivatives one has to take is model-dependent, since additional global symmetries may forbid certain counterterms. Note that the second expression in (3.48) is reminiscent of the Zamolodchikov metric for CFTs [22–29].

Another way to avoid counterterm ambiguities of the partition function is to further constrain the background manifolds, such that the coefficients multiplying the counterterms in (3.38) vanish identically. The rule of thumb is that more symmetries imply fewer counterterms, which allows for more physical observables to exist. For manifolds with the maximum number of four preserved supercharges, the integrability conditions (2.5) imply

$$R = -\frac{4}{3}M\overline{M} - \frac{2}{3}b^\mu b_\mu, \tag{3.49}$$

and hence

$$\alpha_4 = \frac{2}{3}M\overline{M}, \quad \beta_2 = -\overline{M}, \quad \overline{\beta}_2 = -M. \tag{3.50}$$

Following [13], there are two types of backgrounds

- For $M, \overline{M} \neq 0$, the space is locally isometric to the round S^4 or $\mathbb{H}^4$. In this case, the ambiguity (3.46) remains. Since the round sphere is a limiting case of the squashed sphere $\tilde{S}^4$, α_4 cannot vanish identically for $\tilde{S}^4$, so the ambiguity is present in this case as well.

- For $M = \overline{M} = 0$, the background is locally isometric to $\mathcal{M}_3 \times \mathbb{R}$, where $\mathcal{M}_3$ has constant curvature. In this case, the candidate counterterms vanish identically, and there is no obvious obstruction for the finite part of $F = \log Z$ to be a physical observable. To prove that F is indeed physical, one would need to perform a more complete analysis involving also purely gravitational counterterms constructed out of the curvature multiplets of supergravity, along the lines of [30].

It is interesting to compare our result (3.46) to the case of SCFTs on S^4 [24]. In the latter case, there is no mass scale m. However, counterterms that couple marginal operators to the background (e.g. the case $k = l = 0$ in (3.43)) are still present, so there is an ambiguity of the form

$$\log Z \sim \log Z + f(\lambda, \overline{\lambda}), \tag{3.51}$$

where f is an arbitrary function of the marginal couplings, and the finite part of the partition function is completely unphysical.

4 $\mathcal{N} = 1$ theories on manifolds with SU(2) structure

We now turn to the case of manifolds with SU(2) structure, which possess an R-symmetry. In general, the Killing spinor equations (2.3) mix left- and right-handed spinors, so there can be no R-symmetry. However, this mixing is not present whenever either ϵ_L or ϵ_R vanish identically. Without loss of generality, we will assume that there is a supercharge of the form $(\epsilon_L, 0)$. Setting $\epsilon_R = 0$ in (2.3) then yields the Killing spinor equation

$$\nabla_\mu \epsilon_L = \frac{i}{2}b_\mu \epsilon_L - \frac{i}{6}b^\nu \gamma_\mu \gamma_\nu \epsilon_L, \tag{4.1}$$

along with the requirement $\overline{M} = 0$. Note that M has completely dropped out of this expression, so a priori it is an arbitrary function.

Backgrounds with SU(2) structure possess a U(1)$_R$ R-symmetry, under which ϵ_L carries charge 1. Theories with R-symmetry can be naturally coupled to new minimal supergravity [6, 9, 10, 14–16]. In this framework, the six auxiliary degrees of freedom are captured by a conserved vector V_μ and a U(1)$_R$-gauge field A_μ. The conditions for a background to preserve supersymmetry are [6]

$$
\begin{aligned}
D_\mu \epsilon_L &= -\frac{3i}{2} V_\mu \epsilon_L + \frac{i}{2} V^\nu \gamma_\mu \gamma_\nu \epsilon_L, \\
D_\mu \epsilon_R &= \frac{3i}{2} V_\mu \epsilon_R - \frac{i}{2} V^\nu \gamma_\mu \gamma_\nu \epsilon_R,
\end{aligned}
\tag{4.2}
$$

where $D_\mu = \nabla_\mu - ir(A_\mu + \frac{3}{2} V_\mu)$ is an R-covariant derivative. The left- and right-handed supercharges carry R-charges 1 and -1. If we restrict to a subclass of backgrounds with

$$
A_\mu = -\frac{3}{2} V_\mu, \qquad V_\mu \equiv -\frac{1}{3} b_\mu,
\tag{4.3}
$$

we recover (4.1) and its right-handed counterpart. Hence backgrounds with SU(2) structure in old-minimal supergravity are a subclass of the backgrounds of new minimal supergravity.

Let us briefly summarize some known features of the backgrounds $\mathcal{M}$ considered here. From (4.1), we can derive the integrability conditions

$$
\begin{aligned}
R &= 2i\nabla \cdot b - \frac{2}{3} b_\mu b^\mu, \\
\partial_{[\mu} b_{\nu]} &= \frac{1}{2} \epsilon_{\mu\nu\lambda\sigma} \partial^\lambda b^\sigma.
\end{aligned}
\tag{4.4}
$$

As before, we can construct bilinears from the Killing spinor:

$$
f_L = \epsilon_L^\dagger \epsilon_L, \qquad J_{\mu\nu} = \frac{i}{f_L} \epsilon_L^\dagger \gamma_{\mu\nu} \epsilon_L, \qquad \Omega_{\mu\nu} = \epsilon_L^c \gamma_{\mu\nu} \epsilon_L.
\tag{4.5}
$$

Note that there are no invariant vectors in the SU(2) structure case. Using Fierz identities, we have $J_{\mu\nu} J^{\nu\rho} = -\delta_\mu^\rho$, so J defines an almost complex structure. It can be shown that the corresponding Nijenhuis tensor $N^\mu{}_{\nu\rho}$ vanishes identically [13], so the almost complex structure is integrable, and hence $\mathcal{M}$ is a complex manifold. Furthermore, note that the complex structure is metric-compatible, i.e. $g_{\mu\nu} J^\mu{}_\rho J^\nu{}_\sigma = g_{\rho\sigma}$, so $\mathcal{M}$ is hermitian.

To simplify some of our later analysis, we introduce holomorphic coordinates $z^i, \bar{z}^i$ ($i = 1, 2$), such that

$$
J^i{}_j = i\delta^i_j, \qquad J^{\bar{i}}{}_{\bar{j}} = -i\delta^{\bar{i}}_{\bar{j}}.
\tag{4.6}
$$

One can check that $\Omega_{\bar{i}\bar{j}} = 0$, and Ω_{12} is nonvanishing everywhere. Hence Ω defines a nowhere vanishing section of the canonical line bundle $\mathcal{K}$ of (2,0)-forms. To summarize, the supersymmetric backgrounds $\mathcal{M}$ we are considering are hermitian manifolds with SU(2) structure and trivial canonical line bundle $\mathcal{K}$. The only compact 4-manifolds that satisfy those criteria are tori, $K3$ and primary Kodaira surfaces [13, 36].

4.1 General invariants

In section 3.1, we saw that imposing constraints on the number of preserved supercharges can lead to a much richer set of invariants. In this section, we will demonstrate that the same is true when imposing the condition that the supercharges are chiral, i.e. for backgrounds with SU(2) structure.

Setting $\epsilon_R = 0$, the SUSY variations simplify to

$$\delta C = -\epsilon_L^c \psi_L, \tag{4.7}$$

$$\delta \psi_L = -\epsilon_L F, \tag{4.8}$$

$$\delta \psi_R = \frac{1}{2} \gamma^\mu (A_\mu + \nabla_\mu C) \epsilon_L, \tag{4.9}$$

$$\delta F = 0, \tag{4.10}$$

$$\delta \overline{F} = \nabla^\mu (\epsilon_L^c \gamma_\mu \psi_R) - \epsilon_L^c \lambda_L, \tag{4.11}$$

$$\delta A_\mu = -\epsilon_L^c \gamma_\mu \lambda_R + \nabla_\mu (\epsilon_L^c \psi_L), \tag{4.12}$$

$$\delta \lambda_L = \frac{1}{2} \gamma^{\mu\nu} \epsilon_L \nabla_\mu A_\nu - \frac{1}{2} \epsilon_L D, \tag{4.13}$$

$$\delta \lambda_R = 0, \tag{4.14}$$

$$\delta D = \nabla^\mu (\epsilon_L^c \gamma_\mu \lambda_R) + \frac{2i}{3} b_\mu (\epsilon_L^c \gamma^\mu \lambda_R). \tag{4.15}$$

The crucial difference to the case of trivial structure discussed earlier is that the supercharge δ is nilpotent: $\delta^2 = 0$. One fact we can immediately note is that any exact term δV will be δ-closed. In particular, localization seems straightforward. We will further comment on aspects of localization in section 4.3.

To find all bosonic SUSY invariants, we again make the ansatz

$$E = \alpha_1 D + \alpha_2 F + \alpha_3 \overline{F} + \alpha_4 C + \beta^\mu A_\mu, \tag{4.16}$$

with in general nonconstant α_i and β^μ. Demanding that δE is a total derivative, we find the conditions

$$\nabla^i \alpha_1 - \frac{2}{3} i \alpha_1 b^i + \beta^i = 0, \tag{4.17}$$

$$\alpha_4 + \nabla^\mu \beta_\mu = 0, \tag{4.18}$$

$$\alpha_3 = 0. \tag{4.19}$$

Here $i = 1, 2$ denote holomorphic coordinates, and we have used the fact that $\gamma_{\bar{i}} \epsilon_L = 0$, which follows from Fierz identities. We do not attempt to find the complete set of solutions, but instead give three examples of invariants:

- From (4.10), we immediately see that F-terms are invariant. Using (4.8), we can show that these terms are also δ-exact:

$$\alpha_2 F = -\delta \left(\alpha_2 \frac{\epsilon_L^\dagger \psi_L}{f_L} \right). \tag{4.20}$$

In principle, we can allow α_2 to be an arbitrary function.

- A second type of solution can be obtained by setting $\alpha_2 = 0$, and restricting α_1 to be a constant. Since $\alpha_1 = 0$ only leads to a trivial solution, we can set $\alpha_1 = 1$. Then

$$\beta^i = \frac{2i}{3} b^i,$$
$$\alpha_4 = -\nabla^\mu \beta_\mu. \tag{4.21}$$

There are two linearly independent solutions, characterized by the choice of $\beta^{\bar{i}}$. We choose the following linearly independent solutions:

$$\beta_1^\mu = \frac{2i}{3} b^\mu,$$
$$\beta_2^\mu = i\nabla_\nu J^{\mu\nu}. \tag{4.22}$$

Note that with this choice, $\beta_1^i = \beta_2^i = \frac{2i}{3} b^i$ and $\beta_1^{\bar{i}} = \frac{2i}{3} b^{\bar{i}}$, but $\beta_2^{\bar{i}} = \frac{2i}{3} b^{\bar{i}\star}$. The corresponding invariants are

$$E_1 = D + \frac{2i}{3} b^\mu A_\mu - \frac{1}{3} \left(R + \frac{2}{3} b^2 \right) C,$$
$$E_2 = D + i\nabla_\nu J^{\mu\nu} A_\mu. \tag{4.23}$$

It will be convenient to perform a change of basis by letting

$$E_- \equiv E_1 - E_2 = -\frac{4}{3} \mathrm{Im} b^{\bar{i}} A_{\bar{i}} - \frac{1}{2} (R + \frac{2}{3} b^2) C. \tag{4.24}$$

Using (4.9) and integration by parts, we find that

$$E_- = \delta \left[-\frac{4}{3} \mathrm{Im} b_\mu \frac{\epsilon_L^\dagger \gamma^\mu \psi_R}{f_L} \right],$$
$$E_2 = \delta \left[-2 \frac{\epsilon_L^\dagger \lambda_L}{f_L} \right]. \tag{4.25}$$

We conclude that all three invariants are SUSY-exact, and the partition function does not depend on the corresponding coupling constants.

In general, α_1 can be a nontrivial function of the background. In this more general case, we find that

$$E = \alpha_1 D + \left(\frac{2i}{3} \alpha_1 b^\mu - \nabla^\mu \alpha_1 \right) A_\mu - \nabla^\mu \left(\frac{2i}{3} \alpha_1 b_\mu - \nabla_\mu \alpha_1 \right) C$$
$$= \delta \left[-2\alpha_1 \frac{\epsilon_L^\dagger \lambda_L}{f_L} - \frac{4}{3} (\alpha_1 \mathrm{Im} b_\mu + \nabla_\mu \alpha_1) \frac{\epsilon_L^\dagger \gamma^\mu \psi_R}{f_L} \right] + \frac{2}{3} (\Delta_b \alpha_1) C, \tag{4.26}$$

where

$$\Delta_b = -\nabla^\mu \nabla_\mu - i\nabla_\mu J^{\mu\nu} \nabla_\nu. \tag{4.27}$$

E is exact if and only if $\Delta_b \alpha_1 = 0$, which is clearly satisfied for $\alpha_1 = \text{constant}$.

4.2 Chiral invariants

As in the trivial structure case, there are additional chiral and antichiral invariants. These are

$$F, \quad \overline{F}, \quad \overline{\phi}, \tag{4.28}$$

evaluated on chiral/antichiral fields. The first two invariants can be thought of as the special case $M = \overline{M} = 0$ of (3.23), while $\overline{\phi}$ is an additional invariant, due to the form of the SUSY algebra for SU(2) structure. We find that F-terms are exact (see (4.20)) while $\overline{F}$ and $\overline{\phi}$ are not.

4.3 Lagrangians and localization

It is instructive to compare and contrast the SU(2) structure case with the case of trivial structure discussed in section 3. We will do this by analyzing a simple toy-model: consider a pair of chiral and antichiral multiplets $(\Phi, \overline{\Phi})$, with charges $(1, 1)$ and $(-1, -1)$ under the global $U(1) \times U(1)_R$ symmetry. As we saw, on backgrounds with SU(2) structure there is a bigger arsenal of invariants than for the trivial structure case, so there is more freedom in building Lagrangians. Supersymmetric Lagrangians are built by combining superfields into products and taking the corresponding invariants. For SU(2) structure, there is an additional antisymmetric product $S_1 \wedge S_2$ (see appendix A), which gives us even more freedom in constructing Lagrangians. To be concrete, we can consider the following quadratic Lagrangian:

$$e^{-1}\mathcal{L} = \lambda_1 \Phi \times \overline{\Phi}\Big|_{E_-} + \lambda_2 \Phi \times \overline{\Phi}\Big|_{E_2} + \lambda_3 \Phi \times \overline{\Phi}\Big|_F + \lambda_4 \Phi \wedge \overline{\Phi}\Big|_{E_-}$$
$$+ \lambda_5 \Phi \wedge \overline{\Phi}\Big|_{E_2} + \lambda_F \Phi \times \Phi\Big|_F + \overline{\lambda}_F \overline{\Phi} \times \overline{\Phi}\Big|_{\overline{F}}. \tag{4.29}$$

The λ_i are various coupling constants. We have omitted $\overline{\phi}$-terms, which would break R-symmetry explicitly (see section 4.5 for a discussion of these terms).

It turns out that not all of the terms in (4.29) are linearly independent. Using the multiplication rules (A.2) and (A.5), we can write the Lagrangian in component form as

$$e^{-1}\mathcal{L} = t_1 \delta V_1 + t_2 \delta V_2 + t_M \delta V_M + t_b \delta V_b + \lambda_F \delta V_F$$
$$+ \overline{\lambda}_F (2\overline{\phi F} + \psi_R^c \psi_R), \tag{4.30}$$

where

$$V_1 = \frac{1}{f_L}\epsilon_L^\dagger \psi_L \overline{F},$$

$$V_2 = \frac{1}{f_L}\epsilon_L^\dagger \gamma^\mu \psi_R \nabla_\mu \phi,$$

$$V_M = -\frac{1}{3f_L}\epsilon_L^\dagger \psi_L \overline{\phi},$$

$$V_b = \frac{2}{3f_L}\epsilon_L^\dagger \gamma^\mu \psi_R \mathrm{Im}b_\mu,$$

$$V_F = -\frac{2}{f_L}\epsilon_L^\dagger \psi_L \phi, \tag{4.31}$$

and we have chosen a more convenient basis of couplings:

$$t_1 = -2\lambda_5,$$

$$t_2 = 2(\lambda_2 - \lambda_5),$$

$$t_M = \frac{3}{2}(\lambda_2 + \lambda_3),$$

$$t_b = -2(\lambda_1 + \lambda_4). \tag{4.32}$$

If we set $t_i = 1$, $\lambda_F = -m$ and $\overline{\lambda}_F = -\overline{m}$, the Lagrangian reduces to (3.27), with $K = \overline{\Phi}\Phi$ and $W = m\Phi^2$.

The decomposition of (4.30) in terms of SUSY-exact terms makes it manifest that the partition function is independent of all couplings except $\overline{\lambda}_F$. In particular, we are free to take certain linear combinations of couplings to infinity to perform localization. We now show that taking $t \equiv t_1 + t_2 \to \infty$ accomplishes just that.

Evaluating the bosonic part of the corresponding "localization term"

$$t(\delta V_1 + \delta V_2)\big|_{\text{bos.}} = t\left(-F\overline{F} + (g^{\mu\nu} + iJ^{\mu\nu})\,\partial_\mu\overline{\phi}\partial_\nu\phi\right), \tag{4.33}$$

we see that it can be made positive semi-definite by choosing the integration contour $\overline{\Phi} = \Phi^\ddagger$ for the bosonic fields, where $\ddagger$ is the involution

$$(\phi, \overline{\phi}, F, \overline{F})^\ddagger = (\overline{\phi}, \phi, -\overline{F}, -F). \tag{4.34}$$

In the limit $t \to \infty$, the path integral then localizes to bosonic field configurations with $(\delta V_1 + \delta V_2)\big|_{\text{bos.}} = 0$. In our case, the locus is $F = 0$ and $\phi = \phi_0 = \text{constant}$.[5] The partition function is given by a 1-loop integral around the classical locus:[6]

$$Z = \int \mathcal{D}\phi \mathcal{D}\overline{\phi} \mathcal{D}\psi_L \mathcal{D}\psi_R \exp\left[-\int d^4x \sqrt{g}\left(\overline{\phi}\Delta_b\phi + \psi_L^c \Delta_f \psi_R\right)\right]. \tag{4.35}$$

---

[5]A priori, ϕ is allowed to be an anti-holomorphic function. However, on a compact complex manifold this implies that ϕ is a constant [37] .

[6]We neglect the infinite prefactor due to $\int d\phi_0 d\overline{\phi}_0$.

Here we defined

$$\Delta_b = -\nabla^\mu \nabla_\mu - i\nabla_\mu J^{\mu\nu}\nabla_\nu,$$

$$\Delta_f = \gamma^\mu \left(-\gamma^5 \nabla_\mu + \frac{i}{2}b_\mu + \frac{i}{2}\nabla_\nu J^\nu{}_\mu \right). \tag{4.36}$$

4.4 Ambiguities of the partition function

The mere existence of an explicit prescription (4.35) for calculating the partition function on backgrounds with SU(2) structure is not sufficient to conclude that Z is a physical observable. In general, the one-loop determinants that appear need to be regularized, so it is crucial to ask if the final result is regularization scheme independent and thus physical. As we saw, for SU(2) structure the partition function depends nontrivially only on antichiral couplings $\overline{\lambda}_F$. Following our logic in section 3.4, we should then ask what possible finite counterterms could render the partition function ambiguous. The $\overline{F}$-terms in (4.29) can be viewed as a special case of interactions that arise from

$$e^{-1}\mathcal{L}_{\text{int}} = \left[\overline{\Sigma}_m \overline{\Phi}^2 + \overline{\Sigma}_\lambda \overline{\Phi}^3 \right]\big|_{\overline{F}}. \tag{4.37}$$

Here $\overline{\Sigma}_m$ is a spurion that contains a relevant coupling $\overline{m}$ as its lowest component, while $\overline{\Sigma}_\lambda$ contains a marginal coupling $\overline{\lambda}$. Since the non-interacting theory is invariant under $U(1)_R$, we can assign R-charges 0 and +1 to $\overline{\Sigma}_m$ and $\overline{\Sigma}_\lambda$ to restore R-symmetry. The only nonzero counterterm consistent with R-symmetry is

$$e^{-1}\mathcal{L}_{ct} = \overline{\Sigma}_m^4 \big|_{\overline{\phi}}, \tag{4.38}$$

In particular, there are no mixed matter-curvature counterterms, since the expectation value of the curvature superfield (3.44) vanishes identically for SU(2) structure, provided that we consider the R-symmetric case $M = 0$ (see (2.5)). We conclude that there is a quartic ambiguity in the free energy:

$$\log Z \sim \log Z + b(\overline{m}r)^4. \tag{4.39}$$

Any terms in $\log Z$ that depend on terms of order $\overline{m}^5$ or higher are free from ambiguities, or in other words,

$$\frac{\partial^5}{\partial(\overline{m}r)^5}\log Z \tag{4.40}$$

is non-ambiguous.

For certain matter contents, additional symmetries may protect the theory entirely from ambiguities. In fact, this is the case for the toy model discussed in the previous section. To preserve the global U(1) symmetry, we need to assign nonzero U(1)-charges to the spurions. Provided there are no anomalies, the counterterm (4.38) is simply forbidden, as it would break U(1). Therefore, in the particular case at hand, our localization result (4.35) is completely safe from ambiguities, and thus physical.

More generally, the problem of ambiguities is resolved if we consider backgrounds that allow for two chiral supercharges, $(\epsilon_L, 0)$ and $(0, \epsilon_R)$. The problematic $\overline{F}$-terms are now

exact with respect to the additional right-handed supercharge δ_R. As a result, the partition function is completely independent of couplings, and thus physical. Since the Killing spinor equations (2.3) are linear and homogeneous, a pair of non vanishing supercharges $(\epsilon_L, 0)$ and $(0, \epsilon_R)$ can be combined into a single supercharge (ϵ_L, ϵ_R), with the condition that $M = \overline{M} = 0$. Such backgrounds are T^2-fibrations over a Riemann surface, which we encountered in section 3. These backgrounds are therefore ideal candidates to perform localization (see e.g. [38]).

4.5 Breaking R-symmetry

We can explicitly break R-symmetry by adding $\overline{\phi}$-type deformations to our Lagrangian. This corresponds to adding an antiholomorphic potential

$$\mathcal{L} \to \mathcal{L} + \overline{V}(\overline{\phi}). \tag{4.41}$$

In complete analogy to the standard non-renormalization theorems in flat space [39], one can show that this does not introduce any additional finite counterterms involving the couplings $\overline{\lambda}_\phi$ within $\overline{V}$. Notice that this result relies crucially on the fact that even though (4.41) breaks R-symmetry, the background itself is R-symmetric. For example, this would not be the case for theories on S^4.

Since old-minimal supergravity allows for backgrounds with and without R-symmetry, we can also study the explicit breaking of $U(1)_R$ from a supergravity point of view. Looking at (2.3), we can associate the M and $\overline{M}$-terms with the violation of R-symmetry. In the case of SU(2) structure with supercharge $(\epsilon_L, 0)$, we have $\overline{M} = 0$. The function M however is unconstrained and does not appear in the SUSY variations or invariants derived above, yet it is still responsible for breaking R-symmetry: Consider the curved superspace interaction

$$\int d^2\overline{\Theta}^2 2\overline{\epsilon}\,\overline{W}(\overline{\Phi}). \tag{4.42}$$

For $M \neq 0$, the antichiral density is $2\overline{\epsilon} = e\left(1 - \overline{\Theta}^2 M\right)$. Alternatively, we can recast (4.42) as a superspace integral in a background with $M = 0$, and treat $2e^{-1}\overline{\epsilon}$ as a spurious antichiral field. Either way, we find

$$e^{-1}\int d^2\overline{\Theta}^2 2\overline{\epsilon}\,\overline{W}(\overline{\Phi}) = \overline{W}(\overline{\Phi})\big|_{\overline{F}} - M\overline{W}(\overline{\Phi})\big|_{\overline{\phi}}. \tag{4.43}$$

We see that M plays the role of the coupling to the R-violating $\overline{\phi}$-invariant, which we identify as the antiholomorphic potential $\overline{V}$ in (4.41). It is allowed to be an arbitrary function because $\delta\overline{\phi}$ vanishes identically, not just up to total derivatives. Thus turning on a nonzero M corresponds to breaking R-symmetry explicitly.

5 Discussion

In this paper, we have highlighted two unusual features of $\mathcal{N} = 1$ supersymmetry on Euclidean manifolds with S^3-isometry (e.g. the round and squashed S^4); namely, the failure

of localization, and regularization scheme dependent ambiguities of the partition function. Ultimately, both of these features can be traced back to the structure of off-shell supergravity in the old minimal formalism. The Killing spinor equation (2.3) mixes left- and right-handed spinors through the M- and $\overline{M}$-terms. This has the consequence that there are backgrounds that admit only Killing spinors of the form (ϵ_L, ϵ_R), where the left- and right-handed components cannot be "disentangled". This is manifest in the fact that the supercharge squares to a complex generator $\delta^2 \sim \mathcal{L}_K$, with $K^\mu = \epsilon_R^c \gamma^\mu \epsilon_L$ being a complex Killing vector that mixes left and right chiralities. Since δ does not square to an obvious symmetry of the theory, it appears that SUSY-exact terms are in general not SUSY-closed. In this paper, we have proven an equivalent statement, namely that there are no supersymmetric invariants (SUSY-closed terms) that can be written as SUSY-exact terms. We have explicitly identified the obstruction to exactness in terms of the non-vanishing Killing vector $L = [K, K^\star]$, which generates part of the isometry group $SU(2) \times SU(2)$ of S^3.

While the above obstruction might not appear to be very deep at first, it has the important consequence that the partition function must depend nontrivially on the values of all coupling constants. We have discussed two important corollaries: first, since there is no freedom in tuning any of the couplings, the partition function cannot be calculated using localization. A crucial point in arriving at this result was the fact that the usual "deformations" one utilizes to perform localization can equally be thought of as already being part of the complete set of SUSY invariants for a given theory. Since we showed that none of these invariants is exact, the partition function simply does not localize.

Second, we have shown that there are finite supergravity counterterms that introduce scheme-dependent ambiguities into the partition function. Our results extend beyond the previously studied case of SCFTs on S^4 [24] to any four-dimensional supersymmetric background with S^3-isometry. While in the conformal case it was shown that the finite part of the partition function is completely unphysical, our analysis demonstrates that $\log Z$ depends on relevant couplings in such a way that ambiguities are under control: if we expand the free energy in powers of relevant couplings, we find

$$\log Z(m, \lambda) = \log Z(0, \lambda) + \sum_{i=1}^{3} (mr)^i a_i(\lambda) + \tilde{F}(mr, \lambda), \tag{5.1}$$

where the a_i are functions of the marginal couplings, and $\tilde{F}$ may contain all powers of mr except $n = 0, 1, 2, 3$. On S^4, the $\log Z(0, \lambda)$-term can be interpreted as the free energy of the CFT, which is subject to ambiguities, and thus unphysical. As we have shown, the terms up to cubic order in m are ambiguous as well. However, the higher-order part $\tilde{F}$ is free from ambiguities and thus physical.

A similar feature has been observed for $\mathcal{N} = 2^\star$ theories on S^4, where the partition function can been computed using either localization [1] or holographic techniques [40]. It would be interesting to calculate the unambiguous part $\tilde{F}$ of the free energy for the $\mathcal{N} = 1$ case as well, and explicitly confirm some of the results of this paper.

An obvious way to avoid the complications present in the S^3-isometry case is to consider only backgrounds for which the chirality-mixing terms in (2.3) vanish identically. This has

led us to analyze backgrounds with $U(1)_R$ R-symmetry, which possess at least one nilpotent supercharge, $\delta^2 = 0$. In this case, many simplifications occur: with one exception (antichiral $\overline{F}$-terms), the partition function does not depend on the values of couplings in our Lagrangian (4.30), and localization is straightforward. However, the fact that we have found a procedure for calculating the partition function does not necessarily mean that the result will be sensible. As we demonstrated in section 4.4, the partition function is in general subject to antiholomorphic ambiguities. Interestingly, the only ambiguity appears at quartic order in relevant couplings, and thus renormalizes the cosmological constant. This is a special feature of the BRST-like symmetry δ, which provides a trivial extension of the isometry algebra of the background. Some of the standard arguments in Lorentzian supersymmetry, such as the proof of non-renormalization of the vacuum energy, therefore do not apply.

Finally, for backgrounds that preserve two supercharges of opposite chirality, Z is completely independent of all couplings, and there are no ambiguities. Within the framework of old-minimal supergravity, the only manifolds with this property are torus-fibrations over two-dimensional Riemann surfaces. It would be interesting to carry out localization for explicit cases of such backgrounds, presumably paralleling the analysis in [38, 41].

There are two caveats to our analysis of ambiguities of partition functions in sections 3.4 and 4.4, which point towards interesting future directions: first, our classification of possible finite counterterms necessarily requires the existence of a regularization scheme that preserves the symmetries of the theory. As far as we know, there is not yet a satisfactory answer to the question when such a scheme does or does not exist for a supersymmetric theory. If for a given theory there is no supersymmetric regularization scheme, conclusions about the partition function, such as independence of couplings and the physical content, would need to be reexamined. Second, we have only analyzed finite counterterms that involve both matter couplings and curvature invariants at the same time. It would be interesting to also analyze purely gravitational counterterms, which arise as F-type and D-type terms evaluated on the various curvature multiplets of supergravity [30].

Acknowledgments

A Supersymmetric tensor calculus

In order to construct supersymmetric Lagrangians, we need to know the rules for combining superfields [17–19]. Given two multiplets S_1 and S_2, we can form a new multiplet

$$S_1 \times S_2 \equiv \left(C_{12}, \psi_{12L}, \psi_{12R}, F_{12}, \overline{F}_{12}, A_{12\mu}, \lambda_{12L}, \lambda_{12R}, D_{12} \right). \tag{A.1}$$

Demanding that $C_{12} = C_1 C_2$, we can work out the multiplication rules using (2.8):

$$C_{12} = C_1 C_2,$$

$$\psi_{12L} = C_1 \psi_{2L} + C_2 \psi_{1L},$$

$$\psi_{12R} = C_1 \psi_{2R} + C_2 \psi_{1R},$$

$$F_{12} = C_1 F_2 + C_2 F_1 - \psi_{1L}^c \psi_{2L},$$

$$\overline{F}_{12} = C_1 \overline{F}_2 + C_2 \overline{F}_1 + \psi_{1R}^c \psi_{2R},$$

$$A_{12\mu} = C_1 A_{2\mu} + \psi_{1L}^c \gamma_\mu \psi_{2R} + (1 \leftrightarrow 2),$$

$$\lambda_{12L} = C_1 \lambda_{2L} + \overline{F}_1 \psi_{2L} - \frac{1}{2} \gamma^\mu (A_{1\mu} - \nabla_\mu C_1) \psi_{2R} + (1 \leftrightarrow 2),$$

$$\lambda_{12R} = C_1 \lambda_{2R} + F_1 \psi_{2R} + \frac{1}{2} \gamma^\mu (A_{1\mu} + \nabla_\mu C_1) \psi_{2L} + (1 \leftrightarrow 2),$$

$$D_{12} = C_1 D_2 + 2 F_1 \overline{F}_2 + 2 \psi_{1R}^c \lambda_{2R} - 2 \psi_{1L}^c \lambda_{2L} + \psi_{1L}^c \gamma^\mu \left(\nabla_\mu - \frac{i}{2} b_\mu \right) \psi_{2R}$$

$$- \psi_{1R}^c \gamma^\mu (\nabla_\mu + \frac{i}{2} b_\mu) \psi_{2L} + \frac{1}{2} (A_1^\mu A_{2\mu} - \nabla^\mu C_1 \nabla_\mu C_2) + (1 \leftrightarrow 2). \tag{A.2}$$

It is easy to see that the product operator $\times$ is symmetric, i.e. $S_1 \times S_2 = S_2 \times S_1$. This is a result of demanding $C_{12} = C_1 C_2$. A natural question is whether there also exists an antisymmetric product $\wedge$, such that $C_{12} = 0$. We can attempt to derive the multiplication rules in a similar fashion, starting with

$$0 = \delta C_{12} = -\epsilon_L^c \psi_{12L} - \epsilon_R^c \psi_{12R}. \tag{A.3}$$

A quick check reveals that for the trivial structure case, all the components of the product multiplet have to be set to zero, i.e. there is no nontrivial antisymmetric product. In the SU(2) structure case, however, we have more freedom: setting $\epsilon_R = 0$, we see that eq. (A.3) is solved by $\psi_{12L} = 0$, but nonzero ψ_{12R}. In fact, we find that there exists an antisymmetric product

$$S_1 \wedge S_2 \equiv \left(C_{12}, \psi_{12L}, \psi_{12R}, F_{12}, \overline{F}_{12}, A_{12\mu}, \lambda_{12L}, \lambda_{12R}, D_{12} \right), \tag{A.4}$$

with the following multiplication rules:

$$C_{12} = 0,$$

$$\psi_{12L} = 0,$$

$$\psi_{12R} = C_1 \psi_{2R} - C_2 \psi_{1R},$$

$$F_{12} = 0,$$

$$\overline{F}_{12} = C_1 \overline{F}_2 - C_2 \overline{F}_1,$$

$$A_{12\mu} = C_1(A_{2\mu} + \nabla_\mu C_2) + \psi_{1L}^c \gamma_\mu \psi_{2R} - (1 \leftrightarrow 2),$$

$$\lambda_{12L} = C_1 \lambda_{2L} - \overline{F}_1 \psi_{2L} + \gamma^\mu \nabla_\mu C_1 \psi_{2R} - (1 \leftrightarrow 2),$$

$$\lambda_{12R} = C_1 \lambda_{2R} + F_1 \psi_{2R} - \frac{1}{2} \gamma^\mu (A_{1\mu} + \nabla_\mu C_1) \psi_{2L} - (1 \leftrightarrow 2),$$

$$D_{12} = C_1 D_2 + 2 F_1 \overline{F}_2 - 2 \psi_{1L}^c \lambda_{2L} + \psi_{1L}^c \gamma^\mu \left(\nabla_\mu - \frac{i}{2} b_\mu \right) \psi_{2R}$$

$$+ \psi_{1R}^c \gamma^\mu \left(\nabla_\mu + \frac{i}{2} b_\mu \right) \psi_{2L} + A_1^\mu \nabla_\mu C_2 - (1 \leftrightarrow 2). \tag{A.5}$$

Similar expressions can be derived for the case $\epsilon_L = 0$.

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References

[1] V. Pestun, *Localization of gauge theory on a four-sphere and supersymmetric Wilson loops*, *Commun. Math. Phys.* **313** (2012) 71 [`arXiv:0712.2824`] [INSPIRE].

[2] N. Drukker and D.J. Gross, *An exact prediction of $N = 4$ SUSYM theory for string theory*, *J. Math. Phys.* **42** (2001) 2896 [`hep-th/0010274`] [INSPIRE].

[3] J.K. Erickson, G.W. Semenoff and K. Zarembo, *Wilson loops in $N = 4$ supersymmetric Yang-Mills theory*, *Nucl. Phys.* **B 582** (2000) 155 [`hep-th/0003055`] [INSPIRE].

[4] A. Kapustin, B. Willett and I. Yaakov, *Exact results for wilson loops in superconformal Chern-Simons theories with matter*, *JHEP* **03** (2010) 089 [`arXiv:0909.4559`] [INSPIRE].

[5] M. Mariño, *Lectures on localization and matrix models in supersymmetric Chern-Simons-matter theories*, *J. Phys.* **A 44** (2011) 463001 [`arXiv:1104.0783`] [INSPIRE].

[6] G. Festuccia and N. Seiberg, *Rigid supersymmetric theories in curved superspace*, *JHEP* **06** (2011) 114 [`arXiv:1105.0689`] [INSPIRE].

[7] B. Jia and E. Sharpe, *Rigidly supersymmetric gauge theories on curved superspace*, *JHEP* **04** (2012) 139 [`arXiv:1109.5421`] [INSPIRE].

[8] H. Samtleben and D. Tsimpis, *Rigid supersymmetric theories in $4D$ Riemannian space*, *JHEP* **05** (2012) 132 [`arXiv:1203.3420`] [INSPIRE].

[9] C. Klare, A. Tomasiello and A. Zaffaroni, *Supersymmetry on curved spaces and holography*, *JHEP* **08** (2012) 061 [`arXiv:1205.1062`] [INSPIRE].

[10] T.T. Dumitrescu, G. Festuccia and N. Seiberg, *Exploring curved superspace*, *JHEP* **08** (2012) 141 [`arXiv:1205.1115`] [INSPIRE].

[11] J.T. Liu, L.A. Pando Zayas and D. Reichmann, *Rigid supersymmetric backgrounds of minimal off-shell supergravity*, *JHEP* **10** (2012) 034 [`arXiv:1207.2785`] [INSPIRE].

[12] P. de Medeiros, *Rigid supersymmetry, conformal coupling and twistor spinors*, *JHEP* **09** (2014) 032 [`arXiv:1209.4043`] [INSPIRE].

[13] T.T. Dumitrescu and G. Festuccia, *Exploring curved superspace. II*, *JHEP* **01** (2013) 072 [arXiv:1209.5408] [inSPIRE].

[14] M.F. Sohnius and P.C. West, *An alternative minimal off-shell version of $N = 1$ supergravity*, *Phys. Lett.* **B 105** (1981) 353 [inSPIRE].

[15] C. Closset, T.T. Dumitrescu, G. Festuccia and Z. Komargodski, *The geometry of supersymmetric partition functions*, *JHEP* **01** (2014) 124 [arXiv:1309.5876] [inSPIRE].

[16] C. Closset, T.T. Dumitrescu, G. Festuccia and Z. Komargodski, *From rigid supersymmetry to twisted holomorphic theories*, *Phys. Rev.* **D 90** (2014) 085006 [arXiv:1407.2598] [inSPIRE].

[17] S. Ferrara and P. van Nieuwenhuizen, *Tensor calculus for supergravity*, *Phys. Lett.* **B 76** (1978) 404 [inSPIRE].

[18] S. Ferrara and P. Van Nieuwenhuizen, *Structure of supergravity*, *Phys. Lett.* **B 78** (1978) 573 [inSPIRE].

[19] K.S. Stelle and P.C. West, *Tensor calculus for the vector multiplet coupled to supergravity*, *Phys. Lett.* **B 77** (1978) 376 [inSPIRE].

[20] J. Wess and J. Bagger, *Supersymmetry and supergravity*, Princeton University Press, Princeton U.S.A. (1992).

[21] K.S. Stelle and P.C. West, *Relation between vector and scalar multiplets and gauge invariance in supergravity*, *Nucl. Phys.* **B 145** (1978) 175 [inSPIRE].

[22] A.B. Zamolodchikov, *Irreversibility of the flux of the renormalization group in a 2D field theory*, *JETP Lett.* **43** (1986) 730 [inSPIRE].

[23] D. Kutasov, *Geometry on the space of conformal field theories and contact terms*, *Phys. Lett.* **B 220** (1989) 153 [inSPIRE].

[24] E. Gerchkovitz, J. Gomis and Z. Komargodski, *Sphere partition functions and the Zamolodchikov metric*, *JHEP* **11** (2014) 001 [arXiv:1405.7271] [inSPIRE].

[25] H. Jockers, V. Kumar, J.M. Lapan, D.R. Morrison and M. Romo, *Two-sphere partition functions and Gromov-Witten invariants*, *Commun. Math. Phys.* **325** (2014) 1139 [arXiv:1208.6244] [inSPIRE].

[26] J. Gomis and S. Lee, *Exact Kähler potential from gauge theory and mirror symmetry*, *JHEP* **04** (2013) 019 [arXiv:1210.6022] [inSPIRE].

[27] N. Doroud, J. Gomis, B. Le Floch and S. Lee, *Exact results in $D = 2$ supersymmetric gauge theories*, *JHEP* **05** (2013) 093 [arXiv:1206.2606] [inSPIRE].

[28] F. Benini and S. Cremonesi, *Partition functions of $\mathcal{N} = (2,2)$ gauge theories on S^2 and vortices*, *Commun. Math. Phys.* **334** (2015) 1483 [arXiv:1206.2356] [inSPIRE].

[29] N. Doroud and J. Gomis, *Gauge theory dynamics and Kähler potential for Calabi-Yau complex moduli*, *JHEP* **1312** (2013) 99 [arXiv:1309.2305] [inSPIRE].

[30] B. Assel, D. Cassani and D. Martelli, *Supersymmetric counterterms from new minimal supergravity*, *JHEP* **11** (2014) 135 [arXiv:1410.6487] [inSPIRE].

[31] K.S. Stelle and P.C. West, *Minimal auxiliary fields for supergravity*, *Phys. Lett.* **B 74** (1978) 330 [inSPIRE].

[32] S. Ferrara and P. van Nieuwenhuizen, *The auxiliary fields of supergravity*, *Phys. Lett.* **B 74** (1978) 333 [inSPIRE].

[33] M. Sohnius and P.C. West, *The tensor calculus and matter coupling of the alternative minimal auxiliary field formulation of $N = 1$ supergravity*, *Nucl. Phys.* **B 198** (1982) 493 [INSPIRE].

[34] T. Nosaka and S. Terashima, *Supersymmetric gauge theories on a squashed four-sphere*, *JHEP* **12** (2013) 001 [arXiv:1310.5939] [INSPIRE].

[35] D. Lüst, P. Patalong and D. Tsimpis, *Generalized geometry, calibrations and supersymmetry in diverse dimensions*, *JHEP* **01** (2011) 063 [arXiv:1010.5789] [INSPIRE].

[36] W. Barth, C. Peters and A. Van de Ven, *Compact complex surfaces*, Springer, Germany (1984).

[37] M. Nakahara, *Geometry, topology and physics*, Taylor & Francis, U.S.A. (2003).

[38] C. Closset and I. Shamir, *The $\mathcal{N} = 1$ chiral multiplet on $T^2 \times S^2$ and supersymmetric localization*, *JHEP* **03** (2014) 040 [arXiv:1311.2430] [INSPIRE].

[39] N. Seiberg, *Naturalness versus supersymmetric nonrenormalization theorems*, *Phys. Lett.* **B 318** (1993) 469 [hep-ph/9309335] [INSPIRE].

[40] N. Bobev, H. Elvang, D.Z. Freedman and S.S. Pufu, *Holography for $N = 2^*$ on S^4*, *JHEP* **07** (2014) 001 [arXiv:1311.1508] [INSPIRE].

[41] T. Nishioka and I. Yaakov, *Generalized indices for $\mathcal{N} = 1$ theories in four-dimensions*, *JHEP* **12** (2014) 150 [arXiv:1407.8520] [INSPIRE].