Radiative semileptonic $\bar{B}$ decays

Michele Papucci, Tanner Trickle and Mark B. Wise

Walter Burke Institute for Theoretical Physics, California Institute of Technology,
1200 E California Blvd, Pasadena, CA 91125, U.S.A.

E-mail: mpapucci@caltech.edu, ttrickle@caltech.edu, wise@caltech.edu

ABSTRACT: We consider the form factors for the radiative semileptonic decays $\bar{B}(v) \rightarrow D^{(*)}(v')\ell\bar{\nu}_\ell\gamma$ in the kinematic region where the photon momentum, k, is small enough that heavy quark symmetry (HQS) can be applied without the radiated photon changing the heavy quark velocity (i.e., $v^{(\prime)} \cdot k < m_{(b,c)}$). We find that HQS is remarkably powerful, leaving only four new undetermined form factors at leading order in $1/m_{(b,c)}$. In addition, one of them is fixed in terms of the leading order Isgur-Wise function in the kinematic region, $v^{(\prime)} \cdot k < \Lambda_{\text{QCD}}$.

https://doi.org/10.1007/JHEP02(2022)043

Contents

1 Introduction

In the limit where the bottom and charm quark are taken as much heavier than the scale of the strong interactions, QCD contains a heavy quark spin-flavor symmetry (or simply heavy quark symmetry (HQS)) [1–4] that acts on heavy quarks with the same four velocity. These heavy quark spin-flavor symmetries are a powerful tool for understanding some non-perturbative properties of hadrons containing a heavy quark. One of the earliest, and still most important, applications is to the form factors for the semileptonic decays of $\bar{B}$ mesons to D and D^* mesons, i.e., $\bar{B} \to D^{(*)}\ell\bar{\nu}_\ell$, where $\ell \in \{e, \mu, \tau\}$. Lorentz symmetry and parity invariance of the strong interactions imply that these decays are characterized by six Lorentz invariant form factors, but HQS implies that there is just one (the Isgur-Wise function [4]). Furthermore it is normalized to unity at the zero recoil point where the final state charmed meson is at rest in the rest frame of the decaying $\bar{B}$ meson. In addition the matrix elements of the weak vector and axial vector current are independent of the masses of the heavy quarks.

Radiative semileptonic $\bar{B}$ decays to charmed mesons provide a unique laboratory to investigate the predictions of HQS in more complicated kinematic settings, while keeping one initial and one final heavy meson as the only hadrons in the process. The presence of the extra photon provides new kinematic invariants beyond $w \equiv v \cdot v'$, namely $v \cdot k$ and $v' \cdot k$. Here v is the four velocity of the decaying $\bar{B}$ meson, v' is the four velocity of the final state $D^{(*)}$ meson, and k is the four momentum of the final state photon.

Belle II should allow for the study of radiative semileptonic $\bar{B}$ decay at a level of precision approaching what the previous generation of B factories, i.e., Belle [5–12], BaBar [13–19], as well as LHCb [20–22], did for the semileptonic $\bar{B}$ decays.

Moreover, radiative semileptonic $\bar{B}$ decays are an irreducible background to the measurement of non-radiative semileptonic measurements, in the kinematic regions where the photon is not reconstructed. These kinematic regions are dominated by the soft region for Belle II and are more complicated for LHCb, due to the detector geometry and the boost distribution of the $\bar{B}$ mesons. It has been shown [23, 24] that differences in modeling the radiation processes may induce few percent systematic uncertainties in the semileptonic measurements. Having good control of radiative semileptonic processes is therefore necessary for precision studies of the non-radiative semileptonic decays by Belle II and LHCb.

In the $k \to 0$ limit, the radiative and non-radiative processes are related by gauge invariance. Previous work on radiative semileptonic $\bar{B}$ decays has mostly focused in the regime where the photon is soft ($k < \Lambda_{\mathrm{QCD}}$) [25–27]. Away from the soft limit, uncalculable "structure dependent" (SD) contributions arise [28], stemming from the composite nature of the $\bar{B}$ and $D^{(*)}$ mesons. Lorentz invariance and parity invariance of the strong interactions imply that the radiative decays are characterized by 32 scalar form factors that can depend on $v \cdot k$, $v' \cdot k$ and w; a daunting number for gaining knowledge of these SD terms either from experimental measurements or lattice QCD calculations. However, similar to the case of non-radiative semileptonic $\bar{B}$ decays, HQS greatly reduces the number of these form factors.

The purpose of this paper is to initiate this investigation and work out the predictions of HQS for radiative semileptonic $\bar{B}$ decays. Here we will work to leading order in the heavy quark mass expansion, neglecting terms suppressed by powers of $1/m_{b,c}$. Furthermore we work in the kinematic region where radiating the photon does not change the heavy quark velocities. It is in this kinematic region that HQS is most powerful. We find that with these approximations there are only four new (i.e., not proportional to the Isgur-Wise function) Lorentz invariant form factors, and one is normalized to the Isgur-Wise function as $k \to 0$.

In section 2 we begin by calculating the matrix element, $\mathcal{M}$, in terms of general hadronic form factors for $\bar{B} \to D^{(*)} \ell \bar{\nu}_\ell \gamma$ using heavy quark effective theory [29–32]. Then, in section 3, we reduce the number of new, independent form factors to just four, ζ_{1-4}, using the Ward identities. In section 4 we explore the soft photon limit including the leading order behavior in section 4.1 and some terms at sub-leading order in section 4.2. A concluding discussion is given in section 5. Two appendices are included; the first, appendix A works out the general structure of the matrix element, $\mathcal{M}$, for $\bar{B} \to D^{(*)} \ell \bar{\nu}_\ell \gamma$ imposing only Lorentz invariance and the parity symmetry of the strong interactions. The second, appendix B, generalizes the results to new interactions that do not have the usual $V - A$ structure of the weak interactions; potentially useful in the future for new physics signatures.

2 Matrix element

Since $\bar{B} \to D^{(*)} \ell \bar{\nu}_\ell \gamma$ is a radiative process all matrix elements will have a photon polarization vector, ε^μ appended. We factor this out immediately, defining $\mathcal{M} \equiv \varepsilon^*_\mu \mathcal{M}^\mu$ where $\mathcal{M}$ is the total matrix element. In the heavy quark limit there are three classes of Feynman
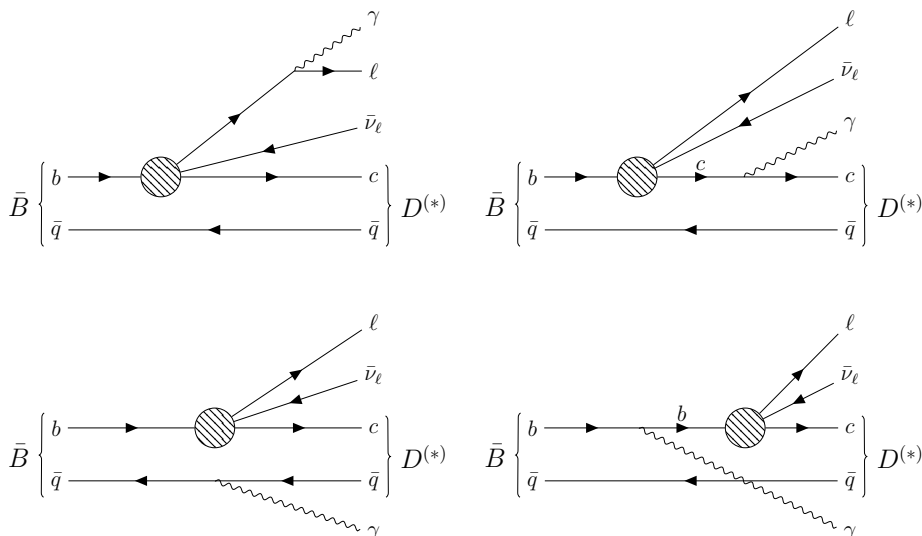
Figure 1. Feynman diagrams contributing to the $\bar{B} \to D^{(*)}\ell\bar{\nu}_\ell\gamma$ process. Diagrams with internal gluon lines and $q\bar{q}$ loops are omitted for illustrative simplicity. *Top left:* $\mathcal{M}_{\text{lepton}}$, radiation from the external lepton. *Top right:* $\mathcal{M}_{\text{heavy}}^c$, radiation from the heavy c quark. *Bottom left:* $\mathcal{M}_{\text{light}}$, radiation from the light degrees of freedom. *Bottom right:* $\mathcal{M}_{\text{heavy}}^b$, radiation from the heavy b quark.

diagrams that can contribute to this process, shown in figure 1. The first, illustrated in the top left panel of figure 1, is from radiation off the external charged lepton line,

$$\frac{\mathcal{M}_{\text{lepton}}^\mu}{\sqrt{m_B m_{D^{(*)}}}} = q_\ell \frac{G_F e V_{cb}}{\sqrt{2}} \frac{(F_V^\nu - F_A^\nu)}{2p_\ell \cdot k} \bar{u}\gamma^\mu(\slashed{p}_\ell + \slashed{k} + m_\ell)\gamma_\nu(1 - \gamma^5)v \tag{2.1}$$

where $q_\ell e$ is the electromagnetic charge of the lepton, p_ℓ is the lepton momentum, and k is the photon momentum. We write this matrix element in terms of hadronic matrix elements, F_Γ^ν, where $\Gamma = V$ implies $\Gamma^\nu = \gamma^\nu$ and $\Gamma = A$ implies $\Gamma^\nu = \gamma^\nu\gamma^5$.[1] This is a textbook example of how HQS severely restricts the number of independent form factors. In the heavy quark limit F_Γ^ν, is proportional to just one overall factor, $\xi(w)$, the Isgur-Wise function, which depends only on $w = v' \cdot v$,

$$F_\Gamma^\nu = -\xi(w) \left\langle D^{(*)}(v') \left| \text{Tr}\left[\bar{H}_{v'}^c \Gamma^\nu H_v^b\right]\right| \bar{B}(v)\right\rangle \tag{2.2}$$

as in ref. [33], where H_v^Q is defined and the traces are explicitly evaluated.

We now turn to radiation off the light degrees of freedom, $\mathcal{M}_{\text{light}}^\mu$, e.g., the Feynman diagram in the bottom left panel of figure 1. The total matrix element is,

$$\frac{\mathcal{M}_{\text{light}}^\mu}{\sqrt{m_B m_{D^{(*)}}}} = \frac{G_F e V_{cb}}{\sqrt{2}} \left(T_V^{\mu\nu} - T_A^{\mu\nu}\right)\bar{u}\gamma_\nu(1 - \gamma^5)v\,, \tag{2.3}$$

---

[1]We will adopt the standard [3, 33] notation and normalization for the states in the heavy quark effective theory (HQET). Relativistic states are labelled with p, e.g., $|\bar{B}(p)\rangle$, and HQET states are labelled with the velocity, v, e.g. $|\bar{B}(v)\rangle$. The normalization of these states differs by a factor of $\sqrt{m}$: $|\bar{B}(p)\rangle = \sqrt{m_B}|\bar{B}(v)\rangle$. This explains the extra factor of $\sqrt{m_B m_{D^{(*)}}}$ in the matrix elements.

where

$$T_\Gamma^{\mu\nu} = \left\langle D^{(*)}(v') \left| \mathrm{Tr}\left[X^\mu \bar{H}_{v'} \Gamma^\nu H_v \right] \right| \bar{B}(v) \right\rangle \tag{2.4}$$

and X^μ describes the effects of the time-ordered product between the light degrees of freedom electromagnetic current and the heavy quarks' weak operator. Again HQS limits the general form of $T_\Gamma^{\mu\nu}$, where the most general form of X^μ is,

$$X^\mu = x^{(1)}v^\mu + x^{(2)}v'^\mu + x^{(3)}\gamma^\mu + ix^{(4)}\sigma^{\mu\nu}k_\nu + ix^{(5)}\epsilon^{\mu\nu\rho\lambda}v_\nu v'_\rho k_\lambda \gamma^5 + x^{(6)}(v^\mu + v'^\mu)\slashed{k}, \tag{2.5}$$

$\sigma^{\mu\nu} \equiv (i/2)[\gamma^\mu, \gamma^\nu]$ and $\epsilon^{0123} = 1$. We omitted terms $\propto k^\mu$, since they give vanishing contribution to the matrix element, as well as terms that do not transform as vectors under parity since QCD and QED are parity invariant theories. HQS forbids X^μ to be between $\bar{H}_{v'}$ and H_v, and therefore the most general form is given by eq. (2.4).[2] The $x^{(j)}$ form factors are scalar functions of w, $v \cdot k$, $v' \cdot k$. X^μ has mass dimension -1, therefore $x^{(1-3)}$ have mass dimensions -1 while $x^{(4-6)}$ have mass dimension -2.

The last class of Feynman diagrams is from radiation off the heavy quarks, shown in the right column of figure 1. There are two contributions here due to radiation from the heavy quark in the initial and final states. The matrix element is given by

$$\frac{\mathcal{M}_{\mathrm{heavy}}^\mu}{\sqrt{m_B m_{D^{(*)}}}} = \frac{G_F e V_{cb}}{\sqrt{2}} \left(q_b v^\mu (G_{V,b}^\nu - G_{V,b}^\nu) + q_c v'^\mu (G_{V,c}^\nu - G_{A,c}^\nu) \right) \bar{u}\gamma_\nu(1-\gamma^5)v\,, \tag{2.6}$$

where

$$G_{\Gamma,Q}^\nu = \left\langle D^{(*)}(v') \left| \mathrm{Tr}\left[Y_Q \bar{H}_{v'} \Gamma^\nu H_v \right] \right| \bar{B}(v) \right\rangle \tag{2.7}$$

and Y_Q describes the time ordered product between the number density of the heavy quarks, $\bar{Q}Q$, and the heavy quarks weak operator. The absence of a γ^μ factor from the heavy quark electromagnetic vertex is because we are working in the heavy quark limit which replaces $\slashed{D} \to v \cdot D$ when acting on the heavy quark fields. The most general form of the Y_Q's allowed by HQS are

$$Y_Q = y_Q^{(1)} + y_Q^{(2)}\slashed{k}\,, \tag{2.8}$$

and $y_Q^{(1)}$ has mass dimension -1, $y_Q^{(2)}$ has mass dimension -2, and both are functions of $w, v \cdot k$ and $v' \cdot k$. Summing the matrix elements in eqs. (2.1), (2.3), (2.6) gives the total matrix element,

$$\frac{\mathcal{M}^\mu}{\sqrt{m_B m_{D^{(*)}}}} = \frac{G_F e V_{cb}}{\sqrt{2}} \Bigg((K_V^{\mu\nu} - K_A^{\mu\nu})\, \bar{u}\gamma_\nu(1-\gamma^5)v$$
$$+ (q_b - q_c)\frac{(F_V^\nu - F_A^\nu)}{2p_\ell \cdot k}\bar{u}\gamma^\mu(\slashed{p}_\ell + \slashed{k} + m_\ell)\gamma_\nu(1-\gamma^5)v \Bigg), \tag{2.9}$$

where

$$K_\Gamma^{\mu\nu} \equiv T_\Gamma^{\mu\nu} + q_b v^\mu G_{\Gamma,b}^\nu + q_c v'^\mu G_{\Gamma,c}^\nu\,, \tag{2.10}$$

and we have replaced q_ℓ with $q_b - q_c$ which will prove convenient in section 3.1.

---

[2]While not immediately obvious, a term $\propto (v^\mu - v'^\mu)\slashed{k}$ in X^μ can in fact be reabsorbed into the other terms present in eq. (2.5).

3 Form factors

In section 2 we showed that HQS limits the number of form factors, for both the $\bar{B} \to D\ell\bar{\nu}_\ell\gamma$ and $\bar{B} \to D^*\ell\bar{\nu}_\ell\gamma$ processes, to ten, $x^{(1-6)}, y_{b,c}^{(1,2)}$. These ten form factors discussed so far are a superset of the number of physical form factors since we have not accounted for the Ward identity (WI). Next we show that the WI limits the number of new, independent form factors to just four. This is significant because, in general, even after applying the WI, the number of form factors is much larger. It has been shown in refs. [28, 34] that there are eight form factors for the $\bar{B} \to D\ell\bar{\nu}_\ell\gamma$ process, and in appendix A we show that the general number of form factors for the $\bar{B} \to D^*\ell\bar{\nu}_\ell\gamma$ process is 24.

3.1 Ward identities

The WI [35] states that $k_\mu\mathcal{M}^\mu$ must vanish. This will highly constrain the form factors in eqs. (2.5), (2.8). While the WI is seemingly only one equation, different terms have to cancel each other individually. For example the $\Gamma = V, A$ terms have different parity, and therefore must cancel individually (as in refs. [25, 26, 28, 34]) giving two equations,

$$k_\mu K_\Gamma^{\mu\nu} + (q_b - q_c)F_\Gamma^\nu = k_\mu T_\Gamma^{\mu\nu} + q_b(k \cdot v)G_{\Gamma,b}^\nu + q_c(k \cdot v')G_{\Gamma,c}^\nu + (q_b - q_c)F_\Gamma^\nu = 0 \quad (3.1)$$

for each $\Gamma \in \{V, A\}$. Moreover, the electromagnetic charges, $q_i e$, give us another way to split the terms. Only $T_\Gamma^{\mu\nu}$ can depend on the charge of the light quark it was radiated from (either sea or valence), and therefore $k_\mu T_\Gamma^{\mu\nu}$ must vanish independently since there is no way to get a cancellation from the other terms which are independent of the light quark charges. Similar reasoning holds for the terms depending on q_b, q_c; all of the electromagnetic charge dependence has been factored out, and therefore G_Γ^ν and F_Γ^ν are independent of q_b, q_c. The terms proportional to q_b, q_c must then cancel individually. The WI becomes six equations,

$$k_\mu T_\Gamma^{\mu\nu} = 0 \quad (3.2)$$

$$(k \cdot v)G_{\Gamma,b}^\nu + F_\Gamma^\nu = 0 \quad (3.3)$$

$$(k \cdot v')G_{\Gamma,c}^\nu - F_\Gamma^\nu = 0 \quad (3.4)$$

where, again, there is a set of equations for $\Gamma \in \{V, A\}$.

We begin by simplifying the $\Gamma = A$ case for eqs. (3.2)–(3.4) for the process $\bar{B} \to D\ell\bar{\nu}_\ell\gamma$. The left side of eqs. (3.3), (3.4) are proportional to a single form factor, and therefore these must be zero. Specifically,

$$y_Q^{(2)} = 0 \quad (3.5)$$

for $Q \in \{b, c\}$. Simplifying eq. (3.2) gives

$$-i\epsilon^{\mu\nu\rho\sigma}k_\nu v_\rho v'_\sigma \left(k \cdot (v + v')x^{(6)} + x^{(3)}\right) = 0 \quad (3.6)$$

and therefore relates $x^{(3)}$ to $x^{(6)}$,

$$x^{(3)} = -k \cdot (v + v')\, x^{(6)}\,. \quad (3.7)$$

Next, we consider the case of $\Gamma = V$, again for the $B \to D\ell\bar{\nu}_\ell\gamma$ scenario, and eqs. (3.2)–(3.4) become

$$-(v^\nu + v'^\nu)(x^{(1)}(k \cdot v) + x^{(2)}(k \cdot v')) = 0 \tag{3.8}$$

$$(v^\nu + v'^\nu)(\xi - y_b^{(1)}(k \cdot v)) = 0 \tag{3.9}$$

$$-(v^\nu + v'^\nu)(\xi + y_c^{(1)}(k \cdot v')) = 0 \tag{3.10}$$

respectively, and therefore

$$x^{(2)} = -\frac{k \cdot v}{k \cdot v'} x^{(1)}$$
$$y_b^{(1)} = \frac{\xi}{k \cdot v}$$
$$y_c^{(1)} = -\frac{\xi}{k \cdot v'} . \tag{3.11}$$

The WI has taken us from ten independent form factors, $x^{(1-6)}$, $y_{b,c}^{(1-2)}$, to just four. We define the independent ones as ζ_{1-4},

$$\zeta_1 \equiv (k \cdot v)x^{(1)}$$
$$\zeta_2 \equiv x^{(4)}$$
$$\zeta_3 \equiv x^{(5)}$$
$$\zeta_4 \equiv x^{(6)} \tag{3.12}$$

(the motivation for the extra factor of $v \cdot k$ will become clear in section 4). ζ_1 is dimensionless while ζ_{2-4} have dimension of mass^{-2}. Written in terms of the five independent form factors (including the Isgur-Wise function) the x, y form factors are

$$x^{(1)} = \frac{\zeta_1}{(k \cdot v)} , \qquad\qquad x^{(2)} = -\frac{\zeta_1}{(k \cdot v')}$$
$$y_b^{(1)} = \frac{\xi}{(k \cdot v)} , \qquad\qquad y_c^{(1)} = -\frac{\xi}{(k \cdot v')}$$
$$x^{(3)} = -k \cdot (v + v')\zeta_4$$
$$y_b^{(2)} = y_c^{(2)} = 0$$
$$x^{(4)} = \zeta_2 , \qquad\qquad x^{(5)} = \zeta_3 , \qquad\qquad x^{(6)} = \zeta_4 \tag{3.13}$$

One can check that with these conditions the WI for the $\bar{B} \to D^*\ell\bar{\nu}_\ell\gamma$ scenario are automatically satisfied. Using these relations, for the $\bar{B} \to D\ell\bar{\nu}_\ell\gamma$ case, the tensor $T_\Gamma^{\mu\nu}$ simplifies to

$$T_V^{\mu\nu} = \zeta_1 \left(\frac{v'^\mu}{k \cdot v'} - \frac{v^\mu}{k \cdot v}\right)(v^\nu + v'^\nu) + \zeta_2 \left(g^{\mu\nu}k \cdot (v' - v) + (v^\mu - v'^\mu)k^\nu\right) \tag{3.14}$$

$$+ \zeta_4 \left((w - 1)\left(g^{\mu\nu}(k \cdot (v + v')) - k^\nu(v^\mu + v'^\mu)\right) + (v'^\nu - v^\nu)\left((k \cdot v)v'^\mu - (k \cdot v')v^\mu\right)\right)$$

$$iT_A^{\mu\nu} = \zeta_2\, \epsilon^{\mu\nu\rho\sigma}k_\rho(v_\sigma + v'_\sigma) + \zeta_3 \epsilon^{\mu\rho\lambda\sigma}k_\rho v_\lambda v'_\sigma(v'^\nu - v^\nu) + \zeta_4\Big((v^\mu + v'^\mu)\epsilon^{\nu\rho\sigma\lambda}k_\rho v_\sigma v'_\lambda \tag{3.15}$$

$$+ k \cdot (v + v')\epsilon^{\mu\nu\rho\sigma}v_\rho v'_\sigma\Big)$$

and $G^\nu_{\Gamma,i}$ are reduced to

$$G^\nu_{V,b} = -\frac{\xi}{k \cdot v}(v^\nu + v'^\nu) \tag{3.16}$$

$$G^\nu_{A,b} = 0 \tag{3.17}$$

$$G^\nu_{V,c} = \frac{\xi}{k \cdot v'}(v^\nu + v'^\nu) \tag{3.18}$$

$$G^\nu_{A,c} = 0 \,. \tag{3.19}$$

For the $\bar{B} \to D^* \ell \bar{\nu}_\ell \gamma$ process, the $T^{*\,\mu\nu}_\Gamma$ read

$$
\begin{aligned}
T^{*\mu\nu}_V =& \zeta_1 \epsilon^{\nu\rho\sigma}{}_\lambda v_\rho v'_\sigma \varepsilon^{*\lambda}_{D^*} \left(\frac{v'^\mu}{k \cdot v'} - \frac{v^\mu}{k \cdot v} \right) \\
&+ \zeta_2 \Big(-\varepsilon^{*\mu}_{D^*} \epsilon^{\nu\rho\sigma\lambda} k_\rho v_\sigma v'_\lambda + v^\nu \epsilon^{\mu\rho\sigma}{}_\lambda k_\rho v'_\sigma \varepsilon^{*\lambda}_{D^*} + v'^\nu \epsilon^{\mu\rho\sigma}{}_\lambda k_\rho v_\sigma \varepsilon^{*\lambda}_{D^*} \\
&\quad - (k \cdot \varepsilon^*_{D^*}) \epsilon^{\mu\nu\rho\sigma} v_\rho v'_\sigma + (w-1) \epsilon^{\mu\nu\rho}{}_\sigma k_\rho \varepsilon^{*\sigma}_{D^*} \Big) + \zeta_3 \epsilon^{\mu\rho\sigma}{}_\lambda k_\rho v_\sigma v'_\lambda \Big((w-1) \varepsilon^{*\nu}_{D^*} - v'^\nu (v \cdot \varepsilon^*_{D^*}) \Big) \\
&+ \zeta_4 \Big((v \cdot \varepsilon^*_{D^*}) \epsilon^{\mu\nu\rho\sigma} k_\rho (v - v')_\sigma - (v^\nu + v'^\nu) \epsilon^{\mu\rho\sigma}{}_\lambda k_\rho (v_\sigma - v'_\sigma) \varepsilon^{*\lambda}_{D^*} \Big) \\
iT^{*\mu\nu}_A =& \zeta_1 \left(\frac{v'^\mu}{k \cdot v'} - \frac{v^\mu}{k \cdot v} \right) \Big((w+1) \varepsilon^{*\nu}_{D^*} - v'^\nu (v \cdot \varepsilon^*_{D^*}) \Big) \\
&- \zeta_2 \Big(k^\nu \big((w+1) \varepsilon^{*\mu}_{D^*} - v'^\mu (v \cdot \varepsilon^*_{D^*}) \big) - g^{\mu\nu} \big((w+1)(k \cdot \varepsilon^*_{D^*}) - (k \cdot v')(v \cdot \varepsilon^*_{D^*}) \big) \\
&\quad + v'^{\{\mu} v^{\nu\}} (k \cdot \varepsilon^*_{D^*}) - v^{\{\mu} \varepsilon^{*\nu\}}_{D^*} (k \cdot v') + v'^{[\mu} \varepsilon^{*\nu]}_{D^*} (k \cdot v) \Big) \\
&+ \zeta_3 \Big((1-w^2) \big((k \cdot \varepsilon^*_{D^*}) g^{\mu\nu} - \varepsilon^{*\mu}_{D^*} k^\nu \big) - (v \cdot \varepsilon_{D^*}) \big(k \cdot (v - wv') g^{\mu\nu} - (v^\mu - wv'^\mu) k^\nu \big) \\
&\quad + v^\nu \big(k \cdot (v - wv') \varepsilon^{*\mu}_{D^*} - (k \cdot \varepsilon^*_{D^*})(v^\mu - wv'^\mu) \big) + v'^\nu (v \cdot \varepsilon^*_{D^*}) \big((k \cdot v) v'^\mu - (k \cdot v') v^\mu \big) \\
&\quad + v'^\nu \big(k \cdot (v' - wv) \varepsilon^{*\mu}_{D^*} - (k \cdot \varepsilon^*_{D^*})(v'^\mu - wv^\mu) \big) \Big) \\
&+ \zeta_4 \Big((v \cdot \varepsilon^*_{D^*}) \big(g^{\mu\nu}(k \cdot (v + v')) - (v^\mu + v'^\mu) k^\nu \big) \\
&\quad + (v'^\nu - v^\nu) \big(\varepsilon^{*\mu}_{D^*}(k \cdot (v + v')) - (k \cdot \varepsilon^*_{D^*})(v^\mu + v'^\mu) \big) \Big)
\end{aligned}
\tag{3.20}
$$

and $G^{*\,\nu}_{\Gamma,i}$ are reduced to

$$
\begin{aligned}
G^{*\nu}_{V,b} &= -\frac{\xi}{k \cdot v} \epsilon^{\nu\rho\sigma}{}_\lambda v_\rho v'_\sigma \varepsilon^{*\lambda}_{D^*} \\
iG^{*\nu}_{A,b} &= -\frac{\xi}{k \cdot v} \big((w+1) \varepsilon^{*\nu}_{D^*} - v'^\nu (v \cdot \varepsilon^*_{D^*}) \big) \\
G^{*\nu}_{V,c} &= \frac{\xi}{k \cdot v'} \epsilon^{\nu\rho\sigma}{}_\lambda v_\rho v'_\sigma \varepsilon^{*\lambda}_{D^*} \\
iG^{*\nu}_{A,c} &= \frac{\xi}{k \cdot v'} \big((w+1) \varepsilon^{*\nu}_{D^*} - v'^\nu (v \cdot \varepsilon^*_{D^*}) \big)
\end{aligned}
\tag{3.21}
$$

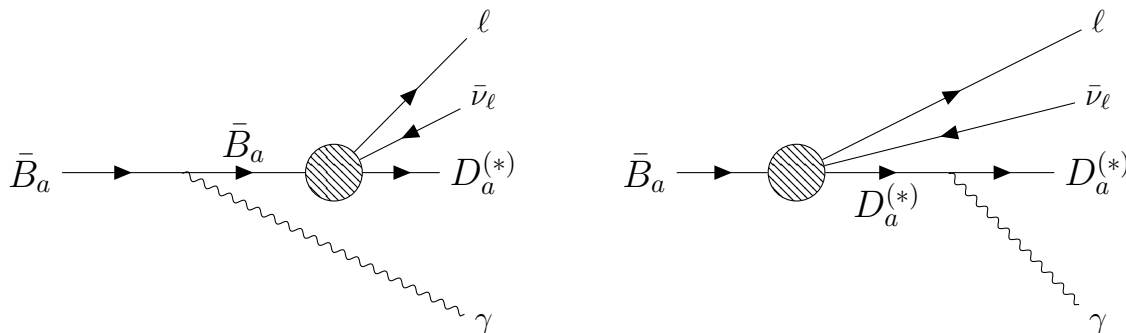**Figure 2**. In the soft limit, $k \ll \Lambda_{\mathrm{QCD}}$, the leading order contributions to the radiative semileptonic $\bar{B}_a \to D_a^{(*)} \ell \bar{\nu}_\ell \gamma$ processes ($\mathcal{M} \sim k^{-1}$) originate from these Feynman diagrams in the heavy hadron effective theory.

where $a^{[\mu} b^{\nu]} \equiv a^\mu b^\nu - b^\mu a^\nu$ (and is straightforwardly generalized for the anti-commutator) and $\varepsilon_{D^*}^\mu$ is the D^* polarization vector.

4 Soft limit

Further insight on the structure of the new form factors ζ_{1-4} derived in section 3.1 can be gained by studying the small k (soft) limit, $k \ll \Lambda_{\mathrm{QCD}}$, where leading and sub-leading soft theorems [36–38] constrain the k^{-1} and k^0 behavior of the ζ_i's, relating the $\bar{B} \to D^{(*)} \ell \bar{\nu}_\ell \gamma$ amplitudes to the amplitudes without photon emission, $\bar{B} \to D^{(*)} \ell \bar{\nu}_\ell$.

The soft limit is a strict subset of the $m_Q \to \infty$ limit, and can be described in terms of hadron fields. This limit is accessed by matching heavy quark effective theory onto a heavy hadron effective theory containing only the lowest multiplets. The effect of higher multiplets are encoded in higher derivative operators. We make the $\mathrm{SU}(3)_V$ transformation properties of the heavy hadron fields explicit in this section.

To understand the leading, and sub-leading order, behavior we expand the matrix element in a power series in k: $\mathcal{M} = \mathcal{M}_{-1} + \mathcal{M}_0 + \mathcal{O}(k)$, where $\mathcal{M}_i \sim \mathcal{O}(k^i)$. In section 4.1 we compute $\mathcal{M}_{-1}$ exactly, and then in section 4.2 we discuss the calculation of $\mathcal{M}_0$ and compute explicitly the contribution to it from the magnetic dipole operator. We find that in both cases the ζ_i's can be written in terms of the Isgur-Wise function.

4.1 Leading soft behavior

To explore the k^{-1} term in the amplitude it is sufficient to study the leading order heavy hadron Lagrangian. At this order, the interaction with the photon is described by the mimimal coupling of the heavy hadron field to the electromagnetic field,

$$(D_\mu H)_a = \partial_\mu H_a - i e A_\mu Q_{ab} H^b \tag{4.1}$$
$$\mathbf{Q} = \mathrm{diag}\,(q_Q - q_u, q_Q - q_d, q_Q - q_s)$$

where a, b are $\mathrm{SU}(3)_V$ flavor indices and $q_u e, q_d e, q_s e$ are the electromagnetic charges of the up, down, and strange quarks, respectively. The expression $e(q_Q - q_a)$ is just the electric

charge of the heavy meson written in terms of the electric charge of the heavy quark, $q_Q e$, and the light valence quark, $q_a e$, as it will be convenient in the matching to the heavy quark limit expressions. The interaction Lagrangian is then

$$\mathcal{L}_{\text{int}} = ev^\mu A_\mu \text{Tr}\left[\bar{H} \cdot \mathbf{Q} \cdot H\right] \tag{4.2}$$

With the interaction Lagrangian, along with the hadron field propagator, $i/(2k \cdot v)$, we can compute the matrix element for $\bar{B}_a \to D_a^{(*)}\ell\bar{\nu}_\ell\gamma$, shown in figure 2. We will use the convention $\bar{B}_a$, $D_a^{(*)}$ with $a = u, d, s$ based on the valence quark flavor to describe different processes. The matrix element, $\mathcal{M}_{-1}$, with just the leading order contribution from radiation off the hadron fields, is

$$\frac{\mathcal{M}_{-1}^\mu}{\sqrt{m_B m_{D^{(*)}}}} = \frac{G_F e V_{cb}}{\sqrt{2}}\left[\left(\frac{(q_a - q_b)v^\mu}{k \cdot v} + \frac{(q_c - q_a)v'^\mu}{k \cdot v'}\right)(F_V^\nu - F_A^\nu)\right]\bar{u}\gamma_\nu(1 - \gamma^5)v + \frac{\mathcal{M}_{-1,\text{lepton}}^\mu}{\sqrt{m_B m_{D^{(*)}}}}. \tag{4.3}$$

where $\mathcal{M}_{\text{lepton}}^\mu$ here is the same as in section 2, and the other two terms describe the photon emission from the $\bar{B}_a$ and the $D_a^{(*)}$ lines, and are collectively $\mathcal{M}_{-1}$. Matching this on to the matrix element found previously, eq. (2.9), expanded in k at this order, it is easy to confirm that the terms $\propto q_b, q_c$ are automatically matched to eqs. (3.16), (3.18), while the q_a dependence must be contained in $T^{\mu\nu}$. Considering the $\bar{B}_a \to D_a\ell\bar{\nu}_\ell\gamma$ case and the terms $\propto q_a$,

$$q_a\xi\left(\frac{v^\mu}{k \cdot v} - \frac{v'^\mu}{k \cdot v'}\right)(v^\nu + v'^\nu) = T_V^{\mu\nu} \tag{4.4}$$

where $T_V^{\mu\nu}$ is given in eq. (3.14). Therefore we see that

$$\zeta_1 \to -q_a\xi + \mathcal{O}\left(\frac{k}{\Lambda_{\text{QCD}}}\right), \tag{4.5}$$

which justifies the extra factor of $v \cdot k$ in the redefinition of $x^{(1)} \to \zeta_1$. We also learn that ζ_{2-4} are unconstrained and can begin at order $\mathcal{O}(1/v \cdot k, 1/v' \cdot k)$, as we will see in the next section. To connect to the literature [25, 26, 28], the emissions from the lepton, the heavy quarks and the leading contribution to ζ_1 correspond to what is conventionally called internal bremsstrahlung (IB) while $\mathcal{O}(k)$ contributions to ζ_1, and ζ_{2-4}, parameterize the structure dependent (SD) contributions.

4.2 Sub-leading soft behavior

To understand the sub-leading soft behavior of ζ_{1-4} we need to study the amplitude at order $\mathcal{O}(k^0)$ and consider higher order operators in the heavy hadron Lagrangian. These are effective operators with one extra covariant derivative on a hadron line, including the magnetic dipole operator,[3]

$$\mathcal{L}_{\text{dipole}} = \frac{ie}{2}F_{\mu\nu}\text{Tr}\left[\sigma^{\mu\nu}\bar{H} \cdot \boldsymbol{\mu} \cdot H\right] \tag{4.6}$$

$$\boldsymbol{\mu} = \text{diag}(\mu_u, \mu_d, \mu_s), \tag{4.7}$$

---

[3]There is also the operator $\text{Tr}\left[\bar{H}D^2H\right]$, but it does not contribute to the on-shell matrix element.
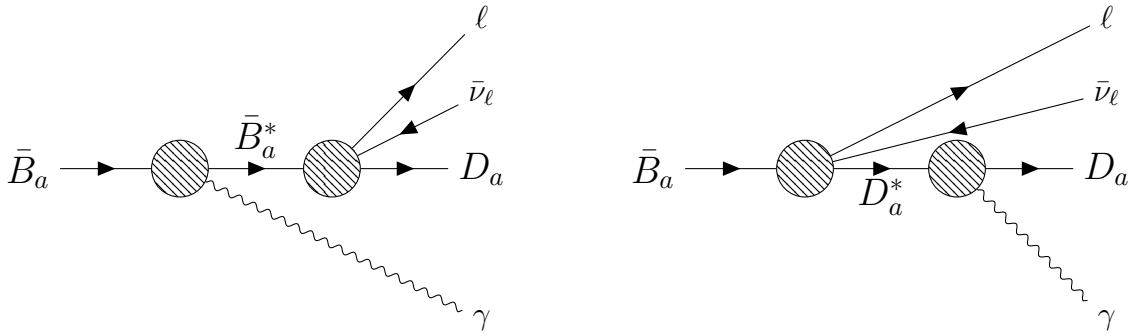
Figure 3. Feynman diagrams contributing to the sub-leading soft, $k \ll \Lambda_{\text{QCD}}$, behavior of the radiative semileptonic process $\bar{B}_a \to D_a \ell \bar{\nu}_\ell \gamma$ within a heavy hadron effective theory. Intermediate excited states, $\bar{B}_a^*$ and D_a^*, arise from the magnetic dipole term in the Lagrangian, eq. (4.6).

which induces a transition between a D_a ($\bar{B}_a$) and D_a^* ($\bar{B}_a^*$). At this order there are also corrections to the weak current operator,

$$\mathcal{L}_{\text{current}} = a_{1,c}(w) \, v^\mu \text{Tr} \left[(D_\mu \bar{H}_c^{(v')}) \Gamma H_b^{(v)} \right] + a_{2,c}(w) \, \text{Tr} \left[(\not{D} \bar{H}_c^{(v')}) \Gamma H_b^{(v)} \right]$$
$$+ a_{1,b}(w) \, v'^\mu \text{Tr} \left[\bar{H}_c^{(v')} \Gamma D_\mu H_b^{(v)} \right] + a_{2,b}(w) \, \text{Tr} \left[\gamma^\mu \bar{H}_c^{(v')} \Gamma D_\mu H_b^{(v)} \right] \tag{4.8}$$

where D_μ is the covariant derivative defined in eq. (4.1). The effect of the current corrections vanishes in the zero recoil limit, $v = v'$, as they can be eliminated using the equations of motion. The coefficient of the magnetic dipole operator can be extracted from the $D^* \to D\gamma$ rate [39] and has been also computed on the lattice [40], while the coefficients $a_{1-2,b-c}$ of the current corrections are currently unknown. The $\mathcal{O}(k^0)$ matrix element reads

$$\frac{\mathcal{M}_0^\mu}{\sqrt{m_{B_a} m_{D_a^{(*)}}}} = \frac{G_F e V_{cb}}{\sqrt{2}} (\Delta T_V^{\mu\nu} - \Delta T_A^{\mu\nu}) \, \bar{u} \gamma_\nu (1 - \gamma^5) v + \frac{\mathcal{M}_{0,\text{lepton}}^\mu}{\sqrt{m_{B_a} m_{D_a^{(*)}}}} \tag{4.9}$$

with $\Delta T_{V,A}$ receiving contribution both from the dipole and the current operators above. To investigate their structure we will focus on the magnetic dipole contribution to the $\bar{B}_a \to D_a \ell \bar{\nu}_\ell \gamma$ amplitude [25]; the $\bar{B}_a \to D_a^* \ell \bar{\nu}_\ell \gamma$ process will be parameterically similar. There are two diagrams contributing, one where the photon is emitted from the $\bar{B}_a$ line (transitioning to a $\bar{B}_a^*$) and one from the D_a (to D_a^*), shown in figure 3. Their sum read:

$$\Delta T_{V,\text{dipole}}^{\mu\nu} = \frac{\mu_a}{2} \xi(w) \Bigg\{ g^{\mu\nu} \left(\frac{k \cdot v}{k \cdot v'} + \frac{k \cdot v'}{k \cdot v} - 2w \right) + \left(\frac{v'^\mu}{k \cdot v'} - \frac{v^\mu}{k \cdot v} \right) k_\rho v^{[\nu} v'^{\rho]}$$
$$+ k^\nu \left[\frac{v^\mu}{k \cdot v} \left(w - \frac{k \cdot v}{k \cdot v'} \right) + \frac{v'^\mu}{k \cdot v'} \left(w - \frac{k \cdot v'}{k \cdot v} \right) \right] \Bigg\}$$
$$\Delta T_{A,\text{dipole}}^{\mu\nu} = \frac{\mu_a}{2} \xi(w) \left[(1 + w) \epsilon^{\mu\nu\rho\sigma} k_\rho \left(\frac{v_\sigma}{k \cdot v} - \frac{v'_\sigma}{k \cdot v'} \right) - \left(\frac{v'^\nu}{k \cdot v'} + \frac{v^\nu}{k \cdot v} \right) \epsilon^{\mu\rho\sigma\lambda} k_\rho v_\sigma v'_\lambda \right] \tag{4.10}$$

Upon matching these results to the $\zeta_{1-4}(w, v \cdot k, v' \cdot k)$ form factors one find that they receive contributions:

$$\zeta_1 = -q_a \xi(w) + \frac{\mu_a}{4}(k \cdot v - k \cdot v') \, \xi(w) + \dots, \quad \zeta_2 = -\frac{\mu_a}{4}(1+w)\left(\frac{1}{k \cdot v} - \frac{1}{k \cdot v'}\right) \xi(w) + \dots,$$

$$\zeta_3 = -\frac{\mu_a}{4}\left(\frac{1}{k \cdot v} - \frac{1}{k \cdot v'}\right) \xi(w) + \dots, \qquad \zeta_4 = \frac{\mu_a}{4}\left(\frac{1}{k \cdot v} + \frac{1}{k \cdot v'}\right) \xi(w) + \dots. \qquad (4.11)$$

where the ellipsis include both contributions from the current corrections at $\mathcal{O}(k^0)$ and terms higher order in k. As expected, the dipole contribution to ζ_1 starts at $\mathcal{O}(k)$, while the contribution to ζ_{2-4} begin at $\mathcal{O}(k^{-1})$, and are all proportional to the leading order Isgur-Wise function.

The presence of $(k \cdot v / k \cdot v')$ and $(k \cdot v' / k \cdot v)$ factors in the amplitude at $\mathcal{O}(k^0)$ is not unexpected. Similar ratios are known to appear in the sub-leading soft limit for effective theories with massless particles [41]. The contributions from the current operators to $\Delta T_{V,A}$ will induce similar contributions to ζ_{1-4} and induce in the amplitude ratios of $v \cdot k$ and $v' \cdot k$.

5 Discussion

We have studied the implications of heavy quark symmetry (HQS) for radiative semileptonic $\bar{B}$ decays. This study is motivated by the increase in data that will be available from Belle II and LHCb. Radiative semileptonic $\bar{B}$ decays are an interesting process since they probe HQS in a different kinematic regime than non-radiative semileptonic processes. Moreover they are a background for other semileptonic $\bar{B}$ decay measurements, such as the extraction of V_{cb} and of the $R(D^{(*)})$ ratios, when the radiated photon goes undetected. We restricted our attention to the kinematic region where radiating the final state photon does not change the heavy quark four velocity. In this region HQS implies a dramatic reduction in the number of independent scalar form factors characterizing the decay amplitudes.

Throughout this paper we have worked to lowest order in the electromagnetic coupling and neglected $1/m_{c,b}$ corrections. There are kinematic regions where this will not be adequate.

For very soft photons there are infrared and collinear (with the electron) divergences that should be resummed, and one loop corrections that should be included. However, these effects are already well studied in the literature. NLO QED effects when the photon is soft have been computed in [26, 27, 42, 43], including resumming the Coulomb corrections, while leading logs are partially resummed in PHOTOS [44, 45] (up to quadratic order) or fully via YFS [46] in other MonteCarlo event generators such as Sherpa [47]. Several numerical comparisons between the different predictions of these calculations have been performed both in the B-factory [23] and LHCb [24] kinematic regimes.

It is therefore important to compare our calculation with the approximations currently used by LHCb and Belle II in simulating semileptonic $\bar{B}$ decays. Experiments are currently using PHOTOS to generate radiated photons in semileptonic $\bar{B}$ decays (with the LO process usually generated by EvtGen)[4] and also include photons from $B \to (D^* \to D\gamma)\ell\bar{\nu}_\ell$ as a sepa-

---

[4]The program BLOR which include several effects neglected in PHOTOS was also available during the past B factory runs [26]. As there are no current plans for Belle II to use it, we will not discuss it any further.

rate process with the D^* (quasi) on-shell (i.e., $|m_{D\gamma} - m_{D^*}| < \text{few} \times \Gamma_{D^*}$). PHOTOS generates (multi-)photon radiation starting from a non-radiative decay event in the leading logarithmic approximation, extrapolated over the whole kinematic space allowed for photon emission, and neglects the interference between amplitudes where the photon is radiated from different charged legs. In the language of this paper, it is equivalent to setting $\zeta_1(w, v \cdot k, v' \cdot k) = -q_a \xi(w)$, $\zeta_{2-4}(w, k \cdot v, k \cdot v') = 0$, neglecting the $\not{k}$ term in $\mathcal{M}_{\text{lepton}}$ and neglecting the interference between $\mathcal{M}_{\text{lepton}}$ and $\mathcal{M}_{\text{light}} + \mathcal{M}_{\text{heavy}}$ in $|\mathcal{M}|^2$. Furthermore, since the photon radiated from a charged $D^{(*)}$ is produced by a deformation of the kinematic configuration of the non-radiative process, for these diagrams the leading order Isgur-Wise function is evaluated at a different recoil parameter point $\bar{w} = (m_B^2 + m_{D^{(*)}}^2 - (p_B - p_{D^{(*)}\gamma})^2)/(2m_B m_{D^{(*)}})$. However the difference $\bar{w} - w$ is $\mathcal{O}(v^{(\prime)} \cdot k/m_{B,D^{(*)}})$ and therefore higher order in $1/m_{b,c}$.

The novel element of this paper is a compact parameterization of contributions to the radiative semileptonic $\bar{B} \to D^{(*)}\ell\bar{\nu}_\ell\gamma$ process in the region where the photon probes the structure of the heavy mesons, in the form of the (currently unknown) functions ζ_{1-4}.

The effect of ζ_{1-4} (and other effects missing in PHOTOS) can be included in current event generators via, e.g., event reweighing with tools such as Hammer. The relevant helicity amplitudes, together with a numerical study of the size of the effects that ζ_{1-4} may induce in semileptonic measurements, along the lines of [24], will be presented elsewhere.

We have also studied the low photon energy, $k < \Lambda_{\text{QCD}}$, behavior of ζ_{1-4} and found that the leading order, in k behavior is determined by the Isgur-Wise function. The sub-leading contributions can be captured by the magnetic moment, μ, and four Wilson coefficients $a_{1-2,b-c}$ which are functions of the recoil parameter w. (μ is constrained by the $D^* \to D\gamma$ width.) In particular, the results of section 4, together with the bookkeeping procedure used by experiments to account for photons originating from on-shell decays of $D^* \to D\gamma$ in separate event samples, imply that, in practice, the magnetic dipole contributions to the SD form factors are never dominant. Away from the D^* peak in the $(D\gamma)$ mass distribution the current corrections are expected to contribute to the same level (or even be parametrically larger when the photon energy drops below the $D - D^*$ $(B - B^*)$ mass gap as the dipole operators will be further suppressed by the mass gap).

There are various avenues to further pursue the study initiated here. For example, translating these results to a form that can be directly used by the experiments, and quantitative estimation of the impact of ζ_{1-4} on the semileptonic analyses as already mentioned above. With enough data Belle II can tag the photon and constrain ζ_{1-4}. In practice a suitable parameterization of the ζ's amenable for fitting should be found. Study of radiative semileptonic transition can also be performed for other heavy hadrons, such as $\Lambda_b \to \Lambda_c \ell\nu$. There are also more interesting field theoretical questions such as how to describe harder photon radiation in heavy quark effective theory. If the photon is sufficiently hard, the heavy quark line will have two cusps, one due to the weak vertex and one due to photon emission.

A General form factor expressions

For $\bar{B} \to D\ell\bar{\nu}_\ell\gamma$ the most general parameterization of the form factors read [28]:

$$K_{V,\text{IB}}^{\alpha\beta} = \frac{1}{2}\left(q_D\frac{v'^\alpha}{k\cdot v'} - q_{\bar{B}}\frac{v^\alpha}{k\cdot v}\right)\left(f_+(w)\left(v^\beta + v'^\beta\right) + f_-(w)\left(v^\beta - v'^\beta\right)\right) \tag{A.1}$$

$$K_{V,\text{SD}}^{\alpha,\beta} = V_1^D\left(k^\beta v^\alpha - g^{\alpha\beta}k\cdot v\right) + V_2^D\left(k^\beta v'^\alpha - g^{\alpha\beta}k\cdot v'\right)$$
$$+ \left(V_3^D v'^\beta + V_4^D v^\beta\right)\left(v^\alpha k\cdot v' - v'^\alpha k\cdot v\right) \tag{A.2}$$

$$K_{A,\text{IB}}^{\alpha\beta} = 0 \tag{A.3}$$

$$iK_{A,\text{SD}}^{\alpha,\beta} = A_1^D\epsilon^{\alpha\beta kv} + A_2^D\epsilon^{\alpha\beta kv'} + A_3^D v^\beta\epsilon^{\alpha kvv'} + A_4^D v'^\beta\epsilon^{\alpha kvv'} \tag{A.4}$$

where we have used $v^\alpha \equiv p_{\bar{B}}^\alpha/m_{\bar{B}}$, $v'^\alpha \equiv p_D^\alpha/m_D$ and the notation for the Levi-Civita tensor $\epsilon^{\beta vv'k} = \epsilon^{\beta\gamma\nu\rho}v^\gamma v'^\nu k^\rho$. In the region $k\cdot v' < m_{(b,c)}$ explored in this work, HQS fixes the new form factors to be:

$$V_1^D = \zeta_2 - (w-1)\zeta_4, \qquad\qquad V_2^D = -\zeta_2 - (w-1)\zeta_4,$$

$$V_3^D = -\frac{\zeta_1^{\text{SD}}}{k\cdot v k\cdot v'} - \zeta_4, \qquad\qquad V_4^D = -\frac{\zeta_1^{\text{SD}}}{k\cdot v k\cdot v'} + \zeta_4, \tag{A.5}$$

$$A_1^D = \zeta_2 - (1+w)\zeta_4, \qquad\qquad A_2^D = \zeta_2 + (1+w)\zeta_4,$$

$$A_3^D = -\zeta_3 + \zeta_4, \qquad\qquad A_4^D = \zeta_3 + \zeta_4\,, \tag{A.6}$$

where we have defined ζ_1^{SD} to be $\zeta_1 - \zeta_1^{\text{IB}}$ with $\zeta_1^{\text{IB}} \equiv q_a[(v'\cdot\varepsilon)/(v\cdot k) - (v\cdot\varepsilon)/(v'\cdot k)]\xi(w)$.

The general parameterization for scalar to vector radiative semileptonic transitions are not present in the literature and we have derived them here. There are 12 structure-dependent form factors for both the vector and axial amplitudes. Therefore, for $\bar{B} \to D^*\ell\bar{\nu}_\ell\gamma$ the most general parameterization of the form factors read:

$$iK_{V,\text{IB}}^{\alpha\beta} = -\left(q_{D^*}\frac{v'^\alpha}{k\cdot v'} - q_{\bar{B}}\frac{v^\alpha}{k\cdot v}\right)g(w)\epsilon^{\beta vv'\varepsilon_{D^*}^*} \tag{A.7}$$

$$iK_{V,\text{SD}}^{\alpha\beta} = V_1^{D^*}\left(v^\alpha\epsilon^{\beta kv\varepsilon_{D^*}^*} + k\cdot v\epsilon^{\alpha\beta v\varepsilon_{D^*}^*}\right) + V_2^{D^*}v^\beta v\cdot\varepsilon_{D^*}^*\epsilon^{\alpha kvv'}$$
$$+ V_3^{D^*}\left(\varepsilon_{D^*}^{*\alpha}\epsilon^{\beta kvv'} + k\cdot\varepsilon_{D^*}^*\epsilon^{\alpha\beta vv'}\right) + V_4^{D^*}\left(v^\alpha k\cdot v' - v'^\alpha k\cdot v\right)\epsilon^{\beta vv'\varepsilon_{D^*}^*}$$
$$+ V_5^{D^*}\epsilon^{\alpha\beta k\varepsilon_{D^*}^*} + \left(V_6^{D^*}\epsilon^{\alpha\beta kv} + V_7^{D^*}\epsilon^{\alpha\beta kv'}\right)k\cdot\varepsilon_{D^*}^*$$
$$+ \left(V_8^{D^*}v^\beta + V_9^{D^*}v'^\beta\right)\epsilon^{\alpha kv\varepsilon_{D^*}^*} + \left(V_{10}^{D^*}v^\beta + V_{11}^{D^*}v'^\beta\right)\epsilon^{\alpha kv'\varepsilon_{D^*}^*}$$
$$+ V_{12}^{D^*}v\cdot\varepsilon_{D^*}^*\epsilon^{\alpha\beta kv'} \tag{A.8}$$

$$K_{A,\text{IB}}^{\alpha\beta} = \frac{1}{2}\left(q_{D^*}\frac{v'^\alpha}{k\cdot v'} - q_{\bar B}\frac{v^\alpha}{k\cdot v}\right)$$
$$\times\left(f(w)\varepsilon_{D^*}^{*\beta} + a_+(w)\left(v^\beta + v'^\beta\right)v\cdot\varepsilon_{D^*}^* + a_-(w)\left(v^\beta - v'^\beta\right)v\cdot\varepsilon_{D^*}^*\right) \quad\text{(A.9)}$$
$$K_{A,\text{SD}}^{\alpha,\beta} = A_1^{D^*}\varepsilon_{D^*}^{*\beta}\left(v^\alpha k\cdot v' - v'^\alpha k\cdot v\right) + A_2^{D^*}\left(\bar g^{\alpha\beta}k\cdot\varepsilon_{D^*}^* - \varepsilon_{D^*}^{*\alpha}k^\beta\right)$$
$$+ A_3^{D^*}k^\beta\left(v^\alpha k\cdot\varepsilon_{D^*}^* - \varepsilon_{D^*}^{*\alpha}k\cdot v\right) + A_4^{D^*}k^\beta\left(v'^\alpha k\cdot\varepsilon_{D^*}^* - \varepsilon_{D^*}^{*\alpha}k\cdot v'\right)$$
$$+ A_5^{D^*}v\cdot\varepsilon_{D^*}^*\left(k^\beta v^\alpha - g^{\alpha\beta}k\cdot v\right) + A_6^{D^*}v\cdot\varepsilon_{D^*}^*\left(k^\beta v'^\alpha - g^{\alpha\beta}k\cdot v'\right)$$
$$+ A_7^{D^*}v^\beta\left(v^\alpha k\cdot\varepsilon_{D^*}^* - \varepsilon_{D^*}^{*\alpha}k\cdot v\right) + A_8^{D^*}v^\beta\left(v'^\alpha k\cdot\varepsilon_{D^*}^* - \varepsilon_{D^*}^{*\alpha}k\cdot v'\right)$$
$$+ A_9^{D^*}v^\beta v\cdot\varepsilon_{D^*}^*\left(v^\alpha k\cdot v' - v'^\alpha k\cdot v\right) + A_{10}^{D^*}v'^\beta\left(v^\alpha k\cdot\varepsilon_{D^*}^* - \varepsilon_{D^*}^{*\alpha}k\cdot v\right)$$
$$+ A_{11}^{D^*}v'^\beta\left(v'^\alpha k\cdot\varepsilon_{D^*}^* - \varepsilon_{D^*}^{*\alpha}k\cdot v'\right) + A_{12}^{D^*}v'^\beta v\cdot\varepsilon_{D^*}^*\left(v^\alpha k\cdot v' - v'^\alpha k\cdot v\right) \quad\text{(A.10)}$$

Matching the amplitudes studied in this work in the region $k\cdot v' < m_{(b,c)}$ to the most general parameterization, one finds that HQS fixes the new form factors in $\bar B \to D^*\ell\bar\nu_\ell\gamma$ to be:

$$V_1^{D^*} = -\zeta_3 + \zeta_4, \qquad\qquad V_3^{D^*} = \zeta_2 - (w-1)\zeta_3$$
$$V_4^{D^*} = \frac{\zeta_1^{\text{SD}}}{k\cdot v\,k\cdot v'} - \zeta_3, \qquad\qquad V_5^{D^*} = (w-1)\zeta_2 - w^2\zeta_3 + \zeta_4$$
$$V_9^{D^*} = -\zeta_2 + w\zeta_3 - \zeta_4, \qquad\qquad V_{10}^{D^*} = -\zeta_2 + w\zeta_3 + \zeta_4,$$
$$V_{11}^{D^*} = -\zeta_3 - \zeta_4, \qquad\qquad V_{12}^{D^*} = \zeta_3 - \zeta_4,$$
$$V_2^{D^*} = V_6^{D^*} = V_7^{D^*} = V_8^{D^*} = 0, \qquad\qquad\qquad\qquad\qquad\text{(A.11)}$$
$$A_1^{D^*} = -(w+1)\frac{\zeta_1^{\text{SD}}}{k\cdot v\,k\cdot v'} + \zeta_2, \qquad A_2^{D^*} = (1+w)\zeta_2 - (w^2-1)\zeta_3,$$
$$A_5^{D^*} = \zeta_3 - \zeta_4, \qquad\qquad A_6^{D^*} = \zeta_2 - w\zeta_3 - \zeta_4,$$
$$A_7^{D^*} = -\zeta_3 + \zeta_4, \qquad\qquad A_8^{D^*} = -\zeta_2 + w\zeta_3 + \zeta_4,$$
$$A_{10}^{D^*} = -\zeta_2 + w\zeta_3 - \zeta_4, \qquad\qquad A_{11}^{D^*} = -\zeta_3 - \zeta_4,$$
$$A_{12}^{D^*} = \frac{\zeta_1^{\text{SD}}}{k\cdot v\,k\cdot v'} - \zeta_3, \qquad\qquad A_3^{D^*} = A_4^{D^*} = A_9^{D^*} = 0. \quad\text{(A.12)}$$

B Matrix elements for other Dirac structures

Here we provide the matrix elements $T_\Gamma^{(*)\,\mu}$ for $\bar B \to D^{(*)}$ for the remaining Dirac structures $\Gamma = 1, \gamma^5, \sigma^{\nu\rho}$, labeled as S, P, T respectively. For $\bar B \to D$ they read:

$$T_S^\mu = (w+1)\zeta_1\left(\frac{v'^\mu}{k\cdot v'} - \frac{v^\mu}{k\cdot v}\right) + \zeta_2\left(v^\mu k\cdot v' - v'^\mu k\cdot v\right)$$
$$-iT_P^\mu = ((w-1)\zeta_3 - \zeta_2)\,\epsilon^{\mu\nu\rho\sigma}k^\nu v^\rho v'^\sigma$$
$$-iT_T^{\mu\nu\rho} = \zeta_1\left(\frac{v'^\mu}{k\cdot v'} - \frac{v^\mu}{k\cdot v}\right)v'^{[\nu}v^{\rho]}$$
$$+ \zeta_2\left(v^\mu v'^{[\nu}k^{\rho]} + v'^\mu v^{[\nu}k^{\rho]} + g^{\mu[\nu}\left(v'^{\rho]}k\cdot v + v^{\rho]}k\cdot v' - (w+1)k^{\rho]}\right)\right)$$
$$+ \zeta_3\left[v'^\mu\left((v'-wv)^{[\nu}k^{\rho]} - (k\cdot v)v'^{[\nu}v^{\rho]}\right) + v^\mu\left((v-wv')^{[\nu}k^{\rho]} - (k\cdot v')v^{[\nu}v'^{\rho]}\right)\right.$$
$$\left. + g^{\mu[\nu}\left(k\cdot v(v-wv')^{\rho]} + k\cdot v'(v'-wv)^{\rho]} + (w^2-1)k^{\rho]}\right)\right]$$
$$+ \zeta_4\left((v+v')^\mu k^{[\nu}(v-v')^{\rho]} - (k\cdot(v+v'))g^{\mu[\nu}(v-v')^{\rho]}\right). \quad\text{(B.1)}$$

and for $\bar{B} \to D^*$:

$$T_S^{*\,\mu} = \zeta_2 \epsilon^{\mu\nu\rho\sigma} k_\nu (v+v')_\rho \varepsilon_{D^*\,\sigma}^* - \zeta_3 (v \cdot \varepsilon_{D^*}^*) \epsilon^{\mu\nu\rho\sigma} k_\nu v_\rho v_\sigma'$$
$$+ \zeta_4 k^\nu (v+v')^{[\nu} \epsilon^{\mu]\rho\sigma\delta} v_\rho v_\sigma' \varepsilon_{D^*\,\delta}^*$$
$$-i T_P^{*\,\mu} = \zeta_1 \left(\frac{v'^\mu}{k \cdot v'} - \frac{v^\mu}{k \cdot v} \right) v \cdot \varepsilon_{D^*}^* + \zeta_2 k^\nu (v-v')^{[\mu} \varepsilon_{D^*}^{*\,\nu]}$$
$$+ \zeta_4 k^\nu \left(v^{[\mu} v'^{\nu]} v \cdot \varepsilon_{D^*}^* - (w-1)(v+v')^{[\mu} \varepsilon_{D^*}^{*\,\nu]} \right)$$
$$-i T_T^{*\,\mu\nu\rho} = \zeta_1 \left(\frac{v'^\mu}{k \cdot v'} - \frac{v^\mu}{k \cdot v} \right) \epsilon^{\nu\rho\sigma\lambda} (v+v')_\sigma \varepsilon_{D^*\,\lambda}^* + \zeta_3 \epsilon^{\mu\alpha\beta\delta} k_\alpha v_\beta v_\delta' (v-v')^{[\nu} \varepsilon_{D^*}^{*\,\rho]}$$
$$+ \zeta_2 k^\sigma \left(\varepsilon_{D^*}^{*\,[\mu} \epsilon^{\sigma]\nu\rho\lambda} (v+v')_\lambda + (v'-v)^{[\nu} \epsilon^{\rho]\mu\sigma\lambda} \varepsilon_{D^*\,\lambda}^* \right) \tag{B.2}$$
$$+ \zeta_4 k^\sigma \varepsilon_{D^*\,\lambda}^* \left[(v+v')^{[\mu} \left(g^{\sigma][\nu} \epsilon^{\rho]\alpha\beta\lambda} v_\alpha v_\beta' + \epsilon^{\sigma]\nu\rho\lambda} - \epsilon^{\sigma]\alpha\lambda[\nu} v^{\rho]} v_\alpha' \right) + v'^{[\mu} v^{\sigma]} \epsilon^{\nu\rho\alpha\lambda} v_\alpha' \right].$$

References

[1] S. Nussinov and W. Wetzel, *Comparison of exclusive decay rates for $b \to u$ and $b \to c$ transitions*, *Phys. Rev. D* **36** (1987) 130 [INSPIRE].

[2] M.A. Shifman and M.B. Voloshin, *On production of d and D^* mesons in B meson decays*, *Sov. J. Nucl. Phys.* **47** (1988) 511 [*Yad. Fiz.* **47** (1988) 801] [INSPIRE].

[3] N. Isgur and M.B. Wise, *Weak decays of heavy mesons in the static quark approximation*, *Phys. Lett. B* **232** (1989) 113 [INSPIRE].

[4] N. Isgur and M.B. Wise, *Weak transition form-factors between heavy mesons*, *Phys. Lett. B* **237** (1990) 527 [INSPIRE].

[5] BELLE collaboration, *Measurement of the decay $B \to D\ell\nu_\ell$ in fully reconstructed events and determination of the Cabibbo-Kobayashi-Maskawa matrix element $|V_{cb}|$*, *Phys. Rev. D* **93** (2016) 032006 [arXiv:1510.03657] [INSPIRE].

[6] BELLE collaboration, *Measurement of the branching ratio of $\bar{B} \to D^{(*)}\tau^-\bar{\nu}_\tau$ relative to $\bar{B} \to D^{(*)}\ell^-\bar{\nu}_\ell$ decays with hadronic tagging at Belle*, *Phys. Rev. D* **92** (2015) 072014 [arXiv:1507.03233] [INSPIRE].

[7] BELLE collaboration, *Measurement of the τ lepton polarization and $R(D^*)$ in the decay $\bar{B} \to D^*\tau^-\bar{\nu}_\tau$*, *Phys. Rev. Lett.* **118** (2017) 211801 [arXiv:1612.00529] [INSPIRE].

[8] BELLE collaboration, *Measurement of the branching ratio of $\bar{B}^0 \to D^{*+}\tau^-\bar{\nu}_\tau$ relative to $\bar{B}^0 \to D^{*+}\ell^-\bar{\nu}_\ell$ decays with a semileptonic tagging method*, *Phys. Rev. D* **94** (2016) 072007 [arXiv:1607.07923] [INSPIRE].

[9] BELLE collaboration, *Measurement of the τ lepton polarization and $R(D^*)$ in the decay $\bar{B} \to D^*\tau^-\bar{\nu}_\tau$ with one-prong hadronic τ decays at Belle*, *Phys. Rev. D* **97** (2018) 012004 [arXiv:1709.00129] [INSPIRE].

[10] BELLE collaboration, *Measurement of the CKM matrix element $|V_{cb}|$ from $B^0 \to D^{*-}\ell^+\nu_\ell$ at Belle*, *Phys. Rev. D* **100** (2019) 052007 [*Erratum ibid.* **103** (2021) 079901] [arXiv:1809.03290] [INSPIRE].

[11] BELLE collaboration, *Measurement of $\mathcal{R}(D)$ and $\mathcal{R}(D^*)$ with a semileptonic tagging method*, arXiv:1904.08794 [INSPIRE].

[12] BELLE collaboration, *Measurement of $\mathcal{R}(D)$ and $\mathcal{R}(D^*)$ with a semileptonic tagging method*, *Phys. Rev. Lett.* **124** (2020) 161803 [arXiv:1910.05864] [INSPIRE].

[13] BABAR collaboration, *A Measurement of the branching fractions of exclusive $\bar{B} \to D^{(*)}(\pi)\ell^-\bar{\nu}_\ell$ decays in events with a fully reconstructed B meson*, *Phys. Rev. Lett.* **100** (2008) 151802 [arXiv:0712.3503] [INSPIRE].

[14] BABAR collaboration, *Determination of the form-factors for the decay $B^0 \to D^{*-}\ell^+\nu_\ell$ and of the CKM matrix element $|V_{cb}|$*, *Phys. Rev. D* **77** (2008) 032002 [arXiv:0705.4008] [INSPIRE].

[15] BABAR collaboration, *Measurement of the decay $B^- \to D^{*0}e^-\bar{\nu}_e$*, *Phys. Rev. Lett.* **100** (2008) 231803 [arXiv:0712.3493] [INSPIRE].

[16] BABAR collaboration, *Measurements of the semileptonic decays $\bar{B} \to D\ell\bar{\nu}$ and $\bar{B} \to D^*\ell\bar{\nu}$ using a global fit to $DX\ell\bar{\nu}$ final states*, *Phys. Rev. D* **79** (2009) 012002 [arXiv:0809.0828] [INSPIRE].

[17] BABAR collaboration, *Measurement of $|V_{cb}|$ and the form-factor slope in $\bar{B} \to D\ell^-\bar{\nu}u$ decays in events tagged by a fully reconstructed B meson*, *Phys. Rev. Lett.* **104** (2010) 011802 [arXiv:0904.4063] [INSPIRE].

[18] BABAR collaboration, *Evidence for an excess of $\bar{B} \to D^{(*)}\tau^-\bar{\nu}_\tau$ decays*, *Phys. Rev. Lett.* **109** (2012) 101802 [arXiv:1205.5442] [INSPIRE].

[19] BABAR collaboration, *Measurement of an excess of $\bar{B} \to D^{(*)}\tau^-\bar{\nu}_\tau$ decays and implications for charged Higgs bosons*, *Phys. Rev. D* **88** (2013) 072012 [arXiv:1303.0571] [INSPIRE].

[20] LHCb collaboration, *Measurement of the ratio of branching fractions $\mathcal{B}(\bar{B}^0 \to D^{*+}\tau^-\bar{\nu}_\tau)/\mathcal{B}(\bar{B}^0 \to D^{*+}\mu^-\bar{\nu}_\mu)$*, *Phys. Rev. Lett.* **115** (2015) 111803 [*Erratum ibid.* **115** (2015) 159901] [arXiv:1506.08614] [INSPIRE].

[21] LHCb collaboration, *Test of lepton flavor universality by the measurement of the $B^0 \to D^{*-}\tau^+\nu_\tau$ branching fraction using three-prong τ decays*, *Phys. Rev. D* **97** (2018) 072013 [arXiv:1711.02505] [INSPIRE].

[22] LHCb collaboration, *Measurement of the ratio of the $B^0 \to D^{*-}\tau^+\nu_\tau$ and $B^0 \to D^{*-}\mu^+\nu_\mu$ branching fractions using three-prong τ-lepton decays*, *Phys. Rev. Lett.* **120** (2018) 171802 [arXiv:1708.08856] [INSPIRE].

[23] F.U. Bernlochner and M. Schonherr, *Comparing different ansatzes to describe electroweak radiative corrections to exclusive semileptonic B meson decays into (pseudo)scalar final state mesons using Monte-Carlo techniques*, arXiv:1010.5997 [INSPIRE].

[24] S. Calí, S. Klaver, M. Rotondo and B. Sciascia, *Impacts of radiative corrections on measurements of lepton flavour universality in $B \to D\ell\nu_\ell$ decays*, *Eur. Phys. J. C* **79** (2019) 744 [arXiv:1905.02702] [INSPIRE].

[25] D. Becirevic and N. Kosnik, *Soft photons in semileptonic $B \to D$ decays*, *Acta Phys. Polon. Supp.* **3** (2010) 207 [arXiv:0910.5031] [INSPIRE].

[26] F.U. Bernlochner and H. Lacker, *A phenomenological model for radiative corrections in exclusive semileptonic B-meson decays to (pseudo)scalar final state mesons*, arXiv:1003.1620 [INSPIRE].

[27] S. de Boer, T. Kitahara and I. Nisandzic, *Soft-photon corrections to $\bar{B} \to D\tau^-\bar{\nu}_\tau$ relative to $\bar{B} \to D\mu^-\bar{\nu}_\mu$*, *Phys. Rev. Lett.* **120** (2018) 261804 [arXiv:1803.05881] [InSPIRE].

[28] J. Gasser, B. Kubis, N. Paver and M. Verbeni, *Radiative K_{e3} decays revisited*, *Eur. Phys. J. C* **40** (2005) 205 [hep-ph/0412130] [InSPIRE].

[29] H.D. Politzer and M.B. Wise, *Effective field theory approach to processes involving both light and heavy fields*, *Phys. Lett. B* **208** (1988) 504 [InSPIRE].

[30] E. Eichten and B.R. Hill, *An effective field theory for the calculation of matrix elements involving heavy quarks*, *Phys. Lett. B* **234** (1990) 511 [InSPIRE].

[31] E. Eichten and B.R. Hill, *Static effective field theory: $1/m$ corrections*, *Phys. Lett. B* **243** (1990) 427 [InSPIRE].

[32] H. Georgi, *An effective field theory for heavy quarks at low-energies*, *Phys. Lett. B* **240** (1990) 447 [InSPIRE].

[33] A.V. Manohar and M.B. Wise, *Heavy quark physics*, *Camb. Monogr. Part. Phys. Nucl. Phys. Cosmol.* **10** (2000) 1 [InSPIRE].

[34] V. Cirigliano and D. Pirjol, *Factorization in exclusive semileptonic radiative B decays*, *Phys. Rev. D* **72** (2005) 094021 [hep-ph/0508095] [InSPIRE].

[35] J.C. Ward, *An identity in quantum electrodynamics*, *Phys. Rev.* **78** (1950) 182 [InSPIRE].

[36] F.E. Low, *Bremsstrahlung of very low-energy quanta in elementary particle collisions*, *Phys. Rev.* **110** (1958) 974 [InSPIRE].

[37] S. Weinberg, *Infrared photons and gravitons*, *Phys. Rev.* **140** (1965) B516 [InSPIRE].

[38] R. Jackiw, *Low-energy theorems for massless bosons: photons and gravitons*, *Phys. Rev.* **168** (1968) 1623 [InSPIRE].

[39] J.F. Amundson et al., *Radiative D^* decay using heavy quark and chiral symmetry*, *Phys. Lett. B* **296** (1992) 415 [hep-ph/9209241] [InSPIRE].

[40] D. Becirevic and B. Haas, *$D^* \to D\pi$ and $D^* \to D\gamma$ decays: axial coupling and magnetic moment of D^* meson*, *Eur. Phys. J. C* **71** (2011) 1734 [arXiv:0903.2407] [InSPIRE].

[41] H. Elvang, C.R.T. Jones and S.G. Naculich, *Soft photon and graviton theorems in effective field theory*, *Phys. Rev. Lett.* **118** (2017) 231601 [arXiv:1611.07534] [InSPIRE].

[42] D. Atwood and W.J. Marciano, *Radiative corrections and semileptonic B decays*, *Phys. Rev. D* **41** (1990) 1736 [InSPIRE].

[43] M. Beneke, P. Böer, G. Finauri and K.K. Vos, *QED factorization of two-body non-leptonic and semi-leptonic B to charm decays*, *JHEP* **10** (2021) 223 [arXiv:2107.03819] [InSPIRE].

[44] E. Barberio, B. van Eijk and Z. Was, *PHOTOS: a universal Monte Carlo for QED radiative corrections in decays*, *Comput. Phys. Commun.* **66** (1991) 115 [InSPIRE].

[45] E. Barberio and Z. Was, *PHOTOS: a universal Monte Carlo for QED radiative corrections. Version 2.0*, *Comput. Phys. Commun.* **79** (1994) 291 [InSPIRE].

[46] D.R. Yennie, S.C. Frautschi and H. Suura, *The infrared divergence phenomena and high-energy processes*, *Annals Phys.* **13** (1961) 379 [InSPIRE].

[47] M. Schonherr and F. Krauss, *Soft photon radiation in particle decays in SHERPA*, *JHEP* **12** (2008) 018 [arXiv:0810.5071] [InSPIRE].