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Power corrections to the CP-violation parameter ε_K

**M. Ciuchini,^a E. Franco,^b V. Lubicz,^{c,a} G. Martinelli,^{d,b} L. Silvestrini^b
and C. Tarantino^{c,a}**

^a*INFN, Sezione di Roma Tre,
Via della Vasca Navale 84, 00146 Rome, Italy*

^b*INFN, Sezione di Roma,
Piazzale Aldo Moro 2, 00185 Rome, Italy*

^c*Dipartimento di Matematica e Fisica, Università Roma Tre,
Via della Vasca Navale 84, 00146 Rome, Italy*

^d*Dipartimento di Fisica, Università La Sapienza,
Piazzale Aldo Moro 2, 00185 Rome, Italy*

E-mail: ciuchini@roma3.infn.it, enrico.franco@roma1.infn.it,
lubicz@fis.uniroma3.it, guido.martinelli@roma1.infn.it,
luca.silvestrini@roma1.infn.it, tarantino@fis.uniroma3.it

ABSTRACT: We present the calculation of the short-distance power corrections to the CP-violation parameter ε_K coming from dimension-8 operators in the $\Delta S = 2$ effective Hamiltonian. A first estimate of this contribution, obtained for large- N_c and in the chiral limit, was provided in ref. [1]. Here we evaluate and include the $\mathcal{O}(m_K^2/m_c^2)$ and $\mathcal{O}(\Lambda_{\text{QCD}}^2/m_c^2)$ contributions that, a priori, could induce $\mathcal{O}(1)$ corrections to previous estimates, as m_K is numerically of order Λ_{QCD} . Our computation shows that there are several dimension-8 operators besides the one considered before. Their effect on ε_K , however, accidentally cancels out to a large extent, leaving the final correction at the level of 1%.

KEYWORDS: CP violation, Kaon Physics

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Contents

1	Introduction	1
2	Outline of the calculation	2
3	Matching	5
4	Matrix elements in the VIA	7
5	A possible approach to compute non-local contributions at $\mathcal{O}(\alpha_s p^2/m_c^2)$	10
6	Result for the power corrections to ε_K	11
A	Useful relations coming from the equations of motion	13
B	Useful Fierz relations	13
C	Matching with the external state made up of four quarks and a gluon	14

1 Introduction

The parameter ε_K , which describes indirect CP-violation in the $K^0 - \bar{K}^0$ system, represents one of the most interesting observables in Flavor Physics. It plays an important role in the Unitarity Triangle Analysis both within the Standard Model and beyond, see [2–4], references therein and [5]. As the phenomenon of $K^0 - \bar{K}^0$ mixing is a loop process further suppressed by the GIM mechanism, ε_K turns out to be a powerful constraint on New Physics models, for which it is important to have experimental and theoretical uncertainties well under control.

On the one hand ε_K is experimentally measured with a 0.4% accuracy, $\varepsilon_K^{\text{exp}} = (2.228 \pm 0.011) \cdot 10^{-3}$ [6], on the other hand, the theoretical accuracy of the SM prediction is approaching a few percent level, mainly thanks to the improvement of the lattice determination of the relevant bag parameter B_K [7]. Given the improved accuracy, the ξ term and the deviation of the phase ϕ_ε from 45° appearing in the theoretical expression

$$\varepsilon_K = e^{i\phi_\varepsilon} \sin \phi_\varepsilon \left(\frac{\text{Im} M_{12}}{\Delta m_K} + \xi \right), \quad (1.1)$$

are not negligible [8]. Furthermore, the dominant long-distance contribution to M_{12} , due to the exchange of two pions, has been evaluated in [9]. The inclusion of these three contributions gives a 6% reduction of the predicted central value of ε_K . Another improvement producing a 2% reduction of the central value is the inclusion of the available NNLO QCD

corrections to the Wilson coefficients of the $\Delta S = 2$ effective Hamiltonian [10, 11]. We have not included these corrections, however, since the relevant matrix element, B_K , computed on the lattice is matched to the $\overline{MS}$ coefficient at the NLO only. While the NNLO result for the Wilson coefficients is a crucial step to reduce the theoretical uncertainty of ε_K , at present there is no gain in evaluating the Wilson coefficient with NNLO accuracy when the accuracy of the matrix element is only at NLO: the overall uncertainty on ε_K remains at the NLO level in any case. In this respect, the perturbative calculation of the NNLO matching of B_K from the non-perturbative RI-MOM/SMOM schemes used in lattice calculations to the $\overline{MS}$ scheme would be welcome. As discussed in ref. [12], the NNLO QCD correction for the charm contribution is actually not needed.

Given the increasing precision of the theoretical ingredients entering ε_K , it is becoming important to include all terms expected to contribute to the theoretical evaluation of ε_K at the percent level. In this paper, we focus on the power corrections due to the finite value of the charm quark mass, denoted as δ_{m_c} , coming from dimension-8 operators in the effective Hamiltonian. The naive dimensional estimate of δ_{m_c} is of $\mathcal{O}(m_K^2/m_c^2) \sim 15\%$. Its size, however, is reduced to about 2–3%, because the dominant top quark contribution to ε_K is unaffected by these corrections. A first estimate of δ_{m_c} , obtained for large- N_c and in the SU(3) chiral limit, has been presented in [1]. In these limits, there is only one dimension-8 operator contributing to δ_{m_c} [13]. Numerically, ref. [1] finds $\delta_{m_c} \sim 0.5 - 1\%$ and it has been neglected in phenomenological analyses up to now. $\mathcal{O}(1)$ corrections to this estimate may, however, be expected, because m_K is numerically of order Λ_{QCD} . In addition, in the large- N_c limit, an operator non vanishing in the chiral limit is also discarded. Indeed the full calculation, presented in this paper, shows that there are several new operators contributing to δ_{m_c} . We find, however, that their effect on δ_{m_c} accidentally cancels out to a large extent, leaving the final correction at the level of 1%.

The paper is organized as follows. In section 2 we discuss the $\Delta S = 2$ effective Hamiltonian relevant for the calculation, including dimension-8 operators. In section 3 we describe the matching procedure followed to compute the Wilson coefficients of the operators entering δ_{m_c} . In section 4 we present the evaluation of the matrix elements of the relevant operators in the vacuum insertion approximation (VIA), on which we rely lacking more precise determinations of the matrix elements at the leading order in chiral perturbation theory. In section 5 we propose a new approach to compute non-local contributions on the lattice, while in section 6 we estimate δ_{m_c} and comment on its impact on the theoretical prediction of ε_K . Some more technical details, such as the relations that have been used in the computation, based on equations of motion and Fierz transformations, are collected in three appendices.

2 Outline of the calculation

In this work we calculate the corrections of $\mathcal{O}(m_K^2/m_c^2)$ and $\mathcal{O}(\Lambda_{\text{QCD}}^2/m_c^2)$, relative to the dominant terms, to the $\Delta S = 2$ effective Hamiltonian describing ε_K . For simplicity, in the following we will denote the ensemble of these $\mathcal{O}(1/m_c^2)$ corrections as δ_{m_c} .

Figure 1. Box diagrams describing $K^0 - \bar{K}^0$ mixing.

The imaginary part of the $\Delta S = 2$ effective Hamiltonian at the leading order in the Operator Product Expansion (OPE) and at zero order in the strong interactions, has the well known expression

$$H_{LT}^{\text{eff}} = \frac{G_F^2}{16\pi^2} M_W^2 \left[\text{Im}(\lambda_c^2) S_0(x_c) + \text{Im}(\lambda_t^2) S_0(x_t) + 2\text{Im}(\lambda_c \lambda_t) S_0(x_c, x_t) \right] (\bar{s} \gamma_L^\mu d) (\bar{s} \gamma_L^\mu d), \quad (2.1)$$

where the subscript LT means dominant (leading) term in the $1/m_c$ expansion; $\lambda_i = V_{id} V_{is}^*$ and the short-distance Inami-Lim functions S_0 's are computed by matching full and effective amplitudes for external states of four quarks with zero momentum [14]. The full amplitude is represented by the box diagrams shown in figure 1.

In eq. (2.1), the CKM factor λ_u has been eliminated by using the unitarity relation $\lambda_u + \lambda_c + \lambda_t = 0$ and, thus, the Inami-Lim functions are linear combinations of the functions (D_{ij}) corresponding to box diagrams with internal i and j quarks, namely

$$\begin{aligned} S_0(x_c) &= D_{uu} + D_{cc} - 2D_{uc}, \\ S_0(x_t) &= D_{uu} + D_{tt} - 2D_{ut}, \\ S_0(x_c, x_t) &= D_{uu} - D_{uc} - D_{ut} + D_{ct}. \end{aligned} \quad (2.2)$$

An equivalent way to express the $\Delta S = 2$ effective Hamiltonian comes from the fact that in the standard convention for the CKM matrix [6] λ_u is real, so that $\text{Im}(\lambda_u^2) = \text{Im}(\lambda_c^2 + \lambda_t^2 + 2\lambda_c \lambda_t) = 0$. By using this relation in eq. (2.1), one can write

$$H_{LT}^{\text{eff}} = \frac{G_F^2}{16\pi^2} M_W^2 \left[\text{Im}(\lambda_c^2) (S_0(x_c) - S_0(x_c, x_t)) + \text{Im}(\lambda_t^2) (S_0(x_t) - S_0(x_c, x_t)) \right] (\bar{s} \gamma_L^\mu d) (\bar{s} \gamma_L^\mu d). \quad (2.3)$$

At the leading order in the $1/m_c$ expansion both the $\text{Im}(\lambda_c^2)$ and $\text{Im}(\lambda_t^2)$ contributions are relevant. The latter, in fact, which is CKM suppressed, is enhanced by the top quark mass, as $S_0(x_t)$ grows as $x_t = m_t^2/M_W^2$ in the large x_t limit, while $S_0(x_c)$ and $S_0(x_c, x_t)$ vanish as $x_c = m_c^2/M_W^2$ in the small x_c limit. For the subleading contributions due to dimension-8 operators, instead, only the $\text{Im}(\lambda_c^2)$ term has some relevance. The $\text{Im}(\lambda_c^2)$ subleading contribution, indeed, represents a correction of $\mathcal{O}(1/m_c^2)$ to the leading term, while in the $\text{Im}(\lambda_t^2)$ term the relative correction is of $\mathcal{O}(1/m_t^2)$ and can safely be neglected.

The calculation of the δ_{m_c} is based on the determination of the Wilson coefficients of dimension-8 operators from matching conditions. To this aim, box diagrams have to be calculated with non vanishing external quark momenta. The $\mathcal{O}(p^2)$ contribution to the full amplitudes is then computed taking arbitrary external momenta and performing derivatives with respect to the external momenta. Since we are evaluating small $\mathcal{O}(1/m_c^2)$ corrections to ε_K , we will use the effective Hamiltonian expanded at zeroth order in the strong interactions. At $\mathcal{O}(\alpha_s)$, apart from the contribution of dimension-8 operators, other contributions cannot be written in terms of local operators, as discussed in the following.

We consider from the beginning only the case of $p^2 < m_c^2$. The matching condition schematically reads

$$\mathcal{A}^{\text{full}}(\Delta S = 2) = \mathcal{A}(\Delta S = 1 \otimes \Delta S = 1) + \sum_i C_i^W \langle O_i \rangle, \quad (2.4)$$

where the first term on the r.h.s. represents the long distance contribution to the amplitude $\mathcal{A}$, coming from the double insertion of the effective $\Delta S = 1$ Hamiltonian, whereas the last term is the contribution that can be expressed as linear combination of local operators multiplied by suitable Wilson coefficients C_i^W . By assuming the charm quark mass to be *large*, that is by neglecting terms of $\mathcal{O}(1/m_c^2)$, the first term on the r.h.s. of eq. (2.4) is absent and the second, in the Standard Model, is simply given by the operator

$$\bar{s} \gamma_L^\mu d \bar{s} \gamma_L^\mu d, \quad (2.5)$$

see eq. (2.3).

When we include the $\mathcal{O}(1/m_c^2)$ corrections, in addition to the presence of local dimension-8 operators, we find a non-local, non-perturbative, non-box amplitude $\mathcal{A}(\Delta S = 1 \otimes \Delta S = 1)$ given by double penguin diagrams of $\mathcal{O}(\alpha_s p^2/m_c^2)$ shown in the left panels of figure 2. The black dots in these figures represent the double insertion of the $\Delta S = 1$ Hamiltonian. Using the unitarity of the CKM matrix one may reduce all the double penguin diagrams to the GIM-like combination of penguins, namely up minus charm and charm minus top, shown in the figure. In general, since the typical scale of these diagrams is of $\mathcal{O}(p^2) \sim \mathcal{O}(\Lambda_{\text{QCD}}^2)$, and the intermediate states propagating in the s or t channels are light, perturbation theory is of no use and $\mathcal{A}(\Delta S = 1 \otimes \Delta S = 1)$ must be evaluated non perturbatively. This is to be contrasted with the case of the box diagrams where the non-perturbative part is encoded in the matrix elements of local operators. The contribution to the mixing amplitude of the double penguin diagrams of figures 2 corresponds to the long distance corrections to ε_K estimated in ref. [9]. Note that this kind of diagrams are also extremely difficult to compute from first principles, in a lattice QCD simulation, even in the simpler case of the K_L - K_S mass difference where only the operators Q_1 and Q_2 of the $\Delta S = 1$ Hamiltonian are considered. Although there is no theoretical issue in the lattice calculation, the difficulty is the standard one of achieving sufficient precision when evaluating disconnected diagrams. For more details see for example the discussion on the *disconnected* diagrams Type 3 and Type 4 of refs. [15, 16], where the reader can also find the definition of the operators Q_1 and Q_2 . For a direct calculation of ε_K see ref. [17]. In the present paper, for our final numerical estimate of the correction at $\mathcal{O}(1/m_c^2)$ to ε_K we will indeed use the estimate of

Figure 2. On the left side the $K^0 - \bar{K}^0$ double penguin diagrams, on the right side their reinter-pretation in terms of connected diagrams. The box represents the insertion of the penguin operator defined in eq. (5.2).

$\mathcal{A}(\Delta S = 1 \otimes \Delta S = 1)$ by ref. [9]. Some further discussion about this non-local contribution will be given in section 5. The diagrams in the right panels of figure 2 will be discussed in section 6.

3 Matching

The Wilson coefficients that we want to determine can be derived by imposing the following matching conditions on amplitudes with four quarks and four quarks plus a gluon external states respectively,

$$\mathcal{A}_{4q}^{\text{full}} = \mathcal{A}_{4q}^{\text{eff}}, \quad \mathcal{A}_{4q1g}^{\text{full}} = \mathcal{A}_{4q1g}^{\text{eff}}, \quad (3.1)$$

for some choice of external particle momenta $p_i \ll m_c$, where “full” and “effective” denote the amplitudes in the full (Standard Model) and effective theories respectively.

The full amplitudes at $\mathcal{O}(\alpha_s^0)$ are calculated from the box diagrams of figure 1, with W propagators expanded at the lowest order in the large W mass, i.e. approximated by 4-fermion contact interactions. By choosing for the matching the color indices of the external quark and anti-quark states in a proper way, one can actually select the contribution of only one of the two box diagrams. For the four quark amplitude, for instance, we choose equal color indices for the quark and anti-quark in the initial state and similarly for the final state, but different between the initial and final states, which corresponds to selecting the box diagram in figure 1 (left panel). In this case, the amplitude has the form

$$\bar{s}^a(p_2)\Gamma^i d^a(p_1)\bar{s}^b(p_3)\Gamma^j d^b(p_4) \quad (3.2)$$

where p_1, p_2 and p_3, p_4 are initial and final momenta respectively and a and b are different color indices. The calculation of the box diagram with the four quarks in the external states

then leads to the expression

$$\mathcal{A}_{4q}^{\text{full}} = -i \frac{G_F^2}{16\pi^2} \text{Im}(\lambda_c^2) \left[\mathcal{A}_{4q}^{\text{full}}|_{\gamma^\mu \gamma^\mu} + \mathcal{A}_{4q}^{\text{full}}|_{q\bar{q}} \right], \quad (3.3)$$

where, by omitting for brevity the external spinors, eliminating the combination $p_4 - p_2$ in favour of $q = p_1 - p_3$ and using the equations of motion,

$$\begin{aligned} \mathcal{A}_{4q}^{\text{full}}|_{\gamma^\mu \gamma^\mu} &= \alpha_1 (p_1 - p_3)^2 \gamma_L^\mu \otimes \gamma_L^\mu = \alpha_1 (m_d^2 + m_s^2 - 2p_1 \cdot p_3) \gamma_L^\mu \otimes \gamma_L^\mu, \\ \mathcal{A}_{4q}^{\text{full}}|_{q\bar{q}} &= \alpha_2 (\not{p}_1 - \not{p}_3)_L \otimes (\not{p}_1 - \not{p}_3)_L \\ &= \alpha_2 [m_d R \otimes \not{p}_1 L - m_d m_s R \otimes L - \not{p}_3 L \otimes \not{p}_1 L + m_s \not{p}_3 L \otimes L], \end{aligned} \quad (3.4)$$

with

$$\alpha_1 = -\frac{5}{9}, \quad \alpha_2 = 2\alpha_1 = -\frac{10}{9}. \quad (3.5)$$

Throughout this paper we use the notation

$$R \equiv 1 + \gamma^5, \quad L \equiv 1 - \gamma^5. \quad (3.6)$$

At $\mathcal{O}(\alpha_s^1)$, the amplitude for the four quarks and a gluon includes the contributions of a gluon emitted either from an internal line or from an external leg. The expression for this amplitude is quite long and tedious, containing about 50 different Dirac structures. For this reason it is not reported here but given in appendix C. Having the full amplitudes at hand, we then need the effective amplitudes in order to impose the matching conditions of eq. (3.1). The computation of the effective amplitudes is based on the effective Hamiltonian

$$H_{1/m_c^2}^{\text{eff}} = \frac{G_F^2}{16\pi^2} \text{Im}(\lambda_c^2) \sum_i C_i O_i, \quad (3.7)$$

where the $C_i^W = G_F^2/(16\pi^2) \text{Im}(\lambda_c^2) C_i$ are the Wilson coefficients that we want to determine and the O_i are the local operators in the expansion [19], namely

$$\begin{aligned} O_1 &= \bar{s} \gamma_L^\nu \mathcal{D}_\mu d \bar{s} \gamma_L^\nu \mathcal{D}^\mu d + \bar{s} \overleftarrow{\mathcal{D}}_\mu \gamma_L^\nu d \bar{s} \overleftarrow{\mathcal{D}}^\mu \gamma_L^\nu d, \\ O_2 &= \bar{s} \gamma_L^\nu \mathcal{D}_\mu d \bar{s} \gamma_L^\mu \mathcal{D}^\nu d + \bar{s} \overleftarrow{\mathcal{D}}_\mu \gamma_L^\nu d \bar{s} \overleftarrow{\mathcal{D}}^\nu \gamma_L^\mu d, \\ O_3 &= \bar{s} \overleftarrow{\mathcal{D}}_\mu \gamma_L^\nu d \bar{s} \gamma_L^\nu \mathcal{D}^\mu d, \\ O_4 &= g_s \bar{s} \gamma_L^\mu \tilde{G}_{\mu\nu} d \bar{s} \gamma_L^\nu d \\ O_5 &= i m_s \bar{s} L \mathcal{D}_\mu d \bar{s} \gamma_L^\mu d - i m_d \bar{s} \overleftarrow{\mathcal{D}}_\mu R d \bar{s} \gamma_L^\mu d, \\ O_6 &= (m_s^2 + m_d^2) \bar{s} \gamma_L^\mu d \bar{s} \gamma_L^\mu d, \\ O_7 &= m_s^2 \bar{s} L d \bar{s} L d + m_d^2 \bar{s} R d \bar{s} R d, \\ O_8 &= m_s m_d \bar{s} L d \bar{s} R d, \end{aligned} \quad (3.8)$$

with $\mathcal{D}_\mu = \partial_\mu - i g_s G_\mu^A t^A$, $\tilde{G}_{\mu\nu}^A = 1/2 \epsilon_{\mu\nu\rho\sigma} G^{\rho\sigma A}$, $G_{\rho\sigma} = G_{\rho\sigma}^A t^A = (\partial_\rho G_\sigma^A - \partial_\sigma G_\rho^A + g_s f^{ABC} G_\rho^B G_\sigma^C) t^A$, implying a sum over repeated color indices, and $\text{Tr}(t^A t^B) = 1/2 \delta^{AB}$.

The effective amplitude for a state of four quarks has then the form

$$\mathcal{A}_{4q}^{\text{eff}} = -i \frac{G_F^2}{16\pi^2} \text{Im}(\lambda_c^2) \sum_i C_i \langle O_i \rangle_{4q}, \quad (3.9)$$

where the matrix elements for the various operators in eq. (3.8) read

$$\begin{aligned}
 \langle O_1 \rangle_{4q} &= 2(p_1 \cdot p_4 + p_2 \cdot p_3) \gamma_L^\mu \otimes \gamma_L^\mu = 2(m_d^2 - m_s^2 + 2p_2 \cdot p_3) \gamma_L^\mu \otimes \gamma_L^\mu, \\
 \langle O_2 \rangle_{4q} &= 2(p_2^\mu p_3^\nu + p_1^\mu p_4^\nu) \gamma_L^\nu \otimes \gamma_L^\mu = \\
 &= 2(-2\not{p}_{3L} \otimes \not{p}_{1L} + m_s \not{p}_{3L} \otimes L - m_d \not{p}_{3L} \otimes R - m_s L \otimes \not{p}_{1L} + m_d R \otimes \not{p}_{1L}), \\
 \langle O_3 \rangle_{4q} &= (p_2 \cdot p_4 + p_1 \cdot p_3) \gamma_L^\mu \otimes \gamma_L^\mu = 2p_1 \cdot p_3 \gamma_L^\mu \otimes \gamma_L^\mu, \\
 \langle O_4 \rangle_{4q} &= 0, \\
 \langle O_5 \rangle_{4q} &= m_s^2 L \otimes L + m_d^2 R \otimes R - 2m_s m_d R \otimes L \\
 &\quad + m_s \not{p}_{3L} \otimes L + m_d \not{p}_{3L} \otimes R + m_s L \otimes \not{p}_{1L} + m_d R \otimes \not{p}_{1L}, \\
 \langle O_6 \rangle_{4q} &= 2(m_s^2 + m_d^2) \gamma_L^\mu \otimes \gamma_L^\mu, \\
 \langle O_7 \rangle_{4q} &= 2m_s^2 L \otimes L + 2m_d^2 R \otimes R, \\
 \langle O_8 \rangle_{4q} &= m_s m_d (L \otimes R + R \otimes L).
 \end{aligned} \tag{3.10}$$

For the state of four quarks plus a gluon, the expressions of the matrix elements are quite long, see appendix C, and here we write the result only for the operator O_4

$$\langle O_4 \rangle_{4q1g} = g_s \epsilon^{\mu\nu\rho\sigma} (i q_\rho) \varepsilon_\sigma^A \left[\gamma_L^\mu t^A \otimes \gamma_L^\nu - \gamma_L^\mu \otimes \gamma_L^\nu t^A \right]. \tag{3.11}$$

The operator O_4 , at variance with the other seven operators, has vanishing matrix element on the four quark state. As a consequence, the determination of the Wilson coefficient C_4 requires the matching between the full and the effective amplitude on a state with a gluon.

From the matching conditions in eq. (3.1) we find

$$\begin{aligned}
 C_1 &= 0, & C_2 &= -\frac{5}{18}, & C_3 &= \frac{5}{9}, & C_4 &= -\frac{13}{9}, \\
 C_5 &= -\frac{5}{9}, & C_6 &= -\frac{5}{18}, & C_7 &= \frac{5}{18}, & C_8 &= 0.
 \end{aligned} \tag{3.12}$$

4 Matrix elements in the VIA

In this section we evaluate the matrix elements of the relevant four fermion operators in the VIA which, lacking a more precise quantitative determination, is expected to provide a reasonable estimate of the matrix elements. In this approximation the four fermion operators are expressed in terms of factored bilinear operators, the matrix elements of which, in this paper, are evaluated by using the chiral effective theory, at the leading order. Given the level of accuracy at which we are working we will consider neither the QCD corrections to the Wilson coefficients of the dimension-8 operators nor the usual problems related to the presence of renormalons in matching the original theory to its OPE at the subleading order [20].

In the chiral effective theory, one starts by defining the meson fields and the explicit symmetry breaking term,

$$U = e^{2i\Pi/f}, \quad \chi = 2BM \tag{4.1}$$

where f is the pseudoscalar decay constant defined such that $f_\pi \sim 130 \text{ MeV}$, $M = \text{diag}(m_u, m_d, m_s)$ and Π is the pseudoscalar meson operator,

$$\Pi = \begin{pmatrix} \eta/\sqrt{6} + \pi^0/\sqrt{2} & \pi^+ & K^+ \\ \pi^- & \eta/\sqrt{6} - \pi^0/\sqrt{2} & K^0 \\ K^- & \bar{K}^0 & -\sqrt{2/3}\eta \end{pmatrix}. \quad (4.2)$$

Under $\text{SU}(3)_L \otimes \text{SU}(3)_R$ chiral transformations, the fields U and χ transform as $U \rightarrow R U L^\dagger$ and $\chi \rightarrow R \chi L^\dagger$.

Next we provide the expressions of the relevant two-quark operators in terms of the chiral fields, at the leading order in the chiral expansion. We start by considering the dimension-3 operators:

$$\bar{q}_i L q_j = -\frac{f^2}{2} B [U^\dagger]_{ji}, \quad (4.3)$$

$$\bar{q}_i R q_j = -\frac{f^2}{2} B [U]_{ji}, \quad (4.4)$$

$$\bar{q}_i \gamma_L^\mu q_j = i \frac{f^2}{2} [(\partial^\mu U^\dagger) U]_{ji}, \quad (4.5)$$

$$\bar{q}_i \gamma_R^\mu q_j = i \frac{f^2}{2} [(\partial^\mu U) U^\dagger]_{ji}, \quad (4.6)$$

$$\bar{q}_i \sigma_L^{\mu\nu} q_j = -i \alpha_T \frac{f^2}{2} [(\partial^\mu U^\dagger) (\partial^\nu U) U^\dagger - (\partial^\nu U^\dagger) (\partial^\mu U) U^\dagger]_{ji}, \quad (4.7)$$

$$\bar{q}_i \sigma_R^{\mu\nu} q_j = -i \alpha_T \frac{f^2}{2} [(\partial^\mu U) (\partial^\nu U^\dagger) U - (\partial^\nu U) (\partial^\mu U^\dagger) U]_{ji}. \quad (4.8)$$

In the chiral expression of the tensor operators the low energy constant α_T is given by $\alpha_T = B_T/(2m_K)$ where B_T is a dimensionless parameter of $\mathcal{O}(1)$ [19]. Note, however, that these operators do not contribute in the VIA, since their matrix elements between the vacuum and a kaon state vanish.

We now list the dimension-4 operators [21] at the lowest order in the chiral expansion:

$$\bar{q}_i \overleftarrow{D}^\alpha \gamma_L^\mu q_j = i \frac{f^2}{4} \left[(\partial^\alpha \partial^\mu U^\dagger) U - \frac{1}{8} g_{\alpha\mu} (\chi^\dagger U + U^\dagger \chi) \right]_{ji} + \dots, \quad (4.9)$$

$$\bar{q}_i \gamma_L^\mu \overrightarrow{D}^\alpha q_j = i \frac{f^2}{4} \left[(\partial^\alpha \partial^\mu U^\dagger) U + \frac{1}{8} g_{\alpha\mu} (\chi^\dagger U + U^\dagger \chi) \right]_{ji} + \dots, \quad (4.10)$$

where on the r.h.s. of eqs. (4.9) and (4.10) we only show explicitly the terms which contribute in the factorized case. Eq. (4.10) can be derived from (4.9) using the identity

$$\partial^\alpha (\bar{q}_i \gamma_L^\mu q_j) = q_i \overleftarrow{D}^\alpha \gamma_L^\mu q_j + \bar{q}_i \gamma_L^\mu \overrightarrow{D}^\alpha q_j. \quad (4.11)$$

The chiral expression for the operators $\bar{q}_i \overrightarrow{D}^\mu L q_j$ and $\bar{q}_i \overleftarrow{D}^\mu R q_j$ can be derived by using the relation:

$$\begin{aligned} \bar{q}_i \overrightarrow{D}^\mu L q_j &= \frac{1}{2} \bar{q}_i (\gamma^\mu \overrightarrow{\not{D}} + \overrightarrow{\not{D}} \gamma^\mu) L q_j = \\ &= -\frac{i}{2} m_j \bar{q}_i \gamma_R^\mu q_j + \frac{1}{2} \partial_\nu (\bar{q}_i \gamma^\nu \gamma_L^\mu q_j) - \frac{1}{2} \bar{q}_i \overleftarrow{\not{D}} \gamma_L^\mu q_j = \\ &= -\frac{i}{2} m_j \bar{q}_i \gamma_R^\mu q_j + \frac{1}{2} \partial^\mu (\bar{q}_i L q_j) - \frac{1}{2} \partial_\nu (\bar{q}_i \sigma_L^{\mu\nu} q_j) - \frac{i}{2} m_i \bar{q}_i \gamma_L^\mu q_j. \end{aligned} \quad (4.12)$$

The chiral expressions for the operators on the r.h.s. of eq. (4.12) are given in eqs. (4.3)–(4.8). Note, however, that while the operator $\partial^\mu(\bar{q}_i L q_j)$ is of $\mathcal{O}(p)$ in the chiral power counting, all other operators on the r.h.s. of eq. (4.12) are of $\mathcal{O}(p^3)$ and can be thus neglected. Therefore, we conclude

$$\begin{aligned}\bar{q}_i \vec{D}^\mu L q_j &= -\frac{f^2}{4} B [\partial^\mu U^\dagger]_{ji}, \\ \bar{q}_i \overleftarrow{D}^\mu R q_j &= -\frac{f^2}{4} B [\partial^\mu U]_{ji}.\end{aligned}\quad (4.13)$$

Finally, we also need the dimension-5 operators $\bar{q}_i \overleftarrow{D}_\alpha \gamma_L^\mu \vec{D}^\alpha q_j$ and $g_s \bar{q}_i \gamma_{\nu L} \tilde{G}^{\nu\mu} q_j$. They are related by the following identity:

$$\begin{aligned}\partial_\alpha \partial^\alpha (\bar{q}_i \gamma_L^\mu q_j) &= \partial_\alpha (\bar{q}_i \overleftarrow{D}^\alpha \gamma_L^\mu q_j + \bar{q}_i \gamma_L^\mu \vec{D}^\alpha q_j) = \\ &= \bar{q}_i \overleftarrow{D}^2 \gamma_L^\mu q_j + \bar{q}_i \gamma_L^\mu \vec{D}^2 q_j + 2 \bar{q}_i \overleftarrow{D}_\alpha \gamma_L^\mu \vec{D}^\alpha q_j = \\ &= -2 g_s \bar{q}_i \gamma_{\nu L} \tilde{G}^{\nu\mu} q_j - (m_i^2 + m_j^2) \bar{q}_i \gamma_L^\mu q_j + 2 \bar{q}_i \overleftarrow{D}_\alpha \gamma_L^\mu \vec{D}^\alpha q_j.\end{aligned}\quad (4.14)$$

Using eq. (4.5) in the above expression, one obtains

$$\bar{q}_i \overleftarrow{D}_\alpha \gamma_L^\mu \vec{D}^\alpha q_j - g_s \bar{q}_i \gamma_{\nu L} \tilde{G}^{\nu\mu} q_j = i \frac{f^2}{4} \left[(\partial_\alpha \partial^\alpha \partial^\mu U^\dagger) U + (m_i^2 + m_j^2) (\partial^\mu U^\dagger) U \right]_{ji}. \quad (4.15)$$

In terms of the chiral fields one can write [19]

$$g_s \bar{q}_i \gamma_{\mu L} \tilde{G}^{\mu\nu} q_j = i \frac{f^2}{2} \delta_K^2 \left[(\partial^\nu U^\dagger) U \right]_{ji}, \quad (4.16)$$

where δ_K^2 is a low-energy constant. A QCD sum rules calculation [19] provides the estimate $\delta_K^2 = 0.12 \pm 0.07 \text{ GeV}^2$. Note that the matrix element of the operator in eq. (4.16), as all the operators of $\mathcal{O}(g_s G_\mu^A)$, is of $\mathcal{O}(\alpha_s^0)$.

We are now ready to compute the matrix elements of the 4-fermion operators in the VIA and at the lowest order in the chiral expansion. We compute always the matrix element between an initial K^0 and a final $\bar{K}^0$ state. Thus $\partial_\mu K^0 = -i p_\mu^K K^0$ and $\partial_\mu \bar{K}^0 = i p_\mu^K \bar{K}^0$. As an example, we report the detailed calculation of the matrix element of the operator O_3 :

$$\begin{aligned}\langle \bar{K}^0 | O_3 | K^0 \rangle &= \langle \bar{K}^0 | (\bar{s}^a \overleftarrow{D}_\mu \gamma_L^\nu d^a) (\bar{s}^b \gamma_L^\nu \vec{D}^\mu d^b) | K^0 \rangle = \\ &= \langle \bar{K}^0 | \bar{s}^a \overleftarrow{D}_\mu \gamma_L^\nu d^a | 0 \rangle \langle 0 | \bar{s}^b \gamma_L^\nu \vec{D}^\mu d^b | K^0 \rangle + \langle \bar{K}^0 | \bar{s}^b \gamma_L^\nu \vec{D}^\mu d^b | 0 \rangle \langle 0 | \bar{s}^a \overleftarrow{D}_\mu \gamma_L^\nu d^a | K^0 \rangle + \\ &\quad + \langle \bar{K}^0 | \bar{s}^a \overleftarrow{D}_\mu \gamma_L^\nu \vec{D}^\mu d^b | 0 \rangle \langle 0 | \bar{s}^b \gamma_L^\nu d^a | K^0 \rangle + \langle \bar{K}^0 | \bar{s}^b \gamma_L^\nu d^a | 0 \rangle \langle 0 | \bar{s}^a \overleftarrow{D}_\mu \gamma_L^\nu \vec{D}^\mu d^b | K^0 \rangle = \\ &= \langle \bar{K}^0 | \bar{s}^a \overleftarrow{D}_\mu \gamma_L^\nu d^a | 0 \rangle \langle 0 | \bar{s}^b \gamma_L^\nu \vec{D}^\mu d^b | K^0 \rangle + \langle \bar{K}^0 | \bar{s}^b \gamma_L^\nu \vec{D}^\mu d^b | 0 \rangle \langle 0 | \bar{s}^a \overleftarrow{D}_\mu \gamma_L^\nu d^a | K^0 \rangle + \\ &\quad + \frac{1}{3} \langle \bar{K}^0 | \bar{s}^a \overleftarrow{D}_\mu \gamma_L^\nu \vec{D}^\mu d^a | 0 \rangle \langle 0 | \bar{s}^b \gamma_L^\nu d^b | K^0 \rangle + \frac{1}{3} \langle \bar{K}^0 | \bar{s}^a \gamma_L^\nu d^a | 0 \rangle \langle 0 | \bar{s}^b \overleftarrow{D}_\mu \gamma_L^\nu \vec{D}^\mu d^b | K^0 \rangle = \\ &= f_K^2 \left[\frac{1}{2} M_K^4 - \frac{1}{8} B^2 (m_s - m_d)^2 \right] - \frac{1}{3} f_K^2 M_K^2 \left[M_K^2 - (m_s^2 + m_d^2) - 2 \delta_K^2 \right] = \\ &= \frac{1}{24} f_K^2 \left[4 M_K^4 - 3 B^2 (m_s - m_d)^2 + 8 M_K^2 (m_s^2 + m_d^2) + 16 M_K^2 \delta_K^2 \right],\end{aligned}\quad (4.17)$$

where, in evaluating the matrix element between kaon states we have replaced the low energy constant f with the kaon decay constant f_K . By neglecting the mass of the down quark, m_d , and thus also replacing $B = M_K^2/m_s$, one finally obtains

$$\langle \bar{K}^0 | O_3 | K^0 \rangle = \frac{1}{24} f_K^2 M_K^2 \left[M_K^2 + 8 m_s^2 + 16 \delta_K^2 \right]. \quad (4.18)$$

The VIA for the matrix elements of the other relevant 4-fermion operators can be derived in a similar way. In the limit $m_d \rightarrow 0$, we obtain

$$\begin{aligned} \langle \bar{K}^0 | O_1 | K^0 \rangle &= \frac{5}{3} f_K^2 M_K^4, \\ \langle \bar{K}^0 | O_2 | K^0 \rangle &= \frac{11}{12} f_K^2 M_K^4, \\ \langle \bar{K}^0 | O_3 | K^0 \rangle &= \frac{1}{24} f_K^2 M_K^2 \left[16 \delta_K^2 + M_K^2 + 8 m_s^2 \right], \\ \langle \bar{K}^0 | O_4 | K^0 \rangle &= 2 f_K^2 M_K^2 \delta_K^2, \\ \langle \bar{K}^0 | O_5 | K^0 \rangle &= -f_K^2 M_K^4, \\ \langle \bar{K}^0 | O_6 | K^0 \rangle &= \frac{8}{3} f_K^2 M_K^2 m_s^2, \\ \langle \bar{K}^0 | O_7 | K^0 \rangle &= -\frac{5}{3} f_K^2 M_K^4, \\ \langle \bar{K}^0 | O_8 | K^0 \rangle &= 0. \end{aligned} \quad (4.19)$$

Note that in the chiral power counting the leading contribution is given by the operators O_3 and O_4 whose matrix elements are of $\mathcal{O}(p^2)$ (for O_3 , however, the $\mathcal{O}(p^2)$ contribution is suppressed by $1/N_c$). The matrix element of O_6 is of $\mathcal{O}(p^6)$ while all other matrix elements are of $\mathcal{O}(p^4)$. In ref. [1] the chiral expansion was performed at the leading order and the large- N_c limit was assumed. As a consequence, only the O_4 contribution was included. We observe that the contribution of the other seven operators, which we include in the present work for the first time, can a priori introduce an $\mathcal{O}(1)$ correction to the O_4 contribution, as m_K is numerically comparable to δ_K ($\delta_K^2/M_K^2 \simeq 0.5$).

5 A possible approach to compute non-local contributions at $\mathcal{O}(\alpha_s p^2/m_c^2)$

Some remark may be useful at this point. Concerning the non-local contributions that appear at $\mathcal{O}(\alpha_s p^2/m_c^2)$, let us take, for example, the diagram on the left upper panel of figure 2 at the lowest perturbative order in the Standard Model, by considering only the CP violating contribution to $K^0 - \bar{K}^0$ mixing. By using unitarity ($\lambda_u + \lambda_c + \lambda_t = 0$), neglecting $\lambda_t \times \mathcal{O}(p^2/m_c^2)$ corrections and without the resummation of large QCD logarithms, we find that there is only one contribution to the amplitude which goes as

$$\mathcal{A}(\Delta S = 1 \otimes \Delta S = 1) \sim \frac{G_F^2}{16 \pi^2} \text{Im}(\lambda_c^2) \frac{\alpha_s}{4\pi} \frac{4}{3} \log\left(\frac{m_t^2}{m_c^2}\right) \left(\Pi[p^2, m_u^2] - \Pi[p^2, m_c^2] \right), \quad (5.1)$$

where α_s is the strong coupling constant and the factor $\log(m_t^2/m_c^2)$ corresponds to the Wilson coefficient of the operator

$$Q = \bar{s} t^A \gamma_L^\mu d \sum_q \bar{q} t^A \gamma_\mu q, \quad (5.2)$$

generated by the heavy charm-top quark GIM penguin. $\Pi[p^2, m_u^2] - \Pi[p^2, m_c^2]$ denotes the hadronic contribution due the contraction of the operator Q with the operator

$$Q_2 = Q_2^u - Q_2^c = \bar{s}\gamma_L^\mu u \bar{u}\gamma_\mu^L d - \bar{s}\gamma_L^\mu c \bar{c}\gamma_\mu^L d, \quad (5.3)$$

which appears at tree level in the effective weak Hamiltonian. $\Pi[p^2, m_u^2]$ corresponds to the diagram on the upper right panel of figure 2 with (light) up quarks in the loop and must be computed non-perturbatively; $\Pi[p^2, m_c^2]$ denotes the same diagram with charm-quarks in the loop and, within the present approximation, can be computed in perturbation theory with a suitable matching to the theory with a propagating charm. GIM is essential to make the final result finite at the lowest non-trivial order in the strong coupling constant. The combination $(\Pi[p^2, m_u^2] - \Pi[p^2, m_c^2])$ is of $\mathcal{O}(p^2)$ and, as expected, it is a correction of $\mathcal{O}(\alpha_s p^2/m_c^2)$ to ε_K since the leading term in $\text{Im}(\lambda_c^2)$, coming from the box diagram of figure 1, goes as $G_F^2/(16\pi^2)m_c^2$.

Thus, by expanding at the leading order in $1/m_t^2$ and $1/m_c^2$ and by using the equations of motion, we may transform the charm-top quark GIM penguin of the Feynman diagram on the left upper panel of figure 2 in the insertion of the operator Q multiplied by its Wilson coefficient, thus obtaining the Feynman diagram on the r.h.s. of upper panel of figure 2, which is a standard *connected* diagram much easier to evaluate in lattice QCD [15, 16]. Following the standard approach, even if the momentum flowing in the charm-top GIM penguin is of $\mathcal{O}(p^2) \sim \mathcal{O}(\Lambda_{\text{QCD}}^2)$, in eq. (5.1) the coefficient of Q can be computed in perturbation theory since both the quarks in the GIM loop are heavy (they are either the top or the charm). Eventually, the most convenient choice is to take $\mu = m_c$ as renormalisation scale. In the calculation of the double penguins, in order take into account higher order QCD corrections, we can imagine to substitute to the Standard Model the double insertion of the low energy $\Delta S = 1$ Hamiltonian and compute on the lattice, with this effective theory, the charm-top and up-charm GIM penguins. This extreme accuracy, although feasible, is probably not necessary since we are evaluating one of the corrections to the leading term and the uncertainties entailed in other terms, specifically in the calculation of the matrix elements of the local dimension-8 operators, are probably of the size, if not larger, than the neglected higher-order QCD effects. The calculation of the other double penguin diagram, shown on the left lower panel of figure 2 could proceed in the same way by replacing it with the connected diagram on the right lower panel of the same figure.

6 Result for the power corrections to ε_K

By combining the results for the Wilson coefficients of the effective Hamiltonian given in section 3 and those for the matrix elements of the four fermion operators evaluated in section 4 in the VIA, we obtain

$$\begin{aligned} \langle \bar{K}^0 | H_{1/m_c^2}^{\text{eff}} | K^0 \rangle &= \frac{G_F^2}{16\pi^2} \text{Im}(\lambda_c^2) \sum_i C_i \langle \bar{K}^0 | O_i | K^0 \rangle = \\ &= \frac{G_F^2}{16\pi^2} \text{Im}(\lambda_c^2) \left[-\frac{1}{108} f_K^2 M_K^2 (272 \delta_K^2 + 15 M_K^2) \right], \end{aligned} \quad (6.1)$$

which represents the main result of this paper. In eq. (6.1) we have neglected small terms of $\mathcal{O}(p^6)$, proportional to $M_K^2 m_s^2$, that contribute for $\sim 1.5\%$ of the total correction. The above result can be compared with the leading¹ contribution to the effective Hamiltonian that can be written in the form

$$\langle \bar{K}^0 | H_{LT}^{\text{eff}} | K^0 \rangle = \frac{G_F^2}{16\pi^2} M_W^2 \left[\text{Im}(\lambda_c^2) (\eta_1 S_0(x_c) - \eta_3 S_0(x_c, x_t)) + \text{Im}(\lambda_t^2) (\eta_2 S_0(x_t) - \eta_3 S_0(x_c, x_t)) \right] \frac{8}{3} f_K^2 M_K^2 \hat{B}_K. \quad (6.2)$$

Therefore, with respect to the term proportional to $\text{Im}(\lambda_c^2)$ in the LT effective Hamiltonian, $\langle \bar{K}^0 | H_{LT}^{\text{eff}} | K^0 \rangle_{\text{Im}(\lambda_c^2)}$, eq. (6.1) represents a relative correction given by

$$\begin{aligned} \frac{\langle \bar{K}^0 | H_{1/m_c^2}^{\text{eff}} | K^0 \rangle}{\langle \bar{K}^0 | H_{LT}^{\text{eff}} | K^0 \rangle_{\text{Im}(\lambda_c^2)}} &= \left[\left(\eta_1 \frac{S_0(x_c)}{x_c} - \eta_3 \frac{S_0(x_c, x_t)}{x_c} \right) \hat{B}_K \right]^{-1} \left(-\frac{17}{18} \frac{\delta_K^2}{m_c^2} - \frac{5}{96} \frac{M_K^2}{m_c^2} \right) \simeq \\ &\simeq -0.51 \left(-\frac{17}{18} \frac{\delta_K^2}{m_c^2} - \frac{5}{96} \frac{M_K^2}{m_c^2} \right) \simeq 0.04. \end{aligned} \quad (6.3)$$

Up to higher order terms, we may then write

$$\text{Im} M_{12} = \langle \bar{K}^0 | H_{LT}^{\text{eff}} | K^0 \rangle (1 + \delta_{BGI} + \delta_{m_c}), \quad (6.4)$$

where $\delta_{BGI} = 0.02$ was computed in ref. [9] and the relative correction with respect to the full LT Hamiltonian is given by

$$\begin{aligned} \delta_{m_c} &\equiv \frac{\langle \bar{K}^0 | H_{1/m_c^2}^{\text{eff}} | K^0 \rangle}{\langle \bar{K}^0 | H_{LT}^{\text{eff}} | K^0 \rangle} = \\ &= \left[\left(\eta_1 \frac{S_0(x_c)}{x_c} - \eta_3 \frac{S_0(x_c, x_t)}{x_c} \right) - \frac{\text{Re}(\lambda_t)}{\text{Re}(\lambda_c)} \left(\eta_2 \frac{S_0(x_t)}{x_c} - \eta_3 \frac{S_0(x_c, x_t)}{x_c} \right) \right]^{-1} \cdot \\ &\cdot \hat{B}_K^{-1} \left(-\frac{17}{18} \frac{\delta_K^2}{m_c^2} - \frac{5}{96} \frac{M_K^2}{m_c^2} \right) \simeq -0.13 \left(-\frac{17}{18} \frac{\delta_K^2}{m_c^2} - \frac{5}{96} \frac{M_K^2}{m_c^2} \right) \simeq 0.010 \pm 0.003, \end{aligned} \quad (6.5)$$

that is the δ_{m_c} correction increases the theoretical prediction for ε_K by 1%. As is evident from eq. (6.5), the $\mathcal{O}(m_K^2/m_c^2)$ correction, included here for the first time, though a priori comparable to the $\mathcal{O}(\delta_K^2/m_c^2)$ term, turns out to be numerically smaller. At the origin of this numerical result there is the larger numerical value of the Wilson coefficient C_4 w.r.t. the coefficients of the other seven operators, and some cancellation occurring among the terms proportional to $f_K^2 m_K^4$. The dominant uncertainty on our result for δ_{m_c} is represented by the VIA. We evaluated it assuming for the matrix elements of the dimension-8 operators either a uniform distribution centered on the VIA with a 50% half width or a Gaussian distribution with $\sigma = 30\%$; in both cases we obtain a 30% error on δ_{m_c} , which keeps a negligible impact on the ε_K total uncertainty.

We conclude by providing an updated theoretical prediction for ε_K , which includes our result for δ_{m_c} of eq. (6.5), the short- and long-distance contributions recently calculated in

¹The effective Hamiltonian in eq. (6.2) is at the leading order in the OPE, while it includes the NLO QCD and EW corrections, which are enclosed in the η 's [22–25].

refs. [8, 9, 25] and uses for the bag-parameter the value $\hat{B}_K = 0.756 \pm 0.016$, obtained by averaging the $N_f = 2 + 1$ and $N_f = 2 + 1 + 1$ FLAG averages [7]. Our prediction is obtained using the UTfit code, in the context of an updated Bayesian Unitarity Triangle Analysis [26], which yields (omitting of course the constraint from ε_K) $\text{Im}(\lambda_t) = (13.44 \pm 0.57) \cdot 10^{-5}$, $\text{Re}(\lambda_t) = (-3.11 \pm 0.11) \cdot 10^{-4}$ and $\text{Re}(\lambda_c) = (-2191.0 \pm 7.5) \cdot 10^{-4}$. All the relevant inputs can be found in [26]. Our theoretical prediction for ε_K reads

$$\varepsilon_K = (1.99 \pm 0.14) \times 10^{-3}. \quad (6.6)$$

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A Useful relations coming from the equations of motion

In the calculation of the contribution of the gluon on an external leg, both in the full and in the effective theory, we can eliminate some structures containing the completely antisymmetric tensor, by using some relations originated from the equations of motion. These relations are obtained from the Chisholm identity

$$\gamma^\alpha \gamma^\beta \gamma_L^\gamma = g^{\alpha\beta} \gamma_L^\gamma - g^{\alpha\gamma} \gamma_L^\beta + g^{\beta\gamma} \gamma_L^\alpha - i \epsilon^{\alpha\beta\gamma\delta} \gamma_L^\delta, \quad (A.1)$$

by saturating one Lorentz index with a momentum which induces the Dirac equation of motion. These relations show that four of the structures involving the completely antisymmetric tensor can be reduced in terms of other structures. They can be written as

$$\epsilon^{\alpha\nu\beta\rho} p_1^\beta \bar{s}(p_2) \gamma_L^\alpha d(p_1) = i p_1^\nu \bar{s}(p_2) \gamma_L^\rho d(p_1) - i p_1^\rho \bar{s}(p_2) \gamma_L^\nu d(p_1) + i m_d \bar{s}(p_2) \sigma_R^{\nu\rho} d(p_1), \quad (A.2)$$

$$\epsilon^{\alpha\nu\beta\rho} p_2^\beta \bar{s}(p_2) \gamma_L^\alpha d(p_1) = -i p_2^\nu \bar{s}(p_2) \gamma_L^\rho d(p_1) + i p_2^\rho \bar{s}(p_2) \gamma_L^\nu d(p_1) - i m_s \bar{s}(p_2) \sigma_L^{\nu\rho} d(p_1), \quad (A.3)$$

$$\epsilon^{\alpha\nu\gamma\rho} p_3^\gamma \bar{s}(p_3) \gamma_L^\alpha d(p_4) = i p_3^\alpha \bar{s}(p_3) \gamma_L^\rho d(p_4) - i p_3^\rho \bar{s}(p_3) \gamma_L^\alpha d(p_4) - i m_s \bar{s}(p_3) \sigma_L^{\alpha\rho} d(p_4), \quad (A.4)$$

$$\begin{aligned} \epsilon^{\beta\gamma\mu\nu} p_2^\gamma \bar{s}(p_3) \gamma_L^\beta d(p_4) &= -i(p_1 + p_2 - 2p_3 - q)^\mu \bar{s}(p_3) \gamma_L^\nu d(p_4) + i(p_1 + p_2 - 2p_3 - q)^\nu \bar{s}(p_3) \gamma_L^\mu d(p_4) \\ &\quad - i m_s \bar{s}(p_3) \sigma_L^{\mu\nu} d(p_4) + i m_d \bar{s}(p_3) \sigma_R^{\mu\nu} d(p_4) \\ &\quad + \epsilon^{\beta\gamma\mu\nu} q^\gamma \bar{s}(p_3) \gamma_L^\beta d(p_4) - \epsilon^{\beta\gamma\mu\nu} p_1^\gamma \bar{s}(p_3) \gamma_L^\beta d(p_4). \end{aligned} \quad (A.5)$$

B Useful Fierz relations

The matrix elements of the operators entering the effective Hamiltonian $H_{1/m_c^2}^{\text{eff}}$ of eq. (3.7) are estimated by using the VIA, in which Fierz transformations play an important role. The Fierz transformations can be obtained starting from the following general formula

$$\begin{aligned} \Gamma_1 \otimes \Gamma_2 &\xrightarrow{\text{Fierz}} \frac{1}{8} \Gamma_1 R \Gamma_2 \otimes R + \frac{1}{8} \Gamma_1 L \Gamma_2 \otimes L + \\ &\quad + \frac{1}{8} \Gamma_1 \gamma_R^\alpha \Gamma_2 \otimes \gamma_L^\alpha + \frac{1}{8} \Gamma_1 \gamma_L^\alpha \Gamma_2 \otimes \gamma_R^\alpha - \\ &\quad - \frac{1}{32} \Gamma_1 \sigma_L^{\alpha\beta} \Gamma_2 \otimes \sigma_L^{\alpha\beta} - \frac{1}{32} \Gamma_1 \sigma_R^{\alpha\beta} \Gamma_2 \otimes \sigma_R^{\alpha\beta}, \end{aligned} \quad (B.1)$$

where we used the definition $\sigma^{\alpha\beta} = [\gamma^\alpha, \gamma^\beta]/2$. From eq. (B.1) one derives, in particular,

$$\gamma_L^\mu \otimes \gamma_L^\nu \xrightarrow{\text{Fierz}} \frac{1}{2} \gamma_L^\mu \otimes \gamma_L^\nu + \frac{1}{2} \gamma_L^\nu \otimes \gamma_L^\mu - \frac{1}{2} g^{\mu\nu} \gamma_L^\rho \otimes \gamma_L^\rho - \frac{i}{2} \varepsilon^{\mu\nu\rho\sigma} \gamma_L^\rho \otimes \gamma_L^\sigma \quad (\text{B.2})$$

$$L \otimes \gamma_L^\mu \xrightarrow{\text{Fierz}} \frac{1}{2} L \otimes \gamma_L^\mu - \frac{1}{2} \sigma_L^{\mu\nu} \otimes \gamma_L^\nu \quad (\text{B.3})$$

$$R \otimes \gamma_L^\mu \xrightarrow{\text{Fierz}} \frac{1}{2} \gamma_L^\mu \otimes R + \frac{1}{2} \gamma_L^\nu \otimes \sigma_R^{\mu\nu} \quad (\text{B.4})$$

It is also useful to report the general formula for the Fierz of color indices. It reads:

$$C_1 \otimes C_2 \xrightarrow{\text{Fierz}} \frac{1}{N} C_1 C_2 \otimes I + 2 C_1 t^A C_2 \otimes t^A, \quad (\text{B.5})$$

where C_1 and C_2 are generic color matrices. From eq. (B.5) one derives, in particular,

$$t^A \otimes I - I \otimes t^A \xrightarrow{\text{Fierz}} -2i f_{ABC} t^B \otimes t^C. \quad (\text{B.6})$$

C Matching with the external state made up of four quarks and a gluon

In this appendix we collect the amplitudes that enter the matching with the external state made up of four quarks and a gluon. As discussed in the main text for the case of the four quark external state, one can choose equal color indices for the quark and anti-quark in the initial state and similarly for the final state, which corresponds to selecting the box diagram in figure 1(a). We do the same also when the gluon is added in the external state. With the additional gluon, moreover, one can impose the matching condition by considering only contributions with the color generator t^A appearing in the initial state bilinear or in the final state bilinear. We perform the first choice, that is we consider the amplitudes having the form

$$\bar{s}^a(p_2) \Gamma^i t_{ab}^A d^b(p_1) \bar{s}^c(p_3) \Gamma^j d^c(p_4). \quad (\text{C.1})$$

By omitting for brevity the external spinors (and the gluon polarization vector), the matrix elements for the various operators in eq. (3.8), read

$$\begin{aligned} \langle O_1 \rangle_{4q1g} = & i \gamma_L^\alpha t^A \otimes \gamma_L^\nu \epsilon^{\alpha\nu\sigma\rho} q^\sigma \left(\frac{(m_d^2 + m_s^2 - 4p_2 \cdot q + 2p_1 \cdot p_2 - 2p_1 \cdot p_3)}{p_2 \cdot q} \right. \\ & + \frac{(m_d^2 + m_s^2 - 2p_2 \cdot q + 2p_3 \cdot q + 2p_1 \cdot p_2 - 2p_1 \cdot p_3)}{p_1 \cdot q} - 2 \frac{p_1 \cdot q}{p_2 \cdot q} \Big) \\ & + \gamma_L^\mu t^A \otimes \gamma_L^\mu \left(- \frac{2p_1^\rho (m_d^2 + m_s^2 - 2p_1 \cdot q - 2p_2 \cdot q + 2p_3 \cdot q + 2p_1 \cdot p_2 - 2p_1 \cdot p_3)}{p_1 \cdot q} \right. \\ & + \frac{2p_2^\rho (m_d^2 + m_s^2 - 2p_1 \cdot q - 2p_2 \cdot q + 2p_1 \cdot p_2 - 2p_1 \cdot p_3)}{p_2 \cdot q} + 4p_3^\rho \Big) \\ & + \gamma_L^\rho t^A \otimes \not{q}_L \left(- \frac{m_d^2 + m_s^2 + 2p_2 \cdot q + 2p_1 \cdot p_2 - 2p_1 \cdot p_3}{p_2 \cdot q} \right. \\ & + \frac{m_d^2 + m_s^2 - 2p_2 \cdot q + 2p_3 \cdot q + 2p_1 \cdot p_2 - 2p_1 \cdot p_3}{p_1 \cdot q} + \frac{2p_1 \cdot q}{p_2 \cdot q} \Big) \end{aligned}$$

$$\begin{aligned}
 & + \not{q}_L t^A \otimes \gamma_L^\rho \left(\frac{m_d^2 + m_s^2 + 2p_2 \cdot q + 2p_1 \cdot p_2 - 2p_1 \cdot p_3}{p_2 \cdot q} \right. \\
 & \left. - \frac{m_d^2 + m_s^2 - 2p_2 \cdot q + 2p_3 \cdot q + 2p_1 \cdot p_2 - 2p_1 \cdot p_3}{p_1 \cdot q} - \frac{2p_1 \cdot q}{p_2 \cdot q} \right), \tag{C.2}
 \end{aligned}$$

$$\begin{aligned}
 \langle O_2 \rangle_{4q1g} = & - \frac{i \gamma_L^\alpha t^A \otimes R m_d (p_1 \cdot q + p_2 \cdot q) \epsilon^{\alpha \nu \sigma \rho} p_3^\nu q^\sigma}{p_1 \cdot q p_2 \cdot q} \\
 & + \frac{i \gamma_L^\alpha t^A \otimes L m_s (p_1 \cdot q + p_2 \cdot q) \epsilon^{\alpha \nu \sigma \rho} p_3^\nu q^\sigma}{p_1 \cdot q p_2 \cdot q} \\
 & - \frac{2i \gamma_L^\alpha t^A \otimes \not{p}_{\not{L}} (p_1 \cdot q + p_2 \cdot q) \epsilon^{\alpha \nu \sigma \rho} p_3^\nu q^\sigma}{p_1 \cdot q p_2 \cdot q} + \frac{2i \gamma_L^\alpha t^A \otimes \not{q}_L \epsilon^{\alpha \nu \sigma \rho} p_3^\nu q^\sigma}{p_1 \cdot q} \\
 & + \not{p}_{\not{L}} t^A \otimes R \left(\frac{2m_d p_1^\rho}{p_1 \cdot q} - \frac{2m_d p_2^\rho}{p_2 \cdot q} \right) + \not{q}_L t^A \otimes R m_d \left(\frac{1}{p_1 \cdot q} - \frac{1}{p_2 \cdot q} \right) p_3^\rho \\
 & + \gamma_L^\rho t^A \otimes R m_d \left(\frac{1}{p_2 \cdot q} - \frac{1}{p_1 \cdot q} \right) p_3 \cdot q + R t^A \otimes \not{p}_{\not{L}} \left(\frac{2m_d p_2^\rho}{p_2 \cdot q} - \frac{2m_d p_1^\rho}{p_1 \cdot q} \right) \\
 & + q^\nu \sigma_R^{\nu \rho} t^A \otimes \not{p}_{\not{L}} m_d \left(\frac{1}{p_2 \cdot q} + \frac{1}{p_1 \cdot q} \right) + \frac{2R t^A \otimes \not{q}_L m_d p_1^\rho}{p_1 \cdot q} - \frac{q^\nu \sigma_R^{\nu \rho} t^A \otimes \not{q}_L m_d}{p_1 \cdot q} \\
 & - 2R t^A \otimes \gamma_L^\rho m_d + L t^A \otimes \not{p}_{\not{L}} \left(\frac{2m_s p_1^\rho}{p_1 \cdot q} - \frac{2m_s p_2^\rho}{p_2 \cdot q} \right) + q^\nu \sigma_L^{\nu \rho} t^A \otimes \not{p}_{\not{L}} \left(-\frac{m_s}{p_1 \cdot q} - \frac{m_s}{p_2 \cdot q} \right) \\
 & - \frac{2L t^A \otimes \not{q}_L m_s p_1^\rho}{p_1 \cdot q} + \frac{q^\nu \sigma_L^{\nu \rho} t^A \otimes \not{q}_L m_s}{p_1 \cdot q} + \not{p}_{\not{L}} t^A \otimes L \left(\frac{2m_s p_2^\rho}{p_2 \cdot q} - \frac{2m_s p_1^\rho}{p_1 \cdot q} \right) \\
 & + \not{q}_L t^A \otimes L m_s \left(\frac{1}{p_2 \cdot q} - \frac{1}{p_1 \cdot q} \right) p_3^\rho + \gamma_L^\rho t^A \otimes L m_s \left(\frac{1}{p_1 \cdot q} - \frac{1}{p_2 \cdot q} \right) p_3 \cdot q + 2L t^A \otimes \gamma_L^\rho m_s \\
 & + \not{p}_{\not{L}} t^A \otimes \not{p}_{\not{L}} \left(\frac{4p_1^\rho}{p_1 \cdot q} - \frac{4p_2^\rho}{p_2 \cdot q} \right) + 2\not{q}_L t^A \otimes \not{p}_{\not{L}} \left(\frac{1}{p_1 \cdot q} - \frac{1}{p_2 \cdot q} \right) p_3^\rho \\
 & + 2\gamma_L^\rho t^A \otimes \not{p}_{\not{L}} \left(\frac{1}{p_2 \cdot q} - \frac{1}{p_1 \cdot q} \right) p_3 \cdot q - \frac{4\not{p}_{\not{L}} t^A \otimes \not{q}_L p_1^\rho}{p_1 \cdot q} - \frac{2\not{q}_L t^A \otimes \not{q}_L p_3^\rho}{p_1 \cdot q} \\
 & + \gamma_L^\rho t^A \otimes \not{q}_L \left(\frac{2p_3 \cdot q}{p_1 \cdot q} - 2 \right) + 4\not{p}_{\not{L}} t^A \otimes \gamma_L^\rho + 2\not{q}_L t^A \otimes \gamma_L^\rho, \tag{C.3}
 \end{aligned}$$

$$\begin{aligned}
 \langle O_3 \rangle_{4q1g} = & \frac{i \gamma_L^\alpha t^A \otimes \gamma_L^\nu (p_1 \cdot p_3 (p_1 \cdot q + p_2 \cdot q) - p_2 \cdot q p_3 \cdot q) \epsilon^{\alpha \nu \sigma \rho} q^\sigma}{p_1 \cdot q p_2 \cdot q} \\
 & + \gamma_L^\mu t^A \otimes \gamma_L^\mu \left(\frac{2p_3 \cdot q p_1^\rho}{p_1 \cdot q} + p_1 \cdot p_3 \left(\frac{2p_2^\rho}{p_2 \cdot q} - \frac{2p_1^\rho}{p_1 \cdot q} \right) - 2p_3^\rho \right) \\
 & + \gamma_L^\rho t^A \otimes \not{q}_L \left(p_1 \cdot p_3 \left(\frac{1}{p_1 \cdot q} - \frac{1}{p_2 \cdot q} \right) - \frac{p_3 \cdot q}{p_1 \cdot q} + 1 \right) \\
 & + \not{q}_L t^A \otimes \gamma_L^\rho \left(p_1 \cdot p_3 \left(\frac{1}{p_2 \cdot q} - \frac{1}{p_1 \cdot q} \right) + \frac{p_3 \cdot q}{p_1 \cdot q} - 1 \right), \tag{C.4}
 \end{aligned}$$

$$\langle O_4 \rangle_{4q1g} = i \gamma_L^\alpha t^A \otimes \gamma_L^\nu \epsilon^{\alpha \nu \sigma \rho} q^\sigma, \tag{C.5}$$

$$\begin{aligned}
 \langle O_5 \rangle_{4q1g} = & \frac{i\gamma_L^\alpha t^A \otimes R m_d(p_1 \cdot q + p_2 \cdot q) \epsilon^{\alpha\nu\sigma\rho} p_3^\nu q^\sigma}{2p_1 \cdot q p_2 \cdot q} \\
 & + \frac{i\gamma_L^\alpha t^A \otimes L m_s(p_1 \cdot q + p_2 \cdot q) \epsilon^{\alpha\nu\sigma\rho} p_3^\nu q^\sigma}{2p_1 \cdot q p_2 \cdot q} \\
 & + q^\nu \sigma_R^{\nu\rho} t^A \otimes L m_d m_s \left(-\frac{1}{p_2 \cdot q} - \frac{1}{p_1 \cdot q} \right) + 2 R t^A \otimes L m_d m_s \left(\frac{p_1^\rho}{p_1 \cdot q} - \frac{p_2^\rho}{p_2 \cdot q} \right) \\
 & + R t^A \otimes R m_d^2 \left(\frac{p_2^\rho}{p_2 \cdot q} - \frac{p_1^\rho}{p_1 \cdot q} \right) + \not{p}_L t^A \otimes R \left(\frac{m_d p_2^\rho}{p_2 \cdot q} - \frac{m_d p_1^\rho}{p_1 \cdot q} \right) \\
 & + \frac{1}{2} \not{q}_L t^A \otimes R m_d \left(\frac{1}{p_2 \cdot q} - \frac{1}{p_1 \cdot q} \right) p_3^\rho + \frac{1}{2} \gamma_L^\rho t^A \otimes R m_d \left(\frac{1}{p_1 \cdot q} - \frac{1}{p_2 \cdot q} \right) p_{3 \cdot q} \\
 & + \frac{q^\nu \sigma_R^{\nu\rho} t^A \otimes R m_d^2(p_1 \cdot q + p_2 \cdot q)}{2p_1 \cdot q p_2 \cdot q} + R t^A \otimes \not{p}_L \left(\frac{m_d p_2^\rho}{p_2 \cdot q} - \frac{m_d p_1^\rho}{p_1 \cdot q} \right) \\
 & + \frac{1}{2} q^\nu \sigma_R^{\nu\rho} t^A \otimes \not{p}_L m_d \left(\frac{1}{p_2 \cdot q} + \frac{1}{p_1 \cdot q} \right) + \frac{R t^A \otimes \not{q}_L m_d p_1^\rho}{p_1 \cdot q} - \frac{q^\nu \sigma_R^{\nu\rho} t^A \otimes \not{q}_L m_d}{2p_1 \cdot q} \\
 & - R t^A \otimes \gamma_L^\rho m_d + \frac{q^\nu \sigma_L^{\nu\rho} t^A \otimes L m_s^2(p_1 \cdot q + p_2 \cdot q)}{2p_1 \cdot q p_2 \cdot q} + L t^A \otimes \not{p}_L \left(\frac{m_s p_2^\rho}{p_2 \cdot q} - \frac{m_s p_1^\rho}{p_1 \cdot q} \right) \\
 & + \frac{1}{2} q^\nu \sigma_L^{\nu\rho} t^A \otimes \not{p}_L m_s \left(\frac{1}{p_2 \cdot q} + \frac{1}{p_1 \cdot q} \right) + \frac{L t^A \otimes \not{q}_L m_s p_1^\rho}{p_1 \cdot q} \\
 & + L t^A \otimes L m_s^2 \left(\frac{p_2^\rho}{p_2 \cdot q} - \frac{p_1^\rho}{p_1 \cdot q} \right) - \frac{q^\nu \sigma_L^{\nu\rho} t^A \otimes \not{q}_L m_s}{2p_1 \cdot q} + \not{p}_L t^A \otimes L \left(\frac{m_s p_2^\rho}{p_2 \cdot q} - \frac{m_s p_1^\rho}{p_1 \cdot q} \right) \\
 & + \frac{1}{2} \not{q}_L t^A \otimes L m_s \left(\frac{1}{p_2 \cdot q} - \frac{1}{p_1 \cdot q} \right) p_3^\rho + \frac{1}{2} \gamma_L^\rho t^A \otimes L m_s \left(\frac{1}{p_1 \cdot q} - \frac{1}{p_2 \cdot q} \right) p_{3 \cdot q} - L t^A \otimes \gamma_L^\rho m_s,
 \end{aligned} \tag{C.6}$$

$$\begin{aligned}
 \langle O_6 \rangle_{4q1g} = & \frac{i\gamma_L^\alpha t^A \otimes \gamma_L^\nu(p_1 \cdot q + p_2 \cdot q)(m_d^2 + m_s^2) \epsilon^{\alpha\nu\sigma\rho} q^\sigma}{p_1 \cdot q p_2 \cdot q} \\
 & + \frac{2\gamma_L^\mu t^A \otimes \gamma_L^\mu(m_d^2 + m_s^2)(p_1 \cdot q p_2^\rho - p_2 \cdot q p_1^\rho)}{p_1 \cdot q p_2 \cdot q} - \frac{\gamma_L^\rho t^A \otimes \not{q}_L(p_1 \cdot q - p_2 \cdot q)(m_d^2 + m_s^2)}{p_1 \cdot q p_2 \cdot q} \\
 & + \frac{\not{q}_L t^A \otimes \gamma_L^\rho(p_1 \cdot q - p_2 \cdot q)(m_d^2 + m_s^2)}{p_1 \cdot q p_2 \cdot q},
 \end{aligned} \tag{C.7}$$

$$\begin{aligned}
 \langle O_7 \rangle_{4q1g} = & 2 R t^A \otimes R m_d^2 \left(\frac{p_2^\rho}{p_2 \cdot q} - \frac{p_1^\rho}{p_1 \cdot q} \right) + q^\nu \sigma_R^{\nu\rho} t^A \otimes R m_d^2 \left(\frac{1}{p_2 \cdot q} + \frac{1}{p_1 \cdot q} \right) \\
 & + q^\nu \sigma_L^{\nu\rho} t^A \otimes L m_s^2 \left(\frac{1}{p_2 \cdot q} + \frac{1}{p_1 \cdot q} \right) + 2 L t^A \otimes L m_s^2 \left(\frac{p_2^\rho}{p_2 \cdot q} - \frac{p_1^\rho}{p_1 \cdot q} \right),
 \end{aligned} \tag{C.8}$$

$$\begin{aligned}
 \langle O_8 \rangle_{4q1g} = & \frac{1}{2} q^\nu \sigma_R^{\nu\rho} t^A \otimes L m_d m_s \left(\frac{1}{p_2 \cdot q} + \frac{1}{p_1 \cdot q} \right) + L t^A \otimes R m_d m_s \left(\frac{p_2^\rho}{p_2 \cdot q} - \frac{p_1^\rho}{p_1 \cdot q} \right) \\
 & + \frac{1}{2} q^\nu \sigma_L^{\nu\rho} t^A \otimes R m_d m_s \left(\frac{1}{p_2 \cdot q} + \frac{1}{p_1 \cdot q} \right) + R t^A \otimes L m_d m_s \left(\frac{p_2^\rho}{p_2 \cdot q} - \frac{p_1^\rho}{p_1 \cdot q} \right).
 \end{aligned} \tag{C.9}$$

We finally report the full amplitude for the four quarks and a gluon

$$\mathcal{A}_{4q1g}^{\text{full}} = -i \frac{G_F^2}{16\pi^2} \text{Im}(\lambda_c^2) \hat{\mathcal{A}}_{4q1g}^{\text{full}}, \quad (\text{C.10})$$

where

$$\begin{aligned} \hat{\mathcal{A}}_{4q1g}^{\text{full}} = & \alpha_1 \left(\frac{2Rt^A \otimes \not{q}_L m_d p_1^\rho}{p_1 \cdot q} - \frac{2\not{p}_{3L} t^A \otimes \not{q}_L p_1^\rho}{p_1 \cdot q} + \not{q}_L t^A \otimes \not{p}_{1L} \left(\frac{1}{p_1 \cdot q} - \frac{1}{p_2 \cdot q} \right) p_3^\rho \right. \\ & + \not{q}_L t^A \otimes L \left(\frac{1}{p_2 \cdot q} - \frac{1}{p_1 \cdot q} \right) m_s p_3^\rho - \frac{\not{q}_L t^A \otimes \not{q}_L p_3^\rho}{p_1 \cdot q} + 2\not{p}_{3L} t^A \otimes \gamma_L^\rho \\ & + \gamma_L^\rho t^A \otimes \not{p}_{1L} \left(\frac{1}{p_2 \cdot q} - \frac{1}{p_1 \cdot q} \right) p_3 \cdot q - \frac{i\gamma_L^\alpha t^A \otimes \not{p}_{1L} (p_1 \cdot q + p_2 \cdot q) \epsilon^{\alpha\nu\sigma\rho} p_3^\nu q^\sigma}{p_1 \cdot q p_2 \cdot q} \\ & + \frac{i\gamma_L^\alpha t^A \otimes \not{q}_L \epsilon^{\alpha\nu\sigma\rho} p_3^\nu q^\sigma}{p_1 \cdot q} - 2Rt^A \otimes \gamma_L^\rho m_d + q^\nu \sigma_R^{\nu\rho} t^A \otimes \not{p}_{1L} \left(\frac{1}{p_2 \cdot q} + \frac{1}{p_1 \cdot q} \right) m_d \\ & - \frac{q^\nu \sigma_R^{\nu\rho} t^A \otimes \not{q}_L m_d}{p_1 \cdot q} + \gamma_L^\rho t^A \otimes L \left(\frac{1}{p_1 \cdot q} - \frac{1}{p_2 \cdot q} \right) p_3 \cdot q m_s \\ & + \frac{1}{2} \not{q}_L t^A \otimes \gamma_L^\rho \left(\frac{m_d^2 + m_s^2 - 2p_1 \cdot p_3 + 4p_2 \cdot q}{p_2 \cdot q} - \frac{m_d^2 + m_s^2 - 2p_1 \cdot p_3 + 2p_3 \cdot q}{p_1 \cdot q} \right) \\ & + \frac{i\gamma_L^\alpha t^A \otimes L (p_1 \cdot q + p_2 \cdot q) \epsilon^{\alpha\nu\sigma\rho} p_3^\nu q^\sigma m_s}{p_1 \cdot q p_2 \cdot q} - q^\nu \sigma_R^{\nu\rho} t^A \otimes L \left(\frac{1}{p_2 \cdot q} + \frac{1}{p_1 \cdot q} \right) m_d m_s \\ & + i\gamma_L^\alpha t^A \otimes \gamma_L^\nu \epsilon^{\alpha\nu\sigma\rho} q^\sigma \left(\frac{m_d^2 + m_s^2 - 2p_1 \cdot p_3 - 2p_2 \cdot q}{2p_2 \cdot q} + \frac{m_d^2 + m_s^2 - 2p_1 \cdot p_3 + 2p_3 \cdot q}{2p_1 \cdot q} \right) \\ & + \frac{1}{2} \gamma_L^\rho t^A \otimes \not{q}_L \left(\frac{m_d^2 + m_s^2 - 2p_1 \cdot p_3 + 4p_3 \cdot q}{p_1 \cdot q} - \frac{m_d^2 + m_s^2 - 2p_1 \cdot p_3 + 4p_2 \cdot q}{p_2 \cdot q} \right) \\ & + \not{p}_{3L} t^A \otimes \not{p}_{1L} \left(\frac{2p_1^\rho}{p_1 \cdot q} - \frac{2p_2^\rho}{p_2 \cdot q} \right) + 2Rt^A \otimes L m_d m_s \left(\frac{p_1^\rho}{p_1 \cdot q} - \frac{p_2^\rho}{p_2 \cdot q} \right) \\ & + Rt^A \otimes \not{p}_{1L} \left(\frac{2m_d p_2^\rho}{p_2 \cdot q} - \frac{2m_d p_1^\rho}{p_1 \cdot q} \right) + \not{p}_{3L} t^A \otimes L \left(\frac{2m_s p_2^\rho}{p_2 \cdot q} - \frac{2m_s p_1^\rho}{p_1 \cdot q} \right) \\ & + \gamma_L^\mu t^A \otimes \gamma_L^\mu \left(-\frac{(m_d^2 + m_s^2 - 2p_1 \cdot p_3 + 2p_3 \cdot q) p_1^\rho}{p_1 \cdot q} + \frac{(m_d^2 + m_s^2 - 2p_1 \cdot p_3) p_2^\rho}{p_2 \cdot q} + 2p_3^\rho \right) \\ & \left. - 2i\gamma_L^\alpha t^A \otimes \gamma_L^\nu \epsilon^{\alpha\nu\sigma\rho} q^\sigma, \right) \quad (\text{C.11}) \end{aligned}$$

with $\alpha_1 = -\frac{5}{9}$.

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