

Words to describe a black hole

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ABSTRACT: We revamp the constructive enumeration of 1/16-BPS states in the maximally supersymmetric Yang-Mills in four dimensions, and search for ones that are not of multi-graviton form. A handful of such states are found for gauge group $SU(2)$ at relatively high energies, resolving a decade-old enigma. Along the way, we clarify various subtleties in the literature, and prove a non-renormalization theorem about the exactness of the cohomological enumeration in perturbation theory. We point out a giant-graviton-like feature in our results, and envision that a deep analysis of our data will elucidate the fundamental properties of black hole microstates.

KEYWORDS: AdS-CFT Correspondence, Black Holes in String Theory, Supersymmetric Gauge Theory, Superspaces

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Contents

1	Introduction	1
2	Letters and words	3
2.1	Review of the cohomology problem	3
2.2	BPS partition function and superconformal index	5
2.3	Multi-gravitons from single-trace cohomology	6
3	A non-renormalization theorem	6
4	Constructive enumeration and discussions	8
4.1	Computational scheme	8
4.2	Counting data and a giant-graviton-like feature	9
4.3	Black hole microstates	13
4.4	Outlook	15

1 Introduction

Recently, the Event Horizon Telescope [1] astounded the world with the first image of a black hole at the center of our galaxy. Instead of pictures, we search for “words” to describe black hole microstates. The seminal work of Strominger and Vafa [2] matching the area of a black hole horizon with the counting of microstates has fundamentally changed our perception of string theory and quantum gravity. Not only did it fortify the legitimacy of string theory as a consistent theory of quantum gravity, but it also revealed that black hole horizons contain a large amount of quantum information, contrary to the classical no-hair theorem. Extracting and decoding this information is key to unveiling profound fundamental aspects of black holes. While [2] sparked bustling activity in *counting* black hole microstates, much less effort has been invested in uncovering the properties of the states themselves.

The AdS/CFT duality provides a rigorous framework to attack this problem. The Bekenstein-Hawking entropy of an asymptotically-AdS black hole is dual to the statistical entropy of a thermal state in the conformal field theory (CFT) living at the boundary of the AdS space. In the classic correspondence between the type IIB superstring theory on $\text{AdS}_5 \times S^5$ and the $\mathcal{N} = 4$ super-Yang-Mills (SYM) theory with gauge group $SU(N)$, the entropy of electrically-charged rotating BPS black holes in AdS_5 [3–7] was expected to be reproduced by the superconformal index of the CFT [8]. However, due to large cancellations between bosonic and fermionic states at nearby charges, the superconformal index at real saddle points is not large enough to account for the growth of the number of black hole microstates. It was then realized in [9], building on prior observation by [10], that the black

hole entropy is related to a complexified Euclidean on-shell action. The complexified bulk geometry identified a set of complex fugacities, and the superconformal index of the CFT with such fugacities exactly reproduced the black hole entropy [9, 11, 12].¹

In light of these triumphs, it is natural to take one step further and try to understand not just the number of black hole microstates, but to find their precise holographic dual, which by the state/operator correspondence are local operators in $\mathcal{N} = 4$ SYM. This problem has two parts:

1. *Enumeration of BPS operators.* This has been studied at weak coupling in [52–55], by organizing them into the cohomology of a supercharge Q on the Fock space of single-trace and multi-trace operators.² The weak-coupling spectrum was then conjectured by [54] to be invariant along the exactly marginal deformation into the strong-coupling regime, which is appropriate for describing black holes. In [55], the Q -cohomology was reformulated as a relative Lie superalgebra cohomology.
2. *Identification of the ones dual to black holes,* and not just a gas of gravitons. To this end, explicit representatives of cohomology classes dual to the multi-gravitons were proposed in [55] and shown to recover the multi-graviton partition function in the AdS at large N . In [55], a sporadic search at low ranks $N = 2, 3$ and reasonably high energies reported a negative result for the existence of states not of multi-graviton form, i.e. candidate black hole microstates. Nevertheless, the recent evaluation of finite N indices strongly suggests that they should be present even at low ranks [39, 40].

The present paper revisits this cohomology problem and performs a systematic search for cohomology classes that are not of multi-graviton form. Our main results can be summarized as follows:

1. We obtain a large collection of fully-refined counting data, which contains much more information than even the fully-refined index, let alone the unrefined index which most explicit evaluations consider.
2. The first non-graviton cohomology class is found for $N = 2$ at energy $E = 19/2$, disproving the conjecture of [55] about their nonexistence.
3. A non-renormalization theorem is proven (assuming the applicability of Leibniz rule of the Q -action) in perturbation theory, which puts the conjecture of [54] on a firmer basis.

¹Earlier, the topologically twisted indices in CFT_3 reproduce the entropy of a class of static dyonic BPS black holes in AdS_4 [13, 14]. Generalizations to other dimensions and other amounts of supersymmetry can be found in [15–51].

²While the Q -closed condition is natural, the unfamiliar reader may at this point wonder about the meaning of Q -exactness. A $1/16$ -BPS operator must also be annihilated by Q^\dagger , which receives quantum corrections. To circumvent this difficulty, one invokes Hodge theory which establishes a bijection between harmonic forms (annihilated by $\Delta \equiv \{Q, Q^\dagger\}$) and the de Rham cohomology (Q -cohomology). Note that Q -exact operators are orthogonal to the $1/16$ -BPS operators, and Q -closed operators with non-zero Δ -eigenvalues are Q -exact.

The remainder of this paper is organized as follows. Section 2 introduces the BPS letters and words, reviews the formulation of the cohomology problem in terms of a BPS superfield, and discusses the BPS partition function that counts the BPS states (without signs). Section 3 proves the non-renormalization conjecture of [54] in perturbation theory, and comments on the non-perturbative extension. The key results of our enumeration are then presented in section 4, with various ramifications discussed.

2 Letters and words

2.1 Review of the cohomology problem

The $\mathcal{N} = 4$ super-Yang-Mills theory has superconformal symmetry $\text{PSU}(2, 2|4)$. Let us denote the 16 supercharges by $Q_\alpha^I, \bar{Q}_{I\dot{\alpha}}$, and the 16 superconformal supercharges by $S_I^\alpha, \bar{S}^{I\dot{\alpha}}$, where the upper (lower) $I = 1, \dots, 4$ is the (anti-)fundamental index of the $\text{SU}(4)_R$ R-symmetry, and $\alpha, \dot{\alpha} = \pm, \dot{\pm}$ are the spinor indices of the Lorentz group $\text{SO}(4) \cong \text{SU}(2)_L \times \text{SU}(2)_R$. The 1/16-BPS operators are defined to be those annihilated by the supercharge $Q \equiv Q_-^4$ and its hermitian conjugate (BPZ conjugate) $Q^\dagger = S \equiv S_4^-$ in radial quantization. They have the commutator

$$\Delta \equiv 2\{Q, Q^\dagger\} = D - 2J_L^3 - q_1 - q_2 - q_3, \tag{2.1}$$

where D is the dilatation operator, J_L^3 is the left $\text{SU}(2)_L$ angular momentum, and q_1, q_2, q_3 are the Cartan generators of the $\text{SO}(6)_R$ R-symmetry.³

By standard arguments (see footnote 2), the space of the 1/16-BPS operators is isomorphic to the cohomology of the supercharge Q [54]. In the weak coupling limit, one can further restrict the Q -cohomology to the classical null space of Δ . In the path integral formalism, local operators are constructed out of gauge-invariant combinations of the fundamental fields, which consist of six scalars Φ_{IJ} in the antisymmetric representation of $\text{SU}(4)_R$ with the reality condition $\Phi_{IJ}^* = \frac{1}{2}\epsilon^{IJKL}\Phi_{KL} \equiv \Phi^{IJ}$, four chiral fermions $\Psi_{I\alpha}$ as well as their complex conjugates $\bar{\Psi}^{I\dot{\alpha}}$, and the gauge field A_μ .

The operators in the classical null space of Δ are constructed using the BPS *letters*, which are fundamental fields and their derivatives of vanishing classical Δ -eigenvalue.⁴ The full set of BPS letters are

$$\phi^i \equiv \Phi^{4i}, \quad \psi_i \equiv -i\Psi_{i+}, \quad \lambda_{\dot{\alpha}} \equiv \bar{\Psi}_{\dot{\alpha}}^4, \quad f \equiv -iF_{++} = F_{\mu\nu}(\sigma^{\mu\nu})_{++}, \tag{2.2}$$

where $i = 1, 2, 3$, and their covariant $D_{\dot{\alpha}}$ -derivatives, where

$$D_{\dot{\alpha}} \equiv (\sigma^\mu)_{+\dot{\alpha}} D_\mu. \tag{2.3}$$

The BPS letters satisfy two relations. The first is the obvious relation between $D_{\dot{\alpha}}$ and f ,

$$[D_{\dot{\alpha}}, D_{\dot{\beta}}] = \epsilon_{\dot{\alpha}\dot{\beta}} f. \tag{2.4}$$

³Our convention is that $R_1 = q_2 + q_3 = R_4^1 - R_1^1$, $R_2 = q_1 - q_2 = R_1^1 - R_2^2$, $R_3 = q_2 - q_3 = R_2^2 - R_3^3$, where R_1, R_2, R_3 are the Cartan generators of $\text{SU}(4)_R$, and $R_1^1, R_2^2, R_3^3, R_4^4$ are the diagonal components of the fundamental representation of $\text{SU}(4)_R$.

⁴This is also referred as the $\text{psu}(1, 2|3)$ subsector [56].

The second is the only equation of motion that is purely made out of the BPS letters,

$$D_{\dot{\alpha}}\lambda^{\dot{\alpha}} = [\phi^i, \psi_i], \quad (2.5)$$

which amounts to the invariance of the path integral under field redefinitions. The gauge-invariant combinations of the BPS letters (2.2) and (2.3) up to the relations (2.4) and (2.5) will be referred as the BPS *words*. The supercharge Q acts on the BPS letters as

$$\begin{aligned} [Q, \phi^i] &= 0, & \{Q, \psi_i\} &= -i\epsilon_{ijk}[\phi^j, \phi^k], \\ \{Q, \lambda_{\dot{\alpha}}\} &= 0, & [Q, f] &= i[\phi^i, \psi_i], \\ [Q, D_{\dot{\alpha}}\zeta] &= -i[\lambda_{\dot{\alpha}}, \zeta] + D_{\dot{\alpha}}[Q, \zeta], \end{aligned} \quad (2.6)$$

where ζ is any fundamental field.

The BPS fields can be assembled into a fermionic ‘‘BPS superfield’’ [55], which is a power series in the Grassmann variables θ_i ,

$$\Psi(z, \theta) = -i[\lambda(z) + 2\theta_i\phi^i(z) + \epsilon^{ijk}\theta_i\theta_j\psi_k(z) + 4\theta_1\theta_2\theta_3f(z)], \quad (2.7)$$

and the component fields $\lambda(z)$, $\phi^i(z)$, $\psi_i(z)$, $f(z)$ are formal power series of the auxiliary variables $z^{\dot{\alpha}}$ [54],

$$\begin{aligned} \phi^i(z) &= \sum_{n=0}^{\infty} \frac{1}{n!} (z^{\dot{\alpha}} D_{\dot{\alpha}})^n \phi^i, & \psi_i(z) &= \sum_{n=0}^{\infty} \frac{1}{n!} (z^{\dot{\alpha}} D_{\dot{\alpha}})^n \psi_i, \\ \lambda(z) &= \sum_{n=0}^{\infty} \frac{1}{(n+1)!} (z^{\dot{\alpha}} D_{\dot{\alpha}})^n (z^{\dot{\beta}} \lambda_{\dot{\beta}}), & f(z) &= \sum_{n=0}^{\infty} \frac{1}{n!} (z^{\dot{\alpha}} D_{\dot{\alpha}})^n f. \end{aligned} \quad (2.8)$$

The BPS superfield satisfies the constraint

$$\Psi(\mathcal{Z} = 0) = 0, \quad (2.9)$$

where we collectively denote all the auxiliary variables as $\mathcal{Z} = (z^+, z^-; \theta_1, \theta_2, \theta_3)$. The BPS letters can be recovered by taking the $\partial_{z^{\pm}}$ and ∂_{θ_i} derivatives of the BPS superfield Ψ and evaluating it at the origin $\mathcal{Z} = 0$ of the superspace $\mathbb{C}^{2|3}$. One does not miss anything by symmetrizing all the $\dot{\alpha}$ indices in (2.8) because the antisymmetric parts can be replaced by the right-hand sides of (2.4) and (2.5). The charges of the BPS superfield and the derivatives are given in table 1, where Y is related to the charge of the ‘‘bonus’’ $U(1)_Y$ symmetry [57, 58].⁵

The supercharge Q acts on the BPS superfield $\Psi(\mathcal{Z})$ as

$$\{Q, \Psi(\mathcal{Z})\} = \Psi(\mathcal{Z})^2. \quad (2.10)$$

It was recognized in [55] that in this formulation the Q -cohomology is nothing but the relative Lie superalgebra cohomology

$$\mathbb{H}^*(\mathcal{G}_N, \mathfrak{sl}_N; \mathbb{C}), \quad (2.11)$$

where $\mathcal{G}_N \equiv \mathbb{C}[z^+, z^-] \otimes \Lambda[\theta_1, \theta_2, \theta_3] \otimes \mathfrak{sl}_N$. For readers not familiar with Lie algebra cohomology, its basic definition can be found in section 3 of [55].

⁵Compared with [57, (5.2)], $Y_{\text{there}} = -2Y_{\text{here}} + 2\sum_{i=1}^3 q_i$.

Charges	J_L^3	J_R^3	q_j	Y
Ψ	$-\frac{1}{2}$	0	$\frac{1}{2}$	1
∂_{z^\pm}	$\frac{1}{2}$	$\pm\frac{1}{2}$	0	0
∂_{θ_i}	$\frac{1}{2}$	0	$\delta_{ij} - \frac{1}{2}$	0

Table 1. Angular momenta and charges of the BPS superfield and derivatives.

2.2 BPS partition function and superconformal index

The BPS partition function is defined as [55]

$$\begin{aligned}
 & Z(x, a, b, u, v, w) \\
 &= \text{Tr}_{\mathcal{H}_{\text{BPS}}} \left[x^Y a^{D-J_L^3+J_R^3-Y} b^{D-J_L^3-J_R^3-Y} u^{-q_2-q_3+Y} v^{-q_1-q_3+Y} w^{-q_1-q_2+Y} \right], \quad (2.12)
 \end{aligned}$$

where x counts the number of the superfield Ψ , a and b count the number of the z -derivatives ∂_{z^\pm} , and u, v, w count the number of θ -derivatives ∂_{θ_i} . The superconformal index defined in [8] is given by specializing the BPS partition function as

$$\begin{aligned}
 I_{\text{SCI}}(t, y, v, w) &= Z(-1, t^3 y, t^3/y, -t^2 v, -t^2 w/v, -t^2/w) \\
 &= \text{Tr} \left[(-1)^F t^{2(D+J_L^3)} y^{2J_R^3} v^{R_2} w^{R_3} \right], \quad (2.13)
 \end{aligned}$$

where $R_1 = q_2 + q_3$, $R_2 = q_1 - q_2$, $R_3 = q_2 - q_3$, and the fermion number F is the sum of the exponents of x, u, v, w , i.e. $F = 3Y - 2q_1 - 2q_2 - 2q_3$. The unflavored index and BPS partition function are defined by

$$\begin{aligned}
 I_N(t) &\equiv I_{\text{SCI}}(t, 1, 1, 1) = Z(-1, t^3, t^3, -t^2, -t^2, -t^2) = \text{Tr} \left[(-1)^F t^{2(D+J_L^3)} \right], \\
 Z_N(t) &\equiv Z(1, t^3, t^3, t^2, t^2, t^2) = \text{Tr}_{\mathcal{H}_{\text{BPS}}} \left[t^{2(D+J_L^3)} \right]. \quad (2.14)
 \end{aligned}$$

Let us expand them as

$$I_N(t) = \sum_n d_N(n) t^n, \quad Z_N(t) = \sum_n \mathbf{d}_N(n) t^n, \quad (2.15)$$

where n is the eigenvalue of $2(D + J_L^3)$. The coefficients $d_N(n)$ and $\mathbf{d}_N(n)$ in the expansions count (up to signs for the case of $d_N(n)$) the BPS states.

There are two important asymptotic limits of the degeneracies $d_N(n)$ and $\mathbf{d}_N(n)$. First, in the $N \rightarrow \infty$ limit with n fixed, the degeneracies only receive contributions from finite energy states, corresponding to (multi-)supergravitons in the AdS bulk. At large n , they behave as [8]

$$\begin{aligned}
 \log |d_\infty(n)| &= \frac{\sqrt{5}\pi}{3} \sqrt{n} + O(\log n), \\
 \log \mathbf{d}_\infty(n) &= \frac{2^{\frac{1}{6}} \sqrt{3}\pi}{5} n^{\frac{5}{6}} + O(\sqrt{n}). \quad (2.16)
 \end{aligned}$$

Second, in the $N, n \rightarrow \infty$ limit with $j = n/N^2$ fixed, the degeneracies receive contributions from states with energy scaling as N^2 , corresponding to black holes in the bulk. In this

limit, the asymptotic behavior for $d_N(n)$ is [9, 11, 12]

$$\lim_{N \rightarrow \infty} N^{-2} \log |d_N(N^2 j)| = \frac{\pi}{2 \cdot 3^{\frac{1}{6}}} j^{\frac{2}{3}} + O(j^{\frac{1}{3}}), \tag{2.17}$$

whereas the asymptotic behavior of $\mathbf{d}_N(n)$ remains an open problem.

For the index, $d_N(n)$ for ranks $N = 2, \dots, 10$ were computed in [39, 40]. For the BPS partition function, we present in section 4 the results of $\mathbf{d}_N(n)$ for ranks $N = 2, 3, 4$.⁶ Note that n , defined as the eigenvalue of $2(D + J_L^3)$, is not to be confused with the energy E , which is the eigenvalue of D .

2.3 Multi-gravitons from single-trace cohomology

The action (2.10) of the supercharge Q is closed in the space of single-trace BPS words. Hence, we can restrict the cohomology problem to this subspace, where all the single-trace Q -cohomology classes were found in [55]. They are represented by the single-graviton operators

$$\partial_{z^+}^{p_1} \partial_{z^-}^{p_2} \partial_{\theta_1}^{q_1} \partial_{\theta_2}^{q_2} \partial_{\theta_3}^{q_3} \text{Tr} \left[(\partial_{z^+} \Psi)^{k_1} (\partial_{z^-} \Psi)^{k_2} (\partial_{\theta_1} \Psi)^{m_1} (\partial_{\theta_2} \Psi)^{m_2} (\partial_{\theta_3} \Psi)^{m_3} \right] \Big|_{Z=0}. \tag{2.19}$$

It is straightforward to check that (2.19) are Q -closed and also not Q -exact in the space of single-trace words.⁷ The single-trace BPS partition function in the large N limit was computed in [55], which matched perfectly with the single-particle partition function in the bulk theory [8].

Products of the single-graviton operators modulo trace relations represent non-trivial cohomology classes and are dual to multi-graviton states. We will refer to such cohomology classes as being *of multi-graviton form*. The partition function counting the multi-graviton cohomology classes in the large N limit is simply given by the plethystic exponential of the single-trace BPS partition function.

3 A non-renormalization theorem

The above counting of 1/16-BPS states is valid in the weak coupling limit. To compare with the entropy of gravitons and black holes in the bulk type IIB superstring theory, one needs to extend the counting to the regime of strong coupling. Let us analyze more precisely the validity regime of the counting in the coupling space using the relative Lie superalgebra cohomology (2.11). The generators of the Cartan subalgebra of the superconformal algebra

⁶The $U(N)$ and $SU(N)$ BPS partition functions are simply related by

$$\frac{Z_{U(N)}(x, a, b, u, v, w)}{Z_{SU(N)}(x, a, b, u, v, w)} = \exp \left[\sum_{n=1}^{\infty} \frac{1}{n} \left(Z_+(x^n, a^n, u^n, v^n, w^n) + (-1)^{n+1} Z_-(x^n, a^n, u^n, v^n, w^n) \right) \right], \tag{2.18}$$

$$Z_{\pm}(x, a, b, u, v, w) = \frac{1}{2} \left(z(x, a, b, u, v, w) \pm z(-x, a, b, -u, -v, -w) \right),$$

$$z(x, a, b, u, v, w) = \frac{(1+u)(1+v)(1+w)}{(1-a)(1-b)} x - x.$$

⁷Switching the ordering of the letters inside a trace results in operators in the same cohomology class.

all have integer or half-integer eigenvalues, except the dilatation operator D , and hence cannot receive quantum corrections. In the perturbation theory, the dilatation operator is expected to take the form

$$D = D_0 + \frac{g_{\text{YM}}^2 N}{8\pi^2} H + \mathcal{O}(g_{\text{YM}}^3), \quad (3.1)$$

where g_{YM} is the gauge coupling, and H is the one-loop Beisert Hamiltonian [59, 60], which is a differential operator acting on the space of all gauge-invariant words. When restricting to the space of BPS words, H is proportional to the anti-commutator of Q and Q^\dagger , i.e.⁸

$$H \propto \{Q, Q^\dagger\}. \quad (3.3)$$

Hence, our discussion in the previous section shows that in the space of BPS words, the null space of the one-loop Beisert Hamiltonian H , is isomorphic to the cohomology $H^*(\mathcal{G}_N, \mathfrak{sl}_N; \mathbb{C})$.

It was conjectured in [54] that the supersymmetric spectrum of the $\mathcal{N} = 4$ super-Yang-Mills theory on S^3 is exactly given by the ground states of the one-loop Beisert Hamiltonian H without any higher-loop or non-perturbative correction. Nontrivial checks of this conjecture by matching the one-loop partition function at infinite N with the supergraviton partition function in the $\text{AdS}_5 \times S^5$ was reported in [53, 55]. Let us prove the part of the conjecture of vanishing perturbative corrections.

The BPS state counting to all orders in perturbation theory is still governed by a relative Lie superalgebra cohomology. We start by figuring out the relative cochain complex and the differential acting on it. Firstly, BPS operators can be written as linear combinations of BPS words with coefficients in the formal power series ring $\mathbb{C}[[g_{\text{YM}}]]$. We do not need to include non-BPS words because operator-mixing in perturbation theory is only among those with the same angular momenta, R-charges, and classical dimension.⁹ Therefore, in terms of the BPS superfield Ψ , the BPS operators reside in the relative cochain complex $C^*(\mathcal{G}_N, \mathfrak{sl}_N; \mathbb{C}[[g_{\text{YM}}]])$.

Next, it is important to note the advantage of our formulation that we only need the knowledge of the supercharge Q , but not the dilatation operator D . Unlike the dilatation operator which is deformed by the coupling g_{YM} , one can choose a regularization scheme such that the action of Q on the BPS superfield is undeformed perturbatively.¹⁰ Using the Leibniz rule, the Q -action on the BPS superfield (2.10) gives the Q -action on the BPS words, or equivalently the differential acting on the relative cochain complex $C^*(\mathcal{G}_N, \mathfrak{sl}_N; \mathbb{C}[[g_{\text{YM}}]])$.

⁸Furthermore, Q and Q^\dagger at leading order in g_{YM} can be represented by

$$Q = \text{Tr} \left(\Psi \Psi \frac{\partial}{\partial \Psi} \right), \quad Q^\dagger = \text{Tr} \left(\Psi \frac{\partial}{\partial \Psi} \frac{\partial}{\partial \Psi} \right). \quad (3.2)$$

⁹The dilatation operator D obviously commutes with angular momenta and R-charges. In perturbation theory, D also commutes with the classical dimension by dimensional analysis, and the classical dimensions of non-BPS words do not satisfy the BPS condition (2.1).

¹⁰For example, one could use the ‘‘dimensional regularization by dimensional reduction’’ (DRED) scheme [61], or the regularization scheme in [62].

Here, we have assumed that the Q -action satisfies the Leibniz rule in perturbation theory. We do not have a proof, but the only known obstruction, the Konishi anomaly [63], is absent in $\mathcal{N} = 4$ SYM [64].¹¹

In summary, the 1/16-BPS operators are classified by the relative Lie superalgebra cohomology

$$H^*(\mathcal{G}_N, \mathfrak{sl}_N; \mathbb{C}[[g_{\text{YM}}]]) , \tag{3.4}$$

which is isomorphic to the tensor product¹²

$$H^*(\mathcal{G}_N, \mathfrak{sl}_N; \mathbb{C}[[g_{\text{YM}}]]) \cong H^*(\mathcal{G}_N, \mathfrak{sl}_N; \mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}[[g_{\text{YM}}]] . \tag{3.5}$$

Consequently, the 1/16-BPS spectrum is invariant to all orders in perturbation theory. Furthermore, by S-duality, our argument can be applied to all the weakly coupled points on the conformal manifold.

Finally, let us comment on the situation at finite couplings. Barring some subtleties, we expect that the BPS operators are linear combinations of BPS words with coefficients being holomorphic functions on the complex plane of the coupling g_{YM} at a finite distance away from the free points. While we do not know whether the cohomology still has an analogous factorization as (3.5), the fact that the factorization exists at every weakly coupled point hints toward the affirmative.

4 Constructive enumeration and discussions

4.1 Computational scheme

We completely restructure and greatly improve the efficiency of the Mathematica code developed in [55]. Our algorithm can be outlined as follows.

1. List all the single-trace BPS words made out of the BPS letters up to a certain value of $n = n_*$. Express each single-trace BPS word in the component form, and take into account the SU traceless condition by substituting away the (N, N) -entry.
2. List all the multi-trace BPS words in the component form up to $n = n_*$ by combining the single-trace BPS words. Eliminate the linearly dependent multi-trace BPS words due to trace relations. Let V_Y denote the resulting vector space in a fixed-charge sector of (J_L^3, J_R^3, q_j, Y) , with Y (denoting the homogeneous degree in Ψ) explicit for later convenience. See section 2.1 for the notations.

¹¹We thank Davide Gaiotto and Justin Kulp for pointing this subtlety out to us.

¹²The universal coefficient theorem implies

$$H^*(\mathcal{G}_N, \mathfrak{sl}_N; \mathbb{C}[[g_{\text{YM}}]]) \cong \text{Hom}_{\mathbb{C}}(H_*(\mathcal{G}_N, \mathfrak{sl}_N; \mathbb{C}), \mathbb{C}[[g_{\text{YM}}]]) .$$

Viewing $\mathbb{C}[[g_{\text{YM}}]]$ as an infinite product $\mathbb{C} \times \mathbb{C} \times \dots$ allows us to write

$$\text{Hom}_{\mathbb{C}}(H_*(\mathcal{G}_N, \mathfrak{sl}_N; \mathbb{C}), \mathbb{C}[[g_{\text{YM}}]]) \cong \text{Hom}_{\mathbb{C}}(H_*(\mathcal{G}_N, \mathfrak{sl}_N; \mathbb{C}), \mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}[[g_{\text{YM}}]] .$$

Finally, we use the fact that

$$\text{Hom}_{\mathbb{C}}(H_*(\mathcal{G}_N, \mathfrak{sl}_N; \mathbb{C}), \mathbb{C}) \cong H^*(\mathcal{G}_N, \mathfrak{sl}_N; \mathbb{C}) .$$

We thank Dingxin Zhang for the discussion on this point.

3. List all the single-graviton operators, express them in the component form, and then list all the multi-graviton operators by combining single-gravitons. Let W_Y denote the resulting vector space.
4. In the above steps, the presence of fermionic BPS letters means that the ordering of letters cannot be completely forgotten. We fix an ordering on the BPS letters, and for each word commute the letters into the fixed ordering, while carefully keeping track of minus signs from fermionic statistics.
5. Act the supercharge Q on V_Y , and compute bases for the spaces

$$V_Y, \quad QV_Y, \quad \text{span}(W_Y \cup QV_{Y-1})$$

by performing numerical QR decomposition in Julia.¹³ This gives matrix representations of the cochain complex and the embedding of $\text{span}(W_Y \cup QV_{Y-1})$ into V_Y ,

$$\begin{array}{ccccccc}
 \cdots & \xrightarrow{Q} & V_{Y-1} & \xrightarrow{Q} & V_Y & \xrightarrow{Q} & V_{Y+1} \xrightarrow{Q} \cdots \\
 & & & \searrow Q & \uparrow & \nearrow_{Q=0} & \\
 & & & & \text{span}(W_Y \cup QV_{Y-1}) & &
 \end{array} \tag{4.1}$$

The representatives of all the cohomology classes can then be *explicitly* constructed.

6. For the purpose of counting, the dimensionality of the Q -cohomology is given by

$$\dim(V_Y) - \dim(QV_Y) - \dim(QV_{Y-1}),$$

and that restricted to the Q -cohomology with representatives of multi-graviton form is given by

$$\dim(\text{span}(W_Y \cup QV_{Y-1})) - \dim(QV_{Y-1}).$$

4.2 Counting data and a giant-graviton-like feature

We have performed the fully-refined cohomological enumeration comprehensively up to the values of n indicated in table 2. While the holographic duality concerns $SU(N)$ gauge group, the index counting literature mostly considers $U(N)$ gauge group. We present the counting data of both for the convenience of the reader, even though the two are simply related by the contribution of a single decoupled D3-brane. Figure 1 depicts the unrefined counting data. The fully-refined counting data can be publicly accessed on <https://github.com/yinhslin/bps-counting>, and will be continually updated as our computation progresses.

¹³We have been extremely careful with numerical stability. In particular, the non-graviton cohomology classes corresponding to candidate black hole microstates discussed in section 4.3 have been verified with exact row reduction in Mathematica.

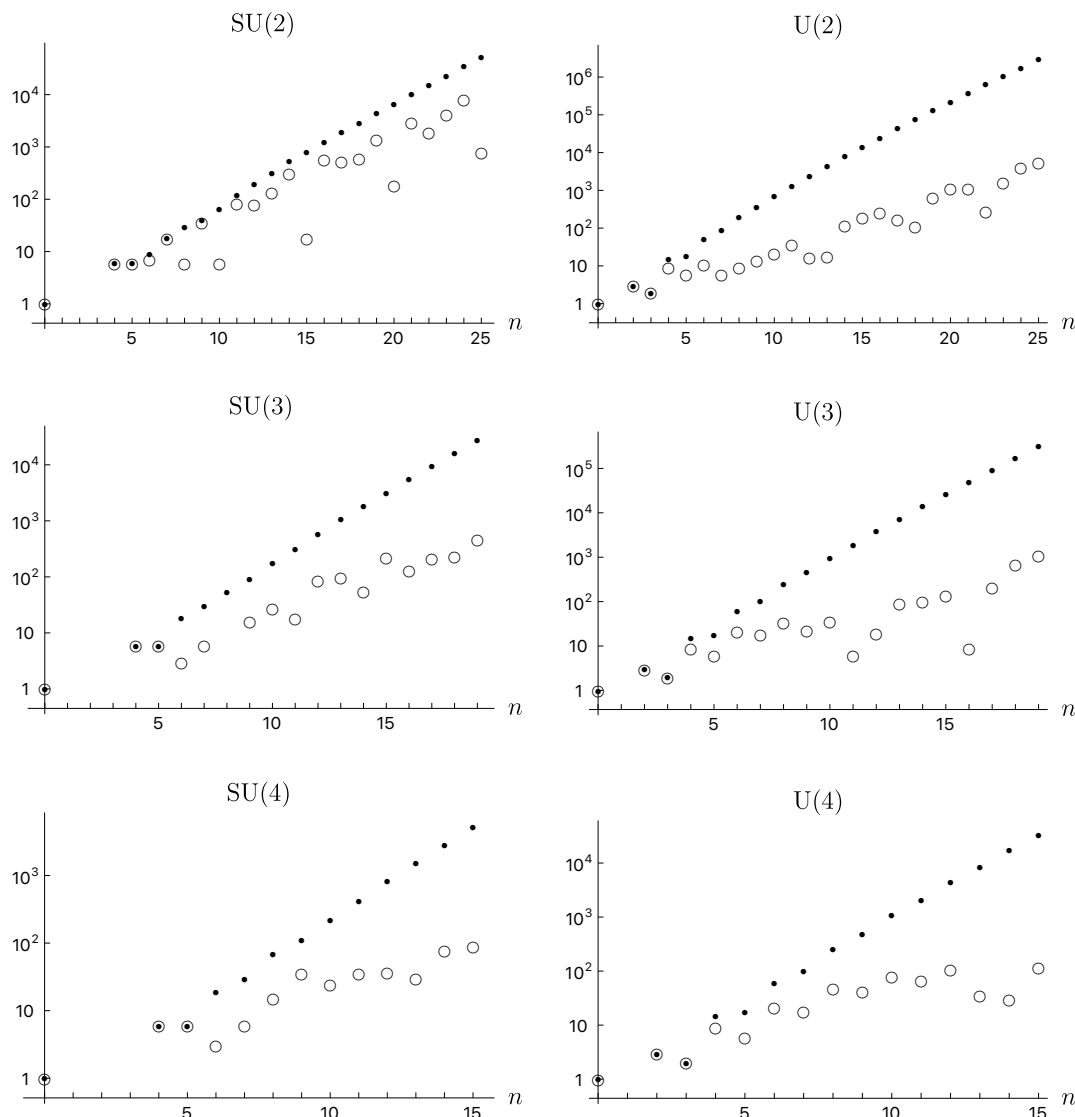


Figure 1. Logarithmic plot of the number of 1/16-BPS states $\mathbf{d}_N(n)$ (dots) and the absolute-valued index $d_N(n)$ (circles) at each n for $SU(N)$ and $U(N)$ gauge groups, $N = 2, 3, 4$.

Inspired by the recent discovery of the giant graviton expansion [65–69], we divide the finite- N BPS partition function (resp. index) by the infinite- N one, and plot the coefficients at each n in figure 2. Note that the counting of 1/16-BPS states at finite- N coincides with infinite- N up to $n = 2N + 1$, as was explained by [39]. A key signature of the giant graviton expansion is the presence of periodic “dips” indicating the contributions of giant gravitons.¹⁴ We see that not only the index but also the partition function exhibits such dips! Such dips come from the sign-changes in the coefficients of $Z_{N=2}/Z_{N=\infty}$. It would be fascinating if a giant graviton expansion of the partition function exists. All in all, we believe that the data we accumulated contains profound information about bulk quantum gravity.

The data underlying figures 1 and 2 are tabulated in tables 3 through 5.

¹⁴We thank Nathan Benjamin and Ji Hoon Lee for a discussion.

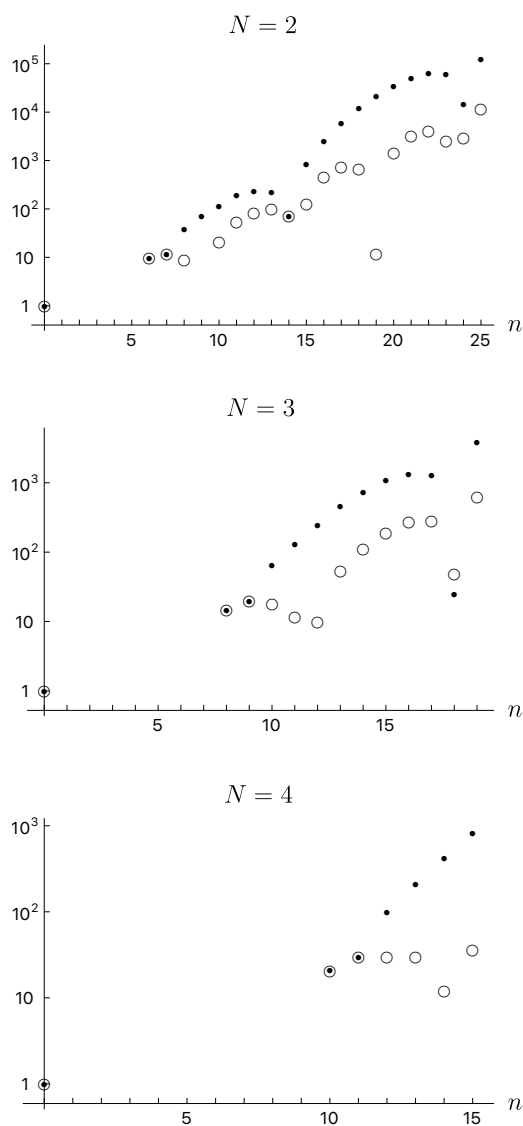


Figure 2. Logarithmic plot of the absolute-valued coefficients of $Z_N/Z_{N=\infty}$ (dots) and $I_N/I_{N=\infty}$ (circles), for $N = 2, 3, 4$.

N	Maximal n
2	25
3	19
4	15

Table 2. The maximal n of comprehensive cohomological enumeration for each N .

n	$\mathbf{d}_{\text{SU}(2)}$	$d_{\text{SU}(2)}$	$\mathbf{d}_{\text{U}(2)}$	$d_{\text{U}(2)}$	$Z_{N=2}/Z_{N=\infty} _{t^n}$	$I_{N=2}/I_{N=\infty} _{t^n}$
0	1	1	1	1	1	1
1	0	0	0	0	0	0
2	0	0	3	3	0	0
3	0	0	2	-2	0	0
4	6	6	15	9	0	0
5	6	-6	18	-6	0	0
6	9	-7	51	11	-10	-10
7	18	18	90	-6	-12	12
8	30	6	195	9	-39	-9
9	40	-36	362	14	-72	0
10	66	6	699	-21	-117	21
11	120	84	1308	36	-198	-54
12	198	-80	2431	-17	-237	83
13	324	-132	4434	-18	-222	-102
14	537	309	8046	114	72	72
15	822	-18	14346	-194	840	128
16	1257	-567	25434	258	2577	-459
17	1944	516	44544	-168	6084	744
18	2959	613	77442	-112	12067	-697
19	4476	-1392	133386	630	21660	12
20	6834	-180	228021	-1089	35166	1440
21	10352	2884	386898	1130	51136	-3240
22	15540	-1926	651843	-273	64368	4182
23	23406	-4242	1091004	-1632	61440	-2580
24	35076	7890	1814578	4104	15129	-2971
25	52020	792	2999724	-5364	-124884	12132

Table 3. Counting of 1/16-BPS states for SU(2) and U(2) gauge groups, and the coefficients of $Z_{N=2}/Z_{N=\infty}$ and $I_{N=2}/I_{N=\infty}$.

n	$\mathbf{d}_{\text{SU}(3)}$	$d_{\text{SU}(3)}$	$\mathbf{d}_{\text{U}(3)}$	$d_{\text{U}(3)}$	$Z_{N=3}/Z_{N=\infty} t^n$	$I_{N=3}/I_{N=\infty} t^n$
0	1	1	1	1	1	1
1	0	0	0	0	0	0
2	0	0	3	3	0	0
3	0	0	2	-2	0	0
4	6	6	15	9	0	0
5	6	-6	18	-6	0	0
6	19	3	61	21	0	0
7	30	6	102	-18	0	0
8	54	0	249	33	-15	-15
9	92	-16	470	-22	-20	20
10	177	27	996	36	-66	-18
11	318	18	1938	6	-132	12
12	595	-87	3865	-19	-246	10
13	1068	96	7458	90	-462	-54
14	1854	54	14373	-99	-735	111
15	3234	-222	27258	138	-1106	-190
16	5610	132	51339	-9	-1371	279
17	9558	210	95562	-210	-1284	-288
18	16329	-235	176594	672	25	49
19	27612	-468	323208	-1116	3870	630

Table 4. Counting of 1/16-BPS states for SU(3) and U(3) gauge groups, and the coefficients of $Z_{N=3}/Z_{N=\infty}$ and $I_{N=3}/I_{N=\infty}$.

4.3 Black hole microstates

For SU(2) gauge group, we found that up to $n = 25$, almost all cohomology elements have a representative that is of multi-graviton form, *except* one element at $E = 19/2$ and $n = 24$ with the total number of derivatives given by $(\#\partial_{z^\pm}, \#\partial_{\theta_i}) = (0, 0, 4, 4, 4)$, as well as six elements at $E = 10$ and $n = 25$ with $(\#\partial_{z^\pm}, \#\partial_{\theta_i}) = (0, 1, 3, 4, 4)$ plus permutations. For $N = 3, 4$, every state is found to be of multi-graviton form, up to the values of n indicated in table 2. The scarcity of candidates for black hole microstates at computationally-accessible charges explains the then-negative result of [55]. Note that the enigma there was due to the lack of any *candidates* for black hole microstates. We are by no means suggesting that every state not of multi-graviton form must be a black hole microstate, but the discovery of viable candidates is reassuring.

n	$\mathbf{d}_{\text{SU}(4)}$	$d_{\text{SU}(4)}$	$\mathbf{d}_{\text{U}(4)}$	$d_{\text{U}(4)}$	$Z_{N=4}/Z_{N=\infty} _{t^n}$	$I_{N=4}/I_{N=\infty} _{t^n}$
0	1	1	1	1	1	1
1	0	0	0	0	0	0
2	0	0	3	3	0	0
3	0	0	2	-2	0	0
4	6	6	15	9	0	0
5	6	-6	18	-6	0	0
6	19	3	61	21	0	0
7	30	6	102	-18	0	0
8	69	15	264	48	0	0
9	112	-36	490	-42	0	0
10	222	24	1086	78	-21	-21
11	420	36	2130	-66	-30	30
12	831	-37	4411	107	-100	-30
13	1530	-30	8676	-36	-210	30
14	2844	78	17280	30	-420	-12
15	5220	88	33670	114	-832	-36

Table 5. Counting of 1/16-BPS states for SU(4) and U(4) gauge groups, and the coefficients of $Z_{N=4}/Z_{N=\infty}$ and $I_{N=4}/I_{N=\infty}$.

For SU(2), the smallest value of $n = 24$ at which a non-graviton state is found coincides with the location of the second dip of $Z_{N=2}/Z_{N=\infty}$ as shown in figure 2. Could these non-graviton states have interpretations in terms of giant gravitons?¹⁵

We stress that our search is comprehensive n -by- n , and not energy-by-energy. By the BPS condition, states with a fixed n have energies bounded below by $E \geq [2n/3]/2$ for $n > 9$. Our exhaustive search for SU(2) up to $n = 25$ proves that there is no black hole microstate for $E < 9$. By comparison, the expectation from the bulk side, valid at large N , is that black hole microstates should show up at energies $E \sim N^2$ or higher [8]. Hence, the actual lowest energy for $N = 2$ is significantly higher than this bulk expectation.

Furthermore, while we have obtained explicit expressions for the space of representatives of each cohomology class, the precise 1/16-BPS operator must be annihilated by Q^\dagger , and is hence a perturbative series in the coupling. Demanding the annihilation by the one-loop Q^\dagger (3.2) will give the weak coupling of the 1/16-BPS operator, but it is unclear whether distinct properties of black hole microstates persist in the transition from strong to weak coupling.

¹⁵One important caveat is that the known giant gravitons have vanishing Lorentz spin, but the non-graviton states we found have nonzero Lorentz spin. We thank Ji Hoon Lee for pointing this out.

4.4 Outlook

What are the salient features of black hole microstates when compared to multi-graviton states? Are there linguistic rules governing the words and letters that underlie information-theoretic properties of black holes, such as chaos? Compared to holographic descriptions of black holes in the canonical ensemble, such as the thermofield double [70], our microcanonical data contains different information and may potentially allow the direct study of things like the eigenstate thermalization hypothesis [71–73]. We envision the computation performed in this paper to be the beginning of a “black hole genome project” that aims to provide key clues for these problems. To this end, a more efficient way of enumerating 1/16-BPS states and black hole microstates is highly desired. The superconformal index admits an integral formula over (special) unitary matrices from the supersymmetric localization of the $\mathcal{N} = 4$ SYM path integral. Could the BPS partition function also be computed by localization techniques?

While there is a closed form expression for the multi-graviton states as products of the single-graviton operators (2.19), we do not have a simple way to write down the operators that represent the non-graviton cohomology classes. A special class of 1/16-BPS operators was written down in determinant form in [52, 74]. It would be interesting to investigate whether those determinant operators are of multi-graviton form, or whether they provide words for non-graviton cohomology classes.

Finally, the one-loop Beisert Hamiltonian in various subsectors has been explored in [75–78], and dubbed as the “spin matrix theory” that describes the dynamics of near-BPS states. We have shown that the Beisert Hamiltonian takes a very simple form (3.2) and (3.3) in terms of the BPS superfield Ψ , which efficiently organizes all the BPS letters. This could help the study of the spin matrix theory in the largest subsector, the $\text{psu}(1, 2|3)$ subsector. On the bulk side, it has been proposed that the near-BPS states describe the near-horizon excitations of black holes, which are captured by the $\mathcal{N} = 2$ Schwarzian theory [79]. It would be fascinating to identify a Schwarzian sector of the spin matrix theory or the quantum mechanics of the all-loop Beisert Hamiltonian.

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References

- [1] EVENT HORIZON TELESCOPE collaboration, *First M87 Event Horizon Telescope Results. IV. Imaging the Central Supermassive Black Hole*, *Astrophys. J. Lett.* **875** (2019) L4 [[arXiv:1906.11241](#)] [[INSPIRE](#)].
- [2] A. Strominger and C. Vafa, *Microscopic origin of the Bekenstein-Hawking entropy*, *Phys. Lett. B* **379** (1996) 99 [[hep-th/9601029](#)] [[INSPIRE](#)].
- [3] J.B. Gutowski and H.S. Reall, *Supersymmetric AdS₅ black holes*, *JHEP* **02** (2004) 006 [[hep-th/0401042](#)] [[INSPIRE](#)].
- [4] J.B. Gutowski and H.S. Reall, *General supersymmetric AdS₅ black holes*, *JHEP* **04** (2004) 048 [[hep-th/0401129](#)] [[INSPIRE](#)].
- [5] Z.-W. Chong, M. Cvetič, H. Lu and C.N. Pope, *General non-extremal rotating black holes in minimal five-dimensional gauged supergravity*, *Phys. Rev. Lett.* **95** (2005) 161301 [[hep-th/0506029](#)] [[INSPIRE](#)].
- [6] Z.W. Chong, M. Cvetič, H. Lu and C.N. Pope, *Five-dimensional gauged supergravity black holes with independent rotation parameters*, *Phys. Rev. D* **72** (2005) 041901 [[hep-th/0505112](#)] [[INSPIRE](#)].
- [7] H.K. Kunduri, J. Lucietti and H.S. Reall, *Supersymmetric multi-charge AdS₅ black holes*, *JHEP* **04** (2006) 036 [[hep-th/0601156](#)] [[INSPIRE](#)].
- [8] J. Kinney, J.M. Maldacena, S. Minwalla and S. Raju, *An Index for 4 dimensional super conformal theories*, *Commun. Math. Phys.* **275** (2007) 209 [[hep-th/0510251](#)] [[INSPIRE](#)].
- [9] A. Cabo-Bizet, D. Cassani, D. Martelli and S. Murthy, *Microscopic origin of the Bekenstein-Hawking entropy of supersymmetric AdS₅ black holes*, *JHEP* **10** (2019) 062 [[arXiv:1810.11442](#)] [[INSPIRE](#)].
- [10] S.M. Hosseini, K. Hristov and A. Zaffaroni, *An extremization principle for the entropy of rotating BPS black holes in AdS₅*, *JHEP* **07** (2017) 106 [[arXiv:1705.05383](#)] [[INSPIRE](#)].
- [11] S. Choi, J. Kim, S. Kim and J. Nahmgoong, *Large AdS black holes from QFT*, [arXiv:1810.12067](#) [SNUTP18-005] [[INSPIRE](#)].
- [12] F. Benini and E. Milan, *Black Holes in 4D $\mathcal{N} = 4$ Super-Yang-Mills Field Theory*, *Phys. Rev. X* **10** (2020) 021037 [[arXiv:1812.09613](#)] [[INSPIRE](#)].
- [13] F. Benini, K. Hristov and A. Zaffaroni, *Black hole microstates in AdS₄ from supersymmetric localization*, *JHEP* **05** (2016) 054 [[arXiv:1511.04085](#)] [[INSPIRE](#)].
- [14] F. Benini, K. Hristov and A. Zaffaroni, *Exact microstate counting for dyonic black holes in AdS₄*, *Phys. Lett. B* **771** (2017) 462 [[arXiv:1608.07294](#)] [[INSPIRE](#)].
- [15] S. Choi, J. Kim, S. Kim and J. Nahmgoong, *Comments on deconfinement in AdS/CFT*, [arXiv:1811.08646](#) [SNUTP18-007] [[INSPIRE](#)].
- [16] M. Honda, *Quantum Black Hole Entropy from 4d Supersymmetric Cardy formula*, *Phys. Rev. D* **100** (2019) 026008 [[arXiv:1901.08091](#)] [[INSPIRE](#)].
- [17] M. Fluder, S.M. Hosseini and C.F. Uhlemann, *Black hole microstate counting in Type IIB from 5d SCFTs*, *JHEP* **05** (2019) 134 [[arXiv:1902.05074](#)] [[INSPIRE](#)].
- [18] A. Arabi Ardehali, *Cardy-like asymptotics of the 4d $\mathcal{N} = 4$ index and AdS₅ blackholes*, *JHEP* **06** (2019) 134 [[arXiv:1902.06619](#)] [[INSPIRE](#)].

- [19] J. Kim, S. Kim and J. Song, *A $4d \mathcal{N} = 1$ Cardy Formula*, *JHEP* **01** (2021) 025 [[arXiv:1904.03455](#)] [[INSPIRE](#)].
- [20] J.P. Gauntlett, D. Martelli and J. Sparks, *Toric geometry and the dual of \mathcal{I} -extremization*, *JHEP* **06** (2019) 140 [[arXiv:1904.04282](#)] [[INSPIRE](#)].
- [21] S.M. Hosseini and A. Zaffaroni, *Geometry of \mathcal{I} -extremization and black holes microstates*, *JHEP* **07** (2019) 174 [[arXiv:1904.04269](#)] [[INSPIRE](#)].
- [22] H. Kim and N. Kim, *Black holes with baryonic charge and \mathcal{I} -extremization*, *JHEP* **11** (2019) 050 [[arXiv:1904.05344](#)] [[INSPIRE](#)].
- [23] A. Cabo-Bizet, D. Cassani, D. Martelli and S. Murthy, *The asymptotic growth of states of the $4d \mathcal{N} = 1$ superconformal index*, *JHEP* **08** (2019) 120 [[arXiv:1904.05865](#)] [[INSPIRE](#)].
- [24] A. Amariti, I. Garozzo and G. Lo Monaco, *Entropy function from toric geometry*, *Nucl. Phys. B* **973** (2021) 115571 [[arXiv:1904.10009](#)] [[INSPIRE](#)].
- [25] D. Gang, N. Kim and L.A. Pando Zayas, *Precision Microstate Counting for the Entropy of Wrapped $M5$ -branes*, *JHEP* **03** (2020) 164 [[arXiv:1905.01559](#)] [[INSPIRE](#)].
- [26] F. Larsen, J. Nian and Y. Zeng, *AdS_5 black hole entropy near the BPS limit*, *JHEP* **06** (2020) 001 [[arXiv:1907.02505](#)] [[INSPIRE](#)].
- [27] G. Kántor, C. Papageorgakis and P. Richmond, *AdS_7 black-hole entropy and $5D \mathcal{N} = 2$ Yang-Mills*, *JHEP* **01** (2020) 017 [[arXiv:1907.02923](#)] [[INSPIRE](#)].
- [28] A. González Lezcano and L.A. Pando Zayas, *Microstate counting via Bethe Ansatz in the $4d \mathcal{N} = 1$ superconformal index*, *JHEP* **03** (2020) 088 [[arXiv:1907.12841](#)] [[INSPIRE](#)].
- [29] A. Lanir, A. Nedelin and O. Sela, *Black hole entropy function for toric theories via Bethe Ansatz*, *JHEP* **04** (2020) 091 [[arXiv:1908.01737](#)] [[INSPIRE](#)].
- [30] S. Choi, C. Hwang and S. Kim, *Quantum vortices, $M2$ -branes and black holes*, [arXiv:1908.02470](#) [SNUTP18-003] [[INSPIRE](#)].
- [31] N. Bobev and P.M. Cricigno, *Universal spinning black holes and theories of class \mathcal{R}* , *JHEP* **12** (2019) 054 [[arXiv:1909.05873](#)] [[INSPIRE](#)].
- [32] J. Nian and L.A. Pando Zayas, *Microscopic entropy of rotating electrically charged AdS_4 black holes from field theory localization*, *JHEP* **03** (2020) 081 [[arXiv:1909.07943](#)] [[INSPIRE](#)].
- [33] A. Cabo-Bizet and S. Murthy, *Supersymmetric phases of $4d \mathcal{N} = 4$ SYM at large N* , *JHEP* **09** (2020) 184 [[arXiv:1909.09597](#)] [[INSPIRE](#)].
- [34] F. Benini, D. Gang and L.A. Pando Zayas, *Rotating Black Hole Entropy from $M5$ Branes*, *JHEP* **03** (2020) 057 [[arXiv:1909.11612](#)] [[INSPIRE](#)].
- [35] K. Goldstein et al., *Probing the EVH limit of supersymmetric AdS black holes*, *JHEP* **02** (2020) 154 [[arXiv:1910.14293](#)] [[INSPIRE](#)].
- [36] A. Arabi Ardehali, J. Hong and J.T. Liu, *Asymptotic growth of the $4d \mathcal{N} = 4$ index and partially deconfined phases*, *JHEP* **07** (2020) 073 [[arXiv:1912.04169](#)] [[INSPIRE](#)].
- [37] M. David, J. Nian and L.A. Pando Zayas, *Gravitational Cardy Limit and AdS Black Hole Entropy*, *JHEP* **11** (2020) 041 [[arXiv:2005.10251](#)] [[INSPIRE](#)].
- [38] A. Cabo-Bizet, D. Cassani, D. Martelli and S. Murthy, *The large- N limit of the $4d \mathcal{N} = 1$ superconformal index*, *JHEP* **11** (2020) 150 [[arXiv:2005.10654](#)] [[INSPIRE](#)].
- [39] S. Murthy, *The growth of the $\frac{1}{16}$ -BPS index in $4d \mathcal{N} = 4$ SYM*, [arXiv:2005.10843](#) [[INSPIRE](#)].

- [40] P. Agarwal et al., *AdS black holes and finite N indices*, *Phys. Rev. D* **103** (2021) 126006 [[arXiv:2005.11240](#)] [[INSPIRE](#)].
- [41] F. Benini et al., *Superconformal indices at large N and the entropy of $AdS_5 \times SE_5$ black holes*, *Class. Quant. Grav.* **37** (2020) 215021 [[arXiv:2005.12308](#)] [[INSPIRE](#)].
- [42] A. González Lezcano, J. Hong, J.T. Liu and L.A. Pando Zayas, *Sub-leading Structures in Superconformal Indices: Subdominant Saddles and Logarithmic Contributions*, *JHEP* **01** (2021) 001 [[arXiv:2007.12604](#)] [[INSPIRE](#)].
- [43] C. Copetti, A. Grassi, Z. Komargodski and L. Tizzano, *Delayed deconfinement and the Hawking-Page transition*, *JHEP* **04** (2022) 132 [[arXiv:2008.04950](#)] [[INSPIRE](#)].
- [44] K. Goldstein et al., *Residues, modularity, and the Cardy limit of the 4d $\mathcal{N} = 4$ superconformal index*, *JHEP* **04** (2021) 216 [[arXiv:2011.06605](#)] [[INSPIRE](#)].
- [45] A. Cabo-Bizet, *From multi-gravitons to Black holes: The role of complex saddles*, [arXiv:2012.04815](#) [[INSPIRE](#)].
- [46] A. Amariti, M. Fazzi and A. Segati, *The SCI of $\mathcal{N} = 4$ $USp(2N_c)$ and $SO(N_c)$ SYM as a matrix integral*, *JHEP* **06** (2021) 132 [[arXiv:2012.15208](#)] [[INSPIRE](#)].
- [47] S. Choi, S. Jeong and S. Kim, *The Yang-Mills duals of small AdS black holes*, [arXiv:2103.01401](#) [SNUTP21-001] [[INSPIRE](#)].
- [48] A. Amariti, M. Fazzi and A. Segati, *Expanding on the Cardy-like limit of the SCI of 4d $\mathcal{N} = 1$ ABCD SCFTs*, *JHEP* **07** (2021) 141 [[arXiv:2103.15853](#)] [[INSPIRE](#)].
- [49] V. Jejjala, Y. Lei, S. van Leuven and W. Li, *$SL(3, \mathbb{Z})$ Modularity and New Cardy limits of the $\mathcal{N} = 4$ superconformal index*, *JHEP* **11** (2021) 047 [[arXiv:2104.07030](#)] [[INSPIRE](#)].
- [50] D. Cassani, J.P. Gauntlett, D. Martelli and J. Sparks, *Thermodynamics of accelerating and supersymmetric AdS_4 black holes*, *Phys. Rev. D* **104** (2021) 086005 [[arXiv:2106.05571](#)] [[INSPIRE](#)].
- [51] N. Ezroua, F. Larsen, Z. Liu and Y. Zeng, *The phase diagram of BPS black holes in AdS_5* , *JHEP* **09** (2022) 033 [[arXiv:2108.11542](#)] [[INSPIRE](#)].
- [52] M. Berkooz, D. Reichmann and J. Simon, *A Fermi Surface Model for Large Supersymmetric AdS_5 Black Holes*, *JHEP* **01** (2007) 048 [[hep-th/0604023](#)] [[INSPIRE](#)].
- [53] R.A. Janik and M. Trzetrzelewski, *Supergravitons from one loop perturbative $N = 4$ SYM*, *Phys. Rev. D* **77** (2008) 085024 [[arXiv:0712.2714](#)] [[INSPIRE](#)].
- [54] L. Grant, P.A. Grassi, S. Kim and S. Minwalla, *Comments on 1/16 BPS Quantum States and Classical Configurations*, *JHEP* **05** (2008) 049 [[arXiv:0803.4183](#)] [[INSPIRE](#)].
- [55] C.-M. Chang and X. Yin, *1/16 BPS states in $\mathcal{N} = 4$ super-Yang-Mills theory*, *Phys. Rev. D* **88** (2013) 106005 [[arXiv:1305.6314](#)] [[INSPIRE](#)].
- [56] T. Harmark, K.R. Kristjansson and M. Orselli, *Decoupling limits of $N = 4$ super Yang-Mills on $R \times S^3$* , *JHEP* **09** (2007) 115 [[arXiv:0707.1621](#)] [[INSPIRE](#)].
- [57] K.A. Intriligator, *Bonus symmetries of $N = 4$ superYang-Mills correlation functions via AdS duality*, *Nucl. Phys. B* **551** (1999) 575 [[hep-th/9811047](#)] [[INSPIRE](#)].
- [58] K.A. Intriligator and W. Skiba, *Bonus symmetry and the operator product expansion of $N = 4$ SuperYang-Mills*, *Nucl. Phys. B* **559** (1999) 165 [[hep-th/9905020](#)] [[INSPIRE](#)].
- [59] N. Beisert, *The complete one loop dilatation operator of $N = 4$ superYang-Mills theory*, *Nucl. Phys. B* **676** (2004) 3 [[hep-th/0307015](#)] [[INSPIRE](#)].

- [60] N. Beisert, *The Dilatation operator of $N = 4$ super Yang-Mills theory and integrability*, *Phys. Rept.* **405** (2004) 1 [[hep-th/0407277](#)] [[INSPIRE](#)].
- [61] W. Siegel, *Supersymmetric Dimensional Regularization via Dimensional Reduction*, *Phys. Lett. B* **84** (1979) 193 [[INSPIRE](#)].
- [62] T. Hayashi, Y. Ohshima, K. Okuyama and H. Suzuki, *Invariant regularization of supersymmetric chiral gauge theory*, *Prog. Theor. Phys.* **100** (1998) 627 [[hep-th/9801062](#)] [[INSPIRE](#)].
- [63] K. Konishi, *Anomalous Supersymmetry Transformation of Some Composite Operators in SQCD*, *Phys. Lett. B* **135** (1984) 439 [[INSPIRE](#)].
- [64] A. Kostyuk and H. Stoecker, *Evidence for psi-prime regeneration in heavy ion collisions*, [nucl-th/0501077](#) [[INSPIRE](#)].
- [65] R. Arai and Y. Imamura, *Finite N Corrections to the Superconformal Index of S -fold Theories*, *PTEP* **2019** (2019) 083B04 [[arXiv:1904.09776](#)] [[INSPIRE](#)].
- [66] R. Arai, S. Fujiwara, Y. Imamura and T. Mori, *Schur index of the $\mathcal{N} = 4$ $U(N)$ supersymmetric Yang-Mills theory via the AdS/CFT correspondence*, *Phys. Rev. D* **101** (2020) 086017 [[arXiv:2001.11667](#)] [[INSPIRE](#)].
- [67] Y. Imamura, *Finite- N superconformal index via the AdS/CFT correspondence*, *PTEP* **2021** (2021) 123B05 [[arXiv:2108.12090](#)] [[INSPIRE](#)].
- [68] D. Gaiotto and J.H. Lee, *The Giant Graviton Expansion*, [arXiv:2109.02545](#) [[INSPIRE](#)].
- [69] J.H. Lee, *Exact stringy microstates from gauge theories*, *JHEP* **11** (2022) 137 [[arXiv:2204.09286](#)] [[INSPIRE](#)].
- [70] J.M. Maldacena, *Eternal black holes in anti-de Sitter*, *JHEP* **04** (2003) 021 [[hep-th/0106112](#)] [[INSPIRE](#)].
- [71] J. M. Deutsch, *Quantum statistical mechanics in a closed system*, *Phys. Rev. A* **43** (1991) 2046.
- [72] M. Srednicki, *Chaos and Quantum Thermalization*, [cond-mat/9403051](#) [[DOI:10.1103/PhysRevE.50.888](#)] [[INSPIRE](#)].
- [73] M. Rigol, V. Dunjko and M. Olshanii, *Thermalization and its mechanism for generic isolated quantum systems*, *Nature* **452** (2008) 854.
- [74] M. Berkooz and D. Reichmann, *Weakly Renormalized Near $1/16$ SUSY Fermi Liquid Operators in $N = 4$ SYM*, *JHEP* **10** (2008) 084 [[arXiv:0807.0559](#)] [[INSPIRE](#)].
- [75] T. Harmark and M. Orselli, *Spin Matrix Theory: A quantum mechanical model of the AdS/CFT correspondence*, *JHEP* **11** (2014) 134 [[arXiv:1409.4417](#)] [[INSPIRE](#)].
- [76] S. Baiguera, T. Harmark and N. Wintergerst, *Nonrelativistic near-BPS corners of $\mathcal{N} = 4$ super-Yang-Mills with $SU(1,1)$ symmetry*, *JHEP* **02** (2021) 188 [[arXiv:2009.03799](#)] [[INSPIRE](#)].
- [77] S. Baiguera, T. Harmark, Y. Lei and N. Wintergerst, *Symmetry structure of the interactions in near-BPS corners of $\mathcal{N} = 4$ super-Yang-Mills*, *JHEP* **04** (2021) 029 [[arXiv:2012.08532](#)] [[INSPIRE](#)].
- [78] S. Baiguera, T. Harmark and Y. Lei, *Spin Matrix Theory in near $\frac{1}{8}$ -BPS corners of $\mathcal{N} = 4$ super-Yang-Mills*, *JHEP* **02** (2022) 191 [[arXiv:2111.10149](#)] [[INSPIRE](#)].
- [79] J. Boruch, M.T. Heydeman, L.V. Iliesiu and G.J. Turiaci, *BPS and near-BPS black holes in AdS_5 and their spectrum in $\mathcal{N} = 4$ SYM*, [arXiv:2203.01331](#) [[INSPIRE](#)].