

Four-loop HQET propagators from the DRA method

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ABSTRACT: We use dimensional recurrence relations and analyticity to calculate four-loop propagator-type master integrals in the heavy-quark effective theory. Compared to previous applications of the DRA method, we apply a new technique of fixing homogeneous solutions from pole parts of integrals evaluated in different rational space-time dimension points. The latter were calculated from the integration-by-parts reduction of finite integrals in shifted space-time dimension and/or with increased propagators powers. We provide results for epsilon expansions of master integrals near $d = 4$ and $d = 3$ using constructed alternative sets of integrals with expansion coefficients having conjectural uniform transcendental weight.

KEYWORDS: Effective Field Theories, Higher-Order Perturbative Calculations, Specific QCD Phenomenology

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1 Introduction

As is well known, massive internal lines in the diagrams bring much complexity in the calculations. However, there are two well-known limiting cases where the analysis simplifies: when the lines are massless and they are infinitely heavy. In particular, this fact justifies the utility of the heavy quark effective theory (HQET) [1]. In HQET, in its simplest form, in addition to a single infinitely heavy particle, one considers massless particles only. Within this theory, the propagator-type integrals are functions with trivial dependence on a single dimensionful parameter, the residual energy ω , and the method of differential equations can not be applied, at least, directly.

Possible applications of such integrals include calculating the heavy quark field anomalous dimension [2, 3], the small-angle expansion of cusp anomalous dimension, and the correlators of various currents in HQET [2, 4]. For example, recently, using integrals calculated in the present paper, the four-loop expression for the heavy quark anomalous dimension and first two terms of the small-angle expansion of the QCD cusp anomalous dimension were calculated in ref. [3].

In three-loop calculations there are only eight propagator-type master integrals, and all of them, except one, are known for arbitrary space-time dimension in terms of hypergeometric functions [5, 6]. The last non-trivial master integral has been calculated up to ϵ^1 terms in ref. [7] using its relation to the three-loop on-shell master integral. Techniques used in three-loop calculation are difficult to apply at four loops due to a large number of master integrals and their grown complexity. Therefore, we choose to switch to a more effective *Dimensional Recurrence and Analyticity* (DRA) technique [8]. This method is based on constructing the general solutions of the dimensional recurrence relations [9] in the form of triangular series and using the analytical properties of the integrals as the functions of space-time dimension d to fix the undetermined periodic functions. The derivation

of dimensional recurrence relations and the construction of their general solutions can be done rather easily using `LiteRed` [10] and `SummerTime` [11] packages. In the present paper we demonstrate that the remaining task of fixing the homogeneous solution can also be accomplished in a quasi-automatic fashion. Our approach is based on deriving required constraints in a specific rational point $d = d_0$ by generating a sufficient number of integrals finite in this point and then reducing them to master integrals. It appears that the finiteness of the initial integrals provides a highly redundant set of constraints on the expansion coefficients of master integrals around d_0 , which can be used not only to fix the periodic functions in general solution, but also to safely crosscheck the obtained results.

The paper is organized as follows. In section 2 we describe some details of the DRA method as applied to our problem, and in section 3 we provide an explicit example of the calculation. Section 4 contains description of the obtained results for general d and for ϵ -expansion near $d = 4$ and $d = 3$. We conclude in section 5.

2 Method of calculation

We consider the propagator-type diagrams obtained by attaching an n -legged graph with massless lines to n points on a single HQET line. For four loops we have $2 \leq n \leq 8$, however for the master integrals identification we can restrict ourselves with $n = 3, 4$, or 5 as other cases reduce to these by partial fractioning. Performing the IBP reduction of the remaining 19 big topologies, we end up with 54 master integrals shown in figure 1. In this figure, the notation $[\text{CL}n]$ with $n = 0, 1, 2, 3, 4$ denotes the complexity level of the integral, i.e. the maximal depth of multiple sums which enter its DRA representation. Note that many of $[\text{CL}0]$ and $[\text{CL}1]$ integrals listed in figure 1 can be obtained from the literature [5, 7, 12].

The dimensional recurrence relations have the form

$$\mathbf{J}(d+2) = L(d)\mathbf{J}(d). \tag{2.1}$$

The application of the DRA method is straightforward provided the matrix $L(d)$ is lower-triangular. This property is obvious for the case when there are no more than one master integral in each sector. In our case the integrals J_{21} and J_{22} belong to the same sector, but, fortunately, the corresponding block in the matrix L is diagonal. The triangular form of the matrix $L(d)$ results to the first-order inhomogeneous difference equation for each J_k :

$$J_k(d+2) = L_{kk}(d)J_k(d) + \sum_{l < k} L_{kl}(d)J_l(d). \tag{2.2}$$

Let us assume that $J_l(d)$ for $l < k$ are already calculated by the same method. Then the general solution of eq. (2.2) can be written as

$$J_k(d) = S^{-1}(d)\omega(d) + \mathcal{R}_k(d), \tag{2.3}$$

where $\mathcal{R}_k(d)$ is a specific solution of inhomogeneous equation, $\omega(d) = \omega(d+2)$ is an arbitrary periodic factor, and the *summing factor* $S(d)$ is a specific solution of

$$S(d) = L_{kk}(d)S(d+2). \tag{2.4}$$

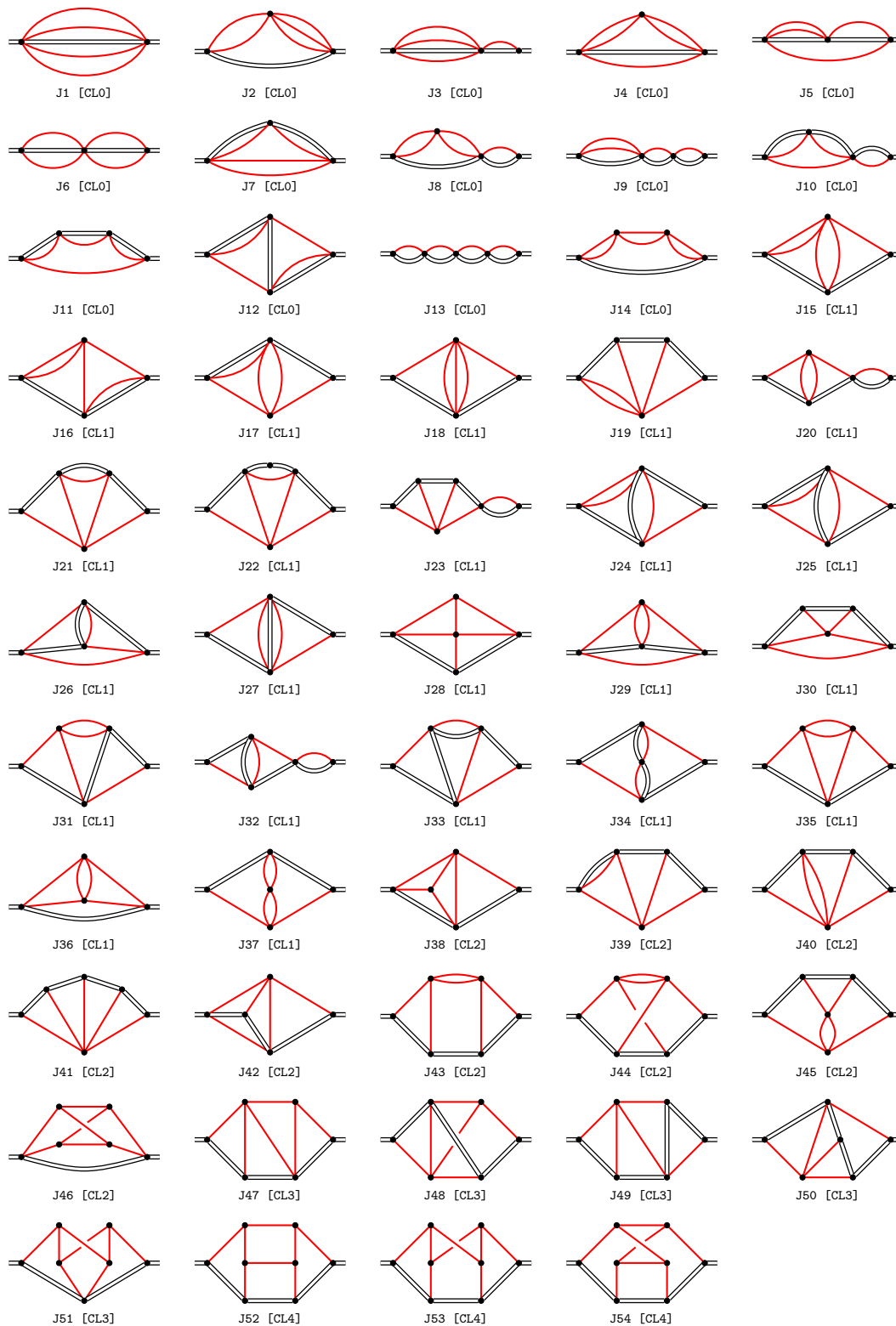


Figure 1. Four-loop propagator-type HQET integrals calculated in this paper. Red solid lines correspond to massless propagators $[-l^2 - i0]^{-1}$, double lines correspond to the HQET propagator $[1 - 2l \cdot n - i0]^{-1}$, (where $n^2 = 1$). The loop integration measure is $\frac{d^d l}{i\pi^{d/2}}$.

The specific inhomogeneous solution \mathcal{R}_k can be expressed in terms of triangular sums, see ref. [8] for details.

In order to fix the function $\omega(z)$ we need to obtain sufficient information about the analytical properties of J_k as a function of d on the chosen *basic stripe* \mathcal{B} — a vertical stripe of width 2 in the complex plane of d . We find it sufficient to use for all master integrals J_k one and the same basic stripe

$$\mathcal{B} = \{d \in \mathbb{C} \mid 0 < \Re d \leq 2\}. \tag{2.5}$$

The conventional approach to obtain the required analytical data is the following. First, one defines the positions of possible singularities on the basic stripe using **Fiesta**'s [13] routine **SDAnalyze**. Then, for each of the found position one tries to find the order of the pole and a few leading coefficients of Laurent expansion near it. For these goals one often needs to derive the Mellin-Barnes representation and to use the MB code [14], although for some simple cases it might be sufficient to use **Fiesta**'s routine **SDEvaluate** alone. This approach usually gives a few first terms of Laurent expansion as limited-precision numbers and should be complemented by the educated guess about their analytical form.

The main drawback of this approach is that the derivation of the required Mellin-Barnes representation requires a substantial amount of manual work. In the present paper we use an alternative approach which appears to provide more than enough information about the analytical properties of the master integrals. First, instead of using **Fiesta**'s **SDAnalyze**, we follow a less specific but much more simple way to restrict possible positions of singularities. Namely, we analyze the position of poles in the matrix of dimensional recurrence relations and assume that the singularities may differ from those poles by a multiple of 2. In this way we obtain the following set of potential positions of poles on the basic stripe:

$$\mathcal{S} = \left\{ \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, 1, \frac{6}{5}, \frac{5}{4}, \frac{4}{3}, \frac{3}{2}, \frac{8}{5}, \frac{5}{3}, \frac{7}{4}, 2 \right\}. \tag{2.6}$$

The main step of our approach is to consider large enough set of integrals in some chosen $d = d_0 - 2\epsilon + 2k$, with $d_0 \in \mathcal{S}$ and $k \in \mathbb{Z}$, *finite* at $\epsilon = 0$ and to reduce them to master integrals J_1, \dots, J_{54} in $d = d_0 - 2\epsilon$ using IBP identities and dimensional recurrence relations. Then, the finiteness of the obtained expression implies some constraints on the ϵ -expansion coefficients of master integrals at $d = d_0 - 2\epsilon$. This approach is very similar to the one that was successfully used for the calculation of four-loop [15] and five-loop [16] massless propagators.¹ The only difference is that for our present purposes we need to consider not a single value of d_0 , but all points in \mathcal{S} . In order to pick a set of finite integrals we use the algorithm of ref. [17] as implemented in the public code **Reduze2** [18]. As the existing implementation supports only even d_0 , we had to slightly modify the routines of **Reduze2** code to support arbitrary rational d_0 .

¹In the calculations of massless propagators it was important to use also the additional Glue-and-Cut symmetry.

3 Example: J_{21} integral

Let us describe our method in some details on the example of integral J_{21} ,

$$J_{21}(d) = \int \frac{dl_1 dl_2 dl_3 dl_4 / (i\pi^{d/2})^4}{[-l_1^2][-l_{13}^2][-l_{23}^2][-l_{34}^2][-l_4^2][1 - 2l_1 \cdot n][1 - 2l_2 \cdot n][1 - 2l_4 \cdot n]}, \quad (3.1)$$

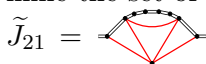
where n is a unit time-like vector, $n^2 = 1$, $l_{ik} = l_i - l_k$, and $[a] = (a - i0)$. The integral J_{21} satisfies the equation

$$J_{21}(d+2) = c_{21}(d)J_{21}(d) + c_7(d)J_7(d) + c_4(d)J_4(d) + c_3(d)J_3(d), \quad (3.2)$$

where $c_k(d)$ are some rational functions of d , in particular,

$$c_{21}(d) = -\frac{3(d-3)(3d-7)(3d-5)}{16(d-1)^3(4d-11)(4d-9)(4d-7)(4d-5)}. \quad (3.3)$$

Analytical properties. In order to discover the analytical properties of J_{21} , we determine the set of finite integrals for each point in \mathcal{S} . For example, we find that the integral $\tilde{J}_{21} =$



is finite in $d = 4$. Reducing this integral in $d = 4 - 2\epsilon$ to the master integrals in $d = 2 - 2\epsilon$, we obtain

$$\begin{aligned} \tilde{J}_{21}(4-2\epsilon) &= \frac{\epsilon(1+2\epsilon)^2 J_{21}(2-2\epsilon)}{16(1-2\epsilon)(1+8\epsilon)(3+8\epsilon)} + \frac{\epsilon(386\epsilon^3 + 395\epsilon^2 + 131\epsilon + 14) J_7(2-2\epsilon)}{48(1-2\epsilon)(1+3\epsilon)(2+5\epsilon)(1+8\epsilon)} \\ &+ \frac{\epsilon(53\epsilon^2 + 37\epsilon + 6)(2\epsilon+1) J_4(2-2\epsilon)}{64(2\epsilon-1)(4\epsilon+1)(5\epsilon+2)(8\epsilon+1)} + \frac{\epsilon(6\epsilon+1)(124\epsilon^2 + 95\epsilon + 18) J_3(2-2\epsilon)}{48(2\epsilon-1)(5\epsilon+2)(8\epsilon+1)(8\epsilon+3)}. \end{aligned} \quad (3.4)$$

Since the left-hand side is finite, so is the right-hand side. Note that in the latter the integral J_{21} is the only nontrivial one, while J_3 , J_4 , J_7 are expressed in terms of Γ -functions. Expanding the right-hand side up to ϵ^{-1} , we obtain the following constraint

$$e^{4\epsilon\gamma_E} J_{21}(2-2\epsilon) = -\frac{10}{\epsilon^4} - \frac{226}{3\epsilon^3} + \left(\frac{286}{3} - 58\pi^2\right) \epsilon^{-2} + \mathcal{O}(\epsilon^{-1}). \quad (3.5)$$

In fact, we can obtain yet more terms of expansion of J_{21} near $d = 2$ once we consider more finite integrals. Finally, we obtain

$$\begin{aligned} e^{4\epsilon\gamma_E} J_{21}(2-2\epsilon) &= -\frac{10}{\epsilon^4} - \frac{226}{3\epsilon^3} + \left(\frac{286}{3} - 58\pi^2\right) \epsilon^{-2} + \left(\frac{5512\zeta(3)}{3} - \frac{166}{3} - \frac{3826\pi^2}{9}\right) \epsilon^{-1} \\ &+ \frac{118336\zeta(3)}{9} - \frac{728}{3} + \frac{4904\pi^2}{9} - \frac{4478\pi^4}{15} + \mathcal{O}(\epsilon), \\ e^{4\epsilon\gamma_E} J_{21}(1-2\epsilon) &= -3072\pi^2 + \left(-\frac{1084928\pi^2}{45} - 24576\pi^2 \log 2\right) \epsilon + \mathcal{O}(\epsilon^2), \\ e^{4\epsilon\gamma_E} J_{21}(2/3-2\epsilon) &= -\frac{14554000\Gamma\left(\frac{4}{3}\right)^5}{189\epsilon} + \mathcal{O}(\epsilon^0), \\ e^{4\epsilon\gamma_E} J_{21}(4/3-2\epsilon) &= \frac{16677\Gamma\left(\frac{5}{3}\right)^5}{10\epsilon} + \mathcal{O}(\epsilon^0), \\ e^{4\epsilon\gamma_E} J_{21}(d_0-2\epsilon) &= \mathcal{O}(\epsilon^0) \text{ in all other points } d_0 \in (0, 2]. \end{aligned} \quad (3.6)$$

The right-hand sides of these constraints are built from the expansion coefficients of simpler integrals $J_{1,2,3,5}$ expressible in terms of Γ -functions.

Summing factor. The summing factor $S(d)$ satisfies

$$S_i^{-1}(d+2) = c_i(d)S_i^{-1}(d), \quad (3.7)$$

where c_{21} is defined in eq. (3.3). It is useful to consider its following decomposition

$$S(d) = S_0(d)\Omega(d)f(d), \quad (3.8)$$

where each of the factors $S_0(d)$, $\Omega(z)$, f has its own meaning. First, we find $S_0(d)$, which is a random solution of eq. (3.7). Then we pick a periodic factor $\Omega(z) = \Omega(e^{i\pi d})$ such as to reduce the number and the orders of singularities of the quantity $S(d)J_{21}(d)$ on the basic stripe. In addition we secure that $S(d)J_{21}(d)/e^{\pi|d|}$ decays when $d \rightarrow \pm i\infty$. Finally, we pick a constant factors f to simplify the leading term of expansion of $S(d)$ at $d = 2 - 2\epsilon$. We have

$$S_0(d) = \frac{2^{4d}\Gamma\left(\frac{11}{2} - \frac{3d}{2}\right)}{\Gamma\left(\frac{13}{2} - 2d\right)\Gamma\left(\frac{3}{2} - \frac{d}{2}\right)^3}, \quad (3.9)$$

$$\Omega(z) = \sin^3\left(\frac{\pi}{2}(d-2)\right)\sin\left(\frac{\pi}{2}\left(d - \frac{4}{3}\right)\right)\sin\left(\frac{\pi}{2}\left(d - \frac{2}{3}\right)\right), \quad (3.10)$$

$$f = \frac{1}{192\pi^{3/2}}. \quad (3.11)$$

Consequently, we have the following properties of $S(d)$:

1. $S(d)$ satisfies eq. (3.7).
2. $S(d)J_{21}(d)$ has no singularities at $d \in (0, 2)$ and is bounded when $\Im d \rightarrow \pm\infty$.
3. $S(2 - 2\epsilon)J_{21}(2 - 2\epsilon) = \frac{10}{\epsilon} + \frac{146}{3} + \mathcal{O}(\epsilon)$.

Two last properties are trivially established from the constraints (3.6).

General and specific solution. We write the general solution of eq. (3.2) as

$$S(d)J_{21}(d) = \mathcal{I}_{21}(d) + \omega(z), \quad (3.12)$$

$$\mathcal{I}_{21}(d) = -\sum_{k=0}^{\infty} S(d_k + 2) [c_7(d_k)J_7(d_k) + c_4(d_k)J_4(d_k) + c_3(d_k)J_3(d_k)], \quad (3.13)$$

where $d_k = d + 2k$ and $\omega(z) = \omega(e^{i\pi d})$ is a periodic function.

Now we have to construct $\omega(z)$ in the right-hand side to fit the analytical properties of the left-hand side of eq. (3.12). We use **SummerTime** package [11] to calculate with high numerical precision the coefficients of ϵ -expansion of the inhomogeneous solution $\mathcal{I}_{21}(d)$

near points $d_0 \in \mathcal{S}$. Then, using some educated guess about the transcendental constants which may appear in the coefficients, we obtain the following analytic expansions

$$\begin{aligned}
 \mathcal{I}_{21}(1-2\epsilon) &\stackrel{\text{PSLQ}}{=} \frac{1}{27\epsilon} + \mathcal{O}(\epsilon^0), & \mathcal{I}_{21}\left(\frac{1}{2}-2\epsilon\right) &\stackrel{\text{PSLQ}}{=} \frac{1}{2\epsilon} + \mathcal{O}(\epsilon^0), \\
 \mathcal{I}_{21}\left(\frac{3}{2}-2\epsilon\right) &\stackrel{\text{PSLQ}}{=} \frac{1}{2\epsilon} + \mathcal{O}(\epsilon^0), & \mathcal{I}_{21}\left(\frac{1}{3}-2\epsilon\right) &\stackrel{\text{PSLQ}}{=} \frac{1}{9\sqrt{3}\pi\epsilon^2} - \frac{14}{27\epsilon} + \mathcal{O}(\epsilon^0), \\
 \mathcal{I}_{21}\left(\frac{5}{3}-2\epsilon\right) &\stackrel{\text{PSLQ}}{=} -\frac{1}{9\sqrt{3}\pi\epsilon^2} - \frac{14}{27\epsilon} + \mathcal{O}(\epsilon^0), & \mathcal{I}_{21}(2-2\epsilon) &\stackrel{\text{PSLQ}}{=} \frac{10}{\epsilon} + \frac{146}{3} + \mathcal{O}(\epsilon^1),
 \end{aligned}
 \tag{3.14}$$

and $\mathcal{I}_{21}(d_0 - 2\epsilon) = \mathcal{O}(\epsilon^0)$ for all other d_0 from \mathcal{S} . Here $\stackrel{\text{PSLQ}}{=}$ means that the analytic coefficients in ϵ -expansion have been determined from their multi-digit numerical values using PSLQ, ref. [19]. Then we construct the function $\omega(z)$ which cancels the poles at $d = 1, \frac{1}{2}, \frac{3}{2}, \frac{1}{3}, \frac{5}{3}$ and preserves the expansion at $d = 2$ and the behavior of $S(d)J_{21}(d)$ at $d \rightarrow \pm i\infty$:

$$\begin{aligned}
 \omega(z) &\stackrel{\text{PSLQ}}{=} \frac{\pi}{9\sqrt{3}} \cot^2 \frac{\pi}{2} \left(d - \frac{5}{3}\right) - \frac{14\pi}{27} \cot \frac{\pi}{2} \left(d - \frac{5}{3}\right) - \frac{\pi}{9\sqrt{3}} \cot^2 \frac{\pi}{2} \left(d - \frac{1}{3}\right) \\
 &\quad - \frac{14\pi}{27} \cot \frac{\pi}{2} \left(d - \frac{1}{3}\right) + \frac{\pi}{2} \cot \frac{\pi}{2} \left(d - \frac{3}{2}\right) + \frac{\pi}{27} \cot \frac{\pi}{2} (d-1) + \frac{\pi}{2} \cot \frac{\pi}{2} \left(d - \frac{1}{2}\right) \\
 &= -\frac{2\pi \sin\left(\frac{\pi d}{2}\right) (1 - 2\cos(2\pi d))}{3(1 - 2\cos(\pi d))^2 \left(\cos\left(\frac{\pi d}{2}\right) + \cos\left(\frac{3\pi d}{2}\right)\right)}.
 \end{aligned}
 \tag{3.15}$$

Thus we obtain the final expression

$$J_{21}(d) = S^{-1}(d) [\mathcal{I}_{21}(d) + \omega(z)], \tag{3.16}$$

where $S(d)$, $\mathcal{I}_{21}(d)$, and $\omega(d)$ are defined in eqs. (3.8)–(3.11), (3.13), and (3.15), respectively. Using the `SummerTime` package to calculate the sum in $\mathcal{I}_{21}(d)$, we obtain the ϵ expansion around $d = 4$ and $d = 3$ with high-precision numeric coefficients. Using PSLQ, we obtain

$$\begin{aligned}
 J_{21}(4-2\epsilon) &= e^{-4\epsilon\gamma_E} \left[-\left(\frac{7}{288} + \frac{\zeta_2}{36}\right) \frac{1}{\epsilon^2} - \left(\frac{667}{1728} + \frac{8\zeta_2}{27} - \frac{5\zeta_3}{72}\right) \frac{1}{\epsilon} - \frac{31993}{10368} - \frac{2725\zeta_2}{1296} + \frac{20\zeta_3}{27} - \frac{49\zeta_2^2}{360} \right. \\
 &\quad \left. - \left(\frac{636223}{62208} + \frac{79321\zeta_2}{7776} + \frac{196\zeta_2^2}{135} - \frac{835\zeta_3}{324} + \frac{293\zeta_2\zeta_3}{108} + \frac{191\zeta_5}{24}\right) \epsilon + \dots \right],
 \end{aligned}
 \tag{3.17}$$

$$\begin{aligned}
 J_{21}(3-2\epsilon) &= e^{-4\epsilon\gamma_E} \pi^2 \left[\frac{7\zeta_3}{\epsilon} + \frac{14}{5}\zeta_2^2 + 32\zeta_{-3,1} + (254\zeta_2\zeta_3 + 357\zeta_5 - 256\zeta_{-3,1,1}) \epsilon \right. \\
 &\quad \left. + \left(\frac{5172}{35}\zeta_2^3 + 192\zeta_{-3,1}\zeta_2 - \frac{2746}{3}\zeta_3^2 - 640\zeta_{-5,1} + 2048\zeta_{-3,1,1,1}\right) \epsilon^2 + \dots \right],
 \end{aligned}
 \tag{3.18}$$

where $\zeta_{a_1\dots a_n}$ is defined in (A.1).

4 Results

Similar to the example in previous section, we derive representations for all master integrals from figure 1 in terms of iterated triangular sums with factorized summands. One

can effectively evaluate these sums as expansions in ϵ with arbitrarily accurate numerical coefficients using the `SummerTime` package [11]. Assuming that we know the basis of transcendental numbers, which may show up in the results, we can use the PSLQ algorithm to recover the analytical form of the coefficients.

With the paper, we provide the results for sums in the `SummerTime` format, admitting the calculation of all considered integrals for arbitrary space-time dimension and/or to arbitrary order in ϵ . Furthermore, we perform PSLQ recognition for $d = 3 - 2\epsilon$ and $d = 4 - 2\epsilon$ to obtain the analytic results, which should be sufficient for any practical application. For $d = 4 - 2\epsilon$ we successfully recognize the analytic result in terms of usual multiple zeta values. For $d = 3 - 2\epsilon$ we use also the alternating Euler-Zagier sums.

The existence of a uniformly transcendental (UT) basis is very remarkable per se, but it is advantageous also for practical reasons as it simplifies the PSLQ recognition. Unfortunately, it is yet unclear how to systematically construct the UT basis for one-scale integrals. Nevertheless, we were able to construct UT bases for both three-dimensional and four-dimensional cases. This was accomplished in a semi-empirical way by checking the integrals which diverge logarithmically in $d = 4$ or $d = 3$. In many cases we observed that such integrals exhibit the property of uniform transcendentality after pulling out a simple rational factor. We have recognized the analytical results for UT integrals up to the weight twelve² for both $d = 4 - 2\epsilon$ and $d = 3 - 2\epsilon$.

The advantage of using UT basis in applications is that it should be expanded exactly up to the transcendentality weight which appear in the physical result. This is in contrast to IBP basis, where some integrals might require higher expansion terms involving higher transcendental weights. We have checked whether it is the case for the master integrals in figure 1 in the following way. Let $\mathbf{J}_{\text{IBP}} = T\mathbf{J}_{\text{UT}}$, where \mathbf{J}_{IBP} is a column of IBP master integrals, depicted in figure 1, \mathbf{J}_{UT} is a UT basis, and T is a transition matrix. Then we evaluate the quantity $T^{-1}(T\mathbf{J}_{\text{UT}})$. Formally, the result coincides with \mathbf{J}_{UT} , however, substituting the *expansions* of \mathbf{J}_{UT} up to a certain transcendental weight and performing the matrix multiplication in the order dictated by braces, we might, in principle, lose some higher terms in the expansions of \mathbf{J}_{UT} . In this case it means that \mathbf{J}_{IBP} should be expanded up to terms involving higher transcendental weight. We find that for $d = 4 - 2\epsilon$ this loss of higher expansion terms does not happen, while for $d = 3 - 2\epsilon$ it does.

The obtained results are attached to the `arXiv` version of the paper. The description of the files attached as supplementary material can be found in appendix A.

Checks of the results. Since our method is quite involved, it is important to perform some crosschecks of our results. First, as we mentioned earlier, the IBP reduction of finite integrals provide an extremely redundant set of constraints which we use not only for fixing the specific solution of dimensional recurrence relations, but also for an extensive crosscheck of the results. Then, we have verified that our results reproduce all terms of ϵ expansion of integrals calculated in [20] for $d = 4 - 2\epsilon$. Unfortunately, for the most complicated integrals in that paper only the divergent part is available. While this paper was being written a

²Note that the expansion of integrals at $d = 3 - 2\epsilon$ appears to have an overall common factor π^2 , so after pooling this factor out we had to recognize only up to t.w. 10 expressions.

work on the numerical calculation of the same set of four-loop integrals appeared [21]. The comparison of the results provided therein with those of the present paper for the cases $d = 4 - 2\epsilon$ has shown only partial agreement.³ We note that the results of ref. [21] for the most complicated integrals J_{52}, J_{53}, J_{54} are also in contradiction with the divergent parts of these integrals calculated in [20].

5 Conclusion

In this paper we have calculated the four-loop HQET propagator-type master integrals using the DRA method [8]. In order to fix the periodic functions in the general solution of the dimensional recurrence relations, we use a novel highly automated approach based on the IBP reduction of finite integrals. In order to pick a sufficient set of the finite integrals we use a criterion from [17] generalized to the case of rational d . Having obtained the expressions for the master integrals in terms of triangular sums treatable by the `SummerTime` package [11], we obtain an ϵ -expansion with high-precision numerical coefficients (up to 10^4) near the most relevant dimensions $d = 4$ and $d = 3$. Using PSLQ algorithm we recognize these high-precision coefficients in terms of multiple zeta values (for $d = 4 - 2\epsilon$) and Euler-Zagier sums (for $d = 3 - 2\epsilon$). The obtained results for even and odd dimensions cover all thinkable practical applications and for the $d = 4 - 2\epsilon$ case were already successfully applied in ref. [3]. The results in $d = 3 - 2\epsilon$ can find their application in perturbative calculations in ABJM theory, see, e.g. [22, 23].

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A Supplementary files description

The main results of the article are available in the form of computer readable files. For alternating Euler-Zagier sums we use the notation

$$\text{mzv}[\mathbf{n1}, \dots, \mathbf{nk}] = \zeta_{n_1, \dots, n_k} = \sum_{i_1 > \dots > i_k > 0} \prod_{l=1}^k \frac{(\text{sign } n_l)^{i_l}}{i_l^{|n_l|}}. \quad (\text{A.1})$$

Short description of files and examples of their usage are provided below.

HQET4l.st

List of arbitrary d results in the `SummerTime` package format. To calculate all integrals in $d = 4 - 2\epsilon$ with 30 digits precision to the order $\mathcal{O}(\epsilon^2)$ one can run:

```
TriangleSumsSeries[#, {ep, 2}, 30] & /@ (Get["HQET4l.st"] /. d -> 4 - 2*ep)
```

³In our notations we found disagreement in integrals $J_{28}, J_{30}, J_{36}, J_{44}, J_{45}, J_{46}, J_{52}, J_{53}, J_{54}$.

HQET4l.ldrv

Matrix $L(d)$ of the lowering dimensional recurrence relation (2.1).

hgetUT4l.3d

List of analytical results for uniform transcendent weight basis functions expanded near $d = 3$ to the transcendent weight 12 in terms of alternating Euler-Zagier sums.

hgetUT4l.4d

List of analytical results for uniform transcendent weight basis functions expanded near $d = 4$ to the transcendent weight 12 in terms of MZV.

ut2mi.3d

Conversion matrix from the set of UT basis functions to integrals calculated in the present paper (figure 1) for $d = 3 - 2\epsilon$. One can obtain list of integrals from basis UT weight functions with:

```
Get["ut2mi.3d"].Get["hgetUT4l.3d"]
```

ut2mi.4d

Conversion matrix from the set of UT basis functions to integrals calculated in the present paper (figure 1) for $d = 4 - 2\epsilon$. One can obtain list of integrals from basis UT weight functions with:

```
Get["ut2mi.4d"].Get["hgetUT4l.4d"]
```

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References

- [1] M. Neubert, *Heavy quark symmetry*, *Phys. Rept.* **245** (1994) 259 [[hep-ph/9306320](#)] [[INSPIRE](#)].
- [2] K.G. Chetyrkin and A.G. Grozin, *Three loop anomalous dimension of the heavy light quark current in HQET*, *Nucl. Phys. B* **666** (2003) 289 [[hep-ph/0303113](#)] [[INSPIRE](#)].
- [3] A.G. Grozin, R.N. Lee and A.F. Pikelner, *Four-loop QCD cusp anomalous dimension at small angle*, *JHEP* **11** (2022) 094 [[arXiv:2208.09277](#)] [[INSPIRE](#)].
- [4] K.G. Chetyrkin and A.G. Grozin, *Correlators of heavy-light quark currents in HQET: OPE at three loops*, *Nucl. Phys. B* **976** (2022) 115702 [[arXiv:2111.14571](#)] [[INSPIRE](#)].
- [5] M. Beneke and V.M. Braun, *Heavy quark effective theory beyond perturbation theory: Renormalons, the pole mass and the residual mass term*, *Nucl. Phys. B* **426** (1994) 301 [[hep-ph/9402364](#)] [[INSPIRE](#)].
- [6] A.G. Grozin, *Calculating three loop diagrams in heavy quark effective theory with integration by parts recurrence relations*, *JHEP* **03** (2000) 013 [[hep-ph/0002266](#)] [[INSPIRE](#)].
- [7] A. Czarnecki and K. Melnikov, *Threshold expansion for heavy light systems and flavor off diagonal current current correlators*, *Phys. Rev. D* **66** (2002) 011502 [[hep-ph/0110028](#)] [[INSPIRE](#)].

- [8] R.N. Lee, *Space-time dimensionality D as complex variable: Calculating loop integrals using dimensional recurrence relation and analytical properties with respect to D* , *Nucl. Phys. B* **830** (2010) 474 [[arXiv:0911.0252](#)] [[INSPIRE](#)].
- [9] O.V. Tarasov, *Connection between Feynman integrals having different values of the space-time dimension*, *Phys. Rev. D* **54** (1996) 6479 [[hep-th/9606018](#)] [[INSPIRE](#)].
- [10] R.N. Lee, *Presenting LiteRed: a tool for the Loop InTEgrals REDuction*, [arXiv:1212.2685](#) [[INSPIRE](#)].
- [11] R.N. Lee and K.T. Mingulov, *Introducing SummerTime: a package for high-precision computation of sums appearing in DRA method*, *Comput. Phys. Commun.* **203** (2016) 255 [[arXiv:1507.04256](#)] [[INSPIRE](#)].
- [12] A.G. Grozin, *Lectures on multiloop calculations*, *Int. J. Mod. Phys. A* **19** (2004) 473 [[hep-ph/0307297](#)] [[INSPIRE](#)].
- [13] A.V. Smirnov, *FIESTA4: Optimized Feynman integral calculations with GPU support*, *Comput. Phys. Commun.* **204** (2016) 189 [[arXiv:1511.03614](#)] [[INSPIRE](#)].
- [14] M. Czakon, *Automatized analytic continuation of Mellin-Barnes integrals*, *Comput. Phys. Commun.* **175** (2006) 559 [[hep-ph/0511200](#)] [[INSPIRE](#)].
- [15] P.A. Baikov and K.G. Chetyrkin, *Four Loop Massless Propagators: An Algebraic Evaluation of All Master Integrals*, *Nucl. Phys. B* **837** (2010) 186 [[arXiv:1004.1153](#)] [[INSPIRE](#)].
- [16] A. Georgoudis, V. Gonçalves, E. Panzer, R. Pereira, A.V. Smirnov and V.A. Smirnov, *Glue-and-cut at five loops*, *JHEP* **09** (2021) 098 [[arXiv:2104.08272](#)] [[INSPIRE](#)].
- [17] A. von Manteuffel, E. Panzer and R.M. Schabinger, *A quasi-finite basis for multi-loop Feynman integrals*, *JHEP* **02** (2015) 120 [[arXiv:1411.7392](#)] [[INSPIRE](#)].
- [18] A. von Manteuffel and C. Studerus, *Reduze 2 — Distributed Feynman Integral Reduction*, ZU-TH-01-12 (2012) [[arXiv:1201.4330](#)] [[INSPIRE](#)].
- [19] H.R.P. Ferguson and D.H. Bailey, *A polynomial time, numerically stable integer relation algorithm*, RNR Technical Report RNR-91-032 (1992).
- [20] A. Grozin, J. Henn and M. Stahlhofen, *On the Casimir scaling violation in the cusp anomalous dimension at small angle*, *JHEP* **10** (2017) 052 [[arXiv:1708.01221](#)] [[INSPIRE](#)].
- [21] Z.-F. Liu and Y.-Q. Ma, *Automatic computation of Feynman integrals containing linear propagators via auxiliary mass flow*, *Phys. Rev. D* **105** (2022) 074003 [[arXiv:2201.11636](#)] [[INSPIRE](#)].
- [22] M.S. Bianchi and A. Mauri, *ABJM θ -Bremsstrahlung at four loops and beyond*, *JHEP* **11** (2017) 173 [[arXiv:1709.01089](#)] [[INSPIRE](#)].
- [23] M.S. Bianchi and A. Mauri, *ABJM θ -Bremsstrahlung at four loops and beyond: non-planar corrections*, *JHEP* **11** (2017) 166 [[arXiv:1709.10092](#)] [[INSPIRE](#)].