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Gravity coupled to a scalar field from a Chern-Simons action: describing rotating hairy black holes and solitons with gauge fields

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Abstract: Einstein gravity minimally coupled to a scalar field with a two-parameter Higgs-like self-interaction in three spacetime dimensions is recast in terms of a Chern-Simons form for the algebra $g^+ \oplus g^-$ where, depending on the sign of the self-interaction couplings, $g^\pm$ can be $so(2,2)$, $so(3,1)$ or $iso(2,1)$. The field equations can then be expressed through the field strength of non-flat composite gauge fields, and conserved charges are readily obtained from boundary terms in the action that agree with those of standard Chern-Simons theory for pure gravity, but with non-flat connections. Regularity of the fields then amounts to requiring the holonomy of the connections along contractible cycles to be trivial. These conditions are automatically fulfilled for the scalar soliton and allow to recover the Hawking temperature and chemical potential in the case of the rotating hairy black holes presented here, whose entropy can also be obtained by the same formula that holds in the case of a pure Chern-Simons theory. In the conformal (Jordan) frame the theory is described by General Relativity with cosmological constant conformally coupled to a self-interacting scalar field, and its formulation in terms of a Chern-Simons form for suitably composite gauge fields is also briefly addressed.

Keywords: Black Holes, Classical Theories of Gravity, AdS-CFT Correspondence

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Contents

1 Introduction

The formulation of three-dimensional General Relativity as a Chern-Simons theory [1, 2] has allowed exploiting time-honored tools available for gauge fields in order to span a wealth of very interesting achievements on classical and quantum aspects of gravitation (for a non-exhaustive list of references, see e.g. [3–16]). Many of these results also extend to supergravity [17–28] as well as for gravitation coupled to higher-spin fields [29–46], since both can be described in terms of a Chern-Simons theory for suitable gauge groups. Ultra and non-relativistic versions thereof have also been developed in [47–53].

The case of General Relativity minimally coupled to a real scalar field appears to be far from that kind of description. Nevertheless, here we show that in the case of a precise two-parameter Higgs-like self-interaction potential given by

$$V(\phi) = \frac{\Lambda}{8}\left(\cosh^6\phi + \nu \sinh^6\phi\right) , \tag{1.1}$$

the theory can be equivalently formulated in terms of a Chern-Simons form that depends on composite gauge fields, so that the field equations no longer imply the vanishing of the field strengths. Thus, the theory and its configuration space can be equivalently described in terms of non-flat connections, so that many of the gauge theory tools open up to analyze their properties.

The self-interaction potential (1.1) enjoys some remarkable properties. Indeed, the first analytic example of a black hole with a minimally coupled scalar field that circumvents the no-hair conjecture was precisely found for $V(\phi)$ in (1.1) in the range $\Lambda < 0$ and $\nu \geq -1$ [54]. Furthermore, the potential in (1.1) falls within the class analyzed in [54], for which the scalar field acquires a slow fall-off at infinity, so that the canonical generators of the asymptotic symmetries acquire an explicit contribution from the scalar field. Thus, it was shown that the Brown-Henneaux boundary conditions [55] for gravity with a localized distribution of matter can be consistently relaxed, so that the asymptotic symmetries are still given by the conformal group in two dimensions with the same central extension. As it was also pointed out in [54], for the self-interaction potential (1.1), once the theory is expressed in the conformal (Jordan) frame, the matter piece of the action becomes conformally invariant, and the action reduces to General Relativity with cosmological constant Λ with a conformally coupled scalar field, whose self-interaction coupling is determined by ν (see section 6).

The plan of the paper is as follows. In the next section, we show that the action

$$I\left[\phi, g_{\mu\nu}\right] = \frac{8}{\kappa}\int d^3x\sqrt{-g}\left(\frac{R}{16} - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V\left(\phi\right)\right), \tag{1.2}$$

with the self-interaction potential given by (1.1), whose field equations read

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8T_{\mu\nu}\,, \tag{1.3}$$

$$\Box\phi - \frac{dV\left(\phi\right)}{d\phi} = 0\,, \tag{1.4}$$

with

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi - g_{\mu\nu}V\left(\phi\right)\,, \tag{1.5}$$

can be expressed in terms of a Chern-Simons form with certain suitable composite gauge fields. In section 3, the Hamiltonian boundary terms required to obtain the conserved charges in terms of the connections are worked out, including the hairy black hole entropy formula in terms of gauge fields. The rotating extension of the hairy black hole in [54] and its global charges are discussed in section 4. Regularity of hairy black holes and the scalar soliton in terms of trivial holonomies is addressed in section 5, including the hairy black hole thermodynamics and the corresponding Cardy formula for its entropy that depends on the global charges of the scalar soliton. Finally, section 6 is devoted to some ending remarks as well as a brief discussion of the formulation of the theory in the conformal frame in terms of a Chern-Simons form for composite gauge fields.

2 Action from a Chern-Simons form with composite gauge fields

Here we show that the action (1.2), up to a boundary term, can be recast using a Chern-Simons form. The gauge fields are defined in terms of the direct sum of the algebras g^+ and g^-, where $g^\pm$ can be the three-dimensional (anti-)de Sitter or Poincaré algebras, depending on the signs of $\Lambda^+ = \Lambda$ and $\Lambda^- = -\nu\Lambda$.

The theory can be formulated through the following composite gauge fields

$$A^+ = \cosh^2\left(\phi\right)e^aP_a^+ + \omega_+^aJ_a^+\,, \tag{2.1}$$

$$A^- = \sinh^2\left(\phi\right)e^aP_a^- + \omega_-^aJ_a^-\,, \tag{2.2}$$

that depend on the scalar field ϕ, the dreibein e^a and additional 1-forms $\omega_\pm^a$. The generators $J_a^\pm$ and $P_a^\pm$ fulfill the following algebra

$$\left[J_a^\pm, J_b^\pm\right] = \epsilon_{abc}J_\pm^c\,, \quad \left[J_a^\pm, P_b^\pm\right] = \epsilon_{abc}P_\pm^c\,, \quad \left[P_a^\pm, P_b^\pm\right] = -\Lambda^\pm\epsilon_{abc}J_\pm^c\,, \tag{2.3}$$

so that each copy corresponds to $so(3,1)$ or $so(2,2)$ for positive or negative signs of $\Lambda^{\pm}$, respectively, or $iso(2,1)$ when Λ^{+} or Λ^{-} vanishes. The field strengths associated to the gauge fields (2.1) and (2.2) are given by

$$F^{\pm} = dA^{\pm} + (A^{\pm})^2 = \mathcal{T}^a_{\pm} P^{\pm}_a + \mathcal{R}^a_{\pm} J^{\pm}_a \,, \tag{2.4}$$

where the components along the $so(2,1)$ generators $J^{\pm}_a$ read

$$\mathcal{R}^a_+ = R^a_+ - \frac{1}{2}\Lambda^+ \cosh^4(\phi)\, \epsilon^{abc} e_b e_c \,, \tag{2.5}$$

$$\mathcal{R}^a_- = R^a_- - \frac{1}{2}\Lambda^- \sinh^4(\phi)\, \epsilon^{abc} e_b e_c \,, \tag{2.6}$$

with $R^a_{\pm} = d\omega^a_{\pm} + \frac{1}{2}\epsilon^{abc}\omega^{\pm}_b \omega^{\pm}_c$, while those along the remaining generators $P^{\pm}_a$ are

$$\mathcal{T}^a_+ = \cosh^2(\phi)\left(T^a_+ + 2\tanh(\phi)\, d\phi\, e^a\right) \,, \tag{2.7}$$

$$\mathcal{T}^a_- = \sinh^2(\phi)\left(T^a_- + 2[\tanh(\phi)]^{-1} d\phi\, e^a\right) \,, \tag{2.8}$$

with $T^a_{\pm} = de^a + \epsilon^{abc}\omega^{\pm}_b e_c$.

We then consider an action principle defined as a combination of two Chern-Simons forms for the composite gauge fields $A^{\pm}$, which reads

$$I\left[\phi, e, \omega^+, \omega^-\right] = \frac{k^+}{4\pi}\int \left\langle A^+ dA^+ + \frac{2}{3}(A^+)^3 \right\rangle + \frac{k^-}{4\pi}\int \left\langle A^- dA^- + \frac{2}{3}(A^-)^3 \right\rangle \,, \tag{2.9}$$

where the nonvanishing components of the invariant bilinear form are given by $\left\langle J^{\pm}_a, P^{\pm}_b \right\rangle = \eta_{ab}$, with η_{ab} standing for the Minkowski metric, and the levels are defined as $k^{\pm} = \pm 2\pi/\kappa$.

The field equations are found varying the action (2.9) with respect to the dynamical fields e^a, ϕ and $\omega^a_{\pm}$

$$\delta I = \frac{1}{2\pi}\int \left\langle \left(k^+ F^+ \frac{\delta A^+}{\delta e^a} + k^- F^- \frac{\delta A^-}{\delta e^a} \right)\delta e^a + \left(k^+ F^+ \frac{\delta A^+}{\delta \phi} + k^- F^- \frac{\delta A^-}{\delta \phi} \right)\delta\phi \right. \tag{2.10}$$

$$\left. + \left(k^+ F^+ \frac{\delta A^+}{\delta \omega^a_+} + k^- F^- \frac{\delta A^-}{\delta \omega^a_+} \right)\delta\omega^a_+ + \left(k^+ F^+ \frac{\delta A^+}{\delta \omega^a_-} + k^- F^- \frac{\delta A^-}{\delta \omega^a_-} \right)\delta\omega^a_- \right\rangle \,, \tag{2.11}$$

so that they read

$$\cosh^2(\phi)\,\mathcal{R}^+_a - \sinh^2(\phi)\,\mathcal{R}^-_a = 0 \,, \tag{2.12}$$

$$\left(\mathcal{R}^+_a - \mathcal{R}^-_a\right) e^a = 0 \,, \tag{2.13}$$

$$T^a_{\pm} + 2[\tanh(\phi)]^{\pm 1} d\phi\, e^a = 0 \,, \tag{2.14}$$

respectively.

Note that the last field equations (2.14) are algebraic for $\omega^a_{\pm}$, which can be solved as

$$\omega^a_{\pm}(e,\phi) = \omega^a(e) - 2[\tanh(\phi)]^{\pm 1} * (e^a d\phi) \,, \tag{2.15}$$

where ω^a is the torsionless Levi-Civita spin connection associated to e^a, and $*$ stands for the Hodge dual, so that $*(e^a d\phi) = \epsilon^{abc}\partial_\nu \phi e^\nu_b e_c$.

A second order action $I(e^a, \phi)$ is obtained by replacing (2.15) in the Chern-Simons action (2.9), which reduces to (1.2) up to a boundary term, where the self-interaction potential is precisely given by (1.1). Analogously, substituting equation (2.15) in (2.12) and (2.13), they reduce respectively to the Einstein and scalar field equations (1.3) and (1.4).

3 Global charges and entropy in terms of gauge fields

In order to deal with conserved charges as well as for the hairy black hole entropy, it is useful to express the action (2.9) in Hamiltonian form. Supplementing the action with a boundary term B, that is required to ensure a well-defined variational principle [73], yields to

$$I = \frac{k^+}{4\pi}\int dtd^2x\epsilon^{ij}\left\langle \dot{A}_i^+ A_j^+ + A_t^+ F_{ij}^+\right\rangle + \frac{k^-}{4\pi}\int dtd^2x\epsilon^{ij}\left\langle \dot{A}_i^- A_j^- + A_t^- F_{ij}^-\right\rangle + \mathcal{B}\,, \tag{3.1}$$

where $F_{ij}^{\pm}$ are the spatial components of the field strengths $F^{\pm}$ in (2.4). Considering that the variations of the gauge fields depend on the scalar field and the dreiben, and making use of (2.15), the variation of the action reads

$$\begin{aligned}\delta I = &\frac{1}{2\kappa}\int dtd^2x\epsilon^{ij}\left\langle\left[\left(-2F_{tj}^+\frac{\delta A_i^+}{\delta e^a}+F_{ij}^+\frac{\delta A_t^+}{\delta e^a}\right)-\left(-2F_{tj}^-\frac{\delta A_i^-}{\delta e^a}+F_{ij}^-\frac{\delta A_t^-}{\delta e^a}\right)\right]\delta e^a\right.\\ &\left.+\left[\left(-2F_{tj}^+\frac{\delta A_i^+}{\delta\phi}+F_{ij}^+\frac{\delta A_t^+}{\delta\phi}\right)-\left(-2F_{tj}^-\frac{\delta A_i^-}{\delta\phi}+F_{ij}^-\frac{\delta A_t^-}{\delta\phi}\right)\right]\delta\phi\right\rangle\\ &+\frac{1}{\kappa}\int dtdS_i\epsilon^{ij}\left\langle\left(A_t^+\frac{\delta A_j^+}{\delta e^a}-A_t^-\frac{\delta A_j^-}{\delta e^a}\right)\delta e^a+\left(A_t^+\frac{\delta A_j^+}{\delta\phi}-A_t^-\frac{\delta A_j^-}{\delta\phi}\right)\delta\phi\right\rangle+\delta\mathcal{B}\,,\end{aligned}$$

where the bulk terms are proportional to the field equations (2.12) and (2.13).

The variation of the boundary term is then given by

$$\begin{aligned}\delta\mathcal{B} &= -\frac{1}{\kappa}\int dtdS_i\epsilon^{ij}\left\langle\left(A_t^+\frac{\delta A_j^+}{\delta e^a}-A_t^-\frac{\delta A_j^-}{\delta e^a}\right)\delta e^a+\left(A_t^+\frac{\delta A_j^+}{\delta\phi}-A_t^-\frac{\delta A_j^-}{\delta\phi}\right)\delta\phi\right\rangle\\ &= -\frac{1}{\kappa}\int dtd\theta\left\langle A_t^+\delta A_\theta^+ - A_t^-\delta A_\theta^-\right\rangle\,,\end{aligned} \tag{3.2}$$

which agrees with that of a pure Chern-Simons theory. Nonetheless, it should be emphasized that in our case the variations of the gauge fields are not fully independent. For the class of configurations that we are interested in, that fulfill the asymptotic conditions spelled out in [54], it can be also shown that the variation of the surface integral in (3.2) integrates as

$$\mathcal{B} = -\frac{1}{2\kappa}\int dtd\theta\left\langle A_t^+A_\theta^+ - A_t^-A_\theta^-\right\rangle\,. \tag{3.3}$$

Intriguingly, if one replaces the explicit form of the composite gauge fields $A^{\pm}$ in (2.1) and (2.2), by virtue of (2.15), the boundary term (3.3) completely gets rid of the scalar field and precisely reduces to that of pure General Relativity in the standard Chern-Simons formulation, given by

$$\mathcal{B} = -\frac{1}{2\kappa}\int dtd\theta\left\langle A_tA_\theta\right\rangle\,, \tag{3.4}$$

where the connection A is defined by setting $\phi = 0$ in the definition of A^+, i.e.,

$$A = A^+\Big|_{\phi=0} = e^aP_a^+ + \omega^aJ_a^+\,, \tag{3.5}$$

stands for the gauge field of pure General Relativity [1, 2].

Analogously, the black hole entropy formula that applies for a generic Chern-Simons theory in [41] (see also [56]), reduces to that of General Relativity

$$
\begin{aligned}
S &= \frac{1}{\kappa}\oint d\theta \left\langle A_\tau^+ A_\theta^+ - A_\tau^- A_\theta^- \right\rangle \Big|_{r_+} \\
&= \frac{1}{\kappa}\oint d\theta \left\langle A_\tau A_\theta \right\rangle \Big|_{r_+} ,
\end{aligned} \tag{3.6}
$$

where $\tau = -it$ is the Euclidean time and the event horizon is located at $r = r_+$.

In sum, the boundary term (3.4) and the entropy (3.6) are found to exclusively depend on the pure gravity gauge field A in (3.5), which noteworthy encodes all of the relevant information, without requiring the contribution of the scalar field in an explicit way. Indeed, it has to be stressed that the pure gravity gauge field (3.5) in our case is generically no longer flat.

4 Rotating hairy black hole

The first analytic example of a black hole solution endowed with minimally coupled scalar hair was found in [54], precisely for the self-interaction under discussion (1.1) in the case of $\Lambda < 0$ and $\nu \geq -1$. Some of its properties have been further analyzed in [57–71] following different approaches, and the rotating extension in the case of $\nu = 0$ was presented in [72].

In this section we extend the rotating hairy black hole solution to the full allowed range of the self-interaction coupling ($\nu > -1$). It can be readily obtained from the static one by performing a Lorentz boost parameterized by ω, with $\omega^2 < 1$, in the $t - \theta$ cylinder.[1] The line element then reads

$$
ds^2 = -N_\infty^2 N(r)^2 dt^2 + \frac{dr^2}{G(r)^2} + R(r)^2 \left(d\theta + N^\theta(r)\, dt\right)^2 , \tag{4.1}
$$

with

$$
N(r)^2 = \frac{r^2 g(r)^2}{R(r)^2} , \tag{4.2}
$$

$$
G(r)^2 = \left(\frac{H(r) + 2B}{H(r) + B}\right)^2 F(r)^2 , \tag{4.3}
$$

$$
R(r)^2 = \frac{r^2 - g(r)^2 \omega^2 \ell^2}{1 - \omega^2} , \tag{4.4}
$$

$$
N^\theta(r) = N_\infty^\theta - \frac{\omega}{\ell}\left(\frac{r^2 - g(r)^2 \ell^2}{r^2 - g(r)^2 \omega^2 \ell^2}\right) N_\infty , \tag{4.5}
$$

where

$$
g(r)^2 = \left(\frac{H(r)}{H(r) + B}\right)^2 F(r)^2 , \qquad F(r)^2 = \frac{H(r)^2}{\ell^2} - (1+\nu)\left(\frac{3B^2}{\ell^2} + \frac{2B^3}{\ell^2 H(r)}\right), \tag{4.6}
$$

[1] The case of $\nu = -1$ is excluded since, as pointed out in [54], the static metric is invariant under this kind of boosts. This configuration shares the causal structure with the massless BTZ black hole, but the null curvature singularity coincides with the "horizon" (NUT) at $r = 0$, possessing vanishing mass, angular momentum, temperature and entropy, regardless the value of the integration constant.

and

$$H(r) = \frac{1}{2}\left(r + \sqrt{r^2 + 4Br}\right) . \tag{4.7}$$

Here the cosmological constant is written in terms of the AdS radius as $\Lambda = -\ell^{-2}$, while N_∞ and N^θ_∞ stand for integration constants that turn out to be related to the Hawking temperature and the chemical potential, respectively (see section 5). The hairy black hole is dressed with a scalar field given by

$$\phi(r) = \text{arctanh}\sqrt{\frac{B}{H(r) + B}} , \tag{4.8}$$

being real provided that $B > 0$. In this case, the scalar field is regular everywhere except at the origin, since it diverges as $\phi_{r\to 0} = -\frac{1}{4}\log(r) + \cdots$, sourcing a null curvature singularity, as it is reflected in the behavior of the Ricci scalar near $r = 0$, which reads

$$R_{r\to 0} = -\frac{4(\nu+1)B^{5/2}}{\ell^2 r^{5/2}} + O(r^{-3/2}) . \tag{4.9}$$

Singularities in the spacetime metric and scalar field at the origin are cloaked by the event horizon located at $r = r_+ = B\Theta_\nu$, with Θ_ν given by

$$\Theta_\nu = 2\left(z\bar{z}\right)^{2/3} \frac{z^{2/3} - \bar{z}^{2/3}}{z - \bar{z}} , \tag{4.10}$$

and $z = 1 + i\sqrt{\nu}$.

It is worth highlighting that, unlike the case of the rotating solution in vacuum, the inner region of the rotating hairy black hole does not possess neither a region with closed timelike curves to be excised nor an inner (Cauchy) horizon. Indeed, the analogue of the latter actually corresponds to the null singularity at the origin. Moreover, the asymptotic behavior of the rotating hairy black hole does not fulfill the Brown-Henneaux boundary conditions [55] because the scalar field (4.8) has a slow fall-off at infinity, generating a strong backreaction on the metric in the asymptotic region. Instead, the solution fits within the relaxed asymptotic behavior described in [54].

In order to describe the rotating hairy black hole in terms of gauge fields, we choose the local frame so that the dreibein reads

$$e^0 = N_\infty N(r)\, dt , \qquad e^1 = \frac{dr}{G(r)} , \qquad e^2 = R(r)\left(d\theta + N^\theta(r)\, dt\right) , \tag{4.11}$$

and hence, the components of the spin connection ω^a are given by

$$\omega^0 = G(r)R(r)'d\theta + \frac{G(r)}{2}\left(2N^\theta(r)R(r)' + R(r)N^\theta(r)'\right) dt , \tag{4.12}$$

$$\omega^1 = \frac{R(r)N^\theta(r)'}{2N_\infty N(r)} dr , \tag{4.13}$$

$$\omega^2 = -\frac{G(r)R(r)^2 N^\theta(r)'}{2N_\infty N(r)} d\theta - \frac{G(r)}{2}\left(\frac{R(r)^2 N^\theta(r) N^\theta(r)'}{N_\infty N(r)} - 2N_\infty N(r)'\right) dt . \tag{4.14}$$

The gauge fields A^+ and A^- in (2.1), (2.2) become then fully specified by virtue of $\omega_\pm^a$ given by (2.15). Nevertheless, as pointed out in the previous section, the pure gravity connection in (3.5) is enough in order to evaluate the boundary term in (3.4), which for the rotating hairy black hole reduces to

$$\mathcal{B} = -(t_2 - t_1)\left[N_\infty\left(\frac{3\pi\,(1+\nu)\,B^2\,(1+\omega^2)}{\kappa\ell^2\,(1-\omega^2)}\right) - N_\infty^\theta\left(\frac{6\pi\,(1+\nu)\,B^2\omega}{\kappa\ell\,(1-\omega^2)}\right)\right]. \tag{4.15}$$

Noteworthy, the result is independent of the radial coordinate r, and hence it can be computed even at finite proper distance ($r = r_0$) without the need of taking the limit $r \to \infty$. The mass M and angular momentum J can then be recognized from the boundary term (4.15) in the following way [73]

$$\mathcal{B} = (t_2 - t_1)\left(-N_\infty M + N_\infty^\theta J\right), \tag{4.16}$$

so that

$$M = \frac{3\pi\,(1+\nu)\,B^2\,(1+\omega^2)}{\kappa\ell^2\,(1-\omega^2)} \qquad \text{and} \qquad J = \frac{6\pi\,(1+\nu)\,B^2\omega}{\kappa\ell\,(1-\omega^2)}. \tag{4.17}$$

The mass and angular momentum in (4.17) reduce to the result in [54] for the static case ($\omega = 0$), as well as to that in [72] for $\nu = 0$.

Note that the following bound is fulfilled

$$\frac{M}{|J|/\ell} = \frac{1+\omega^2}{2|\omega|} \geq 1, \tag{4.18}$$

for $\omega^2 \leq 1$, being saturated at the extremal case ($\omega^2 = 1$). The extremal case can be attained from (4.1) and (4.8) by first rescaling the integration constant B according to $b = \frac{B}{\sqrt{1-\omega^2}}$, and then taking the limit $\omega \to \pm 1$, so that the scalar field vanishes and the line element reduces to that of the extremal rotating BTZ black hole, whose (degenerate) horizon locates at $r_+^2 = 3b^2(1+\nu)$. It is worth noticing that static and stationary extremal cases, respectively for $M = J = 0$, and $M = |J|/\ell$, do not admit scalar hair.

5 Regularity and thermodynamics through non-flat connections

Here we carry out the thermodynamic analysis of the rotating hairy black hole relying on its description in terms of gauge fields in the Euclidean approach, where the period of the thermal cycle is assumed to be determined by the lapse function, so that Euclidean time $\tau = -it$ is normalized with period equals to 1. The Euclidean hairy black hole in the non-extremal case possesses the topology of a solid torus, $\mathbb{R}^2 \times S^1$, where S^1 is the circle parametrized by the angular coordinate θ, and $\mathbb{R}^2$ stands for the $r - \tau$ plane described in polar coordinates centered at $r = r_+$. Since thermal cycles around the horizon are contractible, regularity of the gauge fields at the horizon amounts to require trivial holonomies along them, allowing to fix the Lagrange multipliers at the boundary, given by N_∞ and N_∞^θ, which correspond to the Hawking temperature and the chemical potential.

In concrete, the regularity condition for the Euclidean rotating hairy black hole is obtained from demanding that the corresponding holonomies $\mathcal{H}^{\pm}$ along the thermal cycle at the event horizon to be trivial, i.e.,

$$\mathcal{H}^{\pm} = \exp\left[\int_0^1 A_\tau^{\pm} d\tau\right]_{r_+} = \exp\left[A_\tau^{\pm}\right]_{r_+} = I_c\,, \tag{5.1}$$

where I_c is a suitable element of the center of the gauge group.

One possible way to implement the regularity condition corresponds to the direct diagonalization of the holonomies $\mathcal{H}^{\pm}$ for an appropriate matrix representation of the full algebra $g^{+}\oplus g^{-}$. A simpler option is the one spelled out in [74], being implemented in two steps:

(i) Finding a group element that allows to gauge away the temporal component of the dreibein. In our case this condition is already implemented since the local Lorentz frame has already been chosen, and e_τ vanishes at the horizon provided that $N^{\theta}(r_+) = 0$, so that the chemical potential N_∞^{θ} is fixed in terms of N_∞ as

$$N_\infty^{\theta} = \frac{\omega}{\ell} N_\infty\,, \tag{5.2}$$

and the time component of the gauge fields reduce to $A_\tau^{\pm}(r_+) = \omega_{\pm\tau}(r_+)$, with $\omega_{\pm\tau}^{a}(r_+) = \omega_\tau^{a}(r_+)$, whose non-vanishing component is given by

$$\omega_\tau = N_\infty G(r_+) N'(r_+) J_2^{+}\,. \tag{5.3}$$

Therefore, as naturally expected from the results in section 3, the problem actually reduces to requiring a trivial holonomy for the pure gravity gauge field (3.5), whose time component at the horizon reads $A_\tau = \omega_\tau^{a}(r_+) J_a^{+}$.

(ii) Diagonalizing the remaining components, which in this case reduces to that of the corresponding $so(2,1) \approx sl(2,\mathbb{R})$ subalgebra. The basis can be chosen as

$$J_0^{+} = \frac{1}{2}\left(L_{-1} + L_1\right), \quad J_1^{+} = \frac{1}{2}\left(L_{-1} - L_1\right), \quad J_2^{+} = L_0\,, \tag{5.4}$$

with

$$L_{-1} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad L_0 = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \quad L_1 = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \tag{5.5}$$

fulfilling $[L_n, L_m] = (n-m) L_{n+m}$. Since the $sl(2,\mathbb{R})$ generators are given in the fundamental (spinorial) representation, the suitable element of the center of the group turns out to be $I_c = -\mathbb{I}$.

The diagonalization can be then readily performed, which implies that the eigenvalues of ω_τ must be given by $\pm i\pi$, or equivalently,

$$tr\left[(\omega_\tau)^2\right] = -2\pi^2\,. \tag{5.6}$$

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The latter equation allows to fix the form of N_∞, and hence, by virtue of the former condition in (5.2), the Lagrange multipliers, related to the Hawking temperature and the chemical potential, become fixed as

$$N_\infty = \frac{2\pi\Theta_\nu \ell^2}{3\,(1+\nu)\,B\sqrt{1-\omega^2}} \qquad \text{and} \qquad N_\infty^\theta = \frac{2\pi\Theta_\nu \omega\ell}{3\,(1+\nu)\,B\sqrt{1-\omega^2}}\,. \tag{5.7}$$

Once the Langrange multipliers are fixed as in (5.7), the pure gravity gauge field takes the following form at the horizon

$$A = R(r_+)d\theta P_2^+ + \left(2\pi dt - \frac{G(r_+)^2 R(r_+)^2 N'(r_+) N^{\theta\prime}(r_+)}{4\pi N(r_+)} d\theta\right) J_2^+\,, \tag{5.8}$$

so that the rotating hairy black hole entropy can be directly obtained from (3.6), being given by

$$S = \frac{4\pi^2 B\Theta_\nu}{\kappa\sqrt{1-\omega^2}} = \frac{4\pi^2 R(r_+)}{\kappa} = \frac{A_{\text{hor}}}{4G}, \tag{5.9}$$

with $\kappa = 8\pi G$, which is in full agreement with the Bekenstein-Hawking area law for the entropy. It is then simple to verify that the first law is fulfilled in the grand canonical ensemble, $dS = \beta dM - \beta\Omega dJ$, with M and J determined by (4.17), so that the relationship between the Lagrange multipliers with the Hawking temperature and the angular velocity at the horizon reads

$$\beta = N_\infty \qquad \text{and} \qquad \beta\Omega = N_\infty^\theta\,. \tag{5.10}$$

5.1 Soliton mass, regularity and Cardy formula

An analytic soliton solution endowed with a nontrivial scalar field for the theory under discussion was found in [63]. The line element is given by

$$\begin{aligned} ds^2 = \ell^2 &\left(1 + \frac{1}{\alpha_\nu\,(1+\rho^2)}\right)^{-2} \times \\ &\left[-N_\infty^2 \left[\frac{2\,(1+\rho^2)}{3c_\nu \ell}\right]^2 dt^2 + \frac{4d\rho^2}{2+\rho^2+\frac{c_\nu}{1+\rho^2}} + \left(\frac{2\rho}{2+c_\nu}\right)^2 \left(2+\rho^2+\frac{c_\nu}{1+\rho^2}\right) d\theta^2\right], \end{aligned} \tag{5.11}$$

with $c_\nu = 2\alpha_\nu^{-3}\,(1+\nu)$ and $\alpha_\nu = \frac{1}{2}\left(\Theta_\nu + \sqrt{\Theta_\nu^2 + 4\Theta_\nu}\right)$, while the scalar field reads

$$\phi\,(\rho) = \operatorname{arctanh}\left(\sqrt{\frac{1}{1+\alpha_\nu\,(1+\rho^2)}}\right)\,. \tag{5.12}$$

The radial coordinate has the range $0 \le \rho < \infty$, and the remaining coordinates range precisely as for the hairy black hole metric, so that the solitonic configuration is regular everywhere. As in the case of the rotating hairy black hole, the fall-off at infinity of the soliton also fits that of the relaxed boundary conditions in [54].

Regularity of the soliton at the origin can also be explicitly verified in terms of its corresponding gauge fields in (2.1), (2.2). In order to do that, we follow the same lines as

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in the case of the Euclidean hairy rotating black hole discussed above, adapted to this case. Thus, for the solitonic configuration, the suitable condition turns out to be requiring the holonomy of the gauge fields $A^{\pm}$ around a spatial (angular) cycle that encloses the origin ($\rho = 0$) to be trivial, i.e.,

$$\mathcal{H}^{\pm} = e^{\oint d\theta A_\theta^{\pm}\big|_{\rho=0}} = \mathbb{I}_c\,. \tag{5.13}$$

The local frame can be suitably chosen as in the case of the hairy black hole, so that the angular components of the dreibein e_θ automatically vanish at the origin, without the need of imposing any condition. Thus, the angular components of the gauge fields now fulfill $A_\theta^{\pm}\big|_{\rho=0} = \omega_{\pm\theta}\big|_{\rho=0}$, with $\omega_{\pm\theta}^{a}\big|_{\rho=0} = \omega_\theta^a\big|_{\rho=0}$, so that the problem again reduces to just requiring the trivial holonomy for the pure gravity connection in (3.5), whose angular component at the origin is given by $A_\theta\big|_{\rho=0} = \omega_\theta^a J_a^+\big|_{\rho=0}$. The remaining components can then be diagonalized by choosing the basis and the representation as in (5.4) and (5.5), respectively, so that the holonomy turns out to be trivial provided that the condition in (5.6) with $\omega_\tau \to \omega_\theta$ holds, which automatically does without the need of requiring any additional condition.

The result is reassuring because the soliton is devoid of integration constants, and hence, it has no freedom to be adjusted in order to fulfill the regularity conditions. Besides, as pointed out in [63], this last feature of the soliton suggests that it can be naturally regarded as a ground state of the theory for the sector with non-vanishing scalar fields. Indeed, it is simple to extend the result in [63] to the rotating case, so that the Euclidean hairy black hole turns out to be diffeomorphic to the Euclidean soliton provided that the corresponding modular parameters of the torus are related through S-duality, i.e.,

$$\tau_{\text{sol}} = -\frac{1}{\tau_{\text{hbh}}}$$

with $\tau_{\text{hbh}} = \frac{i\beta(1-i\Omega)}{2\pi}$, precisely as it occurs for the Euclidean BTZ black hole [75, 76] and Euclidean AdS_3 [77, 78].[2]

Thus, assuming that the soliton is the ground state of the hairy sector of the theory, the entropy of the rotating hairy black hole can be seen to be successfully reproduced by the Cardy formula once expressed in terms of left and right ground state energies $\tilde{\Delta}_0^{\pm}$ instead of the central charges [63] (see also [65, 72]). The entropy then reads

$$S = 4\pi\sqrt{-\tilde{\Delta}_0^{+}\tilde{\Delta}^{+}} + 4\pi\sqrt{-\tilde{\Delta}_0^{-}\tilde{\Delta}^{-}}\,, \tag{5.14}$$

where

$$\tilde{\Delta}^{\pm} = \frac{1}{2}\left(M\ell \pm J\right)\,, \tag{5.15}$$

[2]Analogue S-duality relationships between Euclidean three-dimensional black holes and their corresponding diffeomorphic Euclidean solitons are also know to hold for General Relativity on AdS_3 with boundary conditions of KdV type [9], as well as for asymptotically AdS [79] or asymptotically Lifshitz black holes [80] and their corresponding solitons in [81] and [82], respectively in the context of BHT massive gravity [83].

stand for the eigenvalues of the shifted Virasoro operators, $\tilde{L}_0^{\pm} = L_0^{\pm} - \frac{c^{\pm}}{24}$, being related to the mass M and angular momentum J of the hairy black hole in (4.17), while left and right energies of the ground state are determined by the soliton mass according to $\tilde{\Delta}_0^{\pm} = \frac{1}{2}\ell M_{\text{sol}}$.

The mass of the soliton can then be readily obtained by plugging its associated pure gravity gauge field (3.5) into the boundary term in (3.4), which yields to

$$\mathcal{B} = -\frac{1}{2\kappa}\left(t_2 - t_1\right)\oint d\theta \left\langle A_t A_\theta \right\rangle = \left(t_2 - t_1\right) N_\infty \left(\frac{\pi\alpha_\nu^4}{3\kappa\left(1+\nu\right)\left(1+\alpha_\nu\right)^2}\right), \tag{5.16}$$

so that the soliton mass is found to be given by

$$M_{\text{sol}} = -\frac{\pi\alpha_\nu^4}{3\kappa\left(1+\nu\right)\left(1+\alpha_\nu\right)^2}, \tag{5.17}$$

in agreement the the result obtained in [63] through the canonical approach. Therefore, making use of the precise value of the soliton mass in (5.17), it is straightforward to verify that the Cardy formula (5.14) precisely reproduces the rotating hairy black hole entropy in (5.9).

6 Conformal frame and ending remarks

As pointed out in the introduction, the action of a minimally coupled scalar field (1.1) with the self-interaction potential (1.1) relates to that of a conformally coupled self-interacting scalar field. Indeed, changing to the conformal (Jordan) frame according to $\hat{g}_{\mu\nu} = \Omega^{-2} g_{\mu\nu}$, with $\Omega = (1-\varphi^2)$, and redefining the scalar field as $\varphi = \tanh(\phi)$, the action becomes

$$I\left[\hat{g}_{\mu\nu},\varphi\right] = \frac{8}{\kappa}\int d^3x\sqrt{-\hat{g}}\left(\frac{\hat{R}-2\Lambda}{16} - \frac{1}{2}\hat{g}^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - \frac{1}{16}\hat{R}\varphi^2 - \lambda\varphi^6\right), \tag{6.1}$$

with $\lambda = \frac{\Lambda\nu}{8}$, so that the matter piece turns out to be conformally invariant. The field equations then read

$$\hat{G}_{\mu\nu} + \Lambda\hat{g}_{\mu\nu} = 8\hat{T}_{\mu\nu}, \tag{6.2}$$

$$\hat{\Box}\varphi - \frac{1}{8}\hat{R}\varphi - 6\lambda\varphi^5 = 0, \tag{6.3}$$

where the stress-energy tensor

$$\hat{T}_{\mu\nu} = \partial_\mu\varphi\partial_\nu\varphi - \frac{1}{2}\hat{g}_{\mu\nu}\partial_\alpha\varphi\partial^\alpha\varphi - \hat{g}_{\mu\nu}\lambda\varphi^6 + \frac{1}{8}\left(\hat{g}_{\mu\nu}\hat{\Box} - \hat{\nabla}_\mu\hat{\nabla}_\nu + \hat{G}_{\mu\nu}\right)\varphi^2, \tag{6.4}$$

is traceless by virtue of (6.3), implying that the Ricci scalar is constant, $\hat{R} = -6\ell^{-2}$.

The case of a conformally coupled self-interacting scalar field on a fixed background metric can also be described in terms of a Chern-Simons action [84], and here we extend this result to the case of a back-reacting scalar field with a dynamical metric described by (6.1).

It is then useful to define $\Lambda^+ = \Lambda$ and $\Lambda^- = -8\lambda$, as well as the appropriate composite gauge fields as

$$A^+ = \hat{e}^a P_a^+ + \omega_+^a J_a^+, \tag{6.5}$$

$$A^- = \varphi^2\hat{e}^a P_a^- + \omega_-^a J_a^-, \tag{6.6}$$

where according to the signs of $\Lambda^{\pm}$, they take values on the algebras in (2.3) that may correspond to $so(3,1)$, $so(2,2)$ or $iso(2,1)$, precisely as in the case of the minimally coupled scalar field. The field strengths then read

$$F^{\pm} = dA^{\pm} + (A^{\pm})^2 = \mathcal{T}^a_{\pm} P^{\pm}_a + \mathcal{R}^a_{\pm} J^{\pm}_a \,, \tag{6.7}$$

whose components are given by

$$\mathcal{R}^a_+ = R^a_+ - \frac{1}{2}\Lambda^+ \epsilon^{abc} \hat{e}_b \hat{e}_c \,, \tag{6.8}$$

$$\mathcal{R}^a_- = R^a_- - \frac{1}{2}\Lambda^- \varphi^4 \epsilon^{abc} \hat{e}_b \hat{e}_c \,, \tag{6.9}$$

$$\mathcal{T}^a_+ = T^a_+ \,, \tag{6.10}$$

$$\mathcal{T}^a_- = \varphi^2 \left(T^a_- + 2\varphi^{-1} d\varphi \hat{e}^a \right) \,, \tag{6.11}$$

where $R^a_{\pm} = d\omega^a_{\pm} + \frac{1}{2}\epsilon^{abc}\omega^{\pm}_b \omega^{\pm}_c$, and $T^a_{\pm} = d\hat{e}^a + \epsilon^{abc}\omega^{\pm}_b \hat{e}_c$.

Up to a boundary term, the action (6.1) can then be analogously reformulated in terms of a Chern-Simons form as in (2.9), for the composite gauge fields $A^{\pm}$ in (6.5), (6.6) so that $I = I\left[\varphi, \hat{e}, \omega^+, \omega^-\right]$. Varying with respect to the dynamical fields one obtains

$$\frac{\delta I}{\delta \hat{e}^a} : \qquad \mathcal{R}^+_a - \varphi^2 \mathcal{R}^-_a = 0 \,, \tag{6.12}$$

$$\frac{\delta I}{\delta \varphi} : \qquad \left(\mathcal{R}^+_a - \mathcal{R}^-_a \right) \hat{e}^a = 0 \,, \tag{6.13}$$

$$\frac{\delta I}{\delta \omega^a_+} : \qquad T^a_+ = 0 \,, \tag{6.14}$$

$$\frac{\delta I}{\delta \omega^a_-} : \qquad T^a_- + 2\varphi^{-1} d\varphi \hat{e}^a = 0 \,. \tag{6.15}$$

Since the last field equations (6.14) and (6.15) are algebraic for $\omega^a_{\pm}$, they are solved as

$$\omega^a_- \left(\hat{e}, \varphi\right) = \omega^a \left(\hat{e}\right) - 2\varphi^{-1} * \left(\hat{e}^a d\varphi\right) \,. \tag{6.16}$$

and $\omega^a_+ = \omega^a \left(\hat{e}\right)$, where $\omega^a \left(\hat{e}\right)$ stands for the (torsionless) spin connection associated to $\hat{e}^a$. Making use of them, the remaining field equations (6.12) and (6.13) reduce to (6.2) and (6.3), respectively.

A rotating black hole dressed with a conformally coupled scalar field can then be readily obtained from the exact solution discussed in section 4, just by transforming from the Einstein to the conformal frame. In the static case, it reduces to the black holes discussed in [54] and previously in [85] in the case of $\nu = 0$. As in section 5, the analysis of its global charges, regularity and thermodynamics can then also be performed in terms of the gauge fields $A^{\pm}$ in (6.5), (6.6) by virtue of the boundary terms in (3.3), (3.4) and the holonomy condition (5.1). It is also reassuring to verify that the black hole entropy formula in the Chern-Simons approach [41], given by (3.6), reduces to that in the conformal frame

$$S = \left[1 - \varphi^2(r_+)\right] \frac{A_{\text{hor}}}{4G} \,,$$

with $\kappa = 8\pi G$, see e.g., [86, 87], which can also be reproduced from the Cardy formula (5.14) when the scalar soliton in the conformal frame is assumed to be the ground state configuration.

As a final remark, it would be interesting to explore whether the hairy black holes and similar configurations endowed with a non-trivial scalar field that have been found for a variety of gravity theories, with different scalar couplings and self-interactions [88–115], could also be understood in terms of suitable composite gauge fields. Nevertheless, the Chern-Simons formulation for the theory discussed here naively appears to be somewhat rigid, up to a frame choice. In fact, the form of the composite gauge fields in (2.1) and (2.2) can be extended as $A^{\pm} = f_{\pm}^{2}(\phi)\, e^{a} P_{a}^{\pm} + \omega_{\pm}^{a} J_{a}^{\pm}$, so that consistency of the Chern-Simons formulation with a Riemannian (torsionless) geometry implies that $f_{+}^{2} - f_{-}^{2} = 1$, allows to obtain General Relativity minimally coupled to a scalar field with a non-canonical kinetic term and a self-interaction that depend on $f_{+}(\phi)$. However, the scalar field can be appropriately redefined so that in terms of the new scalar field one recovers the original action in (1.2) with canonical kinetic term and precisely the same self-interaction given in (1.1).

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