Superpotentials of D-branes in Calabi-Yau manifolds with several moduli by mirror symmetry and blown-up

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ABSTRACT: We study B-brane superpotentials depending on several closed- and open-moduli on Calabi-Yau hypersurfaces and complete intersections. By blowing up the ambient space along a curve wrapped by B-branes in a Calabi-Yau manifold, we obtain a blow-up new manifold and the period integral satisfying the GKZ-system. Via mirror symmetry to A-model, we calculate the superpotentials and extract Ooguri-Vafa invariants for concrete examples of several open-closed moduli in Calabi-Yau manifolds.

KEYWORDS: D-Branes, Flux Compactifications, Topological Strings

ARXIV EPRINT: [2201.00322](https://arxiv.org/abs/2201.00322)

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1 Introduction

The type IIB compactification with branes can be described by an effective $N = 1$ supergravity theory with a non-trivial superpotential on the open-closed moduli space, because the D-branes wrapping supersymmetric cycles reduce the $N=2$ supersymmetry to $N=1$ supersymmetry. In topological string theory, there are two type D-branes. A-branes wrap special lagrangian cycles and B-branes wrap holomorphic cycles that can be even dimensional in a Calabi-Yau threefold.

On the A-model side, the superpotential is related to the topological string amplitudes, which counts disk instantons [1, 2]. On the B-model side, the topological string is related to holomorphic Chern-Simons theory [3]. The B-brane superpotential is given by a integral over 3-chain with boundary consisting of 2-cycles γ around the B-branes, which is the sections of a line bundle over the open-closed moduli space described by the holomorphic $N=1$ special geometry [4–6]. The B-brane superpotential can be expressed as a linear combination of the integral of the basis of relative period. When considering B-brane wrapping two curves within the same homology class, the superpotential changes on the two sides of the domain wall whose tension is in terms of the Able-Jacobi map.

In physics, the D-brane superpotential is topological sector of the $d=4$, $N=1$ supersymmetric spacetime effective Lagrangian, as well as the generating function of open string topological field theory correlators. It encode the instanton correction and its derivatives determine structure constants of 2d chiral ring and Gauss-Manin connection of the vacuum bundle on the moduli space. The flatness of this connection determines the Picard-Fuchs equations satisfied by period vector. The expansion of superpotentials at large volume phase underlies the Ooguri-Vafa invariants counting the holomorphic disks ending on a lagrangian submanifold on the A-model side. These invariants closely relate to space of the states, non-pertubative effects, and geometric properties of the moduli space.

From the perspective of the deformation theory, the deformations of a curve in Calabi-Yau threefold are given by the sections of the normal bundle. The holomorphic sections lead to massless or light fields in the effective theory and the non-holomorphic sections lead to massive fields whose masses are given by the volume change under infinitesimal deformations. The superpotentials are related to the deformations with masses vanishing at some point in the closed moduli space. In order words, determining the B-brane superpotentials

is equivalent to solve the deformation theory of a pair (X, S) with curve S and Calabi-Yau manifold X . When a D5 brane wrapping a rational curve, the non-trivial superpotential is defined on a family of curves S , whose members are in general non-holomorphic except at some critical points where S is holomorphic curve. The critical locus corresponds to the supersymmetric vacua, other points in moduli space correspond to the obstructed deformation of the rational curve and excitation about the supersymmetric minimum.

The computation of off-shell superpotential for a toric brane, has been presented by local case in [4, 5, 7], and extended to compact Calabi-Yaus in [8–10]. For this brane, the onshell superpotentials and flat coordinates are the solutions to a system of open-closed Picard-Fuchs equations, which arise as a consequence of the $N = 1$ special geometry. These equation can be obtained by Griffiths-Dwork reduction method or GKZ system. When a B-brane wrapping a curve S in X , the blowing up X along S lead to a new manifold X' with an exceptional divisor E [11, 12]. Meanwhile, the deformation theory of (X, S) is equivalent to that of (X', E) .¹ The B-brane superpotential on the Calabi-Yau threefold X can be calculated in terms of period vector of manifold X' . In this note, we first calculate B-brane superpotentials in Calabi-Yau manifolds with several moduli via blowing up method, then extract Ooguri-Vafa invariants from A-model side at large volume phase by mirror symmetry.

The organization of this paper is as follows. In section 2, we introduce the background and formalism. To begin with, we review D-brane superpotential in the Type II string theory and relative cohomology description, recall the basic toric geometry about constructing Calabi-Yau manifold, generalized GKZ system and its local solutions, and outline the procedure to blow up a curve on a Calabi-Yau manifold. In section 3–5, for degree -9 , -8 , -12 Calabi-Yau hypersurface and degree- $(3, 3)$ complete intersection Calabi-Yau manifold, we apply the blow up method to the manifold with a curve on it and obtain a new Kähler manifold with an exceptional divisor. The Picard-Fuchs equations and their solutions are derived by GKZ hypergeometric system from toric data of the enhanced polyhedron. The superpotential are identified as double-logarithmic solutions of the Picard-Fuchs equations and Ooguri-Vafa invariants are extracted at large volume phase. The last section is a brief summary and further discussions. In appendix A, we summarize the GKZ-system for two complete intersections Calabi-Yau manifolds $X_{[4,4]}^{(112|112)}$ and $X_{[6,6]}^{(123|123)}$. In appendix B, we present the compact instanton invariants of above models for first several orders.

2 Toric geometry and blowing up

2.1 Toric geometry and GKZ system

A Calabi-Yau manifold can be defined as a hypersurface or complete intersection of several hypersurfaces in ambient toric variety. We refer to [13] for the background of toric geometry, construction of Calabi-Yau manifolds [14, 15], and GKZ system [16–20].

For a mirror pair of compact hypersurfaces (X^*, X) , one may associate a pair of integral polyhedra (Δ^*, Δ) in a four-dimensional integral lattice and its dual. The n integral points

¹Section 4.4 in [12] and footnote 17 in [11].

of the polyhedron correspond to homogeneous coordinates x_i on the toric ambient space and satisfy linear relations

$$\sum_i l_i^j v_i = 0, \quad a = 1, \dots, h^{2,1}$$

where l_i^j is the i th component of the charge vector l^j . The integral entries of the vectors l^j define the weights l_i^j of the coordinates x_i under the $\mathbb{C}^*$ action

$$x_i \rightarrow (\lambda_j)^{l_i^j} x_i, \quad \lambda_j \in \mathbb{C}^*$$

and l_i^j 's are the $U(1)$ charges of the fields in the gauged linear sigma model (GLSM) [21]. In above description, the mirror Calabi-Yau threefold is determined as a hypersurface in the dual toric ambient space with constraints

$$P = \sum_i a_i y_i, \quad \prod_i y_i^{l_i^j} = z_j \quad (2.1)$$

Here z_j denotes the complex structure moduli of X . In terms of vertices $v_j^* \in \Delta^*, v_i \in \Delta$

$$P(x; a) = \sum_{v_j^* \in \Delta^*} a_j \prod_i x_i^{\langle v_j^*, v_i \rangle + 1} \quad (2.2)$$

The complete intersection Calabi-Yau threefolds can be constructed similarly and we omit the details. Equivalently, Calabi-Yau hypersurfaces is also given by the zero loci of certain sections of the anticanonical bundle. The toric variety contains a canonical Zariski open torus $\mathbb{C}^4$ with coordinates $X = (X_1, X_2, X_3, X_4)$. The sections are

$$P_{\Delta^*}(X, a) = \sum_{v_i^* \in \Delta^* \cap \mathbb{Z}^5} a_i X^{v_i^*} \quad (2.3)$$

After homogenization, above equation is the same as (2.2). On these manifolds, a mirror pairs of branes, defined in [7] by another N charge vectors $\hat{l}^j$. The special Lagrangian submanifold wrapped by the A-brane is described in terms of the vectors $\hat{l}^j$ satisfying

$$\sum_i \hat{l}_i^j |x_i|^2 = c_j, \quad \sum_i \hat{l}_i^j = 0$$

where c_j parametrize the brane position. And the holomorphic submanifold wrapped by mirror B-brane is defined by the following equation

$$\prod_i y_i - \hat{z}_a = 0, \quad \hat{z}_a = \epsilon_a e^{-c_a}$$

The $N = 2$ case, a toric curve, is more interesting to us and it is the geometry setting we are studying in this notes. To handle the toric curve case, we consider the enhanced polyhedron method proposed in [8]. It is possible in a simple manner to construct the enhanced polyhedron from the original polyhedra and the toric curve specified by two charge vectors. We denote the vertices of Δ by v_i , $i = 1, \dots, n$, with v_0 the origin, its

charge vectors by l^i , and two brane vectors by $\hat{l}^1$ and $\hat{l}^2$. We add 4 points to Δ^* to define a new polyhedron Δ' with vertices

$$\begin{aligned} X : \quad & v_i = (1, 0, v_i, 0), & i = 0, \dots, n \\ \hat{l}^1 : \quad & v_{n+1} = (0, 1, v_1^-, -1), & v_{n+2} = (0, 1, v_1^+, -1) \\ \hat{l}^2 : \quad & v_{n+3} = (0, 1, v_2^-, -1), & v_{n+4} = (0, 1, v_2^+, -1) \end{aligned} \quad (2.4)$$

where we use the abbreviation

$$\begin{aligned} v_1^+ &= \sum_{\hat{l}_i^1 > 0} \hat{l}_i^1 v_i, & v_1^- &= - \sum_{\hat{l}_i^1 < 0} \hat{l}_i^1 v_i \\ v_2^+ &= \sum_{\hat{l}_i^2 > 0} \hat{l}_i^2 v_i, & v_2^- &= - \sum_{\hat{l}_i^2 < 0} \hat{l}_i^2 v_i \end{aligned} \quad (2.5)$$

The first line of (2.4) simply embeds the original toric data associated to Δ into Δ' , whereas the second and third line translate the brane data into geometric data of Δ' .

Given the toric data, the GKZ-system on the complex structure moduli space of X is given by the standard formula

$$\begin{aligned} \mathcal{L}_i &= \prod_{l_j^i > 0} \left(\frac{\partial}{\partial a_j} \right)^{l_j^i} - \prod_{l_j^i < 0} \left(\frac{\partial}{\partial a_j} \right)^{-l_j^i}, & i = 1, \dots, h^{2,1} + 2 \\ \mathcal{Z}_i &= \sum_j (\bar{v}_j)^i \vartheta_j - \beta_i, & j = 0, \dots, 6 \end{aligned} \quad (2.6)$$

Here $\beta = (-1, 0, 0, 0, 0)$ is the so-called exponent of GKZ system, $\vartheta_j = a_j \frac{\partial}{\partial a_j}$ are the logarithmic derivative and $\bar{v}_j = (1, v_j)$. The operators $\mathcal{L}_i$'s express the trivial algebraic relations among monomials entering hypersurface constraints, $\mathcal{Z}_0$ expresses the infinitesimal generators of overall rescaling, $\mathcal{Z}_i, i \neq 0$'s are the infinitesimal generators of rescalings of coordinates x_j . All GKZ operators can annihilate the period matrix, thus determine the mirror maps and superpotentials.

This immediately yields a natural choice of complex coordinates given by

$$z^j = (-)^{l_0^j} \prod_i a_i^{l_i^j} \quad (2.7)$$

And from the operators $\mathcal{L}_i$, it is easy to obtain a complete set of Picard-Fuchs operators $\mathcal{D}_i$. Using monodromy information and knowledge of the classical terms, their solution can be associated to integrals over an integral basis of cycles in $H^3(X, \mathbb{Z})$ and given the flux quanta explicit superpotentials can be written down.

For appropriate choice of basis vector l^j , solutions to the GKZ system can be written in terms of the generating functions in these variables

$$\varpi(z; \rho) = \sum \frac{\Gamma(1 - \sum_j l_0^j (n_j + \rho_j))}{\prod_{i>0} \Gamma(1 + \sum_j l_i^j (n_j + \rho_j))} \prod_k z_k^{n_k + \rho_k}$$

then we have a natural basis for the period vector

$$\begin{aligned}\omega_0(z) &= \varpi(z; \rho)|_{\rho \rightarrow 0}, \\ \omega_{1,i}(z) &= \partial_{\rho_i} \varpi(z; \rho)|_{\rho \rightarrow 0}, \\ \omega_{2,i}(z) &= \sum_{j,k} K_{ijk} \partial_{\rho_j} \partial_{\rho_k} \varpi(z; \rho)|_{\rho \rightarrow 0} \\ &\dots\end{aligned}$$

For a maximal triangulation corresponding to a large complex structure point centered at $z = 0$, $\forall a$, $\omega_0(z) = 1 + \mathcal{O}(z)$ and $\omega_{1,i}(z) \sim \log(z_i)$ that define the open-closed mirror maps

$$t_i(z) = \frac{\omega_{1,i}(z)}{\omega_0(z)} = \frac{1}{2\pi i} \log(z_i) + S(z), \quad q_i = e^{2\pi i t_i} \quad (2.8)$$

where $S(z)$ is a series in the coordinates z . In addition, the special solution $\Pi = \mathcal{W}_{\text{open}}(z)$ has further property that its instanton expansion near a large volume/large complex structure point encodes the Ooguri-Vafa invariants of the brane geometry.

$$\mathcal{W}_{\text{inst}}(q) = \sum_{\beta} G_{\beta} q^{\beta} = \sum_{\beta} \sum_{k=1}^{\infty} N_{\beta} \frac{q^{k \cdot \beta}}{k^2} \quad (2.9)$$

2.2 Blowing up and Hodge structure

Blowing up in algebraic geometry is an important tool in this work. Now, we review the construction and properties of blowing up a manifold along its submanifold. Given S be a curve in a Calabi-Yau threefold $X \in Z$ with Z an ambient toric variety, we can blow up along S to obtain a new manifold.

According to section 2.2 (d) in [22], for this case that $X \subset Z$ is a closed irreducible non-singular subvariety of Z and X is transversal to S at every point $S \cap X$, $\pi : Z' \rightarrow Z$ be the blowup of S . Then the subvariety $\pi^{-1}(X)$ consist of two irreducible components,

$$\pi^{-1}(X) = \pi^{-1}(S \cap X) \cup X'$$

and $\pi : X' \rightarrow X$ defines the blow-up of X with center in $S \cap X = S$, i.e. X' is the manifold obtained from blowing up X along S . The subvariety $X' \subset Z'$ is called the *birational transform* of $X \subset Z$ under the blowup.

First, by the local construction, we consider an three dimensional multidisk in X , Δ with holomorphic coordinates $x_i, i = 1, 2, 3$, and V is specified by $x_1 = x_2 = 0$ on each Δ . Then we define the smooth variety

$$\tilde{\Delta} \subset \Delta \times \mathbb{P}^1$$

as follows

$$\tilde{\Delta} = \{(x_1, x_2, x_3, (y_1 : y_2)) \in \Delta \times \mathbb{P}^1 : x_2 y_1 - x_1 y_2 = 0\}$$

Here y_1, y_2 are the homogeneous coordinates on $\mathbb{P}^1$. The projection map $\pi : \tilde{\Delta} \rightarrow \Delta$ on the first factor is clearly an isomorphism away from V , while the inverse image of a point

$z \in V$ is a projective space $\mathbb{P}^1$. The manifold $\tilde{\Delta}$, together with the projection map π is the blow-up of Δ along V ; the inverse image $E = \pi^{-1}(V)$ is an exceptional divisor of the blow-up. For two coordinates patches $U_i = (y_i \neq 0), i = 1, 2$, they have holomorphic coordinates respectively

$$\begin{aligned} z_1^{(1)} &= x_1, & z_2^{(1)} &= \frac{y_2}{y_1} = \frac{x_2}{x_1}, & y_3 &= x_3 \\ z_1^{(2)} &= \frac{y_1}{y_2} = \frac{x_1}{x_2}, & z_2^{(1)} &= x_2, & y_3 &= x_3 \end{aligned}$$

with transition function on $U_1 \cap U_2$ given by $g_{ij} = z_i^{(j)} = \frac{y_i}{y_j} = \frac{x_i}{x_j}$. Next we consider the global construction of the blow-up manifold. Let X be a complex manifold of dimension three and $S \subset X$ be a curve. Let $\{U_\alpha\}$ be a collection of disks in X covering S such that in each disk Δ_α the subvariety $S \cap \Delta_\alpha$ may be given as the locus $(x_1 = x_2 = 0)$, and let $\pi_\alpha : \tilde{\Delta}_\alpha \rightarrow \Delta_\alpha$ be the blow-up of Δ_α along $S \cap \Delta_\alpha$. We then have

$$\pi_{\alpha\beta} : \pi_{\alpha\beta}^{-1}(U_\alpha \cap U_\beta) \rightarrow \pi_\beta^{-1}(U_\alpha \cap U_\beta)$$

and using them, we can patch together the local blow-ups $\tilde{\Delta}_\alpha$ to form a manifold

$$\tilde{\Delta} = \cup_{\pi_{\alpha\beta}} \tilde{\Delta}_\alpha$$

Finally, since π is an isomorphism away from $X \cap (\cup \Delta_\alpha)$, we can take

$$X' = \tilde{\Delta} \cup_\pi X - S$$

X' , together with the projection map $\pi : X' \rightarrow X$ extending π on $\tilde{\Delta}$ and the identity on $X - S$, is called the blow-up of X along S , and the inverse image $\pi^{-1}(S)$ is an exceptional divisor.

From the excision theorem of cohomology in algebraic topology [23],

$$H^3(X, S) \cong H^3(X - S) \cong H^3(X' - E) \cong H^3(X', E) \quad (2.10)$$

which means that the variation of the mixed Hodge structures of $H^3(X, S)$ and $H^3(X', E)$ over the corresponding moduli space are equivalent. The mixed Hodge structure as follow,

$$\phi : H^3(X' - E) \xrightarrow{\sim} \oplus_{p+q=3} H^q(X', \Omega') \quad (2.11)$$

where Ω' denotes the holomorphic p-forms on X' . The filtrations have the form

$$F^m H^3 = \oplus_{p \leq m} H^{3-p}(X', \Omega')$$

and

$$W_{-1} H^3 = 0, \quad W_0 H^3 = H^3(X'), \quad W_1 H^3 = H^3(X' - E)$$

Additionally, the mixed Hodge structure has graded weights

$$Gr_k^W H^3 = Gr_{-k+3}^W H^3 / Gr_{-(k+1)+3}^W H^3$$

that take the following form for the divisor E

$$\begin{aligned} Gr_3^W H^3 &= Gr_0^W H^3 / Gr_{-1}^W H^3 \cong H^3(X'), \\ Gr_2^W H^3 &= Gr_1^W H^3 / Gr_0^W H^3 \cong H^2(E) \end{aligned}$$

The reason to consider these (graded) weights is the following: the mixed Hodge structure is defined such that the Hodge filtration $F^m H^3$ induces a pure Hodge structure on each graded weight, i.e. on $Gr_2^W H^3$ and $Gr_3^W H^3$. Thus, the following two induced filtrations on $Gr_3^W H^3$

$$H^3(X') \cap F^3 H^3 \subset H^3(X') \cap F^2 H^3 \subset H^3(X') \cap F^1 H^3 \subset H^3(X') \cap F^0 H^3 = H^3(X') \quad (2.12)$$

and on $Gr_2^W H^3$

$$H^2(E) \cap F^2 H^3 \subset H^2(E) \cap F^1 H^3 \subset H^2(E) \cap F^0 H^3 = H^2(E) \quad (2.13)$$

lead to pure Hodge structures on $H^3(X')$ and $H^2(E)$. $H^3(X' - E)$ forms a bundle $\mathcal{H}^3$ over the open-closed moduli space $\mathcal{M}$ with the Gauss-Manin connection ∇ satisfying the Griffiths transversality condition

$$\nabla \mathcal{F}^p \in \mathcal{F}^{p-1} \otimes \Omega_{\mathcal{M}}^1 \quad (2.14)$$

The flatness of the Gauss-Manin connection leads to N=1 special geometry and a Picard-Fuchs system of differential equations that govern the mirror maps and superpotentials.

The geometric setting we are interested in is a hypersurface $X : P = 0$ with a curve S on it, $S : P = 0, h_1 = h_2 = 0$. After blowing up along S , the blow-up manifold X' is given globally as the complete intersection in the total space of the projective bundle $\mathcal{W} = \mathbb{P}(\mathcal{O}(D_1) \oplus \mathcal{O}(D_2))$,

$$P = 0, \quad Q \equiv y_1 h_2 - y_2 h_1 = 0 \quad (2.15)$$

where $(y_1, y_2) \sim \lambda(y_1, y_2)$ is the projective coordinates on the $\mathbb{P}^1$ -fiber of the blow-up X' . We have to emphasize that X' is not Calabi-Yau since the first Chern class is nonzero. In addition, the blow-up procedure do not introduce new degrees of freedom associated to deformations of E . Under blowing up map, the open-closed moduli space of (X, S) is mapped into the complex structure deformation of X' . This enable us to calculate the superpotential W_{brane} for B- branes wrapping rational curves via the periods on the complex structure moduli space of X' determined by Picard-Fuchs equations.

3 Two closed and two open moduli case

3.1 Open-closed GKZ-system: branes on $X_9^{(1,1,1,3,3)}$

3.1.1 Five branes wrapping lines

The Calabi-Yau threefold $X_9^{(1,1,1,3,3)}$ is defined as the mirror of the Calabi-Yau hypersurface X^* in $\mathbb{P}_{(1,1,1,3,3)}^4$ with $h^{2,1} = 2$ complex structure moduli and the charge vectors of the GLSM for the A model manifold are given by:

	0	1	2	3	4	5	6
l^1	-3	0	0	0	1	1	1
l^2	0	1	1	1	0	0	-3

The hypersurface constraint for the mirror manifold, written in homogeneous coordinates of $\mathbb{P}_{(1,1,1,3,3)}$, is

$$P = x_1^9 + x_2^9 + x_3^9 + x_4^3 + x_5^3 + \psi(x_1x_2x_3x_4x_5) + \phi(x_1x_2x_3)^3$$

where $\psi = z_1^{-\frac{1}{3}}z_2^{-\frac{1}{9}}$ and $\phi = z_2^{-\frac{1}{3}}$. The Greene-Plesser orbifold group G acts as $x_i \rightarrow \lambda_k^{g_{k,i}} x_i$ with $\lambda_1^9 = \lambda_2^9 = 1$, $\lambda_3^3 = 1$ and weights

$$\mathbb{Z}_9 : g_1 = (1, -1, 0, 0, 0), \quad \mathbb{Z}_9 : g_2 = (1, 0, -1, 0, 0), \quad \mathbb{Z}_3 : g_3 = (0, 0, 0, 1, -1)$$

Next, we add a five-brane wrapping a rational curve on a toric curve S

$$\begin{aligned} S : \quad P &= 0, & h_1 &\equiv \alpha^3\gamma^3x_2^9 - \beta^6(x_1x_2x_3)^3 = 0, & h_2 &\equiv \beta^3\gamma^3x_3^9 - \alpha^6(x_1x_2x_3)^3 = 0 \\ \hat{l}^1 &= (0, 0, 1, 0, 0, 0, -1), & \hat{l}^2 &= (0, 0, 0, 1, 0, 0, -1) \end{aligned} \quad (3.1)$$

An equivalent and convenient form is obtained after simple algebraic manipulations

$$S : P = 0, \quad \alpha^9x_2^9 - \beta^9x_3^9 = 0, \quad \alpha^9x_1^9 - \gamma^9x_3^9 = 0 \quad (3.2)$$

For generic values of the moduli in (3.2), S is an irreducible high genus Riemann surface. But we can make a linearization by following steps: to begin with, we inserted h_1 and h_2 into P ,

$$\tilde{\mathbb{P}}^1 : \quad \eta_1x_4 + \sqrt[3]{x_5^3 + m(x_3, x_4, x_5)} = 0, \quad \eta_2\alpha x_2 - \beta x_3 = 0, \quad \eta_3\alpha x_1 - \gamma x_3 = 0 \quad (3.3)$$

Here $\eta_1^3 = \eta_2^9 = \eta_3^9 = 1$ and

$$m(x_3, x_4, x_5) = \left(\frac{\alpha^9 + \beta^9 + \gamma^9}{\alpha^9} + \frac{\alpha^3\beta^3\gamma^3}{\alpha^9}\phi \right) x_3^9 + \frac{\alpha\beta\gamma}{\alpha^3}\psi x_3^3x_4x_5$$

Due to the non-trivial branching of the roots of unity, (3.3) is non-holomorphic, i.e. it is a non-holomorphic family of rational curves on X .

For special loci of $\mathcal{M}(S)$ where $m(x_1, x_2, x_5)$ vanishes identically,

$$\mathcal{M}_{\mathbb{P}^1}(S) : \quad \alpha^9 + \beta^9 + \gamma^9 + \phi\alpha^3\beta^3\gamma^3 = 0, \quad \psi\alpha\beta\gamma = 0 \quad (3.4)$$

the Riemann surface S in (3.3) degenerates to

$$S : \quad h_0 \equiv x_4^3 + x_5^3, \quad h_1 = \alpha^9x_2^9 - \beta^9x_3^9, \quad h_2 = \alpha^9x_1^9 - \gamma^9x_3^9 \quad (3.5)$$

Under the action of $G = \mathbb{Z}_9^2 \times \mathbb{Z}_3$, (3.5) describes a single line,

$$\mathbb{P}^1 : \quad \eta_1x_4 + x_5 = 0, \quad \alpha x_2 - \beta x_3 = 0, \quad \alpha x_1 - \gamma x_3 = 0 \quad (3.6)$$

In other words, these lines in $\mathbb{P}^4$ have a parametrization by homogeneous coordinates U, V on $\mathbb{P}^1$ as the Veronese mapping

$$\begin{aligned} \mathcal{M}_{\mathbb{P}^1}(S) &\hookrightarrow \mathcal{M}(S) \\ (U, V) &\mapsto (\gamma U, \beta U, \alpha U, -\eta_1 V, V), \quad \eta_1^3 = 1 \end{aligned}$$

	Δ^*				l^1	l^2		$\hat{l}^1$	$\hat{l}^2$
v_0^*	0	0	0	0	-3	0	$x_1x_2x_3x_4x_5$	0	0
v_1^*	-1	-1	-3	-3	0	1	x_1^9	0	0
v_2^*	1	0	0	0	0	1	x_2^9	1	0
v_3^*	0	1	0	0	0	1	x_3^9	0	1
v_4^*	0	0	1	0	1	0	x_4^3	0	0
v_5^*	0	0	0	1	1	0	x_5^3	0	0
v_6^*	0	0	-1	-1	1	-3	$(x_1x_2x_3)^3$	-1	-1

Table 1. Toric Data of A-model side.

Thus all obstructed deformations locate at $\mathcal{M}(S) - \mathcal{M}_{\mathbb{P}^1}(S)$, inducing a non-trivial superpotential, which plays an important role in research on obstruction deformation, especially for a manifold with a submanifold on it. As we know, blowing up is very effective method to handle such case. According to (2.15), we construct the blow-up manifold X' given by the complete intersection in projective bundle

$$X' : P = 0, \quad Q = y_1h_2 - y_2h_1$$

It is obvious from above defining equations that the moduli of S described by the coefficients of the monomials in $h_i, i = 1, 2$ turn into complex structure moduli of X' . We obtain the embedding of the obstructed deformation space of $(X, \tilde{\mathbb{P}}^1)$ into the complex structure moduli space of X' , which is crucial for the following superpotential calculations.

3.1.2 Toric branes and blowing up geometry

Now, we study the A-model manifold, whose toric polyhedron is denoted by Δ^* and charge vectors are denoted by l^1 and l^2 . The integral vertices of polyhedron Δ^* and the charge vectors l^1, l^2 for A-model manifold, $\hat{l}^1, \hat{l}^2$ for A-branes is as table 1.

From above toric data of Δ^* and its dual polyhedron Δ ,

$$\begin{aligned} v_1 &= (-1, -1, -1, -1), & v_2 &= (8, -1, -1, -1), & v_3 &= (-1, 8, -1, -1), \\ v_4 &= (-1, -1, 2, -1), & v_5 &= (-1, -1, -1, 2) \end{aligned}$$

In B-model, the defining equations of the mirror manifold X and the curve S as follow in torus coordinates

$$\begin{aligned} X : \quad P &= a_0 + a_1(X_2X_3X_4^3X_5^3)^{-1} + a_2X_2 + a_3X_3 + a_4X_4 + a_5X_5 + a_6(X_4X_5)^{-1} \\ S : \quad h_1 &= a_7X_2 + a_8(X_4X_5)^{-1}, \quad h_2 = a_9X_3 + a_{10}(X_4X_5)^{-1} \end{aligned} \quad (3.7)$$

where a_i 's are free complex-valued coefficients. With the abbreviation of logarithmic deriva-

tives $\vartheta_i = a_i \frac{\partial}{\partial a_i}$, the GKZ-system of X by (2.7) is,

$$\begin{aligned} \mathcal{Z}_0 &= \sum_{i=0}^6 \vartheta_i + 1, & \mathcal{Z}_i &= -\vartheta_1 + \vartheta_{i+1}, & i &= 1, 2, \\ \mathcal{Z}_i &= -3\vartheta_1 + \vartheta_{i+1} - \vartheta_6, & & & i &= 3, 4, \end{aligned} \quad (3.8)$$

$$\mathcal{L}_1 = \prod_{i=4}^6 \frac{\partial}{\partial a_i} - \left(\frac{\partial}{\partial a_0} \right)^3, \quad \mathcal{L}_2 = \prod_{i=1}^3 \frac{\partial}{\partial a_i} - \left(\frac{\partial}{\partial a_6} \right)^3$$

Here $\mathcal{Z}_0$ represent the invariance of P under overall rescaling and other $\mathcal{Z}_i$'s relate to the invariance of P under the rescaling of torus coordinates X_i 's combined with the rescaling of coefficients a_i 's.

$$\begin{aligned} \mathcal{Z}_i : \quad X_{i+1} &\mapsto \lambda X_{i+1}, & (a_1, a_{i+1}) &\mapsto (\lambda a_1, \lambda^{-1} a_{i+1}), & i &= 1, 2 \\ \mathcal{Z}_i : \quad X_{i+1} &\mapsto \lambda X_{i+1}, & (a_1, a_{i+1}, a_6) &\mapsto (\lambda^3 a_1, \lambda^{-1} a_{i+1}, \lambda a_6), & i &= 3, 4 \end{aligned}$$

Operators $\mathcal{L}_i$'s relate to the symmetries among the Laurent monomials in P (3.7),

$$\begin{aligned} \mathcal{L}_1 : \quad & X_4 X_5 (X_4 X_5)^{-1} = 1 \\ \mathcal{L}_2 : \quad & (X_2^{-1} X_3^{-1} X_4^{-3} X_5^{-3}) X_2 X_3 = ((X_4 X_5)^{-1})^3 \end{aligned}$$

By blowing up X along S , the blow-up manifold X' is obtained.

$$X' : P = 0, \quad Q = y_1(a_9 X_3 + a_{10}(X_4 X_5)^{-1}) - y_2(a_7 X_2 + a_8(X_4 X_5)^{-1}) \quad (3.9)$$

After careful observation on the torus symmetry of X' , we can obtain the infinitesimal generators which are belong to GKZ system associated to X' ,

$$\begin{aligned} \mathcal{Z}'_0 &= \sum_{i=0}^6 \vartheta_i + 1, \quad \mathcal{Z}'_1 = \sum_{i=7}^{10} \vartheta_i, \quad \mathcal{Z}'_2 = -\vartheta_1 + \vartheta_2 + \vartheta_7, \quad \mathcal{Z}'_3 = -\vartheta_1 + \vartheta_3 + \vartheta_9, \\ \mathcal{Z}'_i &= -3\vartheta_1 + \vartheta_i - \vartheta_6 - \vartheta_8 - \vartheta_{10} \quad i = 4, 5, \quad \mathcal{Z}'_6 = -\vartheta_7 - \vartheta_8 + \vartheta_9 + \vartheta_{10} \\ \mathcal{L}'_1 &= \prod_{i=4}^6 \frac{\partial}{\partial a_i} - \left(\frac{\partial}{\partial a_0} \right)^3, \quad \mathcal{L}'_2 = \prod_{i=1}^3 \frac{\partial}{\partial a_i} - \left(\frac{\partial}{\partial a_6} \right)^3 \\ \mathcal{L}'_3 &= \frac{\partial}{\partial a_6} \frac{\partial}{\partial a_7} - \frac{\partial}{\partial a_2} \frac{\partial}{\partial a_8}, \quad \mathcal{L}'_4 = \frac{\partial}{\partial a_3} \frac{\partial}{\partial a_{10}} - \frac{\partial}{\partial a_6} \frac{\partial}{\partial a_9} \end{aligned} \quad (3.10)$$

Here $\mathcal{Z}_0, \mathcal{Z}_1$ are associated with the overall rescaling with respect to $P = 0, Q = 0$ respectively. $\mathcal{Z}_i, i = 2, \dots, 5$ are related to the torus symmetry as before.

$$\begin{aligned} \mathcal{Z}'_2 : \quad X_2 &\mapsto \lambda X_2, & (a_1, a_2, a_7) &\mapsto (\lambda a_1, \lambda^{-1} a_2, \lambda^{-1} a_7) \\ \mathcal{Z}'_3 : \quad X_3 &\mapsto \lambda X_3, & (a_1, a_3, a_9) &\mapsto (\lambda a_1, \lambda^{-1} a_3, \lambda^{-1} a_9) \\ \mathcal{Z}'_4 : \quad X_4 &\mapsto \lambda X_4, & (a_1, a_4, a_6, a_8, a_{10}) &\mapsto (\lambda^3 a_1, \lambda^{-1} a_4, \lambda a_6, \lambda a_8, \lambda a_{10}) \\ \mathcal{Z}'_5 : \quad X_5 &\mapsto \lambda X_5, & (a_1, a_5, a_6, a_8, a_{10}) &\mapsto (\lambda^3 a_1, \lambda^{-1} a_5, \lambda a_6, \lambda a_8, \lambda a_{10}) \end{aligned}$$

In addition, $\mathcal{Z}'_6$ is related to the torus symmetry $(y_1, y_2) \mapsto (\lambda y_1, \lambda^{-1} y_2)$. The new $\mathcal{L}_3, \mathcal{L}_4$ incorporate the parameter $a_7, \dots, a_{10}$ that are associated with the open-closed moduli of

the curve S . All these GKZ operators annihilate the holomorphic three form Ω' on X' that is the pull back of the homomorphic three form Ω on X , i.e. $\Omega' = \pi^*\Omega$.

Now, we formulate the GKZ-system (3.10) on an enhanced polyhedron Δ' , by adding additional vertices on the original polyhedron Δ^* .

	Δ'							l'_1	l'_2	l'_3	l'_4	
v'_0	1	0	0	0	0	0	0	-3	0	0	0	$w'_0 = x_1 x_2 x_3 x_4 x_5$
v'_1	1	0	-1	-1	-3	-3	0	0	1	0	0	$w'_1 = x_1^9$
v'_2	1	0	1	0	0	0	0	1	1	-1	0	$w'_2 = x_2^9$
v'_3	1	0	0	1	0	0	0	0	0	0	1	$w'_3 = x_3^9$
v'_4	1	0	0	0	1	0	0	1	0	0	0	$w'_4 = x_4^3$
v'_5	1	0	0	0	0	1	0	1	0	0	0	$w'_5 = x_5^3$
v'_6	1	0	0	0	-1	-1	0	0	-2	1	-1	$w'_6 = (x_1 x_2 x_3)^3$
v'_7	0	1	1	0	0	0	-1	-1	0	1	0	$w'_7 = y_1 w'_2$
v'_8	0	1	0	0	-1	-1	-1	1	0	-1	0	$w'_8 = y_1 w'_6$
v'_9	0	1	0	1	0	0	1	0	1	0	-1	$w'_9 = y_2 w'_3$
v'_{10}	0	1	0	0	-1	-1	1	0	-1	0	1	$w'_{10} = y_2 w'_6$

(3.11)

where v'_i 's are the integral vertices of Δ' and their corresponding monomials in homogeneous coordinates of $\mathbb{P}^4$ are w'_i . The A-model closed string charge vectors and A-branes charge vectors relate to the maximal triangulation of Δ' and satisfy the relations,

$$l^1 = l'_1 + l'_3, l^2 = l'_2 + l'_4, \hat{l}^1 = l'_3, \hat{l}^2 = l'_4$$

The coordinates z_j by (2.7) on the complex structure moduli space of X' .

$$z_1 = \frac{a_2 a_4 a_5 a_8}{a_0^3 a_7}, \quad z_2 = \frac{a_1 a_2 a_9}{a_6^2 a_{10}}, \quad z_3 = \frac{a_6 a_7}{a_2 a_8}, \quad z_4 = \frac{a_3 a_{10}}{a_6 a_9} \quad (3.12)$$

Next, we convert the $\mathcal{L}_i$ operators to Picard-Fuchs operators $\mathcal{D}_i$, from differential equations about $a_j (j = 0, \dots, 10)$ to those about $z_j (j = 1, \dots, 4)$ of X^b . From table (3.11), we obtain the identity

$$\begin{aligned} \vartheta_0 &= -3\theta_1, & \vartheta_1 &= \theta_2, & \vartheta_2 &= \theta_1 + \theta_2 - \theta_3, & \vartheta_3 &= \theta_4, \\ \vartheta_4 &= \theta_1, & \vartheta_5 &= \theta_1, & \vartheta_6 &= -2\theta_2 + \theta_3 - \theta_4, \\ \vartheta_7 &= -\theta_1 + \theta_3, & \vartheta_8 &= \theta_1 - \theta_3, & \vartheta_9 &= \theta_2 - \theta_4, & \vartheta_{10} &= \theta_2 + \theta_{10} \end{aligned}$$

Inserting above relations between the logarithmic derivatives ϑ_j w.r.t. a_j and the logarithmic derivatives θ_j w.r.t. z_j into $\mathcal{L}$ operators in (3.10), the full set of Picard-Fuchs operators are obtained

$$\begin{aligned} \mathcal{D}_1 &= (\theta_1 + \theta_2 - \theta_3)\theta_1^2(\theta_1 - \theta_3) - z_1(-\theta_1 + \theta_3) \prod_{i=1}^3 (-3\theta_1 - i), \\ \mathcal{D}_2 &= \theta_2(\theta_1 + \theta_2 - \theta_3)(\theta_2 - \theta_4) - z_2(-2\theta_1 + \theta_3 - \theta_4)(-\theta_2 + \theta_4) \\ \mathcal{D}_3 &= (-2\theta_1 + \theta_3 - \theta_4)(-\theta_1 + \theta_3) - z_3(\theta_1 + \theta_2 - \theta_3)(\theta_1 - \theta_3), \\ \mathcal{D}_4 &= \theta_4(-\theta_2 + \theta_4) - z_4(-2\theta_1 + \theta_3 - \theta_4)(\theta_2 - \theta_4) \\ &\dots \end{aligned} \quad (3.13)$$

where $\theta_i = z_i \frac{\partial}{\partial z_i}$'s are the logarithmic derivatives and each operator $\mathcal{D}_a$ corresponds to a linear combination of the charge vectors among l'_1, l'_2, l'_3, l'_4 .

3.1.3 Brane superpotential and disk instantons

Now, we solve the Picard-Fuchs equations (3.13) at $z_i \rightarrow 0$ and identified the mirror maps and superpotentials. By the method introduced in section 2.1, the fundamental period of X as power series solution as follow

$$\omega_0 = 1 + 6z_1z_3 + 90z_1^2z_3^2 + 1680z_1^3z_3^3 + 34650z_1^4z_3^4 + 10080z_1^3z_2z_3^3z_4 + \mathcal{O}(z^8).$$

There are four logarithmic solutions with leading term $\omega_0 \log(Z_i)$

$$\begin{aligned} \omega_{1,1} &= \omega_0 \log(z_1) - 6z_1 + 45z_1^2 - 560z_1^3 + \frac{17325}{2}z_1^4 - \frac{756756}{5}z_1^5 + 21z_1z_3 - 180z_1^2z_3 + 2520z_1^3z_3 \\ &\quad - 46200z_1^4z_3 + \frac{783}{2}z_1^2z_3^2 - 5040z_1^3z_3^2 + \mathcal{O}(z^5) \\ \omega_{1,2} &= \omega_0 \log(z_2) + 12z_1z_3 + 270z_1^2z_3^2 - 180z_1^2z_2z_3^2 - 4z_2z_4 + 6z_1z_3z_4 + 12z_1z_2z_3z_4 + 180z_1^2z_3^2z_4 \\ &\quad + 30z_2^2z_4^2 + \mathcal{O}(z^5) \\ \omega_{1,3} &= \omega_0 \log(z_3) + 6z_1 - 45z_1^2 + 560z_1^3 - \frac{17325}{2}z_1^4 + \frac{756756}{5}z_1^5 - 6z_1z_3 + 180z_1^2z_3 - 2520z_1^3z_3 \\ &\quad + 46200z_1^4z_3 - 135z_1^2z_3^2 + 5040z_1^3z_3^2 + 2z_2z_4 - 6z_1z_2z_3z_4 - 15z_2^2z_4^2 + \mathcal{O}(z^5) \\ \omega_{1,4} &= \omega_0 \log(z_4) + 6z_1z_3 + 135z_1^2z_3^2 + 180z_1^2z_2z_3^2 - 2z_2z_4 - 6z_1z_3z_4 + 6z_1z_2z_3z_4 - 180z_1^2z_3^2z_4 \\ &\quad + 15z_2^2z_4^2 + \mathcal{O}(z^5). \end{aligned}$$

By the definition of the flat coordinates and mirror maps from Kahler moduli space to complex structure moduli space

$$t_j = \frac{\omega_{1,j}}{\omega_0} \quad (3.14)$$

$$q_j = e^{2\pi i t_j} \quad (3.15)$$

we obtain the z_j as a series of q_j upon inversion of the mirror maps

$$\begin{aligned} z_1 &= q_1 + 6q_1^2 + 9q_1^3 + 56q_1^4 - 300q_1^5 - 21q_1^2q_3 - 108q_1^3q_3 - 225q_1^4q_3 + 270q_1^3q_3^2 \\ &\quad + 42q_1^2q_2q_3q_4 + \mathcal{O}(q^5) \\ z_2 &= q_2 - 12q_1q_2q_3 + 54q_1^2q_2q_3^2 + 4q_2^2q_4 - 6q_1q_2q_3q_4 - 132q_1q_2^2q_3q_4 + 2q_2^3q_4^2 + \mathcal{O}(q^5) \\ z_3 &= q_3 - 6q_1q_3 + 27q_1^2q_3 - 164q_1^3q_3 + 1377q_1^4q_3 + 6q_1q_3^2 - 54q_1^2q_3^2 + 414q_1^3q_3^2 + 27q_1^2q_3^3 \\ &\quad - 2q_2q_3q_4 + 12q_1q_2q_3q_4 - 54q_1^2q_2q_3q_4 + 30q_1q_2q_3^2q_4 + \mathcal{O}(q^5) \\ z_4 &= q_4 - 6q_1q_3q_4 + 9q_1^2q_3^2q_4 + 2q_2q_4^2 + 6q_1q_3q_4^2 - 54q_1q_2q_3q_4^2 - q_2^2q_4^3 + \mathcal{O}(q^5) \end{aligned}$$

In addition, we abbreviate the double logarithmic solutions by their leading terms

$$\begin{array}{cccc} \frac{1}{2}\ell_1^2 + 2\ell_3\ell_4, & \frac{1}{2}\ell_2^2, & \frac{1}{2}\ell_3^2 + \ell_3\ell_4, & \frac{1}{2}\ell_4^2, \\ \ell_1\ell_2 + \ell_2\ell_4, & \ell_1\ell_3 - 3\ell_3\ell_4, & \ell_1\ell_4 + \ell_3\ell_4, & \ell_2\ell_3 \end{array}$$

where $\log(z_i)$'s are abbreviated as ℓ_i 's. According to above, a specific linear combination of double logarithmic solutions is constructed,

$$\mathcal{W}_{\text{brane}} = 3t_1t_2 - 3t_3t_4 + \sum_N N_{d_1,d_2,d_3,d_4} \text{Li}_2(q_1^{d_1} q_2^{d_2} q_3^{d_3} q_4^{d_4}) \quad (3.16)$$

where $\text{Li}_2(q_1^{d_1} q_2^{d_2} q_3^{d_3} q_4^{d_4}) = \sum_{n=1}^{\infty} \frac{q_1^{nd_1} q_2^{nd_2} q_3^{nd_3} q_4^{nd_4}}{n^2}$ is the dilogarithm function. We extract the disk instantons N_{d_1,d_2,d_3,d_4} from $\mathcal{W}_{\text{brane}}$ and present first a few invariants of the form $N_{k,m,k,m+n}$ in table 8. When focusing only on the invariants of the form $N_{k,m,k,m+n}$,

$$\mathcal{W}_{\text{brane}} = \cdots + \sum_N N_{k,m+n,k,n} \text{Li}_2((\tilde{q}_1)^k (\tilde{q}_2)^m \tilde{q}_3^n) \quad (3.17)$$

where $\cdots$ are terms independent of invariants, and $\tilde{q}_1 = q_1q_3, \tilde{q}_2 = q_2q_4, \tilde{q}_3 = q_4$. The superpotential (3.17) can be written as

$$\mathcal{W}_{\text{brane}} = \cdots + \sum_N N_{a,b,c} \text{Li}_2((\tilde{q}_1)^a (\tilde{q}_2)^b \tilde{q}_3^c) \quad (3.18)$$

which are essentially the superpotential of the model in [8]. Therefore, our invariants at first several order exactly match with the data of table 5 in [8] and these invariants are marked by blue color in table 8. In addition, we also calculate the invariants at higher order and put them into table 8.

3.1.4 Another curve on X

Following the similar logic, we consider another toric curve S on X with charge vectors

$$\begin{aligned} P &= 0, \\ h_1 &\equiv a_7(x_1x_2x_3x_4x_5) + a_8x_4^3 = 0, & \hat{l}_1 &= (-1, 0, 0, 0, 1, 0, 0) \\ h_2 &\equiv a_9x_3^9 + a_{10}x_4^3 = 0, & \hat{l}_2 &= (0, 0, 0, 1, -1, 0, 0) \end{aligned}$$

After blowing up, the toric data of mirror manifold of X' is determined by charge vectors

$$\begin{aligned} l'_1 &= (-2, 0, 0, 0, 0, 1, 1, -1, 1, 0, 0), & l'_2 &= (0, 1, 1, 0, 1, 0, -3, 0, 0, 1, -1), \\ l'_3 &= (-1, 0, 0, 0, 1, 0, 0, 1, -1, 0, 0), & l'_4 &= (0, 0, 0, 1, -1, 0, 0, 0, 0, -1, 1) \end{aligned} \quad (3.19)$$

satisfying $l^1 = l'_1 + l'_3, l^2 = l'_2 + l'_4$, from which we obtain the local coordinates on complex structure moduli of X'

$$z_1 = \frac{a_5a_6a_8}{a_0^2a_7}, \quad z_2 = \frac{a_1a_2a_4a_9}{a_6^3a_{10}}, \quad z_3 = \frac{a_4a_7}{a_0a_8}, \quad z_4 = \frac{a_6a_{10}}{a_4a_9} \quad (3.20)$$

and Picard-Fuchs operators

$$\begin{aligned} \mathcal{D}_1 &= \theta_1(\theta_1 - 3\theta_2)(\theta_1 - \theta_3) - z_1(-\theta_1 + \theta_3) \prod_{i=1}^2 (-2\theta_1 - \theta_3 - i) \\ \mathcal{D}_2 &= (\theta_2 + \theta_3 - \theta_4)(\theta_2)^2(\theta_2 - \theta_4) - z_2(\theta_1 - 3\theta_2)^3(-\theta_2 - \theta_4) \\ \mathcal{D}_3 &= (\theta_2 + \theta_3 - \theta_4)(-\theta_1 + \theta_3) - z_3(-2\theta_1 - \theta_3 - 1)(\theta_1 - \theta_3) \\ \mathcal{D}_4 &= \theta_4(-\theta_2 + \theta_4) - z_4(\theta_2 + \theta_3 - \theta_4)(\theta_2 - \theta_4) \\ &\cdots \end{aligned} \quad (3.21)$$

As before, we find single logarithmic solutions

$$\begin{aligned}\omega_{1,1} = & \omega_0 \log(z_1) - 2z_1 + 3z_1^2 - \frac{20}{3}z_1^3 + \frac{35}{2}z_1^4 - \frac{252}{5}z_1^5 + z_3 + 10z_1z_3 - 30z_1^2z_3 + 70z_1^3z_3 \\ & - 210z_1^4z_3 - \frac{1}{2}z_3^2 + 12z_1z_3^2 + 171z_1^2z_3^2 - 560z_1^3z_3^2 + \frac{1}{3}z_3^3 - 10z_1z_3^3 + 210z_1^2z_3^3 - \frac{1}{4}z_3^4 \\ & + 10z_1z_3^4 + \frac{1}{5}z_3^5 + 2z_2z_4 - 40z_1^3z_2z_4 - 6z_1z_2z_3z_4 + \mathcal{O}(z^5)\end{aligned}$$

$$\begin{aligned}\omega_{1,2} = & \omega_0 \log(z_2) + 12z_1z_3 + 270z_1^2z_3^2 - 6z_2z_4 + 6z_1z_3z_4 + 18z_1z_2z_3z_4 + 180z_1^2z_3^2z_4 \\ & + 45z_2^2z_4^2 + \mathcal{O}(z^5)\end{aligned}$$

$$\begin{aligned}\omega_{1,3} = & \omega_0 \log(z_3) + 2z_1 - 3z_1^2 + \frac{20}{3}z_1^3 - \frac{35}{2}z_1^4 + \frac{252}{5}z_1^5 - z_3 + 5z_1z_3 + 30z_1^2z_3 - 70z_1^3z_3 \\ & + 210z_1^4z_3 + \frac{1}{2}z_3^2 - 12z_1z_3^2 + \frac{171}{2}z_1^2z_3^2 + 560z_1^3z_3^2 - \frac{1}{3}z_3^3 + 10z_1z_3^3 - 210z_1^2z_3^3 + \frac{1}{4}z_3^4 \\ & - 10z_1z_3^4 - \frac{1}{5}z_3^5 + 40z_1^3z_2z_4 + \mathcal{O}(z^5)\end{aligned}$$

$$\omega_{1,4} = \omega_0 \log(z_4) + 6z_1z_3 + 135z_1^2z_3^2 - 6z_1z_3z_4 - 180z_1^2z_3^2z_4 + \mathcal{O}(z^5)$$

and mirror maps

$$\begin{aligned}z_1 = & q_1 + 2q_1^2 + 3q_1^3 + 4q_1^4 + 5q_1^5 - q_1q_3 - 12q_1^2q_3 - 24q_1^3q_3 - 36q_1^4q_3 + 9q_1^2q_3^2 + 108q_1^3q_3^2 + q_1^2q_3^3 \\ & - 2q_1q_2q_4 - 8q_1^2q_2q_4 - 18q_1^3q_2q_4 + 2q_1q_2q_3q_4 + 102q_1^2q_2q_3q_4 + \mathcal{O}(q^5) \\ z_2 = & q_2 - 12q_1q_2q_3 + 54q_1^2q_2q_3^2 + 6q_2^2q_4 - 6q_1q_2q_3q_4 - 210q_1q_2^2q_3q_4 + 9q_2^3q_4^2 + \mathcal{O}(q^5) \\ z_3 = & q_3 - 2q_1q_3 + q_1^2q_3 + q_3^2 - 7q_1q_3^2 + 12q_1^2q_3^2 - 7q_1^3q_3^2 + q_1^4q_3^2 + q_3^3 - 6q_1q_3^3 + 42q_1^2q_3^3 + q_3^4 \\ & - 6q_1q_3^4 + q_3^5 + 4q_1q_2q_3q_4 - 4q_1^2q_2q_3q_4 + 14q_1q_2z_3^2q_4 + \mathcal{O}(q^5) \\ z_4 = & q_4 - 6q_1q_3q_4 + 9q_1^2q_3^2q_4 + 6q_1q_3q_4^2 + 12q_1q_2q_3q_4^2 + \mathcal{O}(q^5)\end{aligned}$$

as single logarithmic solutions.

The superpotentials is constructed as follow

$$\mathcal{W}_{\text{brane}} = 9t_1^2 + 6t_1t_2 + 6t_1t_3 + \sum_N N_{d_1,d_2,d_3,d_4} \text{Li}_2(q_1^{d_1}q_2^{d_2}q_3^{d_3}q_4^{d_4}) \quad (3.22)$$

as linear combination of double logarithmic solutions. We extract and summarize the Ooguri-Vafa invariants of the form $N_{k,m,k,m+n}$ and $N_{k,m+n,k,n}$ in table 9 and table 10. At first several order, our result marked by blue color exactly agree with table 6 in [8] and we also present higher order result in the table.

3.2 Open-closed GKZ-system: branes on $X_8^{(1,1,2,2,2)}$

3.2.1 Branes wrapping rational curves and blowing up geometry

Now, we study the A-model manifold, whose toric polyhedron is denoted by Δ^* and charge vectors are denoted by l^1 and l^2 . A-model manifold and A-brane are specified by the following toric data. The hypersurface equation for X , written in homogeneous coordinates of $\mathbb{P}_{(1,1,2,2,2)}$, is

$$P = x_1^8 + x_2^8 + x_3^4 + x_4^4 + x_5^4 + \psi(x_1x_2x_3x_4x_5) + \phi(x_1x_2)^4$$

	Δ^*				l^1	l^2		$\hat{l}^1$	$\hat{l}^2$
v_0^*	0	0	0	0	-4	0	$x_1 x_2 x_3 x_4 x_5$	-1	0
v_1^*	-1	-2	-2	-2	0	1	x_1^8	0	-1
v_2^*	1	0	0	0	0	1	x_2^8	0	1
v_3^*	0	1	0	0	1	0	x_3^4	1	0
v_4^*	0	0	1	0	1	0	x_4^4	0	0
v_5^*	0	0	0	1	1	0	x_5^4	0	0
v_6^*	0	-1	-1	-1	1	-2	$(x_1 x_2)^4$	0	0

Table 2. Toric Data of A-model side.

where $\psi = z_1^{-\frac{1}{4}} z_2^{-\frac{1}{8}}$ and $\phi = z_2^{-\frac{1}{4}}$. The toric curve S on X is defined as the complete intersection

$$S : \quad P = 0, \quad h_1 \equiv \gamma^8(x_1 x_2 x_3 x_4 x_5) - \alpha^8 x_3^4 = 0, \quad h_2 \equiv \alpha^8 x_1^8 - \beta^8 x_2^8 = 0$$

The Greene-Plesser orbifold group G acts as $x_i \rightarrow \lambda_k^{g_{k,i}} x_i$ with $\lambda_1^8 = \lambda_2^4 = 1$, $\lambda_3^4 = 1$ and weights

$$\mathbb{Z}_8 : g_1 = (1, -1, 0, 0, 0), \quad \mathbb{Z}_4 : g_2 = (1, 0, -1, 0, 0), \quad \mathbb{Z}_4 : g_3 = (1, 0, 0, -1, 0)$$

Insert h_1 and h_2 into $P = 0$

$$\begin{aligned} \tilde{\mathbb{P}}_1 : \quad \eta_1 x_4 + \sqrt[4]{x_5^4 + m(x_1, x_3)} &= 0, \quad \eta_1^4 = 1 \\ m(x_1, x_3) &= \left(\frac{\alpha^8 + \beta^8}{\alpha^8} + \phi \frac{\beta^8}{\alpha^8} \right) x_1^8 + \left(1 + \frac{\gamma^8}{\alpha^8} \psi \right) x_3^4 \end{aligned} \quad (3.23)$$

$\tilde{\mathbb{P}}^1$ is a non-holomorphic family due to fourth roots of unity. At critical loci of the parameter space α, β, γ

$$\mathcal{M}_{\mathbb{P}^1}(S) : \quad \alpha^8 + \beta^8 + \phi \beta^8 = 0, \quad \alpha^8 + \gamma^8 \psi = 0 \quad (3.24)$$

$m(x_1, x_4)$ vanishes identically and S degenerates to

$$S : \quad h_0 \equiv x_4^4 + x_5^4, \quad h_1 = \gamma^8(x_1 x_2 x_3 x_4 x_5) - \alpha^8 x_3^4 = 0, \quad h_2 = \alpha^8 x_1^8 - \beta^8 x_2^8 = 0 \quad (3.25)$$

Modulo the action of G , (3.23) can be solved holomorphically. Thus the anholomorphic deformation of $\mathbb{P}^1$ (3.25) can be used to described holomorphic deformation of S .

From the toric data in table 2, we obtain the GKZ-system of X by (2.6),

$$\begin{aligned} \mathcal{Z}_0 &= \sum_{i=0}^6 \vartheta_i + 1, & \mathcal{Z}_1 &= -\vartheta_1 + \vartheta_2, & \mathcal{Z}_i &= -2\vartheta_i + \vartheta_{i+1} - \vartheta_6 \quad (i = 2, 3, 4), \\ \mathcal{L}_1 &= \prod_{i=3}^6 \frac{\partial}{\partial a_i} - \left(\frac{\partial}{\partial a_0} \right)^4, & \mathcal{L}_2 &= \frac{\partial}{\partial a_1} \frac{\partial}{\partial a_2} - \left(\frac{\partial}{\partial a_6} \right)^2 \end{aligned} \quad (3.26)$$

	Δ'							l'_1	l'_2	l'_3	l'_4	
v'_0	1	0	0	0	0	0	0	-3	0	-1	0	$w'_0 = x_1 x_2 x_3 x_4 x_5$
v'_1	1	0	-1	-2	-2	-2	0	0	2	0	-1	$w'_1 = x_1^8$
v'_2	1	0	1	0	0	0	0	0	0	0	1	$w'_2 = x_2^8$
v'_3	1	0	0	1	0	0	0	0	0	1	0	$w'_3 = x_3^4$
v'_4	1	0	0	0	1	0	0	1	0	0	0	$w'_4 = x_4^4$
v'_5	1	0	0	0	0	1	0	1	0	0	0	$w'_5 = x_5^4$
v'_6	1	0	0	-1	-1	-1	0	1	-2	0	0	$w'_6 = (x_1 x_2)^4$
v'_7	0	1	0	0	0	0	-1	-1	0	1	0	$w'_7 = y_1 w'_1$
v'_8	0	1	0	1	0	0	-1	1	0	-1	0	$w'_8 = y_1 w'_2$
v'_9	0	1	-1	-2	-2	-2	1	0	-1	0	1	$w'_9 = y_2 w'_0$
v'_{10}	0	1	1	0	0	0	1	0	1	0	-1	$w'_{10} = y_2 w'_3$

Table 3. Toric Data of X' 's Mirror Manifold.

where $\mathcal{Z}_0$ represents the invariant of P under overall rescaling, $\mathcal{Z}_i$'s relate to the torus symmetry, and $\mathcal{L}_i$'s relate to the symmetries among monomials consisting of P . As before, all GKZ operators annihilate the period integrals and determine the mirror maps and superpotentials.

After blowing up X along S , we obtain the blow-up manifold defined by,

$$\begin{aligned} X' : \quad P &= a_0 x_1 x_2 x_3 x_4 x_5 + a_1 x_1^8 + a_2 x_8 + a_3 x_4 + a_4 x_4^4 + a_5 x_5^4 + a_6 (x_1 x_2)^4, \\ Q &= y_1 (a_9 x_1^9 + a_{10} x_2^9) - y_2 (a_7 x_1 x_2 x_3 x_4 x_5 + a_8 x_3^4) \end{aligned} \quad (3.27)$$

As before, the corresponding infinitesimal generators are obtained, which are belong to the GKZ system of X' the GKZ system of X' by observation on the torus symmetry,

$$\begin{aligned} \mathcal{Z}'_0 &= \sum_{i=0}^6 \vartheta_i + 1, \quad \mathcal{Z}'_1 = \sum_{i=7}^{10} \vartheta_i, \quad \mathcal{Z}'_2 = -\vartheta_1 + \vartheta_2 - \vartheta_9 + \vartheta_{10}, \quad \mathcal{Z}'_3 = -2\vartheta_1 + \vartheta_3 - \vartheta_6 + \vartheta_8 - 2\vartheta_9, \\ \mathcal{Z}'_4 &= -2\vartheta_1 + \vartheta_4 - \vartheta_6 - 2\vartheta_9, \quad \mathcal{Z}'_5 = -2\vartheta_1 + \vartheta_5 - \vartheta_6 - 2\vartheta_9, \quad \mathcal{Z}'_6 = -\vartheta_7 - \vartheta_8 + \vartheta_9 + \vartheta_{10} \\ \mathcal{L}'_1 &= \prod_{i=3}^6 \frac{\partial}{\partial a_i} - \left(\frac{\partial}{\partial a_0} \right)^4, \quad \mathcal{L}'_2 = \prod_{i=1}^2 \frac{\partial}{\partial a_i} - \left(\frac{\partial}{\partial a_6} \right)^2 \\ \mathcal{L}'_3 &= \frac{\partial}{\partial a_3} \frac{\partial}{\partial a_7} - \frac{\partial}{\partial a_0} \frac{\partial}{\partial a_8}, \quad \mathcal{L}'_4 = \frac{\partial}{\partial a_2} \frac{\partial}{\partial a_9} - \frac{\partial}{\partial a_1} \frac{\partial}{\partial a_{10}} \end{aligned} \quad (3.28)$$

where $\mathcal{Z}'_0, \mathcal{Z}'_1$ are associated with the overall rescaling with respect to $P = 0, Q = 0$ respectively. $\mathcal{Z}'_i, i = 2, \dots, 6$ are related to the torus symmetry and The $\mathcal{L}'_3, \mathcal{L}'_4$ incorporate the parameter $a_7, \dots, a_{10}$ that are associated with the moduli of the curve S .

Then we formulate above GKZ-system in an enhanced polyhedron Δ' (table 3). Here we present the integral points v'_i of the enhanced polyhedron Δ' and their the corresponding monomials w'_i 's. l'_i 's are the generators of Mori cone l'_i satisfying $l^1 = l'_1 + l'_3$, $l^2 = l'_2 + l'_4$ and they are the maximal triangulation of Δ' .

By definition (2.7), the local coordinates of complex structure moduli space of X' are

$$z_1 = \frac{a_4 a_5 a_6 a_8}{a_0^3 a_7}, \quad z_2 = \frac{a_1^2 a_{10}}{a_6^2 a_9}, \quad z_3 = \frac{a_3 a_7}{a_0 a_8}, \quad z_4 = \frac{a_2 a_9}{a_1 a_{10}} \quad (3.29)$$

Next, we convert the $\mathcal{L}_i$ operators in (3.28) to Picard-Fuchs operators $\mathcal{D}_i$, from differential equations about $a_j (j = 0, \dots, 10)$ to those about $z_j (j = 1, \dots, 4)$,

$$\begin{aligned} \mathcal{D}_1 &= \theta_1^2 (\theta_1 - 2\theta_2)(\theta_1 - \theta_3) - z_1 (-\theta_1 + \theta_3) \prod_{i=1}^3 (-3\theta_1 - \theta_3 - i), \\ \mathcal{D}_2 &= (2\theta_2 - \theta_4)^2 (\theta_2 - \theta_4) - z_2 (\theta_1 - 2\theta_2)^2 (-\theta_2 + \theta_4) \\ \mathcal{D}_3 &= \theta_3 (-\theta_1 + \theta_3) - z_3 (-3\theta_1 - \theta_3 - 1)(\theta_1 - \theta_3), \\ \mathcal{D}_4 &= \theta_4 (-\theta_2 + \theta_4) - z_4 (2\theta_2 - \theta_4)(\theta_2 - \theta_4) \\ &\dots \end{aligned} \quad (3.30)$$

where θ_i 's are the logarithmic derivatives with respect to z_i 's and each $\mathcal{D}_i$ corresponds to a specific linear combination among l'_1, l'_2, l'_3, l'_4 .

3.2.2 Brane superpotential and disk instantons

Now, we solve the Picard-Fuchs equations (3.30) at $z_i \rightarrow 0$ and identified the mirror maps and superpotentials. By the techniques we introduced in section 2.1.

The unique power series solution, as well as the fundamental period of X , is

$$\begin{aligned} \omega_0 &= 1 + 24z_1 z_3 + 2520z_1^2 z_3^2 + 369600z_1^3 z_3^3 + 63063000z_1^4 z_3^4 + 5040z_1^2 z_2 z_3^2 z_4 \\ &\quad + 2217600z_1^3 z_2 z_3^3 z_4 + \mathcal{O}(z^5) \end{aligned}$$

The single logarithmic solutions are

$$\begin{aligned} \omega_{1,1} &= \omega_0 \log(z_1) - 6z_1 + 45z_1^2 - 560z_1^3 + \frac{17325}{2}z_1^4 - \frac{756756}{5}z_1^5 + z_3 + 78z_1 z_3 - 630z_1^2 z_3 \\ &\quad + 8400z_1^3 z_3 - 150150z_1^4 z_3 - \frac{1}{2}z_3^2 + 60z_1 z_3^2 + 9207z_1^2 z_3^2 - 92400z_1^3 z_3^2 + \frac{1}{3}z_3^3 - 60z_1 z_3^3 \\ &\quad + 7560z_1^2 z_3^3 - \frac{1}{4}z_3^4 + 70z_1 z_3^4 + \frac{1}{5}z_3^5 - z_2 z_4 + 90z_1^2 z_2 z_4 - 3360z_1^3 z_2 z_4 + 24z_1 z_2 z_3 z_4 \\ &\quad - 1260z_1^2 z_2 z_3 z_4 - \frac{3}{2}z_2^2 z_4^2 + \mathcal{O}(z^5) \\ \omega_{1,2} &= \omega_0 \log(z_2) + 48z_1 z_3 + 7560z_1^2 z_3^2 - 2520z_1^2 z_2 z_3^2 + 2z_2 z_4 - 48z_1 z_2 z_3 z_4 + 3z_2^2 z_4^2 + \mathcal{O}(z^5) \\ \omega_{1,3} &= \omega_0 \log(z_3) + 6z_1 - 45z_1^2 + 560z_1^3 - \frac{17325}{2}z_1^4 + \frac{756756}{5}z_1^5 - z_3 + 26z_1 z_3 + 630z_1^2 z_3 \\ &\quad - 8400z_1^3 z_3 + 150150z_1^4 z_3 + \frac{1}{2}z_3^2 - 60z_1 z_3^2 + 3069z_1^2 z_3^2 + 92400z_1^3 z_3^2 - \frac{1}{3}z_3^3 + 60z_1 z_3^3 \\ &\quad - 7560z_1^2 z_3^3 + \frac{1}{4}z_3^4 - 70z_1 z_3^4 + \frac{1}{5}z_3^5 - 90z_1^2 z_2 z_4 + 3360z_1^3 z_2 z_4 + 1260z_1^2 z_2 z_3 z_4 + \mathcal{O}(z^5) \\ \omega_{1,4} &= \omega_0 \log(z_4) + 2520z_1^2 z_2 z_3^2 + 1108800z_1^3 z_2 z_3^3 + 378378000z_1^4 z_2 z_3^4 + 252252000z_1^4 z_2^2 z_3^4 z_4 \\ &\quad - 2520z_1^2 z_2 z_3^2 z_4^2 - 1108800z_1^3 z_2 z_3^3 z_4^2 + \mathcal{O}(z^{10}) \end{aligned}$$

such that single logarithmic period of X can be reproduced by $\Pi_1^1 = \omega_{1,1} + \omega_{1,3}$, $\Pi_2^1 \omega_{1,2} + \omega_{1,4}$, and open-closed mirror maps are inverse series of flat coordinates,

$$\begin{aligned}
 z_1 &= q_1 + 6q_1^2 + 9q_1^3 + 56q_1^4 - 300q_1^5 - q_1q_3 - 84q_1^2q_3 - 432q_1^3q_3 - 900q_1^4q_3 + 68q_1^2q_3^2 \\
 &\quad + 4182q_1^3q_3^2 + 12q_1^2q_3^3 + q_1q_2q_4 + 12q_1^2q_2q_4 - 63q_1^3q_2q_4 - q_1q_2q_3q_4 - 264q_1^2q_2q_3q_4 + \mathcal{O}(q^5) \\
 z_2 &= q_2 - 48q_1q_2q_3 - 264q_1^2q_2q_3^2 - 2q_2^2q_4 + 240q_1q_2^2q_3q_4 + 3q_2^3q_4^2 + \mathcal{O}(q^5) \\
 z_3 &= q_3 - 6q_1q_3 + 27q_1^2q_3 - 164q_1^3q_3 + 1377q_1^4q_3 + q_3^2 - 32q_1q_3^2 + 138q_1^2q_3^2 - 184q_1^3q_3^2 + q_3^3 \\
 &\quad - 22q_1q_3^3 + 675q_1^2q_3^3 + q_3^4 - 24q_1q_3^4 + q_3^5 - 6q_1q_2q_3q_4 + 144q_1^2q_2q_3q_4 - 32q_1q_2q_3^2q_4 + \mathcal{O}(q^5) \\
 z_4 &= q_4 - 2520q_1^2q_2q_3^2q_4 - 403200q_1^3q_2q_3^3q_4 - 53701200q_1^4q_2q_3^4q_4 - 403200q_1^3q_2^2q_3^3q_4^2 \\
 &\quad + 2520q_1^2q_2^2q_3^2q_4^3 + 403200q_1^3q_2^2q_3^3q_4^3 + \mathcal{O}(q^{10})
 \end{aligned}$$

The double logarithmic solutions are denoted by their leading term

$$\begin{array}{cccc}
 \frac{1}{2}\ell_1^2, & \frac{1}{2}\ell_2^2, & \frac{1}{2}\ell_3^2 + \ell_1\ell_3, & \frac{1}{2}\ell_4^2 + \ell_2\ell_4, \\
 \ell_1\ell_2, & \ell_1\ell_4, & \ell_2\ell_3, & \ell_3\ell_4
 \end{array}$$

with abbreviations $\ell_i = \log(z_i)$. The brane superpotential is constructed as linear combination of double logarithmic solutions,

$$\mathcal{W}_{\text{brane}} = t_1^2 + \sum_{n_i} N_{d_1, d_2, d_3, d_4} \text{Li}_2(q_1^{d_1} q_2^{d_2} q_3^{d_3} q_4^{d_4}) \quad (3.31)$$

This has the expected integrality properties of the Ooguri-Vafa Li_2 multicover formula. The Ooguri-Vafa invariants of the form $N_{(m,k,m+n,k)}$ are exactly match the data in table 5 in [24]. In addition, we also extract the invariants of the form $N_{(m+n,k,n,k)}$ and summarize them into table 11, where the rows and columns are labelled by m and n , respectively.

3.3 Open-closed GKZ-system: branes on $X_{12}^{(1,1,2,2,6)}$

3.3.1 Five branes wrapping rational curves and blowing up geometry

In this section, we study the A-model manifold, whose toric polyhedron is denoted by Δ^* and charge vectors are denoted by l^1 and l^2 . A-model manifold and A-brane are specified by the following toric data.

The mirror hypersurface X is determined by the constraint,

$$P = x_1^{12} + x_2^{12} + x_3^6 + x_4^6 + x_5^2 + \psi(x_1x_2x_3x_4x_5) + \phi(x_1x_2)^6$$

where x_i 's are homogeneous coordinates in $\mathbb{P}_{(1,1,2,2,6)}$ and $\psi = z_1^{-\frac{1}{6}} z_2^{-\frac{1}{12}}$, $\phi = z_2^{-\frac{1}{2}}$. The Greene-Plesser orbifold group G acts as $x_i \rightarrow \lambda_k^{g_{k,i}} x_i$ with $\lambda_1^6 = \lambda_2^6 = \lambda_3^2 = 1$ and weights

$$\mathbb{Z}_6 : g_1 = (1, 0, -1, 0, 0), \quad \mathbb{Z}_6 : g_2 = (1, 0, 0, -1, 0), \quad \mathbb{Z}_2 : g_3 = (1, 0, 0, 0, -1)$$

The toric curve S is described by the complete intersection

$$S : \quad P = 0, \quad h_1 \equiv \gamma^{12}(x_1x_2x_3x_4x_5) - \alpha^{12}x_5^2 = 0, \quad h_2 \equiv \alpha^{12}x_1^{12} - \beta^{12}x_2^{12} = 0$$

	Δ^*				l^1	l^2		$\hat{l}^1$	$\hat{l}^2$
v_0^*	0	0	0	0	-6	0	$x_1x_2x_3x_4x_5$	-1	0
v_1^*	-1	-2	-2	-6	0	1	x_1^{12}	0	-1
v_2^*	1	0	0	0	0	1	x_2^{12}	0	1
v_3^*	0	1	0	0	1	0	x_3^6	0	0
v_4^*	0	0	1	0	1	0	x_4^6	0	0
v_5^*	0	0	0	1	3	0	x_5^2	1	0
v_6^*	0	-1	-1	-3	1	-2	$(x_1x_2)^6$	0	0

Table 4. Toric Data of A-model side.

Insert h_1 and h_2 into $P = 0$

$$\begin{aligned} \tilde{\mathbb{P}}^1 : \quad \eta_1 x_3 + \sqrt[6]{x_4^6 + m(x_1, x_5)} = 0, \quad \eta_1^6 = 1 \\ m(x_1, x_5) = \left(\frac{\alpha^{12} + \beta^{12}}{\alpha^{12}} + \frac{\beta^{12}}{\alpha^{12}} \phi \right) x_1^{12} + \left(1 + \frac{\gamma^{12}}{\alpha^{12}} \phi \right) x_5^2 \end{aligned} \quad (3.32)$$

Here $\tilde{\mathbb{P}}^1$ is evidently non-holomorphic because of the sixth roots of unity and hence, a non-holomorphic family of rational curves on. However, at special loci

$$\alpha^{12} + \beta^{12} + \phi \beta^{12} = 0, \quad \alpha^{12} + \gamma^{12} \psi = 0 \quad (3.33)$$

we see that S degenerates as follows,

$$\Sigma : \quad h_0 \equiv x_3^6 + x_4^6, \quad h_1 \equiv \gamma^{12}(x_1x_2x_3x_4x_5) - \alpha^{12}x_5^2 = 0, \quad h_2 \equiv \alpha^{12}x_1^{12} - \beta^{12}x_2^{12} = 0 \quad (3.34)$$

Modulo the action of G , (3.32) can be solve holomorphically. Thus the anholomorphic deformation of $\mathbb{P}^1$ (3.34) can be used to describe holomorphic deformation of S .

From the toric data in table 4, the full set of GKZ operators are derived by (2.6)

$$\begin{aligned} \mathcal{Z}_0 = \sum_{i=0}^6 \vartheta_i + 1, \quad \mathcal{Z}_1 = \vartheta_1 - \vartheta_2, \quad \mathcal{Z}_2 = 2\vartheta_1 - \vartheta_3 + \vartheta_6, \\ \mathcal{Z}_3 = 2\vartheta_1 - \vartheta_4 + \vartheta_6, \quad \mathcal{Z}_4 = 6\vartheta_1 - \vartheta_5 + 3\vartheta_6, \\ \mathcal{L}_1 = \frac{\partial}{\partial a_1} \frac{\partial}{\partial a_2} - \left(\frac{\partial}{\partial a_6} \right)^6, \quad \mathcal{L}_2 = \frac{\partial}{\partial a_3} \frac{\partial}{\partial a_4} \left(\frac{\partial}{\partial a_5} \right)^3 \frac{\partial}{\partial a_6} - \left(\frac{\partial}{\partial a_0} \right)^6 \end{aligned} \quad (3.35)$$

where $\mathcal{Z}_0$ represents the invariance of P under overall rescaling, $\mathcal{Z}_i$'s relate to the torus symmetry, and $\mathcal{L}_i$'s relate to the symmetries among monomials consisting of P . And all GKZ operators above annihilate the period matrix and determine the mirror maps and superpotential.

After blowing up X along S , the blow-up manifold X' is obtained as the complete intersection in $\mathcal{W} = \mathbb{P}(\mathcal{O}(12) \oplus \mathbb{P}\mathcal{O}(12))$

$$\begin{aligned} X' : \quad P = a_1x_1^{12} + a_2x_2^{12} + a_3x_3^6 + a_4x_4^6 + a_5x_5^2 + a_0(x_1x_2x_3x_4x_5) + a_6(x_1x_2)^6 = 0 \\ Q = y_1(a_9x_1^{12} + a_{10}x_2^{12}) - y_2(a_7(x_1x_2x_3x_4x_5) + a_8x_3^6) \end{aligned} \quad (3.36)$$

where a_i 's are free complex-valued coefficients. By simple observation, we can obtain the GKZ system of X' as complement to GKZ system of X .

$$\begin{aligned}
 \mathcal{Z}'_0 &= \sum_{i=0}^6 \vartheta_i + 1, & \mathcal{Z}'_1 &= \sum_{i=7}^{10} \vartheta_i, & \mathcal{Z}'_2 &= -\vartheta_1 + \vartheta_2 - \vartheta_9 + \vartheta_{10}, & \mathcal{Z}'_3 &= -2\vartheta_1 + \vartheta_3 - \vartheta_6 - 2\vartheta_9, \\
 \mathcal{Z}'_4 &= -2\vartheta_1 + \vartheta_4 - \vartheta_6 - 2\vartheta_9, & \mathcal{Z}'_5 &= -6\vartheta_1 + \vartheta_5 - 3\vartheta_6 + \vartheta_8 - 6\vartheta_9, & \mathcal{Z}'_6 &= -\vartheta_7 - \vartheta_8 + \vartheta_9 + \vartheta_{10} \\
 \mathcal{L}'_1 &= \prod_{i=1}^3 \frac{\partial}{\partial a_i} - \left(\frac{\partial}{\partial a_0} \right)^3, & \mathcal{L}'_2 &= \prod_{i=4}^6 \frac{\partial}{\partial a_i} - \left(\frac{\partial}{\partial a_3} \right)^3 \\
 \mathcal{L}'_3 &= \frac{\partial}{\partial a_3} \frac{\partial}{\partial a_8} - \frac{\partial}{\partial a_5} \frac{\partial}{\partial a_7}, & \mathcal{L}'_4 &= \frac{\partial}{\partial a_6} \frac{\partial}{\partial a_9} - \frac{\partial}{\partial a_3} \frac{\partial}{\partial a_{10}}
 \end{aligned} \tag{3.37}$$

where $\mathcal{Z}'_0, \mathcal{Z}'_1$ are associated with the overall rescaling with respect to $P = 0, Q = 0$, $\mathcal{Z}'_i, i = 2, \dots, 6$ are related to the torus symmetry, and $\mathcal{L}'_3, \mathcal{L}'_4$ incorporate the parameter $a_7, \dots, a_{10}$ that are associated with the moduli of the curve S .

Then we formulate above GKZ-system on an enhanced polyhedron Δ'

	Δ'							l'_1	l'_2	l'_3	l'_4	
v'_0	1	0	0	0	0	0	0	-1	-3	0	0	$w'_0 = x_1 x_2 x_3 x_4 x_5$
v'_1	1	0	-1	-2	-2	-6	0	0	0	2	-1	$w'_1 = x_1^{12}$
v'_2	1	0	1	0	0	0	0	0	0	0	1	$w'_2 = x_2^{12}$
v'_3	1	0	0	1	0	0	0	0	1	0	0	$w'_3 = x_3^6$
v'_4	1	0	0	0	1	0	0	0	1	0	0	$w'_4 = x_4^6$
v'_5	1	0	0	0	0	1	0	1	0	0	0	$w'_5 = x_5^2$
v'_6	1	0	0	-1	-1	-3	0	0	1	-2	0	$w'_6 = (x_1 x_2)^6$
v'_7	0	1	0	0	0	0	-1	1	-3	0	0	$w'_7 = y_1 w'_1$
v'_8	0	1	0	0	0	1	-1	-1	3	0	0	$w'_8 = y_1 w'_2$
v'_9	0	1	-1	-2	-2	-6	1	0	0	-1	1	$w'_9 = y_2 w'_0$
v'_{10}	0	1	1	0	0	0	1	0	0	1	-1	$w'_{10} = y_2 w'_5$

Here we present the integral points v'_i 's and their corresponding monomials w'_i , as well as l'_i 's as basis of Mori cone satisfying relations $l^1 = 3l'_1 + l'_2$ and $l^2 = l'_3 + l'_4$.

The local coordinates z^i 's on the complex structure moduli space of X' by (2.7),

$$z_1 = -\frac{a_5 a_7}{a_0^3 a_8}, \quad z_2 = \frac{a_3 a_4 a_6 a_8^3}{a_0^3 a_7^3}, \quad z_3 = \frac{a_1^2 a_{10}}{a_6^2 a_9}, \quad z_4 = \frac{a_2 a_9}{a_1 a_{10}} \tag{3.38}$$

Next, we convert the $\mathcal{L}_i$ in (3.37) operators to Picard-Fuchs operators $\mathcal{D}_i$, from differential equations about $a_j (j = 0, \dots, 10)$ to those about $z_j (j = 1, \dots, 4)$,

$$\begin{aligned}
 \mathcal{D}_1 &= \theta_1(\theta_1 - 3\theta_2 - 1) - z_1(-\theta_1 - 3\theta_2)(-\theta_1 + 3\theta_2), \\
 \mathcal{D}_2 &= \theta_2^2(\theta_2 - 2\theta_3)(-\theta_1 + 3\theta_2)^3 - z_2(\theta_1 - 3\theta_2)^3 \prod_{i=1}^3 (-\theta_1 - 3\theta_2 - i)
 \end{aligned}$$

$$\begin{aligned}
 \mathcal{D}_3 &= (2\theta_3 - \theta_4)^2(\theta_3 - \theta_4) - z_3(\theta_2 - 2\theta_3)^2(-\theta_3 + \theta_4), \\
 \mathcal{D}_4 &= \theta_4(-\theta_3 + \theta_4) - z_4(2\theta_3 - \theta_4)(\theta_3 - \theta_4) \\
 &\dots
 \end{aligned}
 \tag{3.39}$$

where $\theta_i = z_i \frac{\partial}{\partial z_i}$'s are the logarithmic derivatives and each $\mathcal{D}_i$ corresponds to a specific linear combination among l'_i 's.

3.3.2 Brane superpotential and disk instantons

Along the line of 2.1, we solve the differential equations (3.38) at $z_i \rightarrow 0$ and identify the mirror maps and superpotential. The fundamental period of X as power series solution is

$$\omega_0 = 1 - 120z_1^3z_2 + 83160z_1^6z_2^2 + 166320z_1^6z_2^2z_3z_4 + \mathcal{O}(z^{10}).$$

The single logarithmic solutions are

$$\begin{aligned}
 \omega_{1,1} &= \omega_0 \log(z_1) + z_1 + \frac{1}{2}z_1^2 + \frac{1}{3}z_1^3 + \frac{1}{4}z_1^4 + \frac{1}{5}z_1^5 + 2z_2 + 12z_1z_2 + 60z_1^2z_2 - 74z_1^3z_2 - 210z_1^4z_2 \\
 &\quad - 15z_2^2 - 126z_1z_2^2 - 630z_1^2z_2^2 - 2520z_1^3z_2^2 + \frac{560}{3}z_2^3 + 2100z_1z_2^3 + 13200z_1^2z_2^3 - \frac{5775}{2}z_2^4 \\
 &\quad - 40950z_1z_2^4 + \frac{252252}{5}z_2^5 - 30z_2^2z_3z_4 - 252z_1z_2^2z_3z_4 + 1120z_2^3z_3z_4 + \mathcal{O}(z^6) \\
 \omega_{1,2} &= \omega_0 \log(z_2) - 3z_1 - \frac{3}{2}z_1^2 - z_1^3 - \frac{3}{4}z_1^4 - \frac{3}{5}z_1^5 - 6z_2 - 36z_1z_2 - 180z_1^2z_2 - 522z_1^3z_2 + 630z_1^4z_2 \\
 &\quad + 45z_2^2 + 378z_1z_2^2 + 1890z_1^2z_2^2 + 7560z_1^3z_2^2 - 560z_2^3 - 6300z_1z_2^3 - 39600z_1^2z_2^3 + \frac{17325}{2}z_2^4 \\
 &\quad + 122850z_1z_2^4 - \frac{756756}{5}z_2^5 - z_3z_4 + 90z_2^2z_3z_4 + 756z_1z_2^2z_3z_4 - 3360z_2^3z_3z_4 - \frac{3}{2}z_3^2z_4^2 + \mathcal{O}(z^6) \\
 \omega_{1,3} &= \omega_0 \log(z_3) - 240z_1^3z_2 + 2z_3z_4 + 240z_1^3z_2z_3z_4 + 3z_3^2z_4^2 + 120z_1^3z_2z_3^2z_4^2 + \frac{20}{3}z_3^3z_4^3 \\
 &\quad + 160z_1^3z_2z_3^3z_4^3 + \frac{35}{2}z_3^4z_4^4 + \frac{252}{5}z_3^5z_4^5 + \mathcal{O}(z^{10}) \\
 \omega_{1,4} &= \omega_0 \log(z_4)
 \end{aligned}$$

such that the single logarithmic periods of X are reproduced by $\Pi_1^1 = \omega_{1,1} + \omega_{1,2}$, $\Pi_2^1 = \omega_{1,3} + \omega_{1,4}$. With single logarithmic solutions, open-closed mirror maps are inverse series of flat coordinates

$$\begin{aligned}
 z_1 &= q_1 - q_1^2 + q_1^3 - q_1^4 + q_1^5 - 2q_1q_2 - 14q_1^2q_2 - 60q_1^3q_2 + 134q_1^4q_2 + 5q_1q_2^2 + 58q_1^2q_2^2 \\
 &\quad + 270q_1^3q_2^2 - 32q_1q_2^3 - 546q_1^2q_2^3 + 286q_1q_2^4 - 2q_1q_2q_3q_4 - 14q_1^2q_2q_3q_4 + 40q_1q_2^2q_3q_4 + \mathcal{O}(q^5) \\
 z_2 &= q_2 + 3q_1q_2 + 3q_1^2q_2 + q_1^3q_2 + 6q_2^2 + 66q_1q_2^2 + 384q_1^2q_2^2 + 1338q_1^3q_2^2 + 9q_2^3 + 222q_1q_2^3 \\
 &\quad + 2940q_1^2q_2^3 + 56q_2^4 + 1350q_1q_2^4 - 300q_2^5 + q_2q_3q_4 + 3q_1q_2q_3q_4 + 3q_1^2q_2q_3q_4 \\
 &\quad + 12q_2^2q_3q_4 + 132q_1q_2^2q_3q_4 - 63q_2^3q_3q_4 + \mathcal{O}(q^5) \\
 z_3 &= q_3 + 240q_1^3q_2q_3 - 2q_3^2q_4 - 1200q_1^3q_2q_3^2q_4 + 3q_3^3q_4^2 + 3120q_1^3q_2q_3^3q_4^2 - 4q_3^4q_4^3 + 5q_3^5q_4^4 + \mathcal{O}(q^{10}) \\
 z_4 &= q_4
 \end{aligned}$$

The double logarithmic solutions are denoted by their leading term

$$\begin{array}{cccc} \frac{3}{2}\ell_1^2 + \ell_1\ell_2, & \frac{1}{2}\ell_2^2, & \frac{1}{2}\ell_3^2, & \frac{1}{2}\ell_4^2 + \ell_3\ell_4, \\ \ell_1\ell_3, & \ell_1\ell_4, & \ell_2\ell_3, & \ell_2\ell_4 \end{array} \quad (3.40)$$

with abbreviations $\ell_i = \log(z_i)$. Then we construct two linear combination of double logarithmic solutions and insert the inverse mirror maps to match the disk instantons in [10].

$$\begin{aligned} \mathcal{W}_{\text{brane}}^I &= \frac{3}{2}t_1^2 + t_1t_2 + \sum_{N^I} N_{d_1,d_2,d_3,d_4}^I \text{Li}_2(q_1^{d_1} q_2^{d_2} q_3^{d_3} q_4^{d_4}) \\ \mathcal{W}_{\text{brane}}^{II} &= \frac{1}{2}t_1t_3 + \sum_{N^{II}} N_{d_1,d_2,d_3,d_4}^{II} \text{Li}_2(q_1^{d_1} q_2^{d_2} q_3^{d_3} q_4^{d_4}). \end{aligned}$$

Invariants of the form $N_{m,n,k,k}$ are summarized in table 12, 13, 14, 15, where the rows and columns are labelled by m and n, respectively.

4 One closed and two open moduli case

4.1 Open-closed GKZ-system: branes on complete intersections $\mathbb{P}_{[3,3]}^{(111|111)}$

4.1.1 Five branes wrapping lines and blowing up geometry

The underlying manifold X^* we are considering in the A-model is the intersection of two cubics in $\mathbb{P}^5$, whose mirror manifold X can be represented a one-parameter family of bicubics, with group $G = \mathbb{Z}_3^2 \times \mathbb{Z}_9$ acting on them,

$$\begin{cases} P_1 = x_1^3 + x_2^3 + x_3^3 + \psi x_4 x_5 x_6 \\ P_2 = x_4^3 + x_5^3 + x_6^3 + \psi x_1 x_2 x_3 \end{cases} \quad (4.1)$$

where ψ is the complex structure modulus.

Turning to the specification of D-brane configurations, we consider the curve S on X

$$\begin{aligned} P_1 &= P_2 = 0, \\ h_1 &\equiv \beta^3(x_1 x_2 x_3) - \alpha \beta \gamma x_2^3 = 0, \\ h_2 &\equiv \gamma^3(x_1 x_2 x_3) - \alpha \beta \gamma x_3^3 = 0 \end{aligned}$$

An equivalent and convenient form is easy to obtained,

$$P_1 = P_2 = 0, \quad \alpha^3 x_2^3 - \beta^3 x_1^3 = 0, \quad \alpha^3 x_3^3 - \gamma^3 x_1^3 = 0 \quad (4.2)$$

For generic values of the moduli, the S is an irreducible higher genus Riemann surface. But we can always make a linearization by inserting h_1 and h_2 into P_1, P_2 ,

$$x_4^3 + x_5^3 + x_6^3 - \Psi x_4 x_5 x_6 = 0, \quad \Psi = \frac{\alpha \beta \gamma}{\alpha^3 + \beta^3 + \gamma^3} \psi^2 \quad (4.3)$$

	Δ^*					l		$\hat{l}^1$	$\hat{l}^2$
v_0^*	0	0	0	0	0	-3	$x_1x_2x_3$	-1	-1
v_0^*	0	0	0	0	0	-3	$x_4x_5x_6$	0	0
v_1^*	-1	-1	-1	-1	-1	1	x_1^3	0	0
v_2^*	1	0	0	0	0	1	x_2^3	1	0
v_3^*	0	1	0	0	0	1	x_3^3	0	1
v_4^*	0	0	1	0	0	1	x_4^3	0	0
v_5^*	0	0	0	1	0	1	x_5^3	0	0
v_6^*	0	0	0	0	1	1	x_6^3	0	0

Table 5. Toric Data of A-model side.

This is an one dimensional family of cubic plane elliptic curves in $\mathbb{P}^2$, called the Hesse pencil. For special value of Ψ , it degenerate into 12 lines [25],

$$x_4 + \eta_1 x_5 + \eta_2 x_6 = 0, \quad \eta_1^3 = \eta_2^3 = 1 \quad (4.4)$$

Upon the action of group G , they are identified as a single line.

Thus the deformation space of (4.4) is embedded in the deformation space of S . And away from that special locus, the obstructed deformation is identified with the unobstructed deformation of S , which means that we can use the obstructed deformation of that line to describe the unobstructed deformation of S .

The polyhedron corresponding to the A-model manifold X^* is denoted as Δ^* , the A-brane charge vectors l , and A-brane charge vectors $\hat{l}^1, \hat{l}^2$. The toric data in A-model side is as table 5.

In toric coordinates, the Calabi-Yau threefold X and the curve S are described as

$$\begin{aligned} X : \quad & P_1 = a_{0,1} + a_1(X_2X_3X_4X_5X_6)^{-1} + a_2X_2 + a_3X_3 = 0, \\ & P_2 = a_{0,2} + a_4X_4 + a_5X_5 + a_6X_6 = 0 \\ S : \quad & h_1 = a_7 + a_8X_2 = 0, \quad h_2 = a_9 + a_{10}X_3 = 0 \end{aligned}$$

with a_i 's are free complex-valued coefficients. The GKZ-system as follow by (2.6),

$$\begin{aligned} \mathcal{Z}_0 &= \sum_{i=0}^6 \vartheta_i + 1, \quad \mathcal{Z}_i = -\vartheta_1 + \vartheta_{i+1}, i = 1, \dots, 5 \\ \mathcal{L}_1 &= \prod_{i=1}^6 \frac{\partial}{\partial a_i} - \left(\frac{\partial}{\partial a_{0,1}} \right)^3 \left(\frac{\partial}{\partial a_{0,2}} \right)^3 \end{aligned} \quad (4.5)$$

where $\vartheta_i = a_i \frac{\partial}{\partial a_i}$'s are the logarithmic derivative. $\mathcal{Z}_0$ represents the invariance of P under overall rescaling and $\mathcal{Z}_i$'s relate to the torus symmetry,

$$\mathcal{Z}_i : \quad X_{i+1} \mapsto \lambda X_{i+1}, \quad (a_1, a_{i+1}) \mapsto (\lambda a_1, \lambda^{-1} a_{i+1}), \quad i = 1, 2, 3, 4$$

For $\mathcal{L}_i$'s, they represents the relations among Laurent monomials in P_1 and P_2 in (4.1).

$$\mathcal{L}_1 : (X_2^{-1})(X_3^{-1})(X_4^{-1})(X_5^{-1})(X_6^{-1})X_2X_3X_4X_5X_6 = 1$$

And all GKZ operators annihilate the period matrix and determine the mirror maps and superpotentials.

After blowing up X along S , the blow-up manifold is

$$X' : P_1 = P_2 = 0, \quad Q = y_1(a_9 + a_{10}X_3) - y_2(a_7 + a_8X_2) \quad (4.6)$$

By careful observations on the defining equations (4.6), the GKZ system of X' is obtained as follow

$$\begin{aligned} \mathcal{Z}'_0 &= \sum_{i=0}^6 \vartheta_i + 1, \quad \mathcal{Z}'_1 = \sum_{i=7}^{10} \vartheta_i, \quad \mathcal{Z}'_2 = -\vartheta_1 + \vartheta_2 + \vartheta_8, \quad \mathcal{Z}'_3 = -\vartheta_1 + \vartheta_3 + \vartheta_{10}, \\ \mathcal{Z}'_i &= -\vartheta_1 + \vartheta_i, \quad i = 4, 5, 6, \quad \mathcal{Z}'_7 = -\vartheta_7 - \vartheta_8 + \vartheta_9 + \vartheta_{10} \\ \mathcal{L}'_1 &= \prod_{i=1}^3 \frac{\partial}{\partial a_i} - \left(\frac{\partial}{\partial a_0} \right)^3, \\ \mathcal{L}'_2 &= \frac{\partial}{\partial a_2} \frac{\partial}{\partial a_7} - \frac{\partial}{\partial a_{0,1}} \frac{\partial}{\partial a_8}, \quad \mathcal{L}'_3 = \frac{\partial}{\partial a_3} \frac{\partial}{\partial a_9} - \frac{\partial}{\partial a_{0,1}} \frac{\partial}{\partial a_{10}} \end{aligned} \quad (4.7)$$

where $\mathcal{Z}'_0, \mathcal{Z}'_1$ are associated with the overall rescaling with respect to $P = 0, Q = 0$ respectively. $\mathcal{Z}'_i, i = 2, \dots, 5$ are related to the torus symmetry.

$$\begin{aligned} \mathcal{Z}'_2 : \quad X_2 &\mapsto \lambda X_2, & (a_1, a_2, a_8) &\mapsto (\lambda a_1, \lambda^{-1} a_2, \lambda^{-1} a_8) \\ \mathcal{Z}'_3 : \quad X_3 &\mapsto \lambda X_3, & (a_1, a_3, a_{10}) &\mapsto (\lambda a_1, \lambda^{-1} a_3, \lambda^{-1} a_{10}) \\ \mathcal{Z}'_i : \quad X_i &\mapsto \lambda X_i, & (a_1, a_i) &\mapsto (\lambda a_1, \lambda^{-1} a_i), \quad i = 4, 5, 6 \end{aligned}$$

In addition, $\mathcal{Z}'_6$ is related to the torus symmetry $(y_1, y_2) \mapsto (\lambda y_1, \lambda^{-1} y_2)$. The $\mathcal{L}'_2, \mathcal{L}'_3$ incorporate the parameter $a_7, \dots, a_{10}$ that are associated with the moduli of the curve S .

Now, we formulate GKZ system (4.7) on an enhanced polyhedron Δ' ,

	Δ'								l'_1	l'_2	l'_3	
v'_0	1	0	0	0	0	0	0	0	0	-1	-1	$w'_{0,1} = x_1 x_2 x_3$
v'_0	1	0	0	0	0	0	0	0	-3	0	0	$w'_{0,2} = x_4 x_5 x_6$
v'_1	1	0	-1	-1	-1	-1	-1	0	1	0	0	$w'_1 = x_1^3$
v'_2	1	0	1	0	0	0	0	0	-1	1	0	$w'_2 = x_2^3$
v'_3	1	0	0	1	0	0	0	0	0	0	1	$w'_3 = x_3^3$
v'_4	1	0	0	0	1	0	0	0	1	0	0	$w'_4 = x_4^3$
v'_5	1	0	0	0	0	1	0	0	1	0	0	$w'_5 = x_5^3$
v'_6	1	0	0	0	0	0	1	0	1	0	0	$w'_6 = x_6^3$
v'_7	0	1	0	0	0	0	0	-1	-2	1	0	$w'_7 = y_1 w'_0$
v'_8	0	1	1	0	0	0	0	-1	2	-1	0	$w'_8 = y_1 w'_2$
v'_9	0	1	0	0	0	0	0	1	-1	0	1	$w'_9 = y_2 w'_0$
v'_{10}	0	1	0	1	0	0	0	1	1	0	-1	$w'_{10} = y_2 w'_3$

where v'_i 's are the integral vertices and w'_i , their corresponding monomials. The A-model closed string charge vectors and A-branes charge vectors satisfy the relations $l^1 = l'_1 + 2l'_2 + l'_3$, $\hat{l}^1 = l'_3$, $\hat{l}^2 = l'_4$.

The coordinates z_i by (2.7) on the complex structure moduli space of X' ,

$$z_1 = \frac{a_1 a_4 a_5 a_6 a_8 a_{10}}{a_{0,2}^3 a_2 a_7^2 a_9}, \quad z_2 = \frac{a_2 a_7}{a_{0,1} a_8}, \quad z_3 = \frac{a_3 a_9}{a_{0,1} a_{10}} \quad (4.8)$$

Next, we convert the $\mathcal{L}_i$ operators in (4.7) to Picard-Fuchs operators $\mathcal{D}_i$, from differential equations about a_j to those about z_j ,

$$\begin{aligned} \mathcal{D}_1 &= \theta_1^4 (2\theta_1 - \theta_2)^2 (\theta_1 - \theta_3) - z_1 (-\theta_1 + \theta_2) (-2\theta_1 + \theta_2)^2 (-\theta_1 + \theta_3) \prod_{i=1}^3 (-3\theta_1 - i), \\ \mathcal{D}_2 &= (-\theta_1 + \theta_2) (-2\theta_1 + \theta_2) - z_2 (-\theta_2 - \theta_3 - 1) (2\theta_1 - \theta_2) \\ \mathcal{D}_3 &= \theta_3 (-\theta_1 + \theta_3) - z_3 (-\theta_2 - \theta_3 - 1) (\theta_1 - \theta_3), \\ &\dots \end{aligned} \quad (4.9)$$

where $\theta_i = z_i \frac{\partial}{\partial z_i}$'s are the logarithmic derivatives and each operator $\mathcal{D}_a$ corresponds to a linear combination among the charge vectors l'_1, l'_2, l'_3 .

4.1.2 Brane superpotential and disk instantons

Now, we solve the Picard-Fuchs equations (4.9) derived in the last section and identified the mirror maps and superpotentials. By the methods introduced in 2.1, at $z_i \rightarrow 0$, the fundamental period of X as series expansion is

$$\omega_0 = 1 + 36z_1 z_2^2 z_3 + 8100z_1^2 z_2^4 z_3^2 + 2822400z_1^3 z_2^6 z_3^3 + 1200622500z_1^4 z_2^8 z_3^4 + \mathcal{O}(z^{16})$$

There are four the single logarithmic solutions

$$\begin{aligned} \omega_{1,1} &= \omega_0 \log(z_1) + 2z_2 - z_2^2 - 12z_1 z_2^2 + \frac{2}{3}z_2^3 - \frac{1}{2}z_2^4 + \frac{2}{5}z_2^5 + z_3 - 24z_1 z_2 z_3 + 90z_1 z_2^2 z_3 \\ &\quad + 144z_1 z_2^3 z_3 - \frac{1}{2}z_3^2 + 72z_1 z_2^2 z_3^2 + \frac{1}{3}z_3^3 - \frac{1}{4}z_3^4 + \frac{1}{5}z_3^5 + \mathcal{O}(z^5) \\ \omega_{1,2} &= \omega_0 \log(z_2) - z_2 + \frac{1}{2}z_2^2 - \frac{1}{3}z_2^3 + \frac{1}{4}z_2^4 - \frac{1}{5}z_2^5 + 12z_1 z_2 z_3 + 30z_1 z_2^2 z_3 - 72z_1 z_2^3 z_3 + \mathcal{O}(z^6) \\ \omega_{1,3} &= \omega_0 \log(z_3) + 12z_1 z_2^2 - 270z_1^2 z_2^4 - z_3 + 30z_1 z_2^2 z_3 + \frac{1}{2}z_3^2 - 72z_1 z_2^2 z_3^2 - \frac{1}{3}z_3^3 + \frac{1}{4}z_3^4 \\ &\quad - \frac{1}{5}z_3^5 + \mathcal{O}(z^5) \end{aligned}$$

such that they are consistent with the single logarithmic periods of X , $\Pi^1 = \omega_{1,1} + 2\omega_{1,2} + \omega_{1,3}$.

By the definition of the flat coordinates and mirror maps

$$t_j = \frac{\omega_{1,j}}{\omega_0} \quad (4.10)$$

$$q_j = e^{2\pi i t_j} \quad (4.11)$$

we obtain the z_j as a series of q_j upon inversion of the mirror maps

$$\begin{aligned} z_1 &= q_1 - 2q_1q_2 + q_1q_2^2 + 12q_1^2q_2^2 - 24q_1^2q_2^3 - q_1q_3 + 2q_1q_2q_3 + 24q_1^2q_2q_3 - q_1q_2^2q_3 \\ &\quad - 150q_1^2q_2^2q_3 - 24q_1^2q_2q_3^2 + \mathcal{O}(q^5) \\ z_2 &= q_2 + q_2^2 + q_2^3 + q_2^4 + q_2^5 - 12q_1q_2^2q_3 - 42q_1q_2^3q_3 + \mathcal{O}(q^5) \\ z_3 &= q_3 - 12q_1q_2^2q_3 + q_3^2 - 42q_1q_2^2q_3^2 + q_3^3 + q_3^4 + q_3^5 + \mathcal{O}(q^5) \end{aligned}$$

In addition, there are also double logarithmic solutions with leading terms

$$\frac{1}{2}\ell_1^2 - \ell_2^2, \quad \frac{3}{2}\ell_2^2 + \ell_1\ell_2, \quad \frac{1}{2}\ell_3^2 + \ell_1\ell_3, \quad \ell_2\ell_3$$

where $\log(z_i)$'s are abbreviated as ℓ_i 's. According to above, a specific linear combination of double logarithmic solutions is constructed and its disk instantons expansions are extracted.

$$\mathcal{W}_{\text{brane}} = 4t_1t_2 + 6t_2^2 + 4t_2t_3 + \sum_N N_{d_1,d_2,d_3} \text{Li}_2(q_1^{d_1}q_2^{d_2}q_3^{d_3}) \quad (4.12)$$

We present first a few invariants of the form $N_{m,m+n,m}$ in table 16.

4.2 Branes on complete intersections $\mathbb{P}_{[4,4]}^{(112|112)}$ and $\mathbb{P}_{[6,6]}^{(123|123)}$

Similar to the last section, we also calculate the superpotential and extract the Ooguri-Vafa invariants at large volume phase for complete intersections Calabi-Yau manifolds of $\mathbb{P}_{[4,4]}^{(112|112)}$ and $\mathbb{P}_{[6,6]}^{(123|123)}$. We summarize the main formulas and tables in appendix A and Ooguri-Vafa invariants for first several orders in appendix C.

5 One closed and three open moduli case

5.1 Open-closed GKZ-system: branes on sextic hypersurface

5.1.1 Branes wrapping rational curves and blowing up geometry

In this section, we consider the A-model manifold X^* with charge vector $l = (-6, 1, 1, 1, 1, 2)$. The corresponding toric polyhedron for X^* consists of following integral vertices

$$(0, 0, 0, 0), \quad (-1, -1, -1, -2), \quad (1, 0, 0, 0), \quad (0, 1, 0, 0), \quad (0, 0, 1, 0), \quad (0, 0, 0, 1) \quad (5.1)$$

The mirror sextic hypersurface X arises as the Calabi-Yau hypersurface is a mirror pair of Calabi-Yau hypersurface in $\mathbb{P}_{(1,1,1,1,2)}^4$. Its defining equation in homogeneous coordinates is

$$P = a_1x_1^6 + a_2x_2^6 + a_3x_3^6 + a_4x_4^6 + a_5x_5^3 + a_0x_1x_2x_3x_4x_5.$$

On X , we consider parallel branes S which are described by

$$S: \quad P = 0, \quad h_1 \equiv a_6(x_1x_2x_3x_4x_5)^2 + a_7x_1x_2x_3x_4x_5^4 + a_8x_5^6 = 0, \quad h_2 \equiv a_9x_1^6 + a_{10}x_5^3 = 0$$

The GKZ operators are derived by (2.6)

$$\begin{aligned} \mathcal{Z}_0 &= \sum_{i=0}^5 \vartheta_i + 1, \quad \mathcal{Z}_i = \vartheta_{i+1} - \vartheta_1, i = 1, 2, 3, \quad \mathcal{Z}_4 = \vartheta_5 - 2\vartheta_1 \\ \mathcal{L}_1 &= \frac{\partial}{\partial a_1} \frac{\partial}{\partial a_2} \frac{\partial}{\partial a_3} \frac{\partial}{\partial a_4} \frac{\partial}{\partial a_5} - \left(\frac{\partial}{\partial a_0} \right)^6 \end{aligned} \quad (5.2)$$

where $\mathcal{Z}_0$ represents the invariance of P under overall rescaling, $\mathcal{Z}_i$'s relate to the torus symmetry, and $\mathcal{L}_i$'s relate to the symmetries among monomials consisting of P . And all GKZ operators above annihilate the period matrix and determine the mirror maps and superpotential.

After blowing up X along S , the blow-up manifold is obtained as the complete intersection in the projective bundle

$$\begin{aligned} X' : \quad P &= a_1 x_1^6 + x_2^6 + x_3^6 + x_4^6 + x_5^3 + a_0(x_1 x_2 x_3 x_4 x_5) = 0 \\ Q &= y_1(a_9 x_1^6 + a_{10} x_5^3) - y_2(a_6(x_1 x_2 x_3 x_4 x_5)^2 + a_7 x_1 x_2 x_3 x_4 x_5^4 + a_8 x_5^6) \end{aligned} \quad (5.3)$$

where a_i 's are free complex-valued coefficients. By observation on the symmetry of above defining equations, we can obtain the GKZ system of X' as complement to GKZ system of X .

$$\begin{aligned} \mathcal{Z}'_0 &= \sum_{i=0}^5 \vartheta_i + 1, \quad \mathcal{Z}'_1 = \sum_{i=6}^{10} \vartheta_i, \quad \mathcal{Z}'_2 = -\vartheta_1 + \vartheta_2 + \vartheta_7 + 2\vartheta_8 - \vartheta_9, \quad \mathcal{Z}'_3 = -1\vartheta_1 + \vartheta_3 - \vartheta_9, \\ \mathcal{Z}'_4 &= -\vartheta_1 + \vartheta_4 - \vartheta_9, \quad \mathcal{Z}'_5 = -2\vartheta_1 + \vartheta_5 - 2\vartheta_6 + \vartheta_8 - 6\vartheta_9, \quad \mathcal{Z}'_6 = -\vartheta_6 - \vartheta_7 - \vartheta_8 + \vartheta_9 + \vartheta_{10} \\ \mathcal{L}'_1 &= \prod_{i=2}^5 \frac{\partial}{\partial a_i} \left(\frac{\partial}{\partial a_7} \right)^2 \frac{\partial}{\partial a_9} - \left(\frac{\partial}{\partial a_0} \right)^4 \frac{\partial}{\partial a_6} \frac{\partial}{\partial a_{10}}, \quad \mathcal{L}'_2 = \frac{\partial}{\partial a_6} \frac{\partial}{\partial a_8} - \left(\frac{\partial}{\partial a_7} \right)^2 \\ \mathcal{L}'_3 &= \frac{\partial}{\partial a_5} \frac{\partial}{\partial a_7} - \frac{\partial}{\partial a_0} \frac{\partial}{\partial a_8}, \quad \mathcal{L}'_4 = \frac{\partial}{\partial a_1} \frac{\partial}{\partial a_{10}} - \frac{\partial}{\partial a_5} \frac{\partial}{\partial a_9} \end{aligned} \quad (5.4)$$

where $\mathcal{Z}'_0, \mathcal{Z}'_1$ are associated with the overall rescaling with respect to $P = 0, Q = 0$, $\mathcal{Z}'_i, i = 2, \dots, 6$ are related to the torus symmetry, and $\mathcal{L}'_3, \mathcal{L}'_4$ incorporate the parameter $a_6, \dots, a_{10}$ that are associated with the moduli of the curve S . The new manifold X' is describe by the following charge vectors

	0	1	2	3	4	5	6	7	8	9	10
l'_1	-4	0	1	1	1	1	-2	2	0	1	-1
l'_2	0	0	0	0	0	0	1	-2	1	0	0
l'_3	-1	0	0	0	0	1	0	1	-1	0	0
l'_4	0	-1	1	0	0	0	0	0	0	1	-1

The local coordinates z^i 's on the complex structure moduli space of X' by (2.7) is,

$$z_1 = \frac{a_2 a_3 a_4 a_5 a_7^2 a_9}{a_0^4 a_6^2 a_{10}}, \quad z_2 = \frac{a_6 a_8}{a_7^2}, \quad z_3 = \frac{a_5 a_7}{a_0 a_8}, \quad z_4 = \frac{a_1 a_{10}}{a_5 a_9} \quad (5.5)$$

Next, we convert the $\mathcal{L}'_i$ in (5.4) operators to Picard-Fuchs operators $\mathcal{D}_i$, from differential equations about $a_j (j = 0, \dots, 10)$ to those about $z_j (j = 1, \dots, 4)$,

$$\begin{aligned} \mathcal{D}_1 &= \theta_1^3 (\theta_1 + \theta_3 - \theta_4) (2\theta_1 - 2\theta_2 + \theta_3)^2 (\theta_1 - \theta_4) - z_1 (-2\theta_1 + \theta_2)^2 (-\theta_1 + \theta_4) \prod_{i=1}^4 (-4\theta_1 - \theta_3 - i), \\ \mathcal{D}_2 &= (-2\theta_1 + \theta_2) (\theta_2 - \theta_3) - z_2 (2\theta_2 - 2\theta_2 + \theta_3)^2 \end{aligned}$$

$$\begin{aligned}
 \mathcal{D}_3 &= (\theta_1 + \theta_3 - \theta_4)(2\theta_1 - 2\theta_2 + \theta_3) - z_3(\theta_2 - \theta_3)(-4\theta_1 - \theta_3 - 1), \\
 \mathcal{D}_4 &= \theta_4(-\theta_1 + \theta_4) - z_4(\theta_1 + \theta_3 - \theta_4)(\theta_1 - \theta_4) \\
 &\dots
 \end{aligned} \tag{5.6}$$

where $\theta_i = z_i \frac{\partial}{\partial z_i}$'s are the logarithmic derivatives.

5.1.2 Brane superpotential and disk instantons

Along the line of 2.1, we solve the differential equations (5.6) at $z_i \rightarrow 0$ and identify the mirror maps and superpotential. The fundamental period of X is

$$\omega_0 = 1 + 360z_1z_2^2z_3^2z_4 + 1247400z_1^2z_2^4z_3^4z_4^2 + 6861254400z_1^3z_2^6z_3^6z_4^3 + \mathcal{O}(z^{10}).$$

The single logarithmic solutions are

$$\begin{aligned}
 \omega_{1,1} &= \omega_0 \log(z_1) - 2z_2 - 3z_2^2 - \frac{20}{3}z_2^3 - \frac{35}{2}z_2^4 - \frac{252}{5}z_2^5 + 2z_2z_3 + 2z_2^2z_3 + 4z_2^3z_3 + 10z_2^4z_3 - z_2^2z_3^2 \\
 &\quad - 120z_1z_2^2z_3^2 - 2z_2^3z_3^2 + 24z_1z_4 - 48z_1z_2z_4 - 24z_1z_2^2z_4 - 48z_1z_2^3z_4 - 240z_1z_2z_3z_4 \\
 &\quad + 240z_1z_2^2z_3z_4 + 1260z_1^2z_4^2 - 5040z_1^2z_2z_4^2 + \mathcal{O}(z^5) \\
 \omega_{1,2} &= \omega_0 \log(z_2) + 2z_2 + 3z_2^2 + \frac{20}{3}z_2^3 + \frac{35}{2}z_2^4 + \frac{252}{5}z_2^5 + z_3 - 2z_2z_3 - 2z_2^2z_3 - 4z_2^3z_3 - 10z_2^4z_3 \\
 &\quad - \frac{1}{2}z_3^2 + z_2z_3^2 + z_2^2z_3^2 + 2z_2^3z_3^2 + \frac{1}{3}z_3^3 - z_2z_3^3 - \frac{1}{4}z_3^4 + z_2z_3^4 - 12z_1z_4 + 24z_1z_2z_4 + 24z_1z_2^2z_4 \\
 &\quad + 48z_1z_2^3z_4 + 120z_1z_2z_3z_4 - 240z_1z_2^2z_3z_4 - 630z_1^2z_4^2 + 2520z_1^2z_2z_4^2 + \mathcal{O}(z^5) \\
 \omega_{1,3} &= \omega_0 \log(z_3) - z_2 - \frac{3}{2}z_2^2 - \frac{10}{3}z_2^3 - \frac{35}{4}z_2^4 - \frac{126}{5}z_2^5 - z_3 + z_2z_3 + z_2^2z_3 + 2z_2^3z_3 + 5z_2^4z_3 + \\
 &\quad \frac{1}{2}z_3^2 - z_2z_3^2 - \frac{1}{2}z_2^2z_3^2 - z_2^3z_3^2 - \frac{1}{3}z_3^3 + z_2z_3^3 + \frac{1}{4}z_3^4 - z_2z_3^4 - \frac{1}{5}z_3^5 - 12z_1z_2^2z_4 - 24z_1z_2^3z_4 + \mathcal{O}(z^5) \\
 \omega_{1,4} &= \omega_0 \log(z_4) + 120z_1z_2^2z_3^2 - 41580z_1^2z_2^4z_3^4 + 180z_1z_2^2z_3^2z_4 + 498960z_1^2z_2^4z_3^4z_4 \\
 &\quad - 360z_1z_2^2z_3^2z_4^2 + 727650z_1^2z_2^4z_3^4z_4^2 + 60z_1z_2^2z_3^2z_4^3 + \mathcal{O}(z^{10})
 \end{aligned}$$

by which the open-closed mirror maps are inverse series of flat coordinates

$$\begin{aligned}
 z_1 &= q_1 + 2q_1q_2 + q_1q_2^2 - 4q_1q_2q_3 - 4q_1q_2^2q_3 + 4q_1q_2^2q_3^2 + \mathcal{O}(q^5) \\
 z_2 &= q_2 - 2q_2^2 + 3q_2^3 - 4q_2^4 + 5q_2^5 - q_2q_3 + 5q_2^2q_3 - 13q_2^3q_3 + 25q_2^4q_3 - 3q_2^2q_3^2 + 18q_2^3q_3^2 + \mathcal{O}(q^5) \\
 z_3 &= q_3 + q_2q_3 + q_3^2 + q_2^2q_3^2 + q_3^3 + q_3^4 + q_3^5 + \mathcal{O}(q^5) \\
 z_4 &= q_4 - 120q_1q_2^2q_3^2q_4 - 180q_1q_2^2q_3^2q_4^2 + \mathcal{O}(q^7)
 \end{aligned}$$

Then we construct a linear combination of double logarithmic solutions and insert the inverse mirror maps.

$$\mathcal{W}_{\text{brane}} = 3t_1^2 + 6t_1t_4 + 3t_4^2 + \sum_N N_{d_1,d_2,d_3,d_4} \text{Li}_2(q_1^{d_1} q_2^{d_2} q_3^{d_3} q_4^{d_4}) \tag{5.7}$$

Invariants of the form $N_{\text{m,m,m,n}}$ are summarized in table 17, where the rows and columns are labelled by m and n, respectively.

When $a_7^2 = a_6 a_8$, the two individual branes coincide. We obtain a new set of charge vectors,

	0	1	2	3	4	5	6	8	9	10
l_1^c	-4	0	1	1	1	1	-1	1	1	-1
l_2^c	-2	0	0	0	0	2	1	-1	0	0
l_3^c	0	-1	1	0	0	0	0	0	1	-1

by which the new complex structure moduli space coordinates are

$$z_1^c = \frac{a_2 a_3 a_4 a_5 a_8 a_9}{a_0^4 a_6 a_{10}}, \quad z_2^c = \frac{a_5^2 a_6}{a_0^2 a_8}, \quad z_3^c = \frac{a_2 a_9}{a_1 a_{10}} \quad (5.8)$$

Similar to the separate case, the superpotential is constructed as linear combination of double logarithmic solutions and Ooguri-Vafa invariants are extracted in table 17.

5.2 Open-closed GKZ-system: branes on sextic hypersurface

5.2.1 Branes wrapping rational curves and blowing up geometry

The mirror octic hypersurface X arises as the Calabi-Yau hypersurfaces in $\mathbb{P}_{(1,1,1,1,4)}^4$

$$P = a_1 x_1^8 + a_2 x_2^8 + a_3 x_3^8 + a_4 x_4^8 + a_5 x_5^2 + a_0 x_1 x_2 x_3 x_4 x_5$$

on which we consider parallel branes which are described by intersections of divisors

$$S: \quad P = 0, \quad h_1 \equiv a_6 (x_1 x_2 x_3 x_4 x_5)^2 + a_7 x_1 x_2 x_3 x_4 x_5^3 + a_8 x_5^4 = 0, \quad h_2 \equiv a_9 x_1^8 + a_{10} x_5^2 = 0$$

The GKZ operators are derived by (2.6)

$$\begin{aligned} \mathcal{Z}_0 &= \sum_{i=0}^5 \vartheta_i + 1, \quad \mathcal{Z}_i = \vartheta_{i+1} - \vartheta_1, \quad i = 1, 2, 3, \quad \mathcal{Z}_4 = \vartheta_5 - 4\vartheta_1 \\ \mathcal{L}_1 &= \frac{\partial}{\partial a_1} \frac{\partial}{\partial a_2} \frac{\partial}{\partial a_3} \frac{\partial}{\partial a_4} \frac{\partial}{\partial a_5} - \left(\frac{\partial}{\partial a_0} \right)^8 \end{aligned} \quad (5.9)$$

where $\mathcal{Z}_0$ represents the invariance of P under overall rescaling, $\mathcal{Z}_i$'s relate to the torus symmetry, and $\mathcal{L}_i$'s relate to the symmetries among monomials consisting of P . And all GKZ operators above annihilate the period matrix and determine the mirror maps and superpotential.

After blowing up X along S , the blow-up manifold X' is obtained as the complete intersection

$$\begin{aligned} X': \quad P &= a_1 x_1^8 + x_2^8 + x_3^8 + x_4^8 + x_5^2 + a_0 (x_1 x_2 x_3 x_4 x_5) = 0 \\ Q &= y_1 (a_9 x_1^8 + a_{10} x_5^2) - y_2 (a_6 (x_1 x_2 x_3 x_4 x_5)^2 + a_7 x_1 x_2 x_3 x_4 x_5^3 + a_8 x_5^4) \end{aligned} \quad (5.10)$$

where a_i 's are free complex-valued coefficients. By observation on above defining equations, GKZ system of X' is obtained as complement to GKZ system of X .

$$\begin{aligned}
 \mathcal{Z}'_0 &= \sum_{i=0}^5 \vartheta_i + 1, \quad \mathcal{Z}'_1 = \sum_{i=6}^{10} \vartheta_i, \quad \mathcal{Z}'_2 = -\vartheta_1 + \vartheta_2 + \vartheta_7 + 2\vartheta_8 - \vartheta_{10}, \quad \mathcal{Z}'_3 = -\vartheta_1 + \vartheta_3 - \vartheta_{10}, \\
 \mathcal{Z}'_4 &= -\vartheta_1 + \vartheta_4 - \vartheta_{10}, \quad \mathcal{Z}'_5 = -4\vartheta_1 + \vartheta_5 - 4\vartheta_{10}, \quad \mathcal{Z}'_6 = -\vartheta_6 - \vartheta_7 - \vartheta_8 + \vartheta_9 + \vartheta_{10} \\
 \mathcal{L}'_1 &= \prod_{i=2}^4 \frac{\partial}{\partial a_i} \left(\frac{\partial}{\partial a_7} \right)^4 \frac{\partial}{\partial a_{10}} - \left(\frac{\partial}{\partial a_0} \right)^3 \left(\frac{\partial}{\partial a_6} \right)^4 \frac{\partial}{\partial a_9}, \quad \mathcal{L}'_2 = \frac{\partial}{\partial a_6} \frac{\partial}{\partial a_8} - \left(\frac{\partial}{\partial a_7} \right)^2 \\
 \mathcal{L}'_3 &= \frac{\partial}{\partial a_5} \frac{\partial}{\partial a_7} - \frac{\partial}{\partial a_0} \frac{\partial}{\partial a_8}, \quad \mathcal{L}'_4 = \frac{\partial}{\partial a_1} \frac{\partial}{\partial a_9} - \frac{\partial}{\partial a_0} \frac{\partial}{\partial a_{10}}
 \end{aligned} \tag{5.11}$$

where $\mathcal{Z}'_0, \mathcal{Z}'_1$ are associated with the overall rescaling with respect to $P = 0, Q = 0$, $\mathcal{Z}'_i, i = 2, \dots, 5$ are related to the torus symmetry, and $\mathcal{L}'_3, \mathcal{L}'_4$ incorporate the parameter $a_6, \dots, a_{10}$ that are associated with the moduli of the curve S . The new manifold X' is describe by the following charge vectors

	0	1	2	3	4	5	6	7	8	9	10
l'_1	-3	0	1	1	1	0	-4	4	0	-1	1
l'_2	0	0	0	0	0	0	1	-2	1	0	0
l'_3	-1	0	0	0	0	1	0	1	-1	0	0
l'_4	-1	1	0	0	0	0	0	0	0	-1	1

The local coordinates z^i 's on the complex structure moduli space of X' by (2.7),

$$z_1 = \frac{a_3 a_4 a_5 a_7^4 a_{10}}{a_0^4 a_9}, \quad z_2 = \frac{a_6 a_8}{a_7^2}, \quad z_3 = \frac{a_5 a_7}{a_0 a_8}, \quad z_4 = \frac{a_1 a_9}{a_0 a_{10}} \tag{5.12}$$

Next, we convert the $\mathcal{L}'_i$ in (3.37) operators to Picard-Fuchs operators $\mathcal{D}_i$, from differential equations about $a_j (j = 0, \dots, 10)$ to those about $z_j (j = 1, \dots, 4)$,

$$\begin{aligned}
 \mathcal{D}_1 &= \theta_1^3 (4\theta_1 - 2\theta_2 + \theta_3)^4 (\theta_1 - \theta_4) - z_1 (-4\theta_1 + \theta_2)^4 (-\theta_1 + \theta_4) \prod_{i=1}^4 (-3\theta_1 - \theta_3 - \theta_4 - i), \\
 \mathcal{D}_2 &= (-2\theta_1 + \theta_2)(\theta_2 - \theta_3) - z_2 (2\theta_2 - 2\theta_2 + \theta_3)^2 \\
 \mathcal{D}_3 &= (\theta_1 + \theta_3 - \theta_4)(2\theta_1 - 2\theta_2 + \theta_3) - z_3 (\theta_2 - \theta_3)(-4\theta_1 - \theta_3 - 1), \\
 \mathcal{D}_4 &= \theta_4(-\theta_1 + \theta_4) - z_4 (\theta_1 + \theta_3 - \theta_4)(\theta_1 - \theta_4) \\
 &\dots
 \end{aligned} \tag{5.13}$$

where $\theta_i = z_i \frac{\partial}{\partial z_i}$'s are the logarithmic derivatives.

5.2.2 Brane superpotential and disk instantons

Along the line of 2.1, we solve the differential equations (5.13) at $z_i \rightarrow 0$ and identify the mirror maps and superpotential. The fundamental period of X is

$$\omega_0 = 1 + 1680 z_1 z_2^4 z_3^4 z_4 + 32432400 z_1^2 z_2^8 z_3^8 z_4^2 + \mathcal{O}(z^{20}).$$

The single logarithmic solutions are

$$\begin{aligned}
 \omega_{1,1} &= \omega_0 \log(z_1) - 4z_2 - 6z_2^2 - \frac{40}{3}z_2^3 + 4z_2z_3 + 4z_2^2z_3 + z_4 + 24z_1z_4 - 96z_1z_2z_4 + 48z_1z_2^2z_4 \\
 &\quad - \frac{1}{2}z_4^2 + \frac{1}{3}z_4^3 + \mathcal{O}(z^3) \\
 \omega_{1,2} &= \omega_0 \log(z_2) + 2z_2 + 3z_2^2 + \frac{20}{3}z_2^3 + z_3 - 2z_2z_3 - 2z_2^2z_3 - \frac{1}{2}z_3^2 + z_2z_3^2 + \frac{1}{3}z_3^3 - 6z_1z_4 \\
 &\quad + 24z_1z_2z_4 + \mathcal{O}(z^3) \\
 \omega_{1,3} &= \omega_0 \log(z_3) - z_2 - \frac{3}{2}z_2^2 - \frac{10}{3}z_2^3 - \frac{35}{4}z_2^4 - z_3 + z_2z_3 + z_2^2z_3 + 2z_2^3z_3 + \frac{1}{2}z_3^2 - z_2z_3^2 \\
 &\quad - \frac{1}{2}z_2^2z_3^2 - \frac{1}{3}z_3^3 + z_2z_3^3 + \frac{1}{4}z_3^4 + \mathcal{O}(z^4) \\
 \omega_{1,4} &= \omega_0 \log(z_4) - z_4 + \frac{1}{2}z_4^2 - \frac{1}{3}z_4^3 + \frac{1}{4}z_4^4 + \mathcal{O}(z^4)
 \end{aligned}$$

by which the open-closed mirror maps are inverse series of flat coordinates

$$\begin{aligned}
 z_1 &= q_1 + 4q_1q_2 + 6q_1q_2^2 + 4q_1q_2^3 + q_1q_2^4 - 8q_1q_2q_3 - 24q_1q_2^2q_3 - q_1q_4 - 24q_1^2q_4 - 4q_1q_2q_4 \\
 &\quad - 72q_1^2q_2q_4 - 6q_1q_2^2q_4 + 8q_1q_2q_3q_4 + 24q_1^2q_4^2 + \mathcal{O}(q^4) \\
 z_2 &= q_2 - 2q_2^2 + 3q_2^3 - 4q_2^4 - q_2q_3 + 5q_2^2q_3 - 13q_2^3q_3 - 3q_2^2q_3^2 + 6q_1q_2q_4 - 24q_1q_2^2q_4 + \mathcal{O}(q^4) \\
 z_3 &= q_3 + q_2q_3 + q_3^2 + q_2^2q_3^2 + q_3^3 + q_3^4 + 6q_1q_2q_3q_4 + \mathcal{O}(q^4) \\
 z_4 &= q_4 + q_4^2 + q_4^3 + q_4^4 + \mathcal{O}(q^4)
 \end{aligned}$$

Then we construct a linear combination of double logarithmic solutions and insert the inverse mirror maps,

$$\mathcal{W}_{\text{brane}} = 2t_1^2 + 4t_1t_4 + 2t_4^2 + \sum_N N_{d_1,d_2,d_3,d_4} \text{Li}_2(q_1^{d_1} q_2^{d_2} q_3^{d_3} q_4^{d_4}) \quad (5.14)$$

Invariants of the form $N_{m,m,m,n}$ are summarized in table 18, where the rows and columns are labelled by m and n, respectively.

When $a_7^2 = a_6a_8$, the two individual branes coincide. We obtain a new set of charge vectors,

	0	1	2	3	4	5	6	8	9	10
l_1^c	-3	0	1	1	1	0	-2	2	-1	1
l_2^c	-2	0	0	0	0	2	1	-1	0	0
l_3^c	-1	1	0	0	0	0	0	0	1	-1

by which the new complex structure moduli space coordinates are

$$z_1^c = \frac{a_2a_3a_4a_8^2a_{10}}{a_0^3a_6^2a_9}, \quad z_2^c = \frac{a_5^2a_6}{a_0^2a_8}, \quad z_3^c = \frac{a_1a_9}{a_0a_{10}} \quad (5.15)$$

Similar to the separate case, the superpotential is constructed as linear combination of double logarithmic solutions and Ooguri-Vafa invariants are extracted in table 18.

6 Summary and conclusions

In this work, we calculate the superpotentials in $d=4$ $N=1$ supersymmetric field theories arising from IIA D6-branes wrapping special Lagrangian three cycles of Calabi-Yau three-fold. The special Lagrangian three-cycles with non-trivial topology are mirror to obstructed rational curves, which correspond to the brane excitation about the supersymmetric minimum. We consider a five brane wrapping a rational curve that coincides with a toric curve S at certain locus of the deformation space $\mathcal{M}(S)$. S is described by the intersection of two divisors $D_1 \cap D_2$ and its unobstructed deformation space match with the obstructed deformation space of the rational curve wrapped by the five brane. After blowing up, the toric curve S is replaced by a exceptional divisor E without introducing new degree of freedom. All the complex structure moduli and brane moduli are embedded into the complex moduli space of the blow-up new manifold, given as the complete intersection in the projective bundle $\mathcal{W} = \mathbb{P}(\mathcal{O}(D_1) \oplus \mathcal{O}(D_2))$.

From observation on the defining equation of X' , we obtain the Picard-Fuchs equations that annihilate the period matrix defined by the natural pairing between the elements of relative homology $H_*(X, S)$ and cohomology $H^*(X, S)$. Via GKZ system of X' , the system of Picard-Fuchs equations are solved at $z_i \rightarrow 0$. The single logarithmic solutions are interpreted as mirror maps and specific linear combinations of double logarithmic solutions are the B-brane superpotentials. Using multi-cover formula and inverse mirror maps, the Ooguri-Vafa invariants are extracted from A-model side and interpreted as counting disk instantons, i.e. holomorphic disks with boundary in a nontrivial homology class on a special Lagrangian submanifold. It would be interesting to directly extract these invariants on the A-model side directly by adequate localization techniques.

Acknowledgments

This work is supported by NSFC(11475178).

A Blow-up geometry of $X_{[4,4]}^{(112|112)}$ and $X_{[6,6]}^{(123|123)}$

A.1 $X_{[4,4]}^{(112|112)}$

Polyhedron vertices and charge vectors associated to A-model manifold of X^* and A-branes on it (table 6): the GKZ system of X as follows

$$\begin{aligned} \mathcal{Z}_0 &= \sum_{i=0}^6 \vartheta_i + 1, & \mathcal{Z}_1 &= -\vartheta_1 + \vartheta_2, & \mathcal{Z}_2 &= -2\vartheta_1 + \vartheta_3, \\ \mathcal{Z}_3 &= -\vartheta_1 + \vartheta_4, & \mathcal{Z}_4 &= -2\vartheta_1 + \vartheta_5, & \mathcal{Z}_5 &= -2\vartheta_1 + \vartheta_6 \end{aligned} \quad (\text{A.1})$$

$$\mathcal{L}_1 = \frac{\partial}{\partial a_1} \frac{\partial}{\partial a_2} \left(\frac{\partial}{\partial a_3} \right)^2 \frac{\partial}{\partial a_4} \frac{\partial}{\partial a_5} \left(\frac{\partial}{\partial a_6} \right)^2 - \left(\frac{\partial}{\partial a_{0,1}} \right)^3 \left(\frac{\partial}{\partial a_{0,2}} \right)^3$$

	Δ^*					l		$\hat{l}^1$	$\hat{l}^2$
$v_{0,1}^*$	0	0	0	0	0	-4	$x_1 x_2 x_3$	-1	-1
$v_{0,2}^*$	0	0	0	0	0	-4	$x_4 x_5 x_6$	0	0
v_1^*	-1	-2	-1	-1	-2	1	x_1^4	0	0
v_2^*	1	0	0	0	0	1	x_2^4	1	0
v_3^*	0	1	0	0	0	2	x_3^4	0	1
v_4^*	0	0	1	0	0	1	x_4^4	0	0
v_5^*	0	0	0	1	0	1	x_5^4	0	0
v_6^*	0	0	0	0	1	2	x_6^2	0	0

Table 6. Toric data of A-model manifold.

After blowing up X along the curve specified by $\hat{l}^1, \hat{l}^2$, the GKZ system of blow-up manifold X' :

$$\begin{aligned}
 \mathcal{Z}'_0 &= \sum_{i=0}^6 \vartheta_i + 1, & \mathcal{Z}'_1 &= \sum_{i=7}^{10} \vartheta_i, & \mathcal{Z}'_2 &= -\vartheta_1 + \vartheta_2 - \vartheta_8 + \vartheta_{10}, & \mathcal{Z}'_3 &= -2\vartheta_1 + \vartheta_3 - 2\vartheta_8, \\
 \mathcal{Z}'_4 &= -\vartheta_1 + \vartheta_4 - \vartheta_8, & \mathcal{Z}'_5 &= -\vartheta_1 + \vartheta_5 - \vartheta_8, & \mathcal{Z}'_6 &= -2\vartheta_1 + \vartheta_6 - 2\vartheta_8, & \mathcal{Z}'_7 &= -\vartheta_7 - \vartheta_8 + \vartheta_9 + \vartheta_{10} \\
 \mathcal{L}'_1 &= \frac{\partial}{\partial a_1} \frac{\partial}{\partial a_2} \left(\frac{\partial}{\partial a_3} \right)^2 \frac{\partial}{\partial a_4} \frac{\partial}{\partial a_5} \left(\frac{\partial}{\partial a_6} \right)^2 - \left(\frac{\partial}{\partial a_{0,1}} \right)^3 \left(\frac{\partial}{\partial a_{0,2}} \right)^3, \\
 \mathcal{L}'_2 &= \frac{\partial}{\partial a_1} \frac{\partial}{\partial a_7} - \frac{\partial}{\partial a_{0,1}} \frac{\partial}{\partial a_8}, & \mathcal{L}'_3 &= \frac{\partial}{\partial a_2} \frac{\partial}{\partial a_9} - \frac{\partial}{\partial a_{0,1}} \frac{\partial}{\partial a_{10}}
 \end{aligned} \tag{A.2}$$

Toric data for the enhanced polyhedron associated to the X' :

	Δ'								l'_1	l'_2	l'_3	
$v'_{0,1}$	1	0	0	0	0	0	0	0	-1	-1	-1	$w'_{0,1} = x_1 x_2 x_3$
$v'_{0,2}$	1	0	0	0	0	0	0	0	-4	0	0	$w'_{0,2} = x_4 x_5 x_6$
v'_1	1	0	-1	-2	-1	-1	-2	0	-1	1	0	$w'_1 = x_1^4$
v'_2	1	0	1	0	0	0	0	0	0	0	1	$w'_2 = x_2^4$
v'_3	1	0	0	1	0	0	0	0	2	0	0	$w'_3 = x_3^4$
v'_4	1	0	0	0	1	0	0	0	1	0	0	$w'_4 = x_4^2$
v'_5	1	0	0	0	0	1	0	0	1	0	0	$w'_5 = x_5^4$
v'_6	1	0	0	0	0	0	1	0	2	0	0	$w'_6 = x_6^4$
v'_7	0	1	0	0	0	0	0	-1	-2	1	0	$w'_7 = y_1 w'_0$
v'_8	0	1	-1	-2	-1	-1	-2	-1	2	-1	0	$w'_8 = y_1 w'_2$
v'_9	0	1	0	0	0	0	0	1	-1	0	1	$w'_9 = y_2 w'_0$
v'_{10}	0	1	1	0	0	0	0	1	1	0	-1	$w'_{10} = y_2 w'_3$

Picard-Fuchs operators from above table:

$$\begin{aligned}
 \mathcal{D}_1 &= (2\theta_1)^2(\theta_1)^2(2\theta_1)^2(2\theta_1 - \theta_2)^2(\theta_1 - \theta_3) \\
 &\quad - z_1(-\theta_1 + \theta_2)(-2\theta_1 + \theta_2)^2(-\theta_1 + \theta_3)(-\theta_1 - \theta_2 - \theta_3 - 1) \prod_{i=1}^4 (-4\theta_1 - i) \\
 \mathcal{D}_2 &= (-\theta_1 + \theta_2)(-2\theta_1 + \theta_2) - z_2(-\theta_1 - \theta_2 - \theta_3 - 1)(2\theta_1 - \theta_2) \\
 \mathcal{D}_3 &= \theta_3(-\theta_1 + \theta_3) - z_3(-\theta_1 - \theta_2 - \theta_3 - 1)(\theta_1 - \theta_3) \\
 &\dots
 \end{aligned} \tag{A.3}$$

Solving above equations by GKZ-system, the unique power series solution, as well as the fundamental period of X , is

$$\begin{aligned}
 \omega_0 &= 1 + 144z_1z_2^2z_3 + 176400z_1^2z_2^4z_3^2 + 341510400z_1^3z_2^6z_3^3 + 811620810000z_1^4z_2^8z_3^4 \\
 &\quad + \mathcal{O}(z^{16})
 \end{aligned} \tag{A.4}$$

and four single logarithmic solutions

$$\begin{aligned}
 \omega_{1,1} &= \omega_0 \log(z_1) + 2z_2 - z_2^2 - 36z_1z_2^2 + \frac{2}{3}z_2^3 - \frac{1}{2}z_2^4 + 3150z_1^2z_2^4 + \frac{2}{5}z_2^5 + z_3 \\
 &\quad - 72z_1z_2z_3 + 492z_1z_2^2z_3 + 720z_1z_2^3z_3 - \frac{1}{2}z_3^2 + 360z_1z_2^2z_3^2 + \frac{1}{3}z_3^3 - \frac{1}{4}z_3^4 \\
 &\quad + \frac{1}{5}z_3^5 + \mathcal{O}(z^5) \\
 \omega_{1,2} &= \omega_0 \log(z_2) - z_2 + \frac{1}{2}z_2^2 - \frac{1}{3}z_2^3 + \frac{1}{4}z_2^4 - \frac{1}{5}z_2^5 + 36z_1z_2z_3 + 156z_1z_2^2z_3 \\
 &\quad - 360z_1z_2^3z_3 + \mathcal{O}(z^5) \\
 \omega_{1,3} &= \omega_0 \log(z_3) + 36z_1z_2^2 - z_3 + 156z_1z_2^2z_3 + \frac{1}{2}z_3^2 - 360z_1z_2^2z_3^2 - \frac{1}{3}z_3^3 \\
 &\quad + 360z_1z_2^2z_3^3 + \frac{1}{4}z_3^4 - \frac{1}{5}z_3^5 + \mathcal{O}(z^5)
 \end{aligned} \tag{A.5}$$

Upon the fundamental period and single logarithmic solutions, the inverse mirror maps are

$$\begin{aligned}
 z_1 &= q_1 - 2q_1q_2 + q_1q_2^2 + 36q_1^2q_2^2 - 72q_1^2q_2^3 - q_1q_3 + 2q_1q_2q_3 + 72q_1^2q_2q_3 - q_1q_2^2q_3 \\
 &\quad - 672q_1^2q_2^2q_3 - 72q_1^2q_2q_3^2 + \mathcal{O}(q^5) \\
 z_2 &= q_2 + q_2^2 + q_2^3 + q_2^4 + q_2^5 - 36q_1q_2^2q_3 - 192q_1q_2^3q_3 + \mathcal{O}(q^5) \\
 z_3 &= q_3 - 36q_1q_2^2q_3 + q_3^2 - 192q_1q_2^2q_3^2 + q_3^3 + \mathcal{O}(q^5)
 \end{aligned} \tag{A.6}$$

The brane superpotential as linear combination of double logarithmic solutions is

$$\mathcal{W}_{\text{brane}} = 4t_1t_2 + 6t_2^2 + 4t_2t_3 + \sum_N N_{d_1,d_2,d_3} \text{Li}_2(q_1^{d_1} q_2^{d_2} q_3^{d_3}) \tag{A.7}$$

and the disk instantons are presented in table 16.

	Δ^*					l		$\hat{l}^1$	$\hat{l}^2$
$v_{0,1}^*$	0	0	0	0	0	-6	$x_1 x_2 x_3$	0	0
$v_{0,2}^*$	0	0	0	0	0	-6	$x_4 x_5 x_6$	0	0
v_1^*	-2	-3	-1	-2	-3	1	x_1^6	-1	-1
v_2^*	1	0	0	0	0	2	x_2^3	1	0
v_3^*	0	1	0	0	0	3	x_3^2	0	1
v_4^*	0	0	1	0	0	1	x_4^6	0	0
v_5^*	0	0	0	1	0	2	x_5^3	0	0
v_6^*	0	0	0	0	1	3	x_6^2	0	0

Table 7. Toric Data of A-model side.

A.2 $X_{[6,6]}^{(123|123)}$

Polyhedron vertices and charge vectors A-model manifold X^* and A-branes on it (table 7): GKZ system of X as follows

$$\begin{aligned}
 \mathcal{Z}_0 &= \sum_{i=0}^6 \vartheta_i + 1, & \mathcal{Z}_1 &= -2\vartheta_1 + \vartheta_2, & \mathcal{Z}_2 &= -3\vartheta_1 + \vartheta_3, \\
 \mathcal{Z}_3 &= -\vartheta_1 + \vartheta_4, & \mathcal{Z}_4 &= -2\vartheta_1 + \vartheta_5, & \mathcal{Z}_5 &= -3\vartheta_5 + \vartheta_6 \\
 \mathcal{L}_1 &= \left(\frac{\partial}{\partial a_1}\right) \left(\frac{\partial}{\partial a_2}\right)^2 \left(\frac{\partial}{\partial a_3}\right)^3 \left(\frac{\partial}{\partial a_4}\right) \left(\frac{\partial}{\partial a_5}\right)^2 \left(\frac{\partial}{\partial a_6}\right)^3 - \left(\frac{\partial}{\partial a_{0,1}}\right)^6 \left(\frac{\partial}{\partial a_{0,2}}\right)^6
 \end{aligned} \tag{A.8}$$

After blowing up along the curve specified by $\hat{l}^1, \hat{l}^2$, the GKZ system of blow-up manifold X' :

$$\begin{aligned}
 \mathcal{Z}'_0 &= \sum_{i=0}^6 \vartheta_i + 1, & \mathcal{Z}'_1 &= \sum_{i=7}^{10} \vartheta_i, & \mathcal{Z}'_2 &= -2\vartheta_1 + \vartheta_2 - 2\vartheta_7 + \vartheta_8 - 2\vartheta_9, \\
 \mathcal{Z}'_3 &= -3\vartheta_1 + \vartheta_3 - 3\vartheta_7 - 3\vartheta_9 + \vartheta_{10}, & \mathcal{Z}'_4 &= -\vartheta_1 + \vartheta_4 - \vartheta_7 - \vartheta_9, & \mathcal{Z}'_5 &= -2\vartheta_1 + \vartheta_5 - 2\vartheta_7 - 2\vartheta_9, \\
 \mathcal{Z}'_6 &= -3\vartheta_1 + \vartheta_6 - 3\vartheta_7 - 3\vartheta_9, & \mathcal{Z}'_7 &= -\vartheta_7 - \vartheta_8 + \vartheta_9 + \vartheta_{10} \\
 \mathcal{L}'_1 &= \left(\frac{\partial}{\partial a_1}\right) \left(\frac{\partial}{\partial a_2}\right)^2 \left(\frac{\partial}{\partial a_3}\right)^3 \left(\frac{\partial}{\partial a_4}\right) \left(\frac{\partial}{\partial a_5}\right)^2 \left(\frac{\partial}{\partial a_6}\right)^3 - \left(\frac{\partial}{\partial a_{0,1}}\right)^6 \left(\frac{\partial}{\partial a_{0,2}}\right)^6, \\
 \mathcal{L}'_2 &= \frac{\partial}{\partial a_1} \frac{\partial}{\partial a_8} - \frac{\partial}{\partial a_3} \frac{\partial}{\partial a_7}, & \mathcal{L}'_3 &= \frac{\partial}{\partial a_1} \frac{\partial}{\partial a_{10}} - \frac{\partial}{\partial a_2} \frac{\partial}{\partial a_9}
 \end{aligned} \tag{A.9}$$

Toric data for the enhanced polyhedron associated to the X' 's mirror manifold:

	Δ'								l'_1	l'_2	l'_3	
$v'_{0,1}$	1	0	0	0	0	0	0	0	-6	0	0	$w'_{0,1} = x_1 x_2 x_3$
$v'_{0,2}$	1	0	0	0	0	0	0	0	-6	0	0	$w'_{0,2} = x_4 x_5 x_6$
v'_1	1	0	-2	-3	-1	-2	-3	0	-2	1	1	$w'_1 = x_1^6$
v'_2	1	0	1	0	0	0	0	0	3	0	-1	$w'_2 = x_2^3$
v'_3	1	0	0	1	0	0	0	0	5	-1	0	$w'_3 = x_3^2$
v'_4	1	0	0	0	1	0	0	0	1	0	0	$w'_4 = x_4^6$
v'_5	1	0	0	0	0	1	0	0	2	0	0	$w'_5 = x_5^3$
v'_6	1	0	0	0	0	0	1	0	3	0	0	$w'_6 = x_6^2$
v'_7	0	1	-2	-3	-1	-2	-3	-1	1	0	-1	$w'_7 = y_1 w'_0$
v'_8	0	1	1	0	0	0	0	-1	-1	0	1	$w'_8 = y_1 w'_2$
v'_9	0	1	-2	-3	-1	-2	-3	1	2	-1	0	$w'_9 = y_2 w'_0$
v'_{10}	0	1	0	1	0	0	0	1	-2	1	0	$w'_{10} = y_2 w'_3$

Picard-Fuchs equations from above table:

$$\begin{aligned}
 \mathcal{D}_1 &= (3\theta_1 - \theta_3)^3 (5\theta_1 - \theta_2)^5 \theta_1 (2\theta_1)^2 (3\theta_1)^3 (2\theta_1 - \theta_2)^2 (\theta_1 - \theta_3) \\
 &\quad - z_1 (-2\theta_1 + \theta_2 + \theta_3)^2 (-2\theta_1 + \theta_2)^2 (-\theta_1 + \theta_3) \prod_{i=1}^6 \prod_{j=1}^6 (-6\theta_1 - i)(-6\theta_1 - j) \\
 \mathcal{D}_2 &= (-2\theta_1 + \theta_2 + \theta_3)(-2\theta_1 + \theta_2) - z_2 (5\theta_1 - \theta_2)(2\theta_1 - \theta_2) \\
 \mathcal{D}_3 &= (-2\theta_1 + \theta_2 + \theta_3)(-\theta_1 + \theta_3) - z_3 (3\theta_1 - \theta_3)(\theta_1 - \theta_3) \\
 &\dots
 \end{aligned} \tag{A.10}$$

Solving above equations by GKZ-system, the unique power series solution, as well as the fundamental period of X , is

$$\begin{aligned}
 \omega_0 &= 1 + 3600 z_1 z_2^2 z_3 + 192099600 z_1^2 z_2^4 z_3^2 + 16679709446400 z_1^3 z_2^6 z_3^3 \\
 &\quad + 1791735431214128400 z_1^4 z_2^8 z_3^4 + \mathcal{O}(z^{16})
 \end{aligned} \tag{A.11}$$

and the single logarithmic solutions are

$$\begin{aligned}
 \omega_{1,1} &= \omega_0 \log(z_1) - 1200 z_1 z_2^2 - 1800 z_1 z_2 z_3 + 29640 z_1 z_2^2 z_3 + 10800 z_1 z_2^3 z_3 \\
 &\quad + 3600 z_1 z_2^2 z_3^2 + \mathcal{O}(z^5) \\
 \omega_{1,2} &= \omega_0 \log(z_2) + 900 z_1 z_2 z_3 + 3000 z_1 z_2^2 z_3 - 5400 z_1 z_2^3 z_3 + \mathcal{O}(z^5) \\
 \omega_{1,3} &= \omega_0 \log(z_3) + 1200 z_1 z_2^2 + 1800 z_1 z_2^2 z_3 - 3600 z_1 z_2^2 z_3^2 + \mathcal{O}(z^5)
 \end{aligned} \tag{A.12}$$

By the fundamental period and single logarithmic solutions, the inverse mirror maps are

$$\begin{aligned}
 z_1 &= q_1 + 1200 q_1^2 q_2^2 - 4243320 q_1^3 q_2^4 + 1800 q_1^2 q_2 q_3 - 29640 q_1^2 q_2^2 q_3 + \mathcal{O}(q^5) \\
 z_2 &= q_2 - 900 q_1 q_2^2 q_3 - 3000 q_1 q_2^3 q_3 + \mathcal{O}(q^5) \\
 z_3 &= q_3 - 1200 q_1 q_2^2 q_3 + 5683320 q_1^2 q_2^4 q_3 - 1800 q_1 q_2^2 q_3^2 + \mathcal{O}(q^5)
 \end{aligned} \tag{A.13}$$

The brane superpotential is

$$\mathcal{W}_{\text{brane}} = 4t_1 t_2 + 6t_2^2 + 4t_2 t_3 + \sum_N N_{d_1, d_2, d_3} \text{Li}_2(q_1^{d_1} q_2^{d_2} q_3^{d_3}) \quad (\text{A.14})$$

and the disk instantons are presented in table [16](#)

B Ooguri-Vafa invariants for two closed and two open moduli

B.1 Ooguri-Vafa invariants for $X_9^{(1,1,3,3,3)}$

k=0: Ooguri-Vafa Invariants $N_{k,m,k,m+n}$						
$N_{0,m,0,m+n}$	$n = 1$	2	3	4	5	6
$m = 0$	3	0	0	0	0	0
1	-3	-3	-3	-3	-3	-3
2	15	21	27	36	45	57
3	-120	-183	-279	-420	-618	*
4	1179	1944	3210	5250	*	*
5	-13572	-22983	-39771	*	*	*

k=1: Ooguri-Vafa Invariants $N_{k,m,k,m+n}$						
$N_{1,m,1,m+n}$	$n = 1$	2	3	4	5	6
$m = 0$	27	0	0	0	0	0
1	90	90	90	90	90	90
2	-684	-954	-1314	-1764	-2304	-2934
3	7470	11736	18486	28620	43218	*
4	-94644	-158022	-267768	-450270	*	*
5	1302120	2254968	3998718	*	*	*

k=2: Ooguri-Vafa Invariants $N_{k,m,k,m+n}$						
$N_{2,m,2,m+n}$	$n = 1$	2	3	4	5	6
$m = 0$	81	108	0	0	0	0
1	-1539	-1377	-1377	-1377	-1377	-1377
2	13554	19728	29079	41112	55917	73404
3	-204120	-333612	-553878	898587	-1412154	*
4	3351969	5781384	10209969	17848764	*	*
5	-56886543	-101222919	-185715828	*	*	*

k=3: Ooguri-Vafa Invariants $N_{k,m,k,m+n}$						
$N_{3,m,3,m+n}$	$n = 1$	2	3	4	5	6
$m = 0$	255	-984	729	0	0	0
1	18150	16764	14796	14796	14796	14796
2	-132492	-221262	-370224	-580212	-864306	-1168506
3	3063876	5345640	9647424	16885548	28230822	*
4	-68640885	-123720696	-231991668	-428397570	*	*
5	1903950048	2812011138	5315645308	*	*	*

Table 8. Ooguri-Vafa Invariants $N_{k,m,k,m+n}$ for $X_9^{(1,1,1,3,3)}$ with brane I at large volume. k and m label the class t_1, t_2 of X_9 and n labels the brane winding. Blue entries agree with table 5 of [8] and entries * exceed in this table the order of our calculation.

k=0: Ooguri-Vafa Invariants $N_{k,m,k,m+n}$						
$N_{0,m,0,m+n}$	$n = 1$	2	3	4	5	6
$m = 0$	54	0	0	0	0	0
1	54	-18	0	0	0	0
2	36	0	0	0	0	0
3	54	0	0	0	0	*
4	54	-18	0	0	*	*
5	36	0	0	*	*	*

k=1: Ooguri-Vafa Invariants $N_{k,m,k,m+n}$						
$N_{1,m,1,m+n}$	$n = 1$	2	3	4	5	6
$m = 0$	0	0	0	0	0	0
1	-108	36	0	0	0	0
2	2772	-1026	0	18	0	0
3	243756	-193050	100548	-33588	6696	*
4	2947320	-2801070	2212272	-1340010	*	*
5	23798376	-22631562	19965852	*	*	*

k=2: Ooguri-Vafa Invariants $N_{k,m,k,m+n}$						
$N_{2,m,2,m+n}$	$n = 1$	2	3	4	5	6
$m = 0$	0	0	0	0	0	0
1	270	-90	0	0	0	0
2	-11160	4104	0	-72	0	0
3	174960	-74358	162	2916	0	*
4	-7067304	6235344	-4430376	2634444	*	*
5	92080530	-71321472	602803296	*	*	*

k=3: Ooguri-Vafa Invariants $N_{k,m,k,m+n}$						
$N_{3,m,3,m+n}$	$n = 1$	2	3	4	5	6
$m = 0$	0	0	0	0	0	0
1	0	0	0	0	0	0
2	0	0	0	0	0	0
3	0	0	12	0	0	*
4	0	0	0	0	*	*
5	0	194724	-541296	*	*	*

Table 9. Ooguri-Vafa Invariants $N_{k,m,k,m+n}$ for $X_9^{(1,1,1,3,3)}$ with brane II at large volume. k and m label the class t_1, t_2 of X_9 and n labels the brane winding. Blue entries agree with table 6 of [8] and entries * exceed in this table the order of our calculation.

k=1: Ooguri-Vafa Invariants $N_{k,m+n,k,n}$						
$N_{1,m+n,1,n}$	$n = 0$	1	2	3	4	5
$m = 1$	72	-1728	-80460	-1075140	-9482724	-65006280
2	-36	17280	340092	3488940	25895214	156528234
3	-1224	-64800	-977832	-8913456	-61353288	-350009424
4	5508	176688	2301588	19441008	127166868	*
5	-15336	-398304	-4742280	-38004912	*	*
6	33948	787968	8853948	*	*	*

k=2: Ooguri-Vafa Invariants $N_{k,m+n,k,n}$						
$N_{2,m+n,2,n}$	$n = 0$	1	2	3	4	5
$m = 1$	-180	7020	-97686	2643372	-37415520	-6097896648
2	108	-5832	-588276	3924954	1797212016	-467071781328
3	-108	133488	368388	-441631440	-17699810580	-467071781328
4	-10944	-411264	75854232	4282520544	107873699400	*
5	61308	-6824952	-730782756	-24887701224	*	*
6	153828	74008728	4105498284	*	*	*

k=3: Ooguri-Vafa Invariants $N_{k,m+n,k,n}$						
$N_{3,m+n,3,n}$	$n = 0$	1	2	3	4	5
$m = 1$	0	0	0	0	0	0
2	0	0	0	0	0	0
3	-8	0	0	192	0	0
4	0	0	0	0	0	*
5	0	0	0	0	*	*
6	4	0	0	*	*	*

Table 10. Ooguri-Vafa Invariants $N_{k,m+n,k,n}$ for $X_9^{(1,1,1,3,3)}$ with brane II at large volume. k and m label the class t_1, t_2 of X_9 and n labels the brane winding. Blue entries agree with table 6 of [8] and entries * exceed in this table the order of our calculation.

B.2 Ooguri-Vafa invariants for $X_8^{(1,1,2,2,2)}$

		k=0: Ooguri-Vafa Invariants $N_{m+n,k,n,k}$			
$N_{0,m+n,0,n}$	$n=0$	1	2	3	4
$m=1$	-18	-216	-4892	-151264	-5681870
2	72	2624	100832	4214112	191601760
3	-486	-33048	-1797228	-94486824	-5033832642
4	4608	449280	31446656	1982842880	120829061120
5	-50850	-6447800	-550590500	-40388001000	*
6	614304	96158016	9679385952	*	*

		k=1: Ooguri-Vafa Invariants $N_{m+n,k,n,k}$			
$N_{1,m+n,1,n}$	$n=0$	1	2	3	4
$m=1$	-42	-6192	-733644	-79606336	-8211597214
2	774	137232	17797280	2037691872	218057272396
3	-14136	-2930112	-416393532	-50570319792	-5639510827530
4	258054	61211088	9500484384	1226211853424	142918383992888
5	-4712292	-1258590240	-212363598836	-29126855887056	*
6	86082552	25561830528	4666034521152	*	*

		k=2: Ooguri-Vafa Invariants $N_{m+n,k,n,k}$			
$N_{2,m+n,2,n}$	$n=0$	1	2	3	4
$m=1$	0	-1368	-1077372	-365059512	-86759396282
2	204	198576	75466800	19307887296	3955750079488
3	-19302	-11637432	-3670150356	-847235304168	-162420251412438
4	903132	485877600	144775961928	32293413751824	6042541334443616
5	-32358618	-16917337440	-4997647584648	-1110817713352920	*
6	995862804	525552662880	157318976664144	*	*

		k=3: Ooguri-Vafa Invariants $N_{m+n,k,n,k}$			
$N_{3,m+n,3,n}$	$n=0$	1	2	3	4
$m=1$	0	0	0	-55820	-150679552
2	0	11744	32336576	243593333568	10806351662272
3	-1566	-5076	-4570885188	-2261841187392	-932100832928880
4	413766	596077920	374635723	152201297231776	57296094052931103
5	-38831778	-42101500720	-22526450326120	-9573014306940528	*
6	2350574802	2241417150432	1107534104102880	*	*

Table 11. Ooguri-Vafa Invariants $N_{k,m+n,k,n}$ for $X_8^{(1,1,2,2,2)}$. k and m label the class t_1, t_2 of X_8 and n labels the brane winding. Entries * exceed in this table the order of our calculation.

B.3 Ooguri-Vafa invariants for $X_{12}^{(1,1,2,2,6)}$

k=0: Ooguri-Vafa Invariants $N_{m,n,k}^I$						
$N_{m,n,0,0}$	$n=0$	1	2	3	4	5
$m=0$	0	3	-12	81	-768	8475
1	6	24	-150	1536	-18876	255192
2	-3	51	-828	12789	-210525	3567759
3	0	0	-2448	62424	-1424304	30845952
4	0	-51	-4284	200175	-6560640	185328198
5	0	-24	-3822	446208	-21903294	824369400
6	0	-3	0	707832	-55037712	2824014780
7	0	0	3822	785040	-106603500	7651196928
8	0	0	4284	527133	-161311872	16706608653

k=1: Ooguri-Vafa Invariants $N_{m,n,k}^I$						
$N_{m,n,1,1}$	$n=0$	1	2	3	4	5
$m=0$	0	7	-129	2356	-43009	785382
1	0	60	-1812	46308	-1087560	24288552
2	0	231	-11688	422769	-12912432	355724019
3	0	0	-44796	2369184	-95605692	3282800160
4	0	-231	-118617	9172275	-495951000	21450278046
5	0	-60	-233400	26397816	-1925808876	105855800388
6	0	-7	0	59555823	-5856840424	411529509612
7	0	0	233400	111805968	-14462333760	1300043526960
8	0	0	118617	179694201	-29956661529	3422531345265

Table 12. Ooguri-Vafa Invariants $N^I(m, n, k, k)$ for $X_{12}^{(1,1,2,2,6)}$ at large volume. Blue result agree with table 1 of [10] and entries * exceed in this table the order of our calculation.

k=2: Ooguri-Vafa Invariants $N_{m,n,k}^I$							
$N_{m,n,2,2}$	$n=0$	1	2	3	4	5	6
$m=0$	0	0	-34	3172	-150522	5393103	-165977134
1	0	0	-438	62580	-3849906	168975888	-6150177030
2	0	0	-2556	576165	-46702074	2531136435	-109516928532
3	0	0	-8592	3273720	-356493366	24094632336	-1247189399850
4	0	0	-20250	12926751	-1922497884	163696266396	-10205528854440
5	0	0	-43854	38066712	-7813282584	846085289856	-63961350090048
6	0	0	0	87483723	-24949597038	3465967380876	-319764160565976
7	0	0	43854	166124856	-64549577412	11584119074256	-1712597040576252
8	0	0	20250	288193197	-139132564554	43650289626	*

k=3: Ooguri-Vafa Invariants $N_{m,n,k}^I$							
$N_{m,n,1,1}$	$n=0$	1	2	3	4	5	6
$m=0$	0	0	0	261	68961	6471963	-391762467
1	0	0	0	4992	-1750680	203030160	-14586604596
2	0	0	0	42489	-20943672	3047732583	-261743302824
3	0	0	0	215676	-156884940	29101608144	-3011532345432
4	0	0	0	738051	-827067552	198520743132	-24958437292476
5	0	0	0	1834944	-3278132892	1031301949	-191124519232428
6	0	0	0	3543912	-10209175968	5273411151753	*
7	0	0	0	5887176	-32716895652	*	*
8	0	0	0	10856700	*	*	*

Table 13. Ooguri-Vafa Invariants $N^I(m, n, k, k)$ for $X_{(1,1,2,2,6)}$. Blue entries agree with table 1 of [10] and entries * in this table exceed the order of our calculation.

k=0: Ooguri-Vafa Invariants $N_{m,n,k}^{II}$							
$N_{m,n,0,0}$	$n=0$	1	2	3	4	5	6
$m=0$	0	2	-11	90	-956	11470	-148104
1	0	18	-144	1728	-23688	347634	-5319648
2	0	90	-864	14850	-268794	4914882	-90121788
3	0	0	-3072	76626	-1869000	43209696	-959687352
4	0	-90	-7983	268938	-8964558	265751964	-7224918462
5	0	-18	-20016	694368	-31713192	1219991850	-41011490232
6	0	-2	0	1418040	-86560046	4358141460	-182848393872
7	0	0	20016	2551068	-189151560	12482067168	-658806706584
8	0	0	7983	5220558	-343903860	29357148258	-1959719830479

k=1: Ooguri-Vafa Invariants $N_{m,n,k}^{II}$							
$N_{m,n,1,1}$	$n=0$	1	2	3	4	5	6
$m=0$	0	-2	0	408	-12976	318240	-7064886
1	0	-18	0	8136	-331560	9921366	-260033616
2	0	-90	0	76698	-4015620	147332466	-4580847000
3	0	0	0	452268	-30639624	1387245600	-51445509768
4	0	90	0	1877238	-165558276	9310082616	-41411436578
5	0	18	0	5834448	-676162008	47508506460	-2548090964232
6	0	2	0	13963950	-2176167772	192171282588	-12488328860112
7	0	0	0	27159444	-5689981368	634779163584	-50184535778280
8	0	0	0	54249498	-12403534176	1752461446878	-169030073870352

Table 14. Ooguri-Vafa Invariants $N_{II}^{II}(m, n, k, k)$ for $X_{12}^{(1,1,2,2,6)}$. Blue entries agree with table 2 of [10] and entries * in this table exceed the order of our calculation.

k=2: Ooguri-Vafa Invariants $N_{m,n,k}^{II}$						
$N_{m,n,2,2}$	$n=0$	1	2	3	4	5
$m=0$	0	0	11	-408	0	539430
1	0	0	144	-8136	0	17027136
2	0	0	864	-76698	0	258298074
3	0	0	3072	-452268	0	2503487904
4	0	0	7983	-1877238	0	17412238368
5	0	0	20016	-5834448	0	92608329600
6	0	0	0	-13963950	0	391990737708
7	0	0	-20016	-27159444	0	1357019627616
8	0	0	-7983	-54249498	0	4136493648546

k=3: Ooguri-Vafa Invariants $N_{m,n,k}^{II}$						
$N_{m,n,3,3}$	$n=0$	1	2	3	4	5
$m=0$	0	0	0	-90	12976	-539430
1	0	0	0	-1728	331560	-17027136
2	0	0	0	-14850	4015620	-258298074
3	0	0	0	-76626	30639624	-2503487904
4	0	0	0	-268938	165558276	-17412238368
5	0	0	0	-694368	676162008	-92608329600
6	0	0	0	-1418040	2176167772	-351774546534
7	0	0	0	-2551068	5200228704	-1014274703904
8	0	0	0	-6391956	8448740559	*

Table 15. Ooguri-Vafa Invariants $N_{m,n,k}^{II}$ for $X_{12}^{(1,1,2,2,6)}$. Blue entries agree with table 2 of [10] and entries * in this table exceed the order of our calculation.

C Ooguri-Vafa invariants for one closed and two open moduli

Ooguri-Vafa Invariants $N_{m,m+n,m}$ for $\mathbb{P}_{1111}[3, 3]$				
$N_{m,m+n,m}$	$n = 0$	1	2	3
$m = 0$	0	-12	0	0
1	120	108	-288	60
2	-2100	11664	9612	-33876
3	70440	-525264	2066280	1751112
4	-3191280	29727324	-139852800	478494708

Ooguri-Vafa Invariants $N_{m,m+n,m}$ for $\mathbb{P}_{11212}[4, 4]$				
$N_{m,m+n,m}$	$n = 0$	1	2	3
$m = 0$	0	-16	0	0
1	552	1120	-2544	1056
2	-39264	281280	539392	-1693632
3	5532984	-52350528	280580256	557960544
4	-1043002176	12346010064	-77352996864	372579061488

Ooguri-Vafa Invariants $N_{m,m+n,m}$ for $\mathbb{P}_{12323}[6, 6]$				
$N_{m,m+n,m}$	$n = 0$	1	2	3
$m = 1$	0	-16	0	0
1	27240	72800	-144240	53280
2	-93024240	697855680	1746803968	-4807419840
3	633116600568	-6085353582144	34589091686304	86687436861792
4	-5741552410002720	68585896764953040	-442591883444977920	2274867202698155760

Table 16. Ooguri-Vafa Invariants for $\mathbb{P}_{[3,3]}^{(111|111)}$, $\mathbb{P}_{[4,4]}^{[112|112]}$ and $\mathbb{P}_{[6,6]}^{[123|123]}$ at large volume. The entries * in this table exceed the order of our calculation.

D Ooguri-Vafa invariants for one closed and three open moduli

U(1) Ooguri-Vafa Invariants $N_{m,m,m,n}$						
$N_{m,m,m,n}$	$n = 0$	1	2	3	4	5
$m = 0$	0	0	0	0	0	0
1	0	-4896	0	0	0	0
2	0	0	1729728	0	0	0
3	0	0	0	-1530550656	0	0
4	0	0	0	0	1882007669502	0

U(2) Ooguri-Vafa Invariants $N_{m,m,n}$						
$N_{m,m,n}$	$n = 0$	1	2	3	4	5
$m = 0$	0	0	0	0	0	0
1	0	4832	0	0	0	0
2	0	0	6266104	0	0	0
3	0	0	0	$\frac{55336185568}{3}$	0	0
4	0	0	0	0	75779426010282	-3443691801600

Table 17. Ooguri-Vafa invariant of Sextic Hypersurface.

U(1) Ooguri-Vafa Invariants $N_{m,m,m,n}$						
$N_{m,m,m,n}$	$n = 0$	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$
$m = 0$	0	0	0	0	0	0
1	0	-3328	0	0	0	0
2	0	0	2030336	0	0	0
3	0	0	0	-3115785728	0	0
4	0	0	0	237081600	*	*

U(2) Ooguri-Vafa Invariants $N_{m,m,n}$						
$N_{m,m,n}$	$n = 0$	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$
$m = 0$	0	0	0	0	0	0
1	0	-5504	0	0	0	0
2	0	0	9520016	0	0	0
3	0	0	0	$\frac{-430429613824}{15}$	0	0
4	0	0	0	0	$\frac{1000605023457608}{7}$	-9744720529920

Table 18. Ooguri-Vafa invariant of Octic Hypersurface.

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