

# Open worldsheets for holographic interfaces<sup>1</sup>

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**ABSTRACT:** Type IIB supergravity admits Janus and multi-Janus solutions with eight unbroken supersymmetries that are locally asymptotic to  $AdS_3 \times S^3 \times M_4$  (where  $M_4$  is either  $T^4$  or  $K_3$ ). These solutions are dual to two or more CFTs defined on half-planes which share a common line interface. Their geometry consists of an  $AdS_2 \times S^2 \times M_4$  fibration over a simply connected Riemann surface  $\Sigma$  with boundary.

In the present paper, we show that regular exact solutions exist also for surfaces  $\Sigma$  which are not simply connected. Specifically, we construct in detail solutions for which  $\Sigma$  has the topology of an annulus. This construction is generalized to produce solutions for any surface  $\Sigma$  with the topology of an open string worldsheet with  $g$  holes.

**KEYWORDS:** AdS-CFT Correspondence, Gauge-gravity correspondence

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**1 Introduction**

One of the best studied examples of the AdS/CFT correspondence [1–3] (for reviews see e.g. [4, 5]) is the duality between  $AdS_3 \times S^3 \times M_4$  vacua of type IIB string theory (where  $M_4$  is either  $T^4$  or  $K_3$ ) and certain two-dimensional superconformal field theories with 16 supersymmetries [6–9].

Deformations of the supersymmetric vacua which preserve half of the supersymmetries are of particular interest. On the CFT side, such deformations can be associated with local operators as well as certain extended objects, such as Wilson loops, interfaces and surface operators. Defects and interfaces that preserve half of the conformal supersymmetry represent an important example of such extended objects for two-dimensional CFTs, and will be the focus of the present paper.

There are several methods for obtaining spacetimes that are dual to superconformal interfaces or defects. First, we can consider a brane intersection where a lower dimensional brane realizes a CFT on its worldvolume while a higher dimensional brane introduces a conformal boundary [10]. In particular, one can add a D3 brane to a D1/D5 brane configuration introducing a 0 + 1-dimensional intersection. Without the additional D3 brane, the D1/D5 branes produce the  $AdS_3 \times S^3 \times M_4$  vacuum in the near-horizon limit. It has been argued [10] that the D3 brane survives this limit and introduces a defect in the dual theory.

Second, probe branes which have an  $AdS_2$  worldvolume inside the  $AdS_3$  space provide a holographic realization of one-dimensional conformal interfaces and defects [10–17]. In general,  $\kappa$ -symmetry of the worldvolume theory may be used to count the number of supersymmetries preserved by the probe and to fix the positions of the probe branes in  $AdS_3 \times S^3 \times M_4$  [18].

Third, in many cases the extended supersymmetry of the interface or defect may be exploited to obtain supergravity solutions in analytic form. It is in this spirit that various regular Janus solutions with 16 supersymmetries were derived in type IIB supergravity [19–22] and M-theory [23–26] to obtain holographic duals of interfaces, defects, Wilson loops and surface operators for super Yang-Mills theories in 2+1 and 3+1 dimensions.<sup>1</sup>

Conformal superalgebras [32] provide a framework in which the above three methods may be understood in a unified way [33]. In this paper, we will focus on the third approach and derive exact supergravity solutions dual to interfaces and defects in 1 + 1-dimensional CFTs.

Exact half-BPS solutions in type IIB supergravity that preserve eight of the 16 supersymmetries of the  $AdS_3 \times S^3 \times M_4$  vacuum and are locally asymptotic to the vacuum solution were constructed in a previous paper [34] (see also [35–38] for earlier related work). The Ansatz for these solutions preserves a  $SO(2, 1) \times SO(3)$  subgroup of the full  $SO(2, 2) \times SO(4)$  isometry group of the vacuum. This  $SO(2, 1) \times SO(3)$  symmetry correctly reflects the global bosonic symmetry for a one-dimensional conformal interface, and uniquely extends to the superalgebra  $SU(1, 1|2)$  which has eight supersymmetries.

The exact solutions of [34] display a rich and interesting moduli space, and have two or more asymptotic  $AdS_3 \times S^3 \times M_4$  regions which may be identified, in the dual CFT, with two-dimensional half-spaces glued together at the one-dimensional interface. In the different asymptotic regions, the dilaton and axion fields approach different constant values, and the D1, D5, NS5 and F1 charges take on different values.

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<sup>1</sup>For closely related work, see also [27–31].

The goal of the present paper is to address a number of important questions which were raised in [34], but were not answered there:

- The solutions of [34] have non-zero D1, D5, NS5, and F1 charges, but all have vanishing D3-brane charge. It would be interesting to find solutions carrying non-trivial D3-brane charge.
- The solutions of [34] have spacetimes given by an  $AdS_2 \times S^2 \times M_4$  fibration over a Riemann surface  $\Sigma$  with boundary. Moreover,  $\Sigma$  is simply-connected, and may be conformally mapped to the upper half-plane. A similar analysis was conducted for half-BPS multi-Janus solutions dual to 2 + 1-dimensional interfaces and defects [20, 21]. In the latter case, the space-time is given by the warped product  $AdS_4 \times S^2 \times S^2 \times \Sigma$ , and regularity appears to require that  $\Sigma$  has only a single boundary component and is simply-connected though no rigorous *theorem* to that effect is yet available. For the lower-dimensional case of a one-dimensional interface or defect considered here, however, this question must be re-examined. We intend to generalize the solutions of [34] to the case where  $\Sigma$  has a more complicated topology, involving more than one boundary component.
- The half-BPS solutions of [34] are completely regular, and have the same symmetries and charges associated with probe branes in the  $AdS_3 \times S^3$  vacuum. They may be viewed as the fully back-reacted solutions where the probe branes have been replaced by geometry and flux. A final question is whether the localized probe branes may be recovered by taking a (possibly singular) limit of the fully back-reacted solution. This reverse process was possible in the original multi-Janus solutions [20, 21], and is expected to be available here as well.

In the present paper, we show that the answers to the first two questions are intimately related. We find that regular type IIB supergravity solutions for which  $\Sigma$  has multiple boundary components do exist, and we construct them explicitly. For  $\Sigma$  with the topology of an annulus (i.e. two boundary components), the construction is carried out explicitly in terms of elliptic functions and their related Jacobi theta functions. For  $\Sigma$  with the topology of a sphere with  $g + 1$  holes, with  $g \geq 2$ , the construction will be given in terms of the higher genus prime forms and theta functions of the double cover of  $\Sigma$ .

The question as to whether regular solutions for which  $\Sigma$  also has handles can be obtained will not be addressed in this paper, though many of the tools needed to examine this question will be developed here.

Finally, the new half-BPS interface and defect solutions we obtain will be an excellent laboratory for considering the probe limit of regular back-reacted solutions.

## 1.1 Organization

The structure of this paper is as follows. In section 2 we briefly review the local half-BPS interface solutions as well as the conditions imposed by global regularity. For more details and the full derivations we refer the reader to [34]. In section 3, we consider the

case of a Riemann surface with two disconnected boundary components (i.e. the annulus). We construct the solutions using theta functions, and solve the constraints imposed by regularity. We show that the solutions carry non-zero D3-brane charge, and that this charge is associated with the non-contractible cycle of the annulus. In section 4, we examine the degeneration of the annulus where one boundary shrinks to a point, and show that, in this limit, extra asymptotic regions with  $AdS_2 \times S^2 \times S^1 \times R$  geometry appear. In section 5, we generalize the annulus solution to the case of a Riemann surface with an arbitrary number of boundary components utilizing the doubling trick to construct the solutions in terms of holomorphic differentials and prime forms on the double Riemann surface. In the final section, we discuss possible generalizations of the solutions found in this paper and list several open questions and directions for future research.

## 2 Local half-BPS interface solutions

In this section, we present a summary of the Ansatz and local half-BPS solutions obtained in [34] which will be used again here. We also review the regularity and boundary conditions needed to promote the local results to globally well-defined solutions.

### 2.1 Ansatz

In the local half-BPS solutions, the spacetime is constructed as a fibration of the product  $AdS_2 \times S^2 \times M_4$  (where  $M_4$  is either  $T^4$  or  $K_3$ ) over a two-dimensional Riemann surface  $\Sigma$  with boundary. This product space is invariant under the global symmetry group  $SO(1, 2) \times SO(3)$ , and the appropriate Ansatz must reflect this symmetry. The metric is given by,

$$ds^2 = f_1^2 ds_{AdS_2}^2 + f_2^2 ds_{S^2}^2 + f_3^2 ds_{M_4}^2 + \rho^2 dzd\bar{z} \tag{2.1}$$

Symmetry requires that all reduced bosonic fields, such as  $f_1, f_2, f_3$ , and  $\rho$ , depend only on  $\Sigma$ . It will be convenient to introduce an orthonormal frame associated with this metric; its components satisfy,

$$\begin{aligned} \eta_{i_1 i_2} e^{i_1} \otimes e^{i_2} &= f_1^2 ds_{AdS_2}^2 & i_{1,2} &= 0, 1 \\ \delta_{j_1 j_2} e^{j_1} \otimes e^{j_2} &= f_2^2 ds_{S^2}^2 & j_{1,2} &= 2, 3 \\ \delta_{k_1 k_2} e^{k_1} \otimes e^{k_2} &= f_3^2 ds_{M_4}^2 & k_{1,2} &= 4, 5, 6, 7 \\ \delta_{ab} e^a \otimes e^b &= \rho^2 dzd\bar{z} & a, b &= 8, 9 \end{aligned} \tag{2.2}$$

The standard one-form  $P$  and composite  $U(1)$  connection  $Q$  may be expressed in terms of the dilaton  $\Phi$  and axion  $\chi$  field, as follows,

$$P = -d\Phi + \frac{i}{2} e^{-2\Phi} d\chi \qquad Q = -\frac{1}{2} e^{-2\Phi} d\chi \tag{2.3}$$

Finally, the  $SO(1, 2) \times SO(3)$  symmetry restricts the three-form  $G$  and five-form  $F_5$  to be given by,

$$\begin{aligned} G &= g_a^{(1)} e^{a01} + g_a^{(2)} e^{a23} \\ F_5 &= h_a e^{a0123} + \tilde{h}_a e^{a4567} \end{aligned} \tag{2.4}$$

Self-duality of  $F_5$  imposes the condition  $h_a = -\epsilon_a{}^b \tilde{h}_b$ . In this Ansatz, only the volume form on  $M_4$  is taken into account, while the other cohomology generators of  $M_4$  are omitted, and the corresponding moduli of these spaces are turned off. This explains why the only dependence on  $M_4$  in the Ansatz for  $G$  and  $F_5$  is through the volume form  $e^{4567}$ .

## 2.2 Local solutions

In [34], the BPS equations and Bianchi identities were reduced to a system of four differential equations which admits a local solution in terms of two harmonic functions,<sup>2</sup>  $H$  and  $K$ , and two holomorphic functions,  $A$  and  $B$ . All supergravity fields of the local solution can be expressed in terms of these functions. The dilaton and axion are given by,

$$e^{4\Phi} = \frac{1}{4K^2} \left( (A + \bar{A})K - (B + \bar{B})^2 \right) \left( (A + \bar{A})K - (B - \bar{B})^2 \right) \quad (2.5)$$

$$\chi = \frac{i}{2K} \left( (A - \bar{A})K - B^2 + \bar{B}^2 \right) \quad (2.6)$$

The metric factors take the following form,

$$f_1^2 = \frac{ce^{-2\Phi} |H|}{2f_3^2 K} \left( (A + \bar{A})K - (B - \bar{B})^2 \right) \quad (2.7)$$

$$f_2^2 = \frac{ce^{-2\Phi} |H|}{2f_3^2 K} \left( (A + \bar{A})K - (B + \bar{B})^2 \right) \quad (2.8)$$

$$f_3^4 = 4 \frac{c^2 e^{2\Phi} K}{A + \bar{A}} \quad (2.9)$$

The constant  $c$  is related to the volume of  $K_3$  and was set to 1 in [34]. The metric on  $\Sigma$  can then be written as

$$\rho^4 = e^{2\Phi} K \frac{|\partial_w H|^4}{H^2} \frac{A + \bar{A}}{|B|^4} \quad (2.10)$$

The following combinations of three-form fluxes and metric factors can be expressed as total derivatives,

$$\begin{aligned} f_2^2 \rho e^{-\Phi} \text{Re}(g^{(2)})_z &= \partial_w b^{(2)}, \\ f_2^2 \rho e^{\Phi} \text{Im}(g^{(2)})_z + \chi f_2^2 \rho e^{-\Phi} \text{Re}(g^{(2)})_z &= \partial_w c^{(2)} \end{aligned} \quad (2.11)$$

There are analogous expressions for  $g_z^{(1)}$  which can be found in [34] and that will not be needed in this paper. The potentials written in terms of our holomorphic and harmonic functions are

$$b^{(2)} = -i \frac{H(B - \bar{B})}{(A + \bar{A})K - (B - \bar{B})^2} + \tilde{h}_1, \quad \tilde{h}_1 = \frac{1}{2i} \int \frac{\partial_w H}{B} + c.c. \quad (2.12)$$

$$c^{(2)} = -\frac{H(A\bar{B} + \bar{A}B)}{(A + \bar{A})K - (B - \bar{B})^2} + h_2, \quad h_2 = \frac{1}{2} \int \frac{A}{B} \partial_w H + c.c. \quad (2.13)$$

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<sup>2</sup>To improve notational clarity, we denote here by  $K$  the function that was denoted by  $\hat{h}$  in [34].

Similarly, a combination of  $F_5$  and metric factors can be written as a total derivative,

$$f_3^4 \rho \tilde{h}_z = \partial_w C_K, \quad C_K = \frac{c^2}{2i} \frac{B^2 - \bar{B}^2}{A + \bar{A}} - \frac{c^2}{2} \tilde{K} \quad (2.14)$$

where  $\tilde{h}_1, \tilde{h}_2, \tilde{K}$  denote the harmonic functions conjugate to  $h_1, h_2, K$  respectively. Finally, we note that it is possible to rescale our functions by a constant  $a$ ,

$$K \rightarrow a^2 K, \quad B \rightarrow aB, \quad H \rightarrow aH, \quad c \rightarrow a^{-1}c \quad (2.15)$$

leaving all the physical fields invariant.

### 2.3 Regularity conditions

Any choice of holomorphic functions  $A, B$  and harmonic functions  $H, K$  will produce bosonic fields which solve the type IIB supergravity field equations and preserve eight real supersymmetries. In general, however, such solutions may either have singularities or be unphysical, e.g. when a real scalar field like the dilaton  $\Phi$  becomes complex. In order to guarantee sensible regular solutions, several additional conditions have to be imposed. One such constraint comes from the fact that the asymptotic regions of the spacetime correspond locally to  $AdS_3 \times S^3 \times M_4$ . The complete list of conditions to be satisfied is as follows:

- The radius of the  $AdS_2$  slice, given by the metric coefficient  $f_1$ , is non-zero and finite everywhere, except at isolated singular points on the boundary of  $\Sigma$ . Each such singular point corresponds to an  $AdS_3 \times S^3 \times M_4$  asymptotic region.
- The radius of the  $S^2$  slice, given by  $f_2$ , is finite in the interior of  $\Sigma$ , and zero on the boundary of  $\Sigma$ . The boundary may be defined as the curve on which  $f_2$  vanishes.
- The radius of the  $M_4$  manifold, given by  $f_3$ , and the dilaton combination  $e^{2\Phi}$  are finite and non-zero everywhere on  $\Sigma$ , including the boundary.

Using the above requirements, it was shown in [34] that the harmonic functions  $H, K, A + \bar{A}$  and  $B + \bar{B}$  must obey vanishing Dirichlet boundary conditions, while the harmonic functions  $i(A - \bar{A}), i(B - \bar{B})$  and  $\tilde{K}$  must obey Neumann boundary conditions. Moreover,  $A, B$  and the meromorphic part of  $K$  can admit only simple poles, and the following regularity conditions need to be respected,

- R1:** All singularities of the harmonic functions  $A + \bar{A}, B + \bar{B}$  and  $K$  must be common, the residues of  $A, B$  and the meromorphic part of  $K$  are related by  $r_A r_K = r_B^2$ ;
- R2:** The functions  $A, B, H, K$  must be regular in the interior of  $\Sigma$ ;
- R3:** The functions  $A + \bar{A}, K$  and  $H$  cannot vanish in the interior of  $\Sigma$ ;
- R4:** All the zeros of  $B$  and  $\partial_w H$  must be common.

Finally, there is an extra condition coming from the requirement that the dilaton field must be real or, more practically, that  $e^{4\Phi}$  must be positive,

**R5:** The following inequality

$$(A + \bar{A})K - (B + \bar{B})^2 > 0 \tag{2.16}$$

must be obeyed throughout  $\Sigma$ , including on the boundary.

For a Riemann surface with a single boundary component, this follows as soon as conditions **R1** to **R4** are satisfied. Similarly, we shall show that this condition does not pose any further constraint on the solutions constructed in this paper.

### 3 Two boundary components: the annulus

In [34] the Riemann surface  $\Sigma$  had a single boundary component. The next simplest choice is a Riemann surface with two boundary components and no handles. By uniformization, it is always possible to map  $\Sigma$  to the annulus, defined as the domain

$$\Sigma \equiv \left\{ w \in \mathbf{C}, 0 \leq \text{Re}(w) \leq 1, 0 \leq \text{Im}(w) \leq \frac{t}{2} \right\} \tag{3.1}$$

where points  $w + 1$  and  $w$  are identified, giving indeed the topology of an annulus. The two boundaries  $\partial\Sigma_{1,2}$  of the annulus are located at  $\text{Im}(w) = 0$  and  $\text{Im}(w) = t/2$ .

The annulus can be constructed from a Riemann surface without boundary, the so called double  $\bar{\Sigma}$ , which is a rectangular torus with pure imaginary modular parameter  $\tau = it$ .

$$\bar{\Sigma} \equiv \{w \in \mathbf{C}, 0 \leq \text{Re}(w) \leq 1, 0 \leq \text{Im}(w) \leq t\} \tag{3.2}$$

where  $w + 1$  is identified with  $w$  and  $w + it$  is identified with  $w$ . The annulus (3.1) is obtained by quotient of the torus (3.2) by an anti-conformal involution

$$\Sigma = \bar{\Sigma}/\mathcal{I}, \quad \mathcal{I}(z) = \bar{z} \tag{3.3}$$

The boundaries  $\partial\Sigma_{1,2}$  of the annulus are the fixed point set of the involution  $\mathcal{I}$ .

#### 3.1 Harmonic and holomorphic functions on the annulus

The construction of the solution for the annulus proceeds in three steps. First, we construct a basic harmonic function with prescribed singularities and boundary conditions. Second, we express  $A$ ,  $H$  and  $K$  using linear combinations of the basic function. Third, we find the meromorphic function  $B$  which satisfies conditions **R1** and **R4**.

In the sequel, we shall make use two Jacobi theta functions, which are defined as follows<sup>3</sup>

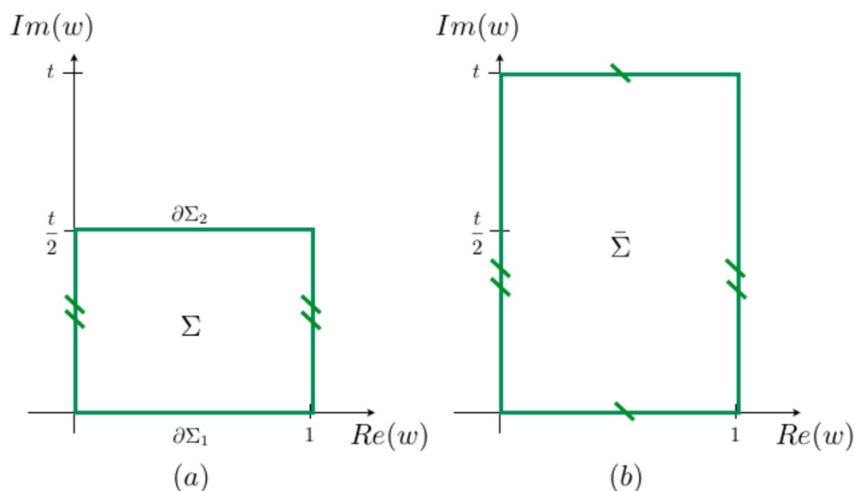
$$\vartheta_1(u|\tau) = 2 \sum_{n=0}^{\infty} (-1)^n q^{(n+1/2)^2} \sin(2n+1)u \tag{3.4}$$

$$\vartheta_4(u|\tau) = 1 + 2 \sum_{n=0}^{\infty} (-1)^n q^{n^2} \cos 2nu \tag{3.5}$$

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<sup>3</sup>We use conventions consistent with [45].





**Figure 1.** (a) The annulus  $\Sigma$  with boundaries at  $\text{Im}(w) = 0, \text{Im}(w) = \frac{t}{2}$ ; (b) the doubled surface  $\bar{\Sigma}$  is a torus with modulus  $\tau = it$ .

where  $q = e^{\pi i \tau} = e^{-\pi t}$ . With these definitions, the quasi-periodicity conditions are,

$$\begin{aligned} \vartheta_1(u + \pi|\tau) &= -\vartheta_1(u|\pi) \\ \vartheta_1(u + \pi\tau|\tau) &= -e^{-2iu} q^{-1} \vartheta_1(u|\tau) \end{aligned} \tag{3.6}$$

Furthermore under shifts by  $\pi\tau/2$  we have,

$$\vartheta_1(u + \pi\tau/2|\tau) = i q^{-\frac{1}{4}} e^{-iu} \vartheta_4(u|\tau) \tag{3.7}$$

Note that  $\vartheta_1$  is an odd function of  $u$  and vanishes linearly as  $u \rightarrow 0$ , while  $\vartheta_4$  is an even function of  $u$  which vanishes at  $\pi\tau/2$ .

### 3.2 Construction of the basic harmonic function

It turns out that the harmonic functions  $A \pm \bar{A}, B \pm \bar{B}, H$ , and  $K$  may be simply constructed by linear superposition of a basic harmonic function, which has simple properties. The basic harmonic function  $h_0$  on the annulus  $\Sigma$  satisfies the following properties:<sup>4</sup>

1.  $h_0$  has a single simple pole on  $\partial\Sigma$ , so that the  $(1,0)$ -form  $\partial h_0$  has one double pole;
2. Away from the pole on  $\partial\Sigma$ ,  $h_0$  satisfies Dirichlet conditions, i.e.  $h_0 = 0$  on  $\partial\Sigma$ .
3.  $h_0 > 0$  in the interior of  $\Sigma$ ;

The harmonic function  $h_0$  can be expressed in terms of Jacobi theta functions,

$$h_0(w, \bar{w}) = i \left( \frac{\partial_w \vartheta_1(\pi w|\tau)}{\vartheta_1(\pi w|\tau)} + \frac{2\pi i w}{\tau} \right) + c.c \tag{3.8}$$

<sup>4</sup>By abuse of notation, we shall refer interchangeably to the poles of the harmonic function  $\mu(w) + \overline{\mu(w)}$  and the poles of its meromorphic part  $\mu(w)$  only.

or as an infinite series of images,

$$h_0(w, \bar{w}) = \frac{\pi i}{\tau} \sin\left(\frac{\pi}{\tau}(w - \bar{w})\right) \sum_{m=-\infty}^{+\infty} \frac{1}{\left|\sin\left(\frac{\pi}{\tau}(w + m)\right)\right|^2} \quad (3.9)$$

The meromorphic part of (3.8) has a simple pole on  $\partial\Sigma_1$  located at  $w = 0$  and hence condition 1 is satisfied. Since the meromorphic part in (3.8) is real when  $\text{Im}(w) = 0$ , the harmonic function  $h_0$  satisfies vanishing Dirichlet boundary conditions at the first boundary  $\partial\Sigma_1$ . Using the transformation property (3.7) one shows that the meromorphic part in (3.8) is real at  $\text{Im}(w) = t/2$  and hence  $h_0$  satisfies vanishing Dirichlet boundary conditions at the second boundary  $\partial\Sigma_2$ . Hence condition 2 is satisfied. This property is actually manifest from the prefactor in (3.9). The harmonic function (3.8) is single valued on the annulus,

$$h_0(w + 1, \bar{w} + 1) = h_0(w, \bar{w}) \quad (3.10)$$

and vanishes on both boundary components. By the maximum principle for harmonic functions it follows that  $h_0 > 0$  in the interior of the annulus.<sup>5</sup> Hence condition 3 is also satisfied.

The basic harmonic function  $h_0$  has a singularity at  $w = 0$  at the first boundary component  $\partial\Sigma_1$ . The location of the singularity can be shifted to any point on the boundary by a real translation so that  $h_0(w - x, \bar{w} - x)$  has a singularity at  $w = x$ . To obtain harmonic functions which have singularities on the second boundary component  $\partial\Sigma_2$ , we define

$$w' \equiv \frac{\tau}{2} - w \quad (3.11)$$

and for a real  $y$ , the harmonic function  $h_0(w' + y, \bar{w}' + y)$  has a pole on the second boundary at  $w = \tau/2 + y$  and satisfies conditions 1-3.

The harmonic function  $\tilde{h}_0$  conjugate to the basic harmonic function  $h_0$  is given by

$$\tilde{h}_0(w, \bar{w}) = \left( \frac{\partial_w \vartheta_1(\pi w | \tau)}{\vartheta_1(\pi w | \tau)} + \frac{2\pi i w}{\tau} \right) + c.c \quad (3.12)$$

Note that, unlike  $h_0$ , the conjugate harmonic function  $\tilde{h}_0$  is not single-valued on the annulus. Its monodromy is given as follows,

$$\tilde{h}_0(w + 1, \bar{w} + 1) = \tilde{h}_0(w, \bar{w}) + \frac{4\pi i}{\tau} \quad (3.13)$$

### 3.3 Construction of the supergravity solution on the annulus

The harmonic functions  $A \pm \bar{A}, B \pm \bar{B}, H$  and  $K$  may now be expressed as linear combinations of basic harmonic functions with a pole at various points on both boundaries. Note that for some of the harmonic functions, both the harmonic function and its harmonic conjugate must be single-valued on the torus. This condition imposes constraints on the coefficients of the linear combination. In this section we will construct the four harmonic functions which determine our solution.

<sup>5</sup>The harmonic function could also be strictly negative, however explicit evaluation of the harmonic function at special points in the interior shows that it takes positive values.

- The function  $H$  must be single-valued and positive on  $\Sigma$ , and obey vanishing Dirichlet boundary conditions. We take  $H$  to have  $N$  poles  $x_{H,\alpha}$  with  $\alpha = 1, \dots, N$  on  $\partial\Sigma_1$ , and  $N'$  poles  $y_{H,\beta}$  with  $\beta = 1, \dots, N'$  on  $\partial\Sigma_2$ . The corresponding residues  $r_{H,\alpha}$  and  $r'_{H,\beta}$  must be positive, but are otherwise left undetermined,

$$H = \sum_{\alpha=1}^N r_{H,\alpha} h_0(w - x_{H,\alpha}, \bar{w} - x_{H,\alpha}) + \sum_{\beta=1}^{N'} r'_{H,\beta} h_0(w' + y_{H,\beta}, \bar{w}' + y_{H,\beta}) \quad (3.14)$$

It is possible to use the translation symmetry of the annulus to fix the position of the first singularity at  $x_{H,1} = 0$ . Since the conjugate harmonic function  $\tilde{H}$  does not appear in any of the expressions for the supergravity fields of the solution,  $\tilde{H}$  does not need to be single-valued on the annulus. Consequently, there are no additional conditions involving the residues and the number of parameters is equal to  $2N + 2N' - 1$ . Note that each pole  $x_{H,\alpha}$  and  $y_{H,\beta}$  of  $H$  corresponds to an asymptotic  $AdS_3 \times S^3$  region. Hence, the full supergravity solution will have a total of  $N + N'$  asymptotic  $AdS_3 \times S^3$  regions.

- The harmonic function  $A + \bar{A}$  must be single-valued and positive in  $\Sigma$ , and obey vanishing Dirichlet boundary conditions. The conjugate function  $-i(A - \bar{A})$  must also be single-valued on  $\Sigma$ . As a result, these functions take on the following form,

$$A + \bar{A} = \sum_{\alpha=1}^M r_{A,\alpha} h_0(w - x_{A,\alpha}, \bar{w} - x_{A,\alpha}) + \sum_{\beta=1}^{M'} r'_{A,\beta} h_0(w' + y_{A,\beta}, \bar{w}' + y_{A,\beta}) \quad (3.15)$$

$$-i(A - \bar{A}) = \sum_{\alpha=1}^M r_{A,\alpha} \tilde{h}_0(w - x_{A,\alpha}, \bar{w} - x_{A,\alpha}) + \sum_{\beta=1}^{M'} r'_{A,\beta} \tilde{h}_0(w' + y_{A,\beta}, \bar{w}' + y_{A,\beta}) \quad (3.16)$$

Single-valuedness of  $-i(A - \bar{A})$  imposes a relation between the residues. Using (3.13), one obtains the monodromy of this function,

$$-i(A - \bar{A})(w + 1, \bar{w} + 1) = -i(A - \bar{A})(w, \bar{w}) + \frac{4\pi i}{\tau} \left( \sum_{\alpha=1}^M r_{A,\alpha} - \sum_{\beta=1}^{M'} r'_{A,\beta} \right) \quad (3.17)$$

Hence,  $-i(A - \bar{A})$  will be single-valued provided the following relation is obeyed,

$$\sum_{\alpha=1}^M r_{A,\alpha} = \sum_{\beta=1}^{M'} r'_{A,\beta} \quad (3.18)$$

This condition brings the number of parameters in the definition of  $A$  down to  $2M + 2M'$ . Note that one of the parameters is given by the constant in the definition of the dual harmonic function  $-i(A - \bar{A})$ .

- The meromorphic function  $B$  must have the same poles as  $A$  (according to the regularity condition **R1**), and the same zeros as  $\partial_w H$  (according to **R4**). Moreover, both  $B + \bar{B}$  and  $-i(B - \bar{B})$  must be single-valued. The functions  $B + \bar{B}$  and  $-i(B - \bar{B})$  are not subject, however, to any positivity condition, and are allowed to change sign inside  $\Sigma$ . We get the following expression for  $B$ :

$$B(w) = \frac{\partial_w H(w)}{\mathcal{N}(w)} \tag{3.19}$$

Here,  $\mathcal{N}(w)dw$  is a single-valued meromorphic form of weight  $(1, 0)$  whose zeros and poles are determined by the requirement that  $B$  and  $\partial_w H$  have common zeros and  $B$  and  $A$  have common poles. Thus,  $\mathcal{N}$  must cancel the (double) poles of  $\partial_w H$ , and reproduce the poles of  $A$ . The resulting form may be expressed in terms of the Jacobi theta function  $\vartheta_1$ ,

$$\mathcal{N}(w) = e^{i\pi\phi w} \frac{\prod_{\alpha=1}^M \vartheta_1\left(\pi(w - x_{A,\alpha}) \middle| \tau\right) \prod_{\beta=1}^{M'} \vartheta_1\left(\pi(w' + y_{A,\beta}) \middle| \tau\right)}{\prod_{\alpha=1}^N \vartheta_1\left(\pi(w - x_{H,\alpha}) \middle| \tau\right)^2 \prod_{\beta=1}^{N'} \vartheta_1\left(\pi(w' + y_{H,\beta}) \middle| \tau\right)^2} \tag{3.20}$$

Here, the exponential prefactor has been included in order to make  $\mathcal{N}$  properly single-valued, and  $\phi$  is a real constant phase. Note that we can fix an overall real constant in the definition of  $B$  using the symmetry (2.15).

It remains to work out the precise conditions required to render  $\mathcal{N}$  single-valued. Using the transformation property (3.6) we see that single-valuedness of  $B$ , namely  $B(w + 1) = B(w)$  requires  $M + M' + \phi$  to be an even integer. Moreover, in order to have  $B + \bar{B}$  obey vanishing Dirichlet boundary conditions, we need  $\mathcal{N}(w)$  to be real for  $w = x$  and  $w = \tau/2 + x$  with  $x$  real. Using (3.7)  $\vartheta_1$  can be re-expressed in terms of  $\vartheta_4$  when evaluated at  $\partial\Sigma_2$ ,

$$\vartheta_1\left(\pi\frac{\tau}{2} - \pi(x - y_{H,\beta}) \middle| \tau\right) = iq^{-1/4} e^{i\pi(x - y_{H,\beta})} \vartheta_4\left(\pi(y_{H,\beta} - x) \middle| \tau\right) \tag{3.21}$$

$$\vartheta_1\left(\pi\frac{\tau}{2} + \pi(x - x_{H,\alpha}) \middle| \tau\right) = iq^{-1/4} e^{-i\pi(x - x_{H,\alpha})} \vartheta_4\left(\pi(x - x_{H,\alpha}) \middle| \tau\right) \tag{3.22}$$

Both  $\vartheta_1$  and  $\vartheta_4$  are real when their argument is real and  $\tau$  is purely imaginary, as we can see from equations (3.4) and (3.5). Using the above expressions, we see that the phases  $\varphi$  of  $\mathcal{N}(x)$  and  $\mathcal{N}(\tau/2 + x)$  are respectively given by:

$$\varphi\left(\mathcal{N}(x)\right) = \pi\left(x(\phi - 2N' + M') - \sum_{\beta=1}^{M'} y_{A,\beta} + 2\sum_{\beta=1}^{N'} y_{H,\beta}\right) - M'\frac{\pi}{2} \tag{3.23}$$

$$\varphi\left(\mathcal{N}(x + \tau/2)\right) = \pi\left(x(\phi + 2N - M) - 2\sum_{\alpha=1}^N x_{H,\alpha} + \sum_{\alpha=1}^M x_{A,\alpha}\right) - M\frac{\pi}{2} \tag{3.24}$$

To cancel the  $x$ -dependence of these phases we need  $\phi = M - 2N = 2N' - M'$ . Upon eliminating  $\phi$ , we have,

$$M + M' = 2(N + N') \tag{3.25}$$

The earlier requirement that  $\phi + M + M'$  is an even integer then implies that  $2N + M'$  must be an even integer, so that both  $M$  and  $M'$  must be even integers. Finally, in order to have real  $\mathcal{N}(w)$  on the boundaries we need the  $x$ -independent parts of the phases to vanish as well, and we have,

$$4 \sum_{\beta=1}^{N'} y_{H,\beta} - 2 \sum_{\beta=1}^{M'} y_{A,\beta} \equiv 0 \pmod{2} \tag{3.26}$$

$$4 \sum_{\alpha=1}^N x_{H,\alpha} - 2 \sum_{\alpha=1}^M x_{A,\alpha} \equiv 0 \pmod{2} \tag{3.27}$$

If the above conditions are satisfied, then  $B + \bar{B}$  obeys Dirichlet boundary conditions. Since  $B + \bar{B}$  is harmonic, single-valued and with the same poles of  $A + \bar{A}$ , it must be possible to express it directly in terms of the basic harmonic function  $h_0$  as well,

$$B + \bar{B} = \sum_{\alpha=1}^M r_{B,\alpha} h_0(w - x_{A,\alpha}, \bar{w} - x_{A,\alpha}) + \sum_{\beta=1}^{M'} r'_{B,\beta} h_0(w' + y_{A,\beta}, \bar{w}' + y_{A,\beta}) \tag{3.28}$$

for some residues  $r_{B,\alpha}$  and  $r'_{B,\beta}$ ,  $\alpha = 1 \dots M$ ,  $\beta = 1 \dots M'$ . Note that these residues do not have necessarily the same sign.

- The harmonic function  $K$  has the same poles as  $A + \bar{A}$  and the residues are given by the condition **R1**. The expression is fixed to be:

$$K = \sum_{\alpha=1}^M \frac{r_{B,\alpha}^2}{r_{A,\alpha}} h_0(w - x_{A,\alpha}, \bar{w} - x_{A,\alpha}) + \sum_{\beta=1}^{M'} \frac{r'_{B,\beta}{}^2}{r_{A,\beta}} h_0(w' + y_{A,\beta}, \bar{w}' + y_{A,\beta}) \tag{3.29}$$

In this expression,  $K$  is manifestly positive everywhere on  $\Sigma$ . Note that the dual harmonic function  $\tilde{K}$  does not need to be single-valued. So far, our solution is dependent on a total number of parameters

$$\text{no. of parameters : } 2(N + N') + 2(M + M') = 6(N + N') \tag{3.30}$$

In this counting the modular parameter of the annulus  $\tau = it$ , the constant  $c$  and a constant in the definition of  $\tilde{K}$  are included.

### 3.4 Regularity conditions

At this stage, the harmonic functions satisfy the regularity conditions **R1**, **R2**, **R3** and **R4**. The last remaining condition is **R5**, namely that

$$K(A + \bar{A}) - (B + \bar{B})^2 > 0 \tag{3.31}$$

everywhere in  $\Sigma$ . We shall now show that the inequality follows from the form of the solution.

We begin by proving the  $\geq$  inequality. To do so, we introduce the following abbreviations,

$$\begin{aligned} h_\alpha &\equiv h_0(w - x_{A,\alpha}, \bar{w} - x_{A,\alpha}) & \alpha = 1, \dots, M \\ h'_\beta &\equiv h_0(w' + y_{A,\beta}, \bar{w}' + y_{A,\beta}) & \beta = 1, \dots, M' \end{aligned} \tag{3.32}$$

For  $w, w'$  in the interior of  $\Sigma$ , both quantities are strictly positive. The precise values taken by these functions will be immaterial in the proof below. To evaluate  $K(A + \bar{A}) - (B + \bar{B})^2$ , we use the expressions for  $(A + \bar{A})$ ,  $(B + \bar{B})$  and  $K$  from (3.15), (3.28) and (3.29) respectively. Since the residues  $r_{A,\alpha}$  and  $r'_{A,\beta}$  are strictly positive, we can define the following combinations,

$$\begin{aligned} v_\alpha &= (r_{A,\alpha} h_\alpha)^{\frac{1}{2}} & u_\alpha &= \frac{r_{B,\alpha} h_\alpha}{v_\alpha} \\ v'_\beta &= (r'_{A,\beta} h'_\beta)^{\frac{1}{2}} & u'_\beta &= \frac{r'_{B,\beta} h'_\beta}{v'_\beta} \end{aligned} \tag{3.33}$$

In terms of these variables we have

$$K(A + \bar{A}) - (B + \bar{B})^2 = \left( \sum_\alpha u_\alpha^2 + \sum_\beta u'_\beta{}^2 \right) \left( \sum_\alpha v_\alpha^2 + \sum_\beta v'_\beta{}^2 \right) - \left( \sum_\alpha u_\alpha v_\alpha + \sum_\beta u'_\beta v'_\beta \right)^2$$

The right hand side is automatically positive or zero in view of Schwartz's inequality, so that we have,

$$K(A + \bar{A}) - (B + \bar{B})^2 \geq 0 \tag{3.34}$$

Equality is obtained if and only if the  $M + M'$  dimensional vectors  $(u_\alpha, u'_\beta)$  and  $(v_\alpha, v'_\beta)$  are proportional to one another, so that  $u_\alpha = \lambda v_\alpha$  and  $u'_\beta = \lambda v'_\beta$  for some real number  $\lambda$ . This proportionality relation implies that  $r_{B,\alpha} h_\alpha = \lambda r_{A,\alpha} h_\alpha$  and  $r'_{B,\beta} h'_\beta = \lambda r'_{A,\beta} h'_\beta$ , as we can see from (3.33). We will rule out this possibility by showing that at least one of the residues  $r_{B,\alpha}, r'_{B,\beta}$  has negative sign. First, we note that  $\partial_w H$  cannot vanish on the boundary of  $\Sigma$ . This follows from the fact that  $H$  can be expanded close to any non-singular point  $x$  on  $\partial\Sigma_{1,2}$  as,

$$H = i\partial_w H(x)(w - x) + \frac{i}{2}\partial_w^2 H(x)(w - x)^2 + \dots + c.c. \tag{3.35}$$

If  $\partial_w H$  has a zero of order  $p$  for  $w = x$ , the above expansion can be rewritten as follows,

$$H = \text{const } \text{Im}(w - x)^{p+1} + \dots = \text{const } r^{p+1} \sin(p + 1)\phi + \dots \tag{3.36}$$

where we have introduced polar coordinates close to  $w = x$ . Because of the sine function, if  $p > 0$  then  $H$  has some zeros in the bulk of  $\Sigma$ . However, this is not possible because  $H$  is strictly positive in the interior of  $\Sigma$  by construction. Hence, the zeros of  $\partial_w H$  cannot be on  $\partial\Sigma_{1,2}$  and must be located in the interior of  $\Sigma$ . The condition **R4** forces  $B$  to have the same zeros as  $\partial_w H$ . Therefore,  $B$  must vanish somewhere in the interior of  $\Sigma$  and some of its residues must be negative. This allows us to exclude that  $r_{B,\alpha} h_\alpha = \lambda r_{A,\alpha} h_\alpha$  and  $r'_{B,\beta} h'_\beta = \lambda r'_{A,\beta} h'_\beta$  for all  $\alpha$  and  $\beta$ . Hence, our solutions satisfy **R5** without any additional condition.

### 3.5 Charges of the solutions

Our solutions display  $N + N'$  non-trivial three-spheres. In general, a three-sphere will correspond to a curve on  $\Sigma$  starting and ending on the boundary. We can construct a basis

of three-spheres by choosing  $N + N' - 1$  curves having support in the neighborhood of singularities of  $H$ , so that each curve starts on the boundary on one side of the pole and ends on the opposite side of the same boundary.

We choose a set of curves  $\mathcal{C}_i$ ,  $i = \{1, 2, \dots, N + N' + 1\}$  as follows (see figure 2 (a)). The curves  $\mathcal{C}_\alpha$ ,  $\{\alpha = 1, 2, \dots, N\}$  are surrounding the pole  $x_{H,\alpha}$  on  $\partial\Sigma_1$ , the curves  $\mathcal{C}_{N+\beta}$ ,  $\{\beta = 1, \dots, N'\}$  are surrounding the poles  $y_{H,\beta}$  on  $\partial\Sigma_2$ . The last curve  $\mathcal{C}_{N+N'+1}$  is chosen to start on the lower boundary and end on the upper boundary.

The flux of the three-form field on each of the spheres gives the total enclosed NS5 charge,

$$\begin{aligned} q_{NS5}(\mathcal{C}) &= \frac{1}{\text{Vol}(S^3)} \int_{\mathcal{C} \times S^2} e^{-\Phi} \text{Re}(G) \\ &= \frac{1}{\pi} \int_{\mathcal{C}} f_2^2 e^{-\Phi} \rho \text{Re}(g^{(2)})_z dw + \frac{1}{\pi} \int_{\mathcal{C}} f_2^2 e^{-\Phi} \rho \text{Re}(g^{(2)})_{\bar{z}} d\bar{w} \\ &= \frac{1}{\pi} \int_{\mathcal{C}} (\partial_w b^{(2)} dw + \partial_{\bar{w}} b^{(2)} d\bar{w}) \end{aligned} \tag{3.37}$$

Similarly, the D5 Maxwell charge is given by<sup>6</sup>

$$\begin{aligned} q_{D5}^M(\mathcal{C}) &= \frac{1}{\text{Vol}(S^3)} \int_{\mathcal{C} \times S^2} e^{\Phi} \text{Im}(G) \\ &= \frac{1}{\pi} \int_{\mathcal{C}} ((\partial_w c^{(2)} - \chi \partial_w b^{(2)}) dw + (\partial_{\bar{w}} c^{(2)} - \chi \partial_{\bar{w}} b^{(2)}) d\bar{w}) \end{aligned} \tag{3.38}$$

The D5 Page charge has a similar expression,

$$q_{D5}^P(\mathcal{C}) = \frac{1}{\text{Vol}(S^3)} \int_{\mathcal{C} \times S^2} (e^{\Phi} \text{Im}(G) - \chi e^{-\Phi} \text{Re}(G)) = \frac{1}{\pi} \int_{\mathcal{C}} (\partial_w c^{(2)} dw + \partial_{\bar{w}} c^{(2)} d\bar{w}) \tag{3.39}$$

The expressions for  $b^{(2)}$  and  $c^{(2)}$  were defined in (2.12), (2.13), while  $V_3$  is the volume of the unit three-sphere. The value of the five-brane charges (3.37) and (3.39) does not change upon deformation of the contours  $\mathcal{C}_i$  as long as one does not cross any of the singularities  $x_{H,\alpha}$  and  $y_{H,\beta}$ . Deforming the contours for the charges associated with the singularities one can show the following relations for the conservation of charges,

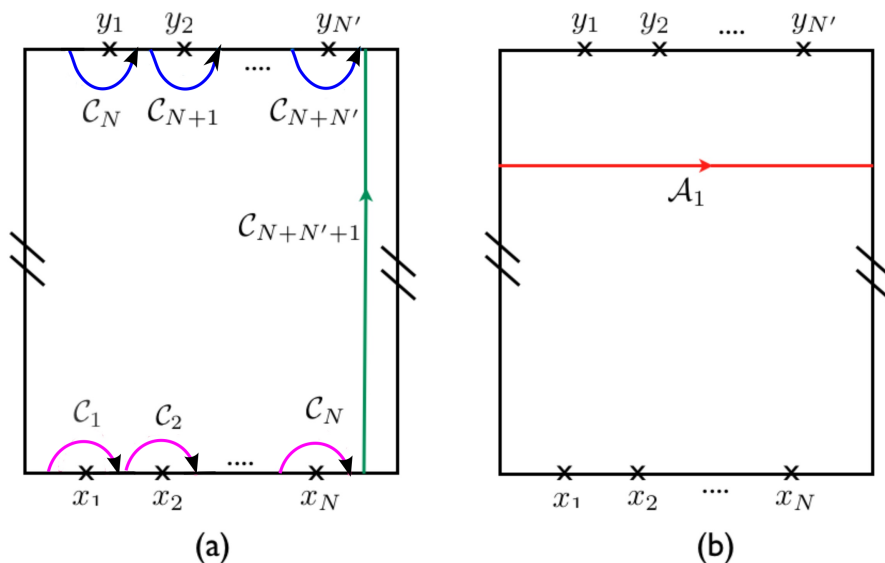
$$\sum_{i=1}^{N+N'} q_{D5}^P(\mathcal{C}_i) = 0, \quad \sum_{i=1}^{N+N'} q_{NS5}(\mathcal{C}_i) = 0 \tag{3.40}$$

Which shows that one of the charges associated with the singularities of  $H$  is linearly dependent.

The explicit expression for the charges associated with the asymptotic regions can be obtained by evaluating the integrals (3.37) for contours which are contracted towards the singularity. For concreteness we consider the singularities on the lower boundary. The

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<sup>6</sup>See [46] for a discussion on the different notions of charges in presence of Chern-Simons terms.



**Figure 2.** (a) Choice of curves  $\mathcal{C}_i$  which are associated with  $N + N'$  linearly independent five-brane charges. (b) Non-contractible cycle  $\mathcal{A}_1$  associated with three-brane charge.

relevant function can be expanded in as  $w \rightarrow x_{H,\alpha}$ ,

$$H \sim \frac{2yr_{H,\alpha}}{|w - x_{H,\alpha}|} \quad (3.41)$$

$$A \sim ia_\alpha^{(0)} + ia_\alpha^{(1)}(w - x_{H,\alpha}) \quad (3.42)$$

$$B \sim ib_\alpha^{(0)} + ib_\alpha^{(1)}(w - x_{H,\alpha}) \quad (3.43)$$

$$K \sim ik_\alpha^{(0)} + ik_\alpha^{(1)}(w - x_{H,\alpha}) + c.c. \quad (3.44)$$

The contribution to the integrals come from the functions  $\tilde{h}_1$  and  $h_2$  in (2.12), (2.13). We get the following expressions,

$$\tilde{h}_1 \sim -\frac{y r_{H,\alpha}}{b_\alpha^{(0)}|w - x_{H,\alpha}|^2} + \frac{1}{2} \frac{r_{H,\alpha} b_\alpha^{(1)}}{(b_\alpha^{(0)})^2} \log \frac{w - x_{H,\alpha}}{\bar{w} - x_{H,\alpha}} \quad (3.45)$$

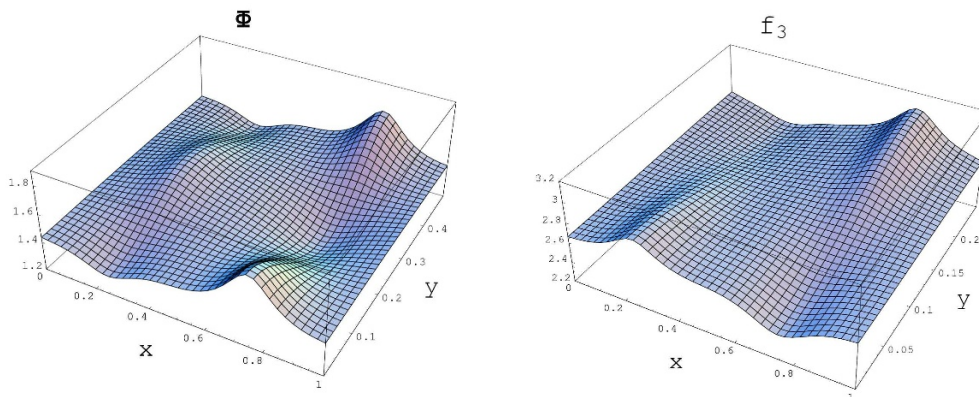
$$h_2 \sim -\frac{y r_{H,\alpha} a_\alpha^{(0)}}{b_\alpha^{(0)}|w - x_{H,\alpha}|^2} + \frac{r_{H,\alpha}}{2} \left( \frac{a_\alpha^{(0)} b_\alpha^{(1)}}{(b_\alpha^{(0)})^2} - \frac{a_\alpha^{(1)}}{b_\alpha^{(0)}} \right) \log \frac{w - x_{H,\alpha}}{\bar{w} - x_{H,\alpha}} \quad (3.46)$$

Since the endpoints of the curves are taken on the boundaries, the terms proportional to  $y$  give a zero contribution and we are left with the following expressions,

$$q_{NS5}(\mathcal{C}_\alpha) = \frac{r_{H,\alpha} b_\alpha^{(1)}}{(b_\alpha^{(0)})^2}, \quad q_{D5}^M(\mathcal{C}_\alpha) = \frac{r_{H,\alpha}}{b_\alpha^{(0)}} \left( \frac{(b_\alpha^{(1)})^2}{k_\alpha^{(0)}} - a_\alpha^{(1)} \right), \quad q_{D5}^P(\mathcal{C}_\alpha) = r_{H,\alpha} \left( \frac{a_\alpha^{(0)} b_\alpha^{(1)}}{(b_\alpha^{(0)})^2} - \frac{a_\alpha^{(1)}}{b_\alpha^{(0)}} \right) \quad (3.47)$$

with analogous equations for the poles on the upper boundary. The expressions for the





**Figure 3.** Dilaton and metric factor  $f_3$  for the example of section 3.6.

charges associated with  $C_{N+N'+1}$  are given by the integrals

$$q_{NS5}(C_{N+N'+1}) = \frac{1}{2i} \int_{C_{N+N'+1}} \left( \frac{1}{B} \partial_w H dw - \frac{1}{\bar{B}} \partial_{\bar{w}} H d\bar{w} \right) = \frac{1}{2i} \int_{C_{N+N'+1}} (\mathcal{N} dw - \bar{\mathcal{N}} d\bar{w})$$

$$q_{D5}^P(C_{N+N'+1}) = \frac{1}{2} \int_{C_{N+N'+1}} \left( \frac{A}{B} \partial_w H dw + \frac{\bar{A}}{\bar{B}} \partial_{\bar{w}} H d\bar{w} \right) = \frac{1}{2} \int_{C_{N+N'+1}} (A\mathcal{N} dw + \bar{A}\bar{\mathcal{N}} d\bar{w})$$

These expressions provide a physical interpretation for the  $(1,0)$ -form  $\mathcal{N}$  introduced in (3.20). Note that these charges are not associated with any asymptotic region.

The three-brane charge of the solution is given by the integral of the self-dual five form over  $K_3 \times \mathcal{C}$  where  $\mathcal{C}$  is a curve in  $\Sigma$ .

$$q_{D3}(\mathcal{C}) = \frac{1}{\text{Vol}(S^5)} \int_{K_3 \times \mathcal{C}} F_5 = \frac{3}{4\pi^3} \int_{\mathcal{C}} f_3^4 \rho (\tilde{h}_z dw + \tilde{h}_{\bar{z}} d\bar{w}) = \frac{3}{4\pi^3} \int_{\mathcal{C}} (\partial_w C_K dw + \partial_{\bar{w}} C_K d\bar{w}) \tag{3.48}$$

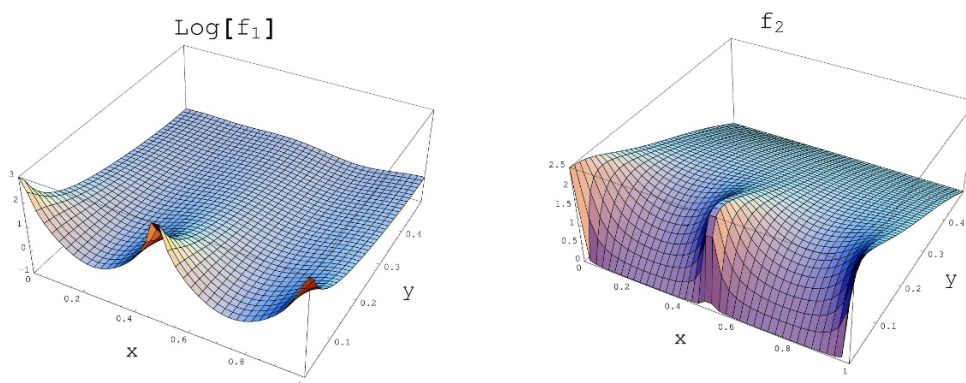
$C_K$  is defined in (2.14). Since the annulus has a non-contractible cycle  $\mathcal{A}_1$  (see figure 2 (b)), the three-brane charge integrated over  $K_3 \times \mathcal{A}_1$  can be nonzero due to the fact that  $\tilde{K}$  has nontrivial monodromy around  $\mathcal{A}_1$ .

$$q_{D3}(\mathcal{A}_1) = \frac{3c^2}{\pi^2 t} \left( \sum_{\alpha=1}^M \frac{r_{B,\alpha}^2}{r_{A,\alpha}} - \sum_{\beta=1}^{M'} \frac{r'_{B,\beta}}{r_{A,\beta}} \right) \tag{3.49}$$

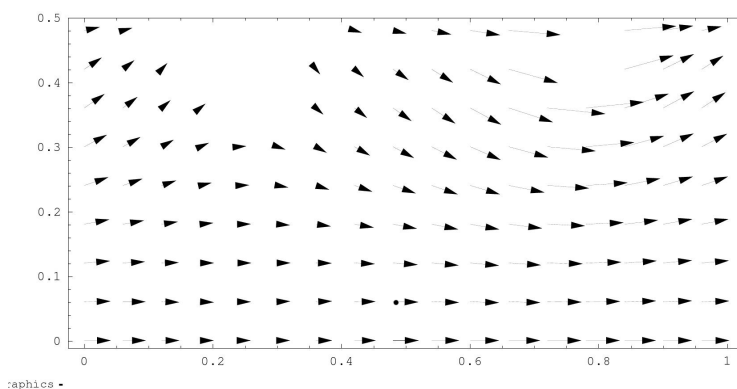
(3.13) and (3.29) were used to derive the above equation.

### 3.6 Examples

In this section we present the solutions with the smallest number of parameters. The conditions which constrain the minimal solutions are the following. First, equation (3.18), together with the positivity of the residues  $r_{A,\alpha}$  and  $r'_{B,\beta}$ , implies that  $M$  and  $M'$  both have to be nonzero. Second, the conditions (3.23) imposed by the fact that  $B$  has to be single-valued, imply that both  $M$  and  $M'$  must be even integers. We can see that the case  $N = 1, N' = 0$ , i.e. there is only one asymptotic region, is impossible as follows:



**Figure 4.** Metric factors  $f_1$  and  $f_2$  for the example of section 3.6.



**Figure 5.** Field lines of  $f_3^4 \rho \tilde{h}_a = \partial_a C_K$  for the example of section 3.6.

$M + M' = 2(N + N')$  implies that  $M + M' = 2$ . The first condition from above implies that  $M = 1, M' = 1$ . However this contradicts the second condition. Hence, in the simplest case, one has at least two asymptotic regions, i.e.  $N + N' = 2$ . It follows from (3.30) that  $M + M' = 4$  and from the second condition above we see that we have  $M = 2, M' = 2$ . There are two distinct cases since the boundaries of the annulus can be exchanged: in the *planar* case ( $N = 2, N' = 0$ ), the two asymptotic regions are on the same boundary and in the *non-planar* case ( $N = 1, N' = 1$ ), the two asymptotic regions are on different boundaries. We present plots of some of the metric functions for the planar case and an annulus with  $\tau = i$ . The poles of  $H$  are located on the lower boundary at  $x = 0$  and  $x = 1/2$  and have unit residues. The poles of  $A$  on the lower boundary are at  $x = 1/4, 3/4$  and have residues 1 and 2 respectively. The poles of  $A$  on the upper boundary are at  $y = 1/4, y = 3/4$  and have residues 2 and 1. The field lines of the combination  $f_3^4 \rho \tilde{h}_z$  are plotted in figure 5.

#### 4 Degeneration and the probe limit

Riemann surfaces with more than one boundary (or with handles) have moduli which correspond to deformations of the surface. It is interesting to consider the behavior of our solutions at the boundary of this moduli space, i.e. when the annulus degenerates. The degeneration we will consider in this section is given by the shrinking of one of the holes of the annulus to zero size. When the inner boundary shrinks to zero size, it is replaced by a puncture in the interior of the disk. Note that each point on the disk can be associated with a particular  $AdS_2 \times S^2$  slice of the geometry. In the parameterization of the annulus defined in (3.1) this limit corresponds to taking  $\tau \rightarrow i\infty$ . It follows from the formula for the D3-brane charge that in this limit  $q_{D3} \rightarrow 0$  in accord with the expectation that in the probe limit the D3 brane charge becomes negligible in comparison to the charges which support the  $AdS_3 \times S^3$  geometry. In the degeneration limit, the fundamental domain will be the upper half-plane with identification  $w \simeq w + 1$ . A useful expression in the expansion is given by

$$\frac{\vartheta'_1(\pi w|\tau)}{\vartheta_1(\pi w|\tau)} = \pi \cot \pi w + 4\pi \sum_{n=1}^{\infty} \frac{q^{2n}}{1 - q^{2n}} \sin(2\pi n w) \tag{4.1}$$

In this limit we have

$$\begin{aligned} \lim_{t \rightarrow \infty} h_0(w - x, \bar{w} - x) &= \pi i (\cot \pi(w - x) - \cot \pi(\bar{w} - x)) \\ \lim_{t \rightarrow \infty} h_0(w' + y, \bar{w}' + y) &= 0 \end{aligned} \tag{4.2}$$

The function  $h_0(w - x, \bar{w} - x)$  still obeys Dirichlet boundary conditions for real  $w$  and has a pole for  $w = x$ . The functions  $A$  and  $H$  assume the following forms in the degeneration limit,

$$H_D = \lim_{t \rightarrow \infty} H = \pi i \sum_{\alpha=1}^N r_{H,\alpha} \cot \pi(w - x_{H,\alpha}) + c.c. \tag{4.3}$$

$$A_D = \lim_{t \rightarrow \infty} A = \pi i \sum_{\alpha=1}^M r_{A,\alpha} \cot \pi(w - x_{A,\alpha}) \tag{4.4}$$

Using (3.28), the meromorphic function  $B$  can be written as,

$$B_D = \lim_{t \rightarrow \infty} B = -\frac{i\pi}{\mathcal{N}} \sum_{\alpha=1}^N \frac{r_{H,\alpha}}{\sin^2 \pi(w - x_{H,\alpha})} \tag{4.5}$$

$$\mathcal{N} = e^{i\pi(M-2N)} \frac{\prod_{\alpha=1}^M \sin \pi(w - x_{A,\alpha})}{\prod_{\alpha=1}^N \sin^2 \pi(w - x_{H,\alpha})} \tag{4.6}$$

It is interesting to note that the contributions to  $B$  from terms with  $h_0(w' + y, \bar{w}' + y)$  will not in general vanish because the residues  $r'_{B,\beta}$  depend on the modular parameter  $\tau$  through the definition (3.28) and may become infinite in the degeneration limit.

At this point, we intend to analyze our solution near the point to which the upper boundary  $\partial\Sigma_2$  has degenerated. Hence, we need to study our basic harmonic functions

when  $\text{Im}w \rightarrow \infty$ . Note that this limit is taken only after the degeneration limit  $\tau \rightarrow i\infty$ . It is easy to see that the harmonic functions  $H$  and  $A + \bar{A}$  become constants. Moreover, it is possible to show that  $\mathcal{N} = \pm 1$  using (3.26), (3.27) and the expansion (4.1). Since  $B$  is defined in terms of  $\partial_w H$ , we get that

$$\lim_{\text{Im}w \rightarrow \infty} B_D = 0 \tag{4.7}$$

Hence, the upper boundary has completely disappeared leaving only a zero of  $B$ . We can map the infinite cylinder to the half plane using the map

$$z = \tan(\pi w) \tag{4.8}$$

Which maps  $w = 0$  to  $z = 0$  and  $w = \frac{1}{2}$  to  $z = \infty$ . The basic harmonic function is expressed in the new coordinates as

$$\lim_{t \rightarrow \infty} h_0(w - x, \bar{w} - x) = i\pi \frac{1 + z\hat{x}}{z - \hat{x}} + c.c. = 2\pi \frac{\text{Im}z(1 + \hat{x}^2)}{|z - \hat{x}|^2} \tag{4.9}$$

Where  $\hat{x} = \tan x$ . In this coordinates, the meromorphic function  $B$  is given by

$$B_D = -\frac{i\pi}{\mathcal{N}}(z^2 + 1) \sum_{\alpha=1}^N r_{H,\alpha} \frac{1 + \hat{x}_{H,\alpha}}{(z - \hat{x}_{H,\alpha})^2} \tag{4.10}$$

$$\mathcal{N} = (1 - iz)^{2N-M} \frac{\prod_{\alpha=1}^M \frac{z - \hat{x}_{A,\alpha}}{\sqrt{\hat{x}_{A,\alpha}^2 + 1}}}{\prod_{\alpha=1}^N \frac{(z - \hat{x}_{H,\alpha})^2}{\hat{x}_{H,\alpha}^2 + 1}} \tag{4.11}$$

We can see that  $B_D$  has a simple zero in the interior of  $\Sigma$  for  $z = i$ . We also note that all the other relevant harmonic functions are strictly positive in the bulk by construction and will have a finite non-zero value at this point. If we expand in polar coordinates on the plane so that  $r = |z - i|$ , equation (2.10) gives the following expression for the two-dimensional metric,

$$\rho^2 dz d\bar{z} \sim \frac{\text{const}}{r^2} (dr^2 + r^2 d\phi^2) = \text{const} \left( \frac{dr^2}{r^2} + d\phi^2 \right) \tag{4.12}$$

Hence, there is an infinite geodesic distance from the boundary to the point at  $z = i$ . The behavior of  $\rho$  is the same encountered in [34] close to points which are zeros of  $B$  but not zeros of  $\partial_z H$ . These points correspond to  $AdS_2 \times S^2 \times S^1 \times R$  regions where the  $AdS_2$  and  $S^2$  spaces have the same radius.

## 5 General multi-boundary solutions

The example of the annulus already displayed the new salient properties of solutions associated with Riemann surfaces  $\Sigma$  with multiple boundary components, namely the presence of homology three spheres and the non-vanishing D3-brane charges due to the fact that

$\tilde{K}$  is not single-valued around the non-contractible cycle of the annulus. Furthermore the degeneration of the Riemann surface has an interesting physical interpretation. Note that many of the objects and techniques we employ in this section are also used in multi-loop string perturbation theory (see [39] for a review). It is an interesting question whether this is a coincidence or points towards a deeper connection of the two subjects.

### 5.1 Doubling the Riemann surface

In this section we shall generalize the construction to Riemann surfaces with an arbitrary number of boundaries but without handles. It is well-known from complex analysis [40] and open string perturbation theory that a Riemann surface  $\Sigma$  with boundaries can be related to a compact Riemann surface  $\bar{\Sigma}$  without boundary which is referred to as the double of  $\Sigma$ . Specifically, the double  $\bar{\Sigma}$  must possess an anti-conformal involution, denoted  $\mathcal{I}$ , which is such that  $\Sigma = \bar{\Sigma}/\mathcal{I}$ . For example, in the case of the annulus, the double is a rectangular torus, and the involution  $\mathcal{I}$  is just complex conjugation.

We generalize the construction as follows [39, 40]. Let  $\Sigma$  be a surface with  $N_B$  disjoint boundaries, and no handles, or equivalently let  $\Sigma$  have the topology of a sphere with  $N_B$  disjoint discs removed. The double  $\bar{\Sigma}$  is a compact Riemann surface of genus  $g = N_B - 1$  without boundary. The anti-conformal involution  $\mathcal{I}$  sets  $\Sigma = \bar{\Sigma}/\mathcal{I}$ . We choose a canonical homology basis for  $H_1(\Sigma)$  of cycles  $A_i$  and  $B_i$  with  $i = 1, \dots, g$  on  $\bar{\Sigma}$  which are adapted to the involution  $\mathcal{I}$ . Specifically, we identify  $N_B - 1$  boundary components with the A-cycles of  $\bar{\Sigma}$ , so that the involution acts on the cycles  $A_i, B_i$  through,

$$\mathcal{I}(A_i) = A_i \qquad \mathcal{I}(B_i) = -B_i \qquad (5.1)$$

Next, we choose a basis of holomorphic differentials  $\omega_i$  on  $\bar{\Sigma}$  satisfying,

$$\mathcal{I}^*(\omega_i(z)) = \bar{\omega}_i(\mathcal{I}(z)) \qquad (5.2)$$

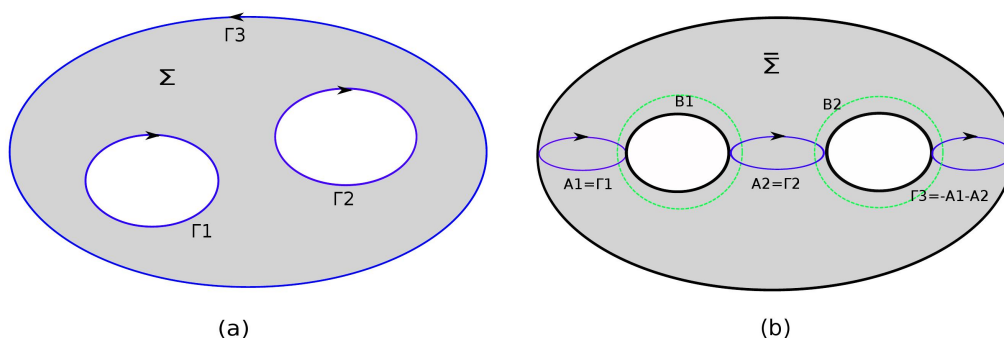
where  $\mathcal{I}^*$  denotes the pull-back of  $\mathcal{I}$  to differential forms. Finally, as usual, we choose the differentials  $\omega_i$  to have canonical normalization on the  $A_i$ -cycles, and define the period matrix  $\tau_{ij}$  in the standard manner,

$$\oint_{A_j} \omega_i = \delta_{ij} \qquad \oint_{B_j} \omega_i = \tau_{ij} \qquad (5.3)$$

Conditions (5.1) and (5.2) imply that the period matrix  $\tau_{ij}$  is purely imaginary. Note that if the surface  $\Sigma$  also had handles, the action of the involution  $\mathcal{I}$  on the cycles as well as on the period matrix, would be more complicated. We will postpone this case for later work.

All necessary functions can be constructed from the theta functions on the double  $\bar{\Sigma}$ . Higher genus theta functions are defined on a complex torus  $\mathbb{C}^g/\Lambda$  with period lattice  $\Lambda = (I, \tau)$ , where  $I$  is the  $g \times g$  identity matrix and  $\tau$  is a  $g \times g$  symmetric matrix such that  $\text{Im}\tau$  is positive. For any  $u \in \mathbb{C}^g/\Lambda$ , the genus  $g$  theta function is defined by,

$$\Theta(u, \tau) = \sum_{N \in \mathbb{Z}^g} \exp 2\pi i \left( \frac{1}{2} N^t \tau N + N^t u \right) \qquad (5.4)$$



**Figure 6.** (a) A Riemann surface  $\Sigma$  with three boundary components, (b) and its double  $\bar{\Sigma}$ . The boundaries are fixed points under the involution  $\mathcal{I}$

Similarly, given  $\epsilon, \epsilon' \in (\mathbb{Z}/2)^g$ , the theta functions with half-integer characteristic  $\alpha = [\epsilon, \epsilon']^t$ , are defined by,

$$\Theta_\alpha(u, \tau) = \sum_{N \in \mathbb{Z}^g} \exp 2\pi i \left( \frac{1}{2}(N + \epsilon)^t \tau (N + \epsilon) + (N + \epsilon)^t (u + \epsilon') \right) \quad (5.5)$$

In order to define theta functions on a genus  $g$  Riemann surface  $\Sigma$ , one needs to introduce the Abel-Jacobi map  $\varphi$ ,

$$\varphi : \Sigma \rightarrow J(\Sigma) = \mathbb{C}^g / \Lambda \quad (5.6)$$

which embeds the Riemann surface  $\Sigma$  into the Jacobian  $J(\Sigma) \equiv \mathbb{C}^g / \Lambda$ . To define the Abel-Jacobi map, one fixes an arbitrary point  $z_0 \in \Sigma$ ,

$$\varphi : z \rightarrow \left( \int_{z_0}^z \omega_1, \dots, \int_{z_0}^z \omega_g \right) \quad \text{mod } \Lambda \quad (5.7)$$

Thus,  $\varphi(z)$  is a  $g$ -dimensional complex vector of Abelian integrals. It is standard to use the notation  $\varphi(z - w) \equiv \varphi(z) - \varphi(w)$ , which defines a quantity that is independent of the choice of  $z_0$ . We are now ready to define the theta functions on  $\Sigma$ ,

$$\begin{aligned} \vartheta(z, \tau) &\equiv \Theta(\varphi(z), \tau) \\ \vartheta_\alpha(z, \tau) &\equiv \Theta_\alpha(\varphi(z), \tau) \end{aligned} \quad (5.8)$$

For each odd half-integer characteristic  $\alpha$ , there exists a unique holomorphic  $(1/2, 0)$  form  $h_\alpha(z)$ , which is given by,

$$h_\alpha(z)^2 = \sum_{j=1}^g \partial_j \Theta_\alpha(0, \tau) \omega_j(z) \quad (5.9)$$

The *prime form*  $E(x, y)$  is then defined as

$$E(z, w) = \frac{\vartheta_\alpha(z - w, \tau)}{h_\alpha(z)h_\alpha(w)} \quad (5.10)$$

The prime form is independent of the choice of odd characteristic  $\alpha$ ; it is antisymmetric  $E(z, w) = -E(w, z)$ ; and vanishes only on the diagonal  $w = z$ . In a local coordinate system near the zero at  $w = z$ , the prime form behaves as

$$\lim_{w \rightarrow z} \frac{E(z, w) \sqrt{dz dw}}{z - w} = 1 + \mathcal{O}((z - w)^2) \tag{5.11}$$

i.e.  $w = z$  is a simple zero. For fixed  $z$ ,  $E(z, w)$  defines a multi-valued holomorphic differential form of weight  $-1/2$ . The prime form is single-valued around  $A_j$ -cycles, and has the following monodromy around  $B_j$  cycles,

$$E(z + B_j, w) = -E(z, w) \exp \{-i\pi\tau_{jj} - 2\pi i\varphi_j(z - w)\} \tag{5.12}$$

A further differential form  $\sigma(z)$  will be needed to construct solutions. It is defined by,

$$\sigma(z) = \exp \left\{ - \sum_{j=1}^g \int_{A_j} \omega_j(y) \log E(y, z) \right\} \tag{5.13}$$

The differential  $\sigma(z)$  has weight  $(g/2, 0)$ , and has neither poles nor zeros on  $\bar{\Sigma}$ . It is single-valued around  $A_j$ -cycles, and has the following monodromy around  $B_j$  cycles:<sup>7</sup>

$$\sigma(z + B_j) = \pm \sigma(z) \exp 2\pi i \left( \frac{g-1}{2} \tau_{jj} - \Delta_j + (g-1)\varphi_j(z) \right) \tag{5.14}$$

where

$$\Delta_j = \frac{1}{2} - \frac{1}{2} \tau_{jj} + \sum_{k \neq j}^g \int_{A_k} \omega_k \varphi_j \tag{5.15}$$

is the vector of Riemann constants. Note that  $\Delta_k$  depends upon the reference point  $z_0$ .

## 5.2 Construction of the basic harmonic function

The key tool in the construction of solution for the case of the annulus was the existence of a basic positive harmonic function  $h_0$ , whose associated meromorphic part is holomorphic in the interior of  $\Sigma$ , and has a single simple pole on the boundary of  $\Sigma$ . All meromorphic and harmonic functions needed for the annulus were then constructed as linear combinations of this basic function.

This method may be generalized to the case of multiple boundary components, whose double is a surface  $\bar{\Sigma}$  of higher genus. Additional care will be needed for higher genus in taking proper account of the weight of differential forms (while for the annulus, all meromorphic differentials could be canonically identified with meromorphic functions). To construct a harmonic function on a genus  $g$  surface, we need a meromorphic function, i.e. a form of weight 0, which has simple poles only. Such objects arise in Riemann surface theory as Abelian integrals of the second kind, obtained as indefinite line integrals of

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<sup>7</sup>The overall sign in the monodromy is immaterial for our purpose, as only  $\sigma(z)^2$  appears in the expressions used in the following section.

Abelian differentials of the second kind, with double poles only. The Abelian differential of the second kind with a double pole at the point  $p$  (but holomorphic elsewhere), and vanishing  $A_i$ -cycles, is unique, and is given in terms of the prime form by,

$$\omega(z, p) = \partial_z \partial_p \ln E(z, p) \quad \int_{A_i} dz \omega(z, p) = 0 \quad (5.16)$$

It is single-valued on  $\bar{\Sigma}$ . The corresponding Abelian integral is given by,

$$\int_q^z dy \omega(y, p) = \partial_p \ln \left( \frac{E(z, p)}{E(q, p)} \right) \quad (5.17)$$

For our purposes, however, the requirement on  $\omega(z, p)$  of vanishing integral around  $A$ -cycles is too restrictive. Relaxing this condition allows us to add to  $\omega(z, p)$  any linear combination of the holomorphic Abelian differentials  $\omega_i$ , with  $i = 1, \dots, g$ . The corresponding Abelian integral is then given by,

$$Z(z|p, q) = \partial_p \ln \left( \frac{E(z, p)}{E(q, p)} \right) + \sum_{i=1}^g C_i(p, q) \int_q^z \omega_i \quad (5.18)$$

for an as yet undetermined  $C_i(p, q)$ . Clearly,  $Z(z|p, q)$  still has a simple pole at  $z = p$ , and a simple zero at  $z = q$ , but now has non-trivial monodromy around both  $A_i$ - and  $B_i$ -cycles,

$$\begin{aligned} Z(z + A_j|p, q) &= Z(z|p, q) + C_j(p, q) \\ Z(z + B_j|p, q) &= Z(z|p, q) + 2\pi i \omega_j(p) + \sum_{i=1}^g C_i(p, q) \tau_{ij} \end{aligned} \quad (5.19)$$

Some care is needed in specifying the derivatives with respect to the locations  $p$  of the poles. Indeed, the poles of the harmonic functions  $A \pm \bar{A}, B \pm \bar{B}, H$  and  $K$  need to be on the boundary of  $\Sigma$ , namely at points  $p$  which are invariant under the involution  $\mathcal{I}(p) = p$ . Therefore, the derivatives with respect to  $p$  which enter into the definition of  $Z(z|p, q)$  must be taken along the boundary only. In a suitable local coordinate system,  $(w, \bar{w})$ , the boundary  $\partial\Sigma$  may be described locally by  $\text{Im}(w) = 0$ . In this coordinate system, the differentiation in  $p$  along the boundary is defined as the derivative with respect to  $\text{Re}(w)$ . All derivatives and differentials in  $p$  will be understood in this manner.

### 5.2.1 Satisfying vanishing Dirichlet boundary conditions

We are now ready to define the basic harmonic functions  $h(z|p)$  on  $\Sigma$ ,

$$h(z|p) \equiv i \left( Z(z|p, q) - Z(\mathcal{I}(z)|p, q) \right) \quad (5.20)$$

where the points  $p, q$  are on the boundary of  $\Sigma$  and satisfy  $\mathcal{I}(p) = p, \mathcal{I}(q) = q$ . We have suppressed the dependence on the point  $q$  in  $h(z|p)$ , because we shall show shortly that, even though  $Z(z|p, q)$  depends upon  $q$ , this dependence cancels out of  $h(z|p)$ . By construction, the harmonic function  $h(z|p)$  is real, and has a single simple pole at  $z = p$ . It remains to determine the  $C_i(p, q)$  so that  $h(z|p)$  consistently obeys vanishing Dirichlet



boundary conditions on all  $g + 1 = N_B$  boundary components of  $\Sigma$ . If  $z, p, q$  are all on the same boundary component of  $\Sigma$ , it is manifest that  $h(z|p) = 0$ . We need to ensure that  $h(z|p) = 0$  continues to hold when  $z$  is on a boundary component different from that of  $p$ . Points on different boundary components may be mapped into one another by moving the points through  $\Sigma$  along sums of half- $B$ -periods.

Recall that  $\partial\Sigma$  has  $N_B = g + 1$  disconnected components  $\Gamma_i$ , and that we have identified  $\Gamma_i = A_i$  for  $i = 1, \dots, g$ . Concretely, let  $z, p, q \in \Gamma_k$ , and let  $z' \in \Gamma_{k'}$ . Comparing the values of  $h(z|p)$  on these different boundary components gives,

$$h(z'|p) - h(z|p) = i \left( \int_z^{z'} - \int_{\mathcal{I}(z)}^{\mathcal{I}(z')} \right) \left( dy \omega(y, q) + \sum_{i=1}^g C_i(p, q) \omega_i \right) \quad (5.21)$$

It suffices to consider nearest neighbor cycles, with  $k' = k + 1$ , for  $k = 1, \dots, g$ , all others being given by linear combinations of these. The difference of the integration contours entering (5.21) then precisely coincides with the cycle  $B_k$ , and the integrals may all be carried out to give,

$$h(z'|p) - h(z|p) = -2\pi\omega_k(p) + i \sum_{j=1}^g C_j(p, q) \tau_{jk} \quad (5.22)$$

The harmonic function  $h(z|p)$  will consistently obey vanishing Dirichlet boundary conditions on all boundary components provided  $h(z'|p) - h(z|p) = 0$  for any pair  $z, z'$  on the boundary of  $\Sigma$ . This gives  $g$  conditions, which determine  $C_i(p, q)$  uniquely,

$$C_i(p, q) = -2\pi i \sum_j (\tau^{-1})_{ij} \omega_j(p) \quad (5.23)$$

Note that  $C_i(p, q)$  is indeed independent of  $q$ , and is a well-defined holomorphic Abelian in  $p$ . Henceforth, we shall also suppress the  $q$ -dependence of  $C_i$  and simply refer to it as  $C_i(p)$ . Since  $\tau_{ij}$  is purely imaginary, the quantities  $C_i(p)$  are real. Since  $C_i(p)$  is real when  $p, q$  are on the boundary of  $\Sigma$ , it follows that all  $q$ -dependence cancels out of  $h(z|p)$ , as promised.

Although  $Z(z|p, q)$  has non-trivial monodromy  $C_k(p)$  around any cycle  $A_k$ , this monodromy is clearly real, and cancels out for  $h(z|p)$ , which is thus single-valued around every  $A_k$ -cycle. The function  $\tilde{h}(z|p)$  is the harmonic dual to  $h(z|p)$ , and is given by

$$\tilde{h}(z|p) = Z(z|p, q) + Z(\mathcal{I}(z)|p, q) \quad (5.24)$$

Its monodromy around an  $A_i$ -cycle is given by  $2C_i(p)$ , and generally is non-vanishing. The harmonic function  $\tilde{h}(z|p)$  is defined only up to an additive  $z$ -independent function, which we choose so as to cancel the  $q$ -dependence of  $\tilde{h}(z|p)$ , as the notation indeed indicates. Note that  $h(z|p)$  transforms as a one-form with respect to  $p$ . This concludes our construction of the basic harmonic functions  $h(z|p)$ , and  $\tilde{h}(z|p)$ .

### 5.3 Construction of the supergravity solution

The supergravity solutions are expressed in terms of the harmonic functions  $A \pm \bar{A}, B \pm \bar{B}, H$ , and  $K$ . In this subsection, we shall now determine these functions for a surface  $\Sigma$  with an arbitrary number  $N_B = g + 1$  of boundary components, but no handles.

**5.3.1 The harmonic function  $H$**

The function  $H$  must be single-valued and positive on  $\Sigma$ , and obey vanishing Dirichlet boundary conditions. We take  $H$  to have  $N_i$  poles  $p_{H,ij}$  on boundary component  $i$ , labeled by  $j = 1, \dots, N_i$ . The corresponding residues  $r_{H,ij}$  must be positive, but are otherwise left undetermined,

$$H = \sum_{i=1}^{N_B} \sum_{j=1}^{N_i} r_{H,ij} h(w|p_{H,ij}) \tag{5.25}$$

Note that each pole  $p_{H,ij}$  of  $H$  corresponds to an asymptotic  $AdS_3 \times S^3$  region. Hence, the full supergravity solution will have a total of  $\sum_{i=1}^{N_B} N_i$  asymptotic  $AdS_3 \times S^3$  regions.

**5.3.2 The harmonic functions  $A \pm \bar{A}$**

The harmonic function  $A + \bar{A}$  must be single-valued and positive in  $\Sigma$ , and obey vanishing Dirichlet boundary conditions. The conjugate function  $-i(A - \bar{A})$  must also be single-valued on  $\Sigma$ . As a result, these functions take on the following form,

$$\begin{aligned} (A + \bar{A}) &= \sum_{i=1}^{N_B} \sum_{j=1}^{M_i} r_{A,ij} h(w|p_{A,ij}) \\ -i(A - \bar{A}) &= \sum_{i=1}^{N_B} \sum_{j=1}^{M_i} r_{A,ij} \tilde{h}(w|p_{A,ij}) \end{aligned} \tag{5.26}$$

Single-valuedness of  $-i(A - \bar{A})$  around each  $A_k$ -cycle imposes a relation between the residues. Using (3.13), one obtains the monodromy of this function,

$$\sum_{i=1}^{N_B} \sum_{j=1}^{M_i} r_{A,ij} C_k(p_{A,ij}) \tag{5.27}$$

Hence,  $-i(A - \bar{A})$  will be single-valued provided the following relation is obeyed,

$$\sum_{i=1}^{N_B} \sum_{j=1}^{M_i} r_{A,ij} \omega_k(p_{A,ij}) = 0 \tag{5.28}$$

As stated before, the basic harmonic function  $h(z|p)$  transforms as a one-form in  $p$ . As a result, the residues  $r_{H,ij}$  and  $r_{A,ij}$  are  $-1$ -forms in  $p_{H,ij}$  and  $p_{A,ij}$  respectively.

**5.3.3 The harmonic functions  $B \pm \bar{B}$**

• The meromorphic function  $B$  must have the same poles as  $A$  (according to the regularity condition **R1**), and the same zeros as  $\partial_w H$  (according to **R4**). Moreover, both  $B + \bar{B}$  and  $-i(B - \bar{B})$  must be single-valued. Thus, these functions can be expressed as

$$\begin{aligned} (B + \bar{B}) &= \sum_{i=1}^{N_B} \sum_{j=1}^{M_i} r_{B,ij} h(w|p_{A,ij}) \\ -i(B - \bar{B}) &= \sum_{i=1}^{N_B} \sum_{j=1}^{M_i} r_{B,ij} \tilde{h}(w|p_{A,ij}) \end{aligned} \tag{5.29}$$

The functions  $B + \bar{B}$  and  $-i(B - \bar{B})$  are not subject, however, to any positivity condition, and are allowed to change sign inside  $\Sigma$ . We get the following expression for  $B$ :

$$B(w) = \frac{\partial_w H(w)}{\mathcal{N}(w)} \tag{5.30}$$

Here,  $\mathcal{N}(w)$  is a single-valued meromorphic form of weight  $(1, 0)$ , whose zeros and poles are determined by the requirement that  $B$  and  $\partial_w H$  have common zeros and  $B$  and  $A$  have common poles. Thus,  $\mathcal{N}$  must cancel the (double) poles of  $\partial_w H$ , and reproduce the poles of  $A$ . Before constructing  $\mathcal{N}$ , we shall first derive the final harmonic function  $K$ .

### 5.3.4 The harmonic function $K$

The harmonic function  $K$  has the same poles as  $A + \bar{A}$  and the residues are given by the condition **R1**. The expression is fixed to be:

$$K = \sum_{i=1}^{N_B} \sum_{j=1}^{M_i} \frac{r_{B,ij}^2}{r_{A,ij}} h(w|p_{A,ij}) \tag{5.31}$$

In this expression,  $K$  is manifestly positive everywhere on  $\Sigma$ . Note that the dual harmonic function  $\tilde{K}$  does not need to be single-valued.

### 5.3.5 Construction of $\mathcal{N}$

The meromorphic form  $\mathcal{N}(z)$  must satisfy the following properties:

1.  $\mathcal{N}(z)$  has the same (double) poles as  $\partial_w H$ , and has zeros where  $A$  has poles.
2.  $\mathcal{N}(z)$  is real on all boundaries.
3.  $\mathcal{N}(z)$  is a  $(1, 0)$ -form so that  $B$  transforms as a scalar.
4.  $\mathcal{N}(z)$  is single-valued around  $A$ -cycles.

To carry out the explicit construction of  $\mathcal{N}(w)$ , it is convenient to work with meromorphic differential one-forms for which all  $2g - 2$  zeros are at prescribed points on  $\Sigma$ . Generally, such one-forms will have monodromy around both  $A$ - and  $B$ -cycles, but it will suffice here to restrict to forms which are single-valued around  $A$ -cycles. These may be conveniently constructed out of the prime form  $E(z, w)$  and the holomorphic form  $\sigma(z)$  of weight  $g/2$  introduced in (5.13). Any holomorphic one-form  $\nu(z)$  with zeros at points  $p_\alpha$ , with  $\alpha = 1, \dots, 2g - 2$ , and vanishing  $A$ -periods may be expressed as follows,

$$\nu(z) = \sigma(z)^2 \prod_{\alpha=1}^{2g-2} E(z, p_\alpha) \tag{5.32}$$

The monodromy around any cycle  $A_k$  vanishes, while around a cycle  $B_k$  it is given by,

$$\nu(z + B_k) = \nu(z) \exp 2\pi i \left\{ \varphi_k(p_1 + \dots + p_{2g-2}) - 2\Delta_k \right\} \tag{5.33}$$

The condition for the vanishing of this monodromy precisely coincides with the requirement that  $p_1 + \dots + p_{2g-2}$  is the canonical divisor, as should have been expected.

We take the following Ansatz:

$$\mathcal{N}(z) = \exp(\pi i \theta^t \varphi(z)) \frac{\prod_{i=1}^{N_B} \prod_{j=1}^{M_i} E(z, p_{A,ij})}{\prod_{i=1}^{N_B} \prod_{j=1}^{N_i} E(z, p_{H,ij})^2} \sigma(z)^2 \tag{5.34}$$

The poles of  $A$  and  $H$ , respectively given by  $p_{A,ij}$  and  $p_{H,ij}$ , are labeled by an index  $i$  which refers to the boundary, and a label  $j$  which refers to the point on boundary  $i$ . The functions  $A$  and  $H$  have respectively  $M_j$  and  $N_j$  poles on boundary  $j$ . It is immediate to see that  $\mathcal{N}$  has the desired poles and zeros, and only those. Furthermore,  $\theta$  is a  $g$ -vector of constants. Recall that  $E(z, w)$  and  $\sigma(z)$  are, and that  $\mathcal{N}$  must be single-valued around  $A$ -cycles. This puts a strong restriction on the components of the vector  $\theta$ ,

$$\theta_k \equiv 0 \pmod{2} \tag{5.35}$$

Next,  $\mathcal{N}(z)$  must be a form of weight  $(1, 0)$  in  $z$ ; this requires the following relation between the number of poles of  $\partial_w H$  and the number of zeros of  $A$ ,

$$2g - 2 = \sum_{i=1}^{N_B} (M_i - 2N_i) \tag{5.36}$$

It remains to ensure that  $\mathcal{N}(z)$  is real on all boundary components.

### 5.3.6 Reality of $\mathcal{N}$ on all boundary components

We start by noting that  $E(z, w)$  is real when both points  $z, w$  are on the same boundary. Next, we will evaluate  $E(z, w)$  when  $z, w$  are on different boundary components. This may be done using the following identity:

$$\Theta \left[ \begin{matrix} \epsilon \\ \epsilon' \end{matrix} \right] (u + \tau \delta, \tau) = \exp \left[ -\pi i \left( \delta^t \tau \delta + 2\delta^t \epsilon' + 2\delta^t u \right) \right] \Theta \left[ \begin{matrix} \epsilon + \delta \\ \epsilon' \end{matrix} \right] (u, \tau) \tag{5.37}$$

with  $u \in \mathbb{C}^g / \Lambda$ , and  $\delta \in (\mathbf{Z}/2)^g$ . This identity is the analog of the half-period relation (3.21)–(3.22). The integral of a holomorphic differential along an half  $B$ -cycle may be expressed in the following from,

$$\int_{B_j^+} \omega_i = R_{ij} + \frac{1}{2} \tau_{ij} \tag{5.38}$$

where  $R$  is a real matrix and  $B_j^+$  is the upper half of the  $B_j$  cycle so that  $B_j = B_j^+ - \mathcal{I}(B_j^+)$ . We can then use (5.37) and (5.38) to find the multiplicative factor acquired by  $E(z, p_{H,jk})$  when  $z$  is taken on boundary component  $i$ , and  $p_{H,jk}$  is on boundary component  $j$ . We construct a curve  $C_{ij}$  going from the  $i$ -th boundary to the  $j$ -th boundary as,

$$C_{ij} = \sum_k c_{ij,k} B_k^+ = c_{ij}^t B^+ \tag{5.39}$$

Here we have introduced a tensor  $c_{ij,k}$ , with  $i, j = 1 \dots N_B = g + 1$  and  $k = 1 \dots g$ .  $C_{ij}$  is constructed as a linear combination of half B-cycles and  $c_{ij,k}$  are the expansion coefficients. More explicitly, the  $c_{ij,k}$  are given by,

$$c_{ij,k} = \begin{cases} 1 & i \leq k < j \\ -1 & j \leq k < i \\ 0 & \text{otherwise} \end{cases} \quad (5.40)$$

It is easy to see that  $C_{i+1 j} = C_{ij} - B_i^+$ . Hence, the tensor  $c$  must obey the identity,

$$c_{i+1 j,k} = c_{ij,k} - \delta_{ik} \quad (5.41)$$

where  $\delta_{ij}$  is the Kronecker delta. With this notation we have that

$$E(x, p_{H,jk}) = \exp \left[ \pi i \left( \frac{c_{ij}^t \tau c_{ij}}{4} + c_{ij}^t \epsilon' + c_{ij}^t \varphi(x - p_{H,jk}) \right) \right] \frac{\Theta \left[ \begin{matrix} \epsilon - \frac{c_{ij}}{2} \\ \epsilon' \end{matrix} \right] \left( \int_{p'_{H,jk}}^x \omega - R c_{ij}, \tau \right)}{h \left[ \begin{matrix} \epsilon \\ \epsilon' \end{matrix} \right] (x) h \left[ \begin{matrix} \epsilon \\ \epsilon' \end{matrix} \right] (p_{H,jk})} \quad (5.42)$$

where  $p_{H,jk}$  and  $p'_{H,jk}$  are the endpoints of  $C_{ij}$  so that  $p'_{H,jk}$  is on the same boundary of  $x$ . The argument of the theta function is manifestly real since it can be evaluated using a path which stays on the  $i$ -th boundary. From the definition (5.5) we see that the theta function is real as long as the argument is real. Moreover, the one-form  $h_\alpha(x)^2$  is real when  $x$  is on one of the boundaries. The phase of  $h_\alpha(x)$  then only depends on the characteristic  $\alpha = [\epsilon, \epsilon']^t$  and must have the effect to cancel any dependence from  $\alpha$  in the exponential of (5.42). Therefore, the phase of  $E(x, p_{H,jk})$  is given by the exponential

$$\exp \{ \pi i c_{ij}^t \varphi(x - p_{H,jk}) \} \quad (5.43)$$

A similar analysis can be repeated for  $\sigma(x)^2$ . Its phase is given by the exponential

$$\exp \left\{ 2\pi i \sum_{j=1}^g \sum_{k=1}^g c_{ij,k} \int_{A_j} \omega_j(y) \varphi_k(y - x) \right\} \quad (5.44)$$

If  $x$  is on the  $i$ -th boundary, we get that the contribution to the phase of  $\mathcal{N}(x)$  coming from the prime forms is given by the following expression,

$$- 2\pi \sum_j^{N_B} \sum_k^{N_j} c_{ij}^t \text{Re} \varphi(x - p_{H,jk}) + \pi \sum_j^{N_B} \sum_k^{M_j} c_{ij}^t \text{Re} \varphi(x - p_{A,jk}) + \pi \theta^t \text{Re} \varphi(x) \quad (5.45)$$

while the contribution coming from  $\sigma(x)^2$  is the following,

$$2\pi \sum_{j=1}^g \sum_{k \neq j}^g c_{ij,k} \text{Re} \left( \int_{A_j} \varphi_k(y) \omega_j(y) - \varphi_k(x) \right) \quad (5.46)$$

In conclusion, to have a real  $\mathcal{N}(x)$  on all boundaries, the position-dependent part of the phase must be identically zero,

$$2 \left( \sum_{j=1}^{N_B} (M_j - 2N_j) c_{ij}^t + \theta^t \right) \text{Re}\varphi(x) - 4 \sum_{j=1}^g \sum_{k \neq j}^g c_{ij,k} \text{Re}\varphi_k(x) = 0 \pmod{2} \quad (5.47)$$

Moreover, the constant part of the phase must add up to an integer multiple of  $\pi$ ,

$$4 \sum_{j=1}^{N_B} \sum_{k=1}^{N_j} c_{ij}^t \text{Re}\varphi(p_{H,jk}) - 2 \sum_{j=1}^{N_B} \sum_{k=1}^{M_j} c_{ij}^t \text{Re}\varphi(p_{A,jk}) + 4 \sum_{j=1}^g \sum_{k \neq j}^g c_{ij,k} \text{Re} \int_{A_j} \varphi_k \omega_j = 0 \pmod{2} \quad (5.48)$$

Furthermore, we note that if  $j \neq 1$  then,

$$c_{1j,k} = \sum_{l=1}^{j-1} \delta_{lk} \quad (5.49)$$

Evaluating equation (5.47) for  $i = 1$  and using the identity (5.49), as well as the fact that the  $\varphi_k(x)$  are linearly independent, one can solve for  $\theta_k$ ,

$$\theta_k = \sum_{j=k+1}^{N_B} (2N_j - M_j) + 2g - 2k \quad (5.50)$$

We note that (5.35) can be satisfied only if all the  $M_j$  are even. Moreover, we subtract equation (5.47) evaluated for  $i$  and  $i + 1$  and use the identity (5.41) to obtain

$$\sum_j^{N_B} M_j = 2 \sum_j^{N_B} N_j + 2g - 2 \quad (5.51)$$

which is exactly (5.36). The identity (5.49) allows us to rewrite the first condition from (5.48) as follows,

$$\sum_{j=2}^{N_B} \sum_{l=1}^{j-1} \left\{ 4 \sum_{k=1}^{N_j} \text{Re}\varphi_l(p_{H,jk}) - 2 \sum_{k=1}^{M_j} \text{Re}\varphi_l(p_{A,jk}) \right\} + 4 \sum_{j=2}^g \sum_{l=1}^{j-1} \text{Re} \int_{A_j} \varphi_l \omega_j = 0 \pmod{2} \quad (5.52)$$

We can also use (5.41) to reduce the remaining  $g$  conditions from (5.48) to

$$4 \sum_{j=1}^{N_B} \sum_{k=1}^{N_j} \text{Re}\varphi(p_{H,jk}) - 2 \sum_{j=1}^{N_B} \sum_{k=1}^{M_j} \text{Re}\varphi(p_{A,jk}) + 4\text{Re}\Delta = 0 \pmod{2} \quad (5.53)$$

where  $\Delta$  is the vector of Riemann constants. These conditions reduce to (3.25), (3.26) and (3.27) in case of the annulus.

## 5.4 Regularity and properties of the solution

The solutions introduced in this sections obey to the regularity conditions R1-R4 by construction. If we re-define,

$$h_\alpha = h(w|p_{A,ij}), \quad \alpha = 1, 2, \dots, \sum_j M_j \quad (5.54)$$

it is possible to repeat the argument of section 3.4 to reduce the regularity inequality R5 to Schwartz's inequality. As seen before, the inequality will hold strictly provided that  $A$  and  $B$  are not proportional, but  $B$  cannot be proportional to  $A$  because it admits zeros in the interior of  $\Sigma$  while  $A$  is strictly positive in  $\Sigma$  by construction. Counting the number of independent parameters of the solution of genus  $g$  is instructive. The harmonic function  $H$  contains  $2 \sum_i N_j$  parameters. The harmonic function  $A + \bar{A}$  contains  $2 \sum_j M_j + 1$  parameters. Using (5.36) to replace  $\sum_j M_j$  gives  $6 \sum_i N_j + 4g - 3$  parameters which are associated with the harmonic functions. The condition that  $A - \bar{A}$  is single-valued around the  $g$  A-cycles (5.28), imposes  $g$  constraints. Furthermore, the reality conditions on  $\mathcal{N}$  (5.53) impose an additional  $g + 1$  conditions. The constant  $c$  and the real additive constant in  $\tilde{K}$  provide two additional parameters. Lastly, there are  $g(g + 1)/2$  real moduli of the Riemann surface. Putting this together we get

$$\text{no. of parameters} : = 6 \sum_i N_j + \frac{g(g + 1)}{2} + 2g - 2 \quad (5.55)$$

We can naturally attribute six parameters to each asymptotic region: F1, D1, NS5 and D5 brane charges and the value of two massless scalars in the asymptotic region. The other parameters are related to the moduli of the Riemann surface and the presence of non-contractible cycles (supporting D3-brane charge) and homological three-spheres (supporting five-brane charge). Note that the number of parameters grows quadratically with the genus of the Riemann surface leading to the possibility of a *bubbling Janus* solution with many boundaries.

## 6 Discussion

In this paper we expanded the class of half-BPS solutions of type IIB string theory that are locally asymptotic to  $AdS_3 \times S^3 \times M_4$  and were constructed in an earlier paper [34].

We constructed solutions in which the Riemann surface  $\Sigma$  has an arbitrary number of boundaries, and studied the simplest case of the annulus in detail. In this case, the double surface is a (square) torus, all harmonic functions can be constructed in terms of Jacobi theta functions and the regularity conditions can be solved explicitly. The requirement that the meromorphic function  $B$  obeys Dirichlet boundary conditions on both boundaries imposes additional constraints on the parameters of the solution. The main new physical feature of the solutions is the presence of non-vanishing three-brane charge. This fact is related to the existence of a non-contractible cycle on the annulus. Moreover, there is an additional three-sphere in the geometry which is not associated with any asymptotic region.

We studied the degeneration limit where the inner boundary shrinks to zero size and disappears leaving the disk with a puncture. The geometry near this puncture is given by an  $AdS_2 \times S^2 \times S^1 \times R$  infinite throat. Note that we did not scale any of the other parameters of the solution in the degeneration limit. The existence of other scaling limits with different asymptotics is an interesting open question.

The solution for the annulus was generalized to Riemann surfaces with an arbitrary number of holes. The construction involves well known functions on higher genus Riemann surfaces, such as theta functions and prime forms. We constructed the harmonic functions and solved the constraints imposed by reality and regularity. In the multi-boundary case, there are many non-contractible cycles coming from the  $A$ -cycles of the double Riemann surface, and each of them can be associated with a non-vanishing D3-brane charge. There are many interesting features of our solutions and possible extensions that deserve further study:

- The supergravity solutions we have found are dual to defect and interface theories in two-dimensional conformal field theories. Even the simplest case in which  $\Sigma$  can be mapped into the disk or upper half-plane has interesting solutions. A possible direction for further research is to investigate the holographic duals for the solutions we have found. In particular, it would be interesting to find out whether the fact that  $H$  can have poles (corresponding to asymptotic AdS regions) on different boundaries have an interpretation in the dual CFT.
- Note that the expression for axion (2.6) in terms of the harmonic functions

$$\chi = \frac{i}{2} \left( (A - \bar{A}) + \frac{\bar{B}^2 - B^2}{K} \right) \tag{6.1}$$

is very similar to the formula for  $C_K$  (2.14) with  $K$  replaced by  $A + \bar{A}$ . Hence, it seems possible to introduce seven-brane charge by dropping the requirement that the harmonic function  $i(A - \bar{A})$  is single-valued. Note that, in this case, the three-form anti-symmetric tensor fields also have non-trivial monodromies. For example, if one chooses the parameters of the harmonic function  $A$  on the annulus such that  $\chi \rightarrow \chi + 1$  as  $w \rightarrow w + 1$ , then one finds that  $F_3 \rightarrow F_3 + H_3$ . Where  $F_3$  and  $H_3$  are the R-R and NS-NS three-form anti-symmetric tensor fields respectively. These are exactly the monodromies one would expect for a single D7-brane [48, 49].

- It is interesting that the machinery of higher loop (open and closed) string perturbation theory was very useful in constructing the holographic duals of interfaces and defects in two-dimensional conformal field theories. This might be a coincidence dictated by the fact that in both cases Riemann surfaces are a basic ingredient. However, it is not inconceivable that there is a deeper connection, possibly indicating a relation between holographic defects and open strings in the effective string description of the D1/D5 system [47]
- In this paper we considered Riemann surfaces without handles. In principle, the methods employed in section 5 can be used also for Riemann surfaces  $\Sigma$  with  $n$



boundaries and  $h$  handles, where the double  $\bar{\Sigma}$  will be a compact Riemann surface of genus  $n - 1 + 2h$ . In this case the period matrix is not purely imaginary, and there are additional constraints from the requirement that our harmonic functions are single-valued. We leave the investigation of this case, and the very interesting question of its physical significance, for future work.

- In section 4 we discussed the degeneration of the annulus where the inner boundary shrinks to zero size and we found that in this limit an extra  $AdS_2 \times S^2 \times S^1 \times R$  asymptotic region appears. We expect that the same interpretation will hold for surfaces with many boundaries where a boundary closes off. It is well known from string perturbation theory (see e.g. [39]) that higher genus Riemann surfaces have other degeneration limits- when an open string loop becomes infinitesimally thin, for example- which may have interesting physical interpretations as well.

We plan to return to some of these questions in the future.

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