Normalization of D instanton amplitudes in two dimensional type 0B string theory

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ABSTRACT: We compute the normalization of the D-instanton amplitudes in type 0B string theory in two dimensions and find perfect agreement with the dual matrix model result.

Contents

1 Introduction

Two dimensional type 0B string theory, whose world-sheet theory consists of a free scalar associated with time direction, its superpartner fermion, a super-Liouville theory with central charge $27/2$ and the usual b,c, β, γ ghost system, is expected to be dual to a matrix theory — theory of non-relativistic free fermions moving under the influence of an inverted harmonic oscillator potential, with fermi level reaching the same height on both sides of the potential [1, 2]. Recently, [3] studied D-instanton contribution to the amplitudes in this type 0B string theory, and found perfect agreement with the results in the dual matrix theory except for one aspect. The overall normalization of the D-instanton induced amplitudes in string theory is controlled by the exponential of the annulus 0-point function, with the boundaries of the annulus lying on the D-instanton. However the annulus amplitude suffers from certain infrared divergences in the open string channel that could not be resolved using the world-sheet formalism. Ref. [3] dealt with this issue by including an unknown constant $\mathcal{N}_D$ in the overall normalization of the amplitudes. $\mathcal{N}_D$ was then determined by comparing the string theory results to the results in the free fermion description of the theory. Once this constant was fixed, there was perfect agreement between the various quantities calculated using the two descriptions.

In recent years string field theory has proved to be a useful tool in dealing with infrared divergences in the open string theory on the D-instanton, including in the annulus zero point function — see e.g. [4–11]. These papers computed similar normalization factors that are left undetermined in the world-sheet description [12–14]. In this paper we use the same approach and get a finite result for the constant $\mathcal{N}_D$. We find perfect agreement between the results obtained this way and the prediction for $\mathcal{N}_D$ obtained by comparison with the results in the dual matrix model.

2 Computation of the normalization constant

We now describe the details of the computation. Open strings living on the D-instanton of type 0B string theory come from only the NS sector. Our starting point is the expression

for the exponential of the annulus partition function on the D-instanton, obtained by taking the $R \to \infty$ limit of eq. (3.14) of [3]:

$$\exp\left[-\int_0^\infty \frac{dt}{2t}\right] . \tag{2.1}$$

This has divergence from the $t \to \infty$ limit. As will be discussed shortly, these can be attributed to propagation of open string zero modes in the Siegel gauge, and can be dealt with by standard method. This also has divergence from the $t \to 0$ limit associated with the closed string channel. This is a physical infrared divergence that indicates that the D-instanton induced processes induce non-trivial changes in the vacuum. Indeed in the language of dual matrix model, these correspond to the transmission of a single fermion or a hole across the inverted harmonic oscillator potential, so that a closed string impinged from one side, represented by a fermion hole pair, does not get reflected as an usual closed string [15] . While this issue may be resolved by expanding the allowed set of asymptotic states, we shall avoid this problem by working with instanton anti-instanton induced amplitudes, for which there is no net transfer of fermion number across the potential barrier and hence the final state can be represented as a collection of usual closed strings. In string theory the normalization associated with such amplitudes, with the instanton and the anti-instanton separated by a distance Δx, is given by [3]:

$$Z = \exp\left[\int_0^\infty \frac{dt}{2t}\left\{-2 + 2\,e^{2\pi t\left(\frac{1}{2} - \frac{1}{2}\left(\frac{\Delta x}{2\pi}\right)^2\right)}\right\}\right] . \tag{2.2}$$

Here the -2 term inside the curly bracket is the contribution from the open strings with both ends on the instanton or both ends on the anti-instanton while the second term is the contribution from open strings with one end on the instanton and the other end on the anti-instanton. The divergence from the $t \to 0$ end has now disappeared as expected, but the divergence from the $t \to \infty$ end remains. Ref. [3] introduced an arbitrary constant $\mathcal{N}_D$ to encode the effect of these divergences and expressed the result as[1]

$$Z = \int dx_1 dx_2 \frac{\mathcal{N}_D^2}{(\Delta x)^2 - (2\pi)^2} , \tag{2.3}$$

where x_1, x_2 are the positions of the instanton and the anti-instanton. Our goal will be to reinterpret these divergences in the language of string field theory and use insights from string field theory to get a finite result for (2.2) and hence determine $\mathcal{N}_D$.

The results (2.2), (2.3) were derived in the $\alpha' = 2$ unit [3]. We shall use some results of [5] that used $\alpha' = 1$ unit. For this it will be useful to convert (2.2), (2.3) into $\alpha' = 1$ unit. This is done by noting that x and α' occur in the combination $x/\sqrt{\alpha'}$. Hence to convert from $\alpha' = 2$ unit to $\alpha' = 1$ unit we need to replace x by $\sqrt{2}x$. This replaces (2.2), (2.3) by

$$Z = \exp\left[\int_0^\infty \frac{dt}{2t}\left\{-2 + 2\,e^{2\pi t\left(\frac{1}{2} - \left(\frac{\Delta x}{2\pi}\right)^2\right)}\right\}\right] , \tag{2.4}$$

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[1]The integrations over x_1, x_2 are supposed to be done after including the contributions from other world-sheet components which also depend on x_1, x_2.

and

$$Z = \int dx_1 dx_2 \frac{\mathcal{N}_D^2}{(\Delta x)^2 - 2\pi^2}\,, \tag{2.5}$$

respectively.

The general strategy for evaluating (2.4) will be the same as those used e.g. in [5, 9, 11]:

1. We first use the identities:

$$\int_0^\infty \frac{dt}{2t}\left(e^{-2\pi h_1 t} - e^{-2\pi h_2 t}\right) = \ln\sqrt{\frac{h_2}{h_1}}\,, \tag{2.6}$$

   and

$$\int \frac{d\phi}{\sqrt{2\pi}}e^{-\frac{1}{2}h\,\phi^2} = \frac{1}{\sqrt{h}},\qquad \int dpdqe^{-hpq} = h\,, \tag{2.7}$$

   for grassmann even variable ϕ and grassmann odd variables p, q, to express (2.4) as path integral over open string fields in the Siegel gauge. The divergence in the $t \to \infty$ limit comes from states with $h = 0$.

2. For open strings with both ends on the D-instanton, one $h = 0$ state comes from the translation mode of the D-instanton and gives a contribution of 1 to the coefficient of $dt/(2t)$ term in the integrand. We also have a pair of grassmann odd zero modes from the ghost sector that contribute -2 to the coefficient of $dt/(2t)$ in the integrand, producing a net factor of $-\int dt/(2t)$ as in (2.1). These grassmann odd modes are the Faddeev-Popov ghost fields associated with Siegel gauge fixing and the vanishing of h indicates that the Siegel gauge choice breaks down for open strings with both ends on the D-instanton (or both ends on the anti-D-instanton).

3. We remedy this by replacing the Siegel gauge fixed path integral by the path integral over the classical open string fields (of ghost number 1) weighted by the exponential of the gauge invariant action, and dividing the result by the volume of the gauge group. The zero modes associated with the translation of the instanton (anti-instanton) are kept unintegrated till the end.

We shall now elaborate on the zero mode integration in a bit more detail. For open strings with both ends on the D-instanton the analysis is the same as for the D-instanton of type IIB string theory and we can borrow the results of that paper keeping in mind that we have now similar contribution also from open strings with both ends on the anti-instanton. We expand the classical open string field with both ends on the instanton as

$$|\psi\rangle = i\phi\beta_{-\frac{1}{2}}\,c_0\,c_1\,|-1\rangle + \xi\,c_1\,d_{-\frac{1}{2}}\,|-1\rangle + \cdots\,, \tag{2.8}$$

where $|-1\rangle$ denote the -1 picture vacuum and $d_{-1/2}$ is the fermionic oscillator associated with the superpartner of the Euclidean time direction. $\cdots$ denote linear combination of states with higher L_0 eigenvalues. There are a similar set of modes of the open strings with both ends on the anti-D-instanton. We shall denote then by $\bar{\phi}$ and $\bar{\xi}$ respectively.

If we consider the open string gauge transformation parameter $|\theta\rangle = i\theta\beta_{-1/2}c_1|-1\rangle$, then under the linearized gauge transformation the string field changes by $Q_B|\theta\rangle$ where

Q_B is the BRST charge. If the state $\beta_{-1/2}c_1| - 1\rangle$ had a non-zero L_0 eigenvalue h then $Q_B|\theta\rangle$ would have a component $ih\theta\beta_{-\frac{1}{2}} c_0 c_1 |-1\rangle$ due to the $c_0 L_0$ term in Q_B. Comparison with (2.8) shows that this could be interpreted as a gauge transformation of ϕ by $\phi \to \phi + h\theta$. We could then fix the Siegel gauge by choosing $\phi = 0$, producing a Jacobian factor of h. This in turn could be represented as the integral over Fadeev-Popov ghost field p and q with action hpq. In the language of string field theory, these ghost fields are the coefficients of the Siegel gauge states $i\gamma_{-\frac{1}{2}} c_1 |-1\rangle$ and $i\beta_{-\frac{1}{2}} c_1 |-1\rangle$ in the expansion of the string field.

In actual practice we have $h = 0$ and this is the origin of the pair of ghost zero modes in the Siegel gauge. This shows that Siegel gauge is not a good choice of gauge and we must go back to the original gauge invariant form of the path integral where we integrate over ϕ and divide by the volume of the gauge group generated by θ. Since the details are identical to those in [5], we shall only quote the result. The analysis of [5] shows that the gauge invariant form of the path integral over the zero modes takes the form

$$\frac{1}{\int d\theta} \int \frac{d\xi}{\sqrt{2\pi}} d\phi \, e^{-\phi^2/4} \frac{1}{\int d\bar{\theta}} \int \frac{d\bar{\xi}}{\sqrt{2\pi}} d\bar{\phi} \, e^{-\bar{\phi}^2/4}, \tag{2.9}$$

where $\bar{\theta}$ is a parameter similar to θ for the anti-D-instanton. Note that the integration measures for $\phi, \bar{\phi}$ are not accompanied by the $1/\sqrt{2\pi}$ factors since these are out of Siegel gauge modes and were not present in the original path integral representation of (2.4) via (2.6), (2.7). Instead, the integration measure over $\phi, \bar{\phi}$ is fixed by demanding that if we had gauge fixed (2.10) using Faddeev-Popov procedure, we would have gotten the integration over the ghost fields with the same integration measure that emerges from the application of (2.6), (2.7) to (2.4). The absence of $1/\sqrt{2\pi}$ factor in the $\phi, \bar{\phi}$ integral can be traced to the absence of such factors in the second equation in (2.7) [5].

We now turn to the rest of the open string modes living on the instanton-anti-instanton system. Unlike in [5], we do not have any R-sector states in the present system. We however have a pair of 'tachyons'[2] with L_0 eigenvalue $-\frac{1}{2} + \left(\frac{\Delta x}{2\pi}\right)^2$ from open strings stretched between the instanton and the anti-instanton, producing the second term inside the curly bracket in (2.4). We shall denote them by $(\psi_1 \pm i\psi_2)/\sqrt{2}$. In this sector there are no subtleties with gauge fixing and we can continue to use Siegel gauge. Besides the states mentioned above, there are infinite towers of higher L_0 states in all sectors, but we see from (2.4) that their contributions cancel between grassmann even and odd modes and so we shall ignore them. Combining (2.9) with the integration over the tachyonic modes on the open string sector connecting the instanton to the anti-instanton, we get

$$Z = \frac{1}{\int d\theta} \int \frac{d\xi}{\sqrt{2\pi}} d\phi \, e^{-\phi^2/4} \frac{1}{\int d\bar{\theta}} \int \frac{d\bar{\xi}}{\sqrt{2\pi}} d\bar{\phi} \, e^{-\bar{\phi}^2/4}$$
$$\times \int \frac{d\psi_1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left\{-\frac{1}{2}+\left(\frac{\Delta x}{2\pi}\right)^2\right\}\psi_1^2} \int \frac{d\psi_2}{\sqrt{2\pi}} e^{-\frac{1}{2}\left\{-\frac{1}{2}+\left(\frac{\Delta x}{2\pi}\right)^2\right\}\psi_2^2}. \tag{2.10}$$

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[2]Even though we call them tachyons, for $\Delta x > \pi\sqrt{2}$ they have positive L_0 eigenvalues.

The analysis of [5] also shows that the variables ξ, $\bar{\xi}$ are related to the shifts in the positions x_1, x_2 of the instanton and the anti-instanton via,

$$\delta x_1 = g_o\, \pi\sqrt{2}\, \xi, \qquad \delta x_2 = g_o\, \pi\sqrt{2}\, \bar{\xi}\,. \tag{2.11}$$

Furthermore the gauge transformation parameters θ is related to the rigid U(1) transformation parameters $\tilde{\theta}$, normalized to have period 2π, via $\theta = 2\tilde{\theta}/g_o$. This gives $\int d\theta = 4\pi/g_o$. A similar result holds for $\int d\bar{\theta}$. Substituting these into (2.10) and carrying out the gaussian integrals over ϕ, $\bar{\phi}$, ψ_1, ψ_2 we get:

$$Z = \left(\frac{g_o}{4\pi}\right)^2 \left(\frac{1}{g_o\pi\sqrt{2}}\right)^2 \frac{1}{2\pi}\,(2\sqrt{\pi})^2 \int dx_1 dx_2 \, \frac{1}{-\frac{1}{2} + \left(\frac{\Delta x}{2\pi}\right)^2} = \frac{1}{4\pi^2}\int dx_1 dx_2 \, \frac{1}{(\Delta x)^2 - 2\pi^2}\,. \tag{2.12}$$

Comparing this with (2.5) we get,[3]

$$\mathcal{N}_D^2 = \frac{1}{4\pi^2}\,. \tag{2.13}$$

This agrees with the value of $\mathcal{N}_D^2$ determined in [3] using comparison with the matrix model results.

3 Finite temperature

For completeness we shall also briefly discuss the finite temperature case, obtained by compactifying X^0 on a circle of radius R. In this case the trace over open string states will include the extra winding modes where the open string wraps the circle w times. From eqs. (3.14) and (3.16) of [3] we see that this corresponds to multiplying the zero temperature result (2.12) by,

$$I \equiv \exp\left[\int_0^\infty \frac{dt}{t} \sum_{w \in \mathbb{Z}-\{0\}} \left\{ e^{-2\pi t\left(-\frac{1}{2} + \left(wR + \frac{\Delta x}{2\pi}\right)^2\right)} - e^{-2\pi t(wR)^2} \right\}\right], \tag{3.1}$$

where we have made the transformations $x \to \sqrt{2}x$, $R \to \sqrt{2}R$ to convert the results from $\alpha' = 2$ unit to $\alpha' = 1$ unit. Using (2.6) we can express this as,

$$I = \prod_{w \in \mathbb{Z}-\{0\}} \frac{(wR)^2}{\left\{-\frac{1}{2} + \left(wR + \frac{\Delta x}{2\pi}\right)^2\right\}}\,. \tag{3.2}$$

This gives, after minor reorganization of the terms,

$$I = \prod_{w \in \mathbb{Z}_+} \frac{(wR)^4}{\left(\frac{1}{\sqrt{2}} + wR - \frac{\Delta x}{2\pi}\right)\left(\frac{1}{\sqrt{2}} - wR - \frac{\Delta x}{2\pi}\right)\left(\frac{1}{\sqrt{2}} + wR + \frac{\Delta x}{2\pi}\right)\left(\frac{1}{\sqrt{2}} - wR + \frac{\Delta x}{2\pi}\right)}$$

$$= \prod_{w \in \mathbb{Z}_+} \frac{1}{\left(1 - \frac{\left(\frac{\Delta x - \sqrt{2}\pi}{2\pi R}\right)^2}{w^2}\right)\left(1 - \frac{\left(\frac{\Delta x + \sqrt{2}\pi}{2\pi R}\right)^2}{w^2}\right)}\,. \tag{3.3}$$

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[3]For this analysis we should regard $\mathcal{N}_D^2$ as one unit instead of assigning separate normalization factors to D-instantons and anti-D-instantons, since these suffer from infrared divergences from the closed string channel.

Using the result,

$$\sin\theta = \theta \prod_{w\in\mathbb{Z}_+}\left(1 - \frac{\theta^2}{\pi^2 w^2}\right), \tag{3.4}$$

we can write

$$I = \frac{(\Delta x)^2 - 2\pi^2}{4\,R^2}\frac{1}{\sin\left(\frac{\Delta x + \sqrt{2}\pi}{2R}\right)\sin\left(\frac{\Delta x - \sqrt{2}\pi}{2R}\right)}. \tag{3.5}$$

Multiplying (2.12) by this factor we get the finite temperature partition function:

$$\frac{1}{4\pi^2}\int dx_1 dx_2\,\frac{1}{4\,R^2\,\sin\left(\frac{\Delta x + \sqrt{2}\pi}{2R}\right)\sin\left(\frac{\Delta x - \sqrt{2}\pi}{2R}\right)}. \tag{3.6}$$

This agrees with eq. (3.17) of [3] after changing $x \to x\sqrt{2}$, $R \to R\sqrt{2}$.

Acknowledgments

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