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Next-to-next-to-leading-order QCD corrections to J/ψ plus η_c production at the B factories

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Abstract: In this paper, we calculate the next-to-next-to-leading-order (NNLO) QCD corrections to $e^+e^- \to J/\psi + \eta_c$ at the B factories. After including the NNLO corrections, the cross section of $e^+e^- \to J/\psi + \eta_c$ is enhanced by about 17%, and the perturbative series of the prediction shows the convergent behavior. It is also found that the contributions from bottom quark starts at the α_s^3-order, which is about 2.4% of the total prediction. The renormalization scale μ_R dependence of the cross section is reduced at the NNLO level, but the prediction is sensitive to the charm quark mass m_c. By considering the uncertainties caused by renormalization scale μ_R, charm quark mass m_c and the NRQCD factorization scale μ_Λ, our prediction shows agreement with the BaBar and Belle measurements within errors.

Keywords: Higher-Order Perturbative Calculations, Quarkonium

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1 Introduction

Since the discovery of J/ψ in 1974, the heavy quarkonium production has been a focus of theoretical and experimental researches, which presents an ideal laboratory for the study of the interaction between quarks in two body systems. It plays an important role in the development of quantum chromodynamics (QCD). Perturbative QCD is essential to calculate the theoretical prediction for the large momentum transfer processes. In order to apply it to the quarkonium production, the colour-evaporation model [1, 2], the color-singlet model [3–5] and the nonrelativistic QCD (NRQCD) factorization formalism [6] have been introduced. Among them, the NRQCD factorization formalism provides a rigorous way to calculate the theoretical prediction perturbatively, whose result can be improved by including higher order corrections of the QCD coupling constant α_s and the heavy-quark relative velocity v. It is very successful when applied to many quarkonium production processes, especially the unpolarized cross section of the J/ψ hadroproduction [7–10]. However, there are still some discrepancies between the NRQCD predictions and the experimental measurements of the heavy quarkonium production. To test the NRQCD factorization, it is significant to investigate more processes about heavy quarkonium production.

In 2002, the total cross section of $e^+e^- \to J/\psi + \eta_c$ measured by Belle at $\sqrt{s} = 10.58\,\text{GeV}$ [11] is $\sigma[J/\psi + \eta_c] \times B^{\eta_c}[\geq 4] = (33^{+7}_{-6} \pm 9)\,\text{fb}$, where $B^{\eta_c}[\geq n]$ denotes the branching ratio of η_c into n or more charged tracks. This measurement was improved as $\sigma[J/\psi + \eta_c] \times B^{\eta_c}[\geq 2] = (25.6 \pm 2.8 \pm 3.4)\,\text{fb}$ [12] in 2004. Later in the year 2005, another independent measurement was finished by BaBar [13], and the total cross section is $\sigma[J/\psi + \eta_c] \times B^{\eta_c}[\geq 2] = (17.6 \pm 2.8^{+1.5}_{-2.1})\,\text{fb}$. Meanwhile, the calculation at NRQCD leading order (LO) of the QCD coupling constant α_s and the charm quark relative velocity v, gives a theoretical prediction for the total cross section $2.3 \sim 5.5\,\text{fb}$ [14–16], which is much smaller than the experimental measurements. A lot of theoretical studies have been performed to explain this large discrepancy. The relativistic corrections have been studied by several groups [17–19]. Some other attempts have also been suggested to solve this discrepancy, such as the light-cone factorization approach [20–22] or light-cone sum rules [23, 24]. The next-to-leading-order (NLO) QCD correction of the process has been regarded as a breakthrough [25, 26], which can greatly enhance the size of cross section and reduce the large discrepancy. The joint NLO QCD and relativistic correction has been investigated in refs. [27, 28]. The improved NLO prediction has been given in ref. [29] by applying the principle of maximum conformality [30–33], which shows excellent agreement with the experimental measurements.

However, the NLO prediction shows very poor convergence, the relative magnitudes of each order is about 1:1. The NNLO QCD correction is still important to verify its perturbative property. In 2019, the challenging NNLO correction of this process was calculated in ref. [34], however the precision of master integrals is not satisfied. In 2022, a powerful algorithm named *Auxiliary Mass Flow* has been pioneered by Liu and Ma [35–37], which can be used to compute the Feynman integrals with very high precision. In this paper, we will calculate the NNLO QCD correction to $e^+e^- \to J/\psi + \eta_c$ with the help of the package `AMFlow` [38] and further include the contribution from bottom quark.[1]

The remaining parts of the paper are organized as follows. In section 2, we will present useful formulas for the process $e^+e^- \to J/\psi + \eta_c$ and give a brief description on the calculation procedures. In section 3, we will show the numerical results and discussions. Section 4 is reserved as a summary.

2 Calculation technology

2.1 Cross section

Under the NRQCD factorization, the cross section for $e^+(k_1)e^-(k_2) \to J/\psi(p_1) + \eta_c(p_2)$ can be written as

$$d\sigma_{e^+e^-\to J/\psi+\eta_c} = d\hat{\sigma}_{e^+e^-\to(c\bar{c})[n_1]+(c\bar{c})[n_2]}\langle\mathcal{O}^{J/\psi}(n_1)\rangle\langle\mathcal{O}^{\eta_c}(n_2)\rangle, \tag{2.1}$$

where $d\hat{\sigma}$ are short-distance coefficients (SDCs) and $\langle\mathcal{O}^{J/\psi}(n_1)\rangle, \langle\mathcal{O}^{\eta_c}(n_2)\rangle$ are long-distance matrix elements (LDMEs). In the lowest-order nonrelativistic approximation, only the color-singlet contribution with $n_1 = {}^3S_1^{[1]}$ and $n_2 = {}^1S_0^{[1]}$ need to be considered, which is discovered to be trivial in present process.

Since LDMEs $\langle\mathcal{O}^{J/\psi}(n_1)\rangle$ and $\langle\mathcal{O}^{\eta_c}(n_2)\rangle$ include the nonperturbative hadronization effects, we start from the cross section of two on-shell $(c\bar{c})$-pairs with quantum number ${}^3S_1^{[1]}$ and ${}^1S_0^{[1]}$, which has same SDC with $e^+e^- \to J/\psi + \eta_c$:

$$d\sigma_{e^+e^-\to(c\bar{c})[n_1]+(c\bar{c})[n_2]} = d\hat{\sigma}_{e^+e^-\to(c\bar{c})[n_1]+(c\bar{c})[n_2]}\langle\mathcal{O}^{(c\bar{c})[n_1]}(n_1)\rangle\langle\mathcal{O}^{(c\bar{c})[n_2]}(n_2)\rangle. \tag{2.2}$$

Here symbols $\langle\mathcal{O}^{(c\bar{c})[n_1]}(n_1)\rangle$ and $\langle\mathcal{O}^{(c\bar{c})[n_2]}(n_2)\rangle$ are related to NRQCD bilinear operators as

$$\langle\mathcal{O}^{(c\bar{c})[n_1]}(n_1)\rangle = |\langle 0|\chi^\dagger\epsilon\cdot\sigma\psi|c\bar{c}(n_1)\rangle|^2, \tag{2.3}$$

$$\langle\mathcal{O}^{(c\bar{c})[n_2]}(n_2)\rangle = |\langle 0|\chi^\dagger\psi|c\bar{c}(n_2)\rangle|^2, \tag{2.4}$$

[1]Recently, the author of ref. [34] update their numerical results in the newest version by using the package `AMFlow` [38] too, where the contribution from bottom quark has been also considered. We will compare their numerical results with ours in the following part.

in which $\langle 0|\chi^\dagger \epsilon\cdot\sigma\psi|c\bar{c}(n_1)\rangle$ and $\langle 0|\chi^\dagger\psi|c\bar{c}(n_2)\rangle$ can be calculated in the NRQCD framework [6, 39–44]. On the other hand, the l.h.s. of eq. (2.2) can be calculated directly in perturbative QCD. Hence, the SDC $d\hat{\sigma}_{e^+e^-\to(c\bar{c})[n_1]+(c\bar{c})[n_2]}$ can be extracted using eq. (2.2). In combination with LDMEs $\langle\mathcal{O}^{J/\psi}(n_1)\rangle$ and $\langle\mathcal{O}^{\eta_c}(n_2)\rangle$, we can obtain the cross section of $e^+e^-\to J/\psi+\eta_c$ with eq. (2.1).

Exclusive production of $e^+e^-\to(c\bar{c})[n_1]+(c\bar{c})[n_2]$ at B factories only involves s-channel contribution. In the proceeding process, the e^+e^- first annihilate into a virtual photon, and then it will decay into two final states. By choosing the Feynman gauge, it is convenient to rewrite the differential cross section as [45]:

$$d\sigma_{e^+e^-\to(c\bar{c})[^3S_1^{[1]}]+(c\bar{c})[^1S_0^{[1]}]}=\frac{1}{4}\frac{1}{2s}\frac{1}{s^2}L^{\mu\nu}H_{\mu\nu}d\Phi_2. \tag{2.5}$$

where $1/4$ comes from the spin average of the initial e^+e^-, $1/2s$ is the flux factor, $1/s^2$ comes from photon propagator and $s=(k_1+k_2)^2$ is the squared center-of-mass energy. $L^{\mu\nu}$ and $H_{\mu\nu}$ are the leptonic tensor and hadronic tensor, respectively. $d\Phi_2$ is the differential phase space for the two-body final state. If we focus on the total cross section, $L^{\mu\nu}$ can be equivalently replaced by $4\pi\alpha s(-\frac{8}{3}g^{\mu\nu})$ [46, 47]. Thus, the complication of calculation can be greatly reduced. The total cross section of the process $e^+e^-\to(c\bar{c})[^3S_1^{[1]}]+(c\bar{c})[^1S_0^{[1]}]$ can be calculated through

$$\sigma_{e^+e^-\to(c\bar{c})[^3S_1^{[1]}]+(c\bar{c})[^1S_0^{[1]}]}=\frac{\alpha}{12s^2}\sqrt{1-\frac{16m_c^2}{s}}H^\mu{}_\mu, \tag{2.6}$$

where $H^\mu{}_\mu$ corresponds to the decay of a virtual photon into $(c\bar{c})[^3S_1^{[1]}]$ plus $(c\bar{c})[^1S_0^{[1]}]$.

2.2 Calculation of the perturbative SDC

In this section, we will give a brief description on the calculation procedures. First, we apply the package `FeynArts` [48] to generate corresponding Feynman diagrams and amplitudes for $\gamma^*\to(c\bar{c})[^3S_1^{[1]}]+(c\bar{c})[^1S_0^{[1]}]$ at NNLO in α_s. Second, we implement the package `FeynCalc` [49, 50] to handle the Lorentz index contraction and Dirac/SU(N_c) traces. Third, we use the Mathematica code developed by Yan-Qing Ma to decompose the Feynman amplitudes into different Feynman integral families. Last, we employ the package `AMFlow` [38] to calculate the Feynman integral families with the help of `Kira` [51], where the package `Kira` is used to do the integration-by-parts (IBP) reduction.

During the calculation, there are nearly 2000 two-loop diagrams for the $\gamma^*\to(c\bar{c})[^3S_1^{[1]}]+(c\bar{c})[^1S_0^{[1]}]$. Some representative Feynman diagrams up to two-loop order are plotted in figure 1, where $p_3=\frac{p_1+q_1}{2}$, $p_4=\frac{p_1-q_1}{2}$, $p_5=\frac{p_2+q_2}{2}$, $p_6=\frac{p_2-q_2}{2}$ and $p_1^2=p_2^2=(2m_c)^2$. The momenta q_1 and q_2 denote the relative momenta between the quark and antiquark in the $(c\bar{c})$-pairs. The last Feynman diagram is also called "light-by-light" Feynman diagram, which denotes a special topology. Such topology shows that a closed quark loop is linked with the virtual photon and three gluons. The contributions of "light-by-light" Feynman diagram from the light quark loops can be ignored, since the sum of electric charges of all the light quarks is zero, i.e., $e_u+e_d+e_s=0$. The total

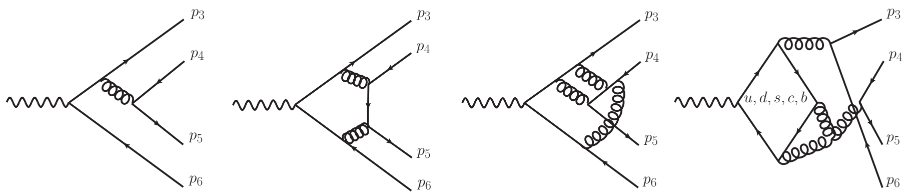

Figure 1. Some representative Feynman diagrams for $\gamma^* \to (c\bar{c})[^3S_1^{[1]}] + (c\bar{c})[^1S_0^{[1]}]$.

NNLO amplitudes can be decomposed into roughly 150 Feynman integral families. These Feynman integral families contain about 10000 Feynman integrals. Due to the powerful package `AMFlow`, we can compute these Feynman integrals with very high precision. We demand 20-digit precision for each Feynman integral family, and the final numerical result achieve at least 10-digit precision.

In order to obtain the final NNLO correction, we adopt the conventional dimensional regularization approach with $d = 4 - 2\epsilon$ to regulate ultraviolet (UV) and infrared (IR) divergences. Feynman diagrams with a virtual gluon line connected with the quark pair in a meson contain Coulomb singularities, which are shown as power divergence in the IR limit of the relative momentum and can be taken into the $c\bar{c}$ wave function renormalization [26, 52]. In our calculation, we set the relative momenta (q_1 and q_2) between the quark and antiquark in the ($c\bar{c}$)-pairs to zero before performing loop integration. The Coulomb divergence vanishes in the calculation under dimensional regularization. The UV divergences are removed through renormalization. We renormalize the heavy quark field and the heavy quark mass in the on-shell (OS) scheme. The coupling constant α_s is renormalized in the $\overline{\text{MS}}$ scheme. More explicitly, the amplitudes are renormalized according to

$$\begin{aligned}&\mathcal{A}(\alpha_s, m_Q) = \\ &Z_{2,c}^2\Big[\mathcal{A}_{\text{bare}}^{0l}(\alpha_{s,\text{bare}}, m_{Q,\text{bare}}) + \mathcal{A}_{\text{bare}}^{1l}(\alpha_{s,\text{bare}}, m_{Q,\text{bare}}) + \mathcal{A}_{\text{bare}}^{2l}(\alpha_{s,\text{bare}}, m_{Q,\text{bare}})\Big],\end{aligned} \tag{2.7}$$

where the $\mathcal{A}_{\text{bare}}^{il}|_{i=0,1,2}$ are the tree, one-loop and two-loop bare amplitudes, respectively. $Z_{2,c}$ is the on-shell wave-function renormalization constant for charm quark. The bare mass is renormalized as $m_{Q,\text{bare}} = Z_{m,Q} m_Q$, where $Z_{m,Q}$ is the on-shell mass renormalization constants for heavy-quarks. The bare coupling constant is renormalized as

$$\alpha_{s,\text{bare}} = \left(\frac{e^{\gamma_E}}{4\pi}\right)^{\epsilon} \mu_R^{2\epsilon} Z_{\alpha_s}^{\overline{\text{MS}}} \alpha_s(\mu_R), \tag{2.8}$$

which corresponds to the $\overline{\text{MS}}$ scheme with n_f active favors. Here μ_R is the renormalization scale and $Z_{\alpha_s}^{\overline{\text{MS}}}$ is the $\overline{\text{MS}}$ coupling constant renormalization constant. The two loop renormalization constants are computed in refs. [53–57]. The renormalized $\mathcal{A}(\alpha_s, m)$ can be obtained by expanding the r.h.s. of eq. (2.7) over renormalized quantities to $\mathcal{O}(\alpha_s^4)$, i.e.,

$$\mathcal{A}(\alpha_s, m_Q) = \mathcal{A}^{0l}(\alpha_s, m_Q) + \left(\frac{e^{\gamma_E}}{4\pi}\right)^{-\epsilon} \mathcal{A}^{1l}(\alpha_s, m_Q) + \left(\frac{e^{\gamma_E}}{4\pi}\right)^{-2\epsilon} \mathcal{A}^{2l}(\alpha_s, m_Q) + \mathcal{O}(\alpha_s^4). \tag{2.9}$$

where the $\mathcal{A}^{il}|_{i=0,1,2}$ are the tree, one-loop and two-loop renormalized amplitudes, respectively. It should be noted that the prefactor $[e^{\gamma_E}/(4\pi)]^{-n\epsilon}$ have been introduced in order to avoid unnecessary $\gamma_E - \ln(4\pi)$ terms. The loop integrals are computed with the measure $\mu_R^{2\epsilon} d^d k/(2\pi)^d$, and the corresponding renormalization constants ($Z_{2,c}$, $Z_{m,Q}$ and $Z_{\alpha_s}^{\overline{\rm MS}}$) can be found in refs. [58, 59]. Thus, the total cross section can be written as

$$
\begin{aligned}
&\sigma_{e^+e^-\to(c\bar{c})[^3S_1^{[1]}]+(c\bar{c})[^1S_0^{[1]}]} \\
&= \kappa \left| \mathcal{A}^{0l}(\alpha_s, m_Q) + \left(\frac{e^{\gamma_E}}{4\pi}\right)^{-\epsilon} \mathcal{A}^{1l}(\alpha_s, m_Q) + \left(\frac{e^{\gamma_E}}{4\pi}\right)^{-2\epsilon} \mathcal{A}^{2l}(\alpha_s, m_Q) + \mathcal{O}(\alpha_s^4) \right|^2 \\
&= \kappa(f_0 + f_1 + f_2) + \mathcal{O}(\alpha_s^5),
\end{aligned}
\tag{2.10}
$$

where $\kappa = \frac{\alpha}{12s^2}\sqrt{1 - \frac{16m_c^2}{s}}$. The $f_i(i = 0, 1, 2,)$ denote the squared amplitudes corresponding to α_s^2-, α_s^3- and α_s^4-orders respectively, which are

$$
f_0 = \left| \mathcal{A}^{0l}(\alpha_s, m_Q) \right|^2, \tag{2.11}
$$

$$
f_1 = \left(\frac{e^{\gamma_E}}{4\pi}\right)^{-\epsilon} 2\mathrm{Re}\left[\mathcal{A}^{1l}(\alpha_s, m_Q) \mathcal{A}^{0l,*}(\alpha_s, m_Q) \right], \tag{2.12}
$$

$$
f_2 = \left(\frac{e^{\gamma_E}}{4\pi}\right)^{-2\epsilon} \left\{ 2\mathrm{Re}\left[\mathcal{A}^{2l}(\alpha_s, m_Q) \mathcal{A}^{0l,*}(\alpha_s, m_Q) \right] + \left| \mathcal{A}^{1l}(\alpha_s, m_Q) \right|^2 \right\}. \tag{2.13}
$$

However, there still remains IR divergence in f_2, which can be canceled by including the two-loop corrections to $\langle \mathcal{O}^{(c\bar{c})[^3S_1^{[1]}]}(^3S_1^{[1]}) \rangle$ and $\langle \mathcal{O}^{(c\bar{c})[^1S_0^{[1]}]}(^1S_0^{[1]}) \rangle$ in $\overline{\rm MS}$ scheme. At the lowest order in velocity expansion, it can be written as

$$
\langle \mathcal{O}^{(c\bar{c})[^3S_1^{[1]}]}(^3S_1^{[1]}) \rangle|_{\overline{\rm MS}} = 2N_c \left[1 - \alpha_s^2(\mu_R) \left(\frac{\mu_\Lambda^2 e^{\gamma_E}}{\mu_R^2 4\pi} \right)^{-2\epsilon} \left(\frac{C_F^2}{3} + \frac{C_F C_A}{2} \right) \frac{1}{2\epsilon} \right], \tag{2.14}
$$

$$
\langle \mathcal{O}^{(c\bar{c})[^1S_0^{[1]}]}(^1S_0^{[1]}) \rangle|_{\overline{\rm MS}} = 2N_c \left[1 - \alpha_s^2(\mu_R) \left(\frac{\mu_\Lambda^2 e^{\gamma_E}}{\mu_R^2 4\pi} \right)^{-2\epsilon} \left(C_F^2 + \frac{C_F C_A}{2} \right) \frac{1}{2\epsilon} \right], \tag{2.15}
$$

which can be obtained from refs. [6, 39–44]. The factor $(\mu_\Lambda^2/\mu_R^2)^{-2\epsilon}$ is derived by running the scale of α_s from factorization scale μ_Λ to renormalization scale μ_R, since the initial results are calculated at the scale μ_Λ. The factor $[e^{\gamma_E}/(4\pi)]^{-2\epsilon}$ comes from the definition of α_s in the $\overline{\rm MS}$ scheme given in eq. (2.8). Therefore, the SDC can be determined as

$$
\begin{aligned}
\hat{\sigma}_{e^+e^-\to(c\bar{c})[^3S_1^{[1]}]+(c\bar{c})[^1S_0^{[1]}]} &= \frac{\sigma_{e^+e^-\to(c\bar{c})[^3S_1^{[1]}]+(c\bar{c})[^1S_0^{[1]}]}}{\langle \mathcal{O}^{(c\bar{c})[^3S_1^{[1]}]}(^3S_1^{[1]}) \rangle|_{\overline{\rm MS}} \langle \mathcal{O}^{(c\bar{c})[^1S_0^{[1]}]}(^1S_0^{[1]}) \rangle|_{\overline{\rm MS}}} \\
&= \frac{\kappa}{(2N_c)^2}(f_0 + f_1 + \tilde{f}_2),
\end{aligned}
\tag{2.16}
$$

where $\tilde{f}_2$ is

$$
\tilde{f}_2 = f_2 + f_0 \alpha_s^2(\mu_R) \left(\frac{\mu_\Lambda^2 e^{\gamma_E}}{\mu_R^2 4\pi} \right)^{-2\epsilon} \left(\frac{4C_F^2}{3} + C_F C_A \right) \frac{1}{2\epsilon}. \tag{2.17}
$$

The resultant $\tilde{f}_2$ is finite, which renders the SDC without any divergences. And it develops an explicit logarithmic dependence on NRQCD factorization scale $\mu_\Lambda \sim \ln \frac{\mu_\Lambda^2}{m^2}$ [34] at the NNLO level simultaneously.

3 Phenomenological results

To do the numerical calculation, the input parameters are taken as follows:

$$m_b = 4.78\,\text{GeV},\;\; \alpha_s(m_{_Z}),= 0.1179 \quad \sqrt{s} = 10.58\,\text{GeV},\;\; \alpha(\sqrt{s}) = 1/130.9, \tag{3.1}$$

$$\langle \mathcal{O}^{J/\psi}(^3S_1^{[1]})\rangle = 0.440^{+0.067}_{-0.056}\,\text{GeV}^3, \qquad \langle \mathcal{O}^{\eta_c}(^1S_0^{[1]})\rangle = 0.437^{+0.111}_{-0.104}\,\text{GeV}^3, \tag{3.2}$$

where the bottom pole mass and running QCD coupling constant at scale $m_{_Z}$ are taken from Particle Data Group [60]. The QED coupling constant and NRQCD LDMEs are taken from ref. [19]. For simplicity, we take the central value of the LDMEs to perform the phenomenological discussion. We use the package `RunDec3` [61] to evaluate the running QCD coupling constant $\alpha_s(\mu_R)$ at three-loop accuracy.

The numerical results of the NNLO QCD corrections to $J/\psi + \eta_c$ production at the B factories with three typical m_c values are

$$\begin{aligned}
\sigma|_{m_c=1.3\,\text{GeV}} = {} & 136.651\alpha_s^2(\mu_R) + \left[(210.237 - 14.4991n_l)\ln\frac{\mu_R^2}{m_c^2} + 16.5325n_l \right.\\
& \left. +297.262\right]\alpha_s^3(\mu_R) + \left[(242.587 - 33.4603n_l + 1.1538n_l^2)\left(\ln\frac{\mu_R^2}{m_c^2}\right)^2 \right.\\
& +(818.692 - 31.0801n_l - 2.63123n_l^2)\ln\frac{\mu_R^2}{m_c^2} - 870.518\ln\frac{\mu_\Lambda^2}{m_c^2}\\
& \left. +124.921 - 87.7298n_l - 2.29574n_l^2\right]\alpha_s^4(\mu_R),
\end{aligned} \tag{3.3}$$

$$\begin{aligned}
\sigma|_{m_c=1.5\,\text{GeV}} = {} & 115.599\alpha_s^2(\mu_R) + \left[(177.849 - 12.2654n_l)\ln\frac{\mu_R^2}{m_c^2} + 10.4752n_l \right.\\
& \left. +215.393\right]\alpha_s^3(\mu_R) + \left[(205.215 - 28.3055n_l + 0.976053n_l^2)\left(\ln\frac{\mu_R^2}{m_c^2}\right)^2 \right.\\
& +(609.319 - 28.6518n_l - 1.66718n_l^2)\ln\frac{\mu_R^2}{m_c^2} - 736.409\ln\frac{\mu_\Lambda^2}{m_c^2}\\
& \left. -109.15 - 74.7989n_l - 2.49917n_l^2\right]\alpha_s^4(\mu_R),
\end{aligned} \tag{3.4}$$

$$\begin{aligned}
\sigma|_{m_c=1.7\,\text{GeV}} = {} & 93.0233\alpha_s^2(\mu_R) + \left[(143.116 - 9.87007n_l)\ln\frac{\mu_R^2}{m_c^2} + 5.9587n_l \right.\\
& \left. +150.238\right]\alpha_s^3(\mu_R) + \left[(165.138 - 22.7776n_l + 0.785436n_l^2)\left(\ln\frac{\mu_R^2}{m_c^2}\right)^2 \right.\\
& +(437.037 - 25.0833n_l - 0.948356n_l^2)\ln\frac{\mu_R^2}{m_c^2} - 592.593\ln\frac{\mu_\Lambda^2}{m_c^2}\\
& \left. -231.418 - 59.3073n_l - 2.29771n_l^2\right]\alpha_s^4(\mu_R),
\end{aligned} \tag{3.5}$$

where n_l is the number of light flavors (u, d and s). Note that we do not distinguish the contribution from the light-by-light Feynman diagrams in eqs. (3.3)–(3.5). To guarantee

		α_s^2-terms	α_s^3-terms	α_s^4-terms	Total
$\mu_\Lambda = m_c$	$\mu_R = 2m_c$	7.40	$7.04 + 0.13$	$2.17 + 0.28$	$16.61 + 0.41$
	$\mu_R = \sqrt{s}/2$	5.06	$5.57 - 0.05$	$3.43 - 0.08$	$14.06 - 0.13$
$\mu_\Lambda = 1\,\mathrm{GeV}$	$\mu_R = 2m_c$	7.40	$7.04 + 0.13$	$4.62 + 0.27$	$19.06 + 0.40$
	$\mu_R = \sqrt{s}/2$	5.06	$5.57 - 0.05$	$4.58 - 0.09$	$15.21 - 0.14$

Table 1. The NNLO cross section (in fb) of $e^+e^- \to J/\psi + \eta_c$ with two typical renormalization scales μ_R under two factorization scale μ_Λ choices. The second numbers appearing in the α_s^3-terms, α_s^4-terms and Total are the contributions from bottom quark.

our result reliable, we have also reproduced the NNLO corrections to $e^+e^- \to \eta_c + \gamma$ process [62].

Assuming $m_c = 1.5\,\mathrm{GeV}$ and the factorization scale $\mu_\Lambda = m_c$ (or $\mu_\Lambda = 1\,\mathrm{GeV}$), we show the total cross section for two typical choices of renormalization scale in table 1. It shows that the contribution from bottom quark is about $0.9 \sim 2.4\%$ of the total prediction. It is also found that the α_s^4-terms of present process is sizable and smaller than α_s^3-terms. The perturbative expansion of the NNLO prediction has exhibited a convergent signature. More explicitly, the relative magnitudes of the α_s^2-terms: α_s^3-terms: α_s^4-terms is about $1 : 97\% : 33\%$ for the case of $\mu_R = 2m_c$ and $1 : 109\% : 66\%$ for the case of $\mu_R = \sqrt{s}/2$. As regards to the renormalization scale μ_R dependence, it is found that $\sigma \in [13.93, 17.02]$ fb for $\mu_R \in [2m_c, \sqrt{s}/2]$ with $\mu_\Lambda = m_c$; i.e., the scale uncertainty of NLO (NNLO) prediction is $\sim 27\%$ (18%), respectively.[2] Nevertheless, the renormalization scale dependence of the total cross section is improved $\sim 9\%$ by including NNLO correction.

To study the factorization scale uncertainty, we also present the numerical NNLO prediction at the factorization scale $\mu_\Lambda = 1\,\mathrm{GeV}$ in table 1. It shows that the α_s^4-terms will change about $34 \sim 99.5\%$ comparing with that of the case $\mu_\Lambda = m_c$. It also indicates that our results are consistent with table I of ref. [34]. The net small difference are caused by the different choices of bottom quark pole mass and the significant digits of $\alpha_s(\sqrt{s}/2)$.

In table 2, we list the numerical results of the NNLO prediction with $m_c = 1.3\,\mathrm{GeV}$, $m_c = 1.5\,\mathrm{GeV}$ and $m_c = 1.7\,\mathrm{GeV}$, respectively. It indicates the predicted cross section is rather sensitive to the charm mass. By taking $m_c = 1.5\,\mathrm{GeV}$ as center value, the relative uncertainties of α_s^2-, α_s^3- and α_s^4-terms are about 59%, 91%, and 197% for $\mu_R = 2m_c$ and 38%, 64%, and 120% for $\mu_R = \sqrt{s}/2$, respectively. It can be found that the prediction with $m_c = 1.5\,\mathrm{GeV}$ is much closer to the BaBar measurements and the prediction with $m_c = 1.3\,\mathrm{GeV}$ is much closer to the Belle measurements.

In figure 2, we plot the μ_R dependence of the predicted cross sections at LO, NLO and NNLO levels, respectively. The three black lines are obtained by taking $m_c = 1.5\,\mathrm{GeV}$. The red bound denotes the uncertainty from m_c within $[1.3, 1.7]\,\mathrm{GeV}$. The factorization scale are taken as $\mu_\Lambda = m_c$ and $\mu_\Lambda = 1\,\mathrm{GeV}$, respectively. Figure 2 shows that: 1) the NNLO

[2]We attempt to eliminate the renormalization scale ambiguity by using the principle of maximum conformality [30–33]. Due to the fact that the effective scale Q_* is found to be $0.37\,\mathrm{GeV}$, which is already in the non-perturbative region. Thus, we have to give up the discussion in present paper.

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		α_s^2-terms	α_s^3-terms	α_s^4-terms	Total($\mu_\Lambda = m_c$)
$m_c = 1.3\,\mathrm{GeV}$	$\mu_R = 2m_c$	9.80	11.10	5.70	26.60
	$\mu_R = \sqrt{s}/2$	5.98	7.46	5.77	19.21
$m_c = 1.5\,\mathrm{GeV}$	$\mu_R = 2m_c$	7.40	7.17	2.45	17.02
	$\mu_R = \sqrt{s}/2$	5.06	5.52	3.35	13.93
$m_c = 1.7\,\mathrm{GeV}$	$\mu_R = 2m_c$	5.42	4.58	0.88	10.88
	$\mu_R = \sqrt{s}/2$	4.07	3.90	1.74	9.71

Table 2. The NNLO cross section (in fb) of $e^+e^- \to J/\psi + \eta_c$ under three choices of charm quark pole mass. The two upper rows, the two middle rows and the two lower rows correspond to $m_c = 1.3\,\mathrm{GeV}$, $m_c = 1.5\,\mathrm{GeV}$ and $m_c = 1.7\,\mathrm{GeV}$, respectively. The $\mu_R = 2m_c$ and $\mu_R = \sqrt{s}/2$ are considered. $\mu_\Lambda = m_c$.

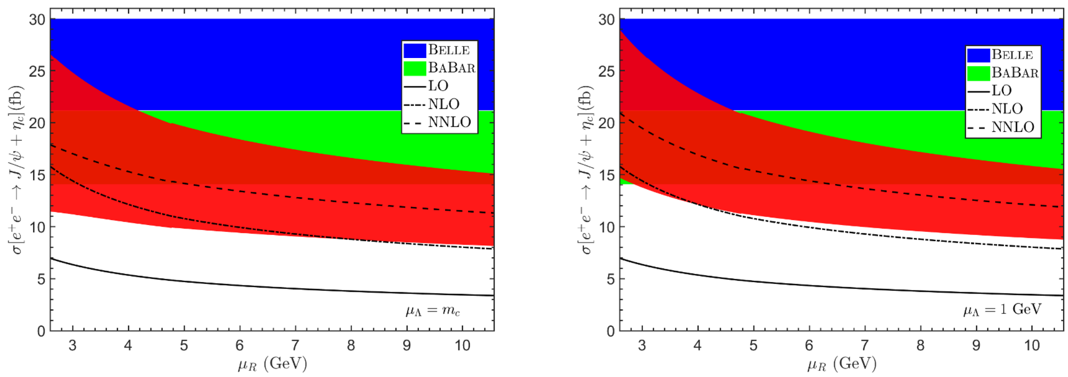

Figure 2. The μ_R dependence of the predicted cross sections at LO, NLO and NNLO levels, respectively. We take $m_c = 1.5\,\mathrm{GeV}$. The red bands represents the uncertainty obtained by varying $m_c \in [1.3, 1.7]\,\mathrm{GeV}$, where the lower bound corresponds to $m_c = 1.7\,\mathrm{GeV}$ and upper bound $m_c = 1.3\,\mathrm{GeV}$. The left figure corresponds to $\mu_\Lambda = m_c$, and the right figure corresponds to $\mu_\Lambda = 1\,\mathrm{GeV}$.

prediction has a milder dependence on the renormalization scale than the NLO prediction in the case of $\mu_\Lambda = m_c$; 2) the NNLO prediction with $\mu_\Lambda = 1\,\mathrm{GeV}$ are much closer to the experimental value, but much larger μ_R dependence than that in the case of $\mu_\Lambda = m_c$; 3) the predicted results with $\mu_R = 2m_c$ agree with the experimental results better than the results with $\mu_R = \sqrt{s}/2$. Comparing our figure 2 with figure 2 of ref. [34], it can be found that our result is smaller than their result in the case of $\mu_\Lambda = m_c$.[3]

4 Summary

As a summary, we have calculated the NNLO QCD corrections of $J/\psi + \eta_c$ production in e^+e^- annihilation at center-of-mass energy $\sqrt{s} = 10.58\,\mathrm{GeV}$. The NNLO corrections to the total cross section for $e^+e^- \to J/\psi + \eta_c$ is sizable, but not comparable to the NLO

[3]It can be found from eqs. (3-11) of ref. [34] that the numerical result of $\sigma_{\mathrm{NNLO},\mu_\Lambda=1\,\mathrm{GeV}}$ is larger than that of $\sigma_{\mathrm{NNLO},\mu_\Lambda=\mathrm{m_c}}$. Figure 2 of our work shows the same conclusion. However, figure 2 of ref. [34] shows that the numerical result of $\sigma_{\mathrm{NNLO},\mu_\Lambda=\mathrm{m_c}}$ is larger than that of $\sigma_{\mathrm{NNLO},\mu_\Lambda=1\,\mathrm{GeV}}$.

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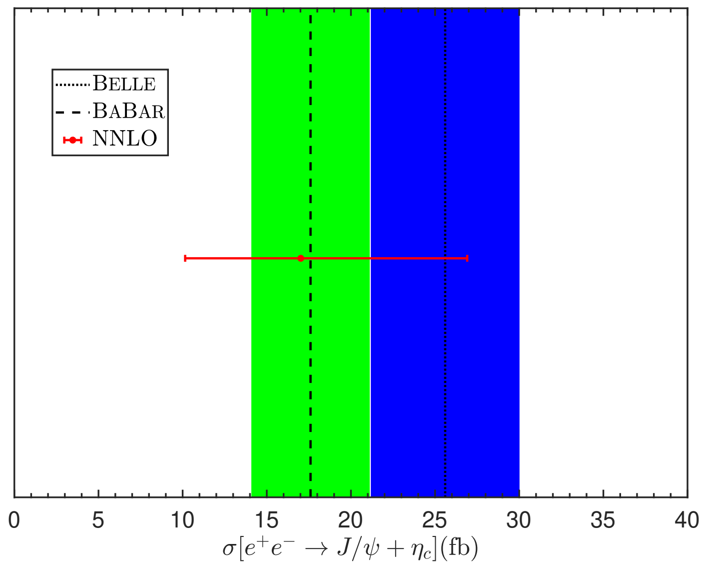

Figure 3. The comparison between the NNLO QCD correction to $e^+e^- \to J/\psi + \eta_c$ and the experimental measurements.

corrections. It exhibits reasonable perturbative convergence behavior. More explicitly, the total cross section for $e^+e^- \to J/\psi + \eta_c$ is enhanced by 17% in the case of $\mu_R = 2m_c$ after including the NNLO corrections. From figure 2, we find that the μ_R dependence of the total cross section is reduced at the NNLO level for $\mu_\Lambda = m_c$, and the predicted cross section is sensitive to the charm quark pole mass. Combining the results shown in table 1 and 2, we obtain the final NNLO QCD corrections to $J/\psi + \eta_c$ production at the B factories, i.e.,

$$\begin{aligned} \sigma_{\text{NNLO}} &= 17.02^{+9.58+0+2.44}_{-6.14-3.09-0} \\ &= 17.02^{+9.89}_{-6.87}\,(\text{fb}) \end{aligned} \tag{4.1}$$

where the central value is obtained by taking $m_c = 1.5\,\text{GeV}$, $\mu_R = 2m_c$ and $\mu_\Lambda = m_c$. The first uncertainty is estimated by varying $m_c \in [1.3, 1.7]\,\text{GeV}$, the second uncertainty is caused by varying $\mu_R \in [2m_c, \sqrt{s}/2]$ and the third uncertainty is obtained by varying $\mu_\Lambda \in [1\,\text{GeV}, m_c]$. Figure 3 shows that this prediction can both overlap with the BaBar and Belle measurements within errors. The following work of this process is focus on improving the μ_R dependence.

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