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Analytic results for Sudakov form factors in QCD

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ABSTRACT: Sudakov form factors appear ubiquitously in factorized cross sections where they allow one to resum large logarithms to all orders in perturbation theory. Their exact evaluation requires numerical integrals over anomalous dimensions, which in practice can hamper efficiency. Alternatively, one can use approximate analytic solutions, which provide fast evaluation at the cost of numerical precision and loss of properties such as renormalization group invariance. We provide an exact analytic expression of the QCD Sudakov form factor which allows one to obtain fast and numerically exact results.

KEYWORDS: QCD Phenomenology

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Contents

1	Introduction	1
2	The Sudakov form factor	3
2.1	Resummation accuracy	4
2.2	The role of the running coupling in retaining RG invariance	5
3	Analytic evaluation of the Sudakov form factor	6
3.1	Indefinite integral over $f(\alpha_s)/\beta(\alpha_s)$	7
3.2	Analytic solutions for η_Γ and K_γ	8
3.3	Analytic solution for K_Γ	9
3.4	Numerical solution of α_s running	10
4	Conclusions	10
A	Coefficients of the partial fraction decomposition	11
A.1	Roots of the β function	11
A.2	Coefficients of the partial fraction decomposition	12

1 Introduction

Differential observables at colliders often resolve additional collinear or soft QCD emissions on top of the underlying Born process. Typically, their perturbative expansion then exhibits Sudakov double logarithms of the form $\alpha_s^n \ln^m(Q/k)$, where $m \leq 2n$, Q is the relevant hard scale, and k is the differential observable. When the two scales become widely separated, $k \ll Q$, these logarithms become large and can eventually spoil the convergence of the perturbative expansion. In this case, they need to be resummed to all orders in α_s to restore the reliability and stability of perturbation theory. The resummation commonly involves Sudakov form factors of the form

$$U(\mu_0, \mu; Q) = \exp \left\{ \int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} \left[\Gamma[\alpha_s(\mu')] \ln \frac{Q}{\mu'} + \gamma[\alpha_s(\mu')] \right] \right\}, \quad (1.1)$$

where Γ is (related to) the cusp anomalous dimension and γ is the noncusp anomalous dimension, and their precise form depends on the observable in question. Eq. (1.1) manifestly exponentiates and thereby resums the large logarithms $\ln(\mu/\mu_0) \sim \ln(Q/k)$.

Sudakov form factors such as eq. (1.1) appear ubiquitously in QCD. They were studied a long time ago in the context of transverse momentum (q_T) resummation [1–3], threshold resummation [4, 5], as well as event shapes [6]. In recent years, there has also been much interest in the analytic resummation of double-differential observables [7–11] and

jet-related observables such as cross sections with jet vetoes [12–14]. Analytic higher-order resummation has also been utilized to achieve NNLO+PS matching [15–20].

For a few observables, resummation has reached next-to-next-next-to-leading logarithmic (N³LL) accuracy, such as event shapes at lepton colliders [21–25], the q_T spectrum in gluon fusion Higgs production [26–29], and the Drell-Yan q_T spectrum [29–36]. Reaching such high perturbative accuracy is particularly important for extracting the strong coupling constant from event shapes [21–24] and for determining nonperturbative transverse momentum distributions from Drell-Yan data [31, 32], as well as to match the (sub-)percent precision of q_T measurements reached at the LHC [37–43].

When aiming for percent level accuracy, it becomes important how to evaluate eq. (1.1). First, there are various ways to specify the resummation accuracy, and in general different prescriptions lead to slightly different results for the Sudakov form factor, see e.g. refs. [44, 45]. However, such differences formally constitute a subleading effect and should be well covered by any reliable estimate of perturbative uncertainties, and hence are not the focus of this paper. Instead, we focus on numerically precise evaluations of the integral in eq. (1.1). A priori, this seems to be a straightforward task, as one can numerically evaluate the integral at the desired precision. However, such numerical integrals may have significant impact on the runtime of computer codes, as they typically appear in the hottest loop. In particular, this can become a concern when the running coupling $\alpha_s(\mu')$ is to be evaluated numerically exact, which beyond LL is not possible in closed form.

In practice, most phenomenological studies employ an analytic approximation of the integral in eq. (1.1) that neglects higher-order terms beyond the desired formal resummation accuracy. Despite retaining formal accuracy, such analytic approximations can break important properties of the Sudakov form factor. For example, ref. [46] explicitly discussed the breaking of the relation

$$U(\mu_0, \bar{\mu}; Q)U(\bar{\mu}, \mu; Q) = U(\mu_0, \mu; Q), \quad (1.2)$$

which holds for any $\bar{\mu}$ as a consequence of eq. (1.1) and encodes RG invariance of the cross section. Ref. [46] traced back the violation of eq. (1.2) to the relation

$$\frac{d\alpha_s(\mu)}{d \ln \mu} = \beta[\alpha_s(\mu)] \quad \Rightarrow \quad \ln \frac{\mu}{\mu_0} = \int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)} \frac{d\alpha_s}{\beta(\alpha_s)}, \quad (1.3)$$

where $\beta(\alpha_s)$ is the QCD β function. Eq. (1.3) is broken by terms beyond the formal resummation accuracy when an approximate solution for the running coupling constant $\alpha_s(\mu)$ is used. This effect leads to an undesired dependency on the (formally arbitrary) scale $\bar{\mu}$ when combining multiple Sudakov form factors.

A direct consequence of eq. (1.2) is the *closure condition*,

$$U(\mu_0, \bar{\mu}; Q)U(\bar{\mu}, \mu_0; Q) = 1, \quad (1.4)$$

which was studied numerically in ref. [45] in the context of coupled gauge theories. There, the authors found that commonly employed analytic approximations of U violate eq. (1.4) at the level of up to a few percent, which can be a prohibitively large effect at N³LL

accuracy. In ref. [47], it was even argued that it may become necessary to consider the non-closure under RG invariance, there dubbed “perturbative hysteresis”, as an additional theory uncertainty. As a direct solution, ref. [45] proposed a seminumerical method, where the integral in eq. (1.1) is evaluated numerically in conjunction with an analytic approximation of the running coupling $\alpha_s(\mu')$, which by construction obeys eqs. (1.2) and (1.4).

Ref. [45] also provided a detailed numerical comparison of several approximate solutions to the numerically exact evaluation of eq. (1.1). While their seminumerical approach yielded errors of only $\mathcal{O}(0.1\%)$ and thus provides a very good approximation to the exact form factor, commonly employed approximations led to unacceptable discrepancies of up to $\mathcal{O}(10\%)$.

In this paper, we overcome the above difficulties by deriving an *exact* analytic solution of eq. (1.1). This allows one to obtain exact results at much better numerical efficiency than numeric integration methods. In particular, our result trivially obeys RG invariance and can, in principle, be evaluated at arbitrary precision.

This paper is structured as follows. In section 2, we discuss the general structure of the form factor and define the logarithmic order counting used to specify the perturbative accuracy of our results. In section 3, we derive the analytic solution for the Sudakov form factor, before concluding in section 4. Useful reference formulas are provided in appendix A.

2 The Sudakov form factor

We first rewrite the Sudakov form factor in eq. (1.1) as

$$\begin{aligned} U(\mu_0, \mu; Q) &= \exp \left\{ \int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} \left[\Gamma[\alpha_s(\mu')] \ln \frac{Q}{\mu'} + \gamma[\alpha_s(\mu')] \right] \right\} \\ &= \exp \left\{ -K_{\Gamma}[\alpha_s(\mu_0), \alpha_s(\mu)] + \ln \frac{Q}{\mu_0} \eta_{\Gamma}[\alpha_s(\mu_0), \alpha_s(\mu)] + K_{\gamma}[\alpha_s(\mu_0), \alpha_s(\mu)] \right\}, \end{aligned} \quad (2.1)$$

where the following basic kernels will be the focus of our study:

$$\begin{aligned} K_{\Gamma}[\alpha_s(\mu_0), \alpha_s(\mu)] &= \int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} \Gamma[\alpha_s(\mu')] \ln \frac{\mu'}{\mu_0} = \int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)} \frac{d\alpha_s}{\beta(\alpha_s)} \Gamma(\alpha_s) \int_{\alpha_s(\mu_0)}^{\alpha_s} \frac{d\alpha'_s}{\beta(\alpha'_s)}, \\ K_{\gamma}[\alpha_s(\mu_0), \alpha_s(\mu)] &= \int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} \gamma[\alpha_s(\mu')] = \int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)} \frac{d\alpha_s}{\beta(\alpha_s)} \gamma(\alpha_s), \\ \eta_{\Gamma}[\alpha_s(\mu_0), \alpha_s(\mu)] &= \int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} \Gamma[\alpha_s(\mu')] = \int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)} \frac{d\alpha_s}{\beta(\alpha_s)} \Gamma(\alpha_s). \end{aligned} \quad (2.2)$$

Here, the second form is obtained by employing eq. (1.3). It makes clear that K_{Γ} , K_{γ} and η_{Γ} depend on the scales μ_0 and μ only through $\alpha_s(\mu_0)$ and $\alpha_s(\mu)$, respectively. It also has the practical advantage that the running coupling has to be evaluated only at the external scales rather than continuously inside the integral, which is particularly useful when it is obtained as external input, e.g. from the parton distribution functions, or by an exact numeric solution of eq. (1.3).

The second form in eq. (2.2) also opens a straightforward way to obtain an analytic approximation by expanding the integrands in α_s to the desired order, after which one can

straightforwardly evaluate the integrals over α_s . For more details and explicit results up to N³LL, see e.g. ref. [45].

2.1 Resummation accuracy

The goal of this paper is to obtain exact analytic formulas for the integrals in eq. (2.2). For this purpose, it is important to *precisely* define the perturbative accuracy with which to evaluate the integrals, i.e. their formal resummation accuracy.

We follow the common definition of classifying the perturbative accuracy of eq. (2.2) through the perturbative order to which the cusp anomalous dimension $\Gamma(\alpha_s)$, the noncusp anomalous dimension $\gamma(\alpha_s)$, and the β function are kept. Their fixed-order expansions read

$$\Gamma(\alpha_s) = \sum_{k=0}^{\infty} \Gamma_k \left(\frac{\alpha_s}{4\pi} \right)^{k+1}, \quad \gamma(\alpha_s) = \sum_{k=0}^{\infty} \gamma_k \left(\frac{\alpha_s}{4\pi} \right)^{k+1}, \quad \beta(\alpha_s) = -2\alpha_s \sum_{k=0}^{\infty} \beta_k \left(\frac{\alpha_s}{4\pi} \right)^{k+1}. \quad (2.3)$$

NⁿLL accuracy is then specified by truncating $\Gamma(\alpha_s)$ at $\mathcal{O}(\alpha_s^{n+1})$, i.e. at $n+1$ loops, while $\gamma(\alpha_s)$ enters at one lower order, i.e. at $\mathcal{O}(\alpha_s^n)$. The running coupling is also evaluated at exactly $n+1$ -loops, i.e. the beta function is calculated at $\mathcal{O}(\alpha_s^{n+2})$. This convention is summarized in table 1. Note that the different orders at which Γ and γ enter the resummed form factor are the reason to treat K_γ and η_Γ separately, even though their formal (all-order) definitions in eq. (2.2) are identical.

In order to illustrate the importance of a precise definition of the resummation accuracy, we note that one can expand the Sudakov form factor as

$$U(\mu_0 = k, \mu = Q; Q) = \exp[L g_{\text{LL}}(a_s L) + g_{\text{NLL}}(a_s L) + a_s g_{\text{NNLL}}(a_s L) + \cdots], \quad (2.4)$$

where for simplicity we chose $\mu_0 = k$ and $\mu = Q$, and defined the abbreviations $a_s = \alpha_s(Q)/(4\pi)$ and $L = \ln(Q/k)$. The first few terms of this expansion read

$$\begin{aligned} g_{\text{LL}}(a_s L) &= \frac{\Gamma_0}{2} a_s L + (a_s L)^2 \frac{2}{3} \Gamma_0 \beta_0 + \mathcal{O}[(a_s L)^3], \\ g_{\text{NLL}}(a_s L) &= \gamma_0 a_s L + (a_s L)^2 \left(\frac{\Gamma_1}{2} + \gamma_0 \beta_0 \right) + \mathcal{O}[(a_s L)^3], \\ g_{\text{NNLL}}(a_s L) &= \gamma_1 a_s L + (a_s L)^2 \left(\frac{\Gamma_2}{2} + 2\gamma_1 \beta_0 + \gamma_0 \beta_1 \right) + \mathcal{O}[(a_s L)^3], \end{aligned} \quad (2.5)$$

where we have expanded eq. (2.5) in $\alpha_s L \ll 1$ to obtain more compact expressions. In practice, one counts $\alpha_s L \sim 1$ and such an expansion is not valid. The corresponding unexpanded expressions can be found e.g. in ref. [48], up to different overall conventions.

Eqs. (2.4) and (2.5) show that in order to exponentiate all leading logarithms, i.e. retaining all terms of $\mathcal{O}(\alpha_s^n L^{n+1})$ in the exponent, one must at least keep Γ_0 and β_0 exact. At NLL, one also needs to keep at least Γ_1 , γ_0 and (not shown in eq. (2.5)) β_1 . Extending this to higher orders results in the classification in table 1. From eq. (2.5), one can also see that formally lower-order terms also appear in the higher-order functions. For example, the term proportional to $\gamma_0 \beta_1$ arising in g_{NNLL} is classified as a NNLL term, but is already induced when keeping the dependence on γ_0 and β_1 exact as specified by our NLL counting.

Accuracy	$\Gamma(\alpha_s)$	$\gamma(\alpha_s)$	$\beta(\alpha_s)/\alpha_s$
LL	$\mathcal{O}(\alpha_s^1)$	—	$\mathcal{O}(\alpha_s^1)$
NLL	$\mathcal{O}(\alpha_s^2)$	$\mathcal{O}(\alpha_s^1)$	$\mathcal{O}(\alpha_s^2)$
NNLL	$\mathcal{O}(\alpha_s^3)$	$\mathcal{O}(\alpha_s^2)$	$\mathcal{O}(\alpha_s^3)$
N ³ LL	$\mathcal{O}(\alpha_s^4)$	$\mathcal{O}(\alpha_s^3)$	$\mathcal{O}(\alpha_s^4)$

Table 1. Classification of the resummation accuracy in terms of the fixed-order expansions of the anomalous dimensions and beta function.

In principle, one may define NⁿLL by keeping only the corresponding terms $g_{N^n\text{LL}}$ in eq. (2.4). As illustrated, this differs from our classification of evaluating each anomalous dimension in eq. (1.1) according to table 1. Both definitions are well defined, and they only differ by higher-order terms in either counting. In practice, as long as the difference between such prescriptions is well covered by an estimate of theory uncertainties, either approach is justified. However, prescriptions truncating the exponent as in eq. (2.4) naturally break the RG invariance (except at LL and N[∞]LL), i.e. eq. (1.2), raising the question whether such a shortcoming should be assigned its own uncertainty [47]. Thus, from a purely theoretical point of view it is advisable to classify the resummation accuracy solely via the integrand in eq. (1.1). The importance of formally defining the logarithmic order-counting via the fixed-order perturbative expansion of the anomalous dimensions, as opposed to retaining towers of logarithms in the solution of the Sudakov, was also highlighted in refs. [44, 49].

2.2 The role of the running coupling in retaining RG invariance

In section 2.1, we specified that for exact NⁿLL accuracy the QCD β function is solved at exactly $n+1$ loop order. In practice, this may not always be feasible. For example, for consistency with external input such as parton distribution functions, one may wish to take the running coupling as external input itself, in which case it may not obey the exact β function due to formally higher-order terms. The same happens if one evaluates the running coupling at higher logarithmic accuracy than in the resummation itself. Both cases are formally well defined as long as they differ from strict NⁿLL accuracy only by higher-order terms in the β function.

Nevertheless, in such a case the two forms in eq. (2.2) are not equivalent any more, and one must choose which one to define as the fundamental quantity. For the practical reasons mentioned before, we define the kernels through the second form in eq. (2.2), i.e. as integrals over α_s rather than μ' . However, in this case one breaks the RG invariance, i.e. eq. (1.2), as already studied in detail in ref. [46]. Since the RG invariance is only broken beyond the formal logarithmic accuracy, this procedure still defines a valid scheme.

Of course, it is still desirable to retain RG invariance as specified by eq. (1.2). Using eq. (2.1), we can rewrite this condition as

$$\begin{aligned}
 & K_\Gamma[\alpha_s(\mu_0), \alpha_s(\mu)] - K_\Gamma[\alpha_s(\mu_0), \alpha_s(\bar{\mu})] - K_\Gamma[\alpha_s(\bar{\mu}), \alpha_s(\mu)] \\
 &= \ln \frac{Q}{\mu_0} \eta_\Gamma[\alpha_s(\mu_0), \alpha_s(\mu)] - \ln \frac{Q}{\mu_0} \eta_\Gamma[\alpha_s(\mu_0), \alpha_s(\bar{\mu})] - \ln \frac{Q}{\bar{\mu}} \eta_\Gamma[\alpha_s(\bar{\mu}), \alpha_s(\mu)] \quad (2.6)
 \end{aligned}$$

for the cusp piece, and

$$K_\gamma[\alpha_s(\mu_0), \alpha_s(\mu)] = K_\gamma[\alpha_s(\mu_0), \alpha_s(\bar{\mu})] + K_\gamma[\alpha_s(\bar{\mu}), \alpha_s(\mu)] \quad (2.7)$$

for the noncusp piece. The latter is trivially true for either form in eq. (2.2). To validate eq. (2.6), we take the derivative with respect to $\bar{\alpha}_s \equiv \alpha_s(\bar{\mu})$, which yields

$$\begin{aligned} & -\frac{\Gamma(\bar{\alpha}_s)}{\beta(\bar{\alpha}_s)} \int_{\alpha_s(\mu_0)}^{\bar{\alpha}_s} \frac{d\alpha'_s}{\beta(\alpha'_s)} + \int_{\bar{\alpha}_s}^{\alpha_s(\mu)} \frac{d\alpha_s}{\beta(\alpha_s)} \frac{\Gamma(\alpha_s)}{\beta(\bar{\alpha}_s)} \\ & = -\ln \frac{Q}{\mu_0} \frac{\Gamma(\bar{\alpha}_s)}{\beta(\bar{\alpha}_s)} + \frac{1}{\beta(\bar{\alpha}_s)} \int_{\bar{\alpha}_s}^{\alpha_s(\mu)} \frac{d\alpha_s}{\beta(\alpha_s)} \Gamma(\alpha_s) + \ln \frac{Q}{\bar{\mu}} \frac{\Gamma(\bar{\alpha}_s)}{\beta(\bar{\alpha}_s)}, \end{aligned} \quad (2.8)$$

which can be simplified to

$$\int_{\alpha_s(\mu_0)}^{\bar{\alpha}_s} \frac{d\alpha'_s}{\beta(\alpha'_s)} = \ln \frac{\bar{\mu}}{\mu_0}. \quad (2.9)$$

As expected, eq. (2.6) is true if and only if the QCD β function is obeyed exactly. Conversely, using a running coupling $\alpha_s(\mu)$ which does not exactly obey the β function at the specified order will violate RG invariance [46].

A possible solution to this problem was already found in ref. [46] by imposing eq. (2.9) at the level of the Sudakov form factor. Specifically, we rewrite eq. (2.1) as

$$\begin{aligned} U(\mu_0, \mu; Q) = \exp \Big\{ & -K_\Gamma[\alpha_s(\mu_0), \alpha_s(\mu)] + L[\alpha_s(\mu_0), \alpha_s(Q)] \eta_\Gamma[\alpha_s(\mu_0), \alpha_s(\mu)] \\ & + K_\gamma[\alpha_s(\mu_0), \alpha_s(\mu)] \Big\}, \end{aligned} \quad (2.10)$$

where we replaced $\ln(Q/\mu_0)$ by¹

$$L[\alpha_s(\mu_0), \alpha_s(Q)] = \int_{\alpha_s(\mu_0)}^{\alpha_s(Q)} \frac{d\alpha_s}{\beta(\alpha_s)} \xrightarrow[\text{running}]{\text{exact } \alpha_s} \ln \frac{Q}{\mu_0}. \quad (2.11)$$

For exact α_s running, this is by definition identical to $\ln(Q/\mu_0)$, while otherwise it differs from it by formally subleading terms. Thus, eq. (2.10) constitutes a valid definition of resummation accuracy on its own.

3 Analytic evaluation of the Sudakov form factor

In this section, we obtain analytic results for the basic building blocks K_Γ , K_γ and η_Γ defined in eq. (2.2), which in turn yield an exact solution of the Sudakov form factor in eq. (1.1). The key ingredient is the generic indefinite integral over a rational function of the form $f(\alpha_s)/\beta(\alpha_s)$, which is discussed in detail in section 3.1. The resulting solutions for η_Γ and K_γ are presented in section 3.2, while the result for K_Γ is more involved and presented in section 3.3. We also briefly comment on the option to numerically solve the QCD β function with our method in section 3.4.

The results presented in this section have been validated against numerical implementations of the RG kernels, and were already used for the N³LL' resummation of the Energy-Energy Correlation in ref. [25] based on an implementation in SCETlib [50].

¹One may be tempted to write this integral as $\eta(\mu_0, Q)$ or $K_\Gamma(\mu_0, Q)$ with $\mathbb{I}(\alpha_s) = 1$, but the corresponding analytic expressions are not applicable as they are derived assuming $\mathbb{I}(\alpha_s) = \mathcal{O}(\alpha_s)$.

3.1 Indefinite integral over $f(\alpha_s)/\beta(\alpha_s)$

We start by considering the generic indefinite integral

$$I_f(\alpha_s) \equiv \int d\alpha_s \frac{f(\alpha_s)}{\beta(\alpha_s)}, \quad (3.1)$$

where $f(\alpha_s)$ and $\beta(\alpha_s)$ are polynomials in α_s , expanded in $a_s \equiv \alpha_s/(4\pi)$ as

$$f(\alpha_s) = \sum_{k=-1}^m f_k a_s^{k+1}, \quad \beta(\alpha_s) = -8\pi a_s \sum_{k=0}^n \beta_k a_s^{k+1}. \quad (3.2)$$

While anomalous dimensions obey $f(\alpha_s) = \mathcal{O}(\alpha_s)$, it will be useful to allow for a non-vanishing constant $f_{-1} \neq 0$ to also cover the case of $L(\mu_0, Q)$ defined in eq. (2.11). The expansions in eq. (3.2) are truncated at $\mathcal{O}(\alpha_s^m)$ and $\mathcal{O}(\alpha_s^n)$, respectively, with m and n given according to table 1.

Our definition of NⁿLL accuracy implies that

$$m \leq n \quad \Leftrightarrow \quad \deg(f) < \deg(\beta), \quad (3.3)$$

such that $\beta(\alpha_s)$ is always a polynomial of higher degree than $f(\alpha_s)$. It follows that $f(\alpha_s)/\beta(\alpha_s)$ is a proper rational function, and as such its partial fraction decomposition has no finite remainder. To obtain it, we first factorize the β function as

$$\beta(\alpha_s) = -8\pi a_s^2 (\beta_0 + \beta_1 a_s + \cdots + \beta_n a_s^n) = -8\pi a_s^2 \beta_n \prod_{i=1}^n (a_s - \delta_i), \quad (3.4)$$

where n is the truncation order specified by eq. (3.2), and we factored out the highest coefficient β_n . The δ_i are the corresponding nonvanishing roots of $\beta(\alpha_s)$, which implicitly depend on the order n of the β function expansion. For later convenience, we also define $\delta_0 \equiv 0$. Recall that the roots of a polynomial are in general complex, even though $\beta(\alpha_s)$ itself is a real function. For $\beta(\alpha_s)$, starting at NNLL ($n = 2$) one indeed encounters complex roots, such that all following expressions are to be understood within complex analysis. Explicit expressions for the δ_i up to N³LL are provided in appendix A.1.

We now write the partial fraction decomposition of $f(\alpha_s)/\beta(\alpha_s)$ as

$$\frac{f(\alpha_s)}{\beta(\alpha_s)} = -\frac{1}{8\pi} \left[\frac{r'_{f,0}}{a_s^2} + \frac{r_{f,0}}{a_s} + \sum_{i=1}^n \frac{r_{f,i}}{a_s - \delta_i} \right], \quad (3.5)$$

which has no finite or purely polynomial contributions thanks to eq. (3.3). We also assume that all δ_i are distinct, i.e. $a_s = 0$ is the only root of higher multiplicity. Using the five-loop result in refs. [51, 52], one can explicitly check that this is fulfilled for the QCD β function at least up to N⁴LL. In eq. (3.5), we have extracted an overall factor $1/(8\pi)$ to obtain more compact expressions for the coefficients $r_{f,i}$. Note that the double pole $r'_{f,0}/a_s^2$ vanishes if $f(\alpha_s) = \mathcal{O}(\alpha_s)$, as is the case for anomalous dimensions.

The coefficients $r'_{f,0}$ and $r_{f,i}$ can be regarded as the coefficient of the corresponding pole of the Laurent series around δ_i , and thus are given in terms of the corresponding

residues,

$$r'_{f,0} = -8\pi \operatorname{Res}_{a_s=0} \left[a_s \frac{f(4\pi a_s)}{\beta(4\pi a_s)} \right], \quad r_{f,i} = -8\pi \operatorname{Res}_{a_s=\delta_i} \left[\frac{f(4\pi a_s)}{\beta(4\pi a_s)} \right]. \quad (3.6)$$

Note that these coefficients depend on the order n of the β function and its roots, and thus differ at different resummation orders. For simplicity, we keep this dependence implicit. The residues can be explicitly evaluated in terms of the coefficients in eqs. (3.2) and (3.4),

$$r'_{f,0} = \frac{f_{-1}}{\beta_0}, \quad r_{f,0} = \frac{f_0}{\beta_0} - \theta_{n>0} \frac{f_{-1}\beta_1}{\beta_0^2}, \quad r_{f,i>0} = -2 \frac{f(4\pi\delta_i)}{\beta'(4\pi\delta_i)}. \quad (3.7)$$

Here, the $\theta_{n>0}$ is necessary, as for $n = 0$ one has $\beta(\alpha_s) = -8\pi a_s^2$, and thus a single pole from β_1 is not yet allowed.

Using eq. (3.5), it is trivial to evaluate the indefinite integral in eq. (3.1),

$$I_f(\alpha_s) = \int d\alpha_s \frac{f(\alpha_s)}{\beta(\alpha_s)} = \frac{r'_{f,0}}{2a_s} - \frac{r_{f,0}}{2} \ln a_s - \frac{1}{2} \sum_{i=1}^n r_{f,i} \ln(a_s - \delta_i). \quad (3.8)$$

The simple expressions in eqs. (3.7) and (3.8) are key ingredients to analytically evaluate the integrals in eq. (2.2).

Before proceeding, we briefly discuss the case of complex roots δ_i . While from eq. (3.7) it is clear that $r'_{f,0}$ and $r_{f,0}$ are manifestly real, the $r_{f,i}$ and δ_i are in general complex. However, by the complex conjugate root theorem, if δ_i is a complex root with coefficient $r_{f,i}$, then its conjugate δ_i^* is also a root with the conjugate coefficient $r_{f,i}^*$. The contribution of such a pair of complex roots to eq. (3.5) is given by

$$\frac{r_{f,i}}{a_s - \delta_i} + \frac{r_{f,i}^*}{a_s - \delta_i^*} = 2 \frac{a_s \Re(r_{f,i}) - \Re(r_{f,i} \delta_i^*)}{[a_s - \Re(\delta_i)]^2 + \Im(\delta_i)^2}, \quad (3.9)$$

where $\Re(x)$ and $\Im(x)$ denote the real and imaginary part of x , respectively. The indefinite integral then becomes

$$I_f(\alpha_s) = \frac{r'_{f,0}}{2a_s} - \sum_{i=0}^n \left\{ \frac{\Re(r_{f,i})}{4} \ln[(a_s - \Re(\delta_i))^2 + \Im(\delta_i)^2] - \frac{\Im(r_{f,i})}{2} \operatorname{atan} \left[\frac{a_s - \Re(\delta_i)}{\Im(\delta_i)} \right] \right\}, \quad (3.10)$$

where we treat real and complex roots on equal footing. If desired, one can thus write eq. (3.8) in a manifestly real fashion to avoid complex-valued logarithms. Eq. (3.10) trivially reproduces eq. (3.8) in the special case that all imaginary parts vanish. In the following, we will always employ eq. (3.8), as it leads to more compact expressions than eq. (3.10).

3.2 Analytic solutions for η_Γ and K_γ

From eq. (2.2), it is obvious that η_Γ and K_γ can be trivially obtained from eq. (3.8). Since they involve anomalous dimensions starting at $\mathcal{O}(\alpha_s)$, we have $r'_{f,0} = 0$, such that we obtain

$$\begin{aligned} \eta_\Gamma(\alpha_0, \alpha_s) &= -\frac{1}{2} \sum_{i=0}^n r_{\Gamma,i} \ln \frac{\alpha_s - 4\pi\delta_i}{\alpha_0 - 4\pi\delta_i}, \\ K_\gamma(\alpha_0, \alpha_s) &= -\frac{1}{2} \sum_{i=0}^n r_{\gamma,i} \ln \frac{\alpha_s - 4\pi\delta_i}{\alpha_0 - 4\pi\delta_i}. \end{aligned} \quad (3.11)$$

We stress again that the residues $r_{f,i}$ and the roots δ_i depend on the truncation order n . From eq. (3.7), we have

$$\begin{aligned} r_{\Gamma,0} &= \frac{\Gamma_0}{\beta_0}, & r_{\Gamma,i} &= -2 \frac{\Gamma(4\pi\delta_i)}{\beta'(4\pi\delta_i)}, & i &= 1, \dots, n, \\ r_{\gamma,0} &= \frac{\gamma_0}{\beta_0} \theta_{n>0}, & r_{\gamma,i} &= -2 \frac{\gamma(4\pi\delta_i)}{\beta'(4\pi\delta_i)}, & i &= 1, \dots, n, \end{aligned} \quad (3.12)$$

where it is important to recall that according to the counting in table 1, γ enters at one lower order than Γ .

3.3 Analytic solution for K_Γ

Recall the definition of K_Γ in eq. (2.2),

$$K_\Gamma(\alpha_0, \alpha_s) = \int_{\alpha_0}^{\alpha_s} d\alpha'_s \frac{\Gamma(\alpha'_s)}{\beta(\alpha'_s)} \int_{\alpha_0}^{\alpha'_s} \frac{d\alpha''_s}{\beta(\alpha''_s)}. \quad (3.13)$$

The inner integral follows from eq. (3.8) by setting $f(\alpha_s) \equiv 1$,

$$L(\alpha_0, \alpha'_s) = \int_{\alpha_0}^{\alpha'_s} \frac{d\alpha''_s}{\beta(\alpha''_s)} = \frac{\bar{r}'_0}{2} \left(\frac{1}{a'_s} - \frac{1}{a_0} \right) - \frac{1}{2} \bar{r}_0 \ln \frac{a'_s}{a_0} - \frac{1}{2} \sum_{i=1}^n \bar{r}_i \ln \frac{a'_s - \delta_i}{a_0 - \delta_i}, \quad (3.14)$$

where as before $a'_s = \alpha'_s/(4\pi)$ and $a_0 = \alpha_0/(4\pi)$, and the $\bar{r}_i$ follow from eq. (3.7) as

$$\bar{r}'_0 = \frac{1}{\beta_0}, \quad \bar{r}_0 = -\frac{\beta_1}{\beta_0^2} \theta_{n>0}, \quad \bar{r}_i = \frac{-2}{\beta'(4\pi\delta_i)}. \quad (3.15)$$

Inserting eq. (3.14) into eq. (3.13) and using the partial fractioning in eq. (3.5), one obtains

$$\begin{aligned} K_\Gamma(\alpha_0, \alpha_s) &= \int_{\alpha_0}^{\alpha_s} d\alpha'_s \frac{\Gamma(\alpha'_s)}{\beta(\alpha'_s)} L(\alpha_0, \alpha'_s) \\ &= -\frac{1}{4} \int_{a_0}^{a_s} da'_s \left[\sum_{i=0}^n \frac{r_{\Gamma,i}}{a'_s - \delta_i} \right] \left[\bar{r}'_0 \left(\frac{1}{a'_s} - \frac{1}{a_0} \right) - \sum_{j=0}^n \bar{r}_j \ln \frac{a'_s - \delta_j}{a_0 - \delta_j} \right], \end{aligned} \quad (3.16)$$

where the $r_{\Gamma,i}$ are the same as in eq. (3.12). Evaluating the remaining integral, we arrive at the final result

$$\begin{aligned} K_\Gamma(\alpha_0, \alpha_s) &= \frac{1}{4} r_{\Gamma,0} \bar{r}'_0 \left(\frac{1}{a_s} + \frac{L_0 - 1}{a_0} \right) + \frac{1}{8} r_{\Gamma,0} \bar{r}_0 L_0^2 \\ &+ \frac{1}{4} \sum_{i=1}^n (r_{\Gamma,i} \bar{r}_0 - r_{\Gamma,0} \bar{r}_i) \left[L_0 \ln \left(1 - \frac{a_0}{\delta_i} \right) + \text{Li}_2 \left(\frac{a_s}{\delta_i} \right) - \text{Li}_2 \left(\frac{a_0}{\delta_i} \right) \right] \\ &+ \frac{1}{4} \sum_{i=1}^n r_{\Gamma,i} \left[\frac{1}{2} \bar{r}_i L_i^2 + \bar{r}_0 L_0 L_i + \bar{r}'_0 \left(\frac{L_0 - L_i}{\delta_i} + \frac{L_i}{a_0} \right) \right] \\ &+ \frac{1}{4} \sum_{\substack{i,j=1 \\ i \neq j}}^n r_{\Gamma,i} \bar{r}_j \left[L_j \ln \frac{\delta_i - a_s}{\delta_i - \delta_j} + \text{Li}_2 \left(\frac{a_s - \delta_j}{\delta_i - \delta_j} \right) - \text{Li}_2 \left(\frac{a_0 - \delta_j}{\delta_i - \delta_j} \right) \right], \end{aligned} \quad (3.17)$$

where we defined the logarithms (again suppressing the dependence on n)

$$L_i = \ln \frac{a_s - \delta_i}{a_0 - \delta_i}. \quad (3.18)$$

Compared to eq. (3.11), eq. (3.17) is significantly more involved. In particular, starting at NNLL one encounters complex roots and residues, and hence the evaluation of eq. (3.17) involves complex dilogarithms. However, these are readily available in most programming languages, and thus do not pose a problem in practice.

3.4 Numerical solution of α_s running

As a side remark, we note that the above results can also be used for a numerical solution of the QCD β function. From eq. (3.14), we have

$$\ln \frac{\mu}{\mu_0} = L[\alpha_s(\mu_0), \alpha_s(\mu)] = \frac{2\pi}{\beta_0} \left[\frac{1}{\alpha_s(\mu)} - \frac{1}{\alpha_s(\mu_0)} \right] - \frac{1}{2} \sum_{i=0}^n \bar{r}_i \ln \frac{\alpha_s(\mu) - 4\pi\delta_i}{\alpha_s(\mu_0) - 4\pi\delta_i}, \quad (3.19)$$

where we already used that $\bar{r}'_0 = 1/\beta_0$, with the remaining $\bar{r}_i$ given in eq. (3.15). The running coupling constant $\alpha_s(\mu)$ follows by numerically inverting eq. (3.19), for example using a root finding algorithm such as Newton's method. In particular, using an analytic approximation of the exact running coupling as the starting point of the root finder greatly improves its convergence. This provides an alternative approach to established methods such as numerically solving the differential equation, for example using the Runge-Kutta method.

4 Conclusions

We have presented an analytic and exact evaluation of the Sudakov form factor that appears ubiquitously in resummed predictions. Our results are based on a partial fraction decomposition of rational functions of the form $f(\alpha_s)/\beta(\alpha_s)$, where $f(\alpha_s)$ is any anomalous dimension and $\beta(\alpha_s)$ is the QCD β function. This allows one to analytically compute the integral with respect to α_s , and explicit expressions for the basic building block of the Sudakov form factor are provided. As a byproduct, we also obtained a method to compute the running coupling constant by means of a numeric root solver.

A key ingredient for our solution is the precise definition of logarithmic accuracy, which we define by the perturbative order of the anomalous dimensions and β function. This definition can not be fulfilled exactly when using a running coupling constant $\alpha_s(\mu)$ that violates the β function by terms beyond formal accuracy. In this case, it is well known that renormalization group invariance can be violated, but it can be restored using a modified definition of the Sudakov form factor which reproduces the standard form in the case of exact running coupling.

Our results are particularly useful for resummed calculations at high accuracy. An important example is q_T resummation, where $\mathcal{O}(1\%)$ accuracy is desired and where the resummation is commonly performed prior to a Fourier and phase-space integral, such that a fast and precise evaluation of the Sudakov form factor is indispensable. Thanks to their

analytic nature, our results can be evaluated at arbitrary precision, and thus can also serve as a well-defined reference result to numerically validate the accuracy of commonly used approximations. It may also prove useful when comparing different resummation codes against each other, as our results can remove one possible source of numerical differences.

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A Coefficients of the partial fraction decomposition

In this appendix, we summarize and provide explicit expressions for the coefficients of the partial fraction decompositions. Following table 1, at $N^n\text{LL}$ accuracy, we expand the cusp and noncusp anomalous dimensions as

$$\Gamma(\alpha_s) = \sum_{k=0}^n \Gamma_k \left(\frac{\alpha_s}{4\pi} \right)^{k+1}, \quad \gamma(\alpha_s) = \sum_{k=0}^{n-1} \gamma_k \left(\frac{\alpha_s}{4\pi} \right)^{k+1}, \quad (\text{A.1})$$

while the QCD β function is expanded as

$$\beta(\alpha_s) \equiv \frac{d\alpha_s}{d \ln \mu} = -2\alpha_s \sum_{k=0}^n \beta_k \left(\frac{\alpha_s}{4\pi} \right)^{k+1}. \quad (\text{A.2})$$

A.1 Roots of the β function

We define the roots δ_i of the β function at $N^n\text{LL}$ accuracy according to eq. (3.2),

$$\beta(\alpha_s) = -8\pi a_s^2 \beta_n \prod_{i=1}^n (a_s - \delta_i), \quad a_s = \frac{\alpha_s}{4\pi}. \quad (\text{A.3})$$

In addition to the double root at $\delta_0 \equiv 0$, one obtains the following roots:

$$\begin{aligned} \text{NLL :} \quad & \delta_1 = -\frac{\beta_0}{\beta_1}, \\ \text{NNLL :} \quad & \delta_{1,2} = \frac{-\beta_1 \pm i\sqrt{4\beta_0\beta_2 - \beta_1^2}}{2\beta_2}, \\ \text{N}^3\text{LL :} \quad & \delta_i = -\frac{1}{3\beta_3} \left(\beta_2 + \xi^i C + \frac{\Delta_0}{\xi^i C} \right), \quad i = 1, 2, 3. \end{aligned} \quad (\text{A.4})$$

Note that at NNLL, $\delta_{1,2}$ are imaginary valued for $n_f \leq 5$. The roots at N^3LL are expressed in terms of the constants

$$C = \sqrt[3]{\Delta_1 + \sqrt{\Delta_1^2 - \Delta_0^3}}, \quad \Delta_0 = \beta_2^2 - 3\beta_1\beta_3, \quad \Delta_1 = \beta_2^3 - \frac{9}{2}\beta_3(\beta_1\beta_2 - 3\beta_0\beta_3), \quad (\text{A.5})$$

and the three different roots are distinguished by different powers of the primitive cube root of unity,

$$\xi = \frac{-1 + \sqrt{-3}}{2} = -\frac{1}{2} + \frac{i}{2}\sqrt{3}. \quad (\text{A.6})$$

This implies that one can take any cubic and square root in C . For $n_f \leq 7$, $\Delta_1^2 - \Delta_0^3 > 0$, such that C is a real number and one has two complex roots and one real root at N^3LL .

A.2 Coefficients of the partial fraction decomposition

We require the partial fraction decompositions

$$\begin{aligned}\frac{\Gamma(\alpha_s)}{\beta(\alpha_s)} &= -\frac{1}{8\pi} \left[\frac{r_{\Gamma,0}}{a_s} + \sum_{i=1}^n \frac{r_{\Gamma,i}}{a_s - \delta_i} \right], \\ \frac{\gamma(\alpha_s)}{\beta(\alpha_s)} &= -\frac{1}{8\pi} \left[\frac{r_{\gamma,0}}{a_s} + \sum_{i=1}^n \frac{r_{\gamma,i}}{a_s - \delta_i} \right], \\ \frac{1}{\beta(\alpha_s)} &= -\frac{1}{8\pi} \left[\frac{\bar{r}'_0}{a_s^2} + \frac{\bar{r}_0}{a_s} + \sum_{i=1}^n \frac{\bar{r}_i}{a_s - \delta_i} \right],\end{aligned}\tag{A.7}$$

where the roots δ_i are given in eq. (A.4). Note that the upper limit n in the second sum is governed by the perturbative order of the β function, even though $\gamma(\alpha_s)$ itself is truncated at one lower order. Furthermore, only $1/\beta$ contains a double pole at $a_s = 0$.

The residues required at NⁿLL are given by

$$\begin{aligned}\bar{r}'_0 &= \frac{1}{\beta_0}, & \bar{r}_0 &= -\frac{\beta_1}{\beta_0^2} \theta_{n>0}, & \bar{r}_i &= \frac{-2}{\beta'(4\pi\delta_i)}, & i &= 1, \dots, n, \\ r_{\Gamma,0} &= \frac{\Gamma_0}{\beta_0}, & r_{\Gamma,i} &= \bar{r}_i \Gamma(4\pi\delta_i), & i &= 1, \dots, n, \\ r_{\gamma,0} &= \frac{\gamma_0}{\beta_0} \theta_{n>0}, & r_{\gamma,i} &= \bar{r}_i \gamma(4\pi\delta_i), & i &= 1, \dots, n.\end{aligned}\tag{A.8}$$

Note that the form of the residues at $a_s = 0$ is independent of the order n , except that $\bar{r}_0$ and $r_{\gamma,0}$ only contribute beyond LL. From eq. (A.1), the cusp and noncusp anomalous dimensions at the appropriate orders are given by

$$\Gamma(4\pi\delta_i) = \sum_{k=0}^n \Gamma_k \delta_i^{k+1}, \quad \gamma(4\pi\delta_i) = \sum_{k=0}^{n-1} \gamma_k \delta_i^{k+1},\tag{A.9}$$

and the derivatives of the β function can be evaluated using

$$\frac{1}{\bar{r}_i} = \frac{\beta'(4\pi\delta_i)}{-2} = \sum_{k=0}^n (k+2) \beta_k \delta_i^{k+1} = \delta_i^2 \beta_n \prod_{\substack{j=1 \\ i \neq j}}^n (\delta_i - \delta_j).\tag{A.10}$$

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