Rigid $6D$ supersymmetry and localization

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ABSTRACT: We construct rigid supersymmetric theories for interacting vector and tensor multiplets on six-dimensional Riemannian spin manifolds. Analyzing the Killing spinor equations, we derive the constraints on these theories. To this end, we reformulate the conditions for supersymmetry as a set of necessary and sufficient conditions on the geometry. The formalism is illustrated with a number of examples, including manifolds that are hermitian, strong Kähler with torsion. As an application, we show that the path integral of pure super Yang-Mills theory defined on a Calabi-Yau threefold $\mathcal{M}_6$ localizes on stable holomorphic bundles over $\mathcal{M}_6$.

Contents

1 Introduction

Recently, supersymmetric field theories on curved backgrounds have regained considerable attention, in particular for the application of localization techniques in the computation of indices, partition functions and Wilson loops, see e.g. [1–5]. A systematic approach to the study of rigid supersymmetry in four-dimensional curved space has been initiated in [6] and further developed in [7–11]. Holographic applications of these theories have been studied in [12, 13] by embedding the curved space at the boundary of an asymptotically AdS space. A similar analysis of supersymmetric theories has been performed for curved five-dimensional spaces in [14, 15] and in [16] for three-manifolds. The construction of these curved theories is based on the existence of an underlying off-shell supergravity in which the full off-shell supergravity multiplet is set to classical background values. The

consistency of this limit requires the existence of solutions of the corresponding Killing spinor equations which in turn poses non-trivial constraints on the background fields.[1]

In the present paper we will focus on rigid supersymmetric theories in six-dimensional Riemannian spin manifolds. We will study the coupling of off-shell Yang-Mills (YM) multiplets to a number of uncharged on-shell tensor multiplets. Such couplings have been constructed in $6D$ flat space-time of Minkowski signature [18]. In view of the applications mentioned above, we shall extend this model to curved background and in Euclidean signature. One way to construct these theories is to start from the Euclidean version of off-shell supergravity coupled to YM and tensor multiplets, and then take the rigid limit in which the fields in the supergravity multiplet are frozen in a manner consistent with supersymmetry. In particular, the vanishing of the fermionic fields in this multiplet would yield the Killing spinor equations. However, while the $N = (1, 0)$ off-shell supergravity theory in $6D$ with Minkowskian signature is well known [19, 20], thanks to the superconformal tensor calculus methods [19], the general couplings to Yang-Mills and multi-tensor multiplets is still missing. Extra complications may arise in the passage to Euclidean signature. This suggests an alternative approach in which we begin by formulating the Euclidean version of the Killing spinor equations with suitable background fields. We distinguish two cases depending on whether these equations descend from an off-shell theory with ($\theta \neq 0$) or without ($\theta = 0$) R-symmetry gauging. In the former case, additional Killing spinor equations have to be imposed. Next, we translate the model of [18] to Euclidean signature and then elevate the supersymmetry parameter to be Killing. Performing a Noether procedure which takes this into account then produces the full dependence of the couplings on the nontrivial background fields. Following this approach we construct the non-trivial supersymmetric tensor/vector couplings on a curved background.

Next, we systematically analyse the Killing spinor equations in six-dimensional Riemannian spin manifolds $\mathcal{M}_6$ using G-structures and we derive the constraints for the existence of non-trivial solutions. We reformulate these constraints as a set of conditions on the geometry. The necessary and sufficient conditions for minimal supersymmetry are given in equations (5.27), (5.35) below as constraints on an SU(3) structure. In particular for a theory whose background descends from a supergravity with R-symmetry gauging ($\theta \neq 0$), these constraints require $\mathcal{M}_6$ to be hermitian, strong Kähler with torsion, but not necessarily conformally balanced. For $\theta = 0$ on the other hand, the manifold $\mathcal{M}_6$ is not necessarily hermitian; we also show that it cannot be (strictly) nearly Kähler.

In either case there is a crucial difference with the structures arising in supersymmetric heterotic compactifications: The Killing spinor equations (2.1), (2.3) below are of the same form as those for the vanishing of the gravitino and dilatino variation in the heterotic theory, provided we set the background field E°_m to zero. For $\theta \neq 0$, this would impose the additional condition that the manifold should be conformally balanced, which is indeed a necessary condition in supersymmetric heterotic compactifications. However as already mentioned for the rigid supersymmetric theories considered here the confor-

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[1]A recent approach for the construction of theories starting from an on-shell formulation of supergravity has been discussed in [17].

mally balanced condition need not be imposed. Indeed as we will see in the following, a rigid (1,1)-supersymmetric theory can be defined on the round $\mathbb{S}^3 \times \mathbb{S}^3$ which is hermitian, strong Kähler with torsion but not conformally balanced. For $\theta = 0$, the manifolds need not to be hermitian, while hermiticity is a necessary condition in supersymmetric heterotic compactifications.

As a special case of our formalism we obtain pure super Yang-Mills (SYM) on a Calabi-Yau threefold $\mathcal{M}_6$ by setting to zero all background fields except for the metric. Applying the localization procedure then yields the result that the SYM path integral localizes on stable holomorphic bundles on $\mathcal{M}_6$.

The rest of the paper is organized as follows: In section 2, we present the Killing spinor equations obtained from off-shell supergravity in the superconformal tensor calculus. In section 3, we construct the Lagrangian and supersymmetry transformation rules for Yang-Mills theory on curved Riemannian manifolds $\mathcal{M}_6$ with background fields admitting solutions to the Killing spinor equations. Section 4 extends the construction to include a number of tensor multiplets interacting by Chern-Simons couplings with the vector fields. In section 5 we analyse the constraints to be satisfied by the background fields in order to admit non-trivial solutions. We reformulate the conditions for minimal supersymmetry as a set of necessary and sufficient conditions on a suitable SU(3) structure. Moreover, we examine the conditions for (1,1) and (1,2) extended supersymmetry. The formalism is illustrated with a number of examples, including Calabi-Yau (CY) threefolds, the round $\mathbb{S}^3 \times \mathbb{S}^3$ and T^2 bundles over noncompact K3's. In section 6 we apply the localization procedure to SYM theory defined on a CY $\mathcal{M}_6$ and show that the path integral localizes on stable holomorphic bundles over $\mathcal{M}_6$.

2 Killing spinor equations I

Our starting point is the off-shell formulation of six-dimensional supergravity obtained from superconformal tensor calculus with the dilaton Weyl multiplet coupled to a linear multiplet after particular gauge fixing [19–21]. The resulting bosonic fields including the space-time metric will constitute the background fields in the rigidly supersymmetric field theories to be constructed in this paper. As discussed in the introduction, when deriving rigid supersymmetric theories from supergravity one has to impose the vanishing of the supersymmetry variation of all fermionic fields from the supergravity multiplet. This defines the Killing spinor and poses constraints on the bosonic background fields. In our conventions (cf. appendix A) and switching from Minkowsi to Euclidean signature, the Killing spinor equations imposing the vanishing of the gravitino fields' variation, read

$$
D_m \epsilon + \frac{1}{8} H^\circ_{mkl} \gamma^{kl} \epsilon = 0 \, ,
$$
$$
D_m \tilde{\zeta} - \frac{1}{8} \tilde{\zeta} \gamma^{kl} H^\circ_{mkl} = 0 \, ,
\tag{2.1}
$$

with covariant spinor derivatives

$$
D_m \epsilon \equiv \partial_m \epsilon + \frac{1}{4} \omega^{\circ\ ab}_m \gamma_{ab} \epsilon + V^\circ_m \epsilon \, ,
\tag{2.2}
$$

etc., a background vector field V_m°, and a closed three-form $H_{mnk}^\circ = 3\partial_{[k}B_{mn]}^\circ$. In passing from Minkowski to Euclidean signature, we have replaced the symplectic Majorana-Weyl spinors by complex Weyl spinors ϵ, ζ, of positive chirality, thus reducing the manifest R-symmetry from $\mathrm{Sp}(1)$ to $\mathrm{U}(1)$. In particular, unlike in Minkowski signature, the spinors ϵ and ζ are no longer related by complex conjugation. If the Killing spinor equations (2.1) descend from a supergravity theory in which the R-symmetry is gauged, there are additional Killing spinor equations to be taken into account. Namely, imposing the vanishing of the remaining fermion variations in that case furthermore implies

$$\gamma^m \epsilon \, \partial_m L^\circ - \frac{1}{6} L^\circ H_{mnk}^\circ \, \gamma^{mnk} \epsilon + \frac{1}{2} E_m^\circ \gamma^m \epsilon \, = \, 0 \; ,$$
$$\tilde\zeta \gamma^m \, \partial_m L^\circ + \frac{1}{6} L^\circ H_{mnk}^\circ \, \tilde\zeta \gamma^{mnk} - \frac{1}{2} E_m^\circ \tilde\zeta \gamma^m \, = \, 0 \; , \tag{2.3}$$

where L° denotes the background dilaton and E_m° is the Hodge dual of the background five-form field strength[2]

$$E^{\circ\, m} \, = \, \frac{1}{24} \, \varepsilon^{mn_1 \ldots n_5} \, \partial_{n_1} E_{n_2 \ldots n_5}^\circ \; . \tag{2.4}$$

In the following we will construct rigidly supersymmetric field theories on backgrounds that allow for non-trivial solutions of the Killing spinor equations (2.1), (2.3). In slight abuse of standard notation, we will refer to a background as $N = (p, q)$ supersymmetric when equations (2.1), (2.3) admit p independent solutions ϵ and q independent solutions $\tilde\zeta$.[3] Flat space with vanishing background fields would thus correspond to $N = (4, 4)$. Note the unlike the analogous structures in four dimensions [6, 22] the Killing spinor equations (2.1), (2.3) for ϵ and $\tilde\zeta$ are entirely decoupled. For the study of minimal supersymmetry we can thus consistently set $\tilde\zeta = 0$.

While we will give a full-fledged analysis of the Killing spinor equations in section 5 below, let us finish this section by deriving a few immediate consequences for backgrounds that are at least $N = (1, 1)$ supersymmetric. Straightforward computation gives rise to the relations

$$\nabla_m \xi_n = -\frac{1}{2} H_{mnk}^\circ \, \xi^k \; , \qquad \nabla_m \xi_{pqr} \, = \, \frac{3}{2} H_{mn[p}^\circ \xi_{qr]}{}^n \; , \tag{2.5}$$

for the Killing spinor bilinears

$$\xi^m \, \equiv \, -\tilde\zeta \gamma^m \epsilon \; , \qquad \xi^{mnk} \, \equiv \, -\tilde\zeta \gamma^{mnk} \epsilon \; . \tag{2.6}$$

In particular, the first equation shows that ξ^m is a Killing vector of the background metric. Furthermore, integrability of the Killing spinor equations (2.1) yields

$$0 = \frac{1}{8} R_{mn}^\circ{}^{ab} \gamma_{ab} \epsilon + \frac{1}{2} V_{mn}^\circ \epsilon + \frac{1}{8} \gamma^{kl} \epsilon \, \nabla_{[m} H_{n]kl}^\circ - \frac{1}{16} H_{mp}^\circ{}^k H_{nqk}^\circ \gamma^{pq} \epsilon \; ,$$
$$0 = \frac{1}{8} R_{mn}^\circ{}^{ab} \tilde\zeta \gamma_{ab} + \frac{1}{2} V_{mn}^\circ \tilde\zeta + \frac{1}{8} \tilde\zeta \gamma^{kl} \, \nabla_{[m} H_{n]kl}^\circ - \frac{1}{16} H_{mp}^\circ{}^k H_{nqk}^\circ \tilde\zeta \gamma^{pq} \; . \tag{2.7}$$

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[2]Since we are in curved background geometry, the completely antisymmetric ε^{mnklpq} is understood to be defined as a tensor, i.e. it carries an implicit factor of $\sqrt{g^\circ}{}^{-1}$.

[3] This notation is not to be confused with the standard notion of chirality in Minkowski space, since here ϵ and ζ are of the same chirality.

Contracting both equations with γ^n and summing them up implies

$$\xi^{nkl} \nabla_m H^\circ_{nkl} = -\frac{3}{2} H^\circ_{mnp} H^{\circ\, p}_{kl} \xi^{nkl} - 12 V^\circ_{mn} \xi^n \ . \tag{2.8}$$

Together with (2.5) we obtain

$$\partial_m \xi_H = -2 V^\circ_{mn} \xi^n \ , \qquad \text{for } \xi_H \equiv \frac{1}{6} \xi^{nkl} H^\circ_{nkl} \ , \tag{2.9}$$

and anticipating the result of the full analysis that the background field strength V°_{mn} vanishes (cf. equation (5.15) below), shows that the scalar combination ξ_H is in fact constant.

3 Vector multiplets

We will now construct rigidly supersymmetric gauge theories on a background that admits non-trivial solutions of the Killing spinor equations (2.1), (2.3). The resulting couplings are motivated by the coupling [21] of a single off-shell vector multiplet to the off-shell supergravity after integrating out the auxiliary fields. However, since the spinor properties, in particular their R-symmetry representation content change with the passage from Minkowski to Euclidean space, we will perform the full construction from scratch.

Consider a set of vector multiplets $\{A^r_m, \lambda^r, \nu^r\}$, labeled by an index r, described by a Lagrangian

$$\mathcal{L}_{\text{vec}} = -\frac{1}{4} F_{mn}{}^r F^{mn\, r} - 2\tilde{\nu}^r \gamma^m D_m \lambda^r + \theta^r E^{\circ\, m} A^r_m$$
$$+ \frac{i}{16} \varepsilon^{mnklpq} B^\circ_{mn} F_{kl}{}^r F_{pq}{}^r + \frac{1}{12} \tilde{\nu}^r \gamma^{mnk} H^\circ_{mnk} \lambda^r \ . \tag{3.1}$$

Here, $F_{mn}{}^r \equiv 2\partial_{[m} A_{n]}{}^r - f_{st}{}^r A_m{}^s A_n{}^t$ is the standard Yang-Mills field strength with structure constants $f_{st}{}^r$, and the spinor covariant derivatives are defined as in (2.2). B°_{mn} is the background two-form potential, the background field $E^{\circ\, m}$ has been defined in (2.4) and satisfies $\nabla_m E^{\circ\, m} = 0$. The θ^r denote a set of coupling constants which single out one of the vector fields and descend from an R-symmetry gauging of the underlying supergravity [21]. Accordingly, their gauge invariance requires an abelian factor within the Yang-Mills gauge group. (For $\theta^r = 0$ on the other hand there is no restriction on the gauge group.) Again, the spinors λ^r and ν^r are not related by complex conjugation, such that the variation of the vector fields and the Lagrangian are in fact complex, as usual in Euclidean supersymmetry, cf. [23–25], see also the discussion in section 6.

It is straightforward to check that the Lagrangian (3.1) is invariant under the supersymmetry transformations rules

$$\delta A_m{}^r = \tilde{\nu}^r \gamma_m \epsilon - \tilde{\zeta} \gamma_m \lambda^r \ ,$$
$$\delta \lambda^r = \frac{1}{4} \gamma^{mn} \epsilon \, F_{mn}{}^r - \theta^r L^\circ \epsilon \ ,$$
$$\delta \tilde{\nu}^r = -\frac{1}{4} \tilde{\zeta} \gamma^{mn} F_{mn}{}^r + \theta^r L^\circ \tilde{\zeta} \ , \tag{3.2}$$

provided the supersymmetry parameters satisfy the Killing spinor equations (2.1), (2.3). The latter equations are only required in case the θ^r are non-vanishing. For $\theta^r = 0$, the only condition on the supersymmetry parameters are equations (2.1). Let us note that variation of the gauge connection in the Dirac term a priori gives rise to a quartic fermion term in the variation of (3.1)

$$\delta_4 \mathcal{L} \propto f_{rst} \left(\tilde{\nu}^r \gamma^m \lambda^t \right) \left(\tilde{\nu}^s \gamma_m \epsilon \right) . \tag{3.3}$$

Fierzing in $\lambda^t) \left(\tilde{\nu}^s \right.$ shows that

$$f_{rst} \left(\tilde{\nu}^r \gamma^m \lambda^t \right) \left(\tilde{\nu}^s \gamma_m \epsilon \right) = f_{rst} \left(\tilde{\nu}^s \gamma^k \lambda^t \right) \left(\tilde{\nu}^r \gamma_k \epsilon \right) , \tag{3.4}$$

upon using that $\gamma^m \gamma_{kln} \gamma_m = 0$. By virtue of antisymmetry of the structure constants, also the quartic variation (3.3) thus vanishes.

It is instructive to work out the $N = (1,1)$ algebra of supersymmetry transformations (3.2) which takes the form[4]

$$[\delta_{\tilde{\zeta}}, \delta_\epsilon] A_m{}^r = \mathcal{L}_\xi A_m^r + D_m(-\xi^n A_n^r) ,$$

$$[\delta_{\tilde{\zeta}}, \delta_\epsilon] \lambda^r = \mathcal{L}_\xi \lambda^r + \frac{1}{2} \xi_H \lambda^r ,$$

$$[\delta_{\tilde{\zeta}}, \delta_\epsilon] \tilde{\nu}^r = \mathcal{L}_\xi \tilde{\nu}^r - \frac{1}{2} \xi_H \tilde{\nu}^r , \tag{3.5}$$

with parameters ξ^n and ξ_H defined in (2.6) and (2.9), respectively, and defining the standard Lie derivative on spinor fields

$$\mathcal{L}_\xi \chi^I \equiv \xi^m \nabla_m \chi^I + \frac{1}{4} \left(\nabla_m \xi_n \right) \gamma^{mn} \chi^I = \xi^m \nabla_m \chi^I - \frac{1}{8} \xi^k H^\circ_{kmn} \gamma^{mn} \chi^I . \tag{3.6}$$

Interestingly, we find that the algebra closes not only into the standard translations and gauge transformation but also into a global U(1) acting on the fermions with parameter ξ_H that is constant due to (2.9).[5]

To summarize, the Lagrangian (3.1) defines non-abelian gauge theory on a curved Euclidean background with supersymmetry parameters defined as solutions of the Killing spinor equations (2.1), (2.3).

4 Tensor multiplets

We will now generalize the construction of the previous section to also include tensor multiplets. To this end, we will first derive the field equations for a set of free tensor multiplets

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[4]Part of these and the following calculations have been facilitated by use of the computer algebra system Cadabra [26, 27].

[5]As we will see in the following the full analysis of the Killing spinor equations shows that the constant ξ_H vanishes in the case of backgrounds for which $E^\circ_m = 0$, but is generically non-vanishing. However, as the Lagrangian (3.1) possesses the global U(1) appearing under closure on the r.h.s. of (3.5), the form of the algebra in any case is consistent, whether or not ξ_H is zero.

on curved background and then proceed to introduce interactions with the vector multiplets. Since supersymmetry is expected to impose self-duality equations for the tensor fields that do not allow for a standard Lagrangian description, we will perform the entire construction on the level of the field equations. Moreover, since there is no off-shell formulation for self-dual tensor fields in six dimensions, their couplings on a curved background cannot directly be read off from an underlying supergravity but have to be constructed from scratch.

Let us consider n tensor multiplets $\{\phi^I, B_{mn}^I, \chi^I, \tilde{\psi}^I\}$, labelled by an index I, with supersymmetry transformations

$$\delta\phi^I = \tilde{\psi}^I\epsilon + \tilde{\zeta}\chi^I \,,$$

$$\delta B_{mn}{}^I = \tilde{\psi}^I\,\gamma_{mn}\,\epsilon - \tilde{\zeta}\gamma_{mn}\chi^I \,,$$

$$\delta\chi^I = \frac{1}{2}\gamma^m\epsilon\,\partial_m\phi^I + \frac{1}{24}\,\gamma^{mnk}\epsilon\left(H_{mnk}{}^I - H_{mnk}^\circ\phi^I\right)\,,$$

$$\delta\tilde{\psi}^I = -\frac{1}{2}\tilde{\zeta}\gamma^m\,\partial_m\phi^I + \frac{1}{24}\,\tilde{\zeta}\gamma^{mnk}\left(H_{mnk}{}^I - H_{mnk}^\circ\phi^I\right)\,. \tag{4.1}$$

where

$$H_{mnk}{}^I \equiv 3\partial_{[m}B_{nk]}{}^I \,, \tag{4.2}$$

denotes the abelian tensor field strength which decomposes into its selfdual and anti-selfdual part according to

$$H_{mnk}^\pm \equiv \frac{1}{2}\left(H_{mnk} \pm \frac{i}{3!}\,\varepsilon_{pqrmnk}H^{pqr}\right)\,. \tag{4.3}$$

Closure of the supersymmetry algebra on the tensor fields gives rise to the self-duality equation

$$(H_{mnk}{}^I)^{(+)} - \phi^I(H_{mnk}^\circ)^{(+)} = 0 \,, \tag{4.4}$$

relating the tensor field strength to the background three-form. Under the supersymmetry transformations (4.1), the self-duality equation transforms into the fermionic field equations

$$0 = D_m\tilde{\psi}^I\gamma^m + \frac{1}{24}\,\tilde{\psi}^I\gamma^{mnk}H_{mnk}^\circ \,,$$

$$0 = \gamma^m D_m\chi^I - \frac{1}{24}\,H_{mnk}^\circ\gamma^{mnk}\chi^I \,, \tag{4.5}$$

provided the supersymmetry parameters ϵ, $\tilde{\zeta}$ satisfy the Killing spinor equations (2.1). In turn, these equations under supersymmetry transform into the scalar field equation

$$0 = \nabla^\mu\partial_\mu\phi^I + \frac{1}{6}\left(H^\circ \cdot H^{I(-)}\right) - \frac{1}{12}\left(H^\circ \cdot H^\circ\right)\phi^I \,, \tag{4.6}$$

where again (2.1) is required. For consistency, we may check the $N = (1,1)$ algebra of supersymmetry transformations (4.1) which close into

$$[\delta_{\tilde{\zeta}}, \delta_\epsilon]\,\phi^I = \mathcal{L}_\xi\phi^I \,,$$

$$[\delta_{\tilde{\zeta}}, \delta_\epsilon]\,B_{mn}{}^I = \mathcal{L}_\xi B_{mn}{}^I + 2\,\partial_{[m}\Lambda_{n]}{}^I \,,$$

$$[\delta_{\tilde{\zeta}}, \delta_\epsilon]\,\chi^I = \mathcal{L}_\xi\chi^I + \frac{1}{2}\,\xi_H\,\chi^I \,, \tag{4.7}$$

with the same translation and U(1) parameters ξ^m, ξ_H, as in (3.5), and the additional tensor gauge parameter $\Lambda_m{}^I \equiv -\xi^n B_{nm}{}^I + \xi_m \phi^I$.

Finally, we may couple the system of tensor multiplets to the non-abelian vector fields introduced in the previous section by introducing Chern-Simons interactions of the form

$$H_{mnk}{}^I \rightarrow \mathcal{H}_{mnk}{}^I \equiv 3\partial_{[m}B_{nk]}{}^I + 6\,d_{rs}^I A_{[m}{}^r \partial_n A_{k]}{}^s - 2f_{pq}{}^s d_{rs}^I A_{[m}{}^r A_n{}^p A_{k]}{}^q \,, \quad (4.8)$$

parametrized by a constant gauge invariant tensor $d_{rs}^I = d_{(rs)}^I$. For simple gauge groups, this implies that $d_{rs}^I = d^I \delta_{rs}$ is expressed in terms of the Cartan-Killing form δ_{rs}. The modified tensor field strength $\mathcal{H}_{mnk}{}^I$ satisfies

$$\partial_{[m}\mathcal{H}_{nkl]}{}^I = \frac{3}{2}\,d_{rs}^I\,\mathcal{F}_{[mn}{}^r\mathcal{F}_{kl]}{}^s \,,$$

$$\delta\,\mathcal{H}_{mnk}{}^I = 3\,\partial_{[m}\left(\delta B_{nk]}{}^I - 2A_n{}^r\,\delta A_{k]}{}^s\,d_{rs}^I\right) + 6\,d_{rs}^I\,\mathcal{F}_{[mn}{}^r\,\delta A_{k]}{}^s \,. \quad (4.9)$$

In presence of d_{rs}^I the supersymmetry transformation rules (4.1) change to

$$\delta B_{mn}{}^I = \tilde{\psi}^I\,\gamma_{mn}\,\epsilon - \tilde{\zeta}\gamma_{mn}\chi^I + 2d_{rs}^I\,A_{[m}{}^r\,\delta A_{n]}{}^s \,,$$

$$\delta\chi^I = \frac{1}{2}\gamma^m\epsilon\,\partial_m\phi^I + \frac{1}{24}\,\gamma^{mnk}\epsilon\left(H_{mnk}{}^I - H_{mnk}^\circ\,\phi^I\right)$$
$$+ \frac{1}{2}d_{rs}^I\gamma^\mu\lambda\left(\tilde{\nu}\gamma_\mu\epsilon - \tilde{\zeta}\gamma_\mu\lambda\right) \,, \quad (4.10)$$

and similar for $\tilde{\psi}$, which again can be verified by checking closure of the algebra. The field equations change into

$$0 = (H_{mnk}{}^I)^{(+)} - \phi^I(H_{mnk}^\circ)^{(+)} + d_{rs}^I\tilde{\nu}^r\gamma_{mnk}\lambda^s \,,$$

$$0 = \gamma^m D_m\chi^I - \frac{1}{24}\,H_{mnk}^\circ\,\gamma^{mnk}\,\chi^I - \frac{1}{2}\,d_{rs}^I F_{mn}^r\gamma^{mn}\lambda^s - 2\,d_{rs}^I\,\theta^r L^\circ\lambda^s \,,$$

$$0 = \nabla^\mu\partial_\mu\phi^I + \frac{1}{6}\left(H^\circ \cdot H^{I(-)}\right) - \frac{1}{12}\left(H^\circ \cdot H^\circ\right)\phi^I$$
$$+ 4\,d_{rs}^I\left(\frac{1}{8}\,F_{mn}{}^r F^{mn\,s} + \tilde{\nu}^r\gamma^m\partial_m\lambda^s + \theta^r\theta^s(L^\circ)^2\right) \,, \quad (4.11)$$

and can be checked to vary into each other under supersymmetry, provided the supersymmetry parameters ϵ, $\tilde{\zeta}$ satisfy the Killing spinor equations (2.1), (2.3). This generalizes the field equations found in flat space [18] for a tensor/vector system interacting by Chern-Simons couplings to curved background. In deriving this result, we had to use the gaugino field equation coming from (3.1). I.e. unlike in flat space where the YM sector can be kept off-shell in the presence of Chern-Simons couplings [18], the non-vanishing background field H_{mnk}° relates the dynamics of the vector and tensor sector.

It would be interesting to further generalize the interactions by allowing for minimal couplings between vector and tensor fields with the latter charged under a non-abelian gauge group. The corresponding supersymmetric system in flat space has been constructed in [28, 29] and allows for an action modulo the standard subtleties concerning self-dual

three-forms. Replacing the constant supersymmetry parameters of that system by solutions to the Killing spinor equations (2.1) should lead to a deformed version of these tensor/vector interactions on curved space. It would also be interesting to analyze if the curved space models can be generalized to include couplings to hypermultiplets, by applying the same procedure to the flat space models of [30, 31].

5 Killing spinor equations II

So far, we have derived the couplings and field equations for vector and tensor multiplets on a curved background under the assumption that the supersymmetry parameters satisfy the Killing spinor equations (2.1), (2.3). In this section, we will analyze these Killing spinor equations and derive the resulting constraints for the background fields, in particular the background geometry. We obtain the necessary and sufficient conditions for the background geometry in order to admit at least one Killing spinor, i.e. for the theory to have (1,0) supersymmetry. In subsections 5.2, 5.3 we derive stronger conditions for the existence of several supercharges. As this section deals exclusively with the background structure we will for notational simplification omit the superscript '°' which so far has distinguished the background structures from the matter fields. I.e. (for this section only) we pass to $H^\circ \to H$, $L^\circ \to L$, etc.

5.1 (1,0) supersymmetry

The Killing spinor equation (2.1) can be thought of as a parallel-transport equation, $\nabla' \epsilon = 0$, with respect to a metric-compatible connection ∇' with torsion given by the background three-form H. Therefore the holonomy of ∇' (in the spinor representation) is a subgroup of G, the stabiliser group of ϵ, and $\mathcal{M}_6$ admits a G-structure [32]. Since ϵ transforms in the **4** of the structure group $\mathrm{Spin}(6) \cong \mathrm{SU}(4)$ of the Riemannian spin manifold $\mathcal{M}_6$, the stabiliser is $G = \mathrm{SU}(3)$ as can be seen by the decomposition $\mathbf{4} \to \mathbf{3} \oplus \mathbf{1}$ under $\mathrm{SU}(4) \to \mathrm{SU}(3)$. Hence $\mathcal{M}_6$ must admit an $\mathrm{SU}(3)$ structure.

The topological condition for an oriented Riemannian six-dimensional manifold $\mathcal{M}_6$ to admit an $\mathrm{SU}(3)$ structure is that it should be spin (see e.g. [33]). Since in the present paper we are assuming that $\mathcal{M}_6$ is spin, there is no additional topological condition imposed by the existence of an $\mathrm{SU}(3)$ structure on $\mathcal{M}_6$. However, as we will see in the following, supersymmetry imposes additional geometrical conditions in the form of constraints on the torsion classes of the $\mathrm{SU}(3)$ structure. In order to systematically analyze these constraints we will now reformulate the $\mathrm{SU}(3)$ structure in terms of certain globally defined forms on $\mathcal{M}_6$.

From equation (2.1) it can easily be seen that either ϵ vanishes identically, or it is nowhere-vanishing. Assuming that ϵ is not identically zero, let us parametrize:

$$\epsilon = e^A \eta \,, \tag{5.1}$$

where η is a unimodular Weyl spinor of positive chirality,

$$\eta^\dagger \eta = 1 \,, \tag{5.2}$$

and A is a function on the six-dimensional manifold. For the purposes of this section we will assume that η is a commuting spinor: we have the freedom to make this choice since the Killing spinor equations are linear in ϵ.

The existence of the unimodular spinor η allows us to define the bilinears

$$
\begin{aligned}
iJ_{mn} &:= \eta^\dagger \gamma_{mn} \eta \\
\Omega_{mnp} &:= \tilde\eta \gamma_{mnp} \eta \,,
\end{aligned}
\tag{5.3}
$$

where our spinor conventions are explained in detail in appendix A. In particular, it follows from the properties of Weyl spinors in six Euclidian dimensions that J is a real two-form and Ω is a complex three-form which is imaginary self-dual.

Using the Fierz identities it can further be shown that $J_m{}^n$ (where we have raised one index with the metric) defines an almost complex structure:

$$
J_m{}^n J_n{}^p = -\delta_m{}^p \,,
\tag{5.4}
$$

with respect to which Ω is $(3,0)$:

$$
(\Pi^+)_m{}^n \Omega_{npq} = \Omega_{mpq}; \qquad (\Pi^-)_m{}^n \Omega_{npq} = 0 \,,
\tag{5.5}
$$

where

$$
(\Pi^\pm)_m{}^n := \frac{1}{2}(\delta_m{}^n \mp iJ_m{}^n) \,,
\tag{5.6}
$$

are the projection operators onto the (1,0) and (0,1) parts. Moreover, it can be seen that

$$
\begin{aligned}
\Omega \wedge J &= 0 \\
\Omega \wedge \Omega^* &= \frac{4i}{3} J^3 \,.
\end{aligned}
\tag{5.7}
$$

The globally defined forms J, Ω subject to the above conditions can be seen to specify an SU(3) structure on $\mathcal{M}_6$ [34].

The intrinsic torsion parametrizes the failure of the spinor η to be covariantly constant. In the case of an SU(3)-structure manifold $\mathcal{M}_6$, the intrinsic torsion decomposes into five modules (torsion classes) $W_1, \ldots, W_5$:

$$
\begin{aligned}
\nabla_m \eta = {}& \frac{1}{2} \left(W_{4m}^{(1,0)} + W_{5m} - \text{c.c} \right) \eta \\
& + \frac{1}{16} \left(4W_1 g_{mn} - 2W_4^p \Omega_{pmn} + 4iW_{2mn} - iW_{3mpq} \Omega^{pq}{}_n \right) \gamma^n \eta^c \,,
\end{aligned}
\tag{5.8}
$$

where W_1 is a complex scalar, W_2 is a complex (1,1)-traceless form, W_3 is a real traceless $(2,1) + (1,2)$ form, W_4 is a real one-form and W_5 is a (1,0) form. In terms of SU(3) representations,

$$
W_1 \sim \mathbf{1} \oplus \mathbf{1}; \qquad W_2 \sim \mathbf{8} \oplus \mathbf{8}; \qquad W_3 \sim \mathbf{6} \oplus \bar{\mathbf{6}}; \qquad W_4 \sim \mathbf{3} \oplus \bar{\mathbf{3}}; \qquad W_5 \sim \mathbf{3} \,.
\tag{5.9}
$$

Equivalently these torsion classes appear in the SU(3) decomposition of the exterior derivatives of J, Ω. Explicitly we have,

$$
\begin{aligned}
dJ &= \frac{3}{2} \text{Im}(W_1^* \Omega) + W_4 \wedge J + W_3 \,, \\
d\Omega &= W_1 J \wedge J + W_2 \wedge J + \Omega \wedge W_5^* \,.
\end{aligned}
\tag{5.10}
$$

Intuitively, the intrinsic torsion characterizes the geometry of $\mathcal{M}_6$ and parameterizes the failure of the manifold to be of special holonomy – which can also be thought of as the failure of the closure of J, Ω.

As a final ingredient before we proceed to the analysis of the Killing spinor equations, we will need the decomposition of the background three-form H from (2.1) with respect to the reduced structure group SU(3). Explicitly we have

$$H_{mnp} = \frac{1}{48}\Omega_{mnp}H^{(0)} + \left(\tilde{H}^{(2,1)}_{mnp} + \frac{3}{4}H^{(1,0)}_{[m}J_{np]} \right) + \text{c.c.}\,, \tag{5.11}$$

where the normalization above has been chosen so that

$$H^{(0)} = \Omega^{*mnp}H_{mnp}\,,$$
$$H^{(1,0)}_m = (\Pi^+)_m{}^s H_{snp}J^{np}\,. \tag{5.12}$$

In terms of SU(3) representations we have,

$$H^{(0)} \sim \mathbf{1}; \qquad H^{(1,0)} \sim \mathbf{3}; \qquad H^{(0,1)} \sim \mathbf{\bar{3}}; \qquad \tilde{H}^{(2,1)} \sim \mathbf{6}; \qquad \tilde{H}^{(1,2)} \sim \mathbf{\bar{6}}\,. \tag{5.13}$$

Plugging (5.8), (5.11) into the Killing spinor equation (2.1) using (B.2) can be seen to impose the following constraints on the torsion classes:

$$W_2 = 0\,,$$
$$2W^{(1,0)}_4 + W_5 = 0\,. \tag{5.14}$$

Moreover it follows that the background vector field is exact,

$$dA + V = 0\,, \tag{5.15}$$

and the irreducible components of H are determined in terms of the torsion classes. Explicitly, in form notation:

$$H^{(0)} = -12W^*_1\,,$$
$$H^{(1,0)} = -4iW^{(1,0)}_4\,,$$
$$\tilde{H}^{(1,2)} = iW^{(1,2)}_3\,. \tag{5.16}$$

5.1.1 $\theta^r \neq 0$

For the remaining analysis, it will make a crucial difference whether the parameters θ^r in the Lagrangian (3.1) are vanishing or not. Only in the latter case do we have to impose the Killing spinor equations (2.3) on the supersymmetry parameters. This leads to further restrictions on the background geometry, to be discussed in this section. The case $\theta^r = 0$ is examined separately in section 5.1.2.

Without loss of generality, we may assume in this section the background field L to be non-vanishing. Otherwise, it is straightforward to see that the Killing spinor equation (2.3) implies that also $E_m = 0$ (taking into account that E_m is real). For the Lagrangian (3.1)

and the supersymmetry transformation rules (3.2), this amounts to setting $\theta^r = 0$, and brings us back to the situation discussed in the next section. With the field L non-zero, the second Killing spinor equation (2.3) implies that the scalar component of H must vanish:

$$H^{(0)} = 0 \ . \tag{5.17}$$

As a consequence of the above and (5.16) there is now an additional constraint on the torsion classes:

$$W_1 = 0 \ . \tag{5.18}$$

Taking the above into account, it follows that the H field is given in terms of the torsions classes:

$$H = -i \left[(W_3^{(2,1)} - W_3^{(1,2)}) + J \wedge (W_4^{(1,0)} - W_4^{(0,1)}) \right] \ . \tag{5.19}$$

By introducing explicit holomorphic coordinates, eq. (5.19) can be rewritten as

$$H = -i(\partial J - \bar{\partial} J) \,, \tag{5.20}$$

where $d = \partial + \bar{\partial}$ and we have noted that

$$(dJ)^{(2,1)} = \partial J \ ; \quad (dJ)^{(1,2)} = \bar{\partial} J \ . \tag{5.21}$$

It follows that the closure of H can be written equivalently as:

$$\partial \bar{\partial} J = 0 \ . \tag{5.22}$$

eq. (5.19) can also be written as:

$$\star H = W_3 - W_4 \wedge J \,, \tag{5.23}$$

where we have used

$$\varepsilon_{m_1 \ldots m_6} = -15 J_{[m_1 m_2} J_{m_3 m_4} J_{m_5 m_6]} \ . \tag{5.24}$$

If W_4 is exact (which need not be true in general), $W_4 = d\phi$, the manifold is called conformally balanced. In that case (5.23) can be written as:

$$\star H = e^{2\phi} d \left(e^{-2\phi} J \right) \ . \tag{5.25}$$

Finally the background field E_m is given by:

$$E_m = -4L W_{4m} - 2\partial_m L \,, \tag{5.26}$$

where in addition $\nabla^m E_m = 0$ has to be imposed, cf. eq. (2.4).

To summarize: the combined Killing spinor equations (2.1), (2.3), the closure of the background three-form and the co-closure of the background one-form imply the following geometric constraints on $\mathcal{M}_6$:

$$\begin{aligned} W_1 = W_2 &= 0 \,, \\ W_4 + \mathrm{Re} W_5 &= 0 \,, \\ \partial \bar{\partial} J &= 0 \\ \nabla^m \left(\partial_m L + 2L W_{4m} \right) &= 0 \ . \end{aligned} \tag{5.27}$$

Given an SU(3)-structure manifold obeying (5.27) there is no obstruction to specifying the profiles for the background fields V, H, E so that the remaining conditions (5.15), (5.19), (5.26) are satisfied. In this sense conditions (5.27) are *necessary and sufficient* for the theory to possess rigid supersymmetry.

Conditions (5.27) have the following geometric interpretation: The vanishing of W_1, W_2 is equivalent to the condition that the almost complex structure is integrable, i.e. the condition that $\mathcal{M}_6$ is hermitian.[6] The second line of (5.27) can be rephrased as the statement of proportionality between the Lee forms L_Ω and L_J of Ω and J respectively. More explicitly the Lee forms are defined by:

$$L_J = J \lrcorner dJ$$
$$L_\Omega = \mathrm{Re}\,(\Omega \lrcorner d\Omega^*) \ , \tag{5.29}$$

where $\varphi \lrcorner \omega$ denotes the contraction of the p-form φ into the $(p+q)$-form ω,

$$\varphi \lrcorner \omega = \frac{1}{p!q!} \varphi^{m_1 \ldots m_p} \omega_{m_1 \ldots m_p n_1 \ldots n_q} dx^{n_1} \wedge \cdots \wedge dx^{n_q} \ . \tag{5.30}$$

On the other hand, taking (5.10) into account, it follows that

$$W_4 = \frac{1}{2} J \lrcorner dJ \ ; \quad \mathrm{Re} W_5 = \frac{1}{8} \mathrm{Re}\,(\Omega \lrcorner d\Omega^*) \ , \tag{5.31}$$

hence the second line of (5.27) can be rewritten in terms of the Lee forms as:

$$L_J + \frac{1}{4} L_\Omega = 0 \ . \tag{5.32}$$

The third line of (5.27) is the condition that the Hermitian manifold $\mathcal{M}_6$ is strong Kähler with torsion (SKT). One immediate consequence of the SKT condition is that $\mathcal{M}_6$ cannot be special hermitian [35].[7]

The first two lines of (5.27) also appear as necessary conditions for supersymmetric heterotic compactifications [37–39]. This is not surprising given the fact that the Killing spinor equations (2.1), (2.3) are of the same form as those for the vanishing of the gravitino and dilatino variation in the heterotic theory, provided we set the background fields V, E to zero and we identify $L = e^{-2\phi}$, where ϕ is the dilaton. Crucially, however, this would impose the additional condition that the torsion class W_4 is exact: $W_4 = d\phi$ as follows from eq. (5.26); in particular the manifold is constrained to be conformally balanced. We stress again that for the rigid supersymmetric theories considered here the conformally balanced condition need not be imposed.

---

[6]A manifold equipped with a Riemannian metric g_{mn} and an almost complex structure $J_m{}^n$ is called almost hermitian if the almost complex structure is an orthogonal transformation with respect to the metric:

$$J_m{}^p J_n{}^q g_{pq} = g_{mn} \ . \tag{5.28}$$

If moreover the almost complex structure is integrable, the manifold is called hermitian.

[7]A special hermitian six-manifold is a manifold which is both half-flat (and therefore can be lifted to a seven-dimensional manifold of G_2 holonomy [36]) and hermitian. Equivalently, a special hermitian six-manifold is a manifold which admits an SU(3) structure whose only non-zero torsion class is W_3.

5.1.2 $\theta^r = 0$

With all parameters θ^r vanishing, the background fields L and E_m completely decouple form the Lagrangian (3.1) and supersymmmetry transformation rules (3.2) and equations (2.3) do no longer have to be imposed on the supersymmetry parameters. As a result, the constraints on the background geometry are relaxed w.r.t. the last section and in particular the torsion class W_1 is no longer zero, cf. (5.18). Equations (5.19), (5.23) for the H field now get modified to:

$$H = -\frac{1}{2}\mathrm{Re}\,(W_1\Omega^*) - i\left[(W_3^{(2,1)} - W_3^{(1,2)}) + J \wedge (W_4^{(1,0)} - W_4^{(0,1)})\right],\tag{5.33}$$

and:

$$\star H = \frac{1}{2}\mathrm{Im}\,(W_1\Omega^*) + W_3 - W_4 \wedge J\,.\tag{5.34}$$

Finally, the necessary and sufficient conditions can be summarized as

$$\begin{aligned} W_2 &= 0\,, \\ W_4 + \mathrm{Re}W_5 &= 0\,, \\ d\left[\mathrm{Re}\,(W_1\Omega^*) + 2\star(W_3 - W_4 \wedge J)\right] &= 0\,. \end{aligned}\tag{5.35}$$

The fact that now W_1 is not necessarily zero implies that the almost complex structure is not integrable in general and the manifold $\mathcal{M}_6$ is not necessarily hermitian. This is a major difference from the $L \neq 0$ case discussed above. The second line above has the same interpretation as in section 5.1.1 as the proportianality between the Lee forms of J and Ω. The last line in (5.35) is the condition of closure of H; for $W_1 = 0$ it reduces to the third line in (5.27).

As we will now show, the system (5.35) does not admit as solution any (strictly) nearly Kähler manifold. Nearly Kähler manifolds are SU(3)-structure manifolds whose only nonvanishing torsion class is W_1. Equivalently, they can be defined as six-dimensional manifolds admitting a Killing spinor, $\nabla_m\eta = \frac{1}{4}W_1\gamma_m\eta^c$, cf. eq. (5.8), so that they are Einstein, $R_{mn} = \frac{5}{4}|W_1|^2 g_{mn}$, and the metric cone over $\mathcal{M}_6$ is a G_2-holonomy manifold [40]. In the case of a nearly Kähler $\mathcal{M}_6$ it can immediately be seen from eqs. (5.10) that W_1 is constant. Performing a constant phase redefinition of Ω we can take $W_1 = iw$, with w a real constant. Eqs. (5.10) then reduce to:

$$dJ = -\frac{3}{2}w\mathrm{Re}\Omega\,, \quad d\mathrm{Im}\Omega = wJ \wedge J\,,\tag{5.36}$$

while $d\mathrm{Re}\Omega = 0$ follows from the above. Moreover the background field H is given by

$$H = -\frac{1}{2}w\mathrm{Im}\Omega\,,\tag{5.37}$$

and it is not closed, unless $w = 0$ and $\mathcal{M}_6$ is a Calabi-Yau.

5.2 (1,1) supersymmetry

In the case of vanishing θ^r, we have seen that the Killing spinor equations (2.3) do no longer have to be imposed. The remaining equations (2.1) are related by simple transposition,

i.e. any solution ϵ of the first equation in (2.1) defines a solution to the second equation by setting $\zeta \equiv \epsilon$. In our notation, this means that the resulting theory in fact is $N = (1,1)$ supersymmetric rather than just $N = (1,0)$.[8]

In contrast, for non-vanishing θ^r, imposing (1,1) supersymmetry in general imposes further constraints on the background geometry, to be discussed in the rest of this section. In this case, we have to look for necessary and sufficient conditions so that the Killing spinor eqs. (2.1), (2.3) are obeyed for a pair of not identically vanishing spinors ϵ, ζ. We have already seen that (1,0) supersymmetry imposes that the background gauge field is exact, cf. eq. (5.15). Hence we may gauge away V by a suitable redefinition of the dynamical fermions of the theory. Setting $V = 0$ in (5.15) then implies that A is constant. In this section we will take $A = 0$ without loss of generality so that, cf. eq. (5.1),

$$\epsilon = \eta \,. \tag{5.38}$$

One immediate consequence of the Killing spinor equations (2.3) is that by taking $\zeta = c\eta$, where c is a constant, the theory is (1,1) supersymmetric if and only if

$$E_m = 0 \,, \tag{5.39}$$

in addition to the conditions derived in section 5.1.1. Combined with (5.26), eq. (5.39) implies the exactness of the W_4 torsion class

$$W_4 = d\phi \,, \tag{5.40}$$

where we have set $L = e^{-2\phi}$. Hence the manifold is constrained to be conformally balanced. As a consequence it also follows that the scalar ξ_H defined in (2.9) vanishes.

More generally, any (commuting) spinor ζ of positive chirality can be written as:

$$\zeta = c\eta + \gamma^m K_m \eta^c \,, \tag{5.41}$$

where K is a (1,0)-form.[9] It can then be seen taking the formulæ of appendix B into account that the Killing spinor equations are equivalent to the following set of conditions:

$$
\begin{aligned}
c = 0 \quad \text{or} \quad E_m &= 0 \\
K \lrcorner dL &= 0 \\
\nabla_{(m} K_{n)} &= 0 \\
dK + K \lrcorner H &= 0 \\
\frac{1}{4}\left(E^{(1,0)} - 2\partial L\right) \wedge K - iLK \lrcorner W_3^{(2,1)} &= 0 \,,
\end{aligned} \tag{5.42}
$$

in addition to the conditions derived in section 5.1.1. Note that the penultimate line above implies that the Lie derivative of the background three-form field vanishes, $\mathcal{L}_K H = 0$; the

---

[8]Note that even when ϵ, ζ are not linearly independent, they still have different action on the fields, cf. (3.2). The supersymmetry thus is (1,1) and not (1,0) or (0,1), which would correspond to one of the two spinor parameters being zero.

[9]With a slight abuse of notation, we will use the same symbol for the form and the vector obtained from it by raising the index with the metric.

second line can also be written as $\mathcal{L}_K L = 0$; the third line is equivalent to K being Killing. It also follows from the above that the norm of K is constant, $\partial_m |K|^2 = 0$. Hence for $K \neq 0$ the triplet (K, J, Ω) defines on $\mathcal{M}_6$ an SU(2) structure.

Normalizing $|K|^2 = 2$ the SU(2) structure can be given equivalently as the triplet (K, j, ω), where [39, 41]:

$$\omega = -\frac{i}{2} K^* \lrcorner \Omega \,, \tag{5.43}$$

is a complex (2,0) form and

$$j = J - \frac{i}{2} K \wedge K^* \,, \tag{5.44}$$

is a real (1,1) form obeying the compatibility conditions

$$\omega \wedge j = 0 \,, \quad \omega \wedge \omega^* = 2 j \wedge j \,. \tag{5.45}$$

If desired, the last line in (5.42) may be expressed in terms of SU(2) torsion classes.

Note again that choosing $E_m = 0$ in the first line of (5.42) implies that the manifold is conformally balanced and the scalar ξ_H defined in (2.9) vanishes.

5.3 (1,2) supersymmetry

Imposing $E_m = 0$ and consequently the conformally balanced condition $W_4 = d\phi$, eqs. (5.42) leave the constant c unconstrained. The spinor ζ in (5.41) can then be written as $\zeta = c\eta + c'\eta'$ where $\eta' = \frac{1}{\sqrt{2}|K|} \gamma^m K_m \eta^c$ is a unimodular spinor orthogonal to η and $c' = \sqrt{2}|K|$. Therefore, ζ is parameterized by two independent parameters c and c', hence the theory is (at least) (1,2)-supersymmetric. In that case it can be seen that the equations in (5.42) reduce to:

$$\begin{aligned}
\partial K &= 0 \,, \\
\mathcal{L}_K \phi &= 0 \,, \\
\nabla_{(m} K_{n)} &= 0 \,, \\
d\left(e^{-2\phi} K \wedge J\right) &= 0 \,,
\end{aligned} \tag{5.46}$$

where we have set $L = e^{-2\phi}$. In deriving the equations above we have used eq. (B.4) and have taken into account that

$$W_3^{(2,1)} = \partial J - J \wedge \partial \phi \,, \tag{5.47}$$

which follows from (5.10) and the exactness of W_4.

5.4 Examples

There are many examples of six-dimensional manifolds obeying the necessary and sufficient conditions (5.27), (5.35). One obvious class is that of Calabi-Yau manifolds; in this case all torsion classes vanish. A compact non-Ricci-flat, non-conformally-balanced example with (1,1) supersymmetry is that of the round $\mathbb{S}^3 \times \mathbb{S}^3$. We will also provide non-compact examples based on the Iwasawa manifold (with (1,1) supersymmetry), as well as certain T^2 bundles over non-compact K3's (with (1,2) supersymmetry). As we will see, however, our

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rigid-supersymmetric theory cannot be defined on the round six-sphere, at least for $\theta^r \neq 0$. The latter conclusion is in accordance with the representation-theoretic classification of Euclidean supersymmetries [42], as pointed out recently in [17]. Note however that the classification of [42] only considers "space-times" whose isometries are simple Lie groups. In particular it does not cover the cases without any isometries, such as e.g. Calabi-Yau manifolds.

The round $\mathbb{S}^3 \times \mathbb{S}^3$. This example is based on an SU(3) structure given in [43]. We identify $\mathbb{S}^3 \times \mathbb{S}^3$ with the group manifold SU(2)×SU(2). The orthonormal frame is given by the SU(2)-invariant one-forms e^a, f^a, $a = 1, 2, 3$, satisying

$$de^a = \frac{1}{2}\varepsilon_{abc}e^b \wedge e^c \quad df^a = \frac{1}{2}\varepsilon_{abc}f^b \wedge f^c \ . \tag{5.48}$$

The SU(3) structure is given by:

$$
\begin{aligned}
J &= e^1 \wedge e^2 - e^4 \wedge e^5 + e^3 \wedge e^6 \\
\Omega &= (e^1 + ie^2) \wedge (e^4 - ie^5) \wedge (e^3 + ie^6) \, ,
\end{aligned}
\tag{5.49}
$$

so that a basis of (1,0)-forms is given by $(e^1 + ie^2)$, $(e^4 - ie^5)$, $(e^3 + ie^6)$. The metric associated with the structure (5.49) is that of the round $\mathbb{S}^3 \times \mathbb{S}^3$:

$$ds^2 = \sum_{a=1}^{3} (e^a \otimes e^a + f^a \otimes f^a) \ . \tag{5.50}$$

Moreover we obtain W_1, $W_2 = 0$ (hence $\mathcal{M}_6$ is hermitian), and

$$
\begin{aligned}
W_3 &= \frac{1}{2}(e^1 \wedge e^2 + f^1 \wedge f^2) \wedge (f^3 - e^3) \\
W_4 &= -\text{Re}W_5 = \frac{1}{2}(e^3 + f^3) \\
W_5 &= e^{\frac{3\pi i}{4}} \frac{e^3 + if^3}{\sqrt{2}} \, ,
\end{aligned}
\tag{5.51}
$$

as can be seen by computing dJ, $d\Omega$ taking (5.48) into account and comparing with eq. (5.10). The H-field can be computed from (5.19) and (5.51):

$$H = -(e^1 \wedge e^2 \wedge e^3 + f^1 \wedge f^2 \wedge f^3) \, , \tag{5.52}$$

which is closed, $dH = 0$ (equivalently $\partial\bar{\partial}J = 0$, hence $\mathcal{M}_6$ is strong Kähler with torsion). Furthermore, setting $L = 0$, $E_m = 0$, the round $\mathbb{S}^3 \times \mathbb{S}^3$ satisfies the necessary and sufficient conditions (5.35) and thus provides a consistent background for a rigid (1,1)-supersymmetric theory. Note in particular that this example is not conformally balanced: indeed it follows from (5.51) that W_4 is not closed and therefore not exact.

The noncompact Iwasawa manifold. The Iwasawa manifold can be viewed as a compact quotient G/Γ where G is the three-dimensional complex Heisenberg group defined as

$$G = \left\{ \begin{pmatrix} 1 & z^1 & -z^3 \\ 0 & 1 & z^2 \\ 0 & 0 & 1 \end{pmatrix}, \quad z^1, z^2, z^3 \in \mathbb{C} \right\}, \tag{5.53}$$

with group product given by ordinary matrix multiplication, and Γ acts as

$$\Gamma : \begin{pmatrix} z^1 \\ z^2 \\ z^3 \end{pmatrix} \longrightarrow \mathbf{M} \cdot \begin{pmatrix} z^1 \\ z^2 \\ z^3 \end{pmatrix} + \mathbf{m}, \tag{5.54}$$

where

$$\mathbf{m} = \begin{pmatrix} m^1 \\ m^2 \\ m^3 \end{pmatrix} \in \mathbb{Z}^3 \oplus i\mathbb{Z}^3, \quad \mathbf{M} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -m^2 & 1 \end{pmatrix}. \tag{5.55}$$

By 'noncompact Iwasawa' we mean the group manifold G (which is homeomorphic to $\mathbb{C}^3$), i.e. before taking the Γ quotient.

The left invariant forms on G are given by dz^1, dz^2, $dz^3 + z^1 dz^2$. We can choose the orthonormal frame of one-forms e^a, $a = 1, \ldots, 6$ so that

$$e^1 + ie^2 = dz^1, \quad e^3 + ie^4 = dz^2, \quad e^5 + ie^6 = dz^3 + z^1 dz^2. \tag{5.56}$$

It follows that:

$$\begin{aligned} de^a &= 0, \quad a = 1, \ldots 4 \\ de^5 &= e^1 \wedge e^3 - e^2 \wedge e^4 \\ de^6 &= e^1 \wedge e^4 + e^2 \wedge e^3. \end{aligned} \tag{5.57}$$

The SU(3) structure is given by

$$\begin{aligned} J &= e^{2\phi} \left(e^1 \wedge e^2 + e^3 \wedge e^4 \right) + e^5 \wedge e^6 \\ \Omega &= e^{2\phi} (e^1 + ie^2) \wedge (e^3 + ie^4) \wedge (e^5 + ie^6), \end{aligned} \tag{5.58}$$

so that the G left-invariant forms given above are $(1,0)$; ϕ is a function of z^1, z^2 but does not depend on z^3. The metric associated with (5.57) reads

$$ds^2 = \sum_{a=1}^{4} e^{2\phi} \left(e^a \otimes e^a \right) + \sum_{a=5}^{6} e^a \otimes e^a. \tag{5.59}$$

We obtain W_1, $W_2 = 0$, and

$$\begin{aligned} W_3 = & \; e^1 \wedge e^3 \wedge e^6 - e^2 \wedge e^4 \wedge e^6 - e^1 \wedge e^4 \wedge e^5 - e^2 \wedge e^3 \wedge e^5 \\ & + \frac{1}{2} d\phi \wedge \left[e^{2\phi} \left(e^1 \wedge e^2 + e^3 \wedge e^4 \right) - e^5 \wedge e^6 \right] \end{aligned} \tag{5.60}$$

$$W_4 = -\mathrm{Re}W_5 = d\phi,$$

as can be seen by computing dJ, $d\Omega$ taking (5.57) into account and comparing with eq. (5.10). Moreover, the H-field can be computed from (5.19) and (5.51):

$$H = -e^2 \wedge e^4 \wedge e^5 + e^1 \wedge e^3 \wedge e^5 + e^2 \wedge e^3 \wedge e^6 + e^1 \wedge e^4 \wedge e^6 - \star \left(d\phi \wedge e^5 \wedge e^6 \right) \ . \quad (5.61)$$

Imposing closure, $dH = 0$, is equivalent to

$$\sum_{i=1}^{2} \frac{\partial}{\partial z^i} \frac{\partial}{\partial \bar{z}^i} e^{2\phi} + 1 = 0 \,, \quad (5.62)$$

which can only be satisfied for the noncompact Iwasawa.

This example is conformally balanced since W_4 is exact. Hence by setting the background field E_m to zero the noncompact Iwasawa satisfies the necessary and sufficient conditions (5.27) and moreover we obtain a (1,1)-supersymmetric theory. Note that although the vector dual to the (1,0)-form $K = e^5 + ie^6$ is Killing (corresponding to isometries along the z^3 direction), K is not ∂-closed: $\partial K = dz^1 \wedge dz^2$. Hence the conditions (5.46) for (1,2) supersymmetry are not satisfied.

T^2 bundles over noncompact K3. This example is based on the work of [44] (see also [45]). In that reference it was shown that six-dimensional manifolds with SU(3) structure can be constructed as T^2 fibrations over K3 surfaces. The metric on the total space is given by

$$ds^2 = e^{2\phi} ds^2(\text{K3}) + (dx + \alpha)^2 + (dy + \beta)^2 \,, \quad (5.63)$$

where $ds^2(\text{K3})$ is the Ricci-flat metric on K3, ϕ is a function on K3, and α, β are local one-forms on K3 which satisfy $d\alpha = \omega_P$, $d\beta = \omega_Q$ with

$$\frac{[\omega_P]}{2\pi}, \ \frac{[\omega_Q]}{2\pi} \in H^2(\text{K3}, \mathbb{Z}) \cap H^{1,1}(\text{K3}) \ . \quad (5.64)$$

The complex SU(3) structure on the total space is given by

$$\begin{aligned}
J &= e^{2\phi} j + (dx + \alpha) \wedge (dy + \beta) \\
\Omega &= e^{2\phi} \omega \wedge [(dx + \alpha) + i(dy + \beta)] \,,
\end{aligned} \quad (5.65)$$

where ω, j are the holomorphic $(2, 0)$ form, the Kähler form on the K3 base[10] respectively. In particular ω, j are both closed. Moreover we will assume that ω_P, ω_Q are anti self-dual $\star \omega_{P,Q} = -\omega_{P,Q}$, so that[11]

$$\omega_P \wedge j = \omega_Q \wedge j = 0 \ . \quad (5.67)$$

---

[10]If the map $\pi : \mathcal{M}_6 \mapsto \mathcal{M}_6/T^2 \simeq \text{K3}$ defines the fibration, we can extend ω, j, α, β from K3 to the total space $\mathcal{M}_6$ by using π^*.

[11]Consider the decomposition

$$H^2(\mathcal{M}_4, \mathbb{R}) = H^+(\mathcal{M}_4, \mathbb{R}) \oplus H^-(\mathcal{M}_4, \mathbb{R}), \quad (5.66)$$

into self-dual and anti self-dual two-forms. For $\mathcal{M}_4$ a compact K3 surface, $H^+(\mathcal{M}_4, \mathbb{R})$ is three-dimensional and is generated by Reω, Imω and j; $H^-(\mathcal{M}_4, \mathbb{R})$ is the 19-dimensional vector space of real (1,1)-forms which are orthogonal to j in the sense of (5.67).

By computing dJ, $d\Omega$ and comparing with (5.10) it can then be seen that the torsion classes are given by

$$W_3 = \frac{1}{2} d\phi \wedge \left[e^{\phi} j - (dx + \alpha) \wedge (dy + \beta) \right] - (dx + \alpha) \wedge \omega_Q + (dy + \beta) \wedge \omega_P$$

$$W_4 = -\text{Re} W_5 = d\phi \,.$$

(5.68)

From the above and (5.23) we find that the H field is given by

$$\star H = -\omega_P \wedge (dy + \beta) + \omega_Q \wedge (dx + \alpha) + d\phi \wedge (dx + \alpha) \wedge (dy + \beta) \,.$$

(5.69)

The closure of H can then be seen to be equivalent to

$$\nabla^2_{\text{K3}} e^{2\phi} + |\omega_P|^2 + |\omega_Q|^2 = 0 \,,$$

(5.70)

where ∇^2_{K3} is the (unwarped) Laplacian on K3, and we have defined $|\omega|^2 = \omega^*_{mn} \omega^{mn}/2$. Similarly to the previous example, the above equation can only be solved for noncompact K3's.

Since W_4 is exact we can set the background field E_m to zero, hence this example satisfies the necessary and sufficient conditions (5.27) and possesses (1,1) supersymmetry. Moreover let us define the one form K by

$$K = dz + (\alpha + i\beta) \,,$$

(5.71)

where $z = x + iy$, which is (1,0) with respect to the complex structure associated with (5.65). Clearly the vector dual to K is Killing for the metric in (5.63). Furthermore $\mathcal{L}_K \phi = 0$ since ϕ only depends on the coordinates of K3 and not on the T^2 coordinate z. We also have $dK = \omega_P + i\omega_Q \in H^{1,1}(\text{K3})$, hence $\partial K = (dK)^{2,0} = 0$. Finally from (5.65) we find

$$d\left(e^{-2\phi} K \wedge J \right) = d\left(K \wedge j \right) = (\omega_P + i\omega_Q) \wedge j = 0 \,.$$

Therefore all the conditions in (5.46) are satisfied and we obtain a rigid (1,2)-supersymmetric theory.

The round sphere $\mathbb{S}^6$. Whether or not the six-dimensional sphere admits a complex structure is still an open question. It is known however that the six-dimensional sphere does not admit an orthogonal complex structure, i.e. one that obeys eq. (5.28) with respect to the standard 'round' metric [46]. In other words the round sphere $\mathbb{S}^6$ is not a hermitian manifold. This implies that $\mathbb{S}^6$ violates the first of the necessary and sufficient conditions in (5.27) therefore our rigid-supersymmetric theory cannot be defined on $\mathbb{S}^6$, at least for $\theta^r \neq 0$.

6 Euclidian 6D SYM and localization

In this section we shall set the parameters θ^r and all other background fields to zero,

$$L^\circ = B^\circ = V^\circ = E^\circ = \theta^r = 0 \,,$$

(6.1)

except for the metric of the curved six-dimensional space $\mathcal{M}_6$. Then the analysis of section 5 implies that all torsion classes vanish and $\mathcal{M}_6$ is a Calabi-Yau threefold.[12] Equivalently, this can be seen by the fact that imposing (6.1), the first Killing spinor equation (2.1) reduces to the condition that the spinor ϵ is covariantly constant, while eq. (2.3) is automatically satisfied. Moreover, as follows from the analysis of section 5.2, this example possesses $(1,1)$ supersymmetry since $\theta^r = 0$. The upshot is that Euclidean rigid $(1,1)$-supersymmetric nonabelian YM theory can be consistently defined on CY threefolds.

6.1 Localization

In order to apply the localization procedure, the theory must be invariant under the action of a fermionic operator Q which is nilpotent, $Q^2 = 0$ (or more generally squares to a symmetry of the theory). Deforming the action by a Q-exact term,

$$S \longrightarrow S + \frac{1}{e^2} Q \cdot U \,, \tag{6.2}$$

leaves invariant the expectation values of Q-closed operators. Hence we may take the limit of zero coupling constant, $e^2 \to 0$, upon which the theory localizes to the set Σ of critical points of $Q \cdot U$ [47]. In this limit the path integral can be performed by restricting S to Σ and computing a one-loop determinant describing the fluctuations normal to Σ. This procedure has been carried out in detail in e.g. [2] for the case of SYM on the round $\mathbb{S}^4$.

The action (3.1) is only invariant on-shell.[13] In order to construct the fermionic operator Q of the previous paragraph we introduce auxiliary scalar bosonic fields K^r. The operator Q is then defined by its action on the fields:

$$\begin{aligned}
Q \cdot A_m{}^r &= \tilde{\nu}^r \gamma_m \eta \,, \\
Q \cdot \lambda^r &= \frac{1}{4} \gamma^{mn} \eta \, F_{mn}{}^r - \frac{1}{2} K^r \eta \\
Q \cdot K &= \tilde{\eta} \gamma^m D_m \nu^r \,,
\end{aligned} \tag{6.3}$$

where η is a commuting unimodular Weyl spinor, as in section 5, which is covariantly constant:

$$\nabla_m \eta = 0 \,. \tag{6.4}$$

Note that by virtue of the fact that η is commuting Q defined in (6.3) is indeed fermionic, i.e. transforms commuting to anticommuting fields and vice-versa.

It can be checked that the operator Q defined in (6.3) is nilpotent off-shell, i.e. $Q^2 = 0$ on all fields without using the equations of motion. Furthermore it can be checked that Q leaves the following Lagrangian invariant up to a total derivative:

$$\mathcal{L}_{\text{off}} = \frac{1}{4} F_{mn}{}^r F^{mn\,r} - 2 \tilde{\nu}^r \gamma^m D_m \lambda^r + \frac{1}{2} K^r K^r \,, \tag{6.5}$$

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[12]We will use the term 'Calabi-Yau threefold' to refer to any six-dimensional manifold whose holonomy is a subset of SU(3), including T^6 and $T^2 \times K3$.

[13]A SYM action with extended off-shell supersymmetry was constructed in [48] (see also [49]).

where $\mathcal{L}_{\text{off}}$ reduces to pure on-shell SYM upon eliminating the auxiliary fields by their equations of motion. Explicitly we have:

$$Q \cdot \mathcal{L}_{\text{off}} = \nabla_m \left[(\tilde{\eta} \gamma^m \nu^r) K^r \right] \ . \tag{6.6}$$

The sign difference between the F^2 term in (3.1) and that in (6.5) above comes from the fact that the spinor parameter η in (6.3) is commuting whereas ϵ in (3.2) is anticommuting.

From (6.5) we see that for convergence of the path integral the fields A_m^r, K^r should be real. On the other hand the Q-transformations of A_m^r, K^r given in (6.3) are complex. This does not affect the localization argument [2]: the action should be thought of as an analytic functional in the space of complexified fields, with the path integral understood as integration over the real subspace thereof.

We now deform the Lagrangian by a Q-exact term:

$$\mathcal{L}_{\text{off}} \longrightarrow \mathcal{L}_{\text{off}} + \frac{1}{e^2} Q \cdot U \,, \tag{6.7}$$

where U is given by

$$U = \lambda^\dagger Q \cdot \lambda \ . \tag{6.8}$$

A straightforward computation using (6.3) then gives

$$\begin{aligned} Q \cdot U &= (Q \cdot \lambda^r)^\dagger (Q \cdot \lambda^r) \\ &= \frac{1}{8} \left(F_{mn}{}^r F^{mn\,r} - \frac{1}{4} \varepsilon^{m_1 \ldots m_6} F_{m_1 m_2}{}^r F_{m_3 m_4}{}^r J_{m_5 m_6} \right) + \frac{1}{4} K^r K^r \,, \end{aligned} \tag{6.9}$$

where we have taken the definition (5.3) into account. Note that since we are in the case where $\mathcal{M}_6$ is a CY threefold and the spinor η is covariantly constant, J_{mn} is a Kähler form and $J_m{}^n$ is a complex structure.

According to the localization procedure, the theory will localize, in the $e^2 \to 0$ limit, to the set Σ of critical points of $Q \cdot U$. To determine Σ we note that the right-hand side of (6.9) can be expressed as a sum of squares:

$$Q \cdot U = \frac{1}{16} \left(F^{mn\,r} - \frac{1}{4} \varepsilon^{mn m_1 \ldots m_4} F_{m_1 m_2}{}^r J_{m_3 m_4} - \frac{1}{2} J^{mn} F_{pq}{}^r J^{pq} \right)^2 + \frac{1}{4} (K^r)^2 \,, \tag{6.10}$$

where we have taken into account (5.4) and the identity:

$$\varepsilon^{mn m_1 \ldots m_4} J_{m_1 m_2} J_{m_3 m_4} = -8 J^{mn} \,, \tag{6.11}$$

which can be proven e.g. by fierzing, taking into account the definition (5.3) of J as a spinor bilinear.

To bring (6.10) to a more familiar form, it is useful to project on the (2,0), (0,2) and (1,1) parts (with respect to the complex structure $J_m{}^n$) using the projectors defined in (5.6). Explicitly we expand:

$$F_{mn}{}^r = F_{mn}^{(2,0)\,r} + F_{mn}^{(0,2)\,r} + \tilde{F}_{mn}^{(1,1)\,r} + \frac{1}{6} F_0^r J_{mn} \,, \tag{6.12}$$

where

$$F_0^r = F_{mn}{}^r J^{mn}$$

$$F_{mn}^{(2,0)\,r} = \left(\Pi^+\right)_m{}^p \left(\Pi^+\right)_n{}^q F_{pq}{}^r$$

$$F_{mn}^{(0,2)\,r} = \left(\Pi^-\right)_m{}^p \left(\Pi^-\right)_n{}^q F_{pq}{}^r = \left(F_{mn}^{(2,0)\,r}\right)^*$$

$$\tilde{F}_{mn}^{(1,1)\,r} = 2 \left(\Pi^+\right)_{[m}{}^p \left(\Pi^-\right)_{n]}{}^q F_{pq}{}^r - \frac{1}{6} F_0^r J_{mn} \,,$$

(6.13)

so that $\tilde{F}_{mn}^{(1,1)\,r}$ is (1,1) and traceless: $\tilde{F}_{mn}^{(1,1)\,r} J^{mn} = 0$. With these definitions, eq. (6.10) can be rewritten as[14]

$$Q \cdot U = \frac{1}{2} \left| F_{mn}^{(2,0)\,r} \right|^2 + \frac{1}{16} \left(F_0^r\right)^2 + \frac{1}{4} \left(K^r\right)^2 \,. \tag{6.15}$$

Hence the set of critical points of $Q \cdot U$ is given by:

$$F_{mn}^{(2,0)\,r} = F_{mn}^{(0,2)\,r} = 0 \,,$$
$$F_0^r = 0 \,, \tag{6.16}$$
$$K^r = 0 \,,$$

while the (1,1)-traceless component of the field strength remains unconstrained. These equations are of course familiar from heterotic compactifications on Calabi-Yau manifolds: The first line of (6.16) defines a holomorphic gauge bundle over $\mathcal{M}_6$; the second line is the Donaldson-Uhlenbeck-Yau equation defining a stable bundle.

In conclusion: the path integral of the SYM theory (6.5) defined on a CY threefold $\mathcal{M}_6$ localizes on stable holomorphic bundles over $\mathcal{M}_6$. Note however that we have restricted our discussion to essentially classical considerations. A quantum-mechanical treatment taking into account the path-integral measure will not be attempted here.

7 Conclusions

In this paper, we have analyzed the conditions for rigid supersymmetry in six dimensions and translated them as a set of necessary and sufficient conditions on the geometry of the six-dimensional manifold. In particular, the background three-form flux H°_{kmn} is given in terms of the torsion classes. Given the existence of Killing spinors on these background structures, we have constructed the explicit Lagrangians and field equations for the interactions of non-abelian vector- and tensor-multiplets on curved space. We have worked out a number of explicit examples, including CY threefolds, the round $\mathbb{S}^3 \times \mathbb{S}^3$, as well as

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[14]More generally, for complex auxiliary fields K^r we find

$$Q \cdot U = \frac{1}{2} \left| F_{mn}^{(2,0)\,r} \right|^2 + \frac{1}{4} \left(\mathrm{Im}K^r - \frac{1}{2}F_0^r\right)^2 + \frac{1}{4} \left(\mathrm{Re}K^r\right)^2 \,. \tag{6.14}$$

As we have already mentioned, for convergence of the euclidean path integral we must take the auxiliary fields to be real.

noncompact examples based on the Iwasawa manifold and T^2 bundles over noncompact K3's. Finally, we have applied the localization procedure to pure SYM theory defined on a CY threefold $\mathcal{M}_6$ and shown that the path integral localizes on stable holomorphic bundles over $\mathcal{M}_6$.

A number of interesting questions remains to be further analyzed. We have fully analyzed the conditions for rigid backgrounds allowing for the minimal amount of supersymmetries, i.e. $N = (1, 0)$ and $N = (1, 1)$ in our notation, and briefly discussed the consequences of $N = (1, 2)$. The existence of more Killing spinors will impose further constraints on the background fields and geometry, just as in the four-dimensional case [6]. With increasing number of supercharges, the structure of the supersymmetry algebra becomes more and more constraining. Let us note that the algebra displayed in (3.5), and (4.7), is soft in the sense that the closure involves dependence on fields, including the background ones. Taking the flat space limit in which the dynamical fields and the background 2-form potential vanishes, yields the rigid superalgebra

$$\{\tilde{Q}_\alpha, Q_\beta\} = (C\gamma^\mu)_{\alpha\beta} P_\mu \,, \tag{7.1}$$

which is the Euclidean Poincaré superalgebra in $6D$ [50]. If we set only the dynamical fields equal to zero, the interpretation of the resulting superalgebra depends on the details of the background fields, such as the Killing spinors and and isometries they may support. In this case, a superalgebra in lower dimensions which also contains internal symmetry generators may be expected. In particular, the analogs of the Poincaré, AdS and conformal superalgebras for space-times in arbitrary dimensions and signatures have been systematically constructed in [51].

Another interesting open question concerns possible anomalies of our model. In Minkowski space-time and with all background fields set to zero, the model reduces to that of [18]. In that model there are gauge anomalies due to the coupling of gauge fields to chiral fermions. For certain gauge groups these anomalies can be cancelled by Green-Schwarz mechanism involving the addition of a one-loop counterterm of the form $\hbar B \wedge F \wedge F$. Supersymmetrization of the model in the presence of this counterterm [52] leads to $\hbar$-corrections to the Yang-Mills field equation, which however is anomalous, as its divergence does not vanish, as expected. This also implies an anomaly term in the closure of the superalgebra. Nonetheless, as shown in [52], introducing a constant α' in front of the YM interactions in the tensor multiplet field equations, and then interchanging the role of α' with $\hbar$ leads to a dual model such that (a) it is anomaly free upon treating the resulting $\hbar$-terms as one-loop counterterm corrections, and (b) the tensor multiplet equations are free, while they arise as source in the field equations of the Yang-Mills multiplet. While the Euclideanization of $6D$ space-time is not expected to upset these results, the consequences of retaining nontrivial background fields from the context of the anomalies are not clear, and remains an interesting open problem. Nonetheless, a direct application of the Noether procedure in the dual formulation that uses the Killing spinor equations is expected to yield a formulation of these models in curved background. It is also worth noting that the tensor-YM system à la [18] as well as its dual formulation, both in $6D$ Minkowski spacetime, were also

obtained from different global limits of the (on-shell) heterotic supergravity compactified on K3, [53].

It should be straightforward to extend the localization results presented here to non-Calabi-Yau manifolds. This would entail modifying the analysis of section 6 to include non-vanishing background fields besides the metric. We should stress however that our treatment of localization was essentailly limited to classical considerations. A fully quantum-mechanical treatment should include the consideration of the path-integral measure and potential anomalies.

Acknowledgments

A 6D conventions

In this section we list some useful relations and explain in more detail our spinor conventions for general even-dimensional Euclidean spaces of dimension $2k$.

The charge conjugation matrix obeys:

$$C^{\mathrm{Tr}} = (-)^{\frac{1}{2}k(k+1)}C \; ; \;\; C^* = (-)^{\frac{1}{2}k(k+1)}C^{-1} \; ; \;\; \gamma_m^{\mathrm{Tr}} = (-)^k C^{-1}\gamma_m C \; . \tag{A.1}$$

The complex conjugate η^c of a spinor η is given by:

$$\eta^c := C\eta^* \, , \tag{A.2}$$

form which it follows that:

$$(\eta^c)^c = (-)^{\frac{1}{2}k(k+1)}\eta \; . \tag{A.3}$$

Covariantly-transforming spinor bilinears must be of the form $(\widetilde{\psi}\gamma_{m_1\ldots m_p}\chi)$, where in any dimension we define:

$$\widetilde{\psi} := \psi^{\mathrm{Tr}}C^{-1} \; . \tag{A.4}$$

One can also show the following useful identity:

$$\gamma^*_{m_1\ldots m_p} = (-)^{kp}C^{-1}\gamma_{m_1\ldots m_p}C \; . \tag{A.5}$$

The case of six-dimensional Euclidean space is obtained by specializing to $k = 3$. The chirality matrix is given by

$$\gamma_{m_1\ldots m_6} = -i\,\varepsilon_{m_1\ldots m_6}\,\gamma^7 \; . \tag{A.6}$$

Since the chiral irreducible representation of Spin(6) is complex, a Weyl spinor η and its complex conjugate η^c have opposite chirality. We also have:

$$\tilde{\eta}^c = \eta^\dagger \; . \tag{A.7}$$

Moreover we find:

$$\left(\widetilde{\psi}_\pm \gamma_{m_1...m_{2l}} \chi_\pm\right) = 0 = \left(\widetilde{\psi}_\pm \gamma_{m_1...m_{2l+1}} \chi_\mp\right), \tag{A.8}$$

where the subscript $\pm$ denotes the chirality. Another useful relation is:

$$\left(\widetilde{\psi}\gamma_{m_1...m_p}\chi\right) = \sigma(-)^{\frac{1}{2}p(p+1)}\left(\widetilde{\chi}\gamma_{m_1...m_p}\psi\right), \tag{A.9}$$

where $\sigma = +1$, $\sigma = -1$ for commuting, anticommuting spinors χ, ψ. As for the spinor fields encountered in our models, the supersymmetry parameters (ϵ, $\tilde{\zeta}$) and the gauginos (λ, $\tilde{\nu}$) have chirality $+$ whereas the tensorinos (χ, $\tilde{\psi}$) have chirality $-$.

B SU(3)-structure identities

Here we summarize several relations which are useful in analyzing the Killing spinor equations. We follow ref. [54] which the reader may consult for further details.

$$0 = (\Pi^+)_m{}^n \gamma_n \eta^c$$

$$\gamma_{mn}\eta = iJ_{mn}\eta + \frac{1}{2}\Omega_{mnp}\gamma^p\eta^c$$

$$\gamma_{mnp}\eta^c = -3iJ_{[mn}\gamma_{p]}\eta^c - \Omega^*_{mnp}\eta . \tag{B.1}$$

The above together with tensor decomposition (5.11) give

$$\gamma^{mnp}H_{mnp}\eta = H^{(0)*}\eta^c + 3iH_t^{(0,1)}\gamma^t\eta$$

$$H_{mpq}\gamma^{pq}\eta = \left(H^{(0)*}g_{mt} + 3\Omega_t{}^{pq}\tilde{H}_{mpq}^{(1,2)} - \frac{3i}{2}\Omega_{mt}{}^p H_p^{(0,1)}\right)\gamma^t\eta^c + 6i(H_m^{(1,0)} + H_m^{(0,1)})\eta$$

$$H_{pqr}\gamma^{pqr}\gamma_m\eta = 3\Omega_t{}^{pq}\tilde{H}_{mpq}^{(1,2)}\gamma^t\eta^c + 6iH_m^{(1,0)} . \tag{B.2}$$

In the above we have taken into account the identity

$$\Omega_{[t}{}^{pq}\tilde{H}_{m]pq}^{(1,2)} = 0 , \tag{B.3}$$

which can easily be shown by direct calculation.

The following identity is useful in deriving the formulæ in section 5.3:

$$(V \lrcorner W) \wedge J = iV \wedge W , \tag{B.4}$$

where V is an arbitrary (1,0)-form and W is any three-form obeying $W \wedge J = 0$.

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