Callan-Rubakov effect and higher charge monopoles

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Abstract: In this paper we study the interaction between magnetic monopoles and massless fermions. In the low energy limit, the monopole's magnetic field polarizes the fermions into purely in-going and out-going modes. Consistency requires that the UV fermion-monopole interaction leads to non-trivial IR boundary conditions that relate the in-going to out-going modes. These non-trivial boundary conditions lead to what is known as the Callan-Rubakov effect. Here we derive the effective boundary condition by explicitly integrating out the UV degrees of freedom for the general class of spherically symmetric SU(N) monopoles coupled to massless fermions of arbitrary representation. We then show that the boundary conditions preserve symmetries without ABJ-type anomalies. As an application we explicitly derive the boundary conditions for the stable, spherically symmetric monopoles associated to the SU(5) Georgi-Glashow model and comment on the relation to baryon number violation.

Contents

1 Introduction and summary

In the 1980's Callan and Rubakov showed that the interaction of massless fermions with a minimal $SU(N)$ monopole is non-trivial in the IR [1–6]. At low energies, the smooth $SU(N)$ monopole is described by a singular $U(1)$ monopole. One then expects that the strong magnetic field near the core of the monopole repels dynamical fields and screens the

UV physics in the monopole core. However, in this theory the fermions have low-energy spin-$j = 0$ modes that penetrate into the core of the monopole. This allows the fermions to interact with the non-abelian degrees of freedom that are confined to the monopole core and induces a non-trivial effective boundary condition on the IR abelian monopole [1–16].

Traditionally, this non-trivial boundary condition is stated in terms of scattering processes where an s-wave fermion scatters off of the monopole. This viewpoint has lead to much confusion in the literature due to the fact that the monopole boundary condition is not stated simply in terms of the fermion fields. The main confusion is how to understand the physical interpretation of the out-state of a typical scattering process [17, 18] which has been cryptically referred to as "propagating pulses of vacuum polarization" [19].

The reason why this fermion-monopole interaction is of great importance is because of its phenomenological implication in the standard model. Originally, Callan and Rubakov studied monopoles in the SU(5) Georgi-Glashow GUT completion of the standard model. There, Callan and Rubakov showed that the fermion-monopole interaction violates baryon-number symmetry in a way that catalyzes proton decay. This effect, which is now referred to as the Callan-Rubakov effect, is clearly important in understanding the phenomenological implications of GUT completions of the standard model.

1.1 Outline and summary of results

Thus far, the Callan-Rubakov effect has only been discussed for theories in which the fermion-monopole interactions reduce to the spherically symmetric SU(2) monopole coupled to fundamental fermions. In this paper we revisit the Callan-Rubakov effect in order to clarify some of the subtle issues described above and we determine the form of the IR boundary conditions for massless fermions of general representation on a generic spherically symmetric SU(N) monopole.

First, in section 2 we discuss the low energy effective theory of massless fermions in the presence of a spherically symmetric monopole in SU(N) gauge theory. In the IR, the smooth UV monopole is described by a singular Dirac monopole. In the Dirac monopole background, the magnetic field polarizes the fermions so that each IR fermion is of definite angular momentum and purely in-going or out-going. These correspond to the fermions in the lowest Landau level when restricted to a spherical shell surrounding the monopole: the scale of the Zeeman splitting is set by the W-boson mass m_W which is on the order of the UV-cutoff scale. Because of the induced energy splitting, the IR theory is well described by the spherical reduction to a 2D theory on the half-plane which describes the radiald propagation of the above modes.

In the effective 2D theory the in-going/out-going fermion modes correspond to 2D chiral/anti-chiral fermions and the monopole defect corresponds to the boundary on the half-plane at $r = 0$. Unitarity then demands that we impose boundary conditions on the monopole that relate the in-going/chiral to out-going/anti-chiral modes. The general boundary conditions for such a system are given by linear relations between the currents of the chiral (J_A) and anti-chiral fermions ($\bar{J}_A$) which can be written as [20–22]:

$$J_A = \mathcal{R}_{AB} \bar{J}_B \big|_{r=0}. \tag{1.1}$$

The Callan-Rubakov effect corresponds to a particular choice of matrix $\mathcal{R}_{AB}$, which can be derived directly from integrating out UV degrees of freedom [1, 23]. This effective boundary condition arises from the interaction of the fermions with a collection of periodic scalar fields (called dyon degrees of freedom) that are localized on the monopole.

In section 3, we derive the origin of a spherically symmetric monopole's dyon degrees of freedom from the UV theory. Spherically symmetric monopoles are smooth, classical Yang-Mills-Higgs field configurations that source a magnetic field. They are specified by a pair of embeddings $\mathrm{SU}(2)_{T,I} \hookrightarrow \mathrm{SU}(N)$ that are generated by the triplets $\vec{T}, \vec{I} \in \mathfrak{su}(N)$. The pair source the asymptotic magnetic field

$$\lim_{r \to \infty} \vec{B} = \frac{\gamma_m}{2r^2}\hat{r} + O(1/r^{2+\delta}) \,, \quad \gamma_m = 2(T_3 - I_3) \,, \tag{1.2}$$

where $\delta > 0$ and $[\vec{I}, \gamma_m] = 0$. The non-trivial Higgs profile asymptotically breaks gauge symmetry and creates a potential well at the monopole that traps the non-abelian gauge field in the broken directions (referred to as W-bosons). This gives rise to a light degree of freedom φ which is localized on the monopole world volume which are the phase modes of the trapped W-bosons.

The dyon degrees of freedom couple to the low energy fermion degrees of freedom through the coupling to the UV gauge field which can be read off explicitly from the spectrum of exact fermion zero-modes. With these solutions in hand, one can then integrate out the dyon degrees of freedom to obtain the effective boundary condition on the long range fermion degrees of freedom.

The resulting boundary conditions are qualitatively similar to the boundary conditions found by Callan and Rubakov: for any in-going particle, there is an out-going fractional flux in multiple fermion channels. This boundary condition encodes the following physical process. When a low energy fermion scatters off of the monopole, it excites the dyon degrees of freedom φ. The scattering deposits charge and energy in the monopole core which is then radiated away by soft, charged modes of the massless fermions:

$$\psi \longrightarrow \tilde{\psi} + \varphi \longrightarrow \tilde{\psi} + J_{\mathrm{elec}} = \tilde{\psi} + \frac{1}{\mathcal{N}} \sum_i c_i \tilde{\psi}_i^\dagger \tilde{\psi}_i \,, \tag{1.3}$$

where $\mathcal{N}$ is some normalization constant. This is illustrated in figure 1 and is the content of section 4.

The general boundary condition for a collection of fermions $\{\psi_{R_i}\}$ that each transform under representations R_i of $\mathrm{SU}(N)$ coupled to a spherically symmetric monopole is given as follows. In total, the number of low energy fermion modes is given by[1] [28]

$$\{\# \text{ of low-energy modes}\} = \sum_{R_i} \sum_{\mu \in S_{R_i}} n_{R_i}(\mu) \langle T_3, \mu \rangle \,, \tag{1.4}$$

where $S_R = \{\mu \in \Delta_R \,|\, \langle \gamma_m, \mu \rangle \times \langle \gamma_m, \mu^* \rangle < 0\}$ for Δ_R the weight system of R, $n_R(\mu)$ is the multiplicity of the weight $\mu \in \Delta_R$, and μ^* is the charge conjugate with respect to the

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[1]Here $\langle \, , \, \rangle$ is the natural pairing $\langle \, , \, \rangle : \mathfrak{g} \times \mathfrak{g}^\vee \to \mathbb{R}$.
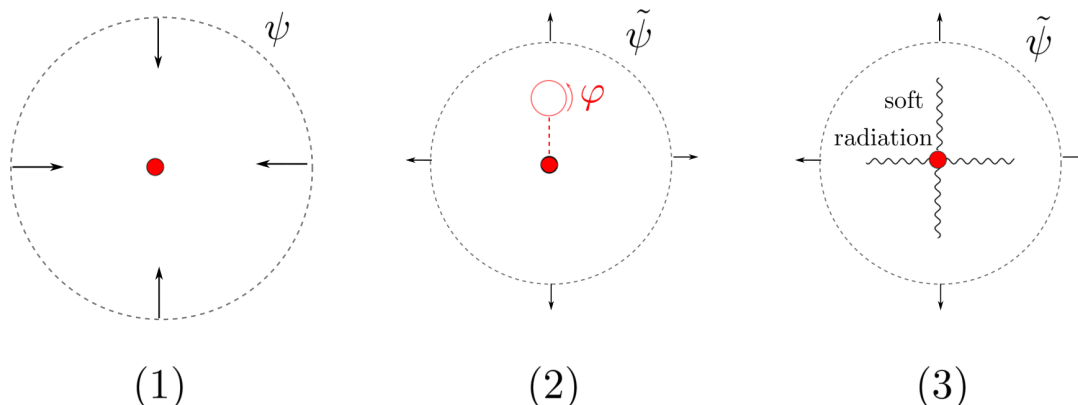
Figure 1. In this figure we illustrate the typical scattering of a low energy fermion mode off of a monopole. (1) shows an incoming spherical fermion wave of ψ which is (2) reflected to an out-going spherical mode of a different fermion $\tilde{\psi}$ while exciting the periodic dyon degrees of freedom φ. The dyon excitation then (3) decays by soft radiation of the massless, charged fermion modes.

$SU(2)_T$ embedding.[2] If we label all low energy modes by a general index A, then we can define the matrix

$$c_A^I = \langle \rho_I, \mu_A \rangle , \quad \rho_I = \sum_{J=1}^{I} J(H_J + H_{N-J}) + I \sum_{J=I+1}^{N-I-1} H_J , \qquad (1.5)$$

where μ_A is the weight of the fermion mode labeled by A and $\{H_I\}$ are simple coroots. The effective boundary condition is then specified by the matrix

$$\mathcal{R}_{AB} = \delta_{AB} - \sum_{I=1}^{N-1} \frac{2}{\sum_C c_C^I \langle H_{N-I}, \mu_C \rangle} c_A^I c_B^I , \qquad (1.6)$$

which is very non-diagonal and leads to fractional fluxes in a generic scattering process. See section 4 for more precise definitions and details.

The existence of fractional flux in a given fermion channel comes from the fact that each dyon couples to a corresponding U(1) gauge current whose unit normalization leads to fractional coefficients as in (1.3). This peculiarity arises from the fact that we have massless charged fermions which each have an anomalous species number symmetry. This means that the species number is not a good quantum number and has no right to be preserved by a scattering process with monopoles which are non-perturbative gauge field configuration. However, the preserved gauge and global symmetries do have good quantum numbers and do have integer fluxes in a generic scattering process. In fact, we show here that the effective monopole boundary conditions explicitly violate global symmetries that are only broken by ABJ-type anomalies. This is the topic of section 5.

Finally, in section 6 we demonstrate our results by explicitly computing the effective boundary conditions for a number of examples. As an application we solve for the effective

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[2]Here the charge conjugate of μ is the image of μ under the generator of the Weyl group of $SU(2)_T \subset SU(N)$.

boundary conditions for the higher charge spherically symmetric monopoles in the SU(5) Georgi-Glashow GUT model SU(5) and show that they preserve B-L symmetry while breaking B- and L-symmetry.

We expect that this work has an interesting extension to the case of massive fermions. In the massive case, the fermion spectra is qualitatively different: there exists an infinite tower of angular momentum modes that are L^2-normalizable/bound to the monopole. Additionally, when the fermions have a mass there is a gap to the continuum of scattering states that prevents the dyon degree of freedom from radiating away its electric charge. We hope to study this scenario in a future work.

2   Low energy monopole-fermion system and the Callan-Rubakov effect

We are interested in understanding the low energy monopole-fermion interactions in UV complete non-abelian gauge theories. In this section we will introduce the Callan-Rubakov effect from the low energy effective theory point of view.

Let us consider a 4D SU(N) gauge theory with an adjoint-valued scalar field Φ and a fermion ψ_R of representation R. Here we will consider the vacuum in which Φ has a vev Φ_∞ that breaks

$$\mathrm{SU}(N) \longrightarrow G_{\Phi_\infty} = \frac{\widetilde{G}_{\Phi_\infty}}{\Gamma} = \frac{\mathrm{SU}(N_1) \times \ldots \times \mathrm{SU}(N_n) \times \mathrm{U}(1)^r}{\Gamma}\,, \tag{2.1}$$

where Γ is a discrete abelian subgroup of the center of $\widetilde{G}_{\Phi_\infty}$. Here we will *not* restrict ourselves to the case where G_{Φ_∞} is abelian.

This theory has a collection of monopoles which are smooth field configurations that solve the Bogomolny equation

$$B_a = D_a\Phi\,, \quad E_a = 0\,, \tag{2.2}$$

where B_a, E_a are the non-abelian magnetic and electric field. Monopole solutions are classified by their asymptotic behavior

$$\lim_{r\to\infty} B_a = \frac{\gamma_m}{2r^2}\hat{r} + O(1/r^3) \qquad \lim_{r\to\infty} \Phi = \Phi_\infty - \frac{\gamma_m}{2r} + O(1/r^2)\,, \tag{2.3}$$

where γ_m is the magnetic charge and Φ_∞ is the Higgs vev. Here we take $\gamma_m = \sum_I n_I h^I$ for $n_I \in \mathbb{Z}_{\geq 0}$ where $\{h^I\}$ are generators of U(1)^r which we take to be normalized $e^{2\pi i h^I} = 1$.

Since this theory is in general non-abelian, the theory has two natural scales associated to it: the UV cutoff scale (set by the Higgs vev) and the confinement scale. Here we will consider the "low energy limit" at intermediate energy scales where the theory is well described by deconfined fermions coupled to the G_{Φ_∞} gauge fields.

In this limit, the smooth monopole is described by a monopole defect operator which sources a gauge field:

$$A_{\mathrm{IR}} = \gamma_m A_{\mathrm{Dirac}} = \frac{\gamma_m}{2}(1 - \cos\theta)d\phi\,, \tag{2.4}$$

in spherical coordinates centered around the monopole.[3] Since this field configuration is singular, the monopole operator is defined by cutting out an infinitesimal 2-sphere around the origin and on which we impose the boundary condition (2.4) on the G_{Φ_∞} fields. In general, this procedure requires boundary conditions for any fields whose long distance behavior does not vanish near the cutoff surface. These boundary conditions will encode the UV physics of a dynamical field's interaction with the UV magnetic monopole.

The magnetic charge γ_m generates a subgroup $U(1)_m \subset G_{\Phi_\infty}$. The representation R of $SU(N)$ then decomposes into $U(1)_m$ representations as

$$R = \bigoplus_i R_i \,, \quad R_i[\gamma_m] = p_i \in \mathbb{Z} \,. \tag{2.5}$$

Here we will find it useful to introduce the notion of representation weights. For a representation R there are a collection of weights $\mu \in \Delta_R$. To each weight μ there is a vector v_μ in the representation space of R and the set of v_μ form a basis of the representation space. Every weight μ can be expressed as a vector in $\mathbb{R}^{N-1}$ and elements of the Cartan subgroup (spanned by simple coroots $\{H_I\}$) have corresponding (dual) vectors so that the charge of v_μ with respect to H_I is given by the Euclidean inner product in $\mathbb{R}^{N-1}$:

$$R(H_I) \cdot v_\mu = \langle H_I, \mu \rangle \, v_\mu \,. \tag{2.6}$$

Consequently, given a choice of Cartan subgroup containing $U(1)_m$, we can decompose

$$R = \bigoplus_{\mu_i \in \Delta_R} R_i \,, \tag{2.7}$$

for weights $\mu_i \in \Delta_R$ in the weight space of R where R_{μ_i} has charge $p_i = \langle \gamma_m, \mu_i \rangle$ under $U(1)_m$. Consequently ψ_R also decomposes

$$\psi_R = \sum_{\mu_i \in \Delta_R} \psi_{\mu_i} v_{\mu_i} \,. \tag{2.8}$$

Here we will assume that the theory has no gauge anomaly. This implies that $\sum_i p_i^3 = 0$ and $\sum_i p_i = 0$.[4]

In the monopole background, the magnetic field polarizes the fermion spin by the Zeeman effect. This can be seen by computing the spectrum of the Dirac operator. As in appendix A, the solutions to the Dirac equation are a spectrum of scattering states built on the zero-mode solutions:

$$
\begin{aligned}
\psi_{\mu_i}(r, \theta, \phi) = {}& c_{+,m} \frac{e^{i\left(m - \frac{p_i}{2}\right)\phi}}{r} r^{\sqrt{(j+1/2)^2 - p_i^2/4}} U(\theta, \phi) \begin{pmatrix} d^j_{-m, \frac{p_i+1}{2}}(\theta) \\ d^j_{-m, \frac{p_i-1}{2}}(\theta) \end{pmatrix} \\
& + c_{-,m} \frac{e^{i\left(m - \frac{p_i}{2}\right)\phi}}{r} r^{-\sqrt{(j+1/2)^2 - p_i^2/4}} U(\theta, \phi) \begin{pmatrix} d^j_{-m, \frac{p_i+1}{2}}(\theta) \\ -d^j_{-m, \frac{p_i-1}{2}}(\theta) \end{pmatrix} \,.
\end{aligned}
\tag{2.9}
$$

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[3]Without loss of generality we will be restricting ourselves to the northern hemisphere.

[4]The fact that $\sum_i p_i = 0$ follows from the fact that all $U(1)$ generators in $SU(N)$ are traceless.

Requiring that the fermion's energy density be finite at $r \to 0, \infty$ then implies that there are no solutions for $p_i = 0$ and restricts us to the solutions with

$$j = j_i := \frac{|p_i| - 1}{2}, \tag{2.10}$$

for $p_i \neq 0$. Explicitly, we find that the finite energy density solutions of the Dirac operator are spanned by

$$\psi_{\mu_i}^{(k,m)} = \frac{U(\theta,\phi)}{r} \hat{\psi}_{\mu_i}^{(k,m)} = \frac{U(\theta,\phi)}{r} \begin{cases} e^{ik(t+r)} e^{i(m-p_i/2)\phi} \begin{pmatrix} 0 \\ d_{-m,j_i}^{j_i}(\theta) \end{pmatrix} & p_i > 0 \\ e^{ik(t-r)} e^{i(m-p_i/2)\phi} \begin{pmatrix} d_{-m,-j_i}^{j_i}(\theta) \\ 0 \end{pmatrix} & p_i < 0 \end{cases} \tag{2.11}$$

which we can express terms of a 2D field $\chi_{\mu_i,m}(t,r)$:

$$\psi_{\mu_i}(r,\theta,\phi) = \frac{U(\theta,\phi)}{r} \sum_m \left(e^{i(m-p_i/2)\phi} d_{-m,\pm j_i}^{j_i}(\theta) \right) \chi_{\mu_i,m}(t,r), \tag{2.12}$$

that lives on the half-plane with coordinates (t, r) with a boundary at $r = 0$.

Note that the fermions are polarized. This may seem like an artifact of restricting to fermions of finite energy density near the origin, however, we can see that this must be true from a simple scaling argument. In the low energy theory, the only mass is the cutoff scale of the IR theory (set by the W-boson mass). Since the 4D fermions ψ_{μ_i} have scaling dimension $\frac{3}{2}$ and 2D fermions χ_{μ_i} have scaling dimension $\frac{1}{2}$, we see that the zero-modes must have scaling $\psi_{\mu_i} \sim \frac{1}{r} f(m_W r)$ so that the higher angular momentum zero-modes are suppressed by the UV scale. It then follows that the tower of scattering states built on these zero-modes also do not contribute to the low energy effective theory.

Notice that this differs from the case of massive fermions which are studied in [24]. In the massive case, the wave function is exponentially suppressed by $\frac{e^{-ur}}{r}$ for mass u. Because of this, the spectrum of low energy modes contains an entire tower of L^2-normalizable higher angular momentum states corresponding to the $c_{+,m}$-solutions in (2.9). We believe that this modified spectrum of low energy modes will lead to interesting effects which we plan to address in a future paper.

2.1 Effective 2D theory and boundary conditions

The dynamics of the low energy dynamics of the fermion can now be described by an effective 2D theory of the $\chi_{\mu_i,m}$ fields. Explicitly, if expand the fermion field operator

$$\psi_{\mu_i}(r,\theta,\phi) = \frac{U(\theta,\phi)}{r} \sum_m \int dk \, c_{i,m}^{(k)} \hat{\psi}_{\mu_i}^{(k,m)}(r,\theta,\phi), \tag{2.13}$$

where the $c_{k,m}^{(i)}$ are creation operators, then we find that spherically reducing ψ_{μ_i} results in the 2D field

$$\chi_{\mu_i,m}(t,r) = \int dk \, \chi_{\mu_i,m}^{(k)}(t,r) = \int dk \, c_{i,m}^{(k)} \begin{cases} e^{ik(t+r)} \chi_+ & p_i > 0 \\ e^{ik(t-r)} \chi_- & p_i < 0 \end{cases} \tag{2.14}$$

which now lives on a half-plane with coordinates (t, r) with a boundary at $r = 0$ and where $\sigma^3 \chi_\pm = \pm \chi_\pm$. Here we see that the fermion polarization in 4D is correlated with the chirality of the modes in the effective 2D theory.

In other words, for each ψ_{μ_i} with $p_i > 0$ we get p_i chiral/left-moving 2D fermions corresponding to the p_i components of the 4D spin-$\frac{p_i-1}{2}$ multiplet. Similarly, for each ψ_{μ_i} with $p_i < 0$ we get $-p_i$ anti-chiral/right-moving 2D fermions corresponding to the $-p_i$ components of the 4D spin-$\frac{-p_i-1}{2}$ multiplet.

We now see that the dynamics of the spherical fermion waves are described by left- and right-moving fermions (chiral/anti-chiral fermions) on the half plane. Since the fermions can reach the boundary, we must impose boundary conditions on the fields there that relate the left- and right-handed fermions.

The question of how to describe fermionic boundary conditions is somewhat subtle. One approach is to simply impose boundary conditions of the form

$$\psi_i = M_{ij} \tilde{\psi}_j \big|_{r=0} \quad \text{or} \quad \psi_i = M_{ij} \tilde{\psi}_j^\dagger \big|_{r=0} \,, \tag{2.15}$$

where ψ_i are chiral and $\tilde{\psi}_i$ are anti-chiral and M_{ij} is an invertible matrix. However, a more general set of boundary conditions are given by the bosonizing the fields

$$\psi_i^\dagger \psi_i = J_i = \partial H_i \,, \quad \tilde{\psi}_i^\dagger \tilde{\psi}_i = \bar{J}_i = \bar{\partial} \widetilde{H}_i \,, \tag{2.16}$$

and then imposing boundary conditions on the bosonized fields $H_i, \widetilde{H}_i$. Given a mode expansion

$$H_i(z) = \sum_n H_{i,n} z^n \,, \quad \widetilde{H}_i(\bar{z}) = \sum_n \widetilde{H}_{i,n} \bar{z}^n \,, \tag{2.17}$$

the general boundary condition is given

$$H_{i,n} = \mathcal{R}_{ij} \widetilde{H}_{j,n} \big|_{r=0} \,, \tag{2.18}$$

which can alternatively be written in terms of the currents

$$J_i = \mathcal{R}_{ij} \bar{J}_j \big|_{r=0} \,. \tag{2.19}$$

For example, the standard Neumann and Dirichlet boundary conditions are given

$$\begin{aligned}
\text{Neumann}: \quad (\partial - \bar{\partial})(H_i + \widetilde{H}_i)\big|_{r=0} = 0 \quad &\Leftrightarrow \quad H_{i,n} = \widetilde{H}_{i,n} \\
\text{Dirichlet}: \quad (H_i + \widetilde{H}_i)\big|_{r=0} = 0 \quad &\Leftrightarrow \quad H_{i,n} = -\widetilde{H}_{i,n}
\end{aligned} \tag{2.20}$$

Then, by identifying the Neumann boundary condition with $(H_i - \widetilde{H}_i)\big|_{r=0}$, we can apply the bosonization map to write the above boundary conditions in terms of the fermion fields:

$$\begin{aligned}
\text{Neumann}: \quad H_i = \widetilde{H}_i \big|_{r=0} \quad &\Rightarrow \quad \psi_i = \tilde{\psi}_i \big|_{r=0} \,, \\
\text{Dirichlet}: \quad H_i = -\widetilde{H}_i \big|_{r=0} \quad &\Rightarrow \quad \psi_i = \tilde{\psi}_i^\dagger \big|_{r=0} \,.
\end{aligned} \tag{2.21}$$

However, more general boundary conditions of the type (2.19) do not have such a simple interpretation in terms of UV fermion fields.

This formalism can be used to construct boundary conditions that preserve gauge and global symmetries [20–22]. Consider the case of N chiral fermions $\{\psi_i\}$ and N anti-chiral fermions $\{\tilde{\psi}_i\}$ in 2D on the half plane with N U(1) symmetries indexed by $a = 1, \ldots, N$ under which the fermions have charges:

	U(1)$_a$
ψ_i	Q_{ai}
$\tilde{\psi}_i$	$\bar{Q}_{ai}$

where we assume that the U(1)$_a$ are independent (i.e. $Q_{ai}, \bar{Q}_{ai}$ are invertible matrices).[5]

For these fermions we can construct the fermion species currents

$$J_i = \psi_i^\dagger \psi_i, \quad \bar{J}_i = \tilde{\psi}_i^\dagger \tilde{\psi}_i, \tag{2.22}$$

with which we can identify the U(1)$_a$ currents

$$\mathcal{J}_a = Q_{ai} J_i, \quad \bar{\mathcal{J}}_a = \bar{Q}_{ai} \bar{J}_i. \tag{2.23}$$

We would now like to construct a boundary condition that preserves the U(1)$_a$ symmetries. Such a boundary condition has the property

$$\left. (\mathcal{J}_a - \bar{\mathcal{J}}_a) \right|_{r=0} = 0. \tag{2.24}$$

Then, using the fact that the $Q_{ai}, \bar{Q}_{ai}$ are invertible, we can relate this to

$$\left. (J_i - \mathcal{R}_{ij} \bar{J}_j) \right|_{r=0} = 0, \quad \mathcal{R}_{ij} = (Q^{-1})_{ia} \bar{Q}_{ja}. \tag{2.25}$$

In the case where the symmetries are non-anomalous, $\mathcal{R}_{ij}$ is a rational, orthogonal matrix. Physically, this means for any incoming fermion of fixed species i the boundary will reflect to an out-going collection of currents corresponding to the non-zero matrix components $\mathcal{R}_{ij}$.

The Callan-Rubakov effect corresponds to a boundary conditions specified by particular choice of $\mathcal{R}_{ij}$. In fact, the idea of bosonizing the spherical modes to obtain effective boundary conditions on the monopole goes all the way back to the original papers of Callan [1, 6].

Previously, only the boundary conditions for N_f fundamental fermions in the presence of a minimal SU(N) monopole were known[6] [1–6, 19, 25, 26]. The low energy theory of this configuration is described by N_f in-going and N_f out-going s-wave fermions. These are described by an effective 2D theory of N_f left- and N_f right-moving fermions on the half-plane. In the above formalism, the monopole boundary condition is specified by

$$\mathcal{R}_{ij} = \begin{cases} 1 - \frac{2}{N_f} & i = j \\ -\frac{2}{N_f} & i \neq j \end{cases} \tag{2.26}$$

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[5]The condition that there are no anomalies among the U(1)$_a$ is given by

$$\sum_i Q_{ai} Q_{bi} - \bar{Q}_{ai} \bar{Q}_{bi} = 0 \quad \forall a, b.$$

Here we will make no assumption about the U(1)$_a$ being non-anomalous.

[6]The boundary conditions for the minimal monopole in the SU(5) Georgi-Glashow model is also known, but it is equivalent to the case of the minimal monopole in SU(N) with fundamental fermions.

Here we see that $\mathcal{R}$ is very non-diagonal so that a generic scattering process will have non-trivial components along all out-going fermion currents. This matrix $\mathcal{R}_{ij}$ corresponds to the boundary condition that preserves a collection of non-anomalous symmetries with charge matrices given by

$$Q_{ai} = \begin{cases} \delta_{a,i} - \delta_{a,i+1} & a < N_f \\ 1 & a = N_f \end{cases}, \qquad \bar{Q}_{ai} = \begin{cases} \delta_{a,i} - \delta_{a,i+1} & a < N_f \\ -1 & a = N_f \end{cases} \qquad (2.27)$$

Here, the $\mathrm{U}(1)_1, \ldots, \mathrm{U}(1)_{N_f-1}$ symmetries are a labeling symmetry where $\mathrm{U}(1)_a$ counts the flux of $\psi_a, \tilde{\psi}_a$ minus the flux of $\psi_{a+1}, \tilde{\psi}_{a+1}$ and $\mathrm{U}(1)_{N_f}$ is the gauge charge. One can explicitly show that the matrix $\mathcal{R}_{ij}$ in (2.26) is given by the formula $\mathcal{R}_{ij} = (\bar{Q}^{-1}Q)_{ij}$ using the above charge matrices.

As we show later, the physics of these boundary conditions comes from the fact that the fermions couple to the dynamical dyon degree of freedom (trapped W-bosons) on the monopole world volume. The boundary conditions then encode the physics of a fermion wave which, in scattering off of the monopole, excites the dyon degree of freedom which then radiates away the accumulated gauge charge and energy into all out-going modes that couple to it [19].

At first glance, it appears strange that the out-going flux for each $\bar{J}_i$ is generically fractional. However, this is simply a fact that the currents $J_i, \bar{J}_i$ are not conserved currents in the quantum theory: each fermion number symmetry is broken by an ABJ anomaly. The magnetic field of the monopole together with the electric field of any low energy charged fermion combine together to turn on a non-trivial gauge background that breaks the conservation of all $J_i, \bar{J}_i$. The fractional flux is simply a reflection of the fact that species number is not a good quantum number in our theory. Rather, the boundary condition is diagonalized by taking a basis of conserved currents rather than a basis of anomalous species currents whose symmetries are broken in the presence of the monopole. It should not be a surprise that non-perturbative effects are reflected in monopole-fermion scattering even at low energy since monopoles are inherently non-perturbative features of non-abelian gauge theory.

The general effective boundary conditions for higher charge monopoles and/or for fermions in arbitrary representations are not known. It is presumed that the physics is very similar to the case of the standard Callan-Rubakov effect. However, this is not immediately obvious since increasing the magnetic field or dimension of the coupled representations naturally leads to fermions of higher angular momentum which experience an angular momentum barrier that repulses their wave function from the core of the monopole. Further, it is generally unknown how to characterize what kinds of symmetries are preserved by monopole boundary conditions although it is presumed that any symmetry which experiences an ABJ-type anomaly will be violated. In the rest of this paper, we will derive the general low energy effective boundary conditions for spherically symmetric monopoles coupled to fermions of arbitrary representation and discuss their interplay with global symmetries.

3 Monopole lines from smooth monopoles

Now we will discuss the general spherically symmetric monopole and derive the origin of their dyon degrees of freedom.

Again, our setting is $SU(N)$ gauge theory with an adjoint-valued Higgs field Φ and a Weyl fermion ψ_R that transforms in representation R of $SU(N)$. We will consider a vacuum in which the Higgs field obtains a vev, $\Phi_\infty \in \mathfrak{su}(N)$, which breaks

$$SU(N) \to G_{\Phi_\infty} = \frac{U(1)^r \times \prod_a SU(N_a)}{\Gamma}, \quad \Gamma \subseteq \prod_a \mathbb{Z}_{N_a}. \tag{3.1}$$

Here we will denote the generators of $U(1)^r$ to be $\{h^I\}$.

This theory has a spectrum of smooth monopoles which are finite energy, time independent gauge field configurations with vanishing electric field that solve the Bogomolny equation

$$B_a = D_a \Phi, \tag{3.2}$$

that have the asymptotic behavior

$$\lim_{r \to \infty} \vec{B} \sim \frac{\gamma_m}{2r^2} \hat{r} + O(1/r^3), \quad \lim_{r \to \infty} \Phi = \Phi_\infty - \frac{\gamma_m}{2r} + O(1/r^2), \tag{3.3}$$

where $\gamma_m = \sum_I n_I h^I$ ($n_I \in \mathbb{Z}_{\geq 0}$) is the magnetic charge.

In the IR limit, these monopoles are well described by a monopole defect as we discussed in the previous section. We will be interested in spherically symmetric monopoles because 1.) they give rise to spherically symmetric boundary conditions in the IR and 2.) there is an explicit construction of their field configurations.

A large class of smooth, spherically symmetric non-abelian monopoles can be constructed by embedding the minimal $SU(2)$ monopole into higher rank non-abelian gauge theories. However, this only a special class of spherically symmetric monopoles. More generally, a monopole is spherically symmetric if it is rotationally invariant up to a gauge transformation. This is equivalent to the statement that the gauge field configuration is invariant under an augmentation of the standard angular momentum generators $\vec{L} = -i\vec{r} \times \vec{\nabla}$ by generators of an $SU(2)$ subgroup $\vec{T}$ (which satisfy $[T_i, T_j] = i\epsilon_{ijk}T_k$): $\vec{K} = \vec{L} + \vec{T}$. Without loss of generality we will restrict to the case where $\vec{T}$ is the maximal, irreducible embedding.

Here we will consider spherically symmetric $SU(N)$ monopoles and we will take the generators $\vec{T}$ to define a subgroup $SU(2)_T \subset SU(N)$. The monopole connection can be written in a spherically symmetric gauge as

$$\vec{A} = \frac{\hat{r}}{r} \times (\vec{\mathfrak{M}} - \vec{T}), \tag{3.4}$$

where $\vec{\mathfrak{M}}$ is a $\mathfrak{su}(N)$-valued vector that satisfies

$$[K_i, \mathfrak{M}_j] = i\epsilon_{ijk}\mathfrak{M}_k. \tag{3.5}$$

Such a $\mathfrak{M}_k$ can be constructed by taking the rotation of a θ, ϕ-invariant vector field along the $\hat{z}$-axis:

$$\mathfrak{M}_i = \Omega \, M_i(r) \, \Omega^{-1} \,, \quad \Omega = e^{-i\phi T_3} e^{-i\theta T_2} \,, \tag{3.6}$$

where $M_3(r) = 0$ and $[T_3, M_\pm(r)] = \pm M_\pm(r)$ for $M_\pm = M_1 \pm iM_2$. We will refer to this as the *vector gauge*.

Following [27], we can then use this ansatz to construct spherically symmetric $\mathrm{SU}(N)$ monopoles. If we parametrize

$$T_3 = \frac{1}{2}\mathrm{diag}\,(N-1, N-3, \ldots, -N+1) \,,$$

$$M_+(r) = \left(M_-(r)\right)^T = \begin{pmatrix} 0 & a_1(r) & & & \\ & 0 & a_2(r) & & \\ & & \ddots & \ddots & \\ & & & 0 & a_{N-1}(r) \\ & & & & 0 \end{pmatrix} \,, \tag{3.7}$$

$$\Phi\big|_{\hat{z}} = \Phi = \mathrm{diag}\,(\phi_1(r) \,, \phi_2(r) - \phi_1(r) \,, \ldots \,, -\phi_{N-1}(r)) \,,$$

where $\Phi(\vec{x}) = \Omega \Phi\big|_{\hat{z}} \Omega^{-1}$, then the Bogomolny equation becomes

$$\begin{aligned} \frac{d\phi_I(r)}{dr} &= \frac{1}{r^2}\Big(a_I^2(r) - I(N+1-I)\Big) \\ \frac{da_I(r)}{dr} &= \frac{1}{2}\Big(2\phi_I(r) - \phi_{I-1}(r) - \phi_{I+1}(r)\Big)a_I(r) \,, \end{aligned} \tag{3.8}$$

which has smooth, finite energy solutions as constructed in [27]. The solutions are specified by an additional embedding $\mathrm{SU}(2)_I \hookrightarrow \mathrm{SU}(N)$ which is generated by $I_i \subset \mathfrak{su}(N)$ ($[I_i, I_j] = i\epsilon_{ijk}I_k$) such that

$$\lim_{r\to 0} M_\pm(r) = T_\pm + O(1) \,, \quad \lim_{r\to\infty} M_\pm(r) = I_\pm + O(1/r^\delta) \quad \delta > 0 \,, \tag{3.9}$$

and $\lim_{r\to 0} \Phi(r) = 0$. Such a choice of $\vec{I}$ defines a monopole with magnetic charge[7]

$$\gamma_m = 2(T_3 - I_3) \,, \tag{3.10}$$

and Higgs vev[8]

$$\Phi_\infty = v\,(T_3 - I_3) \,, \tag{3.11}$$

---

[7]Here we use the convention where $e^{4\pi i I_k}, e^{4\pi i T_k} = \mathbb{1}_{\mathrm{SU}(2)}$ while $e^{2\pi i \gamma_m} = \mathbb{1}_{\mathrm{SU}(N)}$.

[8]In the case where $\vec{T}$ is not a maximal embedding in $\mathrm{SU}(N)$, the Higgs vev can also have a constant term:

$$\Phi_\infty = v\,(T_3 - I_3) + \phi_\infty \,,$$

such that $[\vec{T}, \phi_\infty] = [\vec{I}^{(i)}, \phi_\infty] = 0$, $\forall i$. In this case, the Higgs vev is non-trivial in the monopole core:

$$\lim_{r\to 0} \Phi(r) = \phi_\infty \,.$$

In this paper we will restrict to the case where $\phi_\infty = 0$.

up to a choice of $v \in \mathbb{R}$. Here we additionally assume that $[\vec{I}, \gamma_m] = 0$ and that the decomposition of $\vec{I}$ into irreducible components $\vec{I} = \bigoplus_i \vec{I}^{(i)}$ is non-trivial ($\vec{T} \not\cong \vec{I}$).

The above restrictions on $\vec{I}$ imply that the magnetic charge and Higgs vev are constant along each irreducible component $\vec{I}^{(i)}$ and hence the choice of $\vec{I}$ defines the long range-non-abelian gauge field. If we denote the dimension of the irreducible component of $\vec{I}^{(i)}$ as N_i, then the non-abelian part of the long range gauge symmetry of the theory is given by

$$G_{\text{non-ab.}} = \prod_{i \,|\, N_i \neq 1} \text{SU}(N_i) \,. \tag{3.12}$$

The gauge field for the spherically symmetric monopole is also expressible in a more convenient gauge:

$$A_{\text{can.}} = 2T_3 A_{\text{Dirac}} + i\frac{M_+(r)}{2}e^{-i\phi}(d\theta - i\sin\theta d\phi) - i\frac{M_-(r)}{2}e^{i\phi}(d\theta + i\sin\theta d\phi)\,, \tag{3.13}$$

We will refer to this as the *canonical gauge* which is related to the original gauge by $g = e^{-i\phi T_3}e^{-i\theta T_2}e^{i\phi T_3}$. In this gauge, we can explicitly can compute the field strength:

$$F = \left(T_3 - \frac{[M_+(r), M_-(r)]}{2}\right) d^2\Omega \pm \frac{i}{2}M'_\pm(r)e^{mi\phi}dr \wedge (d\theta \mp i\sin\theta d\phi)\,, \tag{3.14}$$

from which we can read off the magnetic field along the $\hat{z}$-axis:

$$B_r = \frac{2T_3 - [M_+(r), M_-(r)]}{2r^2}\,, \quad B_\pm = \frac{1}{r}\partial_r M_\pm(r)\,. \tag{3.15}$$

The a_I parameterizing $M_\pm(r)$ are divided into two different classes by their asymptotic behavior. If $\lim_{r\to\infty} a_I(r) = 0$, then it has the asymptotic behavior

$$\lim_{r\to\infty} a_I(r) \sim e^{-\Delta_I m_W r}\,, \quad \Delta_I = \frac{2m_I - m_{I-1} - m_{I+1}}{2}\,, \tag{3.16}$$

where $\gamma_m = \sum_I m_I H_I$ and $\{H_I\}$ are a basis of simple coroots such that $[H_I, E_J^\pm] = \pm C_{IJ}E_J^\pm$ and $m_I \in \mathbb{Z}_{\geq 0}$.[9] Note that our construction ensures $\Delta_I \geq 0$. Physically, this means that along broken directions in the gauge group, the W-bosons are exponentially confined to the monopole core as in the case of the minimal SU(2) monopole.

In our following analysis we will also need the m_W dependence of the solution. In particular, since $A_{\text{can.}} = A_\mu dx^\mu$ in equation (3.13) is dimensionless, $a_I(m_W r)$ is a dimensionless function of the dimensionless combination $m_W r$.

---

[9]Here we pick a basis $\{H_I\}$ of the Cartan of SU(N) which in the fundamental representation is given by:

$$R_f(H_I)_{ij} = \delta_{iI}\delta_{jI} - \delta_{i,I+1}\delta_{j,I+1}\,.$$

In this basis we can express

$$T_3 = \frac{1}{2}\sum_{I=1}^{N-1} I(N-I)H_I\,,$$

for the maximal embedding in SU(N).

3.1 IR limit

Now let us study these monopoles in the IR limit. To do so, we restrict to energies $E/m_W \ll 1$ which can be achieved by effectively sending $m_W \to \infty$ while keeping E fixed. This scaling affects the form of the gauge field by projecting to the leading term in an asymptotic expansion in powers of $1/m_W r$:

$$
\begin{aligned}
\lim_{m_W \to \infty} A_{\text{can.}} &= \lim_{m_W \to \infty} 2T_3 A_{\text{Dirac}} \pm i \frac{M_\pm(m_W r)}{2} e^{\mp i\phi}(d\theta \mp i\sin\theta d\phi) \\
&= 2T_3 A_{\text{Dirac}} \pm i\frac{I_\pm}{2} e^{\mp i\phi}(d\theta \mp i\sin\theta d\phi) + O(1/m_W r) \\
&\sim \gamma_m A_{\text{Dirac}} + O(1/m_W r)\,,
\end{aligned}
\tag{3.17}
$$

where the final step is equivalence up to a gauge transformation by $g_I = e^{-i\phi I_3} e^{-i\theta I_2} e^{i\phi I_3}$. Thus we see in the IR that the non-abelian monopole configuration approaches the abelian monopole.

Now we would like to describe the low energy effective gauge theory in the presence of a singular monopole that arises from a $\text{SU}(N)$ gauge theory. As is standard in spontaneous gauge symmetry breaking, the low energy gauge bosons can be described by propagating vector fields for the gauge group G_{Φ_∞}. The gauge fields along the directions in $\mathfrak{su}(N)$ that do not commute with Φ_∞ are massive away from the monopole. However, since $\lim_{r\to 0} \Phi = 0$ we see that the asymptotically massive gauge fields become massless inside the core of the monopole. In essence, the non-trivial Higgs profile acts as a potential well that traps the broken gauge degrees of freedom in the monopole core.

We can now parametrize the perturbations of the gauge field around this background:

$$
A_{\text{SU}(N)} = A_{G_{\Phi_\infty}} + 2T_3 A_{\text{Dirac}} + (W_+ + W_-)\,,
\tag{3.18}
$$

in which we separate the degrees of freedom that obtain a mass from the Higgs field from those that are free-propagating non-abelian degrees of freedom. Here we distinguish between three components: 1.) $A_{G_{\Phi_\infty}}$ are the long range propagating gauge fields of G_{Φ_∞}, 2.) $2T_3 A_{\text{Dirac}}$ is the abelian component of the classical solution. and 3.) the W_+, W_- encode the broken directions in $\text{SU}(N)$. The broken directions in $\text{SU}(N)$ can be further parametrized as

$$
W_\pm = \frac{i}{2} e^{\pm i \sum_I' \varphi_I h^I} M_\pm(r) e^{\mp i\phi}(d\theta \mp i\sin\theta d\phi) + w_\pm(x)\,,
\tag{3.19}
$$

where $W_+ = (W_-)^\dagger$ and the sum $\sum_I'$ runs over the generators of $\text{U}(1)^r$ in G_{Φ_∞}. Here $\varphi_I(x)$ are periodic-valued dynamical fields that parametrize the phase of the $a_I(r)$ for which $\lim_{r\to\infty} a_I = 0$ and $w_\pm$ are complex 1-form fields that take value in the broken directions in $\text{SU}(N)$.

Plugging this parametrization of the gauge field into the action, we find that the $w_\pm$ degrees of freedom obtain a mass:

$$
\begin{aligned}
\mathcal{L} &= \ldots + \frac{1}{2}\Big|[W^+, \Phi]\Big|^2 + \frac{1}{2g^2}\Big|[W^+, W^-]\Big|^2 = \ldots + \frac{1}{2}\Big|[w_+(x), \Phi]\Big|^2 + \frac{1}{2g^2}\Big|[w_+(x), M_-]\Big|^2 \\
&= \ldots + \frac{1}{2}m^2(x)|w_+|^2(x)\,.
\end{aligned}
\tag{3.20}
$$

Here we see that the spatially dependent mass has two components. The first comes from the non-trivial Higgs vev which gives a mass away from the core. The second comes from the profile of $M_\pm(r)$ which contributes to the mass inside the core.[10] Thus, all $w_\pm$ acquire a mass of order m_W and are completely frozen out in the IR. '

Thus, the φ_I, which are massless, are the only dynamical degrees of freedom from the broken directions in SU(N). However, since the φ_I are phases of the $a_I(r)$ that exponentially decay away from the monopole core, the φ_I are themselves confined to the monopole core. This is clear from their kinetic term that comes from the kinetic term for the gauge field:

$$\text{Tr}\left\{(\partial_t W_+)(\partial_t W_-)\right\} = \sum_I a_I^2(r)(\dot\varphi_I)^2\,. \tag{3.21}$$

In the limit $m_W \to \infty$, we see that the above kinetic term is approximately

$$\text{Tr}\left\{(\partial_t W_+)(\partial_t W_-)\right\} \sim \sum_I e^{-2\Delta_I m_W r}(\dot\varphi_I)^2\,. \tag{3.22}$$

Again, we see that this is peaked near $r = 0$ and the integral scales like $1/m_W$ which leads to a kinetic term

$$S_{\text{kin}} = \int \frac{1}{2}\dot\varphi_I^2\;\delta(m_W r)dr\,dt = \int \frac{\dot\varphi_I^2}{2m_W}\delta(r)dr\,dt\,. \tag{3.23}$$

Here $\varphi_I \sim \varphi_I + 2\pi$ is a dimensionless field. The fact that the kinetic term has a prefactor $1/m_W$ implies that we should interpret φ_I as the angular coordinate for a particle of mass m_W that is moving on a circle of radius $1/m_W$. This can be seen from rescaling $\varphi_I \to m_W \varphi_I$ while changing the periodicity $\varphi_I \longmapsto \varphi_I + 2\pi/m_W$:

$$S_{\text{kin}} \longmapsto \int \frac{1}{2}m_W^2 \dot\varphi_I^2\;\delta(m_W r)dr\,dt = \int \frac{1}{2}m_W \dot\varphi_I^2(t)dt\,. \tag{3.24}$$

In addition to the kinetic term for φ_I, the kinetic term of the SU(N) gauge field also contains a coupling to the bulk gauge fields:

$$\text{Tr}\left\{\partial_t W_+[A_0, W_-]\right\} = a_I^2(r)\dot\varphi_I A_0\,, \tag{3.25}$$

where here A_0 is the bulk U(1) gauge field under which φ_I is charged. This leads to a coupling term in the quantum mechanics

$$S_{A_0} = \int A_0\,\dot\varphi_I\;\delta(m_W r)dr\,dt \longmapsto \int A_0\,\dot\varphi_I \delta(r)dr\,dt\,, \tag{3.26}$$

where here we have rescaled φ_I as above.

This coupling induces an electric source term in the bulk equations of motion

$$\nabla \cdot E \approx \dot\varphi_I \delta^{(3)}(x)\,, \tag{3.27}$$

which means that φ_I-excitations elevate the monopole to a dyon. The contribution of the long range electric field to the energy is of order of the cutoff scale which is of order of m_W.

---

[10]The contribution of $M_\pm(r)$ to the mass is always non-trivial due to the fact that $\vec{T}$ generate the maximal SU(2)$_T$ embedding.

This implies that although φ_I is a light field in the sense that its kinetic term is suppressed by $1/m_W$, unlike the $w_\pm$ fields. However, the fluctuations of φ_I source long range electric field that carry a large amount of energy. This implies that the φ_I constitutes a dynamical quantum degree of freedom, but its coupling to the bulk gauge fields energetically forbids long lived excitations.

3.1.1 Example: SU(2) monopole

Now let us explicitly demonstrate the existence of the dyon degree of freedom in the low energy effective theory of SU(2) gauge theory in the minimal monopole background.

Here we parametrize the Lie algebra $\mathfrak{su}(2)$ by $\{H, E^\pm\}$ which satisfy

$$[H, E^\pm] = \pm 2 E^\pm, \quad [E^+, E^-] = H. \tag{3.28}$$

The monopole field configuration is explicitly given by

$$A_{\mathrm{SU}(2)} = H A_{\mathrm{Dirac}} \pm \frac{i}{2}\left(\frac{m_W r}{\mathrm{csch}\,(m_W r)}\right) e^{\mp i\phi}(d\theta - i\sin\theta d\phi)E^\pm,$$
$$\Phi = m_W(\coth(m_W r) - 1/r)H. \tag{3.29}$$

Now we will parametrize the dynamical fields as

$$A_{\mathrm{SU}(2)} = H(A_{\mathrm{Dirac}} + \hat{A}_{\mathrm{U}(1)}) + E^+\left(\hat{w}_+ + \frac{i}{2}\left(\frac{m_W r}{\mathrm{csch}\,(m_W r)}\right) e^{-i\phi + i\hat{\varphi}}(d\theta \mp i\sin\theta d\phi)\right) + c.c.,$$
$$\Phi = H\left(\hat{\rho} + m_W\left(\coth(m_W r) - \frac{1}{m_W r}\right)\right) + \hat{\phi}_+ E^+ + c.c., \tag{3.30}$$

where all dynamical fields are hatted.

Now let us plug in our field parametrization into the action

$$S = \int \frac{1}{4}\mathrm{Tr}\left\{F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}|D_\mu\Phi|^2 + \lambda[\Phi^\dagger, \Phi]^2\right\}d^4x. \tag{3.31}$$

Now we see that $\hat{\phi}_\pm$ from the quartic Φ-interaction and coupling to the W-bosons and $\hat{w}_\pm$ get a mass from the kinetic terms for $A_{\mathrm{SU}(2)}$ and Φ:

$$m_{\phi_\pm}^2, m_{w_\pm}^2 = m_W^2 r^2\left(\coth(m_W r) - \frac{1}{m_W r}\right)^2 + m_W^2 r^2\,\mathrm{csch}\,^2(m_W r) \gtrsim O(m_W^2). \tag{3.32}$$

Thus, these fields are integrated out and the $\hat{w}_\pm$ boson eats the Goldstone mode of $\hat{\phi}_\pm$. We now find that the low energy theory is described by the radial mode $\hat{\rho}$ (which is not important for our story), the long range U(1) gauge field $A_{\mathrm{U}(1)}$, and the dyon mode $\hat{\varphi}$.

The only terms that arise in the action that involve φ arise from the derivatives in the kinetic term:

$$S_\varphi = \int \frac{1}{8\pi}\mathrm{Tr}\left\{\partial_\mu W_+ \partial^\mu W_- + 2g[A^\mu, W_+]\partial_\mu W_-\right\}d^4x$$
$$= \frac{1}{4\pi}\int m_W^2 r^2\,\mathrm{csch}\,^2(m_W r)\left(\frac{1}{2}(\partial_\mu\varphi)^2 + 2g A^\mu \partial_\mu\varphi\right)d^2\Omega\,dr\,dt, \tag{3.33}$$

where $A = A_{\mathrm{U}(1)}$. In the IR limit $m_W \to \infty$, we find that $m_W^2 r^2 \operatorname{csch}^2(m_W r) \to \delta(m_W r)$ and all spatial derivatives of φ are suppressed. This leads to a contribution to the low energy effective action

$$
\begin{aligned}
S_\varphi &= \frac{1}{4\pi} \int m_W^2 r^2 \operatorname{csch}^2(m_W r) \left(\frac{1}{2} \dot{\varphi}^2 + g A_0 \dot{\varphi} \right) d^2\Omega \, d^2 x \\
&= \int \delta(m_W x) \left(\dot{\varphi}^2 + g A_0 \dot{\varphi} \right) d^2 x = \int \left(\frac{1}{2} m_W \dot{\varphi}^2 + g A_0 \dot{\varphi} \right) dt \,,
\end{aligned}
\tag{3.34}
$$

where in the final step we rescaled $\varphi \to m_W \varphi$. Here we see that the dyon degree of freedom contributes to the effective low energy theory as a quantum mechanical rotor localized on the monopole world volume. A similar computation holds for the case of dyon degrees of freedom on a general spherically symmetric monopole.

4 Low energy fermions

Now we would like to derive the interaction between the low energy fermion modes and the dyon degrees of freedom. Due to the fact that the long range electric field suppresses long lived excitations of the dyon field, any fermion-dyon interaction should be integrated out to give an effective boundary condition for the long range fermion fields.

One way to determine the fermion-dyon interaction is from the explicit form of the fermion zero-modes of the full UV Dirac equation. In solving the Dirac equation, we are treating the gauge field as a classical background so that the solutions encode the tree level interaction of the fermions with the monopole. We can then determine the dyon interaction by matching the low energy fermions onto the large r limit of the fermion zero-modes and imposing conservation of gauge symmetry.

4.1 Fermion and angular momentum

We would now like to solve for the fermion zero-modes in the presence of a spherically symmetric monopole. The spherical symmetry implies that the fermions can also be decomposed into representations of the total angular momentum generators which in the vector gauge are written:

$$
\vec{J} = \vec{L} + \vec{S} + \vec{T} = -i\vec{r} \times \vec{\nabla} + \frac{1}{2}\vec{\sigma} + \vec{T}
\tag{4.1}
$$

which generate the group $\mathrm{SU}(2)_J$.

Let us consider a Weyl fermion ψ_R that transforms under $\mathrm{SU}(N)$ with representation R. Since $\vec{J}$ is the sum of three commuting generators, we find that the $\mathrm{SU}(2)_J$-representations will be composed of the tensor products of $\vec{L}, \vec{S}, \vec{T}$. In general, this leads to representations of $\mathrm{SU}(2)_J$ with non-trivial multiplicity for fixed R due to the fact that generically many $\mathrm{SU}(2)_L$ representations can produce a $\mathrm{SU}(2)_J$ representation of fixed spin.

To solve the Dirac equation, we will both gauge transform from the vector to the canonical gauge and implement a frame rotation (See (A.4)). These transformations cause

the rotation generators to transform due to the fact that $\vec{\nabla}, \vec{T}, \vec{S}$ are not gauge/frame invariant:

$$\vec{J} = -i\vec{r} \times \left(\vec{\partial} + 2iT_3 \vec{A}_{\text{Dirac}} - i\sigma^3 \frac{1}{2} \cot\theta \hat{\phi} \right) + \left(T_3 + \frac{1}{2}\sigma_3 \right) \hat{r} \,. \tag{4.2}$$

As discussed in [28], we can construct the representations of $\text{SU}(2)_J$ by diagonalizing J^2, J_3 which have the explicit forms:

$$J^2 = -\frac{1}{2} \left[D^-_{T_3 + \frac{\sigma^3}{2} + 1} D^+_{T_3 + \frac{\sigma^3}{2}} + D^+_{T_3 + \frac{\sigma^3}{2} - 1} D^-_{T_3 + \frac{\sigma^3}{2}} \right] + \left(\frac{1}{2}\sigma_3 + T_3 \right)^2 \,, \tag{4.3}$$

$$J_3 = -i(\partial_\phi + iT_3) \,,$$

where

$$D^\pm_q = \partial_\theta \mp \frac{1}{\sin\theta} \Big(i(\partial_\phi + iT_3) + q\cos\theta \Big) \,. \tag{4.4}$$

To simultaneously diagonalize these, consider a decomposition of the representation R with respect to irreducible representations of $\text{SU}(2)_T$:

$$R = \bigoplus_A R_A \quad \Rightarrow \quad \psi_R = \sum_A \psi_{R_A} \,. \tag{4.5}$$

The basis of the representation space V_{R_A} can be indexed by weights

$$V_{R_A} = \text{span}_{\mathbb{C}}\{v_{\mu_a}\} \,, \quad \mu_a \in \Delta_{R_A} \,, \tag{4.6}$$

where Δ_{R_A} is the weight space of the representation R_A. Now if we decompose

$$\psi_{R_A} = \sum_a \psi^a_{R_A} v_{\mu_a} \,, \tag{4.7}$$

so that the $\psi^a_{R_A}$ have charges q_a where

$$R_A(T_3) \cdot v_{\mu_a} = q_a v_{\mu_a} \,, \quad \langle T_3, \mu_a \rangle = \frac{q_a}{2} \,, \tag{4.8}$$

then we find that the eigenfunctions of J^2, J_3 are given by

$$\psi^{a\,(j)}_{R_A} = e^{i(m - q_a/2)\phi} \begin{pmatrix} f_{a,+}(r)\, d^j_{-m, \frac{q_a+1}{2}}(\theta) \\ f_{a,-}(r)\, d^j_{-m, \frac{q_a-1}{2}}(\theta) \end{pmatrix} \,, \tag{4.9}$$

with quantum numbers

$$J^2 \psi^{(j)}_{R_A} = j(j+1)\psi^{(j)}_{R_A} \,, \quad J_3 \psi^{(j)}_{R_A} = m\psi^{(j)}_{R_A} \,, \tag{4.10}$$

where $d^j_{m,q}(\theta)$ are small Wigner D-functions. These representations are restricted by the fact that $d^j_{m,q}(\theta)$ is only well defined for $|m|, |q| \leq j$ and $m - j, q - j \in \mathbb{Z}$. These restrictions imply that the allowed set of j, m are given by

$$j = \frac{2q_{R_A} + 1}{2} + n_j \geq 0 \,, \quad n_j \in \mathbb{Z} \,, \tag{4.11}$$

$$m = -j, -j+1, \ldots, j-1, j \,.$$

where the representation R_A is a spin-q_{R_A} representation of $SU(2)_T$. Note that when $n_j < 0$ that we must impose additional restrictions on the $f_{a,\pm}(r)$:

$$f_{a,\pm}(r) := 0 \quad \text{if} \quad \left| \frac{q_a \pm 1}{2} \right| > j \,. \tag{4.12}$$

which give $SU(2)_J$ representations a nested structure. To facilitate this we will define the spin-j restricted weight space

$$\Delta_{R_A}^{(j)} = \left\{ \mu_a \in \Delta_{R_A} \ \middle| \ |\langle \mu_a, T_3 \rangle| \leq j + 1/2 \right\} \,, \tag{4.13}$$

which is simply the representation R_A restricted to the subspace with non-trivial $\psi_{R_A}^{(j)}$. In other words, the spin-j mode of ψ_{R_A} is restricted to the component fields $\psi_{R_A}^a$ such that $\left| \frac{q_a}{2} \right| < j$.

One way to make use of the representations of angular momentum is in solving for the fermion modes in abelian monopole background. In order to make contact with the representation theory above, we will gauge transform the standard abelian monopole background:

$$A_{ab.} = \gamma_m A_{\text{Dirac}} \,, \tag{4.14}$$

by $g_I = e^{-i\phi I_3} e^{i\theta I_2} e^{i\phi I_3}$. to the equivalent of the canonical gauge:

$$A_{\text{can.}} = 2T_3 A_{\text{Dirac}} \pm i \frac{I_\pm}{2} e^{\mp i\phi} (d\theta \mp i \sin\theta d\phi) \,. \tag{4.15}$$

This is similar to the full non-abelian gauge field in the previous section except that $M_\pm(r)$ is replaced by its asymptotic value $M_\pm(r) \to I_\pm$ which encodes the long range abelian monopole. Note that this canonical abelian gauge is a good gauge in the IR theory with gauge group G_{Φ_∞}.

The zero-modes of the abelian Dirac operator in the canonical gauge are solved for in [28]. They are somewhat cumbersome due to the fact that they involve Clebsh-Gordon coefficients, but can be simplified by a coordinate independent gauge transformation. Adapting the computation, we find that the spectrum of the Dirac operator can be written

$$\psi_{\mu_a}^{(k,m)} = \frac{U(\theta,\phi)}{r} e^{i(m-q_a/2)\phi} \begin{cases} e^{ik(t+r)} \begin{pmatrix} 0 \\ d_{-m,j_a}^{j_a}(\theta) \end{pmatrix} & q_a > 0 \\ e^{ik(t-r)} \begin{pmatrix} d_{-m,-j_a}^{j_a}(\theta) \\ 0 \end{pmatrix} & q_a < 0 \end{cases} \qquad j_a = \frac{|q_a| - 1}{2} \tag{4.16}$$

where again $d_{m,q}^j(\theta)$ is a small Wigner-D function.

In this gauge, we find that the low energy fermion modes can again be described by the 2D fields $\chi_{\mu_a,m}(t,r)$ corresponding to the expansion

$$\psi_{\mu_a} = \frac{U(\theta,\phi)}{r} \sum_{m=-j_a}^{j_a} \left(e^{i(m-q_a)\phi} d_{-m,j_a}^{j_a}(\theta) \right) \chi_{\mu_a,m}(t,r) \,. \tag{4.17}$$

See appendix A for more details.

4.2 Fermion zero-modes

Using the parametrization of a generic $SU(2)_J$ representation in (4.9) acting on a $SU(2)_T$ representation R_A, the smooth spin-j solutions of the Dirac equation are given by

$$\psi_{R_A}^a = e^{i(m - q_a/2)\phi} \begin{pmatrix} f_{a,+}(r)\, d^j_{-m, \frac{q_a+1}{2}}(\theta) \\ f_{a,-}(r)\, d^j_{-m, \frac{q_a-1}{2}}(\theta) \end{pmatrix}, \tag{4.18}$$

where the functions $f_{a,\pm}(r)$ are given by

$$F(r) = P \exp\left\{-\int \mathcal{D}(r)dr\right\} F_0, \quad F(r) = \sum_{\mu_a \in \Delta^{(j)}_{R_A}} \begin{pmatrix} f_{a,+}(r) \\ f_{a,-}(r) \end{pmatrix} v_{\mu_a}, \tag{4.19}$$

for

$$\mathcal{D}(r) = -\sqrt{(j+1/2)^2 - \frac{q_a^2}{4}}\,\sigma^1 + \sigma^- M_+ + \sigma^+ M_- + 1, \tag{4.20}$$

and F_0 a constant vector. As proven in [28], there exists a unique vector F_0 that corresponds to a smooth, plane-wave normalizable solution of spin-j:

$$\lim_{r \to \infty} r\psi_{R_A} < \infty, \tag{4.21}$$

of the time-independent Dirac equation iff the top and bottom weights of the spin-j representation of R_A:

$$\langle \mu_{\text{top}}, T_3 \rangle = -\langle \mu_{\text{bot}}, T_3 \rangle = 2j + 1, \quad \mu_{\text{top}}, \mu_{\text{bot}} \in \Delta^{(j)}_{R_A}, \tag{4.22}$$

have charge of opposite sign under γ_m:

$$\langle \mu_{\text{top}}, \gamma_m \rangle \langle \mu_{\text{bot}}, \gamma_m \rangle < 0. \tag{4.23}$$

In total, one finds that a fermion ψ_R in representation R of $SU(N)$ has a total number of fermion zero-modes:

$$\dim_{\mathbb{C}}\left[\ker[\bar{\sigma}^\mu D_\mu]\right] = \lim_{\epsilon \to 0^+} \sum_{\mu \in \Delta_R} n_R(\mu)\langle \mu, T_3 \rangle \,\text{sign}(\langle \mu - \epsilon\, w(\mu), T_3 - I_3 \rangle), \tag{4.24}$$

which decompose as $SU(2)_J$ representations as

$$\psi_R \to \bigoplus_{\mu \in S_R} [j_\mu]^{\oplus n_R(\mu)}, \quad j_\mu = \frac{\langle \mu, T_3 \rangle - 1}{2}, \tag{4.25}$$

where $[j]$ is the spin-j representation and the sum runs over the set of weights:

$$S_R = \left\{\mu \in \Delta_R^+ \;\middle|\; \langle \mu, T_3 - I_3 \rangle \times \langle w(\mu), T_3 - I_3 \rangle) < 0 \right\}, \tag{4.26}$$

where w is the unique element of the Weyl group of $\mathfrak{su}(2)_T \subset \mathfrak{su}(N)$ Here we will also use the notation $\mu^* = w(\mu)$. Note that $\mu^*_{\text{top}} = \mu_{\text{bot}}$.

In order to match onto the IR fermion modes, we will also need the explicit asymptotic behavior of the zero-mode solutions. Let us consider the spin-j zero-mode of ψ_{R_A}. The spin-j representation has a highest and lowest weight $\mu_{\text{top}}, \mu_{\text{bot}} \in \Delta_{R_A}^{(j)}$. Up to coordinate independent gauge transformations, the spin-j zero-mode of ψ_{R_A} has asymptotic behavior

$$\lim_{r \to \infty} \psi_{R_A} \sim \frac{1}{r} \left(\psi_{R_A}^+ v_{\mu_{\text{top}}} - \psi_{R_A}^- v_{\mu_{\text{bot}}} \right) + O(1/r^{1+\delta}), \quad \delta > 0, \tag{4.27}$$

where

$$\sigma^0 \bar{\sigma}^r \psi_{R_A}^\pm = \pm \psi_{R_A}^\pm, \tag{4.28}$$

are spin polarized with respect to the radial direction. These $\psi_{R_A}^\pm$ are the top/bottom components of the spin-j representation and hence match onto the zero-momentum scattering modes of $\chi_{\mu_{\text{top/bot}}, m}(t, r)$ in the IR theory. See appendix A for more details.

Interestingly, in the core of the monopole, the solutions to the Dirac equation have the form

$$\lim_{r \to 0} \psi_{R_A} = \sum_\ell r^\ell \psi_{R_A}^{(\ell)}, \quad J^2 \psi_{R_A} = j(j+1) \psi_{R_A}, \tag{4.29}$$

where $\psi_{R_A}^{(\ell)}$ is the spin-j representation with orbital angular momentum ℓ. The minimum ℓ given a fixed j, R_A is

$$\ell_{\min} = \dim[R_A] - j - \frac{1}{2}, \tag{4.30}$$

which implies that fermion zero-modes generically vanish at $r \to 0$ except for solutions with spin-$j = \dim[R_A] - \frac{1}{2}$.

This leads to a confusing point. It is often said that the only the spin-$j = 0$ modes of the fermion interact with the dyon degrees of freedom because their wave function has non-trivial support in the core. However, we see here that this is not exactly true: the modes with spin-$j = \dim[R_A] - \frac{1}{2}$ have no angular momentum barrier — this only corresponds to the s-wave in the case of a minimal monopole and fermions of sufficiently small representation.

In the next section we will show that the emphasis on the angular momentum repulsion from the core is a red herring: even plane-wave normalizable the modes that vanish at the origin still interact the dyon degrees of freedom.

4.3 Low energy effective action

Now we would like to use the explicit form of the fermion zero modes to derive the fermion-dyon coupling. As we discussed in section 2, the low energy limit of the fermions are described by the effective 2D fields $\chi_{\mu,m}(t, r)$.

From the asymptotics of the fermion zero-modes in the previous section, it is clear that the long range degrees of freedom are related by a boundary condition

$$\chi_{\mu,m}(t, r = 0) \leftrightarrow \chi_{\mu^*, m}(t, r = 0), \tag{4.31}$$

where $\chi_{\mu^*, m} = \chi_{w(\mu), m}$ is the charge conjugate of $\chi_{\mu, m}$ with respect to SU(2)$_T$.
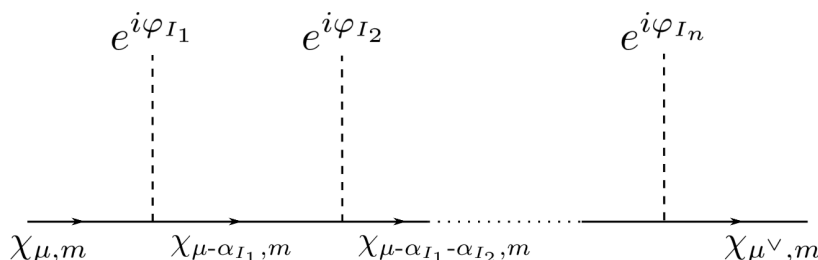
Figure 2. This figure illustrates the Feynman diagram that gives rise to the interaction (4.33). Here $\{\alpha_I\}$ are a set of simple positive roots.

However, the boundary condition cannot be simply $\chi_{\mu,m}(t, r = 0) = \chi_{\mu^*,m}(t, r = 0)$ due to the fact that it does not preserve gauge invariance under the IR gauge group. To fix this, the boundary condition must be supplemented by a charged degree of freedom in the core. Since the 4D fermions only couple to the gauge field, this charged degree of freedom must come from the broken W-bosons. And further, as we demonstrated in the previous section, the only light charged degree of freedom in the core is the dyon degree of freedom. This implies that the boundary condition must be of the form

$$\chi_{\mu,m}(t, r = 0) = e^{ic_\mu^I \varphi_I} \chi_{\mu^*,m}(t, r = 0) \,, \tag{4.32}$$

where c_μ^I is a collection of integers such that the above boundary conditions are gauge invariant. This boundary condition can be encoded by the boundary term:

$$L_{\text{bnd}} = \sum_\mu \int \delta(r) e^{ic_\mu^I \varphi_I} \chi_{\mu,m}^\dagger \chi_{\mu^*,m} \, d^2x + c.c. \,. \tag{4.33}$$

Let us now show how this boundary condition arises. As it turns out, this boundary condition comes from a tree level interaction. If we expand the asymptotic form of the spin-j 4D fermions in $SU(2)_J$-representations

$$\psi_{R_A}^{(j),a} = \frac{U(\theta,\phi)}{r} \sum_m e^{i(m - q_a/2)\phi} \left(d_{-m, \frac{q_a+1}{2}}^j(\theta) \chi_{\mu_a,m}^{(j),+}(t, r) + d_{-m, \frac{q_a+1}{2}}^j(\theta) \chi_{\mu_a,m}^{(j),-}(t, r) \right), \tag{4.34}$$

where $\chi_{\mu_{\text{top}},m}^+, \chi_{\mu_{\text{bot}},m}^- = 0$, then we find upon plugging into the action with the low energy W-boson degrees of freedom (3.19) that the couplings to φ_I are of the form

$$L_{\text{bnd}} = \sum_{I,\mu_i,m} \int \alpha_I(r) e^{i\varphi_I} (\chi_{m,\mu_i}^{(j),+})^\dagger \chi_{m,\mu_i-\alpha_I}^{(j),-} dr + c.c. \,, \tag{4.35}$$

where here the $\chi_{m,\mu_i}^{(j),\pm}$ carry a non-trivial wave function dependence at small $m_W r$.

We know from the explicit zero-mode solutions that only $\chi_{m,\mu_{\text{top}}}^{(j),-}$ and $\chi_{m,\mu_{\text{bot}}}^{(j),+}$ correspond to long range fermion modes. Thus, we should integrate out the other fermion modes. Doing so leads to the interaction (4.33) as shown in the Feynman diagram in figure 2.

This tree level interaction is reflected in the structure of the fermion zero-modes. We can think of the virtual excitations of the fermion zero-modes that give rise to the above

diagram as imprinted in the sub-leading structure of the zero-mode $\psi_{R_A}^{(j)}$ that vanishes at $r \to \infty$:

$$\lim_{r \to \infty} r \psi_{R_A}^{(j)} = \psi_{\mu_{\text{top}}}^{(j)} + \psi_{\mu_{\text{bot}}}^{(j)} + \underbrace{\sum_{\mu_a \neq \mu_{\text{top/bot}}} \frac{1}{r^{\delta_a}} \psi_{\mu_a}^{(j)}}_{\text{virtual}} \quad \delta_a > 0 \,. \tag{4.36}$$

Here, the Feynman diagram clearly shows that the interaction above for fixed R_A and j couples $\psi_{\mu_{\text{top}}}^{(j)} \psi_{\mu_{\text{bot}}}^{(j)}$ to $e^{ic_\mu^I \varphi_I}$ where c_μ^I counts the number of E_I^- required to map $\mu \to \mu^*$

$$\prod_I (E_I^-)^{c_\mu^I} v_\mu = v_{\mu^*} \,, \tag{4.37}$$

and is thus manifestly positive by the highest weight construction. Thus, we generally find interactions of the form

$$S_{\text{int}} = \sum_{m,\mu \in S_{R_A}} \int d^2x \, g_\mu(r) e^{ic_\mu^I \varphi_I} \chi_{\mu,m}^\dagger \chi_{\mu^*,m} + c.c. \tag{4.38}$$

where here $\chi_{\mu,m}$ are the long range fields and $g_\mu(r)$ is a function that encodes the overlap of the 4D fermion modes and the dyon degree of freedom.

We now need to worry about the fact that higher angular momentum modes fermion modes vanish at the origin as we previously alluded to. Recall that the dyon degree of freedom is exponentially localized to the monopole core while the long range fermion modes are generically repulsed:

$$a_I(r) \sim e^{-\Delta_I m_W r} + \dots, \quad f_{\mu_{\text{top}},-}, f_{\mu_{\text{bot}},+}(r) \sim (m_W r)^\ell + \dots. \tag{4.39}$$

However, even though the dyon degree of freedom becomes infinitely localized to the monopole core the wavefunction overlap does not vanish in the limit of $m_W \to \infty$.

To show this, it is sufficient to note that the integral of the coupling function

$$\int g_\mu(r) dr = \int \prod_I (a_I(r))^{c_{\mu_{\text{top}}}^I} f_{\mu_{\text{top}},-}^\dagger(r) f_{\mu_{\text{bot}},+}(r) \frac{dr}{r} \,, \tag{4.40}$$

can be recast as an integral over the dimensionless quantity $x = m_W r$ and therefore is independent of the limit $m_W \to \infty$. Further, since the function can be well approximated by the leading behavior, we can see that the integral is non-zero[11]

$$\int g_\mu(r) dr \simeq \int m_W e^{-c_\mu^I \Delta_I m_W r} (m_W r)^{2\ell} \frac{dr}{r} \approx \frac{\Gamma(2\ell)}{(c_\mu^I \Delta_I)^{2\ell}} \neq 0 \,. \tag{4.41}$$

Since this integral is independent of m_W, non-zero, and the integrand is exponentially peaked near $m_W r = 0$, we see that in the IR limit where $m_W \to \infty$ we can approximate $g_\mu(r) = \delta(r)$. Here the normalization of $g_\mu(r)$ can be fixed by the asymptotic form of the zero-mode spin-j zero-mode relating $\chi_{\mu_{\text{top}},m}^{(j)-}$ and $\chi_{\mu_{\text{bot}},m}^{(j)+}$.

---

[11]In the case where $\ell = 0$ the coupling derived here requires regulation. The reason why the regulated integral makes sense here is that we could alternatively have absorbed the coupling to $\varphi_I c_\mu^I$ into ψ_μ, ψ_{μ^*} so that the singular term in question is simply the coupling to the background gauge field. Then since the ψ_μ are assumed to solve the Dirac equation, we see that the form of the f_μ, a_I must be so that seemingly singular term contributes no singularity. This is an alternative derivation of the coupling (4.57).

4.3.1 Fermion-dyon coupling

Now we will explicitly determine the couplings c_μ^I. As we discussed, c_μ^I is fixed by gauge invariance and can be computed explicitly in terms of $SU(N)$ representation theory.

Let us restrict to an irreducible $SU(2)_T$ component ψ_{R_A} of ψ_R. Generically, only a subset of the positive weights have corresponding fermion zero-modes: those that belong to the subspace S_{R_A}. So, for each weight $\mu \in S_{R_A}$, there exists a fermion zero-mode of spin-$j = \langle \mu, T_3 \rangle - \frac{1}{2}$. The coupling c_μ^I for such a weight is defined to be the number of lowering operators necessary to take $v_\mu \to v_{\mu^*}$ where again μ^* is the charge conjugate of μ with respect to $SU(2)_T$.[12]

To facilitate this calculation, we will use Dynkin indices for representations of $SU(N)$. Here a representation R_Λ of highest weight Λ is denoted by a vector:

$$\Lambda = \sum_{I=1}^{N-1} n_I \lambda^I \longmapsto [n_1, n_2, \dots, n_{N-1}], \quad (\lambda^I, \lambda^J) = C^{IJ}, \tag{4.42}$$

where the λ^I are a basis of positive simple weights of $SU(N)$, C^{IJ} is the inverse Cartan matrix, and $(\,,\,)$ is the Killing form on $\mathfrak{su}(N)$. For $SU(N)$ the raising and lowering operators $E_I^\pm$ act on a vector v_μ by shifting the weights as

$$
\begin{aligned}
E_I^\pm v_\mu &= v_{\mu \pm \alpha_I}\,, \\
\mu &= [n_1, \dots, n_{I-1}, n_I, n_{I+1}, \dots, n_{N-1}]\,, \\
\mu \pm \alpha_I &= [n_1, \dots, n_{I-1} \mp 1, n_I \pm 2, n_{I+1} \mp 1, \dots, n_{N-1}]\,.
\end{aligned}
\tag{4.43}
$$

All of the $SU(N)$ representation R_Λ are expressible by successive action of the E_I^- on the vector of highest weight v_Λ. This is the standard construction of highest weight representations.

It will also be convenient to introduce a notation for the vectors in the dual space (co-root lattice):

$$\widetilde{\Lambda} = [\tilde{n}_1, \dots, \tilde{n}_{N-1}]^\vee\,, \tag{4.44}$$

where $\langle \widetilde{\Lambda}, \Lambda \rangle = (\widetilde{\Lambda}^\vee, \Lambda)$ for

$$\widetilde{\Lambda}^\vee = \sum_{I,J} \tilde{n}_I C_{IJ} \lambda^J\,. \tag{4.45}$$

In this notation, we can also encode T_3, I_3, and γ_m as the vectors

$$T_3 = [t_1, t_2, \dots, t_{N-1}]^\vee\,, \quad I_3 = [i_1, i_2, \dots, i_{N-1}]^\vee\,, \quad \gamma_m = [\gamma_1, \gamma_2, \dots, \gamma_{N-1}]^\vee\,, \tag{4.46}$$

which are related by

$$\gamma_m = T_3 - I_3 = [t_1 - i_1, t_2 - i_2, \dots, t_{N-1} - i_{N-1}]^\vee\,. \tag{4.47}$$

---

[12]More precisely, $\mu^* = w(\mu)$ where w is the unique element of the Weyl group of $SU(2)_T$. This is the "charge conjugate" because it maps $q_\mu = \langle T_3, \mu \rangle \to q_{\mu^*} = -q_\mu$.

The action of the above Lie algebra generators on a vector with weight $\Lambda = [n_1, \ldots, n_{N-1}]$ can easily be computed in this notation: for example

$$\langle T_3, \Lambda \rangle = \sum_I n_I t_I. \tag{4.48}$$

We can now easily compute the couplings to the different W-bosons. Given a weight $\Lambda = [n_1, \ldots, n_{N-1}]$, the charge conjugate weight is given by

$$\Lambda \to \Lambda^* = [-n_{N-1}, \ldots, -n_1]. \tag{4.49}$$

The number of lowering operators required to map a positive weight $\Lambda \to \Lambda^*$ is then given by

$$C_\Lambda^I = \left(C_\Lambda^1, \, C_\Lambda^2, \, C_\Lambda^3, \, \ldots, \, C_\Lambda^{\lfloor \frac{N-1}{2} \rfloor}, \, C_\Lambda^{\lfloor \frac{N-1}{2} \rfloor + 1}, \, \ldots, \, C_\Lambda^{N-1} \right), \tag{4.50}$$

where C_Λ^I is the number of E_I^- required to go from $\Lambda \mapsto \Lambda_\vee$ which are given explicitly by

$$
\begin{aligned}
C_\Lambda^1 &= \sum_I n_I \\
C_\Lambda^2 &= (n_1 + n_{N-1}) + 2 \sum_{I \neq 1, N-1} n_I \\
C_\Lambda^3 &= (n_1 + n_{N-1}) + 2(n_2 + n_{N-2}) + 3 \sum_{I \neq 1, 2, N-1, N-2} n_I \\
&\vdots \\
C_\Lambda^{\lfloor \frac{N-1}{2} \rfloor} &= \sum_M M(n_M + n_{N-M}) \\
C_\Lambda^{\lfloor \frac{N-1}{2} \rfloor + 1} &= \sum_{M \neq \lfloor \frac{N-1}{2} \rfloor} M(n_M + n_{N-M}) + \left(n_{\lfloor \frac{N-1}{2} \rfloor} + n_{N - \lfloor \frac{N-1}{2} \rfloor} \right) \\
&\vdots \\
C_\Lambda^{N-1} &= \sum_I n_I.
\end{aligned}
\tag{4.51}
$$

However, since the low energy effective theory generically has non-abelian gauge symmetry, the E_I^- that are not along broken directions in the SU(N) gauge group in some sense "come for free" due to coupling to long range gluons. Thus, the coupling c_Λ^I to the dynamical dyon degrees of freedom is simply the truncation of the C_Λ^I to the set of I such that $i_I = 0$.

We then find that the low energy effective theory of fermions and dyons is given by

$$
\begin{aligned}
S = \sum_{\mu,m} \int d^2x \Big[& i\chi_{\mu,m}^\dagger \partial_- \chi_{\mu,m} + i\chi_{\mu^*,m}^\dagger \partial_+ \chi_{\mu^*,m} + i\delta(r) e^{ic_I^\mu \varphi_I} \chi_{\mu,m}^\dagger \chi_{\mu^*,m} + c.c. \Big] \\
& + \sum_I \int dt \, \frac{1}{2} m_W \, \dot{\varphi}_I^2.
\end{aligned}
\tag{4.52}
$$

We can interpret the dyon interaction term as a boundary term that imposes the boundary condition

$$\chi_{\mu^*,m} = e^{-ic_I^\mu \varphi_I} \chi_{\mu,m}\big|_{r=0} . \qquad (4.53)$$

We can now "unfold" our theory to recast our theory in a form similar the model studied in [19]. By this, we mean to map the theory with a boundary interaction on the half-plane to a theory with a localized interaction on the full plane. To do this, let us introduce the coordinate

$$x = \begin{cases} -r & x < 0 \\ r & x > 0 \end{cases} \qquad (4.54)$$

and identify the single chiral fermion

$$\Psi_{\mu,m}(t-x) = \begin{cases} \chi_{\mu,m}(t+x) & x < 0 \\ \chi_{\mu^*,m}(t-x) & x > 0 \end{cases} \qquad (4.55)$$

on the (x,t)-plane. We will also use the field redefinition

$$\chi_{\mu,m}\big|_{r=0} \rightarrow e^{i\frac{c_I^\mu}{2}\varphi_I} \chi_{\mu,m}\big|_{r=0} , \quad \chi_{\mu^*,m}\big|_{r=0} \rightarrow e^{-i\frac{c_I^\mu}{2}\varphi_I} \chi_{\mu^{vee},m}\big|_{r=0} \qquad (4.56)$$

to eliminates the above boundary condition/field discontinuity in exchange for the interaction term

$$\mathcal{L}_{\text{int}} = \Theta(r) c_I^{(\mu)} \dot\varphi_I \Psi_{\mu,m}^\dagger \Psi_{\mu,m} , \qquad (4.57)$$

where here $\Theta(r) = \Theta_{\text{heviside}}(r) - \frac{1}{2}$. The unfolded action is now

$$S = \sum_{\mu \in S_R,m} \int d^2x \left[i\Psi_{\mu,m}^\dagger \left(\partial_+ - i\Theta(x)c_\mu^I \dot\varphi_I \right) \Psi_{\mu,m} \right] + \sum_I \int dt \, \frac{1}{2} m_W \dot\varphi_I^2 . \qquad (4.58)$$

4.4 Effective boundary condition

We would now like to examine the dynamics of the effective theory above in order to determine the effective boundary conditions for the fermions. These boundary conditions can be found by simply integrating out the φ_I degrees of freedom. The physics of the dyon-fermion system are very similar to the toy model considered in [19]. Here we will follow a similar analysis to integrate out the φ_I degrees of freedom.

Let us consider the theory described by the action in (4.58). To facilitate the computations in this section we will introduce an index A for the fermions Ψ_A that encodes both $\mu \in S_R$ and $m = -j_\mu, \ldots, j_\mu$. With this notation, the action can be written

$$S = \sum_{\mu \in S_R,m} \int d^2x \left[i\Psi_{\mu,m}^\dagger \left(\partial_+ - i\Theta(x)c_\mu^I \dot\varphi_I \right) \Psi_{\mu,m} \right] + \sum_I \int dt \, \frac{1}{2} m_W \dot\varphi_I^2 . \qquad (4.59)$$

This theory has a series of currents that correspond to a U(1) rotation of each Ψ_A individually:

$$\Psi_B \rightarrow e^{i\phi\delta_{AB}} \Psi_B \quad \Rightarrow \quad J_A(x) = \Psi_A^\dagger(x)\Psi_A(x) . \qquad (4.60)$$

The corresponding symmetries have ABJ-type anomalies and consequently their currents are not conserved. Since the coupling to φ_I appears as a gauge coupling, the conservation equation for the current is given

$$\partial_+ J_A = \dot{\varphi}_I c_A^I \delta(x) \,. \tag{4.61}$$

The action is also manifestly shift invariant under $\varphi_I \to \varphi_I + \beta$ for any constant β. There is a corresponding conserved charge:

$$Q = m_W \dot{\varphi} + \int dx \, \Theta(x) \sum_A J_A(x) \,. \tag{4.62}$$

Together, the conservation equations satisfy:

$$\partial_+ J_A = -\frac{c_A^I}{m_W} \Pi_I \delta(x) \,, \quad \partial_t \Pi_I = \sum_A c_A^I \int dx \, \delta(x) J_A(x) \,, \tag{4.63}$$

where Π_I is the conjugate momentum to φ_I.

Integrating the conservation equations along a line of constant $x - t$ (incoming plane wave) produces

$$
\begin{aligned}
J_A^{(\text{out})} &= J_A^{(\text{in})} - \frac{c_A^I}{m_W} \Pi_I \delta(x) \,, \\
\partial_t \Pi_I &= \sum_A c_A^I J_A^{(\text{in})} - \frac{1}{m_W} \sum_A c_A^I c_A^J \Pi_J \,.
\end{aligned}
\tag{4.64}
$$

which means that the incoming charge differs from the out-going charge by something proportional to the momentum of the periodic scalar.

We can now integrate out the φ_I in which we effectively set

$$\partial_t \langle \Pi_I \rangle = 0 \quad \Rightarrow \quad c_A^I \Pi_I = m_W J_A \,. \tag{4.65}$$

We want to solve this formula for Π_I and plug back into the J_A current conservation equation, but this is in general non-trivial since c_A^I is a non-rectangular matrix. However, the physics of this equation is clear: since c_A^I is the coupling between Π_I, J_A any virtual excitation of the φ_I will decay via the bulk current of charge of the associated gauge charge:

$$S_I = \sum_A c_A^I \int \Theta(x) \dot{\varphi}_I J_A \quad \Rightarrow \quad \varphi_I \to J_I = \frac{1}{N_I} \sum_A c_A^I J_A \,, \tag{4.66}$$

where here N_I is a normalization constant that is fixed by gauge charge conservation.

Each φ_I has charge 2 under the associated $\mathrm{U}(1)_I$ gauge symmetry.[13] For each Ψ_A there is an associated weight $\mu^{(A)} = [\mu_1^{(A)}, \ldots, \mu_{N-1}^{(A)}]$. The charge of the out-going Ψ_A under $\mathrm{U}(1)_I$ is then given by $-\mu_{N-I}^{(A)}$ and the total charge of the current is given by

$$N_I \times Q_I[J_I] = -\sum_A c_A^I \mu_{N-I}^{(A)} \,. \tag{4.67}$$

---

[13]This comes from our choice of convention $[H_I, E_I^\pm] = \pm 2 E_I^\pm$.

Thus, we find that the normalization N_I is given by

$$N_I = -\frac{1}{2} \sum_A c_A^I \mu_{N-I}^{(A)} \,. \tag{4.68}$$

Therefore, the current conservation equation is given by

$$J_A^{(\text{out})} = J_A^{(\text{in})} - \sum_{I,B} \frac{2}{\sum_D c_D^I \mu_{N-I}^{(D)}} c_A^I c_B^I j_B \,. \tag{4.69}$$

Translating into the notation of section 2, the corresponding effective boundary condition is specified by

$$\bar{J}_A = \mathcal{R}_{AB} J_B \,, \quad \mathcal{R}_{AB} = \delta_{AB} - \sum_I \frac{2}{\sum_A c_A^I \mu_{N-I}^{(A)}} c_A^I c_B^I \,. \tag{4.70}$$

Thus, when the dyon is excited by an incoming fermion, the dyonic excitation dissipates through all possible fermion number channels. This can be interpreted as soft, charged radiation made up of fermionic modes that carries away the charge and energy from the monopole. This radiation generically can carry fractional flavor charges since the flavor symmetry is not actually preserved in the quantum theory. This soft radiation is sometimes referred to in the literature as "propagating pulses of vacuum polarization" [19].

5   Preserved symmetries

We would now like to determine what global symmetries are preserved by the boundary conditions specified by a choice of $\mathcal{R}$. We would like to consider a collection of fermions that transform under representations $\{R_i\}$ in the presence of an SU(N) monopole.

Let us consider a U(1) symmetry of the fermion fields that rotates each ψ_{R_i} individually:

$$\psi_{R_i} \to e^{iq_i\phi}\psi_{R_i} \,. \tag{5.1}$$

The charges of the in-going/out-going low energy fermion modes define the vectors $q_A^{(\text{in})}$, $q_A^{(\text{out})}$. The symmetry is preserved iff the in-going charge is equal to the out-going charge:

$$q_A^{(\text{in})} = \mathcal{R}_{AB} \, q_B^{(\text{out})} \,. \tag{5.2}$$

For symmetries that commute with the SU(N) gauge symmetry $q_A^{(\text{in})} = q_A^{(\text{out})} = q_A$ and therefore the associated global symmetry is in fact preserved if q_A is a unit eigenvector of the matrix $\mathcal{R}_{AB}$.

More generally, we can consider IR symmetries that are a linear combination of UV gauge symmetry with UV global symmetries such as $B - L$ symmetry in the standard model. In this case $q_A^{(\text{in})} \neq q_A^{(\text{out})}$ and the condition that the symmetry is preserved by the boundary condition is (5.2).

In general, the boundary conditions specified by a matrix $\mathcal{R}$ can preserve non-abelian symmetries which we can infer by studying the preserved abelian symmetries [22]. Given a choice of $\mathcal{R}$, one can construct an integer lattice $\Lambda_{\mathcal{R}}$ that is generated by the charge vectors

of the preserved abelian symmetries. The boundary conditions specified by $\mathcal{R}$ then preserve a semi-simple non-abelian symmetry group G when $\Lambda_{\mathcal{R}}$ contains the weight lattice of G as a sublattice $\Lambda_{wt}[G] \subset \Lambda_R$.

We would now like to determine under what conditions a global symmetry is violated by the monopole boundary conditions. Due to the UV physics encoded by the monopole boundary conditions it is clear that a global symmetry will be broken if the process

$$\varphi_I \longrightarrow \frac{1}{N_I} \sum_A c_A^I J_A \,, \tag{5.3}$$

violates the global symmetry.

Let us consider a global symmetry $\mathrm{U}(1)_q$ under which a fermion of representation R has charge q_R. Since φ is uncharged under $\mathrm{U}(1)_q$, the violation of $\mathrm{U}(1)_q$ is measured by the charge of the right hand side of (5.3):

$$\Delta Q_q^{(I)} = \frac{1}{N_I} \sum_A q_A^{(\mathrm{out})} c_A^I = \frac{q_R}{N_I} \sum_{\mu \in S_R} m_\mu c_\mu^I \,, \tag{5.4}$$

where m_μ is the number of fermions with weight $\mu \in S_R$. Now, since c_μ^I encodes the gauge charges of the operator $\chi_{\mu,m}^\dagger \chi_{\mu^*,m}$, we can compute c_μ^I as the difference in charge

$$c_\mu^I = Q_I[\psi_\mu] - Q_I[\psi_{\mu^*}] \,, \tag{5.5}$$

under the corresponding $\mathrm{U}(1)_I$ gauge symmetry generated by

$$h^I = \frac{1}{N} \mathrm{diag}\,(\underbrace{N-I, \ldots, N-I}_{I\text{-times}}, \underbrace{-I, \ldots, -I}_{N-I\text{-times}})\,. \tag{5.6}$$

Thus we find that the total charge violation is computed by

$$\Delta Q_q^{(I)} = \frac{1}{N_I} \sum_R \sum_{\mu \in S_R} q_R \left(Q_I[\psi_\mu] - Q_I[\psi_{\mu^*}]\right) m_\mu \,, \tag{5.7}$$

which proportional to the coefficient of the 2D ABJ-type anomaly of $\mathrm{U}(1)_q$. Since the 2D theory is simply the low energy effective theory of the 4D $\mathrm{SU}(N)$ theory, the 2D anomaly must match an ABJ anomaly of the full 4D theory. Therefore, if a global symmetry has no ABJ anomaly, then $\Delta Q_q^{(I)} = 0$ and the scattering process will preserve the symmetry regardless of 't Hooft anomalies.

This relation can be seen explicitly in the case where $\vec{I} = 0$. From [28], we know that the multiplicity of fermions with weight $\mu \in S_R$ is given by

$$m_\mu = 2\langle T_3, \mu \rangle \,, \tag{5.8}$$

and additionally, c_μ^I can be computed by

$$c_\mu^I = \langle \rho_I, \mu \rangle \,, \quad \rho_I = [1, 2, 3, \ldots, \underbrace{I, \ldots, I}_{N-2I+1}, I-1, \ldots, 1]^\vee \,. \tag{5.9}$$

We can now write

$$\sum_{\mu \in S_R} m_\mu c_\mu^I = 2 \sum_{\mu \in \Delta_R^+} \langle T_3, \mu \rangle \langle \rho_I, \mu \rangle = \sum_{\mu \in \Delta_R} \langle T_3, \mu \rangle \langle \rho_I, \mu \rangle \,, \tag{5.10}$$

where here we used the fact that $c_\mu^I \geq 0$ if $\langle T_3, \mu \rangle > 0$ which follows form the fact that $\langle \mu, \rho_I \rangle = c_\mu^I$, which is the number of E_I^- required to go from the weight $\mu \to \mu^*$, is manifestly positive in the construction of highest weight representations.

We can now rewrite the charge violation coming from Ψ_R as

$$\Delta Q_q^{(I)} = \frac{q_R}{N_I} \times \mathrm{Tr}_R[T_3 \rho_I] \,, \tag{5.11}$$

where Tr_R is the trace in the R representation and

$$\rho_I = \mathrm{diag}\,(\underbrace{1, \ldots, 1}_{I}, 0, \ldots, 0, \underbrace{-1, \ldots, -1}_{I}) \,. \tag{5.12}$$

The contribution to global charge violation can then be further simplified to

$$\Delta Q_q^{(I)} = \frac{q_R I_2(R)}{N_I} \times \mathrm{Tr}_f[T_3 \rho_I] \,, \tag{5.13}$$

where the trace is now taken in the fundamental representation. The prefactor $q_R I_2(R)$ matches the anomaly coefficient of an ABJ-anomaly between $\mathrm{U}(1)_q$ and $\mathrm{SU}(N)$ which indicates that the charge violation arises from such anomalies.

Therefore, we see that the fermion boundary conditions on the monopole explicitly violate global symmetries that have ABJ anomalies.

6 Extended examples

Here we will compute the scattering/effective boundary conditions for a variety of examples in order to explicitly demonstrate the above formalism.

6.1 $\mathrm{SU}(2) \to \mathrm{U}(1)$ with arbitrary representations

Let us take $G = \mathrm{SU}(2)$ gauge theory with a collection of fermions of representations of spins $\{j_I\}$. Let us consider the minimal monopole which has

$$\gamma_m = 2T_3 = \mathrm{diag}\,(1, -1) \,, \quad \vec{I} = 0 \,, \tag{6.1}$$

and $\Phi_\infty = vT_3$ for $v \in \mathbb{R}$. This Higgs field breaks the gauge symmetry to $\mathrm{U}(1)$ generated by γ_m. Here we find that the highest weight of the representation of spin-j_I is

$$\Lambda_I = [2j_I], \quad T_3 = \left[\frac{1}{2}\right]^\vee \,. \tag{6.2}$$

Each fermion then has

$$\sum_{i=1}^{[j_I]} 2(j_I + 1 - i) = \begin{cases} j_I(j_I + 1) & j_I \in \mathbb{Z} \\ \left(j_I + \frac{1}{2}\right)^2 & j_I \in \mathbb{Z} + \frac{1}{2} \end{cases} \tag{6.3}$$

total zero-modes. If we index the fermion zero-modes by $\Psi_{i,a}$ where

$$i = 1, \ldots, \lceil j_I \rceil \, , \ a = 1, \ldots, i \, , \tag{6.4}$$

each of which has out-going charge

$$Q[\Psi_{i,a}] = -2(j_I + 1 - i) \tag{6.5}$$

then we find that the dyon decays

$$\varphi \to -\frac{1}{\sum_I \sum_{i=0}^{\lfloor j_I \rfloor} 2(j_I + 1 - i)^2} \sum_{i,a} \bar{\Psi}_{i,a} \Psi_{i,a} \, , \tag{6.6}$$

so that the charges are given

$$-\frac{1}{\sum_I \sum_{i=0}^{\lfloor j_I \rfloor} 2(j_I + 1 - i)^2} \sum_{i,a} Q[\Psi_{i,a}] = 1 \, . \tag{6.7}$$

Relabeling (i, a) by a single index A, we see that the boundary condition are given by

$$J_A \to \mathcal{R}_{AB} \bar{J}_B \, , \quad \mathcal{R}_{AB} = \begin{cases} 1 - \frac{3}{\sum_I j_I(j_I+1)(2j_I+1)} & A = B \\ \frac{3}{\sum_I j_I(j_I+1)(2j_I+1)} & A \neq B \end{cases} \tag{6.8}$$

As an example, for the case of N_f fundamental fermions (where $\{j_I\}_{I=1}^{N_f}$ with $j_I = \frac{1}{2}$), we find

$$\mathcal{R}_{AB} = \begin{cases} 1 - \frac{2}{N_f} & A = B \\ -\frac{2}{N_f} & A \neq B \end{cases} \, , \tag{6.9}$$

matching the known boundary condition of [1–4, 25, 26].

6.2 $\mathrm{SU}(N) \to \mathrm{U}(N-1)$ with N_f fundamental fermions

Let us take $G = \mathrm{SU}(N)$ gauge theory with N_f fundamental fermions. Let us consider a class of monopoles that breaks $\mathrm{SU}(N) \to \frac{\mathrm{SU}(N-1) \times \mathrm{U}(1)}{\mathbb{Z}_{N-1}} = \mathrm{U}(N-1)$. These correspond to $\mathrm{SU}(N)$ fermions where $\vec{T}$ is the maximal embedding and $\vec{I}$ corresponds to the $(N-1)$-dimensional representation of $\mathrm{SU}(2)_I$. Explicitly:

$$\begin{aligned} T_3 &= \frac{1}{2}\mathrm{diag}\,(N-1, N-3, \ldots, -N+3, -N+1) \, , \\ I_3 &= \frac{1}{2}\mathrm{diag}\,(N-2, N-4, \ldots, -N+2, 0) \, , \\ \gamma_m &= \mathrm{diag}\,(1, 1, 1, \ldots, -N+1) \, . \end{aligned} \tag{6.10}$$

Additionally, $\Phi_\infty = v \gamma_m$ for $v \in \mathbb{R}$. This Higgs vev spontaneously breaks $\mathfrak{su}(N)$ to $\mathfrak{su}(N-1) \times \mathfrak{u}(1)$ where the $\mathfrak{u}(1)$ is generated by γ_m. Additionally since $i_I \neq 0$ except for $I = N-1$, only φ_{N-1} is dynamical.

In bracket notation, these are given by

$$T_3 = [t_I]^\vee, \quad t_I = \frac{1}{2}I(N - I),$$
$$I_3 = [i_I]^\vee, \quad i_I = \frac{1}{2}I(N - 1 - I),$$
$$\gamma_m = [I]^\vee.$$
(6.11)

Now, highest weight of the fundamental representation is given by

$$\lambda_1 = [1, 0, \ldots, 0]$$
(6.12)

The weight space is given by

$$\lambda_i = [\underbrace{0, \ldots, 0}_{i-2}, -1, 1, 0, \ldots, 0], \quad i = 2, \ldots, N - 1,$$
(6.13)

along with $\Lambda_N = [0, \ldots, 0, -1]$.

We can now compute the states that have non-trivial zero-modes. To do this, we must first note that

$$\lambda_i^* = \lambda_{N-i+1}.$$
(6.14)

Then, using the fact that

$$\text{sign}(\langle \gamma_m, \lambda_i \rangle) \times \text{sign}(\langle \gamma_m, \lambda_i^* \rangle) > 0 \quad i \neq 1, N,$$
$$\text{sign}(\langle \gamma_m \lambda_1 \rangle) \times \text{sign}(\langle \gamma_m, \lambda_N \rangle) < 0,$$
(6.15)

the are only zero-modes associated to the pair of dual weights λ_1, λ_N which form a spin $j = \frac{N-2}{2}$ multiplet.

For N_f-fundamental fermions, there are $N_T = N_f \times (2j + 1) = N_f(N - 1)$ total low energy modes. The coupling matrix is then given by a N_T-length vector $c_i = 1 \ \forall i$ where here i indexes all $(N - 1)N_f$ low energy fermion currents. For this configuration, the ingoing/outgoing fermion has U(1) charge

$$Q[\Lambda_f] = 1, \quad Q[\Lambda_f^*] = -N + 1,$$
(6.16)

and the U(1) charge of φ_{N-1} is

$$Q[\varphi_{N-1}] = (N - 1) - (-1) = N.$$
(6.17)

This allows us to compute the decay

$$\varphi_{N-1} \to -\frac{N}{N_f(N - 1)^2} \sum_B \bar{J}_B.$$
(6.18)

This leads to the general scattering

$$J_A \to \bar{J}_A + \varphi_{N-1} \to \bar{J}_A - \frac{N}{N_f(N - 1)^2} \sum_B \bar{J}_B,$$
(6.19)

which corresponds to the boundary condition that is specified by the $\mathcal{R}$ matrix:

$$\mathcal{R}_{AB} = \begin{cases} 1 - \frac{N}{N_f(N-1)^2} & A = B \\ -\frac{N}{N_f(N-1)^2} & A \neq B \end{cases}$$
(6.20)

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6.3 $\mathrm{SU}(N) \to \frac{\mathrm{SU}(N-M)\times\mathrm{SU}(M)\times\mathrm{U}(1)}{\mathbb{Z}_{N-M}\times\mathbb{Z}_M}$ with N_f fundamental fermions

Now let us consider the case of $\mathrm{SU}(N)$ gauge theory with N_f fundamental fermions in the presence of a monopole that breaks $\mathrm{SU}(N) \to \frac{\mathrm{SU}(N-M)\times\mathrm{SU}(M)\times\mathrm{U}(1)}{\mathbb{Z}_{N-M}\times\mathbb{Z}_M}$. Here we will take $M < N/2$. This is realized by a $\vec{T}$ which again is the maximal embedding and $\vec{I}$ corresponds to the direct sum of a $(N-M)$- and M-dimensional embedding. Explicitly:

$$
\begin{aligned}
T_3 &= \frac{1}{2}\mathrm{diag}\left(N-1, N-3, \ldots, -N+3, -N+1\right), \\
I_3 &= \mathrm{diag}\left(\frac{N-M-1}{2}, \frac{N-M-3}{2}, \ldots, \frac{-N+M+1}{2}\right) \\
&\quad \oplus \mathrm{diag}\left(\frac{M-1}{2}, \frac{M-3}{2}, \ldots, \frac{-M+3, -M+1}{2}\right), \\
\gamma_m &= \mathrm{diag}\underbrace{\left(M, \ldots, M\right)}_{N-M\ \text{times}} \oplus \mathrm{diag}\underbrace{\left(M-N, \ldots, M-N\right)}_{M\ \text{times}}.
\end{aligned}
\tag{6.21}
$$

Here we use the notation $\oplus$ to denote concatenation. Additionally, $\Phi_\infty = v\,\gamma_m$ for $v \in \mathbb{R}$. This Higgs vev spontaneously breaks the generators $\mathfrak{su}(N)$ to $\mathfrak{su}(N-M) \oplus \mathfrak{su}(M) \oplus \mathfrak{u}(1)$ where the $\mathfrak{u}(1)$ is generated by γ_m.

In bracket notation, these are given by

$$
\begin{aligned}
T_3 &= [t_I]^\vee, \quad t_I = \frac{1}{2}I(N-I), \\
I_3 &= \left[\frac{I(N-M-I)}{2}\right]^\vee \oplus [0] \oplus \left[\frac{I(M-I)}{2}\right]^\vee, \\
\gamma_m &= [IM]^\vee \oplus [(N-M)(M-I+1)]^\vee.
\end{aligned}
\tag{6.22}
$$

Note that since $i_{N-M} = 0$ that only the φ_{N-M} will be dynamical.

Now we can explicitly show that the top M-states have non-trivial zero-modes: they correspond to the positive weights

$$
\lambda_1 = [1, 0, \ldots, 0], \quad \lambda_i = [\underbrace{0, \ldots, 0}_{i-2}, -1, 1, 0, \ldots 0].
\tag{6.23}
$$

These states, have spins

$$
j_i = \frac{i(N-i)-1}{2}, \quad i = 1, \ldots, M,
\tag{6.24}
$$

and have in-going and out-going electric charge:

$$
Q^{(\text{in})} = M, \quad Q^{(\text{out})} = M - N.
\tag{6.25}
$$

Additionally, $Q[\varphi_{N-M}] = N$.

Now, using the fact that the coupling to φ_{N-M} of all fermion zero-modes is exactly 1, and the fact that there are a total of

$$
N_T = N_f \times \sum_{I=1}^{M} 2\frac{I(N-I)-1}{2} = N_f \frac{M(M-1)(3N-2M-1)}{6},
\tag{6.26}
$$

fermion zero-modes which we index by a general label A, we find that the matrix c_A^{N-M} again reduces to a $N_f \frac{M(M-1)(3N-2M-1)}{6}$-dimensional vector

$$c_A^{N-M} = 1\,, \quad \forall A\,. \tag{6.27}$$

Again, we can compute the radiation of the monopole d.o.f.

$$\varphi_{N-M} \to = -\frac{6N}{(N-M)N_f M(M-1)(3N-2M-1)} \sum_A c_A^{N-M} \bar{J}_A\,. \tag{6.28}$$

Again this leads to the boundary condtions specified by

$$\mathcal{R}_{AB} = \begin{cases} 1 - \frac{6N}{(N-M)N_f M(M-1)(3N-2M-1)} & A = B \\ -\frac{6N}{(N-M)N_f M(M-1)(3N-2M-1)} & A \neq B \end{cases} \tag{6.29}$$

which is again a rational matrix.

6.4 $\mathrm{SU}(N) \to \mathrm{U}(N-1)$ with N_f adjoint fermions

Let us now consider the case of an $\mathrm{SU}(N)$ monopole that breaks $\mathrm{SU}(N) \to \frac{\mathrm{SU}(N-1)\times\mathrm{U}(1)}{\mathbb{Z}_{N-1}}$ in a theory with N_f adjoint fermions. Here, the positive weights are given by n-tuples of positive adjacent simple roots:

$$\alpha_I\,, \quad \alpha_I + \alpha_{I+1}\,, \quad \alpha_I + \alpha_{I+1} + \alpha_{I+2} \quad \dots \tag{6.30}$$

Together with the negative n-tuples of adjacent simple roots and $(N-1)$ zero-weights (parameterizing the Cartan algebra) the positive weights generate the (N^2-1)-dimensional adjoint representation. Here we will use the explicit parametrization of simple roots:

$$\alpha_1 = [2, -1, 0, \dots, 0]\,, \quad \alpha_I = [\underbrace{0, \dots, 0}_{I-2}, -1, 2, -1, 0, \dots, 0]\,, \quad \alpha_{N-1} = [0, \dots, 0, -1, 2]\,. \tag{6.31}$$

Their complex conjugate weights are given by

$$\alpha_I^* = -\alpha_{N-I}\,. \tag{6.32}$$

Now by noting that $\gamma_m = [I]^\vee$ and

$$\langle \gamma_m, \alpha_I \rangle = N\,\delta_{I,N-1}\,, \tag{6.33}$$

we see that the low energy modes can only come from fermions with weights that contain α_{N-1}. However, due to the duality of weights (6.32), we see that the only pair of fermions that contributes to the low energy theory is the highest/lowest weight pair:

$$\langle \gamma_m, \mu \rangle \times \langle \gamma_m, \mu^* \rangle < 0 \quad \Rightarrow \quad \mu = \mu_{\text{top}} := \sum_{I=1}^{N-1} \alpha_I\,, \tag{6.34}$$

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which couples to the dyon with weight $c_{\mu_{\text{top}}}^{N-1} = 2$. Using the fact that $T_3 = \frac{1}{2}[I(N-I)]^\vee$ and hence $\langle T_3, \alpha_I \rangle = 1$, the associated low energy modes form a spin-$j = \langle T_3, \mu_{\text{top}} \rangle - \frac{1}{2} = N - \frac{3}{2}$ multiplet. Thus, there are a total

$$N_T = N_f \left(2N - 2\right) , \tag{6.35}$$

low energy fermions which we index by the generalized index A.

We now find that the decay of φ_{N-1} is described by

$$\varphi_{N-1} \to -\frac{1}{2N_f(N-1)} \sum_A \bar{J}_A \tag{6.36}$$

This again gives rise to the boundary conditions

$$J_A \mapsto \mathcal{R}_{AB} \bar{J}_B , \tag{6.37}$$

where $\mathcal{R}_{AB}$ is given by

$$\mathcal{R}_{AB} = \begin{cases} 1 - \frac{1}{N_f(N-1)} & A = B \\ -\frac{1}{N_f(1N-1)} & A \neq B \end{cases} \tag{6.38}$$

6.4.1 Adjoint and fundamental matter

Now we can put together the two sets of results and consider the case where there are N_f-fundamental fermions and N_{ad}-adjoint fermions. Again only φ_{N-1} is dynamical, and its decay is described by

$$\varphi_{N-1} \to \frac{1}{\mathcal{N}} \sum_A n_A \bar{J}_A , \quad n_A = \begin{cases} 1 & \Psi_A \text{ from fundamental fermion} \\ 2 & \Psi_A \text{ from fundamental fermion} \end{cases} \tag{6.39}$$

where the coefficient $\mathcal{N}$ is given explicitly by

$$\begin{aligned} \frac{1}{\mathcal{N}} &= \frac{N}{N_{T,f} \times Q[\psi_f] + 2 \times N_{T,ad} \times Q[\psi_{ad}]} \\ &= -\frac{N}{N_f(N-1) \times (N-1) + N \times 2N_{ad}(N-1)} \end{aligned} \tag{6.40}$$

This leads to boundary conditions described by the matrix

$$\mathcal{R}_{AB} = \begin{cases} 1 - \frac{N}{N_f(N-1)^2 + 4N_{ad}N(N-1)} & A = B \text{ fund.} \\ 1 - \frac{4N}{N_f(N-1)^2 + 4N_{ad}N(N-1)} & A = B \text{ adjoint} \\ -\frac{N}{N_f(N-1)^2 + 4N_{ad}N(N-1)} & A \neq B, \ A, B \text{ fund} \\ -\frac{4N}{N_f(N-1)^2 + 4N_{ad}N(N-1)} & A \neq B \ A, B \text{ adjoint} \\ -\frac{2N}{N_f(N-1)^2 + 4N_{ad}N(N-1)} & A \text{ adjoint}, \ B \text{ fund or fund}\leftrightarrow\text{adjoint} \end{cases} \tag{6.41}$$

6.5 SU(5) GUT monopoles

Now let us consider the case of the spherically symmetric monopoles in the SU(5) Georgi-Glashow GUT theory. In this model, the Higgs vev is given explicitly by

$$\Phi_\infty = v \operatorname{diag}(2, 2, 2, -3, -3), \tag{6.42}$$

for $v \in \mathbb{R}$ which breaks $\mathrm{SU}(5) \to \frac{\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)}{\mathbb{Z}_6}$. Here the U(1) factor is referred to as *hypercharge* and is generated by $Y = \operatorname{diag}(2, 2, 2, -3, -3)$.

In this model, there is a $\mathbb{Z}_6$ clafssification of monopoles that are compatible with this breaking reflecting the $\mathbb{Z}_6$ quotient in the IR gauge group [29]. However, as shown in [30], only the charge 1,2,3 and 6 are stable, spherically symmetric monopoles. Thus, for brevity we will only consider the charge 1,2,3 monopoles.

In the case of a single generation as we will consider here, the matter content of the SU(5) gauge theory is a fermion in the $\bar{5}$- and the 10-representations. These have weight spaces

$$\begin{aligned}
\bar{5}: \quad & \{\mu_i\} = [-1,0,0,0]_1, \ [1,-1,0,0]_2, \ [0,1,-1,0]_3, \ [0,0,1,-1]_4, \ [0,0,0,1]_5, \\
10: \quad & \{\lambda_a\} = [0,1,0,0]_1, \ [1,-1,1,0]_2, \ [-1,0,1,0]_3, \ [1,0,-1,1]_4, \ [-1,1,-1,1]_5, \\
& \quad [1,0,0,-1]_6, \ [-1,1,0,-1]_7, \ [0,-1,0,1]_8, \ [0,-1,1,-1]_9, \ [0,0,-1,0]_{10}.
\end{aligned} \tag{6.43}$$

In terms of standard model fermions, the above weights correspond to

$$\begin{aligned}
\bar{5} &= (d_1^c, d_2^c, d_3^c, \bar{e}, \bar{\nu}), \\
10 &= (u_3^c, u_2^c, u_1^c, u_1, u_2, d_1, d_2, u_3, d_3, e^c).
\end{aligned} \tag{6.44}$$

Importantly, the monopoles we consider here have an additional term to the Higgs vev as described in footnote 8: $\phi_\infty \neq 0$. This is chosen so that Φ_∞ always breaks SU(5) to the standard model regardless of the magnetic charge.

Charge 1 monopole. Let us first consider the charge 1 monopole. This is the simple embedding of the SU(2) monopole into the SU(5) gauge group with

$$\gamma_m = 2T_3 = \operatorname{diag}(0, 0, 1, -1, 0) = [0,0,1,0]^\vee. \tag{6.45}$$

Since $\gamma_m = 2T_3$ and $\vec{I} = 0$, we see that any fermion with positive charge will be in-going and will be paired to an out-going fermion with negative charge. After a simple computation, one can show that the pairs are given by

in-going	out-going	φ-coupling	j
e	d_3^c	$c_1 = 1$	0
d_3	e^c	$c_1 = 1$	0
u_1^c	u_2	$c_1 = 1$	0
u_2^c	u_1	$c_1 = 1$	0

Due to the fact that the fermions all couple to the dyon with equal weight, we find that the decay of the single dyon mode is given by

$$\varphi \to \frac{1}{\mathcal{N}}(d_3^c + e^c + u_2 + u_1)\,, \tag{6.46}$$

where here for notational simplicity we refer to the current by its component fermion field. $\mathcal{N}$ can then be fixed by hypercharge conservation so that the decay of φ is given by

$$\varphi \to -\frac{1}{2}(d_3^c + e^c + u_2 + u_1)\,, \tag{6.47}$$

so that the boundary condition is described by

$$\mathcal{R}_{AB} = \frac{1}{2} \begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix}\,, \tag{6.48}$$

which reproduces the famous Callan-Rubakov effect.

Charge 2 monopole. Now let us consider the charge 2 monopole. This is again a simple embedding of the SU(2) monopole where $\vec{T}$ decomposes as the $2 \oplus 2 \oplus 1$ embedding. This is usually written in the explicit basis

$$\gamma_m = 2T_3 = \operatorname{diag}(1, 1, 0, -1, -1) = [1, 2, 2, 1]^\vee\,. \tag{6.49}$$

Again since $\vec{I} = 0$ the charges of the fermions under T_3 determine the spectrum of low energy fermions.

The low energy fermion modes are paired up as:

in-going	out-going	φ_I-coupling	j
e	d_1^c	$c_1 = 1$	0
u_1^c	d_3	$c_1 = 1$	0
ν	d_2^c	$c_2 = 1$	0
u_2^c	u_3	$c_2 = 1$	0
u_3^c	e^c	$c_1, c_2 = 1$	$\frac{1}{2}$

By matching the charges of φ_I and the out-going fermions we find the decay processes

$$\begin{aligned}
\varphi_1 &\to -\frac{1}{3}\left(d_1^c + d_3 + 2 \times e^c\right)\,, \\
\varphi_2 &\to -\frac{1}{3}\left(d_2^c + u_3 + 2 \times e^c\right)\,,
\end{aligned} \tag{6.50}$$

where again we refer to the currents by their component fermion fields and the multiplicity refers to the spin multiplicity. If we write

$$\begin{aligned}
J_A &= (e, u_1^c, u_2^c, u_{3,+}^c, u_{3,-}^c)\,, \\
\bar{J}_A &= (d_1^c, d_3, d_2^c, u_3, e_+^c, e_-^c)\,,
\end{aligned} \tag{6.51}$$

where $e^c_\pm, u^c_{3,\pm}$ are the $\pm$ states of the $j = \frac{1}{2}$ angular momentum states, then we find that the boundary condition is specified by the matrix

$$\mathcal{R}_{AB} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \end{pmatrix} \tag{6.52}$$

Charge 3 monopole. Now let us consider the charge 3 monopole which is not a simply embedded SU(2) monopole. This is specified by the data

$$\gamma_m = \text{diag}\,(1, 1, 1, -1, -2)\,, \quad T_3 = \frac{1}{2}\text{diag}\,(2, 1, 0, -1, -2)\,, \quad I_3 = \frac{1}{2}\text{diag}\,(1, 0, -1, 0, 0)\,, \tag{6.53}$$

which is written in bracket notation as

$$\gamma_m = [1, 2, 3, 2]^\vee\,, \quad T_3 = \frac{1}{2}[2, 3, 3, 2]^\vee\,, \quad I_3 = \frac{1}{2}[1, 1, 0, 0]^\vee\,. \tag{6.54}$$

For this background the φ_2, φ_3 are dynamical degrees of freedom that interact with the pairs:

in-going	out-going	φ_I-coupling	j
e	d^c_2	$c_2 = 1$	0
$u^c_3 + u^c_1$	e^c	$c_2, c_3 = 1$	1
ν	$d^c_1 + d^c_3$	$c_3 = 1$	$\frac{1}{2}$
u^c_1	d_2	$c_3 = 1$	0
u^c_2	$d_1 + d_3$	$c_3 = 1$	$\frac{1}{2}$

Again, we can compute the decay

$$\begin{aligned} \varphi_2 &\to -\frac{1}{4}(d^c_2 + 3 \times e^c)\,, \\ \varphi_3 &\to -\frac{1}{5}\Big(2 \times d^c_1 + d_2 + 2 \times (d_1 + d_3) + 3 \times e^c\Big)\,, \end{aligned} \tag{6.55}$$

where again we denot the current by its component fermion. The boundary condition is then specified by the matrix

$$\mathcal{R}_{AB} = \begin{pmatrix} 3/4 & -1/4 & -1/4 & -1/4 & 0 & 0 & 0 & 0 & 0 \\ -1/4 & 11/20 & -9/20 & -9/20 & -1/5 & -1/5 & -1/5 & -1/5 & -1/5 \\ -1/4 & -9/20 & 11/20 & -9/20 & -1/5 & -1/5 & -1/5 & -1/5 & -1/5 \\ -1/4 & -9/20 & -9/20 & 11/20 & -1/5 & -1/5 & -1/5 & -1/5 & -1/5 \\ 0 & -1/5 & -1/5 & -1/5 & 4/5 & -1/5 & -1/5 & -1/5 & -1/5 \\ 0 & -1/5 & -1/5 & -1/5 & -1/5 & 4/5 & -1/5 & -1/5 & -1/5 \\ 0 & -1/5 & -1/5 & -1/5 & -1/5 & -1/5 & 4/5 & -1/5 & -1/5 \\ 0 & -1/5 & -1/5 & -1/5 & -1/5 & -1/5 & -1/5 & 4/5 & -1/5 \\ 0 & -1/5 & -1/5 & -1/5 & -1/5 & -1/5 & -1/5 & -1/5 & 4/5 \end{pmatrix}\,. \tag{6.56}$$

6.5.1 $B - L$ symmetry in SU(5) GUT theory

Here we would like to show that the boundary conditions from the charge 1,2,3 monopoles in the SU(5) GUT theory preserves $B - L$ symmetry. $B - L$ symmetry can be realized as

$$Q_{B-L} = Q_Y + Q_D, \quad D = \begin{cases} -3 & \text{on } \bar{5} \\ 1 & \text{on } 10 \end{cases} \tag{6.57}$$

Since we have constructed the boundary conditions to preserve hypercharge, we simply need to check if $\mathrm{U}(1)_D$ is also preserved by the boundary conditions. Since $\mathrm{U}(1)^{(D)}$ commutes with SU(5), it will be preserved by the boundary conditions if the corresponding charge vector is an unit eigenvector of the $\mathcal{R}$-matrix.

For the charge 1 monopole, the D charge vector is given by

$$v_D = (-3, 1, 1, 1), \tag{6.58}$$

which is indeed a unit-eigenvector of $\mathcal{R}_{AB}$.

For the charge 2 monopole, the D charge vector

$$v_D = (-3, 1, -3, 1, 1, 1). \tag{6.59}$$

Again, this is a unit eigenvector of $\mathcal{R}_{AB}$.

For the charge 3 monopole the D charge vector is given by

$$v_D = (-3, 1, 1, 1, 1, -3, -3, 1, 1), \tag{6.60}$$

which is again a unit eigenvector of $\mathcal{R}_{AB}$. A similar computation shows that $B-L$-symmetry is also preserved by the charge 6 monopole. Therefore, $B - L$ symmetry is indeed preserved by the charge monopoles in the SU(5) GUT model which is what we expect from the fact that there is no ABJ-type anomaly for the $\mathrm{U}(1)_{B-L}$.

However, now we can consider baryon number symmetry which does have an ABJ-type anomaly. Here the in-going/out-going symmetries for baryon number are:[14]

$$
\begin{aligned}
&\text{charge 1}: \quad v_B^{(\text{in})} = (0, 1, -1, -1), && v_B^{(\text{out})} = (-1, 0, 1, 1), \\
&\text{charge 2}: \quad v_B^{(\text{in})} = (0, -1, 0, -1, -1, -1), && v_B^{(\text{out})} = (-1, 1, -1, 1, 0, 0), \\
&\text{charge 3}: \quad v_B^{(\text{in})} = (0, -1, -1, -1, -1, 0, 0, -1, -1), \\
&\qquad\qquad\quad v_B^{(\text{out})} = (-1, 0, 0, 0, 1, -1, -1, 1, 1).
\end{aligned}
\tag{6.61}
$$

As expected, none of these vectors satisfy the conservation equation (6.59). Therefore, baryon number symmetry (and therefore lepton number symmetry) is not preserved by the monopole boundary conditions and hence can be violated in scattering processes involving charge 1, 2, and 3 monopoles. Similar results hold for the charge 6 monopole.

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[14]Note that the charges here are $3\times$ the standard charges in the particle physics literature.

Acknowledgments

A Solving the Dirac equation

In this appendix we review the different solutions of the Dirac operator that are discussed in the main body.

A.1 The abelian monopole

Here we would like to solve for the solutions to the Dirac equation coupled to an abelian monopole connection. Here we will take the connection to be

$$A = \gamma_m A_{\text{Dirac}} = \frac{\gamma_m}{2}(1 - \cos\theta)d\phi \,, \tag{A.1}$$

where $\gamma_m \in \mathbb{Z}$ is the magnetic charge and $\sigma = \pm 1$. The time independent Dirac operator can be written

$$\bar{\sigma}^\mu D_\mu = \left(\sigma^{\hat{r}}\partial_r + \frac{1}{r}\sigma^{\hat{\theta}}\partial_\theta + \frac{1}{r\sin\theta}\sigma^{\hat{\phi}}\partial_\phi + \frac{i\gamma_m}{2r\sin\theta}\sigma^{\hat{\phi}}(1 - \cos(\theta))\right) \,, \tag{A.2}$$

where we use the notation

$$\begin{aligned}
\sigma^{\hat{r}} &= \sigma^3\cos\theta + (\sigma^1\cos\phi + \sigma^2\sin\phi)\sin\theta \,, \\
\sigma^{\hat{\theta}} &= -\sigma^3\sin\theta + (\sigma^1\cos\phi + \sigma^2\sin\phi)\cos\theta \,, \\
\sigma^{\hat{\phi}} &= -\sigma^1\sin\phi + \sigma^2\cos\phi \,.
\end{aligned} \tag{A.3}$$

If we now frame rotate by

$$U_\alpha{}^\beta = e^{\frac{i\phi}{2}\sigma^0\bar{\sigma}^3}e^{\frac{i\theta}{2}\sigma^0\bar{\sigma}^2} \,, \tag{A.4}$$

then we find that the Dirac operator becomes

$$U^\dagger\bar{\sigma}^\mu D_\mu U = \sigma^3\left(\partial_r + \frac{1}{r}\right) + \frac{1}{r}\left(\sigma^1\partial_\theta + \frac{1}{\sin(\theta)}\left(\sigma^2\partial_\phi + \frac{i\gamma_m\sigma^2}{2} - \frac{1}{2}(i\sigma^2\gamma_m + \sigma^1)\cos\theta\right)\right). \tag{A.5}$$

Now let us act with the Dirac operator on a fermion ψ_{μ_i} of charge $\langle\gamma_m, \mu_i\rangle = q_i$. Under the field redefinition

$$\psi_{\mu_i} = \frac{e^{i\left(m - \frac{q_i}{2}\right)\phi}}{r}U\hat{\psi}_{\mu_i} \,, \tag{A.6}$$

the Dirac operator acting on $\hat{\psi}_{\mu_i}$ further simplifies to

$$\bar{\sigma}^\mu D_\mu = \left(\sigma^3\partial_r - \frac{\mathcal{K}}{r}\right) \,, \quad \mathcal{K} = \begin{pmatrix} 0 & L^+_{m,\frac{q_i-1}{2}} \\ L^-_{m,\frac{q_i+1}{2}} & 0 \end{pmatrix}^{\dot{\alpha}\alpha} \,, \tag{A.7}$$

where

$$L^{\pm}_{m,m'} = \partial_\theta \pm \frac{1}{\sin\theta}(m - m'\cos\theta).$$ (A.8)

The $L^{\pm}_{m,m'}$ operators are diagonalized by the Wigner small d functions:

$$L^{\pm}_{m,m'} d^j_{m,m'}(\theta) = \mp\sqrt{j(j+1) - m'(m'\pm 1)} d^j_{m,m'\pm 1}(\theta),$$ (A.9)

which are only defined for

$$j \in \frac{1}{2}\mathbb{Z}_{\geq 0}, \quad j - m, j - m' \in \mathbb{Z}, \quad |m|, |m'| \leq j.$$ (A.10)

We can then further reduce the ansatz for the Dirac operator

$$\hat{\psi}_{\mu_i} = \begin{pmatrix} f_+(r)\, d^j_{-m,\frac{q_i+1}{2}}(\theta) \\ f_-(r)\, d^j_{-m,\frac{q_i-1}{2}}(\theta) \end{pmatrix},$$ (A.11)

on which the Dirac operator acts as

$$\bar{\sigma}^\mu D_\mu \hat{\psi}_{\mu_i} = \left(\sigma^3 \partial_r + \frac{i\sigma^2}{r}\sqrt{\left(j+\frac{1}{2}\right)^2 - \frac{q_i^2}{4}}\right)\hat{\psi}_{\mu_i}.$$ (A.12)

We can then solve for the spectrum of the Dirac operator by solving the matrix ODE. While the spectrum is somewhat involved the spectrum of $k = 0$ modes are explicitly written

$$\psi^{(k,m)}_{\mu_i}(r,\theta,\phi) = c_+ \frac{e^{i\left(m-\frac{q_i}{2}\right)\phi}}{r} r^{\sqrt{(j+1/2)^2 - q_i^2/4}} U(\theta,\phi) \begin{pmatrix} d^j_{-m,\frac{q_i+1}{2}}(\theta) \\ d^j_{-m,\frac{q_i-1}{2}}(\theta) \end{pmatrix}$$

$$+ c_- \frac{e^{i\left(m-\frac{q_i}{2}\right)\phi}}{r} r^{-\sqrt{(j+1/2)^2 - q_i^2/4}} U(\theta,\phi) \begin{pmatrix} d^j_{-m,\frac{q_i+1}{2}}(\theta) \\ -d^j_{-m,\frac{q_i-1}{2}}(\theta) \end{pmatrix}.$$ (A.13)

Requiring that the energy density be finite at $r \to 0, \infty$ then implies that there are no solutions for $q_i = 0$ and restricts us to the solutions for which

$$j = j_i := \frac{|q_i| - 1}{2},$$ (A.14)

in the case for $q_i \neq 0$. When restricted to this class of solutions, which are spin polarized, the full spectrum of the Dirac operator is written

$$\psi^{(k,m)}_{\mu_i} = \frac{U(\theta,\phi)}{r}\hat{\psi}^{(k,m)}_{\mu_i} = \frac{U(\theta,\phi)}{r}\begin{cases} e^{ikr}e^{i(m-q_i/2)\phi}\begin{pmatrix} d^{j_i}_{-m,j_i}(\theta) \\ 0 \end{pmatrix} & q_i > 0 \\[20pt] e^{-ikr}e^{i(m-q_i/2)\phi}\begin{pmatrix} 0 \\ d^{j_i}_{-m,-j_i}(\theta) \end{pmatrix} & q_i < 0 \end{cases}$$ (A.15)

We call these modes plane-wave normalizable because the $1/r$ behavior of ψ is the leading term in the partial wave expansion. These solutions have time-dependence of the form

$$\psi_{\mu_i}^{(k,m)} = \frac{U(\theta,\phi)}{r}\hat\psi_{\mu_i}^{(k,m)} = \frac{U(\theta,\phi)}{r}\begin{cases} e^{ik(t+r)}e^{i(m-q_i/2)\phi}\begin{pmatrix} d_{-m,j_i}^{j_i}(\theta) \\ 0 \end{pmatrix} & q_i > 0 \\[1.5em] e^{ik(t-r)}e^{i(m-q_i/2)\phi}\begin{pmatrix} 0 \\ d_{-m,-j_i}^{j_i}(\theta) \end{pmatrix} & q_i < 0 \end{cases} \quad \text{(A.16)}$$

Thus, we see that there are $|q_i|$ polarized, spherical fermion modes in the IR for each $q_i \neq 0$ that are in-going for $q_i > 0$ and out-going for $q_i < 0$.

A.2 The abelian monopole in SU(N)

In order to compare the solutions of the Dirac equation in the abelian monopole background to the fermion zero-modes in the non-abelian monopole background, we need to solve for the fermion modes in the canonical gauge. This can be written as

$$A_{\text{can.}} = T_3(1-\cos\theta)d\phi \pm i\frac{I_\pm}{2}e^{\mp i\phi}(d\theta \mp i\sin\theta d\phi)\,, \quad \text{(A.17)}$$

where here $e^{4\pi i T_3} = \mathbb{1}_{\text{SU}(N)}$. After the frame rotation (A.4) the time independent Dirac operator can be written

$$\left[\sigma^3\left(\partial_r + \frac{1}{r}\right) + \frac{\mathcal{K}}{r} - \left(\sigma^- e^{-i\phi}I_+ - \sigma^+ e^{i\phi}I_-\right)\right]\,, \quad \text{(A.18)}$$

where

$$\mathcal{K} = \sigma^1\left(\partial_\theta + \frac{1}{2}\cot(\theta)\right) + \frac{1}{\sin\theta}\sigma^2\big((\partial_\phi + iT_3) - i\cos(\theta)T_3\big) = \begin{pmatrix} 0 & D_{T_3-\frac{1}{2}}^+ \\ D_{T_3+\frac{1}{2}}^- & 0 \end{pmatrix}, \quad \text{(A.19)}$$

and

$$D_q^\pm = \partial_\theta \pm \frac{1}{\sin\theta}\Big(-i(\partial_\phi + iT_3) - q\cos\theta\Big)\,. \quad \text{(A.20)}$$

Now let us consider the time-independent Dirac equation acting on a fermion ψ_R of representation R of SU(N). Let us decompose ψ_R into weights

$$\psi_R = \sum_{\mu_a \in \Delta_R} \psi_R^a\, v_{\mu_a}\,. \quad \text{(A.21)}$$

$\mathcal{K}$ acts simply on the SU(2)$_J$-representations such that given a spin-j representation:

$$\psi_R^{a\,(j)} = U(\theta,\phi)e^{i(m-q_a/2)\phi}\begin{pmatrix} f_{a,+}(r)\,d_{-m,\frac{q_a+1}{2}}^j(\theta) \\ f_{a,-}(r)\,d_{-m,\frac{q_a-1}{2}}^j(\theta) \end{pmatrix}\,, \quad \mu_a \in \Delta_R^{(j)}\,, \quad \text{(A.22)}$$

where $\langle T_3, \mu_a \rangle = \frac{q_a}{2}$ and $\psi_{R_A}^{(j)}$ is the spin-j solution of ψ_{R_A}. Here, $\mathcal{K}$ acts as

$$\mathcal{K} \cdot \psi_{R_A}^{a\,(j)} = \begin{pmatrix} 0 & -\sqrt{(j+1/2)^2 - \frac{q_a^2}{4}} \\ \sqrt{(j+1/2)^2 - \frac{q_a^2}{4}} & 0 \end{pmatrix} \psi_{R_A}^{a\,(j)}. \tag{A.23}$$

Now, the Dirac equation reduces to a first order matrix ODE for the functions $f_{a,\pm}(r)$:

$$\left[r\partial_r + \mathcal{D} \right] F(r) = 0, \tag{A.24}$$

where

$$\mathcal{D} = \begin{pmatrix} 0 & -\sqrt{(j+1/2)^2 - \frac{q_a^2}{4}} \\ -\sqrt{(j+1/2)^2 - \frac{q_a^2}{4}} & 0 \end{pmatrix} + \sigma^- I_+ + \sigma^+ I_- + 1, \tag{A.25}$$

and

$$F(r) = \sum_{\mu_a \in \Delta_{R_A}^{(j)}} \begin{pmatrix} f_{a,+}(r) \\ f_{a,-}(r) \end{pmatrix} v_{\mu_a}. \tag{A.26}$$

The solutions are given by

$$F(r) = P \exp \left\{ -\int \mathcal{D}(r) dr \right\} F_0, \tag{A.27}$$

for F_0 a constant vector. The set of F_0 that corresponds to plane-wave normalizable solutions

$$\lim_{r \to \infty} r \psi_{R_A}^{(j)} < \infty, \tag{A.28}$$

were solved for in [28]. The solutions decompose with respect to the decomposition of $\vec{I} = \bigoplus_i \vec{I}^{(i)}$. Let us denote the decomposition by the set of spins $\{s_I\}$ of the representations $\{\vec{I}^{(i)}\}$ and R_{s_I} to be the restriction of the representation R to the spin-s_I representation of $\vec{I}^{(i)}$. By construction, for each s_I, the magnetic charge is a constant Q_{s_I} on ψ_R. The s_I-solutions of spin-j only exist for $j_{\min}^{(s_I)} \le j \le j_{\max}^{(s_I)}$ where

$$j_{\min/\max}^{(s_I)} = min/max_{\mu \in \Delta_{R_{s_I}}} |\langle \mu, T_3 \rangle| - \frac{1}{2}. \tag{A.29}$$

The solutions are given explicitly by

$$f_{a,+}(r) = 0, \quad f_{a,-}(r) = \frac{1}{r} \Big\langle j, Q_{s_I} - \frac{1}{2} + s_I - a \Big| |Q_{s_I}| - \frac{1}{2}, Q_{s_I} - \frac{1}{2}; s_I, s_I - a \Big\rangle, \tag{A.30}$$

for $Q_{s_I} > 0$ and

$$f_{a,+}(r) = \frac{1}{r} \Big\langle j, Q_{s_I} + \frac{1}{2} + s_I - a \Big| |Q_{s_I}| - \frac{1}{2}, Q_{s_I} + \frac{1}{2}; s_I, s_I - a \Big\rangle, \quad f_{a,-}(r) = 0, \tag{A.31}$$

for $Q_{s_I} < 0$ where $\mu_a \in \Delta_{R_{s_I}}$ where we have chosen an ordering $\langle I_3, \mu_a \rangle > \langle I_3, \mu_{a+1} \rangle$. Note that there is no solution for modes with $Q_{s_I} = 0$.

Importantly these modes are spin polarized. This means that the spectrum of the Dirac operator is then described by the solutions

$$
\psi_R^{a\,(j)} = \frac{e^{i(m-q_a/2)\phi}}{r} U(\theta,\phi)
\begin{cases}
e^{ik(t+r)} \begin{pmatrix} 0 \\ C_{s_I,a,-}^{(j)}\, d_{-m,\frac{q_a-1}{2}}^{j}(\theta) \end{pmatrix} & Q_{s_I} > 0 \\[4mm]
e^{ik(t-r)} \begin{pmatrix} C_{s_I,a,+}^{(j)}\, d_{-m,\frac{q_a+1}{2}}^{j}(\theta) \\ 0 \end{pmatrix} & Q_{s_I} < 0
\end{cases}
\tag{A.32}
$$

for $\mu_a \in \Delta_{R_{s_I}}^{(j)}$ and

$$
C_{s_I,a,\pm}^{(j)} = \left\langle j, Q_{s_I} \pm \tfrac{1}{2} + s_I - a \,\middle|\, |Q_{s_I}| - \tfrac{1}{2}, Q_{s_I} \pm \tfrac{1}{2}; s_I, s_I - a \right\rangle. \tag{A.33}
$$

Note that in the cases where $\phi_\infty = 0$, as we consider in the bulk of this paper, the theory has a preserved non-abelian gauge symmetry along the irreducible components $\bar{I}^{(i)}$. This means that the above solutions are gauge equivalent to the solutions where $\psi_R^{(j)}$ is purely along a single weight $\mu \in \Delta_{R_{s_I}}$. In the bulk of the paper, we will find it convenient to pick a gauge such that each solution is along the top/bottom weight $\mu_{\text{top/bot}} \in \Delta_{R_{s_I}}$ where $|\langle T_3, \mu_{\text{top/bot}} \rangle| = j + \tfrac{1}{2}$.

A.3 The non-abelian monopole

Now let us solve for solutions to the Dirac operator coupled to the $\mathrm{SU}(N)$ non-abelian monopole connection. Here we take the connection to be

$$
A = T_3(1 - \cos\theta)d\phi + \frac{i}{2}M_+(r)e^{-i\phi}(d\theta - i\sin\theta d\phi) - \frac{i}{2}M_-(r)e^{i\phi}(d\theta + i\sin\theta d\phi). \tag{A.34}
$$

Here we will be interested in solving for the zero-mode solutions which are spanned by the normalizable solutions in the kernel of the time-independent Dirac operator. Again, after performing a frame rotation, the Dirac operator can be written

$$
\left[\sigma^3 \left(\partial_r + \frac{1}{r} \right) + \frac{\mathcal{K}}{r} - \left(\sigma^- e^{-i\phi} M_+ - \sigma^+ e^{i\phi} M_- \right) \right], \tag{A.35}
$$

where

$$
\mathcal{K} = \sigma^1 \left(\partial_\theta + \frac{1}{2}\cot(\theta) \right) + \frac{1}{\sin\theta}\sigma^2 \left((\partial_\phi + iT_3) - i\cos(\theta)T_3 \right) = \begin{pmatrix} 0 & D_{T_3 - \frac{1}{2}}^+ \\ D_{T_3 + \frac{1}{2}}^- & 0 \end{pmatrix}, \tag{A.36}
$$

and

$$
D_q^\pm = \partial_\theta \pm \frac{1}{\sin\theta}\left(-i(\partial_\phi + iT_3) - q\cos\theta \right). \tag{A.37}
$$

Now let us consider the time-independent Dirac equation acting on a fermion ψ_R of representation R of $\mathrm{SU}(N)$. Let us decompose ψ_R into weights

$$
\psi_R = \sum_{\mu_a \in \Delta_R} \psi_R^a \, v_{\mu_a}. \tag{A.38}
$$

$\mathcal{K}$ acts simply on the SU(2)$_J$-representations such that given a spin-j representation:

$$\psi_{R_A}^{a\,(j)} = U(\theta,\phi)e^{i(m-q_a/2)\phi}\begin{pmatrix} f_{a,+}(r)\,d^j_{-m,\frac{q_a+1}{2}}(\theta) \\ f_{a,-}(r)\,d^j_{-m,q_a-\frac{1}{2}}(\theta) \end{pmatrix}\,, \quad \mu_a \in \Delta_{R_A}^{(j)}\,, \tag{A.39}$$

where $\langle T_3, \mu_a \rangle = \frac{q_a}{2}$ and $\psi_R^{(j)}$ is the spin-j solution of ψ_R. Here, $\mathcal{K}$ acts as

$$\mathcal{K}\cdot\psi_{R_A}^{a\,(j)} = \begin{pmatrix} 0 & -\sqrt{(j+1/2)^2 - \frac{q_a^2}{4}} \\ \sqrt{(j+1/2)^2 - \frac{q_a^2}{4}} & 0 \end{pmatrix}\psi_{R_A}^{a\,(j)}\,. \tag{A.40}$$

Now, the Dirac equation reduces to a first order matrix ODE for the functions $f_{a,\pm}(r)$:

$$\left[r\partial_r + \mathcal{D}\right]F(r) = 0\,, \quad F(r) = \sum_{\mu_a \in \Delta_{R_A}^{(j)}}\begin{pmatrix} f_{a,+}(r) \\ f_{a,-}(r) \end{pmatrix}v_{\mu_a}\,, \tag{A.41}$$

where

$$\mathcal{D} = \begin{pmatrix} 0 & -\sqrt{(j+1/2)^2 - \frac{q_a^2}{4}} \\ -\sqrt{(j+1/2)^2 - \frac{q_a^2}{4}} & 0 \end{pmatrix} + \sigma^- M_+ + \sigma^+ M_- + 1\,. \tag{A.42}$$

The smooth solutions given by

$$F(r) = P\exp\left\{-\int\mathcal{D}(r)dr\right\}F_0\,, \tag{A.43}$$

for F_0 a constant vector. As proven in [28], there exists a vector F_0 that corresponds to a smooth, plane-wave normalizable solution

$$\lim_{r\to\infty}r\psi_{R_A}^{(j)} < \infty\,, \tag{A.44}$$

of the time-independent Dirac equation iff the top and bottom weights of the spin-j representation of R_A:

$$\langle\mu_{\rm top}, T_3\rangle = -\langle\mu_{\rm bot}, T_3\rangle = j + \frac{1}{2}\,, \quad \mu_{\rm top}, \mu_{\rm bot} \in \Delta_{R_A}^{(j)}\,, \tag{A.45}$$

have charge of opposite sign under γ_m:

$$\langle\mu_{\rm top}, \gamma_m\rangle\langle\mu_{\rm bot}, \gamma_m\rangle < 0\,. \tag{A.46}$$

Note that since the asymptotic form of the connection approaches the abelian connection in the canonical gauge, the asymptotic form of the fermion zero-modes can be expressed as a linear combination of the zero-momentum solutions found in the previous subsection.

As shown in [28], for each irreducible SU(2)$_T$-representation R_A and j that satisfies (A.46) there is a unique solution which has the asymptotic form of

$$\lim_{r\to\infty}\psi_{R_A}^{(j)} = \psi_{R_A}^{(j)-} - \psi_{R_A}^{(+)}\,, \tag{A.47}$$

where $\psi_{R_A}^{(j)\pm}$ are given by the spin up/down solutions of (A.32).

References

[1] C.G. Callan Jr., *Monopole Catalysis of Baryon Decay*, *Nucl. Phys. B* **212** (1983) 391 [INSPIRE].

[2] C.G. Callan Jr., *Disappearing Dyons*, *Phys. Rev. D* **25** (1982) 2141 [INSPIRE].

[3] C.G. Callan Jr., *Dyon-Fermion Dynamics*, *Phys. Rev. D* **26** (1982) 2058 [INSPIRE].

[4] V.A. Rubakov, *Adler-Bell-Jackiw Anomaly and Fermion Number Breaking in the Presence of a Magnetic Monopole*, *Nucl. Phys. B* **203** (1982) 311 [INSPIRE].

[5] C.G. Callan Jr. and S.R. Das, *Boundary Conditions on the Monopole Dirac Equation*, *Phys. Rev. Lett.* **51** (1983) 1155 [INSPIRE].

[6] C.G. Callan Jr., *The monopole catalysis S matrix*, in the proceedings of *Workshop on Problems in Unification and Supergravity*, (1983), pp. 45–53 [DOI:10.1063/1.34591] [INSPIRE].

[7] B. Grossman, *Does a dyon leak?*, *Phys. Rev. Lett.* **50** (1983) 464 [INSPIRE].

[8] H. Yamagishi, *The fermion monopole system reexamined*, *Phys. Rev. D* **27** (1983) 2383 [INSPIRE].

[9] H. Yamagishi, *The fermion-monopole system reexamined. 2*, *Phys. Rev. D* **28** (1983) 977 [INSPIRE].

[10] H. Yamagishi, *Fermion-monopole system reexamined. 3*, *Phys. Rev. D* **32** (1985) 1576 [INSPIRE].

[11] H. Panagopoulos and H. Yamagishi, *Fermion-monopole system reexamined. 4*, *Phys. Rev. D* **32** (1985) 2113 [INSPIRE].

[12] A. Sen, *Role of Conservation Laws in the Callan-Rubakov Process with Arbitrary Number of Generation of Fermions*, *Phys. Rev. Lett.* **52** (1984) 1755 [INSPIRE].

[13] A. Sen, *Comparison of the Canonical Hamiltonian and the Hamiltonian of Callan and Rubakov for the Monopole Fermion System*, *Phys. Rev. D* **31** (1985) 433 [INSPIRE].

[14] A.P. Balachandran and J. Schechter, *The Missing Charge in the Rubakov Process*, *Phys. Rev. Lett.* **51** (1983) 1418 [INSPIRE].

[15] W. Goldstein, *On the interaction of fermions with a finite sized, nonabelian monopole*, SLAC-PUB-3272 (1983) [INSPIRE].

[16] K. Isler, C. Schmid and C.A. Trugenberger, *Monopole core excitations and the Rubakov-Callan effect*, *Nucl. Phys. B* **294** (1987) 925 [INSPIRE].

[17] J. Preskill, *Magnetic monopoles*, *Ann. Rev. Nucl. Part. Sci.* **34** (1984) 461 [INSPIRE].

[18] R. Kitano and R. Matsudo, *Missing final state puzzle in the monopole-fermion scattering*, *Phys. Lett. B* **832** (2022) 137271 [arXiv:2103.13639] [INSPIRE].

[19] J. Polchinski, *Monopole Catalysis: The Fermion Rotor System*, *Nucl. Phys. B* **242** (1984) 345 [INSPIRE].

[20] P.B. Smith and D. Tong, *Boundary States for Chiral Symmetries in Two Dimensions*, *JHEP* **09** (2020) 018 [`arXiv:1912.01602`] [ɪɴSPIRE].

[21] P.B. Smith and D. Tong, *Boundary RG flows for fermions and the mod 2 anomaly*, *SciPost Phys.* **10** (2021) 010 [`arXiv:2005.11314`] [ɪɴSPIRE].

[22] P.B. Smith and D. Tong, *What Symmetries are Preserved by a Fermion Boundary State?*, `arXiv:2006.07369` [ɪɴSPIRE].

[23] S. Dawson and A.N. Schellekens, *Monopole-Fermion Interactions: The Soliton Picture*, *Phys. Rev. D* **28** (1983) 3125 [ɪɴSPIRE].

[24] G.W. Moore, A.B. Royston and D. Van den Bleeken, *Parameter counting for singular monopoles on* $\mathbb{R}^3$, *JHEP* **10** (2014) 142 [`arXiv:1404.5616`] [ɪɴSPIRE].

[25] J.M. Maldacena and A.W.W. Ludwig, *Majorana fermions, exact mapping between quantum impurity fixed points with four bulk fermion species, and solution of the 'unitarity puzzle'*, *Nucl. Phys. B* **506** (1997) 565 [`cond-mat/9502109`] [ɪɴSPIRE].

[26] I. Affleck and J. Sagi, *Monopole catalyzed baryon decay: A Boundary conformal field theory approach*, *Nucl. Phys. B* **417** (1994) 374 [`hep-th/9311056`] [ɪɴSPIRE].

[27] D. Wilkinson and F.A. Bais, *Exact* SU(N) *Monopole Solutions with Spherical Symmetry*, *Phys. Rev. D* **19** (1979) 2410 [ɪɴSPIRE].

[28] T.D. Brennan, *Index-like theorem for massless fermions in spherically symmetric monopole backgrounds*, *JHEP* **03** (2022) 095 [`arXiv:2106.13820`] [ɪɴSPIRE].

[29] C.L. Gardner, *Selfdual* SU(5) *Monopole solutions*, *Phys. Lett. B* **142** (1984) 379 [ɪɴSPIRE].

[30] C.L. Gardner and J.A. Harvey, *Stable Grand Unified Monopoles With Multiple Dirac Charge*, *Phys. Rev. Lett.* **52** (1984) 879 [ɪɴSPIRE].