

Notes on the BMS group in three dimensions: II. Coadjoint representation

Glenn Barnich and Blagoje Oblak¹

*Physique Théorique et Mathématique,
Université Libre de Bruxelles and International Solvay Institutes,
Campus Plaine C.P. 231, B-1050 Bruxelles, Belgium*

E-mail: gbarnich@ulb.ac.be, boblak@ulb.ac.be

ABSTRACT: The coadjoint representation of the BMS_3 group, which governs the covariant phase space of three-dimensional asymptotically flat gravity, is investigated. In particular, we classify coadjoint BMS_3 orbits and show that intrinsic angular momentum is free of supertranslation ambiguities. Finally, we discuss the link with induced representations upon geometric quantization.

KEYWORDS: Classical Theories of Gravity, Space-Time Symmetries, Differential and Algebraic Geometry, Conformal and W Symmetry

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¹Research Fellow of the Fund for Scientific Research-FNRS Belgium.

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1 Introduction

Coadjoint orbits of semi-direct product groups with an Abelian factor are well-understood [1–5]. Their classification involves the same little groups that appear in the context of “induced representations”, that is, the construction of unitary irreducible representations of the full group from those of the little groups.

The purpose of the present paper is to apply this classification to the centrally extended \widehat{BMS}_3 group and elaborate on its relation with asymptotically flat solutions to Einstein’s equations in three dimensions. Indeed, it follows from the considerations in [6–8] that the reduced phase space of three-dimensional asymptotically flat gravity at null infinity coincides with the coadjoint representation of \widehat{BMS}_3 at fixed central charges $c_1 = 0, c_2 = 3/G$. As a consequence, this solution space consists of coadjoint orbits of \widehat{BMS}_3 . There is thus a close relation between classical gravitational solutions and unitary irreducible representations of their symmetry group, or “ BMS_3 particles” in the terminology of [9]. Our objective here is to make this relation precise.

The plan of the paper is the following. We start, in section 2, by reviewing the coadjoint representation of semi-direct product groups and the classification of their coadjoint orbits. As an application, the case of the Poincaré group in three dimensions is briefly discussed. Section 3 is devoted to a description of the coadjoint representation of \widehat{BMS}_3 and its relation

to three-dimensional asymptotically flat spacetimes at null infinity. In particular, the full understanding of the coadjoint orbits is used to complete the positive energy theorem for asymptotically flat three-dimensional spacetimes [10] by a discussion of angular momentum in this context. We end in section 4 by discussing the link between geometric quantization and induced representations following [1–4], and apply these considerations to the case of the $\widehat{\text{BMS}}_3$ group.

Throughout this work, we will use notations, conventions and results of [9], except for the fact that the dual of the action involved in the semi-direct product, σ^* , will always be written explicitly.

Specific coadjoints orbits of related (conformal) Carroll groups have recently also been discussed in [11, 12].

2 Coadjoint orbits of semi-direct products

In this section we consider a semi-direct product group $H = G \ltimes_{\sigma} A$, with G a Lie group, A an Abelian vector group, and σ a smooth representation of G in A . For simplicity, we will restrict the discussion here to finite-dimensional Lie groups.

2.1 Coadjoint representation

The Lie algebra of H is $\mathfrak{h} = \mathfrak{g} \oplus_{\Sigma} A$ and the adjoint action of H reads

$$\text{Ad}_{(f,\alpha)}(X, \beta) = (\text{Ad}_f X, \sigma_f \beta - \Sigma_{\text{Ad}_f X} \alpha) \quad \forall (f, \alpha) \in H, \quad \forall (X, \beta) \in \mathfrak{h}.$$

The dual space of \mathfrak{h} is $\mathfrak{h}^* = \mathfrak{g}^* \oplus A^*$, whose elements, denoted as (j, p) with $j \in \mathfrak{g}^*$ and $p \in A^*$, are paired with \mathfrak{h} according to¹

$$\langle (j, p), (X, \alpha) \rangle = \langle j, X \rangle + \langle p, \alpha \rangle.$$

Writing down the coadjoint action of H requires some additional notation [1, 4]: a bilinear “cross” product $\times : A \times A^* \rightarrow \mathfrak{g}^* : (\alpha, p) \mapsto \alpha \times p$ is defined by

$$\langle \alpha \times p, X \rangle := \langle p, \Sigma_X \alpha \rangle \quad \forall X \in \mathfrak{g}. \tag{2.1}$$

The notation is justified by the fact that, when H is the Euclidean group in three dimensions, $\alpha \times p$ can be identified with the usual cross product in \mathbb{R}^3 . With this definition, the coadjoint action of H is given by

$$\text{Ad}_{(f,\alpha)}^*(j, p) = (\text{Ad}_f^* j + \alpha \times \sigma_f^* p, \sigma_f^* p), \tag{2.2}$$

where σ^* denotes the dual representation associated with σ , while the Ad^* ’s on the right-hand side denote the coadjoint representation of G . More generally, the notation Ad^* will be reserved for the coadjoint representations of both H and G , the subscript indicating which group we are working with.

¹We use the same notation $\langle \cdot, \cdot \rangle$ for the pairings of \mathfrak{h}^* with \mathfrak{h} , of \mathfrak{g}^* with \mathfrak{g} and of A^* with A .

A special class of such semi-direct products consists of groups of the form $H = G \ltimes_{\text{Ad}} \mathfrak{g}_{\text{ab}}$, with $A = \mathfrak{g}_{\text{ab}}$ seen as an Abelian vector group. In this case, $\alpha \times p = \text{ad}_\alpha^* p$, with an obvious abuse of notation consisting in identifying elements of \mathfrak{g}_{ab} , respectively $\mathfrak{g}_{\text{ab}}^*$, with the corresponding elements of \mathfrak{g} and \mathfrak{g}^* . We use the index “ab” to distinguish the dual space of the Abelian algebra from that of the non-Abelian one. This class includes the Euclidean group in three dimensions, the Poincaré group in three dimensions and the $\widehat{\text{BMS}}_3$ group. The dual of \mathfrak{h} then becomes $\mathfrak{h}^* = \mathfrak{g}^* \oplus \mathfrak{g}_{\text{ab}}^*$ and the coadjoint action (2.2) reduces to

$$\text{Ad}_{(f,\alpha)}^*(j,p) = (\text{Ad}_f^* j + \text{ad}_\alpha^* \text{Ad}_f^* p, \text{Ad}_f^* p).$$

2.2 Coadjoint orbits

The coadjoint orbit of $(j,p) \in \mathfrak{h}^*$ is the set

$$\mathcal{W}_{(j,p)} = \left\{ \text{Ad}_{(f,\alpha)}^*(j,p) \mid (f,\alpha) \in H \right\} \subset \mathfrak{h}^*,$$

with $\text{Ad}_{(f,\alpha)}^*(j,p)$ given by (2.2). The goal is to classify the coadjoint orbits of H , assuming that the orbits and little groups

$$\mathcal{O}_p = \{ \sigma_f^* p \mid f \in G \} \cong G/G_p, \quad G_p = \{ f \in G \mid \sigma_f^* p = p \}$$

of the action σ^* are known. These are the orbits and little groups that play a key role for induced representations of semi-direct product groups. From the second half of the right-hand side of (2.2), involving only $\sigma_f^* p$, it follows that each coadjoint orbit $\mathcal{W}_{(j,p)}$ is a fibre bundle over the orbit \mathcal{O}_p , the fibre above $q = \sigma_f^* p$ being the set

$$\left\{ (\text{Ad}_g^* \text{Ad}_f^* j + \alpha \times q, q) \mid g \in G_q, \alpha \in A \right\} \subset \mathfrak{h}^*.$$

It remains to understand the geometry of these fibres and the relation between fibres at different points.

Preliminary: orbits passing through $j = 0$. Consider the first half of the right-hand side of (2.2),

$$\text{Ad}_f^* j + \alpha \times \sigma_f^* p, \tag{2.3}$$

and take $j = 0$ for now. Then, keeping $q = \sigma_f^* p$ fixed, the set spanned by elements of the form (2.3) is

$$\text{Span}_A := \{ \alpha \times q \mid \alpha \in A \} \subset \mathfrak{g}^*. \tag{2.4}$$

Now, the tangent space of \mathcal{O}_p at q can be identified with the space of “infinitesimal displacements” of q :

$$T_q \mathcal{O}_p = \{ \Sigma_X^* q \mid X \in \mathfrak{g} \} \subset A^*. \tag{2.5}$$

Note that $\Sigma_X^* q = 0$ iff X belongs to the Lie algebra \mathfrak{g}_q of the little group G_q , so that the tangent space (2.5) is isomorphic to the coset space $\mathfrak{g}/\mathfrak{g}_q$. It follows that the cotangent space $T_q^* \mathcal{O}_p$ at q is the annihilator of \mathfrak{g}_q in \mathfrak{g}^* , that is,

$$T_q^* \mathcal{O}_p = \mathfrak{g}_q^0 := \{ j \in \mathfrak{g}^* \mid \langle j, X \rangle = 0 \ \forall X \in \mathfrak{g}_q \} \subset \mathfrak{g}^*.$$

In turn, the latter space coincides with the set (2.4):

$$T_q^* \mathcal{O}_p = \mathfrak{g}_q^0 = \text{Span}_A.$$

Indeed, for all $X \in \mathfrak{g}_q$, one has $\langle \alpha \times q, X \rangle = \langle q, \Sigma_X \alpha \rangle = -\langle \Sigma_X^* q, \alpha \rangle = 0$, so $\alpha \times q \in \mathfrak{g}_q^0 = T_q^* \mathcal{O}_p$ for all $\alpha \in A$. Conversely, any element of \mathfrak{g}_q^0 can be written as $\alpha \times q$ for some $\alpha \in A$. To see this, consider the linear map $\tau_q : A \rightarrow \mathfrak{g}_q^0 : \alpha \mapsto \alpha \times q$. The image of τ_q has dimension $\dim A - \dim \text{Ker}(\tau_q)$. Since $\text{Ker}(\tau_q) = \{\alpha \in A \mid \langle \Sigma_X^* q, \alpha \rangle = 0 \ \forall [X] \in \mathfrak{g}/\mathfrak{g}_q\}$, elements of $\text{Ker}(\tau_q)$ are elements of A constrained by $\dim \mathfrak{g} - \dim \mathfrak{g}_q$ independent conditions. This implies that $\dim \text{Ker}(\tau_q) = \dim A - \dim \mathfrak{g} + \dim \mathfrak{g}_q$, so that $\dim \text{Im}(\tau_q) = \dim \mathfrak{g} - \dim \mathfrak{g}_q = \dim \mathfrak{g}_q^0$. It follows that τ_q is surjective. \square

From this we conclude, more generally, that the orbit passing through $j = 0$ is the cotangent bundle of \mathcal{O}_p :

$$\mathcal{W}_{(0,p)} = \{(\alpha \times \sigma_f^* p, \sigma_f^* p) \mid (f, \alpha) \in H\} = \bigsqcup_{q \in \mathcal{O}_p} T_q^* \mathcal{O}_p = T^* \mathcal{O}_p \subset \mathfrak{h}^*.$$

General case. It remains to understand the role of j in (2.3). Let us therefore fix some $(j, p) \in \mathfrak{h}^*$ and focus for now on elements f belonging to the little group G_p , so that $\sigma_f^*(p) = p$. With this restriction, the set of points reached by the coadjoint action of H on (j, p) is

$$\{(\text{Ad}_f^* j + \alpha \times p, p) \mid f \in G_p, \alpha \in A\} \subset \mathfrak{h}^*, \tag{2.6}$$

where in general $\text{Ad}_f^*(j) \neq j$ because the little group G_p need not be included in the stabilizer of j for the coadjoint action of G . Now, it follows from (2.1) that

$$\text{Ad}_f^*(\alpha \times p) = \sigma_f \alpha \times \sigma_f^* p.$$

Together with the requirement that f belongs to the little group at p , this property allows us to rewrite the set (2.6) as

$$\{(\text{Ad}_f^*(j + \beta \times p), p) \mid f \in G_p, \beta \in A\}. \tag{2.7}$$

Hence, in particular, translations along β allow one to modify at will all components of j that point along directions in the annihilator \mathfrak{g}_p^0 . The only piece of j that is left unchanged by the action of translations is its restriction $j_p := j|_{\mathfrak{g}_p}$ to \mathfrak{g}_p , so the set (2.7) can be rewritten as

$$\underbrace{\{\text{Ad}_f^* j_p \mid f \in G_p\}}_{\mathcal{W}_{j_p}} \times \underbrace{\{\alpha \times p \mid \alpha \in A\}}_{T_p^* \mathcal{O}_p}, \tag{2.8}$$

where $\mathcal{W}_{j_p} \subset \mathfrak{g}_p^*$ denotes the coadjoint orbit of $j_p \in \mathfrak{g}_p^*$ under the little group G_p . Thus, when $\mathcal{W}_{(j,p)}$ is seen as a fibre bundle over \mathcal{O}_p , the fibre above p is the product (2.8) of

the cotangent space of \mathcal{O}_p at p with the coadjoint orbit of the projection j_p of j under the action of the little group of p .

The same construction would hold at any other point q on \mathcal{O}_p , except that the relevant little group would be G_q . Thus, the fibre above any point $q = \sigma_f^* p \in \mathcal{O}_p$ is a product of the cotangent space of \mathcal{O}_p at q with the G_q -coadjoint orbit $\mathcal{W}_{(\text{Ad}_f^* j)_q}$, where $(\text{Ad}_f^* j)_q$ denotes the restriction of $\text{Ad}_f^* j$ to \mathfrak{g}_q . But little groups at different points of \mathcal{O}_p are isomorphic: if one chooses a group element $g_q \in G$ such that $\sigma_{g_q}^*(p) = q$, then $G_q = g_q \cdot G_p \cdot g_q^{-1}$ and $\mathfrak{g}_q = \text{Ad}_{g_q} \mathfrak{g}_p$. Therefore, $\mathcal{W}_{(\text{Ad}_f^* j)_q}$ is diffeomorphic to \mathcal{W}_{j_p} for any $q = \sigma_f^* p \in \mathcal{O}_p$; the relation between the fibres above q and p is given by the coadjoint action of H .

Classification of coadjoint orbits of H . The conclusion of the last paragraph can be used to classify the orbits of H . The *bundle of little group orbits* associated with $(j, p) \in \mathfrak{h}^*$ is defined as

$$\mathcal{B}_{(j,p)} := \left\{ \left((\text{Ad}_f^* j)_{\sigma_f^* p}, \sigma_f^* p \right) \mid f \in G \right\}. \tag{2.9}$$

According to the discussion of the previous paragraph, $\mathcal{B}_{(j,p)}$ is really the same as $\mathcal{W}_{(j,p)}$, except that the cotangent spaces at each point of \mathcal{O}_p are “neglected”. The bundle of little group orbits is thus a fibre bundle over \mathcal{O}_p , the fibre F_q at $q \in \mathcal{O}_p$ being a coadjoint orbit of the corresponding little group G_q . The relation between fibres at different points of \mathcal{O}_p is given by the coadjoint action of H , or explicitly,

$$(k, q) \in F_q \quad \text{iff} \quad \exists f \in G \text{ such that } k = (\text{Ad}_f^* j)_q \text{ and } q = \sigma_f^* p.$$

Conversely, suppose that two elements $p \in A^*$ and $j_0 \in \mathfrak{g}_p^*$ are given. The group G can be seen as a principal G_p -bundle over \mathcal{O}_p , equipped with a natural G_p -action by multiplication from the left in each fibre. In addition, G_p acts on the coadjoint orbit \mathcal{W}_{j_0} , so one can define an action of G_p on $G \times \mathcal{W}_{j_0}$ by

$$(f, k) \in G \times \mathcal{W}_{j_0} \xrightarrow{g \in G_p} (g \cdot f, \text{Ad}_g^*(k)).$$

The corresponding bundle of little group orbits $\mathcal{B}_{(j_0,p)}$ is defined as

$$\mathcal{B}_{(j_0,p)} := (G \times \mathcal{W}_{j_0}) / G_p. \tag{2.10}$$

Thus, from each coadjoint orbit of H , one can build a bundle of little group orbits (2.9); conversely, from each bundle of little group orbits as defined in (2.10), one can build a coadjoint orbit of H by choosing any $j \in \mathfrak{g}^*$ such that $j_p = j_0$ and taking the orbit $\mathcal{W}_{(j,p)}$. In other words, the classification of coadjoint orbits of H is equivalent to the classification of bundles of little group orbits [1, 4].

This yields the complete picture of coadjoint orbits of H : each coadjoint orbit $\mathcal{W}_{(j,p)}$ is a fibre bundle over \mathcal{O}_p , the fibre above $q \in \mathcal{O}_p$ being a product of $T_q^* \mathcal{O}_p$ with a coadjoint orbit of the corresponding little group G_q . Equivalently, $\mathcal{W}_{(j,p)}$ is a fibre bundle over $T^* \mathcal{O}_p$, the fibre above $(q, \alpha \times q) \in T^* \mathcal{O}_p$ being a coadjoint orbit of G_q . To exhaust all coadjoint orbits of H , one proceeds as follows:

1. Pick an element $p \in A^*$ and compute its orbit \mathcal{O}_p under the action σ^* of G ;
2. Find the corresponding little group G_p ;
3. Pick $j_p \in \mathfrak{g}_p^*$ and compute its coadjoint orbit under the action of G_p .

The set of all orbits \mathcal{O}_p and of all coadjoint orbits of the corresponding little groups classifies the coadjoint orbits of H . Put differently, suppose one has classified the following objects:

1. The orbits of G for the action σ^* , with orbit representatives $p_\lambda \in A^*$ and corresponding little groups G_λ , $\lambda \in \mathcal{I}$ being some index such that \mathcal{O}_{p_λ} and $\mathcal{O}_{p_{\lambda'}}$ are disjoint whenever $\lambda \neq \lambda'$;
2. The coadjoint orbits of each G_λ , with orbit representatives $j_{\lambda,\mu} \in \mathfrak{g}_\lambda^*$, $\mu \in \mathcal{J}_\lambda$ being some index such that $\mathcal{W}_{j_{\lambda,\mu}}$ and $\mathcal{W}_{j_{\lambda,\mu'}}$ are disjoint whenever $\mu \neq \mu'$.

Then, the set

$$\{ (j_{\lambda,\mu}, p_\lambda) \mid \lambda \in \mathcal{I}, \mu \in \mathcal{J}_\lambda \} \subset \mathfrak{h}^*$$

forms a complete set of representatives for the collection of disjoint coadjoint orbits $\mathcal{W}_{(j_{\lambda,\mu}, p_\lambda)}$ of H . The (possibly continuous) indices λ, μ label the orbits uniquely.

2.3 Poincaré group in three dimensions

The double cover of the Poincaré group in three dimensions is

$$\mathrm{SL}(2, \mathbb{R}) \ltimes_{\mathrm{Ad}} \mathfrak{sl}(2, \mathbb{R})_{\mathrm{ab}}, \tag{2.11}$$

with $\mathrm{SL}(2, \mathbb{R})$ the double cover of the connected Lorentz group in three dimensions, and $\mathfrak{sl}(2, \mathbb{R})_{\mathrm{ab}}$ isomorphic to the Abelian group of translations. The dual of the Poincaré algebra consists of pairs (j, p) , where both j and p belong to $\mathfrak{sl}(2, \mathbb{R})^*$. One may refer to p as a momentum vector and to j as an angular momentum vector. The projection j_p of j on \mathfrak{g}_p^* is the classical analogue of intrinsic spin, while the components of j that can be varied through translations $\alpha \times p$ represent orbital angular momentum.

The coadjoint orbits of (2.11) are classified by the general results of subsection 2.2. Let therefore $\{p_\lambda \mid \lambda \in \mathcal{I}\}$ be an exhaustive set of representatives for the coadjoint orbits of $\mathrm{SL}(2, \mathbb{R})$, see e.g. [13–15]. For example, take $\mathcal{I} = \mathbb{R} \cup i\mathbb{R}_0^+ \cup \{e^{\pm i\pi/4}\}$ with $\lambda = 0$ corresponding to the vanishing momentum; $p_0 = \lambda \in \mathbb{R}_0^\pm$, $p_i = 0$ corresponding to positive (negative) energy and positive mass squared; $p_0 = 0$, $p_1 = 0$, $p_2 = i\lambda$, $\lambda \in i\mathbb{R}_0^+$, corresponding to negative mass squared; and $p_2 + ip_0 = e^{\pm i\pi/4}$, $p_1 = 0$ corresponding to massless momenta with positive or negative energy.

Whenever $p_\lambda \neq 0$, the little group G_λ is Abelian and one-dimensional, so that $\mathfrak{g}_\lambda^* \cong \mathbb{R}$ and the coadjoint action of G_λ is trivial. The index μ in the set $\{j_{\lambda,\mu}\}$ then runs over all real values, labelling the component of $j_{\lambda,\mu}$ along the direction \mathfrak{g}_λ^* in $\mathfrak{sl}(2, \mathbb{R})^*$. We will denote this component by s , for “spin”. Hence, whenever $p_\lambda \neq 0$, the coadjoint orbit of $(j_{\lambda,\mu}, p_\lambda)$ under the Poincaré group is diffeomorphic to the cotangent bundle $T^*\mathcal{O}_{p_\lambda}$. However, two such orbits having the same index λ , but different indices μ (i.e. different spins s of $j_{\lambda,\mu}$ along \mathfrak{g}_λ^*), are disjoint. The only orbits left are those containing $p = 0$. These are all of the form $\mathcal{W}_j \times \{0\} \cong \mathcal{W}_j$, where \mathcal{W}_j is the coadjoint orbit of $j \in \mathfrak{sl}(2, \mathbb{R})^*$ under $\mathrm{SL}(2, \mathbb{R})$.

3 Coadjoint orbits of BMS_3

As before, we use the notations and conventions of [9], to which we refer for a definition of the BMS_3 group (and its central extension) and the construction of its induced representations. The purpose of this section is to classify the coadjoint orbits of the BMS_3 group and to establish the link of this classification with three-dimensional gravity. In particular, angular momentum is studied in some detail.

3.1 Generalities on the BMS_3 group

The centrally extended $\widehat{\text{BMS}}_3$ group is of the form $G \times_{\text{Ad}} \mathfrak{g}_{\text{ab}}$ with G the universal cover of the Virasoro group. The dual of the $\widehat{\mathfrak{bms}}_3$ algebra is the space $\widehat{\text{Vect}}(S^1)^* \oplus \widehat{\text{Vect}}(S^1)^*_{\text{ab}}$, whose elements are quadruples $(j, ic_1; p, ic_2)$, where c_1 and c_2 are central charges while the supermomentum p and the angular supermomentum j are quadratic differentials on the circle. The pairing with elements of $\widehat{\mathfrak{bms}}_3$ is explicitly given by

$$\langle (j, ic_1; p, ic_2), (X, -ia; \alpha, -ib) \rangle = \int_0^{2\pi} d\phi (j(\phi)X(\phi) + p(\phi)\alpha(\phi)) + c_1 a + c_2 b.$$

Accordingly, the coadjoint action of the $\widehat{\mathfrak{bms}}_3$ algebra is

$$\text{ad}^*_{(X,\alpha)}(j, ic_1; p, ic_2) = (\delta j d\phi^2, 0; \delta p d\phi^2, 0), \tag{3.1}$$

with

$$\delta p = Xp' + 2X'p - \frac{c_2}{24\pi}X''', \quad \delta j = Xj' + 2X'j - \frac{c_1}{24\pi}X''' + \alpha p' + 2\alpha'p - \frac{c_2}{24\pi}\alpha'''. \tag{3.2}$$

The corresponding² coadjoint representation of the $\widehat{\text{BMS}}_3$ group is given by

$$\text{Ad}^*_{(f,\alpha)^{-1}}(j, ic_1; p, ic_2) = \left(\tilde{j}d\phi^2, ic_1; \tilde{p}d\phi^2, ic_2 \right), \tag{3.3}$$

where

$$\tilde{p} = (f')^2 p \circ f - \frac{c_2}{24\pi} S[f], \quad \tilde{j} = (f')^2 \left[j + \alpha p' + 2\alpha'p - \frac{c_2}{24\pi} \alpha''' \right] \circ f - \frac{c_1}{24\pi} S[f], \tag{3.4}$$

and $S[f] = f'''/f' - \frac{3}{2}(f''/f')^2$ denotes the Schwarzian derivative of f .

3.2 Coadjoint orbits

As for the Poincaré group, the coadjoint orbits of $\widehat{\text{BMS}}_3$ are classified according to the general results of subsection 2.2. Due to the structure $G \times \mathfrak{g}_{\text{ab}}$ of $\widehat{\text{BMS}}_3$, the orbits denoted \mathcal{O}_p in section 2 are the well-known coadjoint orbits of the Virasoro group, see e.g. [15–19]. Nevertheless, we will keep calling these orbits “orbits of the action σ^* ” in order to distinguish them from the coadjoint orbits of $\widehat{\text{BMS}}_3$ itself.

For a given $(j, ic_1; p, ic_2) \in \widehat{\mathfrak{bms}}_3^*$, the coadjoint orbit $\mathcal{W}_{(j,ic_1;p,ic_2)}$ is thus a bundle over the cotangent bundle of the orbit $\mathcal{O}_{(p,ic_2)}$, the typical fibre being a coadjoint orbit of the

²More precisely, (3.1)–(3.2) is the differential of (3.3)–(3.4) up to an overall minus sign.

corresponding little group. In the case at hand, each $\mathcal{O}_{(p,ic_2)}$ is a coadjoint orbit of the Virasoro group. Because the structure of Virasoro coadjoint orbits depends crucially on the (non-)vanishing of the central charge c_2 , we will focus first on the case that is relevant for three-dimensional gravity, namely $c_2 \neq 0$.

A generic orbit $\mathcal{O}_{(p,ic_2)}$ then has a one-dimensional (Abelian) little group $G_{(p,ic_2)}$, whose coadjoint representation is trivial. In particular, as in the Poincaré group, little group orbits consist of only one point, specified by the real value (s, ic_1) of $(j, ic_1)_{(p,ic_2)}$ in $\mathfrak{g}_{(p,ic_2)}^*$; the value s can again be considered as the classical analogue of spin. Hence, for generic supermomenta p , the coadjoint orbit $\mathcal{W}_{(j,ic_1;p,ic_2)}$ is diffeomorphic to the cotangent bundle $T^*\mathcal{O}_{(p,ic_2)}$ and is specified by (i) the value of the central charges c_1 and $c_2 \neq 0$, (ii) the supermomentum p , and (iii) the spin s .

Still working at non-zero c_2 , we also need to consider “exceptional” (non-generic) orbits $\mathcal{O}_{(p,ic_2)}$ of constant supermomenta satisfying $p = -n^2 c_2 / 48\pi$ for some positive integer n . For such orbits, the little group is the n -fold cover of $\text{PSL}(2, \mathbb{R})$, so, in contrast to generic orbits, the little group’s coadjoint representation is not trivial. The coadjoint orbit $\mathcal{W}_{(j,ic_1;p,ic_2)}$ then is a fibre bundle over $T^*\mathcal{O}_{(p,ic_2)}$, having a coadjoint orbit of $\text{PSL}^{(n)}(2, \mathbb{R})$ as its typical fibre.

The case of vanishing c_2 is more intricate, because then the little group may have arbitrary dimension (see e.g. the summary in [20], section 2.2). We will not consider this situation in full generality here. Let us only mention one special case: take $c_2 = 0$ and $p = 0$ and consider the coadjoint orbit $\mathcal{W}_{(j,ic_1;0,0)}$, which is diffeomorphic to a coadjoint orbit of the Virasoro group for central charge c_1 and quadratic differential j . The central charge c_1 then plays a crucial role, in contrast to the case $c_2 \neq 0$ discussed above. Such orbits are the BMS_3 analogue of the Poincaré orbits $\mathcal{W}_{(j,0)}$.

3.3 Covariant phase space of asymptotically flat gravity

Preliminary: the AdS case. The general solution of Einstein’s equations in three dimensions with negative cosmological constant $\Lambda = -1/\ell^2$ and Brown-Henneaux boundary conditions [21] can be written as [22, 23]

$$ds^2 = \frac{\ell^2}{r^2} dr^2 - r^2 \left(dx^+ - \frac{8\pi G \ell}{r^2} L^- dx^- \right) \left(dx^- - \frac{8\pi G \ell}{r^2} L^+ dx^+ \right) \quad (3.5)$$

in terms of a radial coordinate $r \in \mathbb{R}^+$ and light-cone coordinates $x^\pm = t/\ell \pm \phi$ on the cylinder, where $L^+(x^+)$ and $L^-(x^-)$ are arbitrary, smooth, 2π -periodic functions. Under the action of conformal transformations of the cylinder at infinity, these functions transform according to the coadjoint representation of the Virasoro group with central charges

$$c^\pm = 3\ell/2G. \quad (3.6)$$

In this sense, the space of solutions of Einstein gravity on AdS_3 coincides with the hyperplane, at fixed central charges (3.6), of the dual space of two copies of the Virasoro algebra. In particular, solutions of AdS_3 gravity are classified by Virasoro coadjoint orbits [24, 25], and this classification defines a symplectic foliation of the space of solutions. From the point

of view of the AdS/CFT correspondence, this property should not appear as a surprise, given that the operator dual to the bulk metric in AdS/CFT is the energy-momentum tensor, which, in two-dimensional conformal field theories, transforms precisely under the coadjoint representation of the Virasoro group.

The flat case. A similar classification can be implemented for asymptotically flat spacetimes. Indeed, in BMS coordinates (r, u, ϕ) , the general solution of Einstein's equations in three dimensions describing asymptotically flat spacetimes at null infinity is given by metrics [7]

$$ds^2 = \Theta du^2 - 2dudr + (2\Xi + u\Theta') dud\phi + r^2 d\phi^2$$

depending on two arbitrary functions on the circle, $\Theta = \Theta(\phi)$ and $\Xi = \Xi(\phi)$. Under finite BMS_3 transformations acting on the cylinder at null infinity, these functions have been shown [8, 26] to transform according to the coadjoint representation (3.1)–(3.4) upon identifying $\Theta = (16\pi G)p$ and $\Xi = (8\pi G)j$, with central charges

$$c_1 = 0, \quad c_2 = 3/G. \tag{3.7}$$

Thus, just as for AdS_3 space-times, the space of solutions of Einstein's equations with suitable flat boundary conditions at null infinity³ is a hyperplane, at fixed central charges (3.7), in the dual space of the $\widehat{\mathfrak{bms}}_3$ algebra. As a consequence, these solutions can be classified according to $\widehat{\text{BMS}}_3$ coadjoint orbits, and this classification again splits solution space into disjoint symplectic leaves. For example, the solution corresponding to $\Theta = -n^2$ and $\Xi = 0$ represents Minkowski spacetime for $n = 1$, and a conical excess of $2\pi n$ for $n > 1$. Its coadjoint orbit $\mathcal{W}_{(0,0;p,ic_2)}$ is diffeomorphic to the cotangent bundle of the Virasoro orbit $\text{Diff}^+(S^1)/\text{PSL}^{(n)}(2, \mathbb{R})$. Other zero-mode solutions (that is, solutions specified by constant Θ and Ξ without $\Theta = -n^2$) represent cosmological solutions, angular defects or angular excesses, depending on the sign of Θ and $\Theta + 1$ [29–33]; when seen as elements of $\widehat{\mathfrak{bms}}_3^*$, their coadjoint orbits are all diffeomorphic to the cotangent bundle of $\text{Diff}^+(S^1)/S^1$.

Surface charge algebra revisited. The identification of the space of solutions with the coadjoint representation of the asymptotic symmetry group can be understood from the expression of the surface charges:⁴ for $(X, \alpha) \in \mathfrak{bms}_3$, the latter are given by [7]

$$Q_{(X,\alpha)}[\Xi, \Theta] = \frac{1}{16\pi G} \int_0^{2\pi} d\phi [2\Xi(\phi)X(\phi) + \Theta(\phi)\alpha(\phi)] = \langle (j, p), (X, \alpha) \rangle, \tag{3.8}$$

when writing $\Theta = 16\pi Gp$ and $\Xi = 8\pi Gj$. In turn, this gives a physical interpretation to the coadjoint vectors p and j of $\widehat{\mathfrak{bms}}_3^*$ in the present context: they represent Bondi mass and angular momentum aspects.

For any Lie group G with Lie algebra \mathfrak{g} , there is a natural Poisson bracket on \mathfrak{g}^* , defined, for any pair of smooth functions $\Phi, \Psi : \mathfrak{g}^* \rightarrow \mathbb{R}$, as

$$\{\Phi, \Psi\}(j) := \langle j, [d\Phi_j, d\Psi_j] \rangle \quad \forall j \in \mathfrak{g}^*. \tag{3.9}$$

³Other boundary boundary conditions are of course possible, leading either to a more restricted symmetry and dynamical structure [27, 28] or to an enhancement with Weyl symmetry [7].

⁴A similar observation also holds in the AdS case.

Alternatively, in terms of coordinates x_a on \mathfrak{g}^* ,

$$\{x_a, x_b\} = f_{ab}^c x_c. \tag{3.10}$$

For the linear maps $\langle (j, ic_1; p, ic_2), (X, -ia; \alpha, -ib) \rangle$ parametrized by Lie algebra elements $(X, -ia; \alpha, -ib)$, the Poisson bracket (3.9) yields

$$\begin{aligned} & \{ \langle (j, ic_1; p, ic_2), (X_1, -ia_1; \alpha_1, -ib_1) \rangle, \langle (j, ic_1; p, ic_2), (X_2, -ia_2; \alpha_2, -ib_2) \rangle \} \\ &= \langle (j, ic_1; p, ic_2), [(X_1, -ia_1; \alpha_1, -ib_1), (X_2, -ia_2; \alpha_2, -ib_2)] \rangle. \end{aligned}$$

When $c_1 = 0, c_2 = 3/G$, these brackets correspond precisely to the Dirac brackets of the surface charges (3.8) as computed in [6]. Equivalently, in terms of coordinates $j(\phi), p(\phi), c_1, c_2$ on $\widehat{\mathfrak{bms}}_3^*$, (3.10) becomes

$$\begin{aligned} \{j(\phi), j(\phi')\} &= (j(\phi) + j(\phi')) \partial_\phi \delta(\phi - \phi') - \frac{c_1}{24\pi} \partial_\phi^3 \delta(\phi - \phi'), \\ \{p(\phi), j(\phi')\} &= (p(\phi) + p(\phi')) \partial_\phi \delta(\phi - \phi') - \frac{c_2}{24\pi} \partial_\phi^3 \delta(\phi - \phi'), \\ \{p(\phi), p(\phi')\} &= 0, \end{aligned}$$

while brackets involving c_1, c_2 vanish.

3.4 Energy and angular momentum

Classifying the space of solutions of three-dimensional gravity according their asymptotic symmetries allows one to study properties of energy and angular momentum, such as boundedness for instance [10, 34] (see also for [35] for other recent considerations). Both in the AdS₃ and in the flat case, energy turns out to be related to a zero-mode of a coadjoint vector of the Virasoro group, and the boundedness properties of this zero-mode are known [15, 18].

In the AdS₃ case, with general solution (3.5) and central charges (3.6), the only solutions that belong to Virasoro orbits whose energy is bounded from below are specified by pairs (L^+, L^-) in which both functions L^\pm belong either to the orbit of a constant $L_{\text{cst}}^\pm \geq -c^\pm/48\pi$, or to the orbit of the “future-directed, massless deformation” of $-c^\pm/48\pi$. This class of solutions contains, in particular, all BTZ black holes, but also conical defects and solutions containing closed time-like curves. On the Virasoro orbits of all such solutions, the zero-mode L_0^\pm of L^\pm is bounded from below either by the value $2\pi L_{\text{cst}}^\pm$, or (in the case of the massless orbit) by the vacuum energy $-c^\pm/24$. In particular, since energy (E) and angular momentum (J) are related to the zero-modes of L^\pm by

$$E = \frac{1}{\ell} (L_0^+ + L_0^-), \quad J = L_0^+ - L_0^-,$$

all solutions of AdS₃ gravity that belong to orbits with energy bounded from below have their angular momentum bounded by

$$|J| \leq \ell E + c^\pm/12. \tag{3.11}$$

Similarly, solutions belonging to the orbit of BTZ black holes, which correspond to the case where L^+ and L^- are positive constants, all satisfy the cosmic censorship bound $|J| \leq \ell E$.

In the case of asymptotically flat gravity, a natural definition of total energy and angular momentum is then also simply given by the zero modes of p and j ,

$$E = \int_0^{2\pi} d\phi p(\phi), \quad J = \int_0^{2\pi} d\phi j(\phi). \quad (3.12)$$

Following [36], chapters 19 and 20, a justification for this definition goes as follows. Total momentum and total angular momentum are the surface charges associated with the translation and the Lorentz Killing vectors of the asymptotically Lorentz frame, for which $u = x^0 - r$, $re^{i\phi} = x^1 + ix^2$ (see section 5.3 of [9] for more details). In particular, total energy and the total rotation vector, which has but one component in three dimensions, are associated to $\partial/\partial x^0 = \partial/\partial u$ and $x^1\partial/\partial x^2 - x^2\partial/\partial x^1 = \partial/\partial\phi$, which yields (3.12) when used in (3.8).

The boundedness properties of total energy on coadjoint orbits of $\widehat{\text{BMS}}_3$ follows from the transformation law (3.3)–(3.4). Because p transforms as a Virasoro coadjoint vector, without any influence of j or c_1 , the energy E has the same boundedness properties as L_0^\pm , but with central charge c_2 given in (3.7). Thus, E is bounded from below on the orbit of p iff this orbit is either that of a constant $p_{\text{cst}} \geq -c_2/48\pi$, or that of the future-directed, massless deformation of $-c_2/48\pi$. In particular, all cosmological solutions and all conical defects in flat space belong to orbits on which energy is bounded from below. Energy is also bounded from below on the orbit of Minkowski space (corresponding to $j = 0$ and $p = -c_2/48\pi$), which realizes the minimum value of energy, $E_{\text{min}} = -c_2/24$.

In order to discuss properties of total angular momentum, we define, as in the Poincaré case, intrinsic angular momentum as total angular computed in the rest frame. A solution labelled by (j, p) is put in its rest frame if the supermomentum $p(\phi)$ is brought to a constant p_{cst} by using a suitable superrotation. This is of course not possible on solutions whose p belongs to a Virasoro coadjoint orbit without constant representative. By integrating on the circle the piece $\alpha \times p = \text{ad}_\alpha^*(p)$ in (3.4), it then follows that:

Intrinsic angular momentum is free from supertranslation ambiguities.

By contrast, whenever both $p(\phi)$ and $\alpha(\phi)$ are non-constant on the circle, meaning in particular that the applied supertranslation is not just a time translation, $\text{ad}_\alpha^*(p)$ has a generally non-vanishing zero-mode and contributes to total angular momentum.

Considering the boundedness properties of the total angular momentum J as such does not make sense. Whenever $p(\phi)$ is non-constant on the circle, the value of J can be tuned at will by acting with supertranslations. In particular, total angular momentum is unbounded from above and from below on all orbits \mathcal{O}_p . This should be contrasted with the completely different situation in AdS_3 spacetimes, where separate boundedness properties for the left- and right-moving energies imply boundedness of total angular momentum, as in eq. (3.11).

As regards intrinsic angular momentum, the situation is different. By construction, supertranslations play no role there. In the rest frame, the only superrotations that are

still allowed are those of the little group $G_{p_{\text{cst}}}$. So for boundedness properties of intrinsic angular momentum, one needs to study its boundedness properties on coadjoint orbits of the little group.

Let us illustrate our purposes with two examples. First, consider the orbit \mathcal{O}_p of a massive BMS_3 solution, that is, an orbit containing a constant supermomentum $p(\phi) = p_{\text{cst}} > -c_2/48\pi$. The corresponding little group $\text{U}(1)$ consists of rigid rotations $f(\phi) = \phi + \text{cst}$, and it follows from (3.4) that intrinsic angular momentum is unaffected by such superrotations.

Second, consider the vacuum supermomentum vector $p = -c_2/48\pi$, whose little group is $\text{PSL}(2, \mathbb{R})$. It follows from the discussion of section 5.3 of [9] that the coadjoint orbits of $\text{PSL}(2, \mathbb{R})$ are the “mass hyperboloids” of the Lorentz group in three dimensions, represented in a three-dimensional space with axes $J_0 = J$, $J_{\pm 1} = \int_0^{2\pi} d\phi j e^{\pm i\phi}$. Depending on the $\text{PSL}(2, \mathbb{R})$ -orbit, the boundedness properties of J are very different. The trivial case of the vacuum orbit $J_0 = J_{\pm 1} = 0$ brings nothing new, as it is left invariant by the whole little group $\text{PSL}(2, \mathbb{R})$; intrinsic angular momentum vanishes and total angular momentum is entirely composed of orbital angular momentum. By contrast, consider the “future-directed, massive orbit” of $\text{PSL}(2, \mathbb{R})$, the upper half of the two-sheeted hyperboloid. Then, little group transformations act non-trivially on J but intrinsic angular momentum is bounded from below by the intersection of the hyperboloid with the J_0 axis. It may, however, take an arbitrarily large value. Similar results hold for the upper conical orbits of $\text{PSL}(2, \mathbb{R})$. For the one-sheeted hyperboloid, the “tachyonic” orbit, intrinsic angular momentum is bounded neither from below, nor from above.

4 Quantization and induced representations

In this section we discuss the relation between classical asymptotically flat solutions and BMS_3 particles, i.e. the link between coadjoint and induced representations of $\widehat{\text{BMS}}_3$, in terms of geometric quantization.

4.1 Geometric quantization for semi-direct products

Generalities on the orbit method. Let G be a Lie group with Lie algebra \mathfrak{g} . The Poisson bracket (3.9) defines a symplectic foliation of \mathfrak{g}^* , the leaf through $j \in \mathfrak{g}^*$ being the coadjoint orbit \mathcal{W}_j of j . For any k belonging to \mathcal{W}_j , the tangent space of \mathcal{W}_j at k can be identified with the space of “infinitesimal displacements” $\text{ad}_X^*(k)$, where $X \in \mathfrak{g}$. The bracket (3.9) then induces, on each orbit \mathcal{W}_j , a G -invariant symplectic form ω given by [37–39]

$$\omega_k(\text{ad}_X^*(k), \text{ad}_Y^*(k)) := k([X, Y]). \tag{4.1}$$

Geometric quantization associates a quantum Hilbert space with the phase space \mathcal{W}_j , proceeding in two steps: prequantization and polarization.

Prequantization turns out to be possible provided the symplectic form satisfies the integrality condition

$$\left[\frac{\omega}{2\pi} \right] \in H_{\text{de Rham}}^2(\mathcal{W}_j, \mathbb{Z}), \tag{4.2}$$

in which case there exists a complex line bundle over \mathcal{W}_j , endowed with a connection whose curvature two-form is $\omega/2\pi$. The pre-quantum Hilbert space consists of all sections of this line bundle that are square-integrable with respect to a Hermitian structure preserved by the connection. This Hilbert space is then reduced to a smaller subspace by choosing a polarization – an appropriate subbundle of the complexified tangent bundle of the symplectic manifold – and restricting both the quantizable observables and the quantum wavefunctions to be compatible with this polarization. The idea of the orbit method is that, when a suitable polarization can be found, geometric quantization should produce an irreducible unitary representation of the corresponding Lie group G . For compact or solvable Lie groups, this procedure actually exhausts all irreducible unitary representations; for other Lie groups, complications may arise, especially if the group is infinite-dimensional.

Semi-direct products. Suppose we want to quantize a coadjoint orbit $\mathcal{W}_{(j,p)}$ of the semi-direct product group H . Since the Lie bracket in \mathfrak{h} reads

$$[(X, \beta), (Y, \gamma)] = ([X, Y], \Sigma_X \gamma - \Sigma_Y \beta),$$

the natural symplectic form (4.1), evaluated at the point $(\text{Ad}_f^* j + \alpha \times \sigma_f^* p, \sigma_f^* p)$ in $\mathcal{W}_{(j,p)}$, is given by⁵

$$\omega_{(\text{Ad}_f^* j + \alpha \times q, q)}((X, \beta), (Y, \gamma)) = \langle \text{Ad}_f^* j, [X, Y] \rangle + \langle \gamma \times q, X \rangle - \langle \beta \times q, Y \rangle + \langle \alpha \times q, [X, Y] \rangle, \quad (4.3)$$

where we write $q = \sigma_f^* p$ for simplicity. In the three last terms of this expression, one may recognize the Liouville symplectic form on the cotangent bundle $T^*\mathcal{O}_p$, provided $\alpha \times q$ is seen as an element of $T_q^*\mathcal{O}_p$. On the other hand, the first term of (4.3) looks just like the natural symplectic form (4.1) on the G -coadjoint orbit of j up to the fact that $\text{Ad}_f^* j$ does *not*, generally, belong to \mathfrak{g}_q^* . Thus, if we see $\mathcal{W}_{(j,p)}$ as a fibre bundle over $T^*\mathcal{O}_p$ with typical fibre the coadjoint orbit \mathcal{W}_{j_p} of the little group, restricting the symplectic form (4.3) to a fibre gives back the symplectic form on the little group’s coadjoint orbit.

This observation actually follows from a more general result, which states that the coadjoint orbits of a semi-direct product are obtained by symplectic induction from the coadjoint orbits of its little groups, see [4, 40] for details. This result is of crucial importance for geometric quantization of $\mathcal{W}_{(j,p)}$. Indeed, since the Liouville symplectic form is exact, it implies that the cohomology class of ω in (4.3) depends only on the class of the natural symplectic form on the appropriate coadjoint orbit of the little group. In other words, the H -coadjoint orbit $\mathcal{W}_{(j,p)}$ is prequantizable iff the corresponding G_p -coadjoint orbit \mathcal{W}_{j_p} is prequantizable. Provided a suitable polarization can be found for this orbit, one obtains a unitary representation \mathcal{R} of the little group G_p , acting on a complex Hilbert space \mathcal{E} whose scalar product we will denote as $(\cdot | \cdot)$. One can then use the representation \mathcal{R} of G_p to “induce” a representation \mathcal{T} of H . To do so, one chooses a real polarization for sections of the trivial line bundle over $T^*\mathcal{O}_p$, such that polarized sections be functions $\mathcal{O}_p \rightarrow \mathbb{C}$.

⁵Here we write the argument of ω^H as a pair of elements of the Lie algebra of H . This is an abuse of notation, being understood that these elements represent tangent vectors of $\mathcal{W}_{(j,p)}$ at the point $(\text{Ad}_f^*(j) + \alpha \times q, q)$ through their coadjoint action on that point, as in (4.1).

Provided a G -quasi-invariant measure μ exists on \mathcal{O}_p , the Hilbert space \mathcal{H} obtained upon quantization of $\mathcal{W}_{(j,p)}$ becomes the space of square-integrable “wavefunctions” $\Psi : \mathcal{O}_p \rightarrow \mathcal{E}$, their scalar product being

$$\langle \Phi | \Psi \rangle := \int_{\mathcal{O}_p} d\mu(q) (\Phi(q) | \Psi(q)). \tag{4.4}$$

The action \mathcal{T} of H on the space of such wavefunctions then coincides with that of an induced representation [40, 41]. Thus, geometric quantization of the coadjoint orbits of a semi-direct product group reproduces induced representations in the sense of Wigner and Mackey.

We should mention, however, that not all induced representations can be recovered in this way. For instance, if the little group G_p is disconnected, it may happen that the only representations of G_p available through quantization are those in which the discrete subgroup of G_p is represented trivially [41]. Furthermore, the construction may suffer from other complications, related for instance to the non-existence of a suitable polarization.

4.2 Gravity and BMS_3 particles

We can apply the procedure outlined in the previous subsection to coadjoint orbits of the $\widehat{\text{BMS}}_3$ group, seen as phase spaces equipped with the natural symplectic form (4.3). As sketched above for the case of finite-dimensional Lie groups, geometric quantization of such orbits produces induced representations. Owing to the discussion of section 3, this relation can be rephrased in terms of solutions of Einstein’s equations in an asymptotically flat space-time in three dimensions: geometric quantization of the orbit corresponding to a solution labelled by the pair (j, p) produces a BMS_3 particle whose supermomenta span the orbit \mathcal{O}_p , and whose spin is determined by j_p . This establishes the link between the considerations of [9] and three-dimensional gravity.

There is, however, an important subtlety: in writing down the scalar product (4.4), we assumed the existence of a quasi-invariant measure μ on \mathcal{O}_p . When \mathcal{O}_p is a finite-dimensional manifold, such a measure always exists [42]. Furthermore, for semi-direct products of the form $G \ltimes_{\text{Ad}} \mathfrak{g}_{\text{ab}}$, the orbits \mathcal{O}_p are coadjoint orbits of G ; they have, therefore, a symplectic form ω given by (4.1), which can be used to define an invariant volume form proportional to $\omega^{d/2}$, where d denotes the dimension of the orbit. But in the case of $\widehat{\text{BMS}}_3$, the orbits \mathcal{O}_p are infinite-dimensional Virasoro orbits, so the question of the existence of a quasi-invariant measure is much more involved, see e.g. [43–47]. We will not study this problem here, but hope that it will be settled in the future.

5 Conclusion

In this work we have shown how the classification of coadjoint orbits of the centrally extended $\widehat{\text{BMS}}_3$ group controls solutions of asymptotically flat Einstein gravity in three dimensions. Upon geometric quantization, these orbits yield BMS_3 particles, i.e. induced representations of the $\widehat{\text{BMS}}_3$ group. This brings the understanding of the relation between group-theoretic aspects of the BMS_3 group and flat space gravity to the same level as has been achieved in the AdS_3 case in [24, 25, 34].

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