

# One-loop partition functions in deformed $\mathcal{N} = 4$ SYM theory

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**ABSTRACT:** We study the thermodynamic behaviour of the real  $\beta$ - and  $\gamma_i$ -deformation of  $\mathcal{N} = 4$  Super Yang-Mills theory on  $\mathbb{R} \times S^3$  in the planar limit. These theories were shown to be the most general asymptotically integrable supersymmetric and non-supersymmetric field-theory deformations of  $\mathcal{N} = 4$  Super Yang-Mills theory, respectively. We calculate the first loop correction to their partition functions using an extension of the dilatation-operator and Pólya-counting approach. In particular, we account for the one-loop finite-size effects which occur for operators of length one and two. Remarkably, we find that the  $\mathcal{O}(\lambda)$  correction to the Hagedorn temperature is independent of the deformation parameters, although the partition function depends on them in a non-trivial way.

**KEYWORDS:** Confinement, AdS-CFT Correspondence,  $1/N$  Expansion, Supersymmetric gauge theory

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**1 Introduction**

It is a long-standing problem in gauge theories to understand the phase transition between the weakly and strongly coupled regimes; e.g. in QCD it corresponds to the confinement-deconfinement transition. In this paper, we address the analogous issue for a class of (asymptotically) integrable deformations of  $\mathcal{N} = 4$  SYM theory on  $\mathbb{R} \times S^3$ .

The maximally supersymmetric  $\mathcal{N} = 4$  SYM theory has seen impressive advances during the last one and a half decades, in particular through the AdS/CFT correspondence [1–3] and integrability in the 't Hooft limit; see [4] for a review on the latter. This has also

spurred interest in less (super) symmetric deformations of this theory that share the properties of integrability and a dual string theory description; see [5, 6] for reviews. The prime example for continuous deformations of  $\mathcal{N} = 4$  SYM theory is the (real)  $\beta$ -deformation, a special case of the  $\mathcal{N} = 1$  exactly marginal deformations which were classified by Leigh and Strassler [7]. It can be obtained by applying a Moyal-like  $*$ -product to the ( $\mathcal{N} = 1$  superspace) action of  $\mathcal{N} = 4$  SYM theory. This  $*$ -product depends on the three Cartan charges of the original  $SU(4)$  R-symmetry and the real parameter  $\beta$ . By generalising this  $*$ -product, the non-supersymmetric  $\gamma_i$ -deformation was proposed [8], which depends on three real parameters. The  $\beta$ - and  $\gamma_i$ -deformation are respectively the most general  $\mathcal{N} = 1$  supersymmetric and non-supersymmetric, asymptotically integrable field-theory deformations of  $\mathcal{N} = 4$  SYM theory [9]. Their conjectured string-theory duals can be found by applying three TsT-transformations, i.e. sequences of T-duality, coordinate shift and T-duality, to the  $S^5$  factor of the  $AdS_5 \times S^5$  background of type IIB string theory [8, 10].<sup>1</sup>

A convenient way to study the thermodynamic behaviour of gauge theories is via the thermal partition function on  $\mathbb{R} \times S^3$ :

$$\mathcal{Z}(T) = \text{tr}_{\mathbb{R} \times S^3} [e^{-H/T}], \tag{1.1}$$

where  $H$  denotes the Hamilton operator,  $T$  the temperature in units of the Boltzmann constant and the trace is taken over all states on  $\mathbb{R} \times S^3$ . In confining theories, the radius  $R$  of the three-sphere serves as an effective IR cutoff that stops the running of the coupling constant; tuning it makes the phase transition accessible to perturbation theory. Hence, this setup allows to study confining and non-confining theories on the same footing. In order to observe a sharp phase transition on the compact space  $S^3$ , the 't Hooft limit has to be taken. Below the critical temperature  $T_H$ , the partition function (1.1) is independent of the number of colours  $N$ . Above  $T_H$ , it scales as  $N^2$  and thus diverges [3].<sup>2</sup> The partition function (1.1) and its phase structure were investigated for several theories via a direct path integral approach on  $\mathbb{R} \times S^3$  [12–15].

For a conformal field theory like  $\mathcal{N} = 4$  SYM theory, a conformal mapping from  $\mathbb{R} \times S^3$  to  $\mathbb{R}^4$  can be used in order to express (1.1) as

$$\mathcal{Z}(T) = \text{tr}_{\mathbb{R}^4} [x^D], \tag{1.2}$$

where  $x = e^{-1/RT}$  and  $D$  is the dilatation operator [3]. In comparison to the path integral approach, this saves one loop order in each calculation.<sup>3</sup> The problem to sum over all states on  $\mathbb{R}^4$ , i.e. gauge-invariant composite operators, then reduces to the enumeration of all graded cyclic spin-chains or necklaces. In free  $\mathcal{N} = 4$  SYM theory, this problem was solved by the means of Pólya theory in [16].<sup>4</sup> A central role in this method is played by

<sup>1</sup>We will address important subtleties in these statements as well as recent developments later.

<sup>2</sup>The critical temperature is also called Hagedorn temperature, named after Rolf Hagedorn, who studied a phase transition in the strong force even before its description in terms of QCD was established [11].

<sup>3</sup>For instance, the contribution with one momentum loop in the path-integral approach can be obtained from the classical, i.e. zero-loop, dilatation operator. In this paper, we follow the counting based on the dilatation-operator approach.

<sup>4</sup>Note the slight mistake in [16] with respect to the grading.

the single-site partition function

$$z(x) = \text{tr}_{\mathcal{A}} \left[ x^{\mathfrak{D}_0} \right], \tag{1.3}$$

where  $\mathfrak{D}_0$  denotes the classical dilatation-operator density and  $\mathcal{A}$  is the alphabet of single-site states from which all single-trace operators can be built. The first loop correction to the  $\mathcal{N} = 4$  SYM theory result was calculated via an extension of Pólya theory in [17]. In addition to  $z(x)$ , it employs the two generalised expectation values

$$\langle \mathfrak{D}_2(x) \rangle = \text{tr}_{\mathcal{A} \times \mathcal{A}} \left[ x^{\mathfrak{D}_0} \mathfrak{D}_2 \right], \tag{1.4}$$

$$\langle P \mathfrak{D}_2(w, y) \rangle = \text{tr}_{\mathcal{A} \times \mathcal{A}} \left[ \mathcal{P} w^{\mathfrak{D}_0} y^{\mathfrak{D}_0} \mathfrak{D}_2 \right], \tag{1.5}$$

where  $\mathfrak{D}_2$  is the one-loop dilatation-operator density of [18] and  $\mathcal{P}$  is the graded permutation operator.

The free result of [16] is also valid for the  $\beta$ - and  $\gamma_i$ -deformation, as the  $*$ -product only affects the interactions. However, if we want to employ the one-loop method of [17], we have to face several important subtleties occurring in the deformed theories that were recently investigated in [19–21].

The  $\beta$ -deformation is only conformally invariant if the gauge group is chosen to be  $SU(N)$ . For gauge group  $U(N)$ , quantum corrections introduce a running double-trace coupling in the component action, which breaks conformal invariance. This coupling is at its non-vanishing fix-point value in the  $SU(N)$  theory [22]. For the purpose of our one-loop calculation, both theories are sufficiently conformally invariant, as the effects of nonconformality only set in at higher loop orders. However, note that at higher loop orders the computation of (1.1) via (1.2) is only valid for the conformal  $SU(N)$  theory. Moreover, the asymptotic one-loop dilatation operator of [9] acquires finite-size corrections, i.e. an explicit dependence on the length of the spin chain. These effects were intensively studied in [20] and at one loop occur for spin chains of length one and two for gauge group  $U(N)$  and  $SU(N)$ , respectively. For gauge group  $SU(N)$ , they can be traced back to the double-trace structure of the propagator and the aforementioned double-trace coupling, whereas they are due to the well-known wrapping effect [23–25] for gauge group  $U(N)$ .

The  $\gamma_i$ -deformation, on the other hand, is not conformally invariant, neither for gauge group  $U(N)$  nor  $SU(N)$  [19], at least not for the candidate proposed in [8] and all its natural Lagrangian extensions. Here, quantum corrections induce running double-trace couplings without a fix point. This poses very interesting and up to now unsolved questions in the context of the AdS/CFT correspondence; see [19] for a discussion of different possibilities. For the purpose of our one-loop calculation in this paper, the nonconformality of the  $\gamma_i$ -deformation itself is no problem. However, a problem arises concerning the uniqueness of the theory: there are many theories that share the single-trace structure of the action proposed in [8] and hence the planar asymptotic dilatation operator of [9].<sup>5</sup> They differ only in the multi-trace structure and thus in finite-size effects. For the sake of definiteness, we will focus on the candidate proposed in [8] with gauge group  $U(N)$  here. We will see

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<sup>5</sup>See the discussion in [19].

later that the temperature of the phase transition is indeed independent of spin chains of small lengths and thus finite-size effects; it is entirely driven by spin chains of large lengths.

This paper is structured as follows. In section 2, we introduce the  $\beta$ - and  $\gamma_i$ -deformation; in particular we provide the one-loop dilatation operator. In section 3, we give a short summary of the method of [17] and modify it to account for finite-size effects. In section 4, we compute the necessary ingredients for the partition function:  $z(x)$ ,  $\langle P\mathfrak{D}_2^{L \geq 3}(w, y) \rangle$ ,  $\langle \mathfrak{D}_2^{L \geq 3}(x) \rangle$ , as well as the finite-size correction term  $Z_{\text{f.s.c.}}^{(1)}(x)$ . In section 5, we discuss the resulting partition function. In particular, we compute the temperature of the phase transition up to and including the first loop order. Section 6 contains our conclusion and outlook. We provide several appendices. In appendix A, we calculate the one-loop anomalous dimensions of the fermionic  $L = 1$  operators, which are required in section 2. In appendix B, we give details on our conventions concerning the spin-chain picture and the required matrix elements of the one-loop dilatation operator of  $\mathcal{N} = 4$  SYM theory. Appendices C–E contain details on the calculations of section 4. In appendix F, we provide a check of our results for gauge group  $U(N)$  via a modification of the method and calculation of [15].

## 2 Deformed $\mathcal{N} = 4$ SYM theory

In this section, we introduce the  $\beta$ - and  $\gamma_i$ -deformation and give the dilatation operator up to the first loop order.

Both deformed gauge theories are closely related to their parent  $\mathcal{N} = 4$  SYM theory. The single-trace part of their respective actions inherits the properties of  $\mathcal{N} = 4$  SYM theory as shown in [26–28].<sup>6</sup> They can be obtained by replacing all products in the action of the undeformed  $\mathcal{N} = 4$  SYM theory by Moyal-like  $*$ -products. For two field  $A$  and  $B$ , the  $*$ -product is defined as

$$A * B = AB e^{\frac{i}{2} \mathbf{q}_A \wedge \mathbf{q}_B}, \tag{2.1}$$

where  $\mathbf{q}_A = (q_A^1, q_A^2, q_A^3)$  and  $\mathbf{q}_B = (q_B^1, q_B^2, q_B^3)$  are the charge vectors of the fields associated with the Cartan subgroup of the  $SU(4)_R$  symmetry group of the undeformed theory, see table 1. The antisymmetric product of the charge vectors is given by

$$\mathbf{q}_A \wedge \mathbf{q}_B = (\mathbf{q}_A)^T \mathbf{C} \mathbf{q}_B, \quad \mathbf{C} = \begin{pmatrix} 0 & -\gamma_3 & \gamma_2 \\ \gamma_3 & 0 & -\gamma_1 \\ -\gamma_2 & \gamma_1 & 0 \end{pmatrix}. \tag{2.2}$$

In concrete applications, it can be conveniently written in terms of the following linear combinations of the deformation parameters:

$$\gamma_i^\pm = \pm \frac{1}{2} (\gamma_{i+1} \pm \gamma_{i+2}), \tag{2.3}$$

where cyclic identification  $i + 3 \sim i$  is understood.

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<sup>6</sup>Note that the properties of multi-trace parts of the actions are not captured by the arguments in [26–28].

$B$	$D_{\alpha\dot{\beta}}$	$\phi^1$	$\phi^2$	$\phi^3$	$\psi_\alpha^1$	$\psi_\alpha^2$	$\psi_\alpha^3$	$\psi_\alpha^4$
$q_B^1$	0	1	0	0	$+\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$+\frac{1}{2}$
$q_B^2$	0	0	1	0	$-\frac{1}{2}$	$+\frac{1}{2}$	$-\frac{1}{2}$	$+\frac{1}{2}$
$q_B^3$	0	0	0	1	$-\frac{1}{2}$	$-\frac{1}{2}$	$+\frac{1}{2}$	$+\frac{1}{2}$

**Table 1.**  $SU(4)_R$  Cartan charges of the fields of  $\mathcal{N} = 4$  SYM theory [9]. The respective anti-fields carry the opposite charges.

The single-trace part of the Euclidean action of the  $\beta$ - and  $\gamma_i$ -deformation can be given as

$$\begin{aligned}
 S_{\text{s.t.}} = \int d^4x \operatorname{tr} & \left( -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - (D^\mu \bar{\phi}_j) D_\mu \phi^j + i \bar{\psi}_A^{\dot{\alpha}} D_{\dot{\alpha}}{}^\alpha \psi_\alpha^A \right. \\
 & \left. + g_{\text{YM}} \left( \frac{i}{2} \epsilon_{ijk} \phi^i \{ \psi^{\alpha j}, \psi_\alpha^k \}_* + \phi^j \{ \bar{\psi}_4^{\dot{\alpha}}, \bar{\psi}_{\dot{\alpha} j} \}_* + \text{h.c.} \right) \right. \\
 & \left. - \frac{g_{\text{YM}}^2}{4} [\bar{\phi}_j, \phi^j] [\bar{\phi}_k, \phi^k] + \frac{g_{\text{YM}}^2}{2} [\bar{\phi}_j, \bar{\phi}_k]_* [\phi^j, \phi^k]_* \right), \tag{2.4}
 \end{aligned}$$

where the spacetime indices  $\mu, \nu = 0, 1, 2, 3$ , spinor indices  $\alpha = 1, 2, \dot{\alpha} = \dot{1}, \dot{2}$ , flavour indices  $i, j, k = 1, 2, 3$  and  $A = 1, 2, 3, 4$  are implicitly summed over via Einstein's summation convention. The deformation parameters enter via the  $*$ -product which occurs in the  $*$ -deformed commutators and anti-commutators. We have kept only those  $*$ -products that do introduce net deformations in the  $\gamma_i$ -deformation. In the  $\beta$ -deformation, the antisymmetric product (2.2) of the gluino charge vector with any other charge vector also vanishes, yielding  $\{ \bar{\psi}_4^{\dot{\alpha}}, \bar{\psi}_{\dot{\alpha} j} \}_* |_\beta = \{ \bar{\psi}_4^{\dot{\alpha}}, \bar{\psi}_{\dot{\alpha} j} \}$ .

In addition to the single-trace terms, the deformed theories also contain multi-trace couplings, which are not inherited from the undeformed parent theory. For the deformations with gauge group  $SU(N)$ , the only additional structures are of double-trace type. For the ones with gauge group  $U(N)$ , also triple- and quadruple-trace structures may occur. The  $\beta$ -deformation with gauge group  $SU(N)$  contains non-vanishing double-trace couplings, which are at the fix-point value such that the theory is conformally invariant [20]. They arise when the auxiliary fields, present in the  $\mathcal{N} = 1$  superspace formulation of the undeformed theory, are integrated out after introducing the deformation, as shown in [19, 29].<sup>7</sup> While these double-trace couplings are absent in the  $U(N)$   $\beta$ -deformation at tree-level, they are induced at loop-level and flow to the  $SU(N)$  fix point in the infrared, which renders the theory non-conformal. The  $\gamma_i$ -deformation is not conformally invariant, neither for  $U(N)$  nor  $SU(N)$  gauge group. Moreover, conformal invariance cannot be restored by extending the theory with any number of multi-trace couplings.<sup>8</sup> The breakdown of conformal invariance originates from running double-trace couplings which have no fix points [19]. At one loop, these couplings also affect the planar anomalous dimensions of  $L = 2$  operators. Hence, starting from two loops, the dilatation operator becomes renormalisation-scheme dependent, as was exemplified in [21].

<sup>7</sup>The double-trace term follows also directly from the procedure mentioned much earlier in [30].

<sup>8</sup>See [19] for details and possible interpretations in the light of the AdS/CFT correspondence.

As already mentioned in the introduction, the breakdown of conformal invariance does not invalidate our approach at one-loop level. However, it leads to an issue of uniqueness. In the case of the  $\mathcal{N} = 1$  supersymmetric  $\beta$ -deformation, a unique conformally invariant theory exists, which has gauge group  $SU(N)$ . In the case of the  $\gamma_i$ -deformation, there is no (known) conformally invariant theory. However, a whole family of  $\gamma_i$ -deformed theories exist that share the single-trace action (2.4) and differ only in the double-trace part of the action. For the sake of definiteness, we focus on the candidate action of the  $\gamma_i$ -deformation that was proposed in [8], i.e. we set the tree-level values of all multi-trace couplings to zero. In the  $\beta$ -deformation with gauge group  $U(N)$ , we equally set the tree-level values of all multi-trace couplings to zero. This leads to the following double-trace part of the Euclidean action of the  $\beta$ -deformation

$$S_{\text{m.t.}}^\beta = \int d^4x \left[ -\frac{s}{N} \frac{g_{\text{YM}}^2}{2} \text{tr}([\bar{\phi}_j, \bar{\phi}_k]_*) \text{tr}([\phi^j, \phi^k]_*) \right], \quad (2.5)$$

where the gauge-group parameter is

$$s = \begin{cases} 0 & \text{for } U(N), \\ 1 & \text{for } SU(N). \end{cases} \quad (2.6)$$

The elementary building blocks on which the dilatation operator acts in the planar limit are single-trace operators.<sup>9</sup> As the deformations do not alter the field content but only the interactions, the single-trace operators are built from the same alphabet as in the undeformed  $\mathcal{N} = 4$  SYM theory:

$$\mathcal{A} = \{D^k \phi^i, D^k \bar{\phi}^i, D^k \psi_\alpha^A, D^k \bar{\psi}_{\dot{\alpha}}^A, D^k \mathcal{F}_{\alpha\beta}, D^k \bar{\mathcal{F}}_{\dot{\alpha}\dot{\beta}}\}, \quad (2.7)$$

where the abbreviation  $D^k \bar{\psi}_{\dot{\alpha}}^2$  stands for expressions with  $k \in \mathbb{N}_0$  covariant derivatives  $D_{\beta\dot{\beta}}$  acting on  $\bar{\psi}_{\dot{\alpha}}^2$  and which are totally symmetric in both kinds of spinor indices. These operators can be mapped to cyclic spin-chain states, see [4] for a review and appendix B for our conventions. The fields in the spin-chain picture can be represented by two sets of bosonic oscillators  $\mathbf{a}^{\dagger\alpha}$  ( $\alpha = 1, 2$ ) and  $\mathbf{b}^{\dagger\dot{\alpha}}$  ( $\dot{\alpha} = 1, 2$ ) and one set of fermionic oscillators  $\mathbf{c}^{\dagger A}$  ( $A = 1, 2, 3, 4$ ). They can be characterised by a vector containing the occupation numbers of each oscillator

$$A_i = (a_{(i)}^1, a_{(i)}^2, b_{(i)}^1, b_{(i)}^2, c_{(i)}^1, c_{(i)}^2, c_{(i)}^3, c_{(i)}^4), \quad (2.8)$$

where the lower index  $i$  labels the site of the spin chain.

In the 't Hooft limit, the dilatation operator admits a perturbative expansion in the effective planar coupling  $g$ :

$$D = D_0 + g^2 D_2 + \mathcal{O}(g^3), \quad g = \frac{\sqrt{\lambda}}{4\pi}, \quad (2.9)$$

where only the classical piece  $D_0$  and the one-loop correction  $D_2$  are shown. Their actions on single-trace operators, which are represented as spin chains, can be written in terms of

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<sup>9</sup>Interactions that split or join traces are suppressed by  $\frac{1}{N}$ .

site-independent densities  $\mathfrak{D}_0$  and  $\mathfrak{D}_2$ , respectively, as

$$D_{2K} = \sum_{i=1}^L \underbrace{\mathbb{1} \otimes \cdots \otimes \mathbb{1}}_{i-1} \otimes \mathfrak{D}_{2K} \otimes \underbrace{\mathbb{1} \otimes \cdots \otimes \mathbb{1}}_{L-i-K}, \quad (2.10)$$

where cyclic identification  $i + L \sim i$  is understood.

The classical dilatation-operator density  $\mathfrak{D}_0$  takes the same form in the deformed theories as in the undeformed one. It yields the classical scaling dimension of the field it acts on. In the spin-chain picture, it takes the diagonal form

$$(\mathfrak{D}_0)_{A_i}^{A_j} = \left( 1 + \frac{1}{2} \sum_{\alpha=1}^2 a_{(i)}^\alpha + \frac{1}{2} \sum_{\dot{\alpha}=1}^2 b_{(i)}^{\dot{\alpha}} \right) \delta_{A_i}^{A_j}. \quad (2.11)$$

In contrast to  $\mathfrak{D}_0$ , the one-loop dilatation-operator density  $\mathfrak{D}_2$  depends on the deformation parameters. In the absence of one-loop finite-size effects, which occur at  $L = 2$  for gauge group  $SU(N)$  and at  $L = 1$  for gauge group  $U(N)$ , it can be expressed in terms of the undeformed density times a phase factor that depends on the order and flavour of the external fields alone [9]:

$$(\mathfrak{D}_2^{L \geq 3})_{A_i A_j}^{A_k A_l} = e^{\frac{i}{2}(\mathbf{q}_{A_k} \wedge \mathbf{q}_{A_l} - \mathbf{q}_{A_i} \wedge \mathbf{q}_{A_j})} (\mathfrak{D}_2^{N=4})_{A_i A_j}^{A_k A_l}. \quad (2.12)$$

In [18],  $\mathfrak{D}_2^{N=4}$  was given in terms of all possible hoppings of the oscillators from site 1 to 2 and vice versa. Each such hopping combination is weighted by the coefficient<sup>10</sup>

$$c_h(n, n_{12}, n_{21}) = \begin{cases} 2h\left(\frac{1}{2}n\right) & \text{if } n_{12} = n_{21} = 0, \\ 2(-1)^{1+n_{12}n_{21}} B\left(\frac{n_{12}+n_{21}}{2}, 1 + \frac{n-n_{12}-n_{21}}{2}\right) & \text{else,} \end{cases} \quad (2.13)$$

where  $h(m)$  denotes the  $m^{\text{th}}$  harmonic number and  $B(a, b)$  is the Euler  $\beta$ -function. This coefficient only depends on the total sum of oscillators at both sites  $n$  and the total sum of oscillators that hop from site  $i$  to  $j$ , denoted by  $n_{ij}$ . In [20], we have given an explicit form of the matrix elements  $(\mathfrak{D}_2^{N=4})_{A_i A_j}^{A_k A_l}$  in terms of the occupation numbers (2.8), which is also suitable for the present context.

For short operators, the asymptotic dilatation-operator density (2.12) of the deformed theories is altered due to finite-size effects. In the  $SU(N)$   $\beta$ -deformation, they stem from the prewrapping effect [20]. It is caused by the double-trace part in the  $SU(N)$  propagator and the double-trace coupling (2.5). This effect can be implemented on the level of the dilatation-operator density at one-loop order via a simple prescription [20]. It is given as

$$(\mathfrak{D}_2^{L=2})_{A_i A_j}^{A_k A_l} = \begin{cases} (\mathfrak{D}_2^{N=4})_{A_i A_j}^{A_k A_l} & \text{if } A_i, A_j, A_k, A_l \in \mathcal{A}_{\text{matter}} \text{ or } A_i, A_j, A_k, A_l \in \bar{\mathcal{A}}_{\text{matter}}, \\ (\mathfrak{D}_2^{L \geq 3})_{A_i A_j}^{A_k A_l} & \text{else,} \end{cases} \quad (2.14)$$

<sup>10</sup>Note that our coefficient  $c_h$  has to be divided by 2 to match the conventions of [18].



where the sub-alphabets of (anti)-matter fields are defined as

$$\begin{aligned} \mathcal{A}_{\text{matter}} &= \{D^k \phi^1, D^k \phi^2, D^k \phi^3, D^k \psi_\alpha^1, D^k \psi_\alpha^2, D^k \psi_\alpha^3\}, \\ \bar{\mathcal{A}}_{\text{matter}} &= \{D^k \bar{\phi}^1, D^k \bar{\phi}^2, D^k \bar{\phi}^3, D^k \bar{\psi}_\alpha^1, D^k \bar{\psi}_\alpha^2, D^k \bar{\psi}_\alpha^3\}. \end{aligned} \tag{2.15}$$

The one-loop finite-size contributions for the  $\beta$ - and  $\gamma_i$ -deformation with gauge group  $U(N)$  only occur for  $L = 1$  operators.<sup>11</sup> They stem from ordinary wrapping corrections and the corresponding wrapping diagrams are of self-energy type. In appendix D of [19], the one-loop self-energy contributions to the scalar propagators were determined. We present an analogous calculation for self-energy contribution to the fermionic propagators in appendix A. The gluon interactions are not affected by the deformations and thus the self-energies for the gluons are undeformed. From the respective contributions, we find the following eigenvalues of  $\mathfrak{D}_2$  on  $L = 1$  operators:

$$\begin{aligned} E_{\text{tr}(D^k \phi^i)} &= E_{\text{tr}(D^k \bar{\phi}^i)} = 4 \left( \sin^2 \frac{\gamma_i^+}{2} + \sin^2 \frac{\gamma_i^-}{2} \right), \\ E_{\text{tr}(D^k \psi_\alpha^i)} &= E_{\text{tr}(D^k \bar{\psi}_\alpha^i)} = 2 \left( \sin^2 \frac{\gamma_i^-}{2} + \sin^2 \frac{\gamma_{i+1}^+}{2} + \sin^2 \frac{\gamma_{i+2}^+}{2} \right), \\ E_{\text{tr}(D^k \psi_\alpha^A)} &= E_{\text{tr}(D^k \bar{\psi}_\alpha^A)} = 2 \sum_{i=1}^3 \sin^2 \frac{\gamma_i^-}{2}, \\ E_{\text{tr}(D^k \mathcal{F}_{\alpha\beta})} &= E_{\text{tr}(D^k \bar{\mathcal{F}}_{\alpha\beta})} = 0, \end{aligned} \tag{2.16}$$

where cyclic identification  $i + 3 \sim i$  is understood. These results agree with the ones of [20] for the  $\beta$ -deformation in the limit  $\gamma_i^+ = -\beta$ ,  $\gamma_i^- = 0$ .

### 3 Partition functions via Pólya theory

In this section, we summarise the approach of [17] to the one-loop partition function of  $\mathcal{N} = 4$  SYM theory via an extension of Pólya theory. We will follow the presentation of [17] and refer the reader there for details. Moreover, we show how this approach can be modified to be applicable in the deformed theories.

In the planar limit, the action of the dilatation operator  $D$  on a multi-trace operators is completely determined by the action of  $D$  on the operator's single-trace factors. Accordingly, we consider single-trace operators first.

#### 3.1 The single-trace partition function

The single-trace partition function is defined in analogy to (1.2) as trace over all single-trace operators:

$$Z(x) = \text{tr}_{\mathbb{R}^4} [x^D], \tag{3.1}$$

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<sup>11</sup>Recall that we are considering the  $U(N)$  theories with zero tree-level values for all double-trace couplings. The dilatation operator in the presence of multi-trace couplings can also be calculated [31], e.g. by an extension of the method of [32].

where  $x = e^{-1/RT}$  as before. Expanding the dilatation operator in the effective planar coupling  $g$  as in (2.9), we obtain the following expansion of (3.1):

$$\begin{aligned} Z(x, g) &= \text{tr} [x^{D_0}] + g^2 \ln x \text{tr} [x^{D_0} D_2] + \mathcal{O}(g^3) \\ &= Z^{(0)}(x) + g^2 \ln x Z^{(1)}(x) + \mathcal{O}(g^3). \end{aligned} \tag{3.2}$$

Up to the first loop order, the dilatation operator does not change the length  $L$  of a single-trace operator. Hence, the respective traces in (3.2) can be expressed as sums of traces with fixed  $L$ . Moreover, the trace in the space of single-trace operator of length  $L$  can be re-expressed as trace in the space of spin chains of length  $L$  if we include the projector on graded cyclically invariant spin-chain states  $\mathcal{P}$ . This projector can be written in terms of the shift operator  $T$  as

$$\mathcal{P} = \frac{1}{L}(1 + T + T^2 + \dots + T^{L-1}). \tag{3.3}$$

Let us first evaluate the contribution from the free theory

$$\text{tr}_L[\mathcal{P}x^{D_0}]. \tag{3.4}$$

Disregarding the existence of fermions, this problem is equivalent to the enumeration of all necklaces of length  $L$  that can be built from a specified alphabet of beads  $\mathcal{A}$ . The latter problem can be solved by the enumeration theorem of Pólya [33]. One subtlety arises due to the existence of fermions, which require the necklaces to be graded cyclically invariant.<sup>12</sup> The final result including this subtlety is

$$\text{tr}_L[\mathcal{P}x^{D_0}] = \frac{1}{L} \sum_{k|L} \varphi(k) \left[ z(\omega^{k+1}x^k) \right]^{L/k}, \tag{3.5}$$

where the sum runs over all divisors  $k$  of  $L$ ,  $\varphi(k)$  is the Euler totient function and

$$z(x) = \sum_{A \in \mathcal{A}} x^{(\mathfrak{D}_0)_A^A} \tag{3.6}$$

is the single-site partition function. The graded cyclic invariance is incorporated by the formal quantity  $\omega$  with the property  $\sqrt{\omega} = -1$ . It exploits the fact that all fermions have half-integer classical scaling dimensions and it can hence be used in all perturbative expansions [17]. The single-trace partition function at zero coupling is obtained by summing (3.5) over all admissible lengths  $L$ . Note that this sum has to start at  $L = 1$  for gauge group  $U(N)$  and at  $L = 2$  for gauge group  $SU(N)$ , since  $SU(N)$  matrices are traceless:

$$Z^{(0)}(x) = \sum_{L=1+s}^{\infty} \text{tr}_L[\mathcal{P}x^{D_0}] = -sz(x) - \sum_{k=1}^{\infty} \frac{\varphi(k)}{k} \ln[1 - z(\omega^{k+1}x^k)], \tag{3.7}$$

with  $s = 0$  for gauge group  $U(N)$  and  $s = 1$  for gauge group  $SU(N)$ , as defined in (2.6).

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<sup>12</sup>If we shift a fermion from the last to the first position in an overall bosonic single-trace operator, we acquire a minus sign.

Now we turn to the first loop order, where we have to evaluate

$$\text{tr}_L[\mathcal{P}x^{D_0}D_2]. \tag{3.8}$$

Compared to the  $\mathcal{N} = 4$  SYM theory case, there is an additional complication due to the fact that the one-loop dilatation-operator density has an explicit  $L$ -dependence in the deformed theories. It occurs at  $L = 1$  and  $L = 2$  and is caused by finite-size effects; cf. section 2. For  $L \geq 3$ , the dilatation-operator density is length-independent and we can use the following result of [17]:<sup>13</sup>

$$\begin{aligned} \text{tr}_L[\mathcal{P}x^{D_2}D_2^{L \geq 3}] &= \sum_{m|L} \varphi(m) [z(\omega^{m+1}x^m)]^{L/m-2} \langle \mathfrak{D}_2^{L \geq 3}(\omega^{m+1}x^m) \rangle \\ &+ \sum_{\substack{k=0 \\ (k,L)=1}}^{L-1} \left[ \langle P\mathfrak{D}_2^{L \geq 3}(\omega^{L-k+1}x^{L-k}, \omega^{k+1}x^k) \rangle \right. \\ &\quad \left. - z(\omega^{L+1}x^L)^{-1} \langle \mathfrak{D}_2^{L \geq 3}(\omega^{L+1}x^L) \rangle \right], \end{aligned} \tag{3.9}$$

where

$$\langle \mathfrak{D}_2^{L \geq 3}(x) \rangle = \sum_{A_1, A_2 \in \mathcal{A}} x^{(\mathfrak{D}_0)_{A_1}^{A_1} + (\mathfrak{D}_0)_{A_2}^{A_2}} (\mathfrak{D}_2^{L \geq 3})_{A_1 A_2}^{A_1 A_2}, \tag{3.10}$$

$$\langle P\mathfrak{D}_2^{L \geq 3}(w, y) \rangle = \sum_{A_1, A_2 \in \mathcal{A}} (-1)^{F(A_1)F(A_2)} w^{(\mathfrak{D}_0)_{A_1}^{A_1}} y^{(\mathfrak{D}_0)_{A_2}^{A_2}} (\mathfrak{D}_2^{L \geq 3})_{A_1 A_2}^{A_2 A_1}, \tag{3.11}$$

with  $F(A_i) = 1$  for fermions and  $F(A_i) = 0$  for bosons and  $\mathfrak{D}_2^{L \geq 3}$  given in (2.12). The rich combinatorial structure of this expression originates from the shift operators in (3.3), and we refer the reader to [17] for its derivation. At length  $L = 2$ , we have

$$\text{tr}_2[\mathcal{P}x^{D_2}D_2^{L=2}] = \langle \mathfrak{D}_2^{L=2}(x) \rangle + \langle P\mathfrak{D}_2^{L=2}(x, x) \rangle, \tag{3.12}$$

and at  $L = 1$

$$\text{tr}_1[\mathcal{P}x^{D_2}D_2^{L=1}] = \sum_{A \in \mathcal{A}} x^{(\mathfrak{D}_0)_A^A} (\mathfrak{D}_2^{L=1})_A^A, \tag{3.13}$$

with  $\mathfrak{D}_2^{L=2}$  and  $\mathfrak{D}_2^{L=1}$  given in (2.14) and (2.16), respectively. The complete one-loop correction to the single-trace partition function is given by the sum of the respective terms over all admissible  $L$ :

$$\begin{aligned} Z^{(1)}(x) &= Z_{\text{f.s.c.}}^{(1)}(x) + \sum_{L=2}^{\infty} \text{tr}_L[\mathcal{P}x^{D_0}D_2^{L \geq 3}] \\ &= Z_{\text{f.s.c.}}^{(1)}(x) + \sum_{n=1}^{\infty} \sum_{\substack{k=0 \\ (k,n)=1}}^{n-1} \left[ \frac{\langle \mathfrak{D}_2^{L \geq 3}(\omega^{n+1}x^n) \rangle}{1 - z(\omega^{n+1}x^n)} + \delta_{n \neq 1} \langle P\mathfrak{D}_2^{L \geq 3}(\omega^{n-k+1}x^{n-k}, \omega^{k+1}x^k) \rangle \right], \end{aligned} \tag{3.14}$$

where the finite-size corrections are included in

$$Z_{\text{f.s.c.}}^{(1)}(x) = (1 - s) \text{tr}_1[\mathcal{P}x^{D_2}D_2^{L=1}] + \text{tr}_2[\mathcal{P}x^{D_2}D_2^{L=2}] - \text{tr}_2[\mathcal{P}x^{D_2}D_2^{L \geq 3}] \tag{3.15}$$

with the dependence of  $\mathfrak{D}_2^{L=2}$  on the gauge group left implicit.

<sup>13</sup>In the second sum,  $(k, L) = 1$  denotes that  $k$  is relatively prime to  $L$ .

### 3.2 The multi-trace partition function

Multi-trace operators can be build as products of single-trace operators obeying the correct statistics, i.e. the Bose-Einstein statistic for bosonic single-trace operators and the Fermi-Dirac statistic for fermionic single-trace operators. For a toy model containing only one single-trace operator  $\mathcal{O}$  with scaling dimension  $\Delta$ , this leads to the well known partition function  $\left(\frac{1}{1-x^\Delta}\right)$  if  $\mathcal{O}$  is bosonic and  $(1+x^\Delta)$  if  $\mathcal{O}$  is fermionic. The result in the complete theory is the product of these expressions over all single-trace operators. For a perturbative expressing, this can be simplified by the use of  $\omega$  introduced below (3.6) to finally arrive at

$$\mathcal{Z}(x) = \exp \left[ \sum_{n=1}^{\infty} \frac{Z(\omega^{n+1}x^n)}{n} \right]. \quad (3.16)$$

Inserting the perturbative expansion of the single-trace partition function (3.2) into (3.16), we obtain

$$\mathcal{Z}(x, g) = \mathcal{Z}^{(0)}(x) + g^2 \ln x \mathcal{Z}^{(1)}(x) + \mathcal{O}(g^3), \quad (3.17)$$

with

$$\mathcal{Z}^{(0)}(x) = \exp \left[ \sum_{n=1}^{\infty} \frac{Z^{(0)}(\omega^{n+1}x^n)}{n} \right], \quad \mathcal{Z}^{(1)}(x) = \mathcal{Z}^{(0)}(x) \sum_{n=1}^{\infty} Z^{(1)}(\omega^{n+1}x^n). \quad (3.18)$$

The final expression for the multi-trace partition function and its one-loop correction can be found by inserting the respective single-trace expressions (3.7) and (3.14) into the above equations. After several steps, we obtain

$$\mathcal{Z}^{(0)}(x) = \exp \left[ -s \sum_{n=1}^{\infty} \frac{z(\omega^{n+1}x^n)}{n} \right] \prod_{m=1}^{\infty} \frac{1}{1 - z(\omega^{m+1}x^m)}, \quad (3.19)$$

and

$$\begin{aligned} \mathcal{Z}^{(1)}(x) = \mathcal{Z}^{(0)}(x) \left[ \sum_{n=1}^{\infty} Z_{\text{f.s.c.}}(\omega^{n+1}x^n) + \sum_{k=1}^{\infty} k \frac{\langle \mathfrak{D}_2^{L \geq 3}(\omega^{k+1}x^k) \rangle}{1 - z(\omega^{k+1}x^k)} \right. \\ \left. + \sum_{k,m=1}^{\infty} \langle P\mathfrak{D}_2^{L \geq 3}(\omega^{k+1}x^k, \omega^{m+1}x^m) \rangle \right]. \quad (3.20) \end{aligned}$$

## 4 Ingredients

In this section, we compute the ingredients for the extended Pólya-theory method:  $z(x)$ ,  $\langle P\mathfrak{D}_2^{L \geq 3}(w, y) \rangle$ ,  $\langle \mathfrak{D}_2^{L \geq 3}(x) \rangle$  and  $Z_{\text{f.s.c.}}^{(1)}(x)$ . We focus on conveying the main ideas and results here and postpone the details to the appendices C, D and E.

### 4.1 The single-site partition function

The single-site partition function defined in (3.6), as well as all other ingredients, can be derived in the oscillator picture, cf. appendix B for details. In this picture, sums over the

full alphabet of the theory (2.7) are expressed in terms of sums over all oscillator occupation numbers (2.8) as

$$\sum_{A_i \in \mathcal{A}} = \sum_{a_{(i)}^1, a_{(i)}^2=0}^{\infty} \sum_{b_{(i)}^1, b_{(i)}^2=0}^{\infty} \sum_{c_{(i)}^1, c_{(i)}^2, c_{(i)}^3, c_{(i)}^4=0}^1 \delta_{C_{(i)}}. \quad (4.1)$$

The occurring Kronecker  $\delta$  guarantees that only those combinations of oscillators contribute that correspond to one of the fields in the alphabet of the theory (2.7). Its argument is the eigenvalue of the central charge operator defined in (B.3).

The single-site partition function only depends on the field content of the theory and thus it is the same for  $\mathcal{N} = 4$  SYM theory and the  $\beta$ - and  $\gamma$ -deformation. Via a direct summation using some of the identities in appendix C, we find

$$z(x) = \sum_{A \in \mathcal{A}} x^{(\mathcal{D}_0)_A} = \frac{2(3 - \sqrt{x})x}{(1 - \sqrt{x})^3}. \quad (4.2)$$

This agrees with the result of [12, 16, 17].

## 4.2 The expectation value $\langle P\mathcal{D}_2^{L \geq 3}(w, y) \rangle$

Next, we compute the permuted expectation value of the asymptotic one-loop dilatation operator. The explicit form of  $\langle P\mathcal{D}_2^{L \geq 3}(w, y) \rangle$  is obtained by inserting (2.12), (4.1) and (B.5) into (3.11). This yields<sup>14</sup>

$$\begin{aligned} \langle P\mathcal{D}_2^{L \geq 3}(w, y) \rangle &= \prod_{i=1}^2 \left( \sum_{a_{(i)}^1, a_{(i)}^2=0}^{\infty} \sum_{b_{(i)}^1, b_{(i)}^2=0}^{\infty} \sum_{c_{(i)}^1, c_{(i)}^2, c_{(i)}^3, c_{(i)}^4=0}^1 \delta_{C_{(i)}} \right) \\ &\times w^{\frac{1}{2}(2 + \sum_{\alpha=1}^2 a_{(1)}^{\alpha} + \sum_{\dot{\alpha}=1}^2 b_{(1)}^{\dot{\alpha}})} y^{\frac{1}{2}(2 + \sum_{\alpha=1}^2 a_{(2)}^{\alpha} + \sum_{\dot{\alpha}=1}^2 b_{(2)}^{\dot{\alpha}})} \\ &\times e^{-i \sum_{l,m=1}^4 c_{(1)}^l c_{(2)}^m} \mathbf{q}_{\psi l} \wedge \mathbf{q}_{\psi m} \\ &\times \prod_{\alpha=1}^2 \left( \sum_{a^{\alpha}=0}^{\infty} \binom{a_{(1)}^{\alpha}}{a^{\alpha}} \binom{a_{(2)}^{\alpha}}{a^{\alpha}} \right) \prod_{\dot{\alpha}=1}^2 \left( \sum_{b^{\dot{\alpha}}=0}^{\infty} \binom{b_{(1)}^{\dot{\alpha}}}{b^{\dot{\alpha}}} \binom{b_{(2)}^{\dot{\alpha}}}{b^{\dot{\alpha}}} \right) \\ &\times \prod_{e=1}^4 \left( \sum_{c^e=0}^1 \binom{c_{(1)}^e}{c^e} \binom{c_{(2)}^e}{c^e} (-1)^{c^e} \right) \\ &\times c_h \left[ \sum_{i=1}^2 \left( \sum_{\alpha=1}^2 a_{(i)}^{\alpha} + \sum_{\dot{\alpha}=1}^2 b_{(i)}^{\dot{\alpha}} + \sum_{e=1}^4 c_{(i)}^e \right), \right. \\ &\quad \sum_{\alpha=1}^2 (a_{(1)}^{\alpha} - a^{\alpha}) + \sum_{\dot{\alpha}=1}^2 (b_{(1)}^{\dot{\alpha}} - b^{\dot{\alpha}}) + \sum_{e=1}^4 (c_{(1)}^e - c^e), \\ &\quad \left. \sum_{\alpha=1}^2 (a_{(2)}^{\alpha} - a^{\alpha}) + \sum_{\dot{\alpha}=1}^2 (b_{(2)}^{\dot{\alpha}} - b^{\dot{\alpha}}) + \sum_{e=1}^4 (c_{(2)}^e - c^e) \right], \end{aligned} \quad (4.3)$$

where we have also used the antisymmetry of  $\mathbf{q}_A \wedge \mathbf{q}_B$ .

<sup>14</sup>Note that in the oscillator picture the fermion number operator takes the form  $F(A) = \sum_{e=1}^4 c^e$ . Hence, the factor  $(-1)^{F(A_1)F(A_2)}$  in (3.11) cancels the respective factor from the matrix element in (B.5).

Performing the above twelve infinite sums is a decisively complicated task due to their entanglement via the central charge constraint and the coefficient  $c_h$  of the harmonic action. We solve it in three steps which are presented in detail in appendix C. In a first step, we use that  $c_h(n, n_{12}, n_{21})$  only depends on the total number of oscillators and the total number of oscillators that change their site. Summation identities of binomial coefficients can hence be employed to cut the number of infinite sums in half. In a second step, we express the coefficient  $c_h(n, n_{12}, n_{21})$  in terms of the following integral representation

$$c_h(n, n_{12}, n_{21}) = \int_0^1 dt (c^{\text{int}}(n, n_{12}, n_{21}) - 1/t\text{-pole}) , \tag{4.4}$$

$$c^{\text{int}}(n, n_{12}, n_{21}) = 2(-1)^{1+n_{12}n_{21}} t^{\frac{1}{2}(n_{12}+n_{21})-1} (1-t)^{\frac{1}{2}(n-n_{12}-n_{21})} ,$$

where the prescription  $-1/t\text{-pole}$  denotes the subtraction of the  $1/t$ -pole that occurs when  $n_{12} = n_{21} = 0$ .<sup>15</sup> This allows us to reduce the entanglement of the infinite sums by defining

$$\langle P\mathfrak{D}_2^{L \geq 3}(w, y) \rangle_{\text{int}} = \langle P\mathfrak{D}_2^{L \geq 3}(w, y) \rangle \Big|_{c_h(n, n_{12}, n_{21}) \rightarrow c^{\text{int}}(n, n_{12}, n_{21})} , \tag{4.5}$$

such that

$$\langle P\mathfrak{D}_2^{L \geq 3}(w, y) \rangle = \int_0^1 dt \left( \langle P\mathfrak{D}_2^{L \geq 3}(w, y) \rangle_{\text{int}} - 1/t\text{-pole} \right) . \tag{4.6}$$

In a third and last step, we perform the remaining six infinite sums in  $\langle P\mathfrak{D}_2^{L \geq 3}(w, y) \rangle_{\text{int}}$ . Two of them can be eliminated via the central charge constraint. In order to further disentangle the remaining four sums, we reexpress the summand as differential and integral operators acting on simpler expressions such that the sums can be performed and the operators applied afterwards. A minimal example of this procedure is the following:

$$\sum_{n=0}^{\infty} (n+1)x^n = \sum_{n=0}^{\infty} \frac{d}{dx} x^{n+1} = \frac{d}{dx} \sum_{n=0}^{\infty} x^{n+1} = \frac{d}{dx} \frac{x}{1-x} = \frac{1}{(1-x)^2} , \tag{4.7}$$

where the feasible sum is the geometric series. In the calculation of  $\langle P\mathfrak{D}_2^{L \geq 3}(w, y) \rangle_{\text{int}}$ , however, the final feasible sum has to be evaluated via the generating functions of Legendre polynomials.

The final result is

$$\begin{aligned} \langle P\mathfrak{D}_2^{L \geq 3}(w, y) \rangle = 4 \left( \frac{wy(1+w^{1/2})^2(1+y^{1/2})^2}{(1-w^{1/2})^2(1-y^{1/2})^2(w^{1/2}+y^{1/2})^2(1+w^{1/2}y^{1/2})^3} f_1(w, y) \right. \\ \left. + \frac{wy}{(1-w)^2(1-y)^2(1+w^{1/2}y^{1/2})(1-wy)} \sum_{i=1}^3 f_2(w, y, \gamma_i^{\pm}) \right. \\ \left. + f_3(w, y) \ln \left[ \frac{1-w}{1-wy} \right] \right) \\ + w \leftrightarrow y , \end{aligned} \tag{4.8}$$

---

<sup>15</sup>A trigonometric version of this integral representation of the harmonic action was proposed in [34].

where

$$f_1(w, y) = 2 - 16w^{1/2} + 7w + 11w^{1/2}y^{1/2} - 16wy^{1/2} + w^{3/2}y^{1/2} + 3wy, \quad (4.9)$$

$$f_2(w, y, \gamma_i^\pm) = \left( \sin^2 \frac{\gamma_i^+}{2} + \sin^2 \frac{\gamma_i^-}{2} \right) \left( 12w^{1/2} - 4w^{3/2} + 4wy^{1/2} - 4w^{3/2}y^{1/2} - 4w^{3/2}y \right. \\ \left. - 4w^2y - 8w^2y^{3/2} + 6w^{1/2}y^{1/2} + 6wy - 2w^{3/2}y^{3/2} - 2w^2y^2 \right) \\ + 4 \sin^2 \frac{\gamma_i^+ + \gamma_i^-}{2} \left( 1 + w^{1/2}y^{1/2} - wy - w^{3/2}y^{3/2} \right), \quad (4.10)$$

$$f_3(w, y) = -\frac{w(w^{1/2} + 3y^{1/2})}{(w^{1/2} + y^{1/2})^3} + \frac{2 - 6y^{1/2}}{(1 - y^{1/2})^3} - \frac{1 + 3w^{1/2}y^{1/2}}{(1 + w^{1/2}y^{1/2})^3}. \quad (4.11)$$

The result for the  $\beta$ -deformation is obtained from (4.8) for  $\gamma_i^- = 0$  and  $\gamma_i^+ = -\beta$ . For  $\gamma_i^\pm = 0$ , the second line of (4.8) drops out and the result of [17] for  $\mathcal{N} = 4$  SYM theory is reproduced.<sup>16</sup>

### 4.3 The expectation value $\langle \mathfrak{D}_2^{L \geq 3}(x) \rangle$

The expectation value of the one-loop dilatation operator density  $\langle \mathfrak{D}_2^{L \geq 3}(x) \rangle$  can be obtained in analogy to  $\langle P\mathfrak{D}_2^{L \geq 3}(x) \rangle$ . In order to apply the above techniques, we define

$$\langle \mathfrak{D}_2^{L \geq 3}(w, y) \rangle = \sum_{A_1, A_2 \in \mathcal{A}} w^{(\mathfrak{D}_0)_{A_1}^{A_1}} y^{(\mathfrak{D}_0)_{A_2}^{A_2}} (\mathfrak{D}_2^{L \geq 3})_{A_1 A_2}^{A_1 A_2}, \quad (4.12)$$

which coincides with the original definition in (3.10) for  $w = y = x$ . Note that the matrix element  $(\mathfrak{D}_2^{L \geq 3})_{A_1 A_2}^{A_1 A_2}$  is independent of the deformation parameters, as can be seen from (2.12). In particular,  $(\mathfrak{D}_2^{L \geq 3})_{A_1 A_2}^{A_1 A_2} = (\mathfrak{D}_2^{L=2})_{A_1 A_2}^{A_1 A_2} = (\mathfrak{D}_2^{\mathcal{N}=4})_{A_1 A_2}^{A_1 A_2}$  such that

$$\langle \mathfrak{D}_2^{L \geq 3}(x) \rangle = \langle \mathfrak{D}_2^{L=2}(x) \rangle = \langle \mathfrak{D}_2^{\mathcal{N}=4}(x) \rangle. \quad (4.13)$$

We find<sup>17</sup>

$$\langle \mathfrak{D}_2^{L \geq 3}(x) \rangle = 4 \left( \frac{(1 + \sqrt{x})^2}{(1 - \sqrt{x})^6} \left[ -(1 - 4\sqrt{x} + x)^2 \ln(1 - x) - x(1 - 8\sqrt{x} + 2x) \right] \right). \quad (4.15)$$

This result agrees with the one of [17], which was obtained by means of the representation theory of  $\text{PSL}(4|4)$ .<sup>18</sup>

<sup>16</sup>Note that our conventions for  $\mathfrak{D}_2$  and hence also for  $\langle P\mathfrak{D}_2^{L \geq 3}(w, y) \rangle$  differ by a factor of 4 with respect to [17].

<sup>17</sup>The matrix element  $(\mathfrak{D}_2^{\mathcal{N}=4})_{A_1 A_2}^{A_1 A_2}$  can be obtained from [20] by setting  $A_3 = A_1$  and  $A_4 = A_2$ . In addition, we have to shift the summation variables according to

$$\tilde{a}^\alpha = a_{(1)}^\alpha - a^\alpha, \quad \tilde{b}^\alpha = b_{(1)}^\alpha - b^\alpha, \quad \tilde{c}^e = c_{(1)}^e - c^e, \quad (4.14)$$

which amounts to summing over oscillators that hop from one site to the other instead of oscillators that stay at their original position.

<sup>18</sup>Recall the factor of 4 difference between our convention for  $\mathfrak{D}_2$  and the one of [17].

#### 4.4 The finite-size contributions $Z_{\text{f.s.c.}}^{(1)}(x)$

Finally, we need to account for the one-loop finite-size effects mentioned in section 2 and calculate  $Z_{\text{f.s.c.}}^{(1)}(x)$  defined in (3.15).

For the  $\gamma_i$ -deformation with gauge group  $U(N)$  and vanishing tree-level multi-trace couplings, the only finite-size contributions stem from the  $L = 1$  operators whose anomalous dimensions were given in (2.16). According to (3.13), their contributions to the partition function are

$$Z_{\text{f.s.c. } U(N)}^{(1)}(x, \gamma_i^\pm) = \sum_{A \in \mathcal{A}} x^{(\mathcal{D}_0)_A} (\mathcal{D}_2^{L=1})_A = 8 \sum_{i=1}^3 \left( \sin^2 \frac{\gamma_i^+}{2} + \sin^2 \frac{\gamma_i^-}{2} \right) \frac{x - x^3 + x^{\frac{3}{2}} - x^{\frac{5}{2}}}{(1-x)^4}, \quad (4.16)$$

which can be calculated in a similar way as  $z(x)$  in subsection 4.1. The special case of the  $U(N)$   $\beta$ -deformation can be obtained by setting the deformation parameters  $\gamma_i^- = 0$  and  $\gamma_i^+ = -\beta$ .

In the  $\beta$ -deformation with gauge group  $SU(N)$ , the prewrapping corrections have to be accounted for, i.e. instead of the asymptotic density (2.12) the finite-size corrected density (2.14) has to be taken at  $L = 2$ . Inserting (3.12) into (3.15) and using (4.13), we find

$$Z_{\text{f.s.c. } SU(N)}^{(1)}(x, \beta) = \langle P\mathcal{D}_2^{L=2}(x, x) \rangle - \langle P\mathcal{D}_2^{L \geq 3}(x, x) \rangle = -6 \frac{(x + x^{\frac{3}{2}})^2}{(1-x)^4} \left( 8 \sin^2 \frac{\beta}{2} \right), \quad (4.17)$$

where the calculation of  $\langle P\mathcal{D}_2^{L=2}(x, x) \rangle$  is presented in appendix D.

In the  $\beta$ -deformation, (4.16) and (4.17) can directly be understood in terms of the anomalous dimensions and characters of the supermultiplets that were identified as affected by the finite-size effects in [20].

## 5 Partition function and Hagedorn temperature

The one-loop partition function of the  $\beta$ -deformation with gauge group  $U(N)$  and  $SU(N)$  and the  $\gamma_i$ -deformation with gauge group  $U(N)$  can be obtained by assembling the ingredients from the previous section. In particular, the final result for the multi-trace partition function is obtained by inserting (3.19), (4.8), (4.15) and (4.17) into (3.20). As it does not allow for significant simplifications, we refrain from showing it.<sup>19</sup> We have checked the perturbative expansion in  $x = e^{-1/RT}$  for gauge group  $U(N)$  by modifying the approach of [15]. The details of this modification are shown in appendix F and the results of both methods agree. Next, we calculate the Hagedorn temperature.

The Hagedorn temperature  $T_H = T_H(g = 0)$  of free planar  $\mathcal{N} = 4$  SYM theory was first calculated in [16]. As  $z(x)$  is a monotonically increasing function of  $x$ , the partition

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<sup>19</sup>The terms in the perturbative expansion of the partition function can be understood via the multiplets of single-trace operators and their one-loop anomalous dimensions. For the  $\beta$ -deformation, the latter were determined for all operators with classical scaling dimension  $\Delta_0 \leq 4.5$  in [20].



function (3.19) first diverges if  $z(x) = 1$ . From  $z(x_H) = 1$  and (4.2), we find

$$T_H = \frac{1}{\ln(7 + 4\sqrt{3})} \frac{1}{R}. \tag{5.1}$$

This temperature is independent of the deformation parameters, as the free  $\beta$ - and  $\gamma_i$ -deformation coincide with the free  $\mathcal{N} = 4$  SYM theory. Remarkably, the one-loop correction to the Hagedorn temperature is also independent of the deformation parameters — as we will show in the following.

For  $\mathcal{N} = 4$  SYM theory, the one-loop correction to the Hagedorn temperature was calculated in [17]. At this temperature, the partition function has a simple pole:

$$\mathcal{Z}(x) \sim \frac{C}{x_H - x}, \tag{5.2}$$

with a constant  $C$ . Expanding it around the pole, one finds

$$\frac{C}{x_H + \delta x_H - x} = \frac{C}{x_H - x} \left[ 1 - \frac{\delta x_H}{x_H - x} + \dots \right]. \tag{5.3}$$

For the deformed theories, we follow the reasoning of [17] and compare the expansion (5.3) to the multi-trace partition function (3.20). As in the undeformed case, we find that only the  $k = 1$  term in the second sum of (3.20) contributes to the double pole which shifts the Hagedorn temperature at one loop. In principle, the first and third sum could also develop divergences when evaluated. In the  $\mathfrak{su}(2)$ ,  $\mathfrak{so}(6)$  and  $\mathfrak{su}(2|3)$  subsectors, this does, however, not occur; the first and third sum can be evaluated in a closed form and their contribution is finite for finite temperatures. Numerical studies suggest that their contribution in the full theory remains finite at  $T_H$  as well.

As  $\langle \mathfrak{D}_2^{L \geq 3}(x) \rangle$  and  $z(x)$  are undeformed in the  $\beta$ - and  $\gamma_i$ -deformation, so is the one-loop correction to the Hagedorn temperature. From the residue of the aforementioned  $k = 1$  term, we obtain

$$\delta x_H = - \lim_{x \rightarrow x_H} \left[ g^2 (x_H - x) \ln x \frac{\langle \mathfrak{D}_2^{L \geq 3}(x) \rangle}{1 - z(x)} \right] = - \frac{2}{3} g^2 x_H \ln x_H \langle \mathfrak{D}_2^{L \geq 3}(x_H) \rangle. \tag{5.4}$$

Inserting (5.1) into (4.15) yields  $\langle \mathfrak{D}_2^{L \geq 3}(x_H) \rangle = 3$ . Thus,

$$\frac{\delta x_H}{x_H} = -2g^2 \ln x_H \tag{5.5}$$

and

$$\frac{\delta T_H}{T_H} = - \frac{1}{\ln x_H} \frac{\delta x_H}{x_H} = 2g^2. \tag{5.6}$$

Hence, the one-loop Hagedorn temperature of  $\beta$ - and  $\gamma_i$ -deformed  $\mathcal{N} = 4$  SYM theory is given by

$$T_H(g) = T_H (1 + 2g^2 + \dots), \quad g^2 = \frac{\lambda}{(4\pi)^2} = \frac{Ng_{\text{YM}}^2}{(4\pi)^2}, \tag{5.7}$$

which is identical to the one-loop Hagedorn temperature of  $\mathcal{N} = 4$  SYM theory as computed in [17].

## 6 Conclusion and outlook

In this paper, we have calculated the one-loop partition function of the  $\beta$ - and  $\gamma_i$ -deformation of  $\mathcal{N} = 4$  SYM theory on  $\mathbb{R} \times \mathbb{S}^3$ .

For the computation of the partition function, we have used an extension of the generalised Pólya-theory method of [17], which is based on the dilatation operator of the theory on  $\mathbb{R}^4$ . Separating off the combinatorial problem, the thermal one-loop partition function of  $\mathcal{N} = 4$  SYM theory can be given in terms of three physical ingredients alone: the single-site partition function  $z(x)$  and the generalised expectation values  $\langle \mathfrak{D}_2^{L \geq 3}(x) \rangle$  and  $\langle P\mathfrak{D}_2^{L \geq 3}(w, y) \rangle$ . In the  $\beta$ - and  $\gamma_i$ -deformed theories, the thermal one-loop partition function in addition depends on the finite-size contributions which arise from the length-dependent contributions of the respective dilatation operator. We have calculated all four ingredients in the deformed theories in a purely combinatorial approach. In the limit of vanishing deformation parameters, we reproduce the results of [17] and also provide an analytic derivation for  $\langle P\mathfrak{D}_2^{L \geq 3}(w, y) \rangle$ .

We have checked our result for the one-loop partition function in the case of gauge group  $U(N)$  by modifying the direct path-integral/Feynman-diagram calculation [15] on  $\mathbb{R} \times \mathbb{S}^3$  to include the  $\beta$ - and  $\gamma_i$ -deformation. The results of both methods agree.

The space-time manifold  $\mathbb{R} \times \mathbb{S}^3$ , with small radius  $R$ , enables a meaningful treatment of thermodynamic phenomena in gauge theories via perturbation theory. The maximally supersymmetric  $\mathcal{N} = 4$  SYM theory as well its  $\beta$ - and  $\gamma_i$ -deformations exhibit a Hagedorn transition that was identified with the confinement-deconfinement phase transition in the free theory. While the temperature of this phase transition is trivially independent of the deformations in the free theory, this independence is not guaranteed at the  $\mathcal{O}(\lambda)$  correction, where the dilatation operator becomes deformation-parameter dependent. Remarkably, we have found that the  $\mathcal{O}(\lambda)$  correction to the Hagedorn temperature is independent of the deformation parameters as well.

In [35], the thermodynamic properties of a certain one-parameter deformed background in string theory were investigated. The Hagedorn temperature was found to be undeformed although the partition function depends non-trivially on the deformation parameter. In subsequent research, it was tried to repeat this analysis for the string-theory dual to the  $\beta$ -deformation, but the analysis was only successful in sectors that do not allow for non-trivial tests [36]. Motivated by our results and the findings on the string theory side, it is tempting to propose that the Hagedorn temperature of the  $\beta$ -deformation is independent of the deformation parameters at all values of  $\lambda$ . Further investigations are clearly required.

It would also be interesting to compute the partition function and the Hagedorn temperature in  $\mathcal{N} = 4$  SYM theory to two-loop order. In [37], it was shown how to generalise the method of [17] to two loops in the  $\mathfrak{su}(2)$  sector and a generalisation to two or even more loops in the full theory seems feasible. Once the complete two-loop dilatation operator of  $\mathcal{N} = 4$  SYM theory becomes available, computing the two-loop partition function of  $\mathcal{N} = 4$  SYM theory reduces to a combinatorial exercise. Moreover, it would be interesting to determine the order of the phase transition in  $\mathcal{N} = 4$  SYM theory like it was done in [13] for pure Yang-Mills theory on  $\mathbb{R} \times \mathbb{S}^3$ .

In [17], it was remarked that the Pólya-theory approach to partition functions makes no apparent use of the integrability observed in  $\mathcal{N} = 4$  SYM theory. Naively, this is not surprising: the Bethe ansatz is a powerful tool for finding eigenvalues of  $D$  and hence of  $x^D$ , i.e. roots of the characteristic polynomial, while the partition function is the trace of  $x^D$ , which is determined by the second to highest term of the characteristic polynomial alone. However, the whole spectrum of  $\mathcal{N} = 4$  SYM theory is — in principle — known via integrability and by knowing the spectrum the partition function can be constructed immediately. It would thus be very interesting to develop a formalism that directly employs the methods of integrability to derive the thermal partition function of  $\mathcal{N} = 4$  SYM theory. This might even lead to a closed all-loop expression.

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### A The one-loop anomalous dimensions of fermionic $L = 1$ operators

In this appendix, we calculate the one-loop self-energies of the fermions, from which we obtain the anomalous dimensions of the fermionic  $L = 1$  operators in the deformed theories with gauge group  $U(N)$ .

Using the Feynman rules of [19] and Fermi-Feynman gauge, we arrive at the following divergent one-loop contributions to the fermionic propagator

$$\begin{aligned}
 K \left[ \alpha A a \rightarrow \text{---} \overset{\curvearrowright}{\text{---}} \text{---} \beta B b \right] &= -\frac{g^2}{\varepsilon} \delta_A^B (1 - \delta_A^4) \left[ 2 \text{tr}(\mathbf{T}^a \mathbf{T}^b) - \frac{1}{N} \text{tr}(\mathbf{T}^a) \text{tr}(\mathbf{T}^b) \sum_{\substack{k=1 \\ k \neq A}}^3 \cos \gamma_k^+ \right] p_{\dot{\beta}}^\alpha, \\
 K \left[ \alpha A a \rightarrow \text{---} \overset{\curvearrowright}{\text{---}} \text{---} \beta B b \right] &= -\frac{g^2}{\varepsilon} \delta_A^B \left[ (1 + 2\delta_A^4) \text{tr}(\mathbf{T}^a \mathbf{T}^b) - \frac{1}{N} \text{tr}(\mathbf{T}^a) \text{tr}(\mathbf{T}^b) \sum_{k=1}^3 (\delta_A^k + \delta_A^4) \cos \gamma_k^- \right] p_{\dot{\beta}}^\alpha, \\
 K \left[ \alpha A a \rightarrow \text{---} \text{---} \text{---} \beta B b \right] &= -\frac{g^2}{\varepsilon} \delta_A^B \left[ \text{tr}(\mathbf{T}^a \mathbf{T}^b) - \frac{1}{N} \text{tr}(\mathbf{T}^a) \text{tr}(\mathbf{T}^b) \right] p_{\dot{\beta}}^\alpha,
 \end{aligned} \tag{A.1}$$

where the operator  $K$  extracts the UV divergence of the respective diagram and  $g$  was defined in (2.10). Taking the sum of the three contributions yields the counterterm for the  $U(1)$  component of the fermionic fields<sup>20</sup>

$$\delta \mathcal{Z}_{\psi^A, U(1)}^{(1)} = -\frac{g^2}{\varepsilon} \left( 3 - \sum_{k=1}^3 \left( \cos \gamma_k^+ + (\delta_A^k + \delta_A^4) (\cos \gamma_k^- - \cos \gamma_k^+) \right) \right). \quad (\text{A.2})$$

The trace of an  $L = 1$  operator projects to the  $U(1)$  component and the covariant derivatives reduce to the ordinary ones in this case. Moreover, all diagrams contributing to the anomalous dimension of such operators are of one-particle-reducible type. Hence, the operator renormalisation constant is simply given by

$$\mathcal{Z}_{\text{tr}(D^k \psi^A)} = \mathcal{Z}_{\psi^A, U(1)}^{-\frac{1}{2}}. \quad (\text{A.3})$$

Accordingly, the required one-loop anomalous dimensions are given by

$$\begin{aligned} \gamma_{\text{tr}(D^k \psi^A)}^{(1)} &= \lim_{\varepsilon \rightarrow 0} \left( g \varepsilon \frac{d}{dg} \ln \mathcal{Z}_{\psi^A, U(1)}^{-\frac{1}{2}} \right)^{(1)} \\ &= 2 \sum_{k=1}^3 \left[ \sin^2 \frac{\gamma_k^+}{2} + (\delta_A^k + \delta_A^4) \left( \sin^2 \frac{\gamma_k^-}{2} - \sin^2 \frac{\gamma_k^+}{2} \right) \right]. \end{aligned} \quad (\text{A.4})$$

## B The oscillator picture

In this appendix, we present our conventions for the oscillator picture [18, 38, 39] and the action of the undeformed one-loop dilatation-operator density therein.

The fields from the alphabet (2.7) can be represented via two sets of bosonic oscillators  $\mathbf{a}^{\dagger\alpha}$  ( $\alpha = 1, 2$ ) and  $\mathbf{b}^{\dagger\dot{\alpha}}$  ( $\dot{\alpha} = 1, 2$ ) and one set of fermionic oscillators  $\mathbf{c}^{\dagger A}$  ( $A = 1, 2, 3, 4$ ). These oscillators obey the usual (anti-)commutation relations:

$$[\mathbf{a}_\alpha, \mathbf{a}^{\dagger\beta}] = \delta_\alpha^\beta, \quad [\mathbf{b}_{\dot{\alpha}}, \mathbf{b}^{\dagger\dot{\beta}}] = \delta_{\dot{\alpha}}^{\dot{\beta}}, \quad \{\mathbf{c}_A, \mathbf{c}^{\dagger B}\} = \delta_A^B, \quad (\text{B.1})$$

with all other (anti-)commutators vanishing. In terms of the oscillators, the fields are

$$\begin{aligned} D^k \mathcal{F} &\hat{=} (\mathbf{a}^\dagger)^{k+2} (\mathbf{b}^\dagger)^k |0\rangle, \\ D^k \psi^A &\hat{=} (\mathbf{a}^\dagger)^{k+1} (\mathbf{b}^\dagger)^k \mathbf{c}^{\dagger A} |0\rangle, \\ D^k \varphi^{AB} &\hat{=} (\mathbf{a}^\dagger)^k (\mathbf{b}^\dagger) \mathbf{c}^{\dagger A} \mathbf{c}^{\dagger B} |0\rangle, \\ D^k \bar{\psi}^{ABC} &\hat{=} (\mathbf{a}^\dagger)^k (\mathbf{b}^\dagger)^{k+1} \mathbf{c}^{\dagger A} \mathbf{c}^{\dagger B} \mathbf{c}^{\dagger C} |0\rangle, \\ D^k \bar{\mathcal{F}} &\hat{=} (\mathbf{a}^\dagger)^k (\mathbf{b}^\dagger)^{k+2} \mathbf{c}^{\dagger 1} \mathbf{c}^{\dagger 2} \mathbf{c}^{\dagger 3} \mathbf{c}^{\dagger 4} |0\rangle, \end{aligned} \quad (\text{B.2})$$

where  $\bar{\psi}_D = \frac{1}{3!} \varepsilon_{ABCD} \bar{\psi}^{ABC}$  and we have suppressed all spinor indices. All physical fields fulfil the central charge constraint, i.e. the action of the following operator on them vanishes:

$$C = \sum_{\alpha=1}^2 \mathbf{a}^{\dagger\alpha} \mathbf{a}_\alpha - \sum_{\dot{\alpha}=1}^2 \mathbf{b}^{\dagger\dot{\alpha}} \mathbf{b}_{\dot{\alpha}} + \sum_{A=1}^4 \mathbf{c}^{\dagger A} \mathbf{c}_A - 2. \quad (\text{B.3})$$

<sup>20</sup>Note that  $\text{tr}(T^a T^b) = \delta^{ab}$ ,  $\text{tr}(T^a) = \sqrt{N} \delta^{a0}$  and the inverse of the propagator  $-\frac{p_{\dot{\alpha}}^\beta}{p^2}$  is  $\frac{p_{\dot{\beta}}^\alpha}{2}$ .

A generic  $n$ -site state is given by  $n$  families of oscillators  $(\mathbf{a}_{(i)}^\dagger, \mathbf{b}_{(i)}^\dagger, \mathbf{c}_{(i)}^\dagger)$  individually satisfying (B.3). It can be represented in terms of its oscillator occupation numbers

$$A_i = (a_{(i)}^1, a_{(i)}^2, b_{(i)}^1, b_{(i)}^2, c_{(i)}^1, c_{(i)}^2, c_{(i)}^3, c_{(i)}^4), \quad (\text{B.4})$$

where the index  $i = 1, \dots, n$  specifies the site on which the corresponding oscillators act.

In [18], the harmonic action is given as a weighted sum over all reorderings of the oscillators at two neighbouring sites with weight (2.13). An explicit expression in terms of the occupation numbers (B.4) can be found in [20]. Restricting the expression (C.4) in [20] to the permuted diagonal element, i.e.  $A_3 = A_2$  and  $A_4 = A_1$ , we find

$$\begin{aligned} (\mathfrak{D}_2^{N=4})_{A_2 A_1}^{A_1 A_2} &= \prod_{\alpha=1}^2 \left( \sum_{a^\alpha=0}^{\infty} \binom{a_{(1)}^\alpha}{a^\alpha} \binom{a_{(2)}^\alpha}{a_{(2)}^\alpha - a^\alpha} \right) \prod_{\dot{\alpha}=1}^2 \left( \sum_{b^{\dot{\alpha}}=0}^{\infty} \binom{b_{(1)}^{\dot{\alpha}}}{b^{\dot{\alpha}}} \binom{b_{(2)}^{\dot{\alpha}}}{b_{(2)}^{\dot{\alpha}} - b^{\dot{\alpha}}} \right) \\ &\times \prod_{e=1}^4 \left( \sum_{c^e=0}^1 \binom{c_{(1)}^e}{c^e} \binom{c_{(2)}^e}{c_{(2)}^e - c^e} \right) (-1)^{\sum_{e=1}^4 \sum_{l=1}^4 c_{(1)}^e c_{(2)}^l + c^e} \\ &\times c_h \left[ \sum_{i=1}^2 \left( \sum_{\alpha=1}^2 a_{(i)}^\alpha + \sum_{\dot{\alpha}=1}^2 b_{(i)}^{\dot{\alpha}} + \sum_{e=1}^4 c_{(i)}^e \right), \right. \\ &\quad \sum_{\alpha=1}^2 (a_{(1)}^\alpha - a^\alpha) + \sum_{\dot{\alpha}=1}^2 (b_{(1)}^{\dot{\alpha}} - b^{\dot{\alpha}}) + \sum_{e=1}^4 (c_{(1)}^e - c^e), \\ &\quad \left. \sum_{\alpha=1}^2 (a_{(2)}^\alpha - a^\alpha) + \sum_{\dot{\alpha}=1}^2 (b_{(2)}^{\dot{\alpha}} - b^{\dot{\alpha}}) + \sum_{e=1}^4 (c_{(2)}^e - c^e) \right], \end{aligned} \quad (\text{B.5})$$

where  $a^\alpha$ ,  $b^\alpha$  and  $c^e$  are the numbers of oscillators that stay at their initial sites and  $c_h$  is given in (2.13).

### C The calculation of $\langle P\mathfrak{D}_2^{L \geq 3}(w, y) \rangle$

In section 4, we have sketched the computation of  $\langle P\mathfrak{D}_2^{L \geq 3}(w, y) \rangle$ . In this appendix, we present the details of this computation in three steps, starting from (4.3).

In the first step, we use that the harmonic action  $c_h$  is insensitive to the kind of oscillator that is hopping. We rewrite the occurring bosonic summations in terms of the variables

$$a_{(i)} = \sum_{\alpha=1}^2 a_{(i)}^\alpha, \quad a = \sum_{\alpha=1}^2 a^\alpha, \quad b_{(i)} = \sum_{\dot{\alpha}=1}^2 b_{(i)}^{\dot{\alpha}}, \quad b = \sum_{\dot{\alpha}=1}^2 b^{\dot{\alpha}}, \quad (\text{C.1})$$

for the sites  $i = 1, 2$ . For each pair  $(a_{\bullet}^1, a_{\bullet}^2) \in \{(a_{(1)}^1, a_{(1)}^2), (a_{(2)}^1, a_{(2)}^2), (a^1, a^2)\}$ , we use the summation identity

$$\sum_{a_{\bullet}^1, a_{\bullet}^2=0}^{\infty} f(a_{\bullet}^1, a_{\bullet}^2) = \sum_{a_{\bullet}=0}^{\infty} \sum_{\tilde{a}_{\bullet}=0}^{a_{\bullet}} f(\tilde{a}_{\bullet}, a_{\bullet} - \tilde{a}_{\bullet}), \quad (\text{C.2})$$

which is valid for any function  $f$ , to express all occurrences of  $a_{\bullet}^{\alpha}$  in terms of  $a_{\bullet}$  and  $\tilde{a}_{\bullet}$ . In the resulting expressions, the sums over  $\tilde{a}_{(i)}$  can be performed via the identity

$$\sum_{\tilde{a}_{(i)}=0}^{a_{(i)}} \binom{\tilde{a}_{(i)}}{\tilde{a}} \binom{a_{(i)} - \tilde{a}_{(i)}}{a - \tilde{a}} = \binom{a_{(i)} + 1}{a + 1}, \quad (\text{C.3})$$

and the remaining sum over  $\tilde{a}$  simply yields a factor of  $(a + 1)$ . This procedure allows us to directly perform three of the six original sums over **a**-type oscillators. The sums involving **b**-type oscillators can be treated analogously. While the coefficient  $c_{\text{h}}$  is independent of the kind of **c**-type oscillator that is hopping, the phase factor which incorporates the deformation depends on it. We define

$$G^{\gamma_i}(c_{(1)}, c_{(2)}, c) = \prod_{e=1}^4 \left( \sum_{c_{(1)}^e, c_{(2)}^e, c^e=0}^1 \binom{c_{(1)}^e}{c^e} \binom{c_{(2)}^e}{c^e} (-1)^{c^e} \right) e^{-i \sum_{l,m=1}^4 c_{(1)}^l c_{(2)}^m \mathbf{q}_{\psi^l} \times \mathbf{q}_{\psi^m}} \quad (\text{C.4})$$

$$\times \delta_{(c_{(1)} - \sum_{e=1}^4 c_{(1)}^e)} \delta_{(c_{(2)} - \sum_{e=1}^4 c_{(2)}^e)} \delta_{(c - \sum_{e=1}^4 c^e)}.$$

With the above simplifications, (4.3) turns into

$$\begin{aligned} \langle P\mathcal{D}_2^{L \geq 3}(w, y) \rangle &= \sum_{a_{(1)}, a_{(2)}, a=0}^{\infty} \sum_{b_{(1)}, b_{(2)}, b=0}^{\infty} \sum_{c_{(1)}, c_{(2)}, c=0}^4 \delta_{(a_{(1)} - b_{(1)} + c_{(1)} - 2)} \delta_{(a_{(2)} - b_{(2)} + c_{(2)} - 2)} \\ &\times w^{\frac{1}{2}(2+a_{(1)}+b_{(1)})} y^{\frac{1}{2}(2+a_{(2)}+b_{(2)})} G^{\gamma_i}(c_{(1)}, c_{(2)}, c) \\ &\times (a+1) \binom{a_{(1)}+1}{a+1} \binom{a_{(2)}+1}{a+1} (b+1) \binom{b_{(1)}+1}{b+1} \binom{b_{(2)}+1}{b+1} \\ &\times c_{\text{h}} \left[ \sum_{i=1}^2 (a_{(i)} + b_{(i)} + c_{(i)}), \right. \\ &\quad (a_{(1)} - a) + (b_{(1)} - b) + (c_{(1)} - c), \\ &\quad \left. (a_{(2)} - a) + (b_{(2)} - b) + (c_{(2)} - c) \right]. \end{aligned} \quad (\text{C.5})$$

We can further use the Kronecker  $\delta$ 's from the central charge constraint to eliminate two of the remaining sums, say those over  $b_{(1)}$  and  $b_{(2)}$ .

In the second step, we employ the integral representation of the harmonic action (4.4) to replace (C.5) by the respective integrand defined in (4.5):

$$\begin{aligned} \langle P\mathcal{D}_2(w, y) \rangle_{\text{int}} &= -2 \sum_{c_{(1)}, c_{(2)}, c=0}^4 \sum_{a, b=0}^{\infty} \sum_{a_{(1)}=\max\{0, 2-c_{(1)}\}}^{\infty} \sum_{a_{(2)}=\max\{0, 2-c_{(2)}\}}^{\infty} \\ &\times G^{\gamma_i}(c_{(1)}, c_{(2)}, c) w^{a_{(1)} + \frac{1}{2}c_{(1)}} y^{a_{(2)} + \frac{1}{2}c_{(2)}} \\ &\times (a+1) \binom{a_{(1)}+1}{a+1} \binom{a_{(2)}+1}{a+1} \\ &\times (b+1) \binom{a_{(1)}+c_{(1)}-1}{b+1} \binom{a_{(2)}+c_{(2)}-1}{b+1} \\ &\times t^{a_{(1)}+c_{(1)}+a_{(2)}+c_{(2)}-3} \left( \frac{t-1}{t} \right)^{a+b+c}. \end{aligned} \quad (\text{C.6})$$

This brings us to the third step. The combinatorial coefficients occurring in (C.6) can be rewritten in terms of differential and integral operators acting on the thermal weights  $w$  and  $y$ :

$$\begin{aligned}
\langle P\mathcal{D}_2(w, y) \rangle_{\text{int}} &= -2 \sum_{c_{(1)}, c_{(2)}, c=0}^4 \sum_{a, b=0}^{\infty} \sum_{a_{(1)}=\max\{0, 2-c_{(1)}\}}^{\infty} \sum_{a_{(2)}=\max\{0, 2-c_{(2)}\}}^{\infty} \\
&\times G^{\gamma_i}(c_{(1)}, c_{(2)}, c) \frac{1}{a!(a+1)!} \frac{1}{b!(b+1)!} \\
&\times \left( w^{-\frac{1}{2}c_{(1)}} \int_0^w dw w^{\frac{1}{2}c_{(1)}-1} \right) \left( w^{-\frac{1}{2}c_{(1)}} w^{b+2} \frac{d^{b+2}}{dw^{b+2}} w^{\frac{1}{2}c_{(1)}} \right) \\
&\times \left( w^{\frac{1}{2}c_{(1)}-1} w^{a+1} \frac{d^{a+1}}{dw^{a+1}} w^{1-\frac{1}{2}c_{(1)}} \right) w^{a_{(1)}+\frac{1}{2}c_{(1)}} \\
&\times \left( y^{-\frac{1}{2}c_{(2)}} \int_0^y dy y^{\frac{1}{2}c_{(2)}-1} \right) \left( y^{-\frac{1}{2}c_{(2)}} y^{b+2} \frac{d^{b+2}}{dy^{b+2}} y^{\frac{1}{2}c_{(2)}} \right) \\
&\times \left( y^{\frac{1}{2}c_{(2)}-1} y^{a+1} \frac{d^{a+1}}{dy^{a+1}} y^{1-\frac{1}{2}c_{(2)}} \right) y^{a_{(2)}+\frac{1}{2}c_{(2)}} \\
&\times t^{a_{(1)}+c_{(1)}+a_{(2)}+c_{(2)}-3} \left( \frac{t-1}{t} \right)^{a+b+c},
\end{aligned} \tag{C.7}$$

where both differential and integral operators act on everything on their right.<sup>21</sup> Since those operators do not explicitly depend on  $a_{(1)}$  and  $a_{(2)}$ , we can now perform the sums over these two variables using the well known result for geometric series:

$$\begin{aligned}
\langle P\mathcal{D}_2(w, y) \rangle_{\text{int}} &= -2 \sum_{c_{(1)}, c_{(2)}, c=0}^4 \sum_{a, b=0}^{\infty} G^{\gamma_i}(c_{(1)}, c_{(2)}, c) \frac{1}{a!(a+1)!} \frac{1}{b!(b+1)!} \\
&\times \left( w^{-\frac{1}{2}c_{(1)}} \int_0^w dw w^{\frac{1}{2}c_{(1)}-1} \right) \left( w^{-\frac{1}{2}c_{(1)}} w^{b+2} \frac{d^{b+2}}{dw^{b+2}} w^{\frac{1}{2}c_{(1)}} \right) \\
&\times \left( w^{\frac{1}{2}c_{(1)}-1} w^{a+1} \frac{d^{a+1}}{dw^{a+1}} w^{1-\frac{1}{2}c_{(1)}} \right) w^{\frac{1}{2}c_{(1)}} \frac{(wt)^{\max\{0, 2-c_{(1)}\}}}{1-wt} \\
&\times \left( y^{-\frac{1}{2}c_{(2)}} \int_0^y dy y^{\frac{1}{2}c_{(2)}-1} \right) \left( y^{-\frac{1}{2}c_{(2)}} y^{b+2} \frac{d^{b+2}}{dy^{b+2}} y^{\frac{1}{2}c_{(2)}} \right) \\
&\times \left( y^{\frac{1}{2}c_{(2)}-1} y^{a+1} \frac{d^{a+1}}{dy^{a+1}} y^{1-\frac{1}{2}c_{(2)}} \right) y^{\frac{1}{2}c_{(2)}} \frac{(yt)^{\max\{0, 2-c_{(2)}\}}}{1-yt} \\
&\times t^{c_{(1)}+c_{(2)}-3} \left( \frac{t-1}{t} \right)^{a+b+c}.
\end{aligned} \tag{C.8}$$

Defining an abbreviation for the second half of the respective  $w$ - and  $y$ -dependent lines,

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<sup>21</sup>Note that, in a slight abuse of notation, we have labelled the integration variable with the same symbol as the upper integration boundary.

we find

$$\begin{aligned}
 O(x, a, c_{(i)}) &= x^{\frac{1}{2}c_{(i)}} \left( x^{\frac{1}{2}c_{(i)}-1} x^{a+1} \frac{d^{a+1}}{dx^{a+1}} x^{1-\frac{1}{2}c_{(i)}} \right) x^{\frac{1}{2}c_{(i)}} \frac{(xt)^{\max\{0, 2-c_{(i)}\}}}{1-xt} \\
 &= \frac{(a+1)!(tx)^a}{(1-tx)^{a+2}} x^{c_{(i)}} - \delta_{c_{(i)}} (\delta_a + 2tx\delta_a + 2tx\delta_{(a-1)}) - \delta_{(c_{(i)}-1)}\delta_a x,
 \end{aligned} \tag{C.9}$$

for  $(i, x) \in \{(1, w), (2, y)\}$ . This allows us to perform the sum over  $a$  via the following identity<sup>22</sup>

$$\begin{aligned}
 &\sum_{a=0}^{\infty} \frac{1}{a!(a+1)!} \left( \frac{t-1}{t} \right)^a O(w, a, c_{(1)}) O(y, a, c_{(2)}) \\
 &= \frac{w^{c_{(1)}} y^{c_{(2)}}}{(1-t(w+y-wy))^2} \\
 &\quad - \frac{y^{c_{(2)}}}{(1-ty)^3} \left[ \delta_{c_{(1)}} (1+t(2w-y-2wy)) - \delta_{(c_{(1)}-1)} w(1-ty) \right] \\
 &\quad - \frac{w^{c_{(1)}}}{(1-tw)^3} \left[ \delta_{c_{(2)}} (1+t(2y-w-2wy)) - \delta_{(c_{(2)}-1)} y(1-tw) \right] \\
 &\quad + wy\delta_{(c_{(1)}-1)}\delta_{(c_{(2)}-1)} + (1+2t(w+y-wy+twy))\delta_{c_{(1)}}\delta_{c_{(2)}} \\
 &\quad + y(1+2tw)\delta_{c_{(1)}}\delta_{(c_{(2)}-1)} + w(1+2ty)\delta_{c_{(2)}}\delta_{(c_{(1)}-1)}.
 \end{aligned} \tag{C.10}$$

Inserting this identity into (C.8), the last four lines of (C.10) drop out, as they are at most linear in either  $w$  or  $y$ . This leaves us with

$$\begin{aligned}
 \langle P\mathcal{D}_2(w, y) \rangle_{\text{int}} &= -2 \sum_{c_{(1)}, c_{(2)}, c=0}^4 \sum_{b=0}^{\infty} G^{\gamma_i}(c_{(1)}, c_{(2)}, c) \frac{1}{b!(b+1)!} \\
 &\quad \times \left( w^{-\frac{1}{2}c_{(1)}} \int_0^w dw w^{\frac{1}{2}c_{(1)}-1} \right) \left( y^{-\frac{1}{2}c_{(2)}} \int_0^y dy y^{\frac{1}{2}c_{(2)}-1} \right) \\
 &\quad \times t^{c_{(1)}+c_{(2)}-3} \left( \frac{t-1}{t} \right)^{b+c} w^{b+2-\frac{1}{2}c_{(1)}} y^{b+2-\frac{1}{2}c_{(2)}} \\
 &\quad \times \frac{d^{b+2}}{dw^{b+2}} \frac{d^{b+2}}{dy^{b+2}} \frac{w^{c_{(1)}} y^{c_{(2)}}}{(1-t(w+y-wy))^2}.
 \end{aligned} \tag{C.11}$$

Writing the last factor in (C.11) as a power series in the variables  $\hat{w} = w-1$  and  $\hat{y} = y-1$  as

$$\frac{w^{c_{(1)}} y^{c_{(2)}}}{(1-t(w+y-wy))^2} = \sum_{\alpha, \beta=0}^4 \binom{c_{(1)}}{\alpha} \binom{c_{(2)}}{\beta} \sum_{n=0}^{\infty} \frac{n+1}{(1-t)^2} \left( \frac{t}{t-1} \right)^n \hat{w}^{n+\alpha} \hat{y}^{n+\beta}, \tag{C.12}$$

we can apply the remaining derivative operators. Combining all  $b$ -dependent terms and

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<sup>22</sup>This identity can be found with the help of Mathematica.



substituting  $l = b + 2$ , we finally obtain

$$\begin{aligned}
 \langle PD_2(w, y) \rangle_{\text{int}} &= -2 \sum_{c_{(1)}, c_{(2)}, c=0}^4 \sum_{\alpha, \beta=0}^4 \binom{c_{(1)}}{\alpha} \binom{c_{(2)}}{\beta} G^{\gamma_i}(c_{(1)}, c_{(2)}, c) \\
 &\times \left( w^{-\frac{1}{2}c_{(1)}} \int_0^w dw w^{-1} \right) \left( y^{-\frac{1}{2}c_{(2)}} \int_0^y dy y^{-1} \right) \\
 &\times t^{c_{(1)}+c_{(2)}-c-1} (t-1)^{c-4} (w-1)^\alpha (y-1)^\beta \\
 &\times \xi_{\alpha\beta} \left( \frac{t}{t-1} (w-1)(y-1), \frac{t-1}{t} \frac{wy}{(w-1)(y-1)} \right),
 \end{aligned} \tag{C.13}$$

where the function  $\xi_{\alpha\beta}(X, Y)$  is defined as

$$\xi_{\alpha\beta}(X, Y) = \sum_{l=0}^{\infty} \sum_{n=0}^{\infty} (l-1)l^2(n+1) \binom{n+\alpha}{l} \binom{n+\beta}{l} Y^n X^l. \tag{C.14}$$

The evaluation of this function is presented in appendix E.

Anticipating the results of appendix E, the expression (C.13) does no longer contain any infinite sums. The remaining finite sums and integrals can be evaluated with the help of *Mathematica*. Assembling everything in (4.6), we find the result (4.8)–(4.11).

## D The calculation of $Z_{\text{f.s.c.}}^{(1)}(\mathbf{x})$

In this appendix, we compute the  $L = 2$  finite-size corrections to the single-trace partition function of the  $\beta$ -deformation with gauge group  $SU(N)$ . They arise when using the finite-size-corrected dilatation-operator density (2.14) instead of the asymptotic version (2.12). The calculation of  $\langle P\mathfrak{D}_2^{L=2}(w, y) \rangle$  differs from that of  $\langle P\mathfrak{D}_2^{L \geq 3}(w, y) \rangle$  only in the sums over fermionic oscillators. Hence, it is sufficient to give the appropriate definition of the fermionic occupation number function  $G_{L=2}^\beta(c_{(1)}, c_{(2)}, c)$ , which has to replace the asymptotic function  $G^{\gamma_i}(c_{(1)}, c_{(2)}, c)$  in the derivation of appendix C.

According to the prescription (2.14) for the finite-size-corrected dilatation-operator density  $\mathfrak{D}_2^{L=2}$ , the deformation parameter  $\beta$  in  $\mathfrak{D}^{L \geq 3}$  has to be set to zero whenever the fields  $A_i$  at site 1 and 2 are either taken from the subalphabet  $\mathcal{A}_{\text{matter}}$  or from  $\bar{\mathcal{A}}_{\text{matter}}$ , which were defined in (2.15). In the oscillator picture, these restrictions translate to the constraints

$$A_i \in \mathcal{A}_{\text{matter}} \Leftrightarrow \sum_{e=1}^3 c_{(i)}^e = 1, \quad A_i \in \bar{\mathcal{A}}_{\text{matter}} \Leftrightarrow \sum_{e=1}^3 c_{(i)}^e = 2, \tag{D.1}$$

which have to be included in the fermionic occupation number function  $G^{\gamma_i}(c_{(1)}, c_{(2)}, c)$  of (C.4). This yields

$$\begin{aligned}
 G_{L=2}^\beta(c_{(1)}, c_{(2)}, c) &= \prod_{e=1}^4 \left( \sum_{c_{(1)}^e, c_{(2)}^e, c^e=0}^1 \binom{c_{(1)}^e}{c^e} \binom{c_{(2)}^e}{c^e} (-1)^{c^e} \right) \\
 &\times e^{-i \sum_{l,m=1}^4 c_{(1)}^l c_{(2)}^m \mathbf{q}_\psi^l \times \mathbf{q}_\psi^m} \left| \begin{array}{l} \beta=0 \text{ if } \sum_{e=1}^3 c_{(1)}^e = \sum_{e=1}^3 c_{(2)}^e = 1 \\ \text{or if } \sum_{e=1}^3 c_{(1)}^e = \sum_{e=1}^3 c_{(2)}^e = 2 \end{array} \right. \\
 &\times \delta_{(c_{(1)} - \sum_{e=1}^4 c_{(1)}^e)} \delta_{(c_{(2)} - \sum_{e=1}^4 c_{(2)}^e)} \delta_{(c - \sum_{e=1}^4 c^e)}.
 \end{aligned} \tag{D.2}$$

Inserting  $G_{L=2}^\beta(c_{(1)}, c_{(2)}, c)$  into (C.5) and following the derivation of appendix C yields  $\langle P\mathcal{D}_2^{L=2}(w, y) \rangle$ . Combining it with the asymptotic expression  $\langle P\mathcal{D}_2^{L \geq 3}(w, y) \rangle$ , we obtain the finite-size correction (4.17).

## E Summation identities

In this appendix, we derive summation identities for

$$\xi_{\alpha\beta}(X, Y) = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} (i-1)i^2(j+1) \binom{j+\alpha}{i} \binom{j+\beta}{i} Y^i X^j, \quad (\text{E.1})$$

with  $\alpha, \beta = 0, 1, 2, 3, 4$ . This can be achieved by applying a finite number of derivative and integral operators and using the following identities:

$$\sum_{i=0}^j \binom{j}{i}^2 t^i = (1-t)^j P_j \left( \frac{1+t}{1-t} \right), \quad \sum_{n=0}^{\infty} P_n(x) z^n = (1-2xz+z^2)^{-1/2}, \quad (\text{E.2})$$

where  $P_n(x)$  denotes the  $n^{\text{th}}$  Legendre polynomial.

Note that  $\xi_{\alpha\beta}$  is symmetric in  $\alpha$  and  $\beta$ . Assuming that  $\alpha \geq \beta$ , we find

$$\begin{aligned} & (i-1)i^2(j+1) \binom{j+\alpha}{i} \binom{j+\beta}{i} Y^i X^j \\ &= (i-1)i^2(j+1) \prod_{\gamma=\beta+1}^{\alpha} \left( 1 - \frac{i}{j+\gamma} \right) \binom{j+\alpha}{i}^2 Y^i X^j \\ &= \left( Y^2 \frac{d^2}{dY^2} \right) \left( Y \frac{d}{dY} \right) \left( \frac{d}{dX} X \right) \\ & \quad \prod_{\gamma=\beta+1}^{\alpha} \left[ 1 - \left( Y \frac{d}{dY} \right) \left( \frac{1}{X^\gamma} \int_0^X dX X^{\gamma-1} \right) \right] \binom{j+\alpha}{i}^2 Y^i X^j. \end{aligned} \quad (\text{E.3})$$

Using (E.2), we have

$$\begin{aligned} \sum_{i,j=0}^{\infty} \binom{j+\alpha}{i}^2 Y^i X^j &= \frac{1}{X^\alpha} \sum_{j=0}^{\infty} (X(1-Y))^{j+\alpha} P_{j+\alpha} \left( \frac{1+Y}{1-Y} \right) \\ &= \frac{1}{X^\alpha} \left[ \frac{1}{(1-2X(1+Y) + X^2(1-Y)^2)^{1/2}} - \sum_{k=0}^{\alpha-1} (X(1-Y))^k P_k \left( \frac{1+Y}{1-Y} \right) \right]. \end{aligned} \quad (\text{E.4})$$

Thus, combining (E.3) with (E.4) allows to express  $\xi_{\alpha\beta}(X, Y)$  in a form explicitly solvable with **Mathematica**:

$$\begin{aligned} \xi_{\alpha\beta}(X, Y) &= \left( Y^2 \frac{d^2}{dY^2} \right) \left( Y \frac{d}{dY} \right) \left( \frac{d}{dX} X \right) \prod_{\gamma=\beta+1}^{\alpha} \left( 1 - \left( Y \frac{d}{dY} \right) \left( \frac{1}{X^\gamma} \int_0^X dX X^{\gamma-1} \right) \right) \\ & \quad \frac{1}{X^\alpha} \left[ \frac{1}{(1-2X(1+Y) + X^2(1-Y)^2)^{1/2}} - \sum_{k=0}^{\alpha-1} (X(1-Y))^k P_k \left( \frac{1+Y}{1-Y} \right) \right]. \end{aligned} \quad (\text{E.5})$$

For example for  $(\alpha, \beta) = (2, 1)$ , we obtain

$$\xi_{21}(X, Y) = \frac{24XY^2}{(1 - 2X(1 + Y) + X^2(1 - Y)^2)^{9/2}} \left[ 1 - X^2(5 - 12Y + 5Y^2) + X^3(5 - 6Y - 6Y^2 + 5Y^3) - 9X^4(1 - Y)^2Y - X^5(1 - Y)^4(1 + Y) \right]. \tag{E.6}$$

The remaining expressions give no further insights, and hence we refrain from showing them.

## F Path-integral calculation

In this appendix, we provide a non-trivial test of our Pólya-theoretical approach by comparing the one-loop partition function of the deformed  $\mathcal{N} = 4$  SYM theory with gauge group  $U(N)$  with the one obtained from a Feynman-diagrammatic approach. We briefly present the results of the latter approach for the undeformed  $\mathcal{N} = 4$  SYM theory [15] and provide the necessary modifications for  $\beta$ - and  $\gamma_i$ -deformed theories with gauge group  $U(N)$ . We find agreement for both approaches for gauge group  $U(N)$  and comment on the generalisation for gauge group  $SU(N)$ .

In [15], a Feynman-diagrammatic approach<sup>23</sup> was used to calculate the one-loop partition function<sup>24</sup> of a class of Yang-Mills theories on  $\mathbb{R} \times S^3$  with matter fields transforming in the adjoint representation of the gauge group  $U(N)$ . This calculation employs the vacuum path integral of the Euclidean theory on  $S^1 \times S^3$ , where the circumference of  $S^1$  is  $1/RT = -\ln x$ . The Euclidean action<sup>25</sup> with  $N_s$  scalars  $\Phi^a$ ,  $a = 1, \dots, N_s$ , and  $N_f$  fermions  $\Psi^I$ ,  $I = 1, \dots, N_f$  is assumed to take the form

$$S_E = \int_0^{\frac{1}{RT}} dt \int_{S^3} d\Omega \operatorname{tr} \left\{ \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \Phi^a (-D^2 + 1) \Phi^a + i \Psi^{\dagger I} \sigma^\mu D_\mu \Psi_I - \frac{1}{4} \tilde{g}_{\text{YM}}^2 \mathbf{Q}^{abcd} \Phi_a \Phi_b \Phi_c \Phi_d + \frac{1}{2} \tilde{g}_{\text{YM}} (\rho^{a\dagger})^{IJ} \Psi_I \varepsilon[\Phi^a, \Psi_J] + \frac{1}{2} \tilde{g}_{\text{YM}} \rho_{IJ}^a \Psi^{\dagger I} \varepsilon[\Phi^a, \Psi^{\dagger J}] \right\}, \tag{F.1}$$

where  $\varepsilon$  is used to contract the spinor indices and the coupling tensors  $\mathbf{Q}^{abcd}$  and  $\rho_{IJ}^a$  are general but assumed to stem from commutator interactions. After gauge fixing and integrating out the non-zero modes in the path integral, one obtains an effective action depending on the effective remaining zero-mode  $U$  and  $x = e^{-1/RT}$ . Up to order  $\tilde{g}_{\text{YM}}^2$ , it has the form  $S_{\text{eff}}(U, x) = S_{\text{eff}}^{1\text{-loop}} + S_{\text{eff}}^{2\text{-loop}}$ , with

$$S_{\text{eff}}^{1\text{-loop}} = - \sum_{n=1}^{\infty} \frac{1}{n} z_n(x) \operatorname{tr}(U^n) \operatorname{tr}(U^{\dagger n}), \tag{F.2}$$

<sup>23</sup>The underlying Feynman-diagrammatic formalism for gauge theories on  $\mathbb{R} \times S^3$  was developed in [12–14].

<sup>24</sup>Note that our notion of loops refers to divergent loops in two-point functions or powers of  $g^2$ , whereas the notion of loops in [12–15] refers to loops in vacuum bubbles. As a consequence, ‘two-loop’ in the counting of [15] corresponds to ‘one-loop’ in our counting.

<sup>25</sup>Note that the normalisation of the coupling constant  $\tilde{g}_{\text{YM}}$  of [15] differs by a factor of  $\sqrt{2}$  from our  $g_{\text{YM}}$ .

$$S_{\text{eff}}^{2\text{-loop}} = -\tilde{g}_{\text{YM}}^2 \ln x \left[ N \sum_{n=1}^{\infty} f_n(x) \left( \text{tr}(U^n) \text{tr}(U^{\dagger n}) - 1 \right) + \sum_{n,m=1}^{\infty} f_{nm}(x) \left( \text{tr}(U^n) \text{tr}(U^m) \text{tr}(U^{-n-m}) + c.c. - 2N \right) \right]. \quad (\text{F.3})$$

The coefficients  $f_n(x)$  and  $f_{nm}(x)$  have to be calculated via Feynman diagrams and  $z_n(x) = z(\omega^{n+1}x^n)$  has the same interpretation as the single-site partition function (3.6).

In the large  $N$  limit and for a temperature below the critical temperature, the Euclidean path-integral can be solved in the saddle point approximation; see [12] for details. For the  $U(N)$  theory, the final result for the multi-trace partition function is

$$\mathcal{Z}_{U(N)}(x) = x^{\tilde{\lambda} \tilde{F}_2^{np}(x)} \prod_{n=1}^{\infty} \frac{x^{-\tilde{\lambda} f_n(x)}}{1 - z_n(x) - \tilde{\lambda} n f_n(x) \ln x}, \quad (\text{F.4})$$

where  $\tilde{F}_2^{np}(x) = -2 \sum_{n,m=1}^{\infty} f_{nm}(x)$  and the 't Hooft coupling takes the form  $\tilde{\lambda} = N \tilde{g}_{\text{YM}}^2$ . The explicit Feynman-diagram calculation of [15] yields the coefficients:

$$f_n(x) = f_{1,+}(x^n) + (-1)^{n+1} f_{1,-}(x^n), \quad (\text{F.5})$$

with

$$\begin{aligned} f_{1,+}(x) = & \frac{1}{\pi^2} \left[ \left( -\frac{3}{8} N_s + \frac{1}{8} \text{tr}(\rho^{a\dagger} \rho^a) \right) x \right. \\ & + \left( \frac{1}{4} + \frac{1}{2} N_f + \frac{1}{8} N_s - \frac{1}{8} \mathbf{Q}^{abba} + \frac{1}{2} \text{tr}(\rho^{a\dagger} \rho^a) \right) x^2 \\ & + \left( 4 + 2N_f + \frac{25}{8} N_s - \frac{1}{2} \mathbf{Q}^{abba} + \frac{11}{8} \text{tr}(\rho^{a\dagger} \rho^a) \right) x^3 \\ & + \left( \frac{55}{4} + 5N_f + \frac{41}{4} N_s - \frac{5}{4} \mathbf{Q}^{abba} + 3 \text{tr}(\rho^{a\dagger} \rho^a) \right) x^4 \\ & + \left( 32 + 10N_f + \frac{185}{8} N_s - \frac{5}{2} \mathbf{Q}^{abba} + \frac{45}{8} \text{tr}(\rho^{a\dagger} \rho^a) \right) x^5 \\ & \left. + \left( \frac{245}{4} + \frac{35}{2} N_f + \frac{347}{8} N_s - \frac{35}{8} \mathbf{Q}^{abba} + \frac{19}{2} \text{tr}(\rho^{a\dagger} \rho^a) \right) x^6 + \dots \right], \quad (\text{F.6}) \end{aligned}$$

$$\begin{aligned} f_{1,-}(x) = & \frac{1}{\pi^2} \left[ \frac{1}{4} \text{tr}(\rho^{a\dagger} \rho^a) x^{\frac{3}{2}} + (3N_f + \frac{5}{4} \text{tr}(\rho^{a\dagger} \rho^a)) x^{\frac{5}{2}} \right. \\ & + (12N_f + \frac{7}{2} \text{tr}(\rho^{a\dagger} \rho^a)) x^{\frac{7}{2}} + (30N_f + \frac{15}{2} \text{tr}(\rho^{a\dagger} \rho^a)) x^{\frac{9}{2}} \\ & \left. + (60N_f + \frac{55}{4} \text{tr}(\rho^{a\dagger} \rho^a)) x^{\frac{11}{2}} + \dots \right], \quad (\text{F.7}) \end{aligned}$$

$$\begin{aligned}
 \tilde{F}_2^{np}(x) &= \frac{\mathbf{Q}^{aabb}}{8\pi^2} x^2 - \frac{\text{tr}(\rho^{a\dagger}\rho^a)}{\pi^2} x^{\frac{5}{2}} \\
 &+ \frac{[4\mathbf{Q}^{aabb} - 2\text{tr}(\rho^{a\dagger}\rho^a) - 6N_f - 9N_s]}{4\pi^2} x^3 - \frac{4\text{tr}(\rho^{a\dagger}\rho^a)}{\pi^2} x^{\frac{7}{2}} \\
 &+ \frac{18 + 23\mathbf{Q}^{aabb} - 48N_s}{8\pi^2} x^4 + \frac{3(N_f - 3\text{tr}(\rho^{a\dagger}\rho^a))}{\pi^2} x^{\frac{9}{2}} \\
 &+ \frac{14\mathbf{Q}^{aabb} - \text{tr}(\rho^{a\dagger}\rho^a) - 9N_f - 39N_s}{2\pi^2} x^5 - \frac{20\text{tr}(\rho^{a\dagger}\rho^a)}{\pi^2} x^{\frac{11}{2}} \\
 &+ \frac{42 + 50\mathbf{Q}^{aabb} - 6\text{tr}(\rho^{a\dagger}\rho^a) - 18N_f - 117N_s}{4\pi^2} x^6 + \dots, \tag{F.8}
 \end{aligned}$$

where we have to set  $N_f = 4$  and  $N_s = 6$  to obtain the result for  $\mathcal{N} = 4$  SYM theory and its deformations. With these findings, the authors of [15] were able to reproduce the result derived from the Pólya-theoretic approach [17] for the undeformed  $\mathcal{N} = 4$  SYM theory.

The result for the multi-trace partition function (F.4) can be modified<sup>26</sup> to also capture the  $\beta$ - and  $\gamma_i$ -deformation of  $\mathcal{N} = 4$  SYM, which have non-commutator interactions. To this end, the simplifications which are only justified for commutator interactions have to be undone. In (F.3), the terms which are independent of the zero-mode  $U$  stem from non-planar contributions. For non-commutator interactions, they have coefficients  $\hat{f}_n(x)$  and  $\hat{f}_{nm}(x)$  which are independent of the respective planar coefficients. Hence, we have to generalise (F.3) to

$$\begin{aligned}
 S_{\text{eff}}^{2\text{-loop}} &= -\tilde{g}_{\text{YM}}^2 \ln x \left[ N \sum_{n=1}^{\infty} \left( f_n(x) \text{tr}(U^n) \text{tr}(U^{\dagger n}) - \hat{f}_n(x) \right) \right. \\
 &\quad \left. + \sum_{n,m=1}^{\infty} \left( f_{nm}(x) (\text{tr}(U^n) \text{tr}(U^m) \text{tr}(U^{-n-m}) + c.c.) - 2N \hat{f}_{nm}(x) \right) \right]. \tag{F.9}
 \end{aligned}$$

Defining  $\hat{F}_2^{np}(x) = -2 \sum_{n,m=1}^{\infty} \hat{f}_{nm}(x)$  and performing the saddle point approximation then yields

$$\mathcal{Z}_{\text{U}(N)}(x) = x^{\tilde{\lambda} \hat{F}_2^{np}(x)} \prod_{n=1}^{\infty} \frac{x^{-\tilde{\lambda} \hat{f}_n(x)}}{1 - z_n(x) \tilde{\lambda} n f_n(x) \ln x}. \tag{F.10}$$

While the planar contributions  $f_n(x)$  and  $f_{nm}(x)$  only contain the contracted coupling tensors  $\text{tr}(\rho^{a\dagger}\rho^a)$  and  $\mathbf{Q}^{aabb} = \mathbf{Q}^{abba}$ , the non-planar contributions  $\hat{f}_n(x)$  and  $\hat{f}_{nm}(x)$  must contain  $\text{tr}(\rho^{a*}\rho^a)$  and  $\mathbf{Q}^{abab}$ . In the case of commutator interactions, the non-planar contributions can be expressed in terms of planar ones via  $\rho^{a\dagger} = (\rho^{aT})^* = -\rho^{a*}$  and  $\mathbf{Q}^{abab} = -2\mathbf{Q}^{abba}$ .<sup>27</sup> Thus, to reconstruct  $\hat{f}_n(x)$  and  $\hat{f}_{nm}(x)$  from  $f_n(x)$  and  $f_{nm}(x)$ , we

<sup>26</sup>We thank Ofer Aharony and Ran Yacobi for a very helpful discussion on the exact nature of the necessary modifications.

<sup>27</sup>Mind the typo in footnote 21 of [15].

have to make the following replacements:

$$\begin{aligned} \text{tr}(\rho^{a\dagger}\rho^a) &\rightarrow -\text{tr}(\rho^{a*}\rho^a), \\ \mathbf{Q}^{abba} &\rightarrow -\frac{1}{2}\mathbf{Q}^{abab}, \quad \mathbf{Q}^{aabb} \rightarrow -\frac{1}{2}\mathbf{Q}^{abab}. \end{aligned} \tag{F.11}$$

The coupling tensors  $\mathbf{Q}^{abcd}$  and  $\rho_{IJ}^a$  can be determined by comparing the action of  $\beta$ - and  $\gamma_i$ -deformed  $\mathcal{N} = 4$  SYM theory with gauge group  $U(N)$  (2.4) with the action in (F.1). To this end, we expand the complex scalars in (2.4) as  $\phi^i = (\Phi^i + i\Phi^{i+3})/\sqrt{2}$  and cyclically symmetrise the fields in the quartic scalar vertex. Recall that the coupling constant is  $\tilde{g}_{\text{YM}} = g_{\text{YM}}/\sqrt{2}$ , and that the action (F.1) appears as  $e^{-S}$  in the path integral, whereas the action (2.4) appears as  $e^S$  in the path integral. The relevant contractions of the coupling tensors are

$$\begin{aligned} \mathbf{Q}^{aabb} = \mathbf{Q}^{abba} &= -30, & \mathbf{Q}^{abab} &= 60 - 24 \sum_{i=1}^3 \sin^2 \frac{\gamma_i^+ + \gamma_i^-}{2}, \\ \text{tr}(\rho^{a\dagger}\rho^a) &= 24, & \text{tr}(\rho^{a*}\rho^a) &= 24 - 8 \sum_{i=1}^3 \left( \sin^2 \frac{\gamma_i^+}{2} + \sin^2 \frac{\gamma_i^-}{2} \right). \end{aligned} \tag{F.12}$$

The power series obtained from (F.10) for the multi-trace partition function confirms our result for gauge group  $U(N)$ .<sup>28</sup>

Generalising the above analysis to the  $\beta$ - and  $\gamma_i$ -deformation with gauge group  $SU(N)$  is considerably harder, since the calculation of [15] manifestly uses the Feynman rules derived for gauge group  $U(N)$ . For commutator interactions, the  $U(1)$  fields are free and their contribution to the partition function can simply be divided out in the final result. This is no longer true in the case of non-commutator interactions; in the  $\beta$ - and  $\gamma_i$ -deformation the  $U(1)$  matter modes couple to the  $SU(N)$  matter modes as explicitly shown in [19]. Therefore, to obtain the one-loop partition function in the  $SU(N)$  case, modifications at the level of Feynman diagrams would be necessary. These modifications would have to include the  $SU(N)$  propagators of all fields and in case of the  $\beta$ -deformation the contributions from the double-trace coupling (2.5).

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<sup>28</sup>We verified this up to order 6 in  $x$ , which is the maximum order given in [15].

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