Published for SISSA by Springer

Received: *January 25, 2015*
Accepted: *February 17, 2015*
Published: *March 18, 2015*

States in non-associative quantum mechanics: uncertainty relations and semiclassical evolution

Martin Bojowald,[a] Suddhasattwa Brahma,[a] Umut Büyükçam[a] and Thomas Strobl[b]

[a] *Institute for Gravitation and the Cosmos, The Pennsylvania State University, 104 Davey Lab, University Park, PA 16802, U.S.A.*

[b] *Institut Camille Jordan, Université Claude Bernard Lyon 1, 43 boulevard du 11 novembre 1918, 69622 Villeurbanne cedex, France*

E-mail: bojowald@gravity.psu.edu, sxb1012@psu.edu, uxb101@psu.edu, strobl@math.univ-lyon1.fr

Abstract: A non-associative algebra of observables cannot be represented as operators on a Hilbert space, but it may appear in certain physical situations. This article employs algebraic methods in order to derive uncertainty relations and semiclassical equations, based on general properties of quantum moments.

Keywords: Differential and Algebraic Geometry, Flux compactifications

ArXiv ePrint: 1411.3710

doi:10.1007/JHEP03(2015)093

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Contents

1 Introduction

Quantum mechanics represents the classical Poisson algebra of basic variables q_j and p_k, $\{q_j, p_k\} = \delta_{jk}$, as an operator algebra acting on a Hilbert space, so that the Poisson bracket is turned into the commutator $[\hat{q}_j, \hat{p}_k] = i\hbar\delta_{jk}$ of basic operators.[1] The Jacobi identity satisfied by a Poisson bracket has an analog in the associativity of the operator product: a simple calculation shows that

$$\epsilon^{ijk}[[\hat{O}_i, \hat{O}_j], \hat{O}_k] = 2\epsilon^{ijk}[\hat{O}_i, \hat{O}_j, \hat{O}_k] \tag{1.1}$$

where the 3-bracket on the right-hand side is used to denote the associator of the product of the quantum observables, $[\hat{O}_1, \hat{O}_2, \hat{O}_3] := (\hat{O}_1\hat{O}_2)\hat{O}_3 - \hat{O}_1(\hat{O}_2\hat{O}_3)$. If the Jacobiator of the

[1] This latter equation, as usual, holds on a dense subspace only. From a mathematical perspective, it is convenient to consider bounded operators obtained after exponentiating $\hat{q}_i$ and $\hat{p}_j$, resulting in the Weyl algebra. In the present paper, however, we focus on conceptual questions of the construction of non-associative quantum mechanics and related consequences in possible physical applications, postponing more mathematical issues to later work.

classical bracket — or the Jacobiator of the operator product introduced on the left-hand side of (1.1) — vanishes for all triples of operators $\hat{O}_i$, an associative operator algebra is consistent with Dirac's basic quantization rule relating Poisson brackets to commutators. (These concepts have been formalized mathematically in different ways, for instance in the frameworks of group-theoretical quantization [1], geometric quantization [2], and deformation quantization.)

For classical systems with modified brackets, such as twisted Poisson structures [3–5], the Jacobi identity may no longer hold true and be replaced by a non-zero Jacobiator $\epsilon^{ijk}\{\{O_i, O_j\}, O_k\} \neq 0$. As the quantum analog, there must be a non-zero associator $[\hat{O}_1, \hat{O}_2, \hat{O}_3] \neq 0$ of a non-associative operator algebra. Such an algebra cannot be represented on a Hilbert space in the standard way, and alternatives making use, for instance, of non-associative $*$-products must be developed. In this paper, we focus on the general aspects of states on a non-associative operator algebra and see how the basic notions familiar from quantum mechanics can be derived in representation-independent terms. In some respects (and unless extra conditions on states are imposed) the results seem to differ from existing constructions using non-associative $*$-products [6–10].

Non-associative structures have recently gained interest in the context of certain flux compactifications of string theory and double-field theory [6, 11–14]. They have played a role in the understanding of gauge anomalies, and also appear in "standard" quantum mechanics if a charged particle is coupled to a density of magnetic monopoles [15]. These monopoles need not be fundamental, and therefore the systems may describe realistic physics in some analog models of condensed-matter systems (see for instance [16]). A related version is realized in chiral gauge theories [17–19]. We briefly review how non-Poisson brackets or non-associative algebras appear, which will present the main example to keep in mind throughout this article.

In the presence of a magnetic field with vector potential A_i, the canonical momentum of a particle with mass m and charge e is $\pi_i = mv_i + eA_i$ in terms of the velocity $v_i = \dot{q}_i$. While the momentum components obey canonical Poisson brackets with the position variables and have zero brackets with one another, the velocity components or the kinematical momentum components $p_i = mv_i$ have brackets related to the magnetic field:

$$\{p_i, p_j\} = e\epsilon_{ijk}B^k \,. \tag{1.2}$$

(We use the convention that repeated indices are summed over.) These brackets define a Poisson structure provided the magnetic field is divergence free.

If the divergence is non-zero, the magnetic field no longer has a vector potential, but one may still use (1.2) as the definition of a bracket on phase space (together with $\{q_i, p_j\} = \delta_{ij}$ and antisymmetry). One then computes a non-zero Jacobiator

$$\epsilon^{ijk}\{\{p_i, p_j\}, p_k\} = -2e\partial_l B^l \,. \tag{1.3}$$

The corresponding quantum mechanics cannot be represented by an associative operator algebra acting on a Hilbert space.

For a constant monopole density, the bracket is twisted Poisson [3–5] and can be realized as a Malcev algebra [20, 21]. The $*$-product constructions of [6–10] simplify for

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a constant density and allow several explicit results to be derived, but they hold more generally. Our calculations here are complementary and allow for $\partial_l B^l$ to be non-constant even in some explicit results. The existence of relevant algebras and states based on our relations alone is more difficult to show, but if they are assumed to exist, several properties can be derived efficiently by considering expectation-value functions $\omega\colon \mathcal{A} \to \mathbb{C}$.

From the physical perspective, this example is of interest because the existence of a magnetic monopole density, fundamental or effective, *somewhere* within the system necessitates a modification of very fundamental aspects of quantum mechanics. The notion of a Hilbert space is a non-local one, for instance in the sense that wave functions in the Schrödinger representation are normalized by an integration over all of space. Nevertheless, for meaningful experiments it must be possible to construct a local description of quantum physics outside the magnetic monopole density, where it has to reproduce the established and experimentally verified quantum properties (at least to a very high precision). This (perhaps hypothetical) physical system thus provides an interesting playground and a test for the development of non-associative quantum mechanics.

2 Properties of states

As briefly derived in the example of a magnetic monopole density, we assume that we have an algebra $\mathcal{A}$ of observables, which includes elements $\hat{q}_i$ and $\hat{p}_j$ (as well as a unit $\mathbb{I}$) and obeys the relations

$$[\hat{q}_i, \hat{q}_j] = 0 \tag{2.1}$$

$$[\hat{q}_i, \hat{p}_j] = i\hbar\delta_{ij} \tag{2.2}$$

$$[\hat{p}_i, \hat{p}_j] = i\hbar e\epsilon_{ijk}\hat{B}^k \tag{2.3}$$

$$[\hat{p}_i, \hat{p}_j, \hat{p}_k] = -\hbar^2 e\epsilon_{ijk}\widehat{\partial_l B^l} \tag{2.4}$$

where $i, j, k \in \{1, 2, 3\}$. Summation over double indices is assumed, and, as before, the 3-bracket denotes the associator. We further assume that, other than (2.4), all associators between the fundamental variables $\hat{q}_i$ and $\hat{p}_j$ vanish. Among basic operators, only the associator (2.4) is non-zero because of the non-vanishing Jacobiator (1.3).

In (2.3) and (2.4), $\hat{B}^k \in \mathcal{A}$ (and similarly $\widehat{\partial_l B^l}$) is obtained by inserting $\hat{q}_i$ in the classical function $B^k(q_i)$. Since the $\hat{q}_i$ commute and associate with one another, $\hat{B}^k$ is well-defined for polynomial B^k. For non-polynomial magnetic fields, we assume that $\hat{B}^k$ can be defined by a formal power series. With the magnetic field and also its divergence allowed to be functions of q_i, the relations may be non-linear. If the magnetic-field components are assumed to be analytic functions, then also their derivatives are well-defined, which we will use in some semiclassical expansions. The assumption of analyticity may have to be weakened in some physical situations because it is not consistent with a monopole density of compact support. For the algebra, we need the first derivatives of B^i, so that these functions should at least be differentiable.

At this point, we encounter the first existence question. In associative cases, it is known that $\hat{q}_i$ and $\hat{p}_j$ are not bounded in Hilbert-space representations. It is then more

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convenient to use exponentiated (Weyl) operators for explicit constructions of algebra representations. In the present case, Hilbert-space representations cannot exist at all, and existence questions are more complicated. In this article, we take a pragmatic view and assume that an algebra with the relations (2.1)–(2.4) (as well as a $*$-relation introduced below) exists. Our aim is to derive properties of states which are of interest for physical questions and can be obtained using only the given relations. This view is akin to the one taken in particle physics, where it is difficult to show that interacting quantum field theories do indeed exist, but powerful computational methods are still available and can be compared with observations.

The relations (2.1)–(2.3) are direct translations of basic brackets, the first two of standard form and (2.3) derived from (1.2). The non-zero Jacobiator (1.3) implies that there must be a non-zero associator. However, (1.1) shows that only the totally antisymmetric part of the associator is determined by the correspondence between classical brackets and commutators. Contributions to the associator which are not totally antisymmetric can be considered as quantization choices, which one may be able to fix so as to realize certain simplifications. For now, we will assume simplifications which appear to be consistent with the equations (2.1)–(2.4), postponing a more precise construction of $\mathcal{A}$ to later work.

We could assume the associator between any three elements of the algebra $\mathcal{A}$ to be completely antisymmetric, or equivalently

$$A(BB) = (AB)B \tag{2.5}$$
$$(AA)B = A(AB) \tag{2.6}$$
$$(AB)A = A(BA) \tag{2.7}$$

for all $A, B, C \in \mathcal{A}$. Any algebra satisfying these conditions (two of which imply the third one) is called an *alternative algebra*. For such an algebra, we have additional relations between algebra elements which are not as strong as assocativity but will turn out to be useful: an alternative algebra satisfies the Moufang identities [22]

$$C(A(CB)) = (CAC)B \tag{2.8}$$
$$((AC)B)C = A(CBC) \tag{2.9}$$
$$(CA)(BC) = C(AB)C\,. \tag{2.10}$$

(If (2.5) holds, we do not need to set further paranthesis in (2.10).) These identities are also useful for an extension of some of the measurement axioms of quantum mechanics to non-associative versions [23]. The algebras constructed by $*$-products in [7–10], with the same basic associator (2.4), are *not* alternative.[2] Our explicit results derived in the rest of this paper only require (2.4) to be totally antisymmetric, and the corresponding Moufang identity for A, B and C linear in the $\hat{p}_i$. They will therefore also hold for the known $*$-algebra realizations of (2.4), but there may be deviations at higher moments or $\hbar$-orders.

We turn $\mathcal{A}$ into a $*$-algebra by requiring $\hat{q}_i$ and $\hat{p}_j$ to be self-adjoint. (We then have the usual relations, such as $(\lambda\hat{p}_1)^* = \lambda^*\hat{p}_1$ for all $\lambda \in \mathbb{C}$, and $(AB)^* = B^*A^*$ for all $A, B \in \mathcal{A}$.)

[2] We are grateful to Peter Schupp and Richard Szabo for pointing this out to us.

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This requirement is consistent with (2.4) thanks to the alternative nature of the basic associator: for self-adjoint $\hat{p}_i^* = \hat{p}_i$, we then have

$$[\hat{p}_1, \hat{p}_2, \hat{p}_3]^* = \hat{p}_3(\hat{p}_2\hat{p}_1) - (\hat{p}_3\hat{p}_2)\hat{p}_1 = -[\hat{p}_3, \hat{p}_2, \hat{p}_1] = [\hat{p}_1, \hat{p}_2, \hat{p}_3] \tag{2.11}$$

so that both sides of (2.4) are self-adjoint.

2.1 The Cauchy-Schwarz inequality and uncertainty relations

In treatments of algebra theory relevant for quantum mechanics it is often assumed that one is dealing only with associative algebras. Several important results no longer apply in the non-associative case. However, a notable exception is the Cauchy-Schwarz inequality. It is important in quantum mechanics because it leads to the uncertainty relation, and fortunately, this result is still available for non-associative algebras. Even the standard proof can be used without modifications, which we sketch here for completeness.

For any complex-valued, positive linear functional ω on the algebra $\mathcal{A}$ (that is, $\omega(A^*A) \geq 0$ for all $A \in \mathcal{A}$), we would like to prove that

$$\omega(A^*A)\omega(B^*B) \geq |\omega(B^*A)|^2 \tag{2.12}$$

for any two elements A and B in $\mathcal{A}$. We define a new element $A' := A\exp(-i\mathrm{arg}\omega(B^*A))$, so that $|\omega(B^*A)| = \omega(B^*A')$, and compute

$$\begin{aligned} 0 &\leq \omega\left((\sqrt{\omega(B^*B)}A' - \sqrt{\omega(A'^*A')}B)^*(\sqrt{\omega(B^*B)}A' - \sqrt{\omega(A'^*A')}B)\right) \\ &= 2\omega(B^*B)\omega(A'^*A') - \sqrt{\omega(B^*B)\omega(A'^*A')}\left(\omega(A'^*B) + \omega(B^*A')\right) \\ &= 2\omega(B^*B)\omega(A'^*A') - 2\sqrt{\omega(B^*B)\omega(A'^*A')}|\omega(B^*A)|\,. \end{aligned} \tag{2.13}$$

Therefore,

$$|\omega(B^*A)| \leq \sqrt{\omega(B^*B)}\sqrt{\omega(A'^*A')} = \sqrt{\omega(B^*B)\omega(A^*A)}\,.$$

One can then derive the standard uncertainty relation for basic operators by applying the Cauchy-Schwarz inequality to $A = \hat{q}_i - \omega(\hat{q}_i)\mathbb{I}$ and $B = \hat{p}_j - \omega(\hat{p}_j)\mathbb{I}$: we have $\omega(A^*A) = (\Delta q_i)^2$, $\omega(B^*B) = (\Delta p_j)^2$, and $\omega(B^*A)$ can be split into its real part, which equals the covariance $C_{q_ip_j} := \frac{1}{2}\omega(\hat{q}_j\hat{p}_i + \hat{p}_i\hat{q}_j) - \omega(\hat{q}_j)\omega(\hat{p}_i)$, and its imaginary part proportional to the commutator $[\hat{q}_i, \hat{p}_j]$. The inequality (2.12) then implies

$$(\Delta q_j)^2(\Delta p_i)^2 \geq \frac{\hbar^2}{4}\delta_{ij} + C^2_{q_jp_i} \geq \frac{\hbar^2}{4}\delta_{ij}\,. \tag{2.14}$$

In the present case, there is a new uncertainty relation for different components of p^i thanks to the non-zero commutator (2.3):

$$(\Delta p_i)^2(\Delta p_j)^2 \geq \frac{\hbar^2 e^2}{4}(\epsilon_{ijk}\omega(\hat{B}^k))^2\delta_{ij} + C^2_{p_ip_j} \geq \frac{\hbar^2 e^2}{4}(\epsilon_{ijk}\omega(\hat{B}^k))^2\delta_{ij}\,. \tag{2.15}$$

These relations depend only on commutators, and therefore are equivalent to those given in [7–10] based on $*$-products.

At this stage, we note a difference with the $*$-product treatment of non-associative algebras. When one constructs an analog of a Hilbert-space representation of wave functions ψ acted on by $\mathcal{A}$ using a non-associative $*$-product, one assigns to any $A \in \mathcal{A}$ a map $\psi \mapsto A * \psi$ on a set of wave functions ψ instead of the usual associative action of operators. (The constructions in [9] are more general and consider also density states.) In deriving the uncertainty relation, one applies two such multiplications of the form $A*(B*\psi)$. This product is sensitive to non-associativity, and indeed the derivation of an uncertainty relation is non-trivial. In [9], the problem has been solved by introducing modified (and associative) composition maps derived but different from the original algebra product:[3] $\circ$ is obtained from $(A \circ B) * C = A * (B * C)$ for all A, B, C, and $\bar{\circ}$ from $C * (A \bar{\circ} B) = (C * A) * B$. For the $*$-product action on states, a Cauchy-Schwarz inequality holds for $\circ$ but not for the original $*$. However, as derived in detail in [9], the $\circ$-commutator acts by $(\hat{p}_i \circ \hat{p}_j - \hat{p}_j \circ \hat{p}_i) * \psi = \psi * \hat{K}$, with $\hat{K}$ corresponding to the right-hand side of the commutator (2.3), but now acting from the right. Accordingly, the resulting uncertainty relation is not of the standard form, unless an additional "symmetry" condition is imposed on wave functions, or $\rho * C = C * \rho$ on density states ρ. The general derivation of the Cauchy-Schwarz inequality, on the other hand, makes use of products of at most two operators and is not sensitive to non-associativity. It implies an uncertainty relation that is equivalent to the one obtained using $*$-products only if the symmetry condition is imposed. We view this observation as an additional argument that wave functions should indeed obey the symmetry condition (as already suggested in [9]).

2.2 Failure of the GNS construction

Given an *associative* $*$-algebra $\mathcal{A}$ and a positive linear functional ω on it, one can construct a Hilbert-space representation by making use of the GNS construction. (See for instance [24, 25].) It is clear that the construction must fail in the non-associative case because such an algebra cannot act by standard operator multiplication on a Hilbert space. Nevertheless, it is interesting to see where exactly the construction breaks down.

In the GNS construction, one starts with the algebra $\mathcal{A}$ as a linear space and constructs a Hilbert space from it. Multiplication in the algebra then implies an action of the algebra on the Hilbert space. In order to derive the Hilbert space, one introduces a (degenerate) scalar product on $\mathcal{A}$ by $\langle A|B\rangle := \omega(A^* B)$ for all $A, B \in \mathcal{A}$. The scalar product is positive semidefinite because ω is assumed to be a positive linear functional, but it has a kernel spanned by all $C \in \mathcal{A}$ for which $\omega(C^* C) = 0$. Assuming the algebra to be associative, the kernel is a left-ideal in $\mathcal{A}$ and can be factored out, leaving a linear space with a positive definite scalar product which can be completed to a Hilbert space.

In this last step, associativity is important. In order to show that the kernel is a left-ideal, one makes use of the Cauchy-Schwarz inequality and computes (using associativity only at this place in the present paper)

$$|\omega((AC)^*(AC))|^2 = |\omega(C^* A^* AC)|^2 \leq \omega(C^* C)\omega((A^* AC)^* A^* AC) = 0\,, \tag{2.16}$$

[3] These composition maps are important for the construction of states obeying the positivity condition [9].

so that AC is in the kernel for any $A \in \mathcal{A}$ and C in the kernel. For a non-associative algebra, (2.16) is not available and it is in general impossible to factor out the kernel consistently in order to obtain a Hilbert space.

It would be possible to obtain a left ideal from the kernel of ω if all C in the kernel would be self-adjoint (or anti-selfadjoint). For an alternative algebra, we could then proceed as in (2.16) thanks to the Moufang identity

$$C(AB)C = (CA)(BC) \tag{2.17}$$

which allows us to write

$$|\omega((AC)^*(AC))|^2 = |\omega((CA^*)(AC))|^2 = |\omega(C(A^*A)C)|^2 \tag{2.18}$$

$$\leq \omega(C^*C)\omega(((A^*A)C)^*((A^*A)C)) = 0 \tag{2.19}$$

if $C^* = \pm C$. A real Hilbert space would follow from the GNS construction if the algebra could be restricted to only (anti-)self-adjoint elements. Unfortunately, however, a closed algebra of (anti-)self-adjoint elements can be obtained only with (anti-)commutative multiplication.

The GNS construction plays an important role in algebraic approaches to quantum mechanics and quantum field theory because it shows that Hilbert-space representations do exist. In particular, using all states in a Hilbert-space representation, one is assured that sufficiently many positive linear functionals exist on the algebra, allowing one to derive potential measurement results. A quantum system would not be considered meaningful if it does not allow sufficiently many states, for instance when $\omega(\hat{q}_1) = \omega(\hat{q}_2)$ for all states ω. For every point $(\bar{q}_1, \bar{q}_2, \bar{q}_3; \bar{p}_1, \bar{p}_2, \bar{p}_3)$ in the classical phase space in which we expect a semiclassical quantum description to be available, we should require that there is a state ω such that $\omega(\hat{q}_i) = \bar{q}_i$ and $\omega(\hat{p}_j) = \bar{p}_j$. The classical freedom of choosing initial values then remains unrestricted after quantization.

If sufficiently many states exist, general features of expectation-value functionals can be employed to derive generic properties which are independent of which specific representation is used. For non-associative algebras, we cannot have standard Hilbert-space representations, and we are not aware of an alternative version of the GNS construction that could guarantee the existence of sufficiently many positive linear functionals on the algebra. The methods of [9] show that states can be constructed with an action of the algebra given by a $*$-product, and positivity properties have been demonstrated. However, as shown by the discussion of uncertainty relations in the preceding section, the general algebraic results we make use of here agree with those found by non-associative $*$-products only when the class of states is restricted by an additional symmetry condition. To the best of our knowledge, it is not clear whether sufficiently many positive linear functionals obeying the symmetry condition do exist. In what follows, we will have to assume that there are such states, some of whose properties we will be able to derive.

2.3 Moments

Without a Hilbert space, we cannot describe states by wave functions. However, we can use an alternative set of variables which describes a positive linear functional ω on $\mathcal{A}$ in terms of expectation values $\omega(O)$ and moments of the form $\omega((O-\omega(O)\mathbb{I})^n)$ for O one of the basic operators. More generally, covariance parameters in which ω is applied to products of $O_i - \omega(O_i)\mathbb{I}$ for different values of i are also required. We introduce these variables and determine some of their algebraic relations after switching to a physics-oriented notation in which $\omega(A)$ is written as the expectation value $\omega(A) = \langle A \rangle$ of an operator A. These expectation values, by definition, refer to a state as a positive linear functional on the algebra; they do not require wave functions or a Hilbert space. Moreover, we will omit explicit insertions of the unit operator $\mathbb{I}$ and assume that it is understood in expressions such as $\hat{A} - \langle \hat{A} \rangle$.

In the associative case, the definition of the moment variables is as follows:

$$\begin{aligned}\Delta(p_x^{a_1} q_x^{a_2} p_y^{b_1} q_y^{b_2} p_z^{c_1} q_z^{c_2}) := \; & \langle (\hat{p}_x - \langle \hat{p}_x \rangle)^{a_1} (\hat{q}_x - \langle \hat{q}_x \rangle)^{a_2} \\ & \times (\hat{p}_y - \langle \hat{p}_y \rangle)^{b_1} (\hat{q}_y - \langle \hat{q}_y \rangle)^{b_2} \\ & \times (\hat{p}_z - \langle \hat{p}_z \rangle)^{c_1} (\hat{q}_z - \langle \hat{q}_z \rangle)^{c_2} \rangle_{\text{Weyl}} \end{aligned} \tag{2.20}$$

with totally symmetric or Weyl ordering indicated by the subscript "Weyl." Weyl ordering makes sure that we do not count as different moments which can be obtained from each other by simple applications of the commutator. Moreover, the moments of Weyl ordered products in an associative algebra are defined as real numbers. It turns out to be useful to define them as expectation values of products of the differences $\hat{A} - \langle \hat{A} \rangle$ as opposed to products just of basic operators because a semiclassical state can then be defined as one in which moments of order $a_1 + a_2 + b_1 + b_2 + c_1 + c_2 =: n$ are of the order $O(\hbar^{n/2})$. In this way, one generalizes the family of Gaussian states in which this order relationship can be confirmed by an explicit calculation. We make use of the $\hbar$-orders in our semiclassical equations derived in sections 3 and 4.

For a non-associative algebra, we have to be careful with the order in which the products are performed. We define the moments by declaring that products of operators in them are to be evaluated from the left, that is

$$\Delta(p_x p_y p_z) = \langle \{ (\hat{p}_x - \langle \hat{p}_x \rangle)(\hat{p}_y - \langle \hat{p}_y \rangle) \} \, (\hat{p}_z - \langle \hat{p}_z \rangle) \rangle_{\text{Weyl}} \,. \tag{2.21}$$

A bracket on the space of expectation values and moments is defined via the commutator

$$\{ \langle \hat{O}_1 \rangle, \langle \hat{O}_2 \rangle \} = \frac{\langle [\hat{O}_1, \hat{O}_2] \rangle}{i\hbar} \tag{2.22}$$

combined with the Leibniz rule for products of expectation values. For an associative algebra, this definition gives rise to a Poisson bracket;[4] for a non-associative one, the associator is turned into a non-zero Jacobiator of (2.22). Evaluating the bracket on basic variables gives

$$\{ \langle \hat{q}_i \rangle, \langle \hat{q}_j \rangle \} = \frac{1}{i\hbar} \langle [\hat{q}_i, \hat{q}_j] \rangle = 0 \tag{2.23}$$

$$\{ \langle \hat{q}_i \rangle, \langle \hat{p}_j \rangle \} = \frac{1}{i\hbar} \langle [\hat{q}_i, \hat{p}_j] \rangle = \delta_{ij} \tag{2.24}$$

[4]The resulting Poisson manifold is much larger than the classical phase space, and in fact infinite-dimensional owing to infinitely many independent moments.

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$$\{\langle\hat{p}_i\rangle,\langle\hat{p}_j\rangle\} = \frac{1}{i\hbar}\langle[\hat{p}_i,\hat{p}_j]\rangle = e\epsilon_{ijk}\langle\hat{B}^k\rangle\,. \tag{2.25}$$

For a magnetic field B^k linear in the q_i, the right-hand side of the last relation is a function of basic expectation values, which from now on we will abbreviate as $q_i = \langle\hat{q}_i\rangle$. For a quadratic function, such as $B^k(q_i) = C(q_i)^2$ with a constant C, we have $\langle\hat{B}^k\rangle = C\langle\hat{q}_i^2\rangle = C(q_i)^2 + C\Delta(q_i^2)$ with a moment contribution. In general, if the magnetic field is non-linear, we may further expand

$$\langle\hat{B}^k\rangle = \langle B^k(q_i + (\hat{q}_i - q_i))\rangle = B^k(q_i) + \sum_{a,b,c}\frac{1}{a!b!c!}\frac{\partial^{a+b+c}B^k}{\partial q_x^a\partial q_y^b\partial q_z^c}\Delta(q_x^a q_y^b q_z^c) \tag{2.26}$$

with a series of moment contributions. There will be an infinite number of terms if B is non-polynomial. Such an expansion is usually asymptotic and gives rise to semiclassical or effective equations following the methods of [26, 27].

2.4 Volume uncertainty and uncertainty volume

Moments are subject to uncertainty relations and cannot be assigned arbitrary values. For covariances and fluctuations (2.20) with $a_1 + a_2 + b_1 + b_2 + c_1 + c_2 = 2$, the standard uncertainty relation follows from the Cauchy-Schwarz inequality with $\hat{A} = \widehat{\Delta O}_1 = \hat{O}_1 - \langle\hat{O}_1\rangle$ and $\hat{B} = \widehat{\Delta O}_2 = \hat{O}_2 - \langle\hat{O}_2\rangle$ linear in basic operators $\hat{O}_1$ and $\hat{O}_2$. Moments of higher order are restricted by uncertainty relations that follow from the Cauchy-Schwarz inequality with $\hat{A}$ and $\hat{B}$ polynomial in $\widehat{\Delta O}_i$. (See for instace [28, 29].)

A non-zero commutator between two observables $\hat{O}_1$ and $\hat{O}_2$ provides a lower bound for the product of their fluctuations ΔO_1 and ΔO_2. For a set of n canonical pairs $(\hat{q}_i, \hat{p}_i)$, the lower bound of $\prod_{i=1}^n(\Delta q_i \Delta p_i) \geq (\hbar/2)^n$ may then be interpreted as an elementary chunk of phase-space volume. For non-canonical commutators, different lower bounds may be realized for a subset of the phase-space variables or even the configuration (or momentum) variables among themselves. For instance, (2.3) suggests that areas in momentum space have lower bounds given by, for instance, $\Delta p_x \Delta p_y \geq \frac{1}{2}\hbar e\langle\hat{B}^z\rangle$, depending on the magnetic field and therefore, possibly, on the position. A new suggestion, going back to [11] and further analyzed in [9], is that a non-zero associator may provide an independent lower bound for triple products of fluctuations, such as $\Delta p_x \Delta p_y \Delta p_z$ for (2.4). Non-associativity in position space may then, intriguingly, imply spatial discreteness. (However, even if there is a lower bound for quantum fluctuations, the relation to discrete structures is not obvious: in [30], uncertainty relations have been computed for a discrete system, given by the cotangent space of a circle, and no lower bound for fluctuations of the discrete momentum was obtained. As discussed there, such lower bounds could rather be taken as an indication for extended fundamental objects, as would be appropriate for lower bounds found in models of string theory.)

It is not obvious how such uncertainty relations may be derived in a general way. The Cauchy-Schwarz inequality quite naturally leads to commutators by expressing the expectation value $\langle A^*B\rangle$ in terms of symmetric and antisymmetric combinations of A and B. It is more difficult to see how the associator might appear in uncertainty relations as an intrinsic quantity (as opposed to a quantity derived from the commutator which happens to resemble the associator). For instance, given a non-trivial uncertainty relation between momentum components, such as

$$(\Delta p_x)^2(\Delta p_y)^2 \geq \frac{1}{4}\hbar^2 e^2 \langle \hat{B}^z\rangle^2 \,,$$

and the standard uncertainty relation between Δq_z and Δp_z, a magnetic field with $\partial^2 B^z/\partial q_z^2 \neq 0$ would imply a non-trivial lower bound for the triple product

$$(\Delta p_x)^2(\Delta p_y)^2(\Delta p_z)^2 \geq \frac{1}{4}\hbar^2 e^2 \langle \hat{B}^z\rangle^2 (\Delta p_z)^2 \tag{2.27}$$

$$= \frac{1}{4}\hbar^2 e^2 \left(B^z(\langle \hat{q}_j\rangle)^2 + B^z(\langle \hat{q}_j\rangle)\frac{\partial^2 B^z(\langle \hat{q}_j\rangle)}{\partial \langle \hat{q}_z\rangle^2}(\Delta q_z)^2 + \cdots \right)(\Delta p_z)^2$$

$$\geq \frac{1}{16}\hbar^4 e^2 B^z \frac{\partial^2 B^z}{\partial q_z^2} + \frac{1}{4}\hbar^2 e^2 (B^z)^2 (\Delta p_z)^2 + \cdots \tag{2.28}$$

However, such an uncertainty relation is neither simple enough to suggest a universal and state-independent lower bound, nor does it follow directly from the associator. Moreover, there would be no lower bound for a linear magnetic field, or a constant associator. (For a semiclassical state, the second term in (2.28) would be dominant, so that the inequality would just amount to the momentum uncertainty relation for Δp_x and Δp_y, multiplied with an additional factor of Δp_z on both sides.)

A direct definition of volume uncertainty would be the uncertainty $\Delta V = \sqrt{\langle \hat{V}^2\rangle - \langle \hat{V}\rangle}$ of the volume operator $\hat{V} := ((\hat{p}_x\hat{p}_y)\hat{p}_z)_{\rm Weyl}$. (The definition $\hat{V} := (\hat{p}_x(\hat{p}_y\hat{p}_z))_{\rm Weyl}$ would result in the same operator for an alternative algebra.) However, an uncertainty relation follows from the Cauchy-Schwarz inequality only when ΔV is combined with the fluctuation of another observable not commuting with $\hat{V}$. No universal lower bound for ΔV itself would be implied.

One can introduce different quantities which may capture some of the intuition that may be associated with the notion of "volume uncertainty." For instance, the quantity $(\widehat{\Delta p}_x\widehat{\Delta p}_y)\widehat{\Delta p}_z$ could be related to the associator. In what follows, we call this triple product of uncertainties the uncertainty volume, in order to distinguish it from the uncertainty of the volume operator. (As noted in [9], the antisymmetrized uncertainty volume is related to the associator, but it is not clear to us how this quantity may appear as an upper or lower bound.)

Although the uncertainty volume does appear in some uncertainty relations, it turns out that it is subject to an upper rather than lower bound by higher-order uncertainty relations. Choosing $\hat{A} = \frac{1}{2}(\widehat{\Delta p}_x\widehat{\Delta p}_y + \widehat{\Delta p}_y\widehat{\Delta p}_x)$ and $\hat{B} = \widehat{\Delta p}_z$, one can compute

$$\langle \hat{A}^*\hat{A}\rangle = \Delta(p_x^2p_y^2) - \frac{1}{4}\langle[\widehat{\Delta p}_x,\widehat{\Delta p}_y]^2\rangle + \frac{1}{6}\langle \widehat{\Delta p}_y[[\widehat{\Delta p}_x,\widehat{\Delta p}_y],\widehat{\Delta p}_x] - \widehat{\Delta p}_x[[\widehat{\Delta p}_x,\widehat{\Delta p}_y],\widehat{\Delta p}_y]\rangle$$

$$= \Delta(p_x^2p_y^2) + \frac{e^2\hbar^2}{4}\langle \hat{B}^z\rangle + \frac{e^2\hbar^2}{6}\left\langle \widehat{\Delta p}_x\widehat{\partial_y B^z} - \widehat{\Delta p}_y\widehat{\partial_x B^z}\right\rangle \,. \tag{2.29}$$

(For a linear magnetic field, the last term is zero.) We obtain $\langle\hat{B}^*\hat{B}\rangle = (\Delta p_z)^2$, as usual, and $\langle\hat{A}^*\hat{B}\rangle$ contains in its real part the fluctuation volume:

$$\langle\hat{A}^*\hat{B}\rangle = \langle(\widehat{\Delta p_x}\widehat{\Delta p_y})\widehat{\Delta p_z}\rangle + \frac{1}{2}\hbar^2 e\left\langle\widehat{\partial_z B^z}\right\rangle - \frac{1}{4}i\hbar e\langle\hat{B}^z\widehat{\Delta p_z} + \widehat{\Delta p_z}\hat{B}^z\rangle\,. \tag{2.30}$$

Therefore, the uncertainty relation for the fluctuation volume $f := \langle(\widehat{\Delta p_x}\widehat{\Delta p_y})\widehat{\Delta p_z}\rangle$ is of the form

$$\left(f + \frac{1}{2}\hbar^2 e\left\langle\widehat{\partial_z B^z}\right\rangle\right)^2 \leq \Delta(p_x^2 p_y^2)\Delta(p_z^2) + \cdots \tag{2.31}$$

However, the associator again does not play a direct role in the derivation.

3 Algebra of second-order moments

We now calculate some of the brackets between second-order moments, providing characteristic examples in which different features of non-associative algebras appear. These brackets are useful for Hamiltonian equations of motion once the dynamics is specified, which we will explore in the next section.

3.1 Application of Moufang identities

We begin with an example in which the identity (2.17) (for A, B and C linear in the $\hat{p}_i$) plays an important role. For the bracket of two covariances of different momentum components, we have

$$\begin{aligned} P_1 &:= \{\Delta(p_x p_y), \Delta(p_y p_z)\} \\ &= \frac{1}{4i\hbar}\langle[(\hat{p}_x - p_x)(\hat{p}_y - p_y) + (\hat{p}_y - p_y)(\hat{p}_x - p_x), \\ &\qquad (\hat{p}_y - p_y)(\hat{p}_z - p_z) + (\hat{p}_z - p_z)(\hat{p}_y - p_y)]\rangle \\ &= \frac{1}{4i\hbar}\langle[i\hbar e\hat{B}^z + 2(\hat{p}_y - p_y)(\hat{p}_x - p_x), i\hbar e\hat{B}^x + 2(\hat{p}_z - p_z)(\hat{p}_y - p_y)]\rangle \end{aligned} \tag{3.1}$$

using the non-zero commutator (2.3). We continue and write out the commutator explicitly,

$$\begin{aligned} P_1 = \frac{1}{i\hbar}\langle &((\hat{p}_y - p_y)(\hat{p}_x - p_x))((\hat{p}_z - p_z)(\hat{p}_y - p_y)) \\ &-((\hat{p}_z - p_z)(\hat{p}_y - p_y))((\hat{p}_y - p_y)(\hat{p}_x - p_x)) \\ &+\frac{1}{2}i\hbar e(\hat{B}^z(\hat{p}_z - p_z)(\hat{p}_y - p_y) - (\hat{p}_z - p_z)(\hat{p}_y - p_y)\hat{B}^z) \\ &+\frac{1}{2}i\hbar e(-\hat{B}^x(\hat{p}_y - p_y)(\hat{p}_x - p_x) + (\hat{p}_y - p_y)(\hat{p}_x - p_x)\hat{B}^x)\rangle\,. \end{aligned} \tag{3.2}$$

The Moufang identity can be used in the first line, but not in the second line in its present form. Two additional applications of commutators bring the momentum factors of the second line into the form of (2.10):

$$\begin{aligned} P_1 = \frac{1}{i\hbar}\langle &((\hat{p}_y - p_y)(\hat{p}_x - p_x))((\hat{p}_z - p_z)(\hat{p}_y - p_y)) \\ &-((\hat{p}_y - p_y)(\hat{p}_z - p_z) - i\hbar\hat{B}^x)((\hat{p}_x - p_x)(\hat{p}_y - p_y) - i\hbar\hat{B}^z) \end{aligned}$$

$$+\frac{1}{2}i\hbar e(\hat{B}^z(\hat{p}_z - p_z)(\hat{p}_y - p_y) - (\hat{p}_z - p_z)(\hat{p}_y - p_y)\hat{B}^z)$$
$$+\frac{1}{2}i\hbar e(-\hat{B}^x(\hat{p}_y - p_y)(\hat{p}_x - p_x) + (\hat{p}_y - p_y)(\hat{p}_x - p_x)\hat{B}^x)\rangle\,. \tag{3.3}$$

Now distributing the second term and using (2.10), and collecting the middle terms of the first and second terms into a commutator, we obtain

$$\begin{aligned} P_1 = {}& e\left\langle -(\hat{p}_y - p_y)\hat{B}^y(\hat{p}_y - p_y)\right\rangle \\ &+\frac{e}{2}\left\langle \hat{B}^x(\hat{p}_x - p_x)(\hat{p}_y - p_y) + (\hat{p}_y - p_y)(\hat{p}_x - p_x)\hat{B}^x\right\rangle \\ &+\frac{e}{2}\left\langle \hat{B}^z(\hat{p}_z - p_z)(\hat{p}_y - p_y) + (\hat{p}_y - p_y)(\hat{p}_z - p_z)\hat{B}^z\right\rangle\,. \end{aligned}$$

We can now expand $\langle \hat{B}^i\rangle$ as in (2.26) in order to express this expectation value in terms of moments. If we keep up to second-order moments for semiclassical equations, we obtain

$$\{\Delta(p_xp_y), \Delta(p_yp_z)\} = -eB^y\Delta(p_y^2) + eB^x\Delta(p_xp_y) + eB^z\Delta(p_yp_z)\,.$$

3.2 Application of the associator

Another example in which a combination of commutators and the associator can be used directly is

$$\begin{aligned} P_2 :={}& \{\Delta(p_xq_z), \Delta(p_yp_z)\} \\ ={}& \frac{1}{2i\hbar}\langle[(\hat{p}_x - p_x)(\hat{q}_z - q_z), (\hat{p}_y - p_y)(\hat{p}_z - p_z) + (\hat{p}_z - p_z)(\hat{p}_y - p_y)]\rangle \\ ={}& \frac{1}{2i\hbar}\langle((\hat{q}_z - q_z)(\hat{p}_x - p_x))((\hat{p}_y - p_y)(\hat{p}_z - p_z)) \\ &-((\hat{p}_y - p_y)(\hat{p}_z - p_z))((\hat{p}_x - p_x)(\hat{q}_z - q_z))\rangle + (y \leftrightarrow z \text{ only in } p\text{-terms})\,. \end{aligned} \tag{3.4}$$

The goal here is to bring the triple product of $\hat{p}_i$ in the first term to the form of the second term; for this reason we use the associator first, and concentrate only on the first term (omitting the $(\hat{q}_z - q_z)$ term for now):

$$(\hat{p}_x - p_x)((\hat{p}_y - p_y)(\hat{p}_z - p_z)) = ((\hat{p}_x - p_x)(\hat{p}_y - p_y))(\hat{p}_z - p_z) + \hbar^2 e\widehat{\partial_i B^i}\,. \tag{3.5}$$

After using the commutator in the parantheses of the first term on the right-hand side we arrive at

$$(\hat{p}_x-p_x)((\hat{p}_y-p_y)(\hat{p}_z-p_z)) = ((\hat{p}_y-p_y)(\hat{p}_x-p_x))(\hat{p}_z-p_z)+i\hbar e\hat{B}^z(\hat{p}_z-p_z)+\hbar^2 e\widehat{\partial_i B^i}\,. \tag{3.6}$$

Once again, we use the associator followed by the commutator, writing

$$\begin{aligned} (\hat{p}_x - p_x)((\hat{p}_y - p_y)(\hat{p}_z - p_z)) ={}& (\hat{p}_y - p_y)((\hat{p}_z - p_z)(\hat{p}_x - p_x)) \\ &-i\hbar e(\hat{p}_y - p_y)\hat{B}^y + i\hbar e\hat{B}^z(\hat{p}_z - p_z) + 2\hbar^2 e\widehat{\partial_i B^i}\,. \end{aligned} \tag{3.7}$$

Applying this procedure one last time on the very first term in (3.7) yields

$$((\hat{p}_y - p_y)(\hat{p}_z - p_z))(\hat{p}_x - p_x) - i\hbar e(\hat{p}_y - p_y)\hat{B}^y + i\hbar e\hat{B}^z(\hat{p}_z - p_z) + 3\hbar^2 e\widehat{\partial_i B^i}\,. \tag{3.8}$$

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So far we have looked only at the first term in (3.4). Observe that we brought it to the same form as the second term in (3.4) up to the position of $(\hat{q}_z - q_z)$ to the left and right, respectively, which can be combined into a commutator with $(\hat{p}_z - p_z)$ to yield an $i\hbar$. Now doing the same calculation with the third and fourth term in (3.4), which we did not spell out explicitly, yields the bracket wherein the contributions from the associator terms drop out

$$\begin{aligned}\{\Delta(p_xq_z), \Delta(p_yp_z)\} = &\frac{1}{2}\langle(\hat{p}_y - p_y)(\hat{p}_x - p_x) + (\hat{p}_x - p_x)(\hat{p}_y - p_y)\rangle \\ &+\frac{e}{2}\left\langle \hat{B}^z(\hat{q}_z - q_z)(\hat{p}_z - p_z) + (\hat{q}_z - q_z)(\hat{p}_z - p_z)\hat{B}^z\right\rangle \\ &-\frac{e}{2}\left\langle \hat{B}^y(\hat{q}_z - q_z)(\hat{p}_y - p_y) + (\hat{q}_z - q_z)(\hat{p}_y - p_y)\hat{B}^y\right\rangle \\ &-\frac{i\hbar e}{2}\langle\hat{B}^z\rangle\,. \end{aligned} \tag{3.9}$$

Expanding to second order in moments, we obtain

$$\begin{aligned}\{\Delta(p_xq_z), \Delta(p_yp_z)\} = &\Delta(p_xp_y) + eB^z\Delta(p_zq_z) - eB^y\Delta(p_yq_z) \\ &-\frac{i\hbar e}{2}\left(B^z + \frac{1}{2}\frac{\partial^2 B^z}{\partial q_i \partial q_j}\Delta(q_iq_j)\right) + \cdots\,. \end{aligned} \tag{3.10}$$

Here and in what follows, the dots indicate terms having moments higher than second order, or terms of order larger than $\hbar$ in a semiclassical state.

3.3 Application of commutator identities

In our third example of the brackets it is sufficient to use standard commutator identities:

$$\begin{aligned}\{\Delta(p_xq_y), \Delta(p_yq_x)\} &= \frac{1}{i\hbar}\langle[(\hat{p}_x - p_x)(\hat{q}_y - q_y), (\hat{p}_y - p_y)(\hat{q}_x - q_x)]\rangle \qquad (3.11)\\ &= \langle(\hat{p}_x - p_x)(\hat{q}_x - q_x) - (\hat{p}_y - p_y)(\hat{q}_y - q_y) + e\hat{B}^z(\hat{q}_y - q_y)(\hat{q}_x - q_x)\rangle \\ &= \Delta(p_xq_x) - \Delta(p_yq_y) + eB^z\Delta(q_xq_y)\end{aligned}$$

expanded up to second order in moments.

In general, however, one should be careful with the usual identity $[\hat{A}, \hat{B}\hat{C}] = [\hat{A}, \hat{B}]\hat{C} + \hat{B}[\hat{A}, \hat{C}]$ when the algebra is not associative, as has already been pointed out in [9] in the context of Heisenberg equations of motion, no longer given by a derivation $[\cdot, \hat{H}]$. In fact, the equation is not valid in general: we have

$$[\hat{A}, \hat{B}\hat{C}] = \hat{A}(\hat{B}\hat{C}) - (\hat{B}\hat{C})\hat{A}$$

and

$$[\hat{A}, \hat{B}]\hat{C} + \hat{B}[\hat{A}, \hat{C}] = (\hat{A}\hat{B})\hat{C} - (\hat{B}\hat{A})\hat{C} + \hat{B}(\hat{A}\hat{C}) - \hat{B}(\hat{C}\hat{A})\,.$$

The two terms in the middle of the last equation cancel out only when the multiplication of three given operators is associative. In our example, we have at most two momentum components, so that this requirement is satisfied. In general, one can write the difference of the usual two expressions as a combination of associators:

$$[\hat{A}, \hat{B}\hat{C}] - [\hat{A}, \hat{B}]\hat{C} - \hat{B}[\hat{A}, \hat{C}] = -[\hat{A}, \hat{B}, \hat{C}] - [\hat{B}, \hat{C}, \hat{A}] + [\hat{B}, \hat{A}, \hat{C}]\,. \tag{3.12}$$

For an alternative algebra, all three terms are equal and the difference is three times the negative associator $[\hat{A}, \hat{B}, \hat{C}]$.

3.4 Brackets

Having shown a few explicit calculations, we give here a list of some more brackets of generic type, including their expansions up to second-order moments:

$$\{\Delta(p_yp_x),\Delta(q_yq_x)\} = -\Delta(p_xq_x)-\Delta(p_yq_y) \tag{3.13}$$

$$\{\Delta(p_xq_y),\Delta(q_xq_y)\} = -\Delta(q_y^2) \tag{3.14}$$

$$\begin{aligned}\{\Delta(p_xq_y),\Delta(p_yq_x)\} &= \Delta(p_xq_x)-\Delta(p_yq_y)+e\langle\hat{B}^z(\hat{q}_y-q_y)(\hat{q}_x-q_x)\rangle\\ &= \Delta(p_xq_x)-\Delta(p_yq_y)+eB^z\Delta(q_xq_y)+\cdots\end{aligned} \tag{3.15}$$

$$\begin{aligned}\{\Delta(p_xq_y),\Delta(p_zq_z)\} &= -e\langle\hat{B}^y(\hat{q}_y-q_y)(\hat{q}_z-q_z)\rangle\\ &= -eB^y\Delta(q_yq_z)+\cdots\end{aligned} \tag{3.16}$$

$$\begin{aligned}\{\Delta(p_xq_x),\Delta(p_yq_y)\} &= e\langle\hat{B}^z(\hat{q}_x-q_x)(\hat{q}_y-q_y)\rangle\\ &= eB^z\Delta(q_xq_y)+\cdots\end{aligned} \tag{3.17}$$

$$\begin{aligned}\{\Delta(p_xq_z),\Delta(p_yp_z)\} &= \Delta(p_xp_y)+\frac{e}{2}\langle\hat{B}^z(\hat{q}_z-q_z)(\hat{p}_z-p_z)+(\hat{q}_z-q_z)(\hat{p}_z-p_z)\hat{B}^z\rangle\\ &\quad -\frac{e}{2}\langle(\hat{q}_z-q_z)\hat{B}^y(\hat{p}_y-p_y)+(\hat{q}_z-q_z)(\hat{p}_y-p_y)\hat{B}^y\rangle-\frac{i\hbar e}{2}\langle\hat{B}^z\rangle\\ &= \Delta(p_xp_y)+eB^z\Delta(p_zq_z)-eB^y\Delta(p_yq_z)\\ &\quad -\frac{i\hbar e}{2}\left(B^z+\frac{1}{2}\frac{\partial^2B^z}{\partial q_i\partial q_j}\Delta(q_iq_j)\right)+\cdots\end{aligned} \tag{3.18}$$

$$\begin{aligned}\{\Delta(p_xq_x),\Delta(p_yp_z)\} &= \frac{e}{2}\langle\hat{B}^z(\hat{q}_x-q_x)(\hat{p}_z-p_z)+(\hat{q}_x-q_x)(\hat{p}_z-p_z)\hat{B}^z\rangle\\ &\quad -\frac{e}{2}\langle(\hat{q}_x-q_x)\hat{B}^y(\hat{p}_y-p_y)+(\hat{q}_x-q_x)(\hat{p}_y-p_y)\hat{B}^y\rangle\\ &= eB^z\Delta(p_zq_x)-eB^y\Delta(p_yq_x)+\cdots\end{aligned} \tag{3.19}$$

For a non-constant magnetic field, some of the brackets of basic expectation values and moments are non-zero as well:

$$\{p_x,\Delta(p_y^2)\} = e\langle\hat{B}^z(\hat{p}_y-p_y)+(\hat{p}_y-p_y)\hat{B}^z\rangle \tag{3.20}$$

$$= 2e\frac{\partial B^z}{\partial q_i}\Delta(p_yq_i)+\cdots\,. \tag{3.21}$$

4 Semiclassical dynamics of a charged particle in a magnetic monopole density

For algebraic states, the dynamics is defined in terms of a flow of positive linear functionals ω_t, $t\in\mathbb{R}$ on $\mathcal{A}$ with respect to a Hamiltonian $H\in\mathcal{A}$:

$$\frac{\mathrm{d}\omega_t(O)}{\mathrm{d}t} := \frac{1}{i\hbar}\omega_t([O,H]) = \{\omega_t(O),\omega_t(H)\} \tag{4.1}$$

in terms of the bracket (2.22). This definition agrees with the standard Schrödinger or Heisenberg flow in the case of an associative algebra of operators represented on a Hilbert space, but it does not require this additional structure. (It is also insensitive to the commutator $[\cdot,\hat{H}]$ no longer being a derivation, which had been noted in [9].) We can therefore apply it to the example of a non-associative algebra studied here.

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4.1 General magnetic field

We first choose the "free-particle" Hamiltonian

$$\hat{H} = \frac{1}{2m}(\hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2)\,, \tag{4.2}$$

so that we will be considering the motion of a charged particle in a background magnetic field without additional forces. Interactions between the charged particle and the magnetic field are represented by the non-associativity of the algebra or the Jacobiator of the bracket of expectation values and moments, rather than terms in the Hamiltonian. We obtain the quantum Hamiltonian as

$$H_Q := \langle \hat{H} \rangle = p_x^2 + p_y^2 + p_z^2 + \Delta(p_x^2) + \Delta(p_y^2) + \Delta(p_z^2) \tag{4.3}$$

which generates Hamiltonian equations of motion as per (4.1), now writing $\omega(\hat{H}) = \langle \hat{H} \rangle$.

As an example we look at the x-components of the equations of motion,

$$\dot{q}_x = \frac{1}{m} p_x \tag{4.4}$$

$$\begin{aligned} \dot{p}_x &= \frac{1}{2m}\{p_x, p_y^2 + p_z^2\} + \frac{1}{2m}\{p_x, \langle \hat{p}_y^2 \rangle - p_y^2 + \langle \hat{p}_z^2 \rangle - p_z^2\} \\ &= \frac{e}{2m}(\langle \hat{p}_y \hat{B}_z + \hat{B}_z \hat{p}_y \rangle - \langle \hat{p}_z \hat{B}_y + \hat{B}_y \hat{p}_z \rangle)\,. \end{aligned} \tag{4.5}$$

In this expression we expand the magnetic field as

$$\hat{B}^z(\hat{q}) = B^z(q) + (\hat{q}_i - q_i)\frac{\partial B^z}{\partial q_i} + \frac{1}{2}(\hat{q}_j - q_j)(\hat{q}_i - q_i)\frac{\partial^2 B^z}{\partial q_i \partial q_j} + \cdots \tag{4.6}$$

and insert it in the first term of (4.5):

$$\langle \hat{p}_y \hat{B}^z + \hat{B}^z \hat{p}_y \rangle = 2B^z p_y + 2\frac{\partial B^z}{\partial q_i}\Delta(p_y q_i) + p_y \frac{\partial^2 B^z}{\partial q_i \partial q_j}\Delta(q_i q_j) + \cdots\,. \tag{4.7}$$

Here we just added the (vanishing) contribution $\langle p_y(\hat{q}_i - q_i)\rangle$. Similarly expanding the second term in (4.5) and using the definitions of moments we get

$$\begin{aligned} m\ddot{q}_x &= e(B^z v_y - B^y v_z) \\ &\quad + \frac{e}{m}\frac{\partial B^z}{\partial q_i}\Delta(p_y q_i) - \frac{e}{m}\frac{\partial B^y}{\partial q_i}\Delta(p_z q_i) + \frac{e}{2m}\left(p_y \frac{\partial^2 B^z}{\partial q_i \partial q_j} - p_z \frac{\partial^2 B^y}{\partial q_i \partial q_j}\right)\Delta(q_i q_j) + \cdots\,. \end{aligned} \tag{4.8}$$

The first term is the classical Lorentz force. The additional terms are quantum corrections to the equation of motion, which vanish for a constant magnetic field. Indeed, it is well-known that a charged particle in a constant magnetic field can be described by a harmonic-oscillator Hamiltonian, and the harmonic oscillator does not give rise to quantum corrections in Ehrenfest equations.

The moments that appear in (4.8) are themselves subject to dynamical equations of motion with respect to the effective Hamiltonian. We have

$$\dot{\Delta}(p_y q_i) = 2\Delta(p_y p_x)\delta_{ix} + 2\Delta(p_y^2)\delta_{iy} + 2\Delta(p_y p_z)\delta_{iz} - 2eB^z\Delta(p_x q_i)$$

$$+2eB^x\Delta(p_zq_i) + 2eq_i\frac{\partial B^z}{\partial q_j}\Delta(p_xq_j) + eq_ip_x\frac{\partial^2 B^z}{\partial q_j\partial q_k}\Delta(q_jq_k)$$
$$-2eq_i\frac{\partial B^x}{\partial q_j}\Delta(p_zq_j) - eq_ip_z\frac{\partial^2 B^x}{\partial q_j\partial q_k}\Delta(q_jq_k) + \cdots . \tag{4.9}$$

The equation for $\Delta(p_zq_i)$ is analogous to the equation above. The remaining moment in (4.8) has an equation of motion of the form

$$\dot{\Delta}(q_iq_j) = 2\Delta(p_xq_i)\delta_{jx} + 2\Delta(p_xq_j)\delta_{ix} + 2\Delta(p_yq_i)\delta_{jx} + 2\Delta(p_yq_j)\delta_{ix}$$
$$+2\Delta(p_zq_i)\delta_{jx} + 2\Delta(p_zq_j)\delta_{ix} + \cdots . \tag{4.10}$$

For a closed set of equations, we need an equation of motion for moments of the form $\Delta(p_yp_x)$, which appears in (4.9). This calculation turns out to be more challenging, but it can be handled by using the associator as well as the defining identities (2.5) for an alternative algebra

$$\dot{\Delta}(p_yp_x) = 2e\Bigg[- B^z\Delta(p_x^2) + B^z\Delta(p_y^2) + p_x\frac{\partial B^z}{\partial q_i}\Delta(p_xq_i) + \frac{p_x^2}{2}\frac{\partial^2 B^z}{\partial q_i\partial q_j}\Delta(q_iq_j)$$
$$-p_y\frac{\partial B^z}{\partial q_i}\Delta(p_yq_i) - \frac{p_y^2}{2}\frac{\partial^2 B^z}{\partial q_i\partial q_j}\Delta(q_iq_j) - B^y\Delta(p_yp_z) + B^x\Delta(p_xp_z)$$
$$-p_z\frac{\partial B^y}{\partial q_i}\Delta(p_yq_i) + p_z\frac{\partial B^x}{\partial q_i}\Delta(p_xq_i)$$
$$-i\hbar\frac{\partial^2 B^j}{\partial q_i\partial q_j}\Delta(p_zq_i) - \frac{i\hbar p_z}{2}\left(\frac{\partial B^j}{\partial q_j} + \frac{1}{2}\frac{\partial^3 B^j}{\partial q_j q_i q_k}\Delta(q_iq_k)\right)\Bigg] + \cdots \tag{4.11}$$

We now have a closed system of equations for the moments up to second order.

4.2 Canonical variables in the absence of a magnetic charge density

In order to test the quantum corrections for a non-constant magnetic field, we use moment expansions in a derivation of semiclassical equations for the canonical variables q_i and $\pi_j = m\dot{q}_j + eA_j$ (with $\{q_i, \pi_j\} = i\hbar$). These variables can be used only in the absence of a magnetic charge density, in which case we can compare their dynamics with (4.8).

In canonical variables, the Hamiltonian operator (4.2) is

$$\hat{H} = \frac{1}{2m}(\hat{\pi} - eA)^2 = \frac{1}{2m}\delta^{ij}(\pi_i - eA_i)(\pi_j - eA_j)\,. \tag{4.12}$$

To second order in moments, it implies a quantum Hamiltonian

$$H_Q = \langle\hat{H}\rangle = \frac{1}{2m}\delta^{ij}\pi_i\pi_j - \frac{e}{m}\delta^{ij}\pi_iA_j + \frac{e^2}{2m}\delta^{ij}A_iA_j \tag{4.13}$$
$$+\frac{1}{2m}\delta^{ij}\Delta(\pi_i\pi_j) - \frac{e}{m}\delta^{ik}\frac{\partial A_i}{\partial q_j}\Delta(\pi_kq_j)$$
$$-\frac{e}{2m}\delta^{il}\left((\pi_i - eA_i)\frac{\partial^2 A_l}{\partial q_j\partial q_k} - e\frac{\partial A_i}{\partial q_j}\frac{\partial A_l}{\partial q_k}\right)\Delta(q_jq_k)$$

where A_i is understood as the classical function $A_i(\langle\hat{q}_j\rangle)$ evaluated at expectation values.

We compute Hamiltonian equations of motion

$$\begin{aligned}\dot{q}_i &= \frac{1}{m}\pi_i - \frac{e}{m}A_i - \frac{e}{2m}\frac{\partial^2 A_i}{\partial q_k \partial q_l}\Delta(q_k q_l) \\ &= \frac{1}{m}(\pi_i - e\langle \hat{A}_i \rangle)\end{aligned} \tag{4.14}$$

and

$$\begin{aligned}\dot{\pi}_i &= \frac{e}{m}\delta^{jk}\pi_j\frac{\partial A_k}{\partial q_i} - \frac{e^2}{m}\delta^{jk}A_j\frac{\partial A_k}{\partial q_i} \\ &+\frac{e}{m}\delta^{jk}\frac{\partial^2 A_j}{\partial q_i \partial q_l}\delta(\pi_k q_l) \\ &+\frac{q}{2m}\delta^{jk}\left((\pi_j - eA_j)\frac{\partial^3 A_k}{\partial q_i \partial q_m \partial q_n}\right. \\ &\left.-e\left(\frac{\partial A_j}{\partial q_i}\frac{\partial^2 A_k}{\partial q_m \partial q_n} + \frac{\partial \dot{\Delta}(p_y p_x)A_j}{\partial q_m}\frac{\partial^2 A_k}{\partial q_i \partial q_n} + \frac{\partial A_j}{\partial q_n}\frac{\partial^2 A_k}{\partial q_i \partial q_m}\right)\right)\Delta(q_m q_n) + \cdots .\end{aligned} \tag{4.15}$$

We will also need the equations of motion for some moments:

$$\dot{\Delta}(q_m q_n) = \frac{1}{m}\left(\Delta(\pi_m q_n) + \Delta(\pi_n q_m)\right) - \frac{e}{m}\left(\frac{\partial A_m}{\partial q_l}\Delta(q_l q_n) + \frac{\partial A_n}{\partial q_l}\Delta(q_l q_m)\right) + \cdots . \tag{4.16}$$

With these results, we can rewrite the Hamiltonian equations of motion as second-order differential equations for the components q_i:

$$\begin{aligned}m\ddot{q}_i &= \frac{e}{m}\pi_j\left(\delta^{jk}\delta_{il}\frac{\partial A_k}{\partial q_l} - \frac{\partial A_l}{\partial q_j}\right) - \frac{e^2}{m}A_j\left(\delta^{jk}\delta_{il}\frac{\partial A_k}{\partial q_l} - \frac{\partial A_l}{\partial q_j}\right) \\ &+\frac{e}{m}\frac{\partial}{\partial q_m}\left(\delta^{jk}\delta_{il}\frac{\partial A_k}{\partial q_l} - \frac{\partial A_l}{\partial q_j}\right)\Delta(\pi_j q_m) \\ &+\frac{e}{2m}\left((\pi_j - eA_j)\frac{\partial^2}{\partial q_m \partial q_n}\left(\delta^{jk}\delta_{il}\frac{\partial A_k}{\partial q_l} - \frac{\partial A_l}{\partial q_j}\right) - e\frac{\partial^2 A_j}{\partial q_m \partial q_n}\left(\delta^{jk}\delta_{il}\frac{\partial A_k}{\partial q_l} - \frac{\partial A_l}{\partial q_j}\right)\right. \\ &\left.-2e\frac{\partial A_j}{\partial q_m}\frac{\partial}{\partial q_n}\left(\delta^{jk}\delta_{il}\frac{\partial A_k}{\partial q_l} - \frac{\partial A_l}{\partial q_j}\right)\right)\Delta(q_m q_n) + \cdots .\end{aligned} \tag{4.17}$$

After several simplifications, we can bring this equation into the form

$$\begin{aligned}m\ddot{q}_i &= \frac{e}{m}(\pi_j - e\langle \hat{A}_j \rangle)\left(\delta^{jk}\delta_{il}\frac{\partial A_k}{\partial q_l} - \frac{\partial A_l}{\partial q_j}\right) \\ &\quad+\frac{e}{m}\frac{\partial}{\partial q_m}\left(\delta^{jk}\delta_{il}\frac{\partial A_k}{\partial q_l} - \frac{\partial A_l}{\partial q_j}\right)\Delta((\pi_j - eA_j)q_m) \\ &\quad+\frac{e}{2m}(\pi_j - eA_j)\frac{\partial^2}{\partial q_m \partial q_n}\left(\delta^{jk}\delta_{il}\frac{\partial A_k}{\partial q_l} - \frac{\partial A_l}{\partial q_j}\right)\Delta(q_m q_n) + \cdots\end{aligned} \tag{4.18}$$

$$\begin{aligned}&= \frac{e}{m}(\pi_j - e\langle \hat{A}_j \rangle)\left\langle \delta^{jk}\delta_{il}\frac{\partial A_k}{\partial q_l} - \frac{\partial A_l}{\partial q_j}\right\rangle \\ &\quad+\frac{e}{m}\frac{\partial}{\partial q_m}\left(\delta^{jk}\delta_{il}\frac{\partial A_k}{\partial q_l} - \frac{\partial A_l}{\partial q_j}\right)\Delta((\pi_j - eA_j)q_m) + \cdots .\end{aligned} \tag{4.19}$$

This equation agrees with (4.8), but is valid only in the absence of a magnetic charge density.

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4.3 Potential and magnetic charge density

If there is a position-dependent potential in addition to the magnetic field, the effective Hamiltonian is

$$\begin{aligned} H_Q &= \frac{1}{2m}(p_x^2+p_y^2+p_z^2)+V(q_i) \\ &\quad +\frac{1}{2m}(\Delta(p_x^2)+\Delta(p_y^2)+\Delta(p_z^2))+\frac{1}{2}\frac{\partial^2 V}{\partial q_i\partial q_j}\Delta(q_iq_j)+\cdots . \end{aligned} \tag{4.20}$$

The potential implies the usual additional terms $-\partial V/\partial q_i$ and $-\frac{1}{2}(\partial^3 V/\partial q_i\partial q_j\partial q_k)\Delta(q_jq_k)$ in the equation of motion for $m\ddot{q}_i$.

$$\begin{aligned} m\ddot{q}_x &= e(B^z v_y - B^y v_z) - \frac{\partial V}{\partial q_x} \\ &\quad +\frac{e}{m}\frac{\partial B^z}{\partial q_i}\Delta(p_yq_i)-\frac{e}{m}\frac{\partial B^y}{\partial q_i}\Delta(p_zq_i)+\frac{e}{2m}\left(p_y\frac{\partial^2 B^z}{\partial q_i\partial q_j}-p_z\frac{\partial^2 B^y}{\partial q_i\partial q_j}\right)\Delta(q_iq_j) \\ &\quad -\frac{1}{2}\frac{\partial^3 V}{\partial q_i\partial q_j\partial q_k}\Delta(q_jq_k)+\cdots . \end{aligned} \tag{4.21}$$

Equations of motion for moments (which appear in the above equation) in this case are modified as follows:

$$\begin{aligned} \dot{\Delta}(p_yq_i) &= 2\Delta(p_yp_x)\delta_{ix}+2\Delta(p_y^2)\delta_{iy}+2\Delta(p_yp_z)\delta_{iz}-2eB^z\Delta(p_xq_i) \\ &\quad +2eB^x\Delta(p_zq_i)+2eq_i\frac{\partial B^z}{\partial q_j}\Delta(p_xq_j)+eq_ip_x\frac{\partial^2 B^z}{\partial q_j\partial q_k}\Delta(q_jq_k) \\ &\quad -2eq_i\frac{\partial B^x}{\partial q_j}\Delta(p_zq_j)-eq_ip_z\frac{\partial^2 B^x}{\partial q_j\partial q_k}\Delta(q_jq_k) \\ &\quad -\frac{1}{2}\frac{\partial^2 V}{\partial q_j\partial q_k}\left[\Delta(q_iq_j)\delta_{ky}+\Delta(q_iq_k)\delta_{jy}\right]+\cdots \end{aligned} \tag{4.22}$$

$$\begin{aligned} \dot{\Delta}(q_iq_j) &= 2\Delta(p_xq_i)\delta_{jx}+2\Delta(p_xq_j)\delta_{ix}+2\Delta(p_yq_i)\delta_{jx}+2\Delta(p_yq_j)\delta_{ix} \\ &\quad +2\Delta(p_zq_i)\delta_{jx}+2\Delta(p_zq_j)\delta_{ix}+\cdots . \end{aligned} \tag{4.23}$$

For completeness, we also note

$$\begin{aligned} \dot{\Delta}(p_yp_x) &= 2e\Bigg[-B^z\Delta(p_x^2)+B^z\Delta(p_y^2)+p_x\frac{\partial B^z}{\partial q_i}\Delta(p_xq_i)+\frac{p_x^2}{2}\frac{\partial^2 B^z}{\partial q_i\partial q_j}\Delta(q_iq_j) \\ &\quad -p_y\frac{\partial B^z}{\partial q_i}\Delta(p_yq_i)-\frac{p_y^2}{2}\frac{\partial^2 B^z}{\partial q_i\partial q_j}\Delta(q_iq_j)-B^y\Delta(p_yp_z)+B^x\Delta(p_xp_z) \\ &\quad -p_z\frac{\partial B^y}{\partial q_i}\Delta(p_yq_i)+p_z\frac{\partial B^x}{\partial q_i}\Delta(p_xq_i) \\ &\quad -i\hbar\frac{\partial^2 B^j}{\partial q_i\partial q_j}\Delta(p_zq_i)-\frac{i\hbar p_z}{2}\left(\frac{\partial B^j}{\partial q_j}+\frac{1}{2}\frac{\partial^3 B^j}{\partial q_jq_iq_k}\Delta(q_iq_k)\right)\Bigg] \\ &\quad -\frac{1}{2}\frac{\partial^2 V}{\partial q_i\partial q_j}\left(\Delta(p_yq_i)\delta_{jx}+\Delta(p_yq_j)\delta_{ix}+\Delta(p_xq_i)\delta_{jy}+\Delta(p_xq_j)\delta_{iy}\right)+\cdots . \end{aligned} \tag{4.24}$$

As is evident from (4.9), (4.10), (4.11), (4.22) and (4.24) the equations of motion for $\Delta(p_y q_i)$ (and $\Delta(p_z q_i)$) and $\Delta(p_y p_x)$ get some additional terms due to the potential , whereas that for $\Delta(q_i q_j)$ remains the same.

5 Conclusions

Taking a pragmatic view that leaves aside existence questions, we have shown that moment methods are efficient for a derivation of some aspects of non-associative quantum mechanics, regarding in particular uncertainty relations and semiclassical equations of motion. This general fact is not surprising because these algebraic methods are representation independent and do not require a Hilbert space, a property which is useful in some quantum-gravity models as well in which such methods had been explored and developed first.

Still, we did encounter several non-trivial steps in this new application, which are likely to recur when one tries to extend our results to higher orders. In several explicit examples, we exploited the existence of Moufang identities in alternative algebras (which, interestingly, also help to generalize the axioms of quantum mechanics [23]). Based on the algebraic relations alone, it seems likely that a more complete version of formal quantum mechanics can be developed, of which we have given here semiclassical properties.

Nevertheless, there are several interesting mathematical questions left open. For instance, while we explicitly used only the antisymmetric associator $[\hat{p}_1, \hat{p}_2, \hat{p}_3]$ of basic momentum components in our semiclassical derivations, some higher-order terms would require relationships for associators of products of momenta. Moufang identities would be available only if all these associators are totally antisymmetric, that is for an alternative algebra. The $*$-products of [9], with the same basic relations as used here, do not provide an alternative algebra. Our semiclassical results should still hold in these cases because the associator of basic momenta is totally antisymmetric, but there may be differences at higher orders. Properties of moments may therefore allow one to distinguish between different versions of algebras realizing the basic relations (2.1)–(2.4).

Acknowledgments

This work was supported in part by NSF grant PHY-1307408. It is a pleasure to thank Dieter Lüst, Peter Schupp and Richard Szabo for valuable comments on a draft of this article, and Stefan Waldmann for discussions.

References

[1] C.J. Isham, *Topological and global aspects of quantum theory*, in *Relativity, groups and topology II — Lectures given at the* 1983 *Les Houches Summer School on Relativity, Groups and Topology*, B.S. DeWitt and R. Stora eds., North-Holland, Amsterdam The Netherlands (1983) [INSPIRE].

[2] N.M.J. Woodhouse, *Geometric quantization*, Oxford mathematical monographs, Clarendon, U.K. (1992).

[3] J.-S. Park, *Topological open p-branes*, hep-th/0012141 [INSPIRE].

[4] C. Klimčík and T. Strobl, *WZW-Poisson manifolds*, *J. Geom. Phys.* **43** (2002) 341 [math/0104189] [INSPIRE].

[5] P. Ševera and A. WEinstein, *Poisson geometry with a 3 form background*, *Prog. Theor. Phys. Suppl.* **144** (2001) 145 [math/0107133] [INSPIRE].

[6] R. Blumenhagen, A. Deser, D. Lüst, E. Plauschinn and F. Rennecke, *Non-geometric fluxes, asymmetric strings and nonassociative geometry*, *J. Phys.* **A 44** (2011) 385401 [arXiv:1106.0316] [INSPIRE].

[7] D. Mylonas, P. Schupp and R.J. Szabo, *Membrane σ-models and quantization of non-geometric flux backgrounds*, *JHEP* **09** (2012) 012 [arXiv:1207.0926] [INSPIRE].

[8] I. Bakas and D. Lüst, 3*-cocycles, non-associative star-products and the magnetic paradigm of R-flux string vacua*, *JHEP* **01** (2014) 171 [arXiv:1309.3172] [INSPIRE].

[9] D. Mylonas, P. Schupp and R.J. Szabo, *Non-geometric fluxes, quasi-Hopf twist deformations and nonassociative quantum mechanics*, *J. Math. Phys.* **55** (2014) 122301 [arXiv:1312.1621] [INSPIRE].

[10] D. Mylonas, P. Schupp and R.J. Szabo, *Nonassociative geometry and twist deformations in non-geometric string theory*, PoS(ICMP2013)007 [arXiv:1402.7306] [INSPIRE].

[11] D. Lüst, *T-duality and closed string non-commutative (doubled) geometry*, *JHEP* **12** (2010) 084 [arXiv:1010.1361] [INSPIRE].

[12] R. Blumenhagen and E. Plauschinn, *Nonassociative gravity in string theory?*, *J. Phys.* **A 44** (2011) 015401 [arXiv:1010.1263] [INSPIRE].

[13] D. Lüst, *Twisted Poisson structures and non-commutative/non-associative closed string geometry*, PoS(CORFU2011)086 [arXiv:1205.0100] [INSPIRE].

[14] R. Blumenhagen, M. Fuchs, F. Haßler, D. Lüst and R. Sun, *Non-associative deformations of geometry in double field theory*, *JHEP* **04** (2014) 141 [arXiv:1312.0719] [INSPIRE].

[15] H.J. Lipkin, W.I. Weisberger and M. Peshkin, *Magnetic charge quantization and angular momentum*, *Annals Phys.* **53** (1969) 203 [INSPIRE].

[16] M.J.P. Gingras, *Observing monopoles in a magnetic analog of ice*, *Science* **326** (2009) 375 [arXiv:1005.3557].

[17] K. Johnson and F.E. Low, *Current algebras in a simple model*, *Prog. Theor. Phys. Suppl.* **37** (1966) 74 [INSPIRE].

[18] F. Buccella, G. Veneziano, R. Gatto and S. Okubo, *Necessity of additional unitary-antisymmetric q-number terms in the commutators of spatial current components*, *Phys. Rev.* **149** (1966) 1268.

[19] S.G. Jo, *Commutators in an anomalous non-Abelian chiral gauge theory*, *Phys. Lett.* **B 163** (1985) 353 [INSPIRE].

[20] M. Günaydin and B. Zumino, *Magnetic charge and non-associative algebras*, in *Symposium to honor G.C. Wick*, Pisa Italy (1984) [INSPIRE].

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[21] M. Günaydin and D. Minic, *Nonassociativity, Malcev algebras and string theory*, *Fortsch. Phys.* **61** (2013) 873 [arXiv:1304.0410] [INSPIRE].

[22] R. Moufang, *Alternativekörper und der Satz vom vollständigen Vierseit* (in German), *Abh. Math. Sem. Univ. Hamburg* **9** (1933) 207.

[23] M. Günaydin, C. Piron and H. Ruegg, *Moufang plane and octonionic quantum mechanics*, *Commun. Math. Phys.* **61** (1978) 69 [INSPIRE].

[24] R. Haag, *Local quantum physics*, Springer-Verlag, Berlin, Heidelberg Germany and New York U.S.A. (1992).

[25] W. Thirring, *Quantum mathematical physics*, Springer, New York U.S.A. (2002).

[26] M. Bojowald and A. Skirzewski, *Effective equations of motion for quantum systems*, *Rev. Math. Phys.* **18** (2006) 713 [math-ph/0511043] [INSPIRE].

[27] M. Bojowald and A. Skirzewski, *Quantum gravity and higher curvature actions*, in *Proceedings of "Current Mathematical Topics in Gravitation and Cosmology" (42nd Karpacz Winter School of Theoretical Physics)*, A. Borowiec and M. Francaviglia eds., *eConf* **C 0602061** (2006) 03 [*Int. J. Geom. Meth. Mod. Phys.* **4** (2007) 25] [hep-th/0606232] [INSPIRE].

[28] M. Bojowald and A. Tsobanjan, *Effective Casimir conditions and group coherent states*, *Class. Quant. Grav.* **31** (2014) 115006 [arXiv:1401.5352] [INSPIRE].

[29] D. Brizuela, *Statistical moments for classical and quantum dynamics: formalism and generalized uncertainty relations*, *Phys. Rev.* **D 90** (2014) 085027 [arXiv:1410.5776] [INSPIRE].

[30] M. Bojowald and A. Kempf, *Generalized uncertainty principles and localization of a particle in discrete space*, *Phys. Rev.* **D 86** (2012) 085017 [arXiv:1112.0994] [INSPIRE].

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