RECEIVED: August 26, 2021
REVISED: December 13, 2021
ACCEPTED: January 16, 2022
PUBLISHED: February 1, 2022

Entanglement in descendants

Barsha G. Chowdhury and Justin R. David

*Centre for High Energy Physics, Indian Institute of Science,
C.V. Raman Avenue, Bangalore 560012, India*

E-mail: barsha@iisc.ac.in, justin@iisc.ac.in

ABSTRACT: We study the single interval entanglement and relative entropies of conformal descendants in 2d CFT. Descendants contain non-trivial entanglement, though the entanglement entropy of the canonical primary in the free boson CFT contains no additional entanglement compared to the vacuum, we show that the entanglement entropy of the state created by its level one descendant is non-trivial and is identical to that of the U(1) current in this theory. We determine the first sub-leading corrections to the short interval expansion of the entanglement entropy of descendants in a general CFT from their four point function on the n -sheeted plane. We show that these corrections are determined by multiplying squares of appropriate dressing factors to the corresponding corrections of the primary. Relative entropy between descendants of the same primary is proportional to the square of the difference of their dressing factors. We apply our results to a class of descendants of generalized free fields and descendants of the vacuum and show that their dressing factors are universal.

KEYWORDS: Conformal Field Theory, Conformal and W Symmetry, Field Theories in Lower Dimensions

ARXIV EPRINT: [2108.00898](https://arxiv.org/abs/2108.00898)

Contents

1	Introduction	1
2	Entanglement and relative entropy of excited states	4
2.1	Free boson CFT	8
2.2	Entanglement entropy of ∂e^{iLX}	10
3	Short distance expansion	17
3.1	Primaries	18
3.2	Descendants: leading contribution to entanglement entropy	25
3.3	Descendants: sub-leading contribution to entanglement entropy	28
3.3.1	Level 1	29
3.3.2	Level 2	32
3.3.3	Level 3	35
3.3.4	Global descendants $\partial^l \mathcal{O}$	36
3.4	Relative entropy between descendants	37
4	Applications	39
4.1	Generalised free fields	39
4.2	Descendants of the vacuum	41
5	Conclusions	44
A	$2n$-point function of ∂e^{iLX} on the uniformised plane	45
B	Properties of descendants	51
B.1	Conformal transformations of descendants	51
B.2	Correlators with descendants	54

1 Introduction

Ideas from information theory have played a key role in our recent understanding of holography, quantum gravity and black hole physics. The most well studied and useful information theoretic quantity has proven to be entanglement entropy and relative entropy. In theories which admit a holographic duals, entanglement entropy can be evaluated from areas of extremal surfaces [1, 2]. Relative entropy between two nearby excited states is equal to the corresponding bulk relative entropy [3] Relative entropy has been applied to obtain a precise version of the Bekenstein bound and the second law of black hole thermodynamics [4–6].

Quantum field theories generally admit symmetries for example conformal field theories in 2 dimensions admit Virasoro symmetry. It is interesting to study how such a symmetry is reflected in information theoretic quantities like entanglement entropy and relative entropy. In particular since the global part of the Virasoro algebra corresponds to isometries of AdS_3 , there should be simple modifications of extremal surfaces or relative entropies when one considers these quantities in the bulk for states excited by such a symmetry. As far as we are aware this simple question has not been investigated in the literature.

In this paper we study this question by evaluating single interval entanglement entropy and relative entropies of conformal descendants of primaries in 2 dimensional conformal field theories. Information theoretic properties of conformal descendants have been studied earlier [7–10], but it has been confined to only the 2nd Rényi entropy and the trace square distance. However to understand how information theoretic quantities of descendants can be read out from extremal surfaces in holography it is important to study entanglement entropies and relative entropies. In fact as we will see in this paper, there are certain simplifications when one considers entanglement entropy or relative entropy. One might expect that descendants do not contain interesting information theoretic properties. This is easily dismissed by first considering relative entropy. We know that relative entropies is always positive and is a measure of the distance between states, see [11] for a review of information theoretic measures in quantum field theory. Therefore descendants of the same primary, and in fact distinct descendants of a given primary at the same level should differ in their relative entropy.

To begin, we show that descendants contain non-trivial entanglement by considering the free boson theory. It is known that the single interval entanglement entropy of its canonical excitation e^{iLX} is identical to that of the vacuum [12, 13]. In this paper we evaluate the single interval entanglement entropy of the level one descendant ∂e^{iLX} exactly. We use the replica trick developed in [12–15] to evaluate single interval entanglement entropy. Evaluating the $2n$ point function of the descendant on the uniformized plane we show that the single interval entropy is identical to that of the $U(1)$ current in this theory. As far as we are aware this is the first exact result of entanglement entropy of a descendant.

For a general conformal field theories it is not easy to evaluate the single interval entanglement entropy of excited states exactly since it involves knowing the $2n$ point function of the corresponding operator on the uniformized plane involved in implementing the replica trick. However we can study entanglement entropies or relative entropies in the short interval expansion. The leading term in the short distance expansion of the entanglement entropy of any primary in 2 dimension conformal field theory was obtained in [12]. This contribution to the short interval expansion is obtained by factoring the $2n$ point function into n 2-point functions where the pairs of operators are located on each wedge of the uniformized plane as in figure 6. A method to obtain the first sub-leading corrections to the entanglement entropy and relative entropy was developed in [15]. In this paper we revisit this method and simplify the evaluation of the first sub-leading corrections. We show that the sub-leading correction can be extracted from factorizing the $2n$ -point function into 2-point functions on $n - 2$ wedges in the uniformized plane and a four point function valued on 4 operators on the remaining pair of wedges as shown in figure 7.

Then one needs to sum over different pairs of wedges. This view of the first sub-leading correction allows us to generalise these computations to conformal descendants. We obtain the following expression for the short interval expansion of the entanglement entropy of the primary¹

$$\begin{aligned}\hat{S}(\rho_{\mathcal{O}}) &= 2h(1 - \pi x \cot \pi x) - \frac{8h^2}{15c}(\sin \pi x)^4 \\ &\quad - C_{\mathcal{O}\mathcal{O}\mathcal{O}_p} C_{\mathcal{O}\mathcal{O}}^{\mathcal{O}_p} (\sin \pi x)^{2h_p} \frac{\Gamma(\frac{3}{2})\Gamma(h_p+1)}{2\Gamma(h_p+\frac{3}{2})} + \dots\end{aligned}\quad (1.1)$$

Here we are considering only the holomorphic sector of the conformal field theory, the primary $\mathcal{O}$ has conformal dimension $(h, 0)$. h_p is the conformal dimension of the lowest lying primary in the theory and $C_{\mathcal{O}\mathcal{O}\mathcal{O}_p}$ is the corresponding OPE coefficient. Further more among the two sub-leading terms only one contributes depending on the certain conditions discussed after equation (3.10). For example, the term proportional to $1/c$ which arises from the stress tensor exchange is negligible for low lying primaries in large c conformal field theories.

We develop methods that enable the evaluation of the sub-leading corrections to short interval expansion of entanglement entropy of conformal descendants in 2d conformal field theories efficiently. After studying all descendants till level 3 and the class of descendants $\partial^l \mathcal{O}$, we find the leading contributions to the single interval entropy of descendants take the form

$$\begin{aligned}\hat{S}(\rho_{\mathcal{O}^{[-l]}}) &= 2(h+l)(1 - \pi x \cot \pi x) - \frac{8h^2}{15c}[D_{\mathcal{O}^{[l]}}(h, 2)]^2(\sin \pi x)^4 \\ &\quad - C_{\mathcal{O}\mathcal{O}\mathcal{O}_p} C_{\mathcal{O}\mathcal{O}}^{\mathcal{O}_p} [D_{\mathcal{O}^{[-l]}}(h, h_p)]^2 \frac{\Gamma(\frac{3}{2})\Gamma(h_p+1)}{2\Gamma(h_p+\frac{3}{2})} (\sin \pi x)^{2h_p} + \dots,\end{aligned}\quad (1.2)$$

where $\mathcal{O}^{[-l]}$ is the level l descendant of the primary $\mathcal{O}$. The leading term behaves as though the weight has shifted by $h \rightarrow h + l$. The sub-leading terms are modified compared to the expression for the corresponding primary in (1.1) by the appearance of a square of a pre-factor $D_{\mathcal{O}^{[-l]}}(h, h_p)$ which we call the dressing factor. This factor depends on the level and nature of the descendant, weight of the primary and central charge. $C_{\mathcal{O}\mathcal{O}\mathcal{O}_p}$ is the OPE coefficient. We evaluate the dressing factor for all states till level 3 and for the class $\partial^l \mathcal{O}$. From the computations which lead us to (1.2), we see that the dressing factor is the ratio of a deformed norm of the descendant, to the usual norm. The notion of the deformed norm is explained in section 3.3.

We show that the leading contributions to the relative entropy between descendants at level l, l' of primaries with weight h, h' respectively is given by

$$\begin{aligned}S(\rho_{\mathcal{O}^{[-l]}} | \rho_{\mathcal{O}'^{[-l']}}) &= \frac{8}{15c} \left[h D_{\mathcal{O}^{[-l]}}(h, 2) - h' D_{\mathcal{O}'^{[-l']}}(h, 2) \right]^2 (\sin \pi x)^4 \\ &\quad + \left\| (C_{\mathcal{O}\mathcal{O}\mathcal{O}_p} D_{\mathcal{O}^{[-l]}}(h, h_p) - C_{\mathcal{O}'\mathcal{O}'\mathcal{O}_p} D_{\mathcal{O}'^{[-l']}}(h, h_p)) \right\|^2 \frac{\Gamma(\frac{3}{2})\Gamma(h_p+1)}{2\Gamma(h_p+\frac{3}{2})} (\sin \pi x)^{2h_p} + \dots\end{aligned}\quad (1.3)$$

¹(1.1) is the difference of the entanglement entropy of the excited state and the vacuum state.

Here $|| \cdot ||^2$ refers to the fact that we need to take the square of its argument together with the index $\mathcal{O}_p$ in one of the term raised by the Zamolodchikov metric. Again, only one of the 2 terms is the leading contribution. It is easy to see that the relative entropy of descendants of the same primary is proportional to the square of the difference of their dressing factors. We test the results (1.2) and (1.3) using the exact results from the free boson theory.

Finally we apply our results to a class of descendants of generalized free fields and the vacuum. Generalized free fields in holographic conformal field theories. are known to be dual to minimally coupled scalars The entanglement entropy of $\partial^l \mathcal{O}$ where $\mathcal{O}$ is a generalised free field is given by

$$\hat{S}(\rho_{\partial^l \mathcal{O}}) = 2(h+l)(1 - \pi x \cot \pi x) - (\sin \pi x)^{4h} \frac{\Gamma(\frac{3}{2})\Gamma(2h+1)}{\Gamma(2h+\frac{3}{2})} \times \left(\frac{\mathcal{N}_l}{(l!)^2} \right)^2. \quad (1.4)$$

Here $\mathcal{N}_l$ is the norm of the global descendant $\partial^l \mathcal{O}$ at level l . For primaries that is $l = 0$, the sub-leading correction has been reproduced from the bulk [16]. This involved not only corrections to the Ryu-Takayanagi surface, but also evaluation of the bulk entanglement across this surface. The operator $\partial^l \mathcal{O}$ corresponds to the state $(L_{-1})^l |h\rangle$. which is obtained by the action of an isometry in the bulk on the excitation of the minimally coupled scalar. Therefore it should be possible to reproduce the result in (1.4) for $l \neq 0$ by generalising the methods of [16]. For the class of descendants $L_{-n}|0\rangle$ we show that the leading contributions to the single interval entanglement is given by

$$\begin{aligned} \hat{S}(\rho_{L_{-(l+2)}|0\rangle}) &= 2(l+2)(1 - \pi x \cot \pi x) - \frac{8(l+2)^2}{15c} \sin^4 \pi x \\ &\quad - \left[\frac{(l+3)(l+2)(l+1)}{3!} \right]^2 \frac{128}{315} \sin^8 \pi x + \dots \end{aligned} \quad (1.5)$$

Again only one among the last two terms contribute. For holographic conformal field theories and for low lying excitations, it is only the third term that contributes at the sub-leading order. As expected the entanglement entropy of descendants of the vacuum is universal. It should be possible to verify (1.5) using holography.

The organization of the paper is as follows. In section 2 we briefly review the set up involved in evaluating entanglement properties of low lying states. We use it to evaluate the entanglement entropy of the level one descendant of e^{ilX} exactly in the free boson theory. In section 3 we develop a simplified method to obtain the entanglement entropy of low lying states in the short interval expansion and then apply it to conformal descendants. We also evaluate the relative entropy between various descendants. We apply these results to a class of descendants of generalized free fields and the vacuum in section 4. Section 5 contains our conclusions. Appendix A contains the details for evaluating the $2n$ point function of the level one descendant of e^{ilX} on the uniformized plane. Appendix B discusses the properties of conformal descendants needed to arrive at the results in the paper.

2 Entanglement and relative entropy of excited states

We briefly review the set up of evaluating entanglement and relative entropy of excited states. Consider a 2d CFT on a cylinder parametrised by (t, ϕ) where t is the Euclidean

Figure 1. The path integral for the density matrix $\rho_{\mathcal{O}}$.

time and $\phi \sim \phi + 2\pi$ is the spatial coordinate. We excite the CFT by placing an operator $\mathcal{O}$, not necessarily a primary at $t \rightarrow -\infty$, let us call refer to this state as $|\mathcal{O}\rangle$. In this paper we will restrict our discussions only to the holomorphic sector of the CFT to keep the discussion uncluttered. All conclusions obtained in this paper can easily be generalised to include the anti-holomorphic sector. We wish to consider the reduced density matrix of $\rho_{\mathcal{O}}$ obtained by tracing over the complement of the interval $[0, 2\pi x]$ at the $t = 0$ time slice.

$$\rho_{\mathcal{O}} = \text{Tr}_{[0, 2\pi x]^c}(|\mathcal{O}\rangle\langle\mathcal{O}|). \quad (2.1)$$

To clarify, the bi-partite system consists of the interval $[0, 2\pi x]$ and its complement $[0, 2\pi x]^c$. The partial trace is performed over the complement, therefore the sub-system of interest is the interval of length $2\pi x$. The ratio of the size of the sub-system to the system is x . The path integral picture of this reduced density matrix is given in figure 1. As we mentioned the state $|\mathcal{O}\rangle$ is obtained by placing the operator $\mathcal{O}$ at $t \rightarrow -\infty$, while the state $\langle\mathcal{O}|$ is its adjoint obtained by placing the operator at $t \rightarrow +\infty$. The partial trace over the complement of $[0, 2\pi x]$ leaves a cut at this interval. To evaluate the entanglement entropy of the reduced density matrix $\rho_{\mathcal{O}}$, we would need to evaluate $\text{Tr}(\rho_{\mathcal{O}}^n)$. The path integral then involves n copies of the cylinder in figure 1 glued along the cut with the last cylinder glued to the first. It is convenient to use the cylinder to plane map given in

$$z = e^{t+i\phi}, \quad (2.2)$$

to equivalently regard the path integral on an n sheeted plane Σ_n . In this picture, n copies of the operator $\mathcal{O}$ are placed at the origin as well at ∞ on each sheet as shown in figure 2. Using the state operator correspondence on the plane, the state $|\mathcal{O}\rangle$ at the origin can be written as

$$|\mathcal{O}\rangle = \mathcal{O}(0)|0\rangle, \quad (2.3)$$

while the state at infinity is given by

$$\langle\mathcal{O}| = \langle 0|\mathcal{O}^\dagger(\infty) = \lim_{z \rightarrow 0} \langle 0|I \circ \mathcal{O}^*(z). \quad (2.4)$$

Figure 2. The path integral for $\rho_{\mathcal{O}}^n$ on the n -sheeted plane obtained using the cylinder to the plane map. The entangling interval is on the arc between $v = 1$ and $u = e^{2\pi ix}$ on the unit circle. The operators are mapped to the origin and infinity on each plane.

Here $I \circ \mathcal{O}$ refers to the action of the inversion which takes the origin to ∞ .

$$I(z) = -\frac{1}{z}, \quad (2.5)$$

which is a map in $\text{SL}(2, \mathbb{Z})$. If $\mathcal{O}$ is a primary of weight h .

$$I \circ \mathcal{O}(z) = (\partial_z I(z))^h \mathcal{O}(I(z)) = z^{-2h} \mathcal{O}\left(-\frac{1}{z}\right). \quad (2.6)$$

However if $\mathcal{O}$ is a descendent one needs to use the appropriate conformal transformation. For example let $\partial\mathcal{O}$ be the level 1 descendant, then we have

$$I \circ \partial\mathcal{O}(z) = [\partial_z I(z)]^{h+1} \mathcal{O}(I(z)) + h \partial_z^2 I(z) [\partial_z I(z)]^{h-1} \mathcal{O}(I(z)). \quad (2.7)$$

Using the map (2.2) we see that the cut along the interval on the cylinder lies on the unit circle at the points

$$v = 1, \quad u = e^{2\pi ix}. \quad (2.8)$$

From the path integral on the n sheeted plane, the trace of the reduced density matrix $\rho_{\mathcal{O}}^n$ can be written as

$$\frac{\text{Tr} \rho_{\mathcal{O}}^n}{\text{Tr} \rho_{(0)}^n} = \left\langle \prod_{k=0}^{n-1} \mathcal{O}(0_k) \prod_{k'=0}^{n-1} \mathcal{O}^*(\infty_{k'}) \right\rangle_{\Sigma_n}. \quad (2.9)$$

Here by 0_k we mean the origin and ∞_k we mean ∞ at the k -th cover. Note to obtain ∞_k , we can take $z \rightarrow 0$ in the inversion map (2.5). The reduced density matrix $\rho_{(0)}$ refers to the path integral on Σ_n without any operator insertions. To make the evaluation of the correlation function tractable, we use the uniformization map which takes the branched cover Σ_n to the plane

$$w(z) = \left(\frac{z - u}{z - v} \right)^{\frac{1}{n}}. \quad (2.10)$$

Figure 3. The uniformized plane with $2n$ operator insertions. The figure shows the example of $n = 8$. Each sheet of figure 2 is mapped to a wedge on the uniformized plane.

The uniformized plane is shown in figure 3. For points at ∞ we compose this map with the inversion and use

$$\hat{w}(z) = \left(\frac{-\frac{1}{z} - u}{-\frac{1}{z} - v} \right)^{\frac{1}{n}}. \quad (2.11)$$

The maps in (2.10) and (2.11) take the operators placed on the branched planes to the following locations on the uniformized plane

$$0_k \rightarrow w_k = e^{\frac{2\pi i(k+x)}{n}}, \quad \infty_k \rightarrow \hat{w}_k = e^{\frac{2\pi i k}{n}}. \quad (2.12)$$

Using the uniformization map, the reduced density matrix is given by

$$\frac{\text{Tr} \rho_{\mathcal{O}}^n}{\text{Tr} \rho_{(0)}^n} = \left\langle \prod_{k=0}^{n-1} w \circ \mathcal{O}(w_k) \prod_{k'=0}^{n-1} \hat{w} \circ \mathcal{O}^*(\hat{w}_{k'}) \right\rangle. \quad (2.13)$$

Here $w \circ \mathcal{O}$ and $\hat{w} \circ \mathcal{O}^*$ refers to the action of the conformal transformation given in (2.10) and (2.11) respectively.² The Rényi entropies and entanglement entropies are then obtained by evaluating

$$S_n(\rho_{\mathcal{O}}) = \frac{1}{1-n} \log(\text{Tr}(\rho_{\mathcal{O}}^n)), \quad S(\rho_{\mathcal{O}}) = \lim_{n \rightarrow 1} S_n(\rho_{\mathcal{O}}). \quad (2.14)$$

Therefore these entropies can be computed provided the $2n$ point function of the operator $\mathcal{O}$ on the uniformized plane can be evaluated. Since the single interval Rényi entropies of the vacuum of the CFT is known and universal, we will be interested in the difference of

²The complex conjugation on the operator $\mathcal{O}$ refers to taking complex conjugates of the parameters in the operator, but keeping holomorphic z . eg. $(e^{iLX(z)})^* = e^{-iLX(z)}$. This action arises on taking the adjoint since we are interested in working only in the holomorphic sector.

the entanglement entropy of the excited state from that of the vacuum. To evaluate this difference, it is sufficient to examine the ratio of the traces of the density matrices given in (2.13). We refer to this as the entanglement entropy of the excited state and it is given by

$$\hat{S}_n(\rho_{\mathcal{O}}) = \frac{1}{1-n} \log \left(\frac{\text{Tr} \rho_{\mathcal{O}}^n}{\text{Tr} \rho_{(0)}^n} \right), \quad S(\rho_{\mathcal{O}}) = \lim_{n \rightarrow 1} \hat{S}_n(\rho_{\mathcal{O}}). \quad (2.15)$$

Let us now consider the relative entropy between the reduced density matrix $\rho_{\mathcal{O}}$ and that corresponding to another operator $\rho_{\mathcal{O}'}$ which is given by

$$S(\rho_{\mathcal{O}} || \rho_{\mathcal{O}'}) = \text{Tr}(\rho_{\mathcal{O}} \log \rho_{\mathcal{O}}) - \text{Tr}(\rho_{\mathcal{O}} \log \rho_{\mathcal{O}'}). \quad (2.16)$$

To re-write the relative entropy in terms of correlators we use the identity

$$\begin{aligned} S(\rho_{\mathcal{O}} || \rho_{\mathcal{O}'}) &= \lim_{n \rightarrow 1} S_n(\rho_{\mathcal{O}} || \rho_{\mathcal{O}'}), \\ &= \lim_{n \rightarrow 1} \frac{1}{n-1} \left[\log(\text{Tr} \rho_{\mathcal{O}}^n) - \log(\text{Tr} \rho_{\mathcal{O}} \rho_{\mathcal{O}'}^{n-1}) \right]. \end{aligned} \quad (2.17)$$

Going through a similar analysis for the trace $\text{Tr} \rho_{\mathcal{O}} \rho_{\mathcal{O}'}^{n-1}$ we see that the path integral involves the n -branched plane in which one plane say the 0-th plane contains the operator $\mathcal{O}$ and the rest $n-1$ planes contain the operator $\mathcal{O}'$. Therefore we have

$$\begin{aligned} \frac{\text{Tr}(\rho_{\mathcal{O}} \rho_{\mathcal{O}'}^{n-1})}{\text{Tr}(\rho_{(0)})^n} &= \left\langle \mathcal{O}(0_0) \mathcal{O}(\infty_0) \prod_{k=1}^{n-1} [\mathcal{O}'(0_k) \mathcal{O}'^*(\infty_k)] \right\rangle_{\Sigma_n}, \\ &= \left\langle w \circ \mathcal{O}(w_0) \hat{w} \circ \mathcal{O}^*(\hat{w}_0) \prod_{k=1}^{n-1} [w \circ \mathcal{O}'(w_k) \hat{w} \circ \mathcal{O}'^*(\hat{w}_k)] \right\rangle. \end{aligned} \quad (2.18)$$

Using the definition of $S_n(\rho_{\mathcal{O}} || \rho_{\mathcal{O}'})$ in (2.17) and (2.18) we obtain

$$S_n(\rho_{\mathcal{O}} || \rho_{\mathcal{O}'}) = \frac{1}{n-1} \log \frac{\langle \prod_{k=0}^{n-1} w \circ \mathcal{O}(w_k) \prod_{k'=0}^{n-1} \hat{w} \circ \mathcal{O}^*(\hat{w}'_k) \rangle}{\langle w \circ \mathcal{O}(w_0) \hat{w} \circ \mathcal{O}^*(\hat{w}_0) \prod_{k=1}^{n-1} [w \circ \mathcal{O}'(w_k) \hat{w} \circ \mathcal{O}'^*(\hat{w}_k)] \rangle}. \quad (2.19)$$

We begin by applying these formulae to the case of the free boson. We evaluate single interval entanglement entropies and relative entropies excited by the primary $\mathcal{O} = e^{ilX}$ and the descendant $\partial\mathcal{O} = \partial e^{ilX}$. We will see that in this particular case, the entanglement entropy of the state the descendant can be exactly evaluated.

2.1 Free boson CFT

Consider the CFT of a single boson X and $\mathcal{O} = e^{ilX}$, $\mathcal{O}^* = e^{-ilX}$. Then from (2.13) we see that to evaluate the Rényi entropy we would need to evaluate the following $2n$ point function for this operator

$$\begin{aligned} \left\langle \prod_{k=0}^{n-1} w \circ \mathcal{O}(w_k) \prod_{k'=0}^{n-1} \hat{w} \circ \mathcal{O}^*(\hat{w}_{k'}) \right\rangle &= \prod_{k=0}^{n-1} \left[\left. \frac{\partial w}{\partial z} \right|_{0_k} \left. \frac{\partial \hat{w}}{\partial z} \right|_{\infty_k} \right]^{\frac{l^2}{2}} f(w; \hat{w}), \\ f(w, \hat{w}) &= \left[\prod_{k, k'=0; k' > k}^{n-1} (w_k - w_{k'}) (\hat{w}_k - \hat{w}_{k'}) \right]^{l^2} \left[\prod_{k', k=0}^{n-1} (w_k - \hat{w}_{k'}) \right]^{-l^2}. \end{aligned} \quad (2.20)$$

Here the products over k, k' run over $0, 1, \dots, n-1$. From the uniformization map (2.10) and (2.11) we see that

$$w_k = e^{\frac{2\pi i(k+x)}{n}}, \quad \hat{w}_k = e^{\frac{2\pi i k}{n}}, \quad (2.21)$$

and

$$\begin{aligned} \left. \frac{\partial w}{\partial z} \right|_{0_k} &= \frac{1}{n} e^{\frac{2\pi i(x+k)}{n}} (1 - e^{-2\pi i x}), & \left. \frac{\partial \hat{w}}{\partial z} \right|_{\infty_k} &= -\frac{1}{n} e^{\frac{2\pi i k}{n}} (1 - e^{2\pi i x}), \\ &\equiv B_k, & &\equiv \hat{B}_k. \end{aligned} \quad (2.22)$$

The correlator (2.20) have been evaluated before in [13, 14], but here we reevaluate it so as to develop the methods which will enable to evaluate the corresponding correlator for the level one descendant of $\mathcal{O}$. To proceed, we would need the following algebraic identities

$$\prod_{k'=0}^{n-1} \left(z - e^{\frac{2\pi i k'}{n}} \right) = z^n - 1, \quad (2.23)$$

$$\prod_{k' \neq k, k'=0}^{n-1} \left(z - e^{\frac{2\pi i k'}{n}} \right) = \frac{z^n - 1}{z - e^{\frac{2\pi i k}{n}}}. \quad (2.24)$$

From these identities, it is easy to derive the relations.

$$\prod_{k'=0}^{n-1} \left(e^{\frac{2\pi i(k+x)}{n}} - e^{\frac{2\pi i k'}{n}} \right) = e^{2\pi i x} - 1, \quad (2.25)$$

$$\prod_{k'; k' \neq k} \left(e^{\frac{2\pi i k}{n}} - e^{\frac{2\pi i k'}{n}} \right) = n e^{-\frac{2\pi i k}{n}}. \quad (2.26)$$

To obtain (2.26) we can substitute $z = e^{\frac{2\pi i(k+x)}{n}}$ and then take the $x \rightarrow 0$ limit. Using these identities we can simplify the first set of terms that occur in the function $f(w, \hat{w})$ of (2.20).

$$\begin{aligned} \left[\prod_{k, k'=0; k' > k}^{n-1} (w_k - w_{k'}) (\hat{w}_k - \hat{w}_{k'}) \right] &= \prod_{k, k'=0; k' > k}^{n-1} \left(e^{\frac{2\pi i(k+x)}{n}} - e^{\frac{2\pi i(k'+x)}{n}} \right) \left(e^{\frac{2\pi i k}{n}} - e^{\frac{2\pi i k'}{n}} \right), \\ &= e^{\pi i(n-1)x} \prod_{k=0}^{n-1} \prod_{k'; k' \neq k} \left(e^{\frac{2\pi i k}{n}} - e^{\frac{2\pi i k'}{n}} \right), \\ &= e^{\pi i(n-1)x} \prod_{k=0}^{n-1} n e^{-\frac{2\pi i k}{n}} = e^{\pi i(n-1)x} n^n e^{-i\pi(n-1)}. \end{aligned} \quad (2.27)$$

Substituting this result in (2.20), we obtain

$$\begin{aligned} \left\langle \prod_{k=0}^{n-1} w \circ \mathcal{O}(w_k) \prod_{k'=0}^{n-1} \hat{w} \circ \mathcal{O}^*(\hat{w}_{k'}) \right\rangle &= \left(\frac{1}{n} \right)^{nl^2} e^{\pi i x l^2} e^{\pi i(n-1)l^2} (e^{i\pi})^{\frac{nl^2}{2}} (2 \sin \pi x)^{nl^2} \\ &\quad \times \left(e^{\pi i(n-1)x} n^n e^{-i\pi(n-1)} \right)^{l^2} \\ &\quad \times (2i e^{\pi i x} \sin \pi x)^{-nl^2}, \\ &= 1. \end{aligned} \quad (2.28)$$

The first line arises from the conformal transformations, the second line from using (2.28) in (2.20) and the third line arises from terms of (2.25). Therefore we have the relation

$$\prod_{k=0}^{n-1} (B_k \hat{B}_k)^{\frac{i^2}{2}} f(w, \hat{w}) = 1, \quad (2.29)$$

where $B_k, \hat{B}_k$ and $f(w, \hat{w})$ are defined in (2.22) and (2.20) respectively.

Using the result for the correlator (2.20) in (2.13) we obtain

$$\frac{\text{Tr} \rho_{e^{iLX}}^n}{\text{Tr} \rho_{(0)}^n} = 1. \quad (2.30)$$

Therefore from (2.14) we see that the single interval Rényi entropies as well as the entanglement entropy of the state $e^{ikX}|0\rangle$ is identical to the vacuum. To be explicit we write

$$\hat{S}(\rho_{e^{iLX}}) = 0, \quad (2.31)$$

where $\hat{S}$ is defined in (2.15).

2.2 Entanglement entropy of ∂e^{iLX}

Let us proceed to evaluate the single interval entanglement entropy of the level one descendant ∂e^{iLX} . From (2.13), we see that we need the conformal transformation of the level one descendant. This is given by

$$f \circ \partial \mathcal{O}(z) = [\partial_z f(z)]^{h+1} \partial \mathcal{O}(f(z)) + h \partial_z^2 f(z) [\partial_z f(z)]^{h-1} \mathcal{O}(f(z)). \quad (2.32)$$

Thus we would need the second derivatives of the conformal transformation in (2.10) and which is given by

$$\begin{aligned} \left. \frac{\partial^2 w}{\partial z^2} \right|_{0_k} &= e^{\frac{2\pi i(k+x)}{n}} \frac{1}{n} (1 - e^{-2\pi i x}) \left[(1 + e^{-2\pi i x}) + \frac{1}{n} (1 - e^{-2\pi i x}) \right], \\ &= B_k A, \end{aligned} \quad (2.33)$$

where

$$A \equiv \left[(1 + e^{-2\pi i x}) + \frac{1}{n} (1 - e^{-2\pi i x}) \right]. \quad (2.34)$$

Similarly the derivative of the conformal transformation (2.11) is given by

$$\begin{aligned} \left. \frac{\partial^2 \hat{w}}{\partial z^2} \right|_{\infty_k} &= \frac{1}{n} e^{\frac{2\pi i k}{n}} (1 - e^{2\pi i x}) \left[(1 + e^{2\pi i x}) + \frac{1}{n} (1 - e^{2\pi i x}) \right], \\ &= \hat{B}_k \hat{A}, \end{aligned} \quad (2.35)$$

where

$$\hat{A} \equiv - \left[(1 + e^{2\pi i x}) + \frac{1}{n} (1 - e^{2\pi i x}) \right]. \quad (2.36)$$

Using this input into the conformal transformation of the descendant ∂e^{ilX} given in (2.32) we see that the relevant $2n$ point correlator can be written as

$$\mathcal{C}_{2n} = \prod_{k=0}^{n-1} (B_k \hat{B}_k)^{\frac{l^2}{2}} \left[\frac{l^2}{2} A + B_k \partial_{u_k} \right] \left[\frac{l^2}{2} \hat{A} + \hat{B}_k \partial_{\hat{u}_k} \right] f(u, \hat{u}) \Big|_{(u, \hat{u})=(w, \hat{w})}. \quad (2.37)$$

In (2.37) we first act the derivatives on the function

$$f(u, \hat{u}) = \left[\prod_{k, k'=0; k' > k}^{n-1} (u_k - u_{k'}) (\hat{u}_k - \hat{u}_{k'}) \right]^{l^2} \left[\prod_{k', k=0}^{n-1} (u_k - \hat{u}_{k'}) \right]^{-l^2}, \quad (2.38)$$

and then set $(u, \hat{u})$ to the point $(w, \hat{w})$, which are the points on the uniformized plane. The operator ∂e^{ilX} is not normalized. Evaluating the norm using the adjoint defined in (2.4), (2.6) we obtain

$$\begin{aligned} \langle \partial e^{ilX} | \partial e^{ilX} \rangle &= \lim_{z \rightarrow 0} \langle I \circ \partial e^{ilX}(z) e^{ilX}(0) \rangle, \\ &= -l^2. \end{aligned} \quad (2.39)$$

Note that the reason we obtain a negative sign is due to our definition of the adjoint. It is easy to see that according to this definition, the adjoint of the Virasoro generator $(L_n)^\dagger = (-1)^n L_{-n}$,³ therefore we get

$$\langle h | L_1^\dagger L_1 | h \rangle = -\langle h | L_{-1} L_1 | h \rangle = -2h. \quad (2.40)$$

Here $|h\rangle$ is a primary of weight h . The above equation agrees with the explicit calculation in (2.39).

Before we proceed we will need the following identities. From (2.23) we can obtain

$$\sum_{k=0}^{n-1} \frac{1}{z - e^{\frac{2\pi i k}{n}}} = \frac{nz^{n-1}}{z^n - 1}. \quad (2.41)$$

On taking the limit that z is one of the n -th root of unity we obtain

$$\sum_{j=0, j \neq k}^{n-1} \frac{1}{e^{\frac{2\pi i(k+x)}{n}} - e^{\frac{2\pi i(j+x)}{n}}} = \frac{n-1}{2} e^{-\frac{2\pi i(k+x)}{n}}. \quad (2.42)$$

Then from the obvious substitutions in (2.41) we obtain

$$\sum_{j=0}^{n-1} \frac{1}{e^{\frac{2\pi i(k+x)}{n}} - e^{\frac{2\pi i j}{n}}} = \frac{ne^{-\frac{2\pi i(k+x)}{n}}}{1 - e^{-2\pi i x}}, \quad (2.43)$$

$$\sum_{j=0}^{n-1} \frac{1}{e^{\frac{2\pi i k}{n}} - e^{\frac{2\pi i(j+x)}{n}}} = \frac{ne^{-\frac{2\pi i k}{n}}}{1 - e^{2\pi i x}}. \quad (2.44)$$

³This can be seen from performing the inversion transformation on $T(z) = \sum_n \frac{L_n}{z^{n+2}}$.

Now examining (2.37) we see that there are derivatives acting on the function $f(u, \hat{u})$. Consider a single derivative

$$\begin{aligned}\partial_{u_k} f(u, \hat{u}) &= f(u, \hat{u}) g_k(u, \hat{u}), \\ g_k(u, \hat{u}) &= l^2 \left[\sum_{j=0, j \neq k}^{n-1} \frac{1}{u_k - u_j} - \sum_{j=0}^{n-1} \frac{1}{u_k - \hat{u}_j} \right], \\ \partial_{\hat{u}_k} f(u, \hat{u}) &= f(u, \hat{u}) h_k(u, \hat{u}), \\ h_k(u, \hat{u}) &= l^2 \left[\sum_{j=0, j \neq k}^{n-1} \frac{1}{\hat{u}_k - \hat{u}_j} - \sum_{j=0}^{n-1} \frac{1}{\hat{u}_k - u_j} \right].\end{aligned}\tag{2.45}$$

Now setting $(u, \hat{u}) = (w, \hat{w})$ we obtain

$$\begin{aligned}\partial_{u_k} f(u, \hat{u})|_{(u, \hat{u})=(w, \hat{w})} &= l^2 f(w, \hat{w}) \left[\sum_{k \neq j} \frac{1}{e^{\frac{2\pi i(x+j)}{n}} - e^{\frac{2\pi i(x+k)}{n}}} - \sum_k \frac{1}{e^{\frac{2\pi i(x+j)}{n}} - e^{\frac{2\pi i k}{n}}} \right], \\ &= l^2 f(w, \hat{w}) \left[\frac{n-1}{2} e^{-\frac{2\pi i(x+j)}{n}} - \frac{n}{1 - e^{-2\pi i x}} e^{-\frac{2\pi i(x+j)}{n}} \right].\end{aligned}\tag{2.46}$$

After substituting the points on the uniformized plane, in the second line we have used the identities (2.42) and (2.43). In (2.37), the derivative at u_k occurs with B_k . Therefore let us evaluate

$$\begin{aligned}B_k \partial_{u_k} f(u, \hat{u})|_{(u, \hat{u})=(w, \hat{w})} &= g_k(u, \hat{u}) f(u, \hat{u})|_{(u, \hat{u})=(w, \hat{w})}, \\ &= -l^2 f(w, \hat{w}) \left(\frac{1}{2} (1 + e^{-2\pi i x}) + \frac{1}{2n} (1 - e^{-2\pi i x}) \right), \\ &= -\frac{l^2}{2} f(w, \hat{w}) A.\end{aligned}\tag{2.47}$$

Note that this relation does not involve a sum over k . This relation helps us simplify the various terms which involve the derivatives. Similarly we have the following relation for the derivative with respect to $\hat{u}_k$.

$$\begin{aligned}\hat{B}_k \partial_{\hat{u}_k} f(u, \hat{u})|_{(u, \hat{u})=(w, \hat{w})} &= g_k(u, \hat{u}) f(u, \hat{u})|_{(u, \hat{u})=(w, \hat{w})}, \\ &= -\frac{l^2}{2} f(w, \hat{w}) \hat{A}.\end{aligned}\tag{2.48}$$

We are now ready to evaluate the correlator (2.37). To illustrate the manipulations involved we first consider the case $n = 2$. Let us organise the terms according to the number of derivatives. The term involving no derivatives is given by

$$\begin{aligned}\mathcal{C}_2^{(0)} &= \left(\prod_{k=0}^1 (B_k \hat{B}_k)^{\frac{l^2}{2}} \left(\frac{l^2}{2} \right)^2 A \hat{A} \right) f(w, \hat{w}), \\ &= \left(\frac{l^2}{2} \right)^4 (A \hat{A})^2.\end{aligned}\tag{2.49}$$

The superscript indicates the number of derivatives. To arrive at the second line in the above equation we have used the relation (2.29) with $n = 2$. All the terms with a single derivative results in

$$\begin{aligned} \mathcal{C}_2^{(1)} &= \prod_{k=0}^1 (B_k \hat{B}_k)^{\frac{l^2}{2}} \left(A \hat{A}^2 B_j \sum_{j=0}^1 \partial_{u_j} + A^2 \hat{A} \sum_{j=0}^1 \hat{B}_j \partial_{\hat{u}_j} \right) f(u, \hat{u}) \Big|_{(u, \hat{u})=(w, \hat{w})}, \\ &= -4 \left(\frac{l^2}{2} \right)^4 (A \hat{A})^2. \end{aligned} \quad (2.50)$$

Here we have used (2.47), (2.48) to simplify the action of the derivatives and (2.29) to arrive at the last line. The terms involving 2 derivatives can be simplified using the following relations

$$\begin{aligned} \partial_{u_j} \partial_{u_l} f(u, \hat{u}) &= f(u, \hat{u}) g_j(u, \hat{u}) g_l(u, \hat{u}) + l^2 \frac{f(u, \hat{u})}{(u_j - u_l)^2}, \\ \partial_{\hat{u}_j} \partial_{\hat{u}_l} f(u, \hat{u}) &= f(u, \hat{u}) h_j(u, \hat{u}) h_l(u, \hat{u}) + l^2 \frac{f(u, \hat{u})}{(\hat{u}_j - \hat{u}_l)^2}, \\ \partial_{\hat{u}_j} \partial_{u_l} f(u, \hat{u}) &= f(u, \hat{u}) g_j(u, \hat{u}) h_l(u, \hat{u}) - l^2 \frac{f(u, \hat{u})}{(u_l - \hat{u}_j)^2}, \end{aligned} \quad (2.51)$$

where $j \neq l$. Then using these formulae for the derivatives together with the relations (2.47), (2.48) and (2.29) we obtain

$$\begin{aligned} \mathcal{C}_2^{(2)} &= \left(\frac{l^2}{2} \right)^4 A^2 \hat{A} + \left(\frac{l^2}{2} \right)^2 \hat{A}^2 \frac{l^2}{(w_0 - w_1)^2} B_0 B_1 \\ &\quad + \left(\frac{l^2}{2} \right)^4 A^2 A'^2 + \left(\frac{l^2}{2} \right)^2 A^2 \frac{l^2}{(\hat{w}_0 - \hat{w}_1)^2} \hat{B}_0 \hat{B}_1 \\ &\quad + 4 \left(\frac{l^2}{2} \right)^4 A^2 \hat{A}^2 - \left(\frac{l^2}{2} \right)^2 A \hat{A} \left(\frac{l^2 B_0 \hat{B}_0}{(w_0 - \hat{w}_0)^2} + \frac{l^2 B_0 \hat{B}_1}{(w_0 - \hat{w}_1)^2} + \frac{l^2 B_1 \hat{B}_0}{(w_1 - \hat{w}_0)^2} + \frac{l^2 B_1 \hat{B}_1}{(w_1 - \hat{w}_1)^2} \right). \end{aligned} \quad (2.52)$$

The last line in the above equation is due to the contribution of the mixed derivative $\partial_{u_j} \partial_{u_l} f(u, \hat{u})$. Similarly using the expressions for 3 derivatives in (A.14) we obtain

$$\begin{aligned} \mathcal{C}_2^{(3)} &= -2 \left(\frac{l^2}{2} \right)^4 A^2 A'^2 - 2 \left(\frac{l^2}{2} \right)^2 \hat{A}^2 \frac{l^2 B_0 B_1}{(w_0 - w_1)^2} \\ &\quad + \left(\frac{l^2}{2} \right)^2 A \hat{A} \left(\frac{l^2 B_0 \hat{B}_0}{(w_0 - \hat{w}_0)^2} + \frac{l^2 B_0 \hat{B}_1}{(w_0 - \hat{w}_1)^2} + \frac{l^2 B_1 \hat{B}_0}{(w_1 - \hat{w}_0)^2} + \frac{B_1 \hat{B}_1}{(w_1 - \hat{w}_1)^2} \right) \\ &\quad - 2 \left(\frac{l^2}{2} \right)^4 A^2 \hat{A}^2 - 2 \left(\frac{l^2}{2} \right)^2 A^2 \frac{l^2 \hat{B}_0 \hat{B}_1}{(\hat{w}_0 - \hat{w}_1)^2} \\ &\quad + \left(\frac{l^2}{2} \right)^2 A \hat{A} \left(\frac{l^2 B_0 \hat{B}_0}{(w_0 - \hat{w}_0)^2} + \frac{l^2 B_0 \hat{B}_1}{(w_0 - \hat{w}_1)^2} + \frac{l^2 B_1 \hat{B}_0}{(w_1 - \hat{w}_0)^2} + \frac{B_1 \hat{B}_1}{(w_1 - \hat{w}_1)^2} \right). \end{aligned} \quad (2.53)$$

The first two lines arise from derivatives of the type $\partial_{\hat{u}_i} \partial_{u_j} \partial_{u_k}$, while the last two lines arise from derivatives of the type $\partial_{u_i} \partial_{\hat{u}_j} \partial_{\hat{u}_k}$. Finally using (A.15) the terms arising from the

action of 4 derivatives are given by

$$\begin{aligned} \mathcal{C}_2^{(4)} = & \left(\frac{l^2}{2}\right)^4 A^2 A'^2 + \left(\frac{l^2}{2}\right)^2 A^2 \frac{l^2 \hat{B}_0 \hat{B}_1}{(\hat{w}_0 - \hat{w}_1)^2} + \left(\frac{l^2}{2}\right)^2 \hat{A}^2 \frac{l^2 B_0 B_1}{(w_0 - w_1)^2} \\ & - \left(\frac{l^2}{2}\right)^2 A \hat{A} \left(\frac{l^2 B_0 \hat{B}_0}{(w_0 - \hat{w}_0)^2} + \frac{l^2 B_0 \hat{B}_1}{(w_0 - \hat{w}_1)^2} + \frac{l^2 B_1 \hat{B}_0}{(w_1 - \hat{w}_0)^2} + \frac{l^2 B_1 \hat{B}_1}{(w_1 - \hat{w}_1)^2} \right) \\ & + l^4 B_0 B_1 \hat{B}_0 \hat{B}_1 \left(\frac{1}{(w_0 - \hat{w}_0)^2 (w_1 - \hat{w}_1)^2} + \frac{1}{(w_0 - \hat{w}_1)^2 (w_1 - \hat{w}_0)^2} + \frac{1}{(w_0 - w_1)^2 (\hat{w}_0 - \hat{w}_1)^2} \right). \end{aligned} \quad (2.54)$$

Adding (2.49), (2.50), (2.52), (2.53), (2.54), we obtain the relevant 4 point function of the descendant to be

$$\begin{aligned} \mathcal{C}_2 = & \sum_{a=0}^4 \mathcal{C}_2^{(a)}, \\ = & l^4 B_0 B_1 \hat{B}_0 \hat{B}_1 \left(\frac{1}{(w_0 - \hat{w}_0)^2 (w_1 - \hat{w}_1)^2} + \frac{1}{(w_0 - \hat{w}_1)^2 (w_1 - \hat{w}_0)^2} + \frac{1}{(w_0 - w_1)^2 (\hat{w}_0 - \hat{w}_1)^2} \right). \end{aligned} \quad (2.55)$$

Note that all powers of l^2 greater than 2 cancel, or in another words all terms containing A or $\hat{A}$ cancel. Also observe that the final result is identical to the four point function of the U(1) current $i\partial X$ on the uniformized plane together with the slopes of the uniformization maps (2.10), (2.11). Let us finally evaluate the normalised correlator by dividing by the norm evaluated in (2.39). This removes the factor l^4 from the correlator and we obtain

$$\begin{aligned} \frac{\text{Tr}(\rho_{\partial e^{iX}}^2)}{\text{Tr}(\rho_{(0)}^n)} = & B_0 B_1 \hat{B}_0 \hat{B}_1 \\ & \times \left(\frac{1}{(w_0 - \hat{w}_0)^2 (w_1 - \hat{w}_1)^2} + \frac{1}{(w_0 - \hat{w}_1)^2 (w_1 - \hat{w}_0)^2} + \frac{1}{(w_0 - w_1)^2 (\hat{w}_0 - \hat{w}_1)^2} \right). \end{aligned} \quad (2.56)$$

The result in (2.56) implies that the single interval 2nd Rényi entropy of the descendant ∂e^{iX} is identical to that of the holomorphic current ∂X . This observation can be generalized to the n -th Rényi entropy. The details of this proof is given in appendix A. The result of evaluating (2.37) for arbitrary n is given by

$$\mathcal{C}_{2n} = l^{2n} (-1)^n \left(\prod_{k=0}^{n-1} (B_k \hat{B}_k) \right) \left(\prod_{k=0}^{n-1} \frac{1}{(w_k - \hat{w}_k)} + \text{distinct permutations} \right). \quad (2.57)$$

Here $w_k, \hat{w}_k$ are given in (2.12) and the slopes of the conformal transformations are given in (2.22). As remarked for the $n = 2$ case, this correlator is identical to the $2n$ point function of the primary $i\partial X$ on the uniformized plane. To evaluate the entanglement entropy we need to consider the normalized operator for which we divide (2.57) by the norm. Therefore using (2.15) we obtain that the n -th Rényi entropy of the level one descendant ∂e^{iX} is given by

$$\hat{S}_n(\rho_{\partial e^{iX}}) = \frac{1}{1-n} \log \left(\frac{\mathcal{C}_{2n}}{(-l^2)^n} \right). \quad (2.58)$$

Since we have shown that $\mathcal{C}_{2n}$ is identical to that of the U(1) current, we can use the earlier results of both the Rényi and entanglement entropy for the state excited by the U(1)

current obtained in [17, 18]. After substituting for $w_k, \hat{w}_k$ from (2.12) and the slopes of the conformal transformations from (2.22) we can write the correlator as

$$\frac{\mathcal{C}_{2n}}{(-1)^n l^{2n}} = \left(\frac{\sin \pi x}{n} \right)^{2n} \text{Hf}(M). \quad (2.59)$$

The Hafnian is defined by

$$\text{Hf}(M) \equiv \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} \prod_{i=0}^{n-1} \frac{1}{\left(\sin \frac{\theta_{\sigma(2i)} - \theta_{\sigma(2i+1)}}{2} \right)^2}, \quad (2.60)$$

where

$$\theta_k - \theta_l = \begin{cases} \frac{2\pi}{n}(k-l), & k, l \leq n-1 \\ \frac{2\pi}{n}(k+x-l), & k \leq n-1, l > n-1, \\ \frac{2\pi}{n}(k-x-l), & l \leq n-1, k > n-1, \\ \frac{2\pi}{n}(k-l), & k, l > n-1, \end{cases} \quad (2.61)$$

and M is the matrix whose entries are given by

$$M_{kl} = \frac{1}{\left(\sin \frac{\theta_k - \theta_l}{2} \right)^2}. \quad (2.62)$$

The Hafnian can be simplified using the methods of [17, 18] and can be obtained in terms of known functions

$$\text{Hf}(M) = 4^n \left(\frac{\Gamma(\frac{1+n+n \csc \pi x}{2})}{\Gamma(\frac{1-n+n \csc \pi x}{2})} \right)^2. \quad (2.63)$$

Substituting this in the equation of the Rényi entropies of the excited state in (2.15), we obtain

$$\hat{S}_n(\rho_{\partial e^{i l X}}) = \frac{1}{1-n} \log \left[\left(2 \frac{\sin \pi x}{n} \right)^{2n} \left(\frac{\Gamma(\frac{1+n+n \csc \pi x}{2})}{\Gamma(\frac{1-n+n \csc \pi x}{2})} \right)^2 \right]. \quad (2.64)$$

Taking the $n \rightarrow 1$ limit we obtain the entanglement entropy

$$\begin{aligned} \hat{S}(\rho_{\partial e^{i l X}}) &= -2 \left(\log(2 \sin \pi x) + \psi \left(\frac{\csc \pi x}{2} \right) + \sin \pi x \right), \\ &= \frac{2}{3} \pi^2 x^2 + O(x^4), \end{aligned} \quad (2.65)$$

where

$$\psi(x) = \frac{d}{dx} \Gamma(x). \quad (2.66)$$

As we mentioned earlier, this result is identical to the entanglement entropy of the U(1) current since the $2n$ point function of the descendant $\partial e^{i l X}$ on the uniformized plane coincides with that of the U(1) current $i \partial X$. The plot of this entanglement entropy as a function of x is given in figure 4. At this point we can also emphasise that the

Figure 4. Single interval entanglement entropy of the descendant ∂e^{iLX} . Note that as expected the entanglement entropy interval x is the same as its complement $1 - x$.

entanglement entropy of the primary e^{iLX} vanishes (2.31), however its level one descendant shows interesting entanglement structure. In the next subsection, we will extract some of the universal relations which determine the entanglement entropy of the descendant given the entanglement entropy of the primary in the short distance $x \ll 1$ limit.

We can proceed and evaluate various relative entropies associated with the level one descendant with respect to the vacuum and the primary.

$S(\rho_{\partial e^{iLX}} | \rho_{(0)})$. Consider the relative entropy $S(\rho_{\partial e^{iLX}} | \rho_{(0)})$ where $\rho_{(0)}$ is the reduced density matrix of a single interval in vacuum. From (2.19), we see that the correlator in the numerator is given by $\mathcal{C}_{2n}$. The correlator in the denominator consists of $n - 1$ insertions of the identity and one insertion of the descendant. Therefore we need to evaluate the following

$$\mathcal{D} = \langle w \circ \partial e^{iLX(w_0)} \hat{w} \circ \partial e^{-iLX(\hat{w}_0)} \rangle. \quad (2.67)$$

This is just the two point function on the uniformized plane in which both the operators are on a single wedge. It is easily evaluated and the normalised correlator is given by

$$\mathcal{D} = \left(\frac{\sin \pi x}{n \sin \frac{\pi x}{n}} \right)^{2(\frac{l^2}{2} + 1)} \left[1 + l^2 \left(\cos \frac{\pi x}{n} - n \cot \pi x \sin \frac{\pi x}{n} \right)^2 \right]. \quad (2.68)$$

Substituting this and $\mathcal{C}_{2n}$ into the expression for the relative Rényi entropies in (2.19) we obtain

$$\begin{aligned} S_n(\rho_{\partial e^{iLX}} | \rho_{(0)}) = & \frac{1}{n-1} \left\{ \log \left[\left(2 \frac{\sin \pi x}{n} \right)^{2n} \left(\frac{\Gamma(\frac{1+n+n \csc \pi x}{2})}{\Gamma(\frac{1-n+n \csc \pi x}{2})} \right)^2 \right] \right. \\ & \left. - 2 \left(\frac{l^2}{2} + 1 \right) \log \left(\frac{\sin \pi x}{n \sin \frac{\pi x}{n}} \right) - \log \left[1 + l^2 \left(\cos \frac{\pi x}{n} - n \cot \pi x \sin \frac{\pi x}{n} \right)^2 \right] \right\}. \end{aligned} \quad (2.69)$$

The $n \rightarrow 1$ limit results in the relative entropy of the primary and the descendant

$$S(\rho_{\partial e^{iX}} | \rho_{(0)}) = 2 \left(\log(2 \sin \pi x) + \psi \left(\frac{\csc \pi x}{2} \right) + \sin \pi x \right) + 2 \left(\frac{l^2}{2} + 1 \right) (1 - \pi x \cot \pi x). \quad (2.70)$$

$S(\rho_{\partial e^{iX}} | \rho_{e^{iX}})$. From (2.19) we see that to evaluate the relative entropy of the descendant with the primary we need the following correlator which occurs in the denominator

$$\mathcal{D} = \langle w \circ \partial e^{iX(w_0)} \hat{w} \circ \partial e^{iX(\hat{w}_0)} \prod_{k=1}^{n-1} w \circ \partial e^{iX(w_k)} \hat{w} \circ \partial e^{iX(\hat{w}_k)} \rangle. \quad (2.71)$$

From the conformal transformation of the descendant we see that this correlator is given by

$$\mathcal{D} = \left(\prod_{k=0}^{n-1} (B_k \hat{B}_k)^{\frac{l^2}{2}} \right) \left[\frac{l^2}{2} A + B_0 \partial_{u_0} \right] \left[\frac{l^2}{2} \hat{A} + \hat{B}_0 \partial_{\hat{u}_0} \right] f(u, \hat{u}) \Big|_{(u, \hat{u})=(w, \hat{w})}. \quad (2.72)$$

We can evaluate this correlator using the methods developed in appendix A, this results in

$$\mathcal{D} = \frac{-B_0 \hat{B}_0 l^2}{(w_0 - \hat{w}_0)^2}. \quad (2.73)$$

As expected this is the two point function of U(1) currents on one wedge of the uniformised plane. Substituting the normalised correlators in (2.19) we obtain

$$S_n(\rho_{\partial e^{iX}} | \rho_{e^{iX}}) = \frac{1}{n-1} \left\{ \log \left[\left(2 \frac{\sin \pi x}{n} \right)^{2n} \left(\frac{\Gamma(\frac{1+n+n \csc \pi x}{2})}{\Gamma(\frac{1-n+n \csc \pi x}{2})} \right)^2 \right] - 2 \log \left(\frac{\sin \pi x}{n \sin \frac{\pi x}{n}} \right) \right\}. \quad (2.74)$$

We obtain the relative entropy by taking the $n \rightarrow 1$ limit, which is given by

$$S(\rho_{\partial e^{iX}} | \rho_{e^{iX}}) = 2 \left(\log(2 \sin \pi x) + \psi \left(\frac{\csc \pi x}{2} \right) + \sin \pi x \right) + 2(1 - \pi x \cot \pi x). \quad (2.75)$$

Figure 5 plots this relative entropy, as expected it is positive and it increases monotonically with respect to the interval length x .

3 Short distance expansion

In the previous section we have seen the example of level one descendant of the operator e^{iX} in the free boson theory for which the entanglement entropies and certain relative entropies could be evaluated exactly. In general it is not possible to evaluate the $2n$ -point functions involved in evaluating the entanglement entropies or the relative entropies (2.13), (2.19). In this section we develop a systematic approximation that enables us to obtain these quantities in the short interval approximation. This approximation was developed in [15]. Here we first revisit it and formulate it so that it can be used easily when the excitations involve descendants.

Figure 5. Relative entropy of the descendant ∂e^{iLX} with respect to the primary e^{iLX} .

3.1 Primaries

Before we proceed let us discuss the case of primaries in the $2n$ point function (2.13). The positions of the operators in the uniformised plane is shown in figure 3. In the short distance limit, the distance between the 2 operators on the same wedge is small and therefore the leading contribution is obtained by considering the factorisation of the $2n$ -point function into n two point functions as shown in figure 6. We write this leading term as

$$\begin{aligned} \mathcal{C}_{2n} &= \left\langle \prod_{k=0}^{n-1} w \circ \mathcal{O}(w_k) \hat{w} \circ \mathcal{O}^*(\hat{w}_k) \right\rangle, \\ &\simeq \prod_{k=0}^{n-1} \langle w \circ \mathcal{O}(w_k) \hat{w} \circ \mathcal{O}^*(\hat{w}_k) \rangle + \dots \end{aligned} \quad (3.1)$$

When the operator $\mathcal{O}$ is a primary, the two point function is easily evaluated and is given by

$$\langle w \circ \mathcal{O}(w_k) \hat{w} \circ \mathcal{O}^*(\hat{w}_k) \rangle = \left(\frac{\sin \pi x}{n \sin \frac{\pi x}{n}} \right)^{2h}. \quad (3.2)$$

Here we have used the conformal transformation (2.10) and (2.11) and the corresponding slopes (2.22) at w_k and $\hat{w}_k$. Note that the 2-point function on the same wedge is independent of the wedge. Now substituting (3.2) into the factorized approximation (3.1) of the $2n$ point function, we can evaluate the Rényi entropies and the entanglement entropy.

$$\begin{aligned} \hat{S}_n(\rho_{\mathcal{O}}) &= \frac{1}{1-n} \log \left(\frac{\sin \pi x}{n \sin \frac{\pi x}{n}} \right)^{2nh} + \dots, \\ \hat{S}(\rho_{\mathcal{O}}) &= \lim_{n \rightarrow 1} \hat{S}_n(\rho_{\mathcal{O}}) = 2h(1 - \pi x \cot \pi x) + \dots \end{aligned} \quad (3.3)$$

The leading short interval contribution is universal, it just depends on the dimension of the operator and is independent of the details of the conformal field theory. This contribution has been studied earlier both in CFT and in holography [12, 14–16, 19, 20]

Let us proceed to evaluate the sub-leading corrections to the short interval limit. The approach used in [15] involves using the description of the $2n$ point function as a 4-point

Figure 6. Factorisation of the $2n$ -point functions into n 2-pt functions on the same wedge. The figure shows the example of $n = 8$.

function between $\mathcal{O}, \mathcal{O}^*$ and two twist operators. This 4-point function is rewritten as a sum over states in the $\mathbb{Z}_n$ orbifold theory. A similar approach was also used in [21] to obtain the universal short distance corrections to the Rényi entropy of the excited state.⁴ In this work, the universal OPE coefficients of two twist operators with the stress tensor, and its descendants was used to extract the universal sub-leading corrections to the short distance limit. We will compare our results with that of [21] subsequently.

Here we will obtain these corrections directly from the $2n$ point function. The leading approximation in (3.1) involved the factorisation of the $2n$ point functions into n two point functions on the same wedge. The sub-leading correction found by [15] arises by examining the lightest states formed by primaries in 2 copies of the orbifold theory. Thinking of the $2n$ -point function on the uniformized plane, this correction arises from the factorisation into $n - 2$ pairs of 2-point functions on the same wedge. The remaining 4 operators are contracted using the connected 4-point function.⁵ The figure 7 illustrates this contraction. The operators shaded in blue are contracted using the connected four point function. One can choose any 2 wedges among the n wedges in which these 4 operators lie. Therefore we should sum over all such pairs. This discussion leads to the following approximation of the $2n$ point function

$$\begin{aligned} \mathcal{C}_{2n} \simeq & \prod_{k=0}^{n-1} \langle w \circ \mathcal{O}(w_k) \hat{w} \circ \mathcal{O}^*(\hat{w}_k) \rangle \\ & + \left(\frac{\sin \pi x}{n \sin \frac{\pi x}{n}} \right)^{2(n-2)h} \times \sum_{j,k=0, j \neq k}^{n-1} \langle w \circ \mathcal{O}(w_j) \hat{w} \circ \mathcal{O}^*(\hat{w}_j) w \circ \mathcal{O}(w_k) \hat{w} \circ \mathcal{O}^*(\hat{w}_k) \rangle_c + \dots \end{aligned} \quad (3.4)$$

The subscript ‘ c ’ refers to the connected 4 point function.

⁴We thank the referee for bringing the reference [21] to our notice.

⁵The connected four point function does not contain the pair wise contraction of these operators on the same wedge. This contribution is already accounted.

Figure 7. Subleading contribution to the short distance expansion from the factorisation of the $2n$ -point function to $(n - 2)$ two point functions on the same wedge and a 4-point function of operators on a pair of wedges. These operators are shaded in blue. One needs to sum all pairs of wedges which contains the 4-point function. The figure shows the example of $n = 8$.

We can evaluate the connected 4-point function using its conformal block decomposition which is given by, (see for example in [22]).

$$\langle w \circ \mathcal{O}(w_j) \hat{w} \circ \mathcal{O}^*(\hat{w}_j) w \circ \mathcal{O}(w_k) \hat{w} \circ \mathcal{O}^*(\hat{w}_k) \rangle_c = (B_j \hat{B}_j B_k \hat{B}_k)^h \times \quad (3.5)$$

$$\frac{1}{(w_j - \hat{w}_j)^{2h} (w_k - \hat{w}_k)^{2h}} \left(\sum_{q=1}^{\infty} \chi_{\text{vac}, q} w^2 {}_2F_1(q, q, 2q, w) + \sum_p C_{\mathcal{O}\mathcal{O}\mathcal{O}_p} C_{\mathcal{O}\mathcal{O}}^{\mathcal{O}_p} w^{h_p} \mathcal{F}(c, h, h_p; w) \right),$$

where the cross ratio w is given by

$$w = \frac{(w_j - \hat{w}_j)(w_k - \hat{w}_k)}{(w_j - w_k)(\hat{w}_j - \hat{w}_k)} = \left(\frac{\sin \frac{\pi x}{n}}{\sin \frac{\pi}{n}(j-k)} \right)^2. \quad (3.6)$$

The hypergeometric functions ${}_2F_1(q, q, 2q, w)$ are the global $\text{SL}(2, C)$ blocks and the sum over q expresses the Virasoro block corresponding to the exchange of the stress tensor and its descendants in terms of these global blocks. Note that this is the vacuum block without the $q = 0$ term since we are interested in the connected 4 point function. $\mathcal{F}(c, h, h_p; w)$ is the Virasoro block corresponding to the exchange of the primary $\mathcal{O}_p$ with weight h_p and its descendants. The leading term in the expansion of the Virasoro block is unity

$$\mathcal{F}(c, h, h_p; w) = 1 + O(w), \quad (3.7)$$

$C_{\mathcal{O}\mathcal{O}\mathcal{O}_p}C_{\mathcal{O}\mathcal{O}}^{\mathcal{O}_p}$ is the product of the structure constants, the raised superscript $\mathcal{O}$ refers to the fact that this index in the structure constants are raised by the Zamolodchikov metric. The coefficients χ have been evaluated in [22]. The first two coefficients are of importance to us

$$\chi_1 = 0, \quad \chi_2 = \frac{2h^2}{c}, \quad (3.8)$$

where c is the central charge of the CFT. It is clear from the conformal block expansion in (3.5), the expression for the crossratio (3.6) and the property (3.7), (3.8) that the leading terms in short interval contribution to the four point function is given by

$$\langle w \circ \mathcal{O}(w_j) \hat{w} \circ \mathcal{O}^*(\hat{w}_j) w \circ \mathcal{O}(w_k) \hat{w} \circ \mathcal{O}^*(\hat{w}_k) \rangle_c = (B_j \hat{B}_j B_k \hat{B}_k)^h \times \quad (3.9)$$

$$\frac{1}{(w_j - \hat{w}_j)^{2h} (w_k - \hat{w}_k)^{2h}} \left\{ \chi_2 w^2 [1 + O(w)] + C_{\mathcal{O}\mathcal{O}\mathcal{O}_p} C_{\mathcal{O}\mathcal{O}}^{\mathcal{O}_p} w^p [1 + O(w)] \right\}.$$

Here we have kept the leading term from the vacuum block and the exchange corresponding to the primary with the lowest weight. We will simply refer to this primary as $\mathcal{O}_p$ with weight h_p . Substituting for w we obtain

$$\langle w \circ \mathcal{O}(w_j) \hat{w} \circ \mathcal{O}^*(\hat{w}_j) w \circ \mathcal{O}(w_k) \hat{w} \circ \mathcal{O}^*(\hat{w}_k) \rangle_c = (B_j \hat{B}_j B_k \hat{B}_k)^h \times \quad (3.10)$$

$$\frac{1}{(w_j - \hat{w}_j)^{2h} (w_k - \hat{w}_k)^{2h}} \left[\chi_2 \left(\frac{\sin \frac{\pi x}{n}}{\sin \frac{\pi}{n} (j - k)} \right)^4 + C_{\mathcal{O}\mathcal{O}\mathcal{O}_p} C_{\mathcal{O}\mathcal{O}}^{\mathcal{O}_p} \left(\frac{\sin \frac{\pi x}{n}}{\sin \frac{\pi}{n} (j - k)} \right)^{2h_p} \right] + \dots$$

Though we have kept leading terms from stress tensor exchange together with that of the lowest primary, only one of the terms contribute depending on the weight h_p and the central charge c . We note the following

1. For conformal field theories at finite central charge c , when $h_p < 2$, the contribution is from the lowest primary and the 4 point function is sensitive to the details of the CFT through the structure constants. However when $h_p > 2$, the contribution from the stress tensor contribution is dominant.
2. Finally for large c CFT's and when $h \sim O(c)$, then $\chi_2 \sim O(c)$. Then the leading term arises from the stress tensor exchange.
3. For large c CFT's and $h \sim O(1)$ we have $\chi_2 \rightarrow 0$, therefore the contribution due to the stress tensor exchange is negligible and the leading term is due to exchange of the lowest primary h_p . An example in which this situation occurs is in generalised free field theory. Then $h_p = 2h$, we will discuss this case in some detail in section 4.1.

Substituting the conformal block decomposition of (3.10) in (3.4) we obtain

$$\mathcal{C}_{2n} = \left(\frac{\sin \pi x}{n \sin \frac{\pi x}{n}} \right)^{2hn} \times \left[1 + \chi_2 \left(\sin \frac{\pi x}{n} \right)^4 \sum_{j,k=0, j \neq k}^{n-1} \left(\frac{1}{\sin \frac{\pi}{n} (j - k)} \right)^4 \right. \quad (3.11)$$

$$\left. + C_{\mathcal{O}\mathcal{O}\mathcal{O}_p} C_{\mathcal{O}\mathcal{O}}^{\mathcal{O}_p} \left(\sin \frac{\pi x}{n} \right)^{2h_p} \sum_{j,k=0, j \neq k}^{n-1} \left(\frac{1}{\sin \frac{\pi}{n} (j - k)} \right)^{2h_p} \right],$$

where it is understood this result for the $2n$ point function includes only the leading and the first sub-leading corrections. After changing variables, one of the sums can be performed and we obtain

$$\mathcal{C}_{2n} = \left(\frac{\sin \pi x}{n \sin \frac{\pi x}{n}} \right)^{2hn} \left[1 + \chi_2 \left(\sin \frac{\pi x}{n} \right)^4 \sum_{l=1}^{n-1} \frac{n-l}{(\sin \frac{\pi l}{n})^4} \right. \quad (3.12)$$

$$\left. + C_{\mathcal{O}\mathcal{O}\mathcal{O}_p} C_{\mathcal{O}\mathcal{O}}^{\mathcal{O}_p} \left(\sin \frac{\pi x}{n} \right)^{2h_p} \sum_{l=1}^{n-1} \frac{n-l}{(\sin \frac{\pi l}{n})^{2h_p}} \right].$$

To perform the last sum we use [23]

$$\begin{aligned} f(\alpha, n) &= \sum_{l=1}^{n-1} \frac{n-l}{(\sin \frac{\pi l}{n})^{2\alpha}}, \\ &= (n-1) \frac{\Gamma(\frac{3}{2})\Gamma(\alpha+1)}{2\Gamma(\alpha+\frac{3}{2})} + O((n-1)^2). \end{aligned} \quad (3.13)$$

Using the above approximation in (3.12) and then substituting the expansion of the $2n$ point function in the expression for the entanglement entropy (2.15), we obtain

$$\begin{aligned} \hat{S}(\rho_{\mathcal{O}}) &= 2h(1 - \pi x \cot \pi x) - \frac{8h^2}{15c} (\sin \pi x)^4 \\ &\quad - C_{\mathcal{O}\mathcal{O}\mathcal{O}_p} C_{\mathcal{O}\mathcal{O}}^{\mathcal{O}_p} (\sin \pi x)^{2h_p} \frac{\Gamma(\frac{3}{2})\Gamma(h_p+1)}{2\Gamma(h_p+\frac{3}{2})} + \dots \end{aligned} \quad (3.14)$$

Let us recall that h_p is the lowest dimension primary in the CFT and among the two sub-leading terms only one contributes depending on the conditions discussed after equation (3.10).

For completeness we discuss 2 applications of the result (3.14). Let us consider the free boson theory for which $c = 1$ with $\mathcal{O} = e^{i\ell X}$ and $h = \frac{\ell^2}{2}$. The lightest primary is the U(1) current $\mathcal{O}_p = i\partial X$ with $h_p = 1 < 2$. Therefore the 3rd term in (3.14) is the dominant contribution and we should trust this equation to $O(x^2)$. Expanding the first term in (3.14) to $O(x^2)$ together with the 3rd term we obtain

$$\hat{S}(\rho_{e^{i\ell X}}) = \frac{1}{3}(\ell^2 - C_{\mathcal{O}\mathcal{O}\mathcal{O}_p} C_{\mathcal{O}\mathcal{O}}^{\mathcal{O}_p})x^2 + O(x^4). \quad (3.15)$$

The OPE coefficient can be evaluated easily and is given by $C_{\mathcal{O}\mathcal{O}\mathcal{O}_p} C_{\mathcal{O}\mathcal{O}}^{\mathcal{O}_p} = \ell^2$. Therefore we obtain $\hat{S}(\rho_{e^{i\ell X}}) = 0 + O(x^4)$ which is consistent with the result in (2.31).

For the second application we consider a CFT with large central charge c CFT with $h \sim O(c)$, then the single interval entanglement entropy of the excited state can be obtained by considering the heavy-heavy-light-light correlator [24].

$$S(\rho_{\mathcal{O}}) = \frac{c}{6} \log \left(\frac{2\alpha}{\epsilon} \sin \frac{\pi x}{\alpha} \right), \quad \alpha = \frac{1}{\sqrt{\frac{24h}{c} - 1}}. \quad (3.16)$$

Here ϵ is the UV cutoff. Note that we have the factor $\frac{c}{6}$ rather than $\frac{c}{3}$ since we are in the holomorphic sector of the CFT.⁶ The entanglement entropy of the vacuum is given by taking the $h = 0$ limit in (3.16)

$$S(\rho_0) = \frac{c}{6} \log \left(\frac{2}{\epsilon} \sin \pi x \right). \quad (3.17)$$

Evaluating the entanglement entropy of the excited state we obtain

$$\begin{aligned} \hat{S}_{\rho_{\mathcal{O}}} &= S(\rho_{\mathcal{O}}) - S(\rho_0), \\ &= \frac{2h}{3} \pi^2 x^2 + \left(\frac{2h}{45} - \frac{8h^2}{15c} \right) \pi^4 x^4 + O(x^6). \end{aligned} \quad (3.18)$$

⁶The $\frac{c}{6}$ arises from the conformal dimension of twist operator $\frac{c}{24}(n - \frac{1}{n})$ in the holomorphic sector.

From (3.14) see that when c is large and $h \sim O(c)$ it is only the first two terms which contribute

$$\begin{aligned}\hat{S}(\rho_{\mathcal{O}}) &= 2h(1 - \pi x \cot \pi x) - \frac{8h^2}{15c}(\sin \pi x)^4, \\ &= \frac{2h}{3}\pi^2 x^2 + \left(\frac{2h}{45} - \frac{8h^2}{15c}\right)\pi^4 x^4 + O(x^6).\end{aligned}\quad (3.19)$$

As expected the above equation precisely agrees with (3.18).

As mentioned earlier, the reference [21] obtained the short distance expansion of the Rényi entropy of a primary using the universal OPE coefficients of the twist operator with the vacuum block. We see S_n^{NL} in equation (3.2) of [21] precisely agrees with the first term in (3.14) upon taking the limit $n \rightarrow 1$ and identifying x with the ratio $\frac{\ell}{L}$. The term proportional to $\frac{x^4}{c}$ in (3.14) agrees with the leading term of S_n^{NNL} , at order $(\frac{\ell}{L})^4$. Reference [21] goes further at this order and evaluates the term proportional to $(\frac{\ell}{L})^6$, the leading contribution at $O(\frac{1}{c^2})$ is also obtained. However the method of [21] is not sensitive to the details of the conformal field theory, since the universal OPE coefficients are used and therefore the sub-leading corrections do not contain the term proportional to the structure constants in (3.14).

Our aim in the next subsections is to derive a similar expression to that in (3.14) for the entanglement entropy when the excited state is a descendant of $\mathcal{O}$ in a generic CFT.

Relative entropy between primaries. Let us use the short distance expansion developed to evaluate the relative entropy between a primary $\mathcal{O}, \mathcal{O}'$ of weight h and of weight h' respectively. From (2.19), we see that we would need to evaluate the following correlator in the short distance expansion

$$\hat{\mathcal{C}}_{2n} = \left\langle w \circ \mathcal{O}(w_0) \hat{w} \circ \mathcal{O}^*(\hat{w}_0) \prod_{k=1}^{n-1} \left[w \circ \mathcal{O}'(w_k) \hat{w} \circ \mathcal{O}'^*(\hat{w}_k) \right] \right\rangle. \quad (3.20)$$

Again the leading contribution to this correlator in the short distance expansion arises from factorisation of the $2n$ point function into n two point functions,

$$\begin{aligned}\hat{\mathcal{C}}_{2n} &\simeq \langle w \circ \mathcal{O}'(w_k) \hat{w} \circ \mathcal{O}^*(\hat{w}_0) \rangle \prod_{k=1}^{n-1} \langle w \circ \mathcal{O}'(w_k) \hat{w} \circ \mathcal{O}'^*(\hat{w}_k) \rangle + \dots, \\ &= \left(\frac{\sin \pi x}{n \sin \frac{\pi x}{n}} \right)^{2(h+(n-1)h')} + \dots.\end{aligned}\quad (3.21)$$

Following the same arguments as in the case of the correlator $\mathcal{C}_{2n}$, we see that the leading corrections arise from the following

$$\begin{aligned}\hat{\mathcal{C}}_{2n} &\simeq \left(\frac{\sin \pi x}{n \sin \frac{\pi x}{n}} \right)^{2(h+(n-1)h')} \\ &+ \left(\frac{\sin \pi x}{n \sin \frac{\pi x}{n}} \right)^{2(h+(n-3)h')} \times \sum_{j,k=1, j \neq k}^{n-1} \langle w \circ \mathcal{O}'(w_j) \hat{w} \circ \mathcal{O}'^*(\hat{w}_j) w \circ \mathcal{O}'(w_k) \hat{w} \circ \mathcal{O}'^*(\hat{w}_k) \rangle_c \\ &+ \left(\frac{\sin \pi x}{n \sin \frac{\pi x}{n}} \right)^{2(n-2)h'} \times \sum_{j=1}^{n-1} \langle w \circ \mathcal{O}(w_0) \hat{w} \circ \mathcal{O}^*(\hat{w}_0) w \circ \mathcal{O}'(w_j) \hat{w} \circ \mathcal{O}'^*(\hat{w}_j) \rangle_c + \dots.\end{aligned}\quad (3.22)$$

The second line contains the sub-leading correction that arises from the factorisation into the connected 4 point functions of only $\mathcal{O}'$ and $n - 2$ pairs of 2-point functions which includes the pair $\mathcal{O}$. The third line contains the sub-leading correction that arises from the factorisation into the connected 4 points functions containing a pair of $\mathcal{O}'$ and $\mathcal{O}$, while the remaining operators are contracted pairwise on the same wedge. Using the conformal block decomposition for these four point functions we obtain

$$\begin{aligned} \langle w \circ \mathcal{O}'(w_j) \hat{w} \circ \mathcal{O}'^*(\hat{w}_j) w \circ \mathcal{O}'(w_k) \hat{w} \circ \mathcal{O}'^*(\hat{w}_k) \rangle_c &= \frac{(B_j \hat{B}_j B_k \hat{B}_k)^{h'}}{(w_j - \hat{w}_j)^{2h'} (w_k - \hat{w}_k)^{2h'}} \\ &\times \left[\frac{2h'^2}{c} \left(\frac{\sin \frac{\pi x}{n}}{\sin \frac{\pi}{n}(j-k)} \right)^4 + C_{\mathcal{O}'\mathcal{O}'\mathcal{O}_p} C_{\mathcal{O}'\mathcal{O}'}^{\mathcal{O}_p} \left(\frac{\sin \frac{\pi x}{n}}{\sin \frac{\pi}{n}(j-k)} \right)^{2h_p} \right] + \dots \end{aligned} \quad (3.23)$$

The analysis leading upto this equation is the same as that of (3.10) except that the operator $\mathcal{O}$ is replaced by $\mathcal{O}'$, here h_p is the lowest weight of the primary that appears in the OPE of $\mathcal{O}'$. Similarly the leading contributions from the four point function involving a pair of $\mathcal{O}$ and a pair of $\mathcal{O}'$ is given by

$$\begin{aligned} \langle w \circ \mathcal{O}(w_0) \hat{w} \circ \mathcal{O}'^*(\hat{w}_0) w \circ \mathcal{O}'(w_j) \hat{w} \circ \mathcal{O}'^*(\hat{w}_j) \rangle_c &= \frac{(B_0 \hat{B}_0)^h (B_j \hat{B}_j)^{h'}}{(w_0 - \hat{w}_0)^{2h} (w_j - \hat{w}_j)^{2h'}} \\ &\times \left[\frac{2hh'}{c} \left(\frac{\sin \frac{\pi x}{n}}{\sin \frac{\pi j}{n}} \right)^4 + C_{\mathcal{O}\mathcal{O}\mathcal{O}_p} C_{\mathcal{O}'\mathcal{O}'}^{\mathcal{O}_p} \left(\frac{\sin \frac{\pi x}{n}}{\sin \frac{\pi j}{n}} \right)^{2h_p} \right] + \dots \end{aligned} \quad (3.24)$$

Here we have assumed that the lowest weight of the primary that appears in the OPE of $\mathcal{O}$ as well as $\mathcal{O}'$ is the same operator $\mathcal{O}_p$ with weight h_p . Now we substitute the expressions for the four point functions (3.23) and (3.24) into (3.22) to obtain

$$\begin{aligned} \hat{\mathcal{C}}_{2n} &\simeq \left(\frac{\sin \pi x}{n \sin \frac{\pi x}{n}} \right)^{2(h+(n-1)h')} \times \\ &\left[1 + \left(\sin \frac{\pi x}{n} \right)^4 \left(\frac{2h'^2}{c} \sum_{j,k=1, j \neq k}^{n-1} \frac{1}{\left(\sin \frac{\pi}{n}(j-k) \right)^4} + \frac{2hh'}{c} \sum_{j=1}^{n-1} \frac{1}{\left(\sin \frac{\pi j}{n} \right)^4} \right) \right. \\ &\left. + \left(\sin \frac{\pi x}{n} \right)^{2h_p} \left(C_{\mathcal{O}'\mathcal{O}'\mathcal{O}_p} C_{\mathcal{O}'\mathcal{O}'}^{\mathcal{O}_p} \sum_{j,k=1, j \neq k}^{n-1} \frac{1}{\left(\sin \frac{\pi}{n}(j-k) \right)^{2h_p}} + C_{\mathcal{O}\mathcal{O}\mathcal{O}_p} C_{\mathcal{O}'\mathcal{O}'}^{\mathcal{O}_p} \sum_{j=1}^{n-1} \frac{1}{\left(\sin \frac{\pi j}{n} \right)^{2h_p}} \right) \right]. \end{aligned} \quad (3.25)$$

Using the result for the sum in (3.13), the two sums that occur in (3.25) can be performed and they are given by

$$\begin{aligned} \sum_{j,k=1, j \neq k}^{n-1} \left(\frac{1}{\sin \frac{\pi}{n}(j-k)} \right)^{2\alpha} &= -(n-1) \frac{\Gamma(\frac{3}{2})\Gamma(\alpha+1)}{2\Gamma(\alpha+\frac{3}{2})} + O((n-1)^2), \\ \sum_{j=1}^{n-1} \left(\frac{1}{\sin \frac{\pi j}{n}} \right)^{2\alpha} &= 2(n-1) \frac{\Gamma(\frac{3}{2})\Gamma(\alpha+1)}{2\Gamma(\alpha+\frac{3}{2})} + O((n-1)^2). \end{aligned} \quad (3.26)$$

Substituting these expressions for the sums into (3.25) and using the similar approximation for the correlator $\mathcal{C}_{2n}$ given in (3.12) in the expression for the relative entropy in (2.19) we

obtain

$$S(\rho_{\mathcal{O}}|\rho_{\mathcal{O}'}) = \frac{8}{15c}(h-h')^2(\sin \pi x)^4 \quad (3.27)$$

$$+ (C_{\mathcal{O}\mathcal{O}\mathcal{O}_p} - C_{\mathcal{O}'\mathcal{O}'\mathcal{O}_p}) \left(C_{\mathcal{O}\mathcal{O}}^{\mathcal{O}_p} - C_{\mathcal{O}'\mathcal{O}'}^{\mathcal{O}_p} \right) \frac{\Gamma(\frac{3}{2})\Gamma(h_p+1)}{2\Gamma(h_p+\frac{3}{2})} (\sin \pi x)^{2h_p} + \dots$$

Here we have assumed that h_p is the lowest weight primary that occurs in the OPE's. Again we mention that depending on the conditions discussed after equation (3.10) it is only one of the terms in (3.27) which contributes to the leading approximation for the relative entropy. Note that the expression for relative entropy is manifestly positive.

3.2 Descendants: leading contribution to entanglement entropy

As discussed in section 3.1 the leading contribution is obtained by approximating the $2n$ point function as n factorised 2-point functions on the uniformised plane. Therefore we begin by evaluate the two point function of the level-1 descendant $\partial\mathcal{O}$. Using the conformal transformation of the level descendant given in (B.1) we obtain

$$\langle w \circ \partial\mathcal{O}(w_k) \hat{w} \circ \partial\mathcal{O}(\hat{w}_k) \rangle = (-2h) \left(\frac{\sin \pi x}{n \sin \frac{\pi x}{n}} \right)^{2(h+1)} \left[1 + 2h \left(\cos \frac{\pi x}{n} - n \cot \pi x \sin \frac{\pi x}{n} \right)^2 \right]. \quad (3.28)$$

The important point to notice is that the prefactor is identical to that of a primary of weight $h+1$, however there is a correction given by the second factor, which has the property

$$\left[1 + 2h \left(\cos \frac{\pi x}{n} - n \cot \pi x \sin \frac{\pi x}{n} \right)^2 \right] = 1 + O((n-1)^2). \quad (3.29)$$

Indeed due to this property, the leading contribution to the entanglement entropy in the short distant limit is given by

$$S(\rho_{\partial\mathcal{O}}) = 2(h+1)(1 - \pi x \cot \pi x) + \dots \quad (3.30)$$

To obtain this we have to normalise the correlator in (3.28) and use (2.15).

Now this property we have just demonstrated for the level one descendant can be shown to be true for all descendants. It simplifies both the computation of entanglement and relative entropies at the leading order as well as the first sub-leading order. Let us first demonstrate this property for all global descendants of $\mathcal{O}$ of the form $\partial^l\mathcal{O}$.

Global descendants. The two point function for these descendants on the same wedge can be obtained by

$$\langle w \circ \partial^l\mathcal{O}(w_k) \partial^l\mathcal{O}^*(\hat{w}_k) \rangle = \partial_z^l \partial_{\hat{z}}^l G(z, \hat{z}, n)|_{(z, \hat{z})=(0_k, \hat{0}_k)}, \quad (3.31)$$

$$G(z, \hat{z}, n) = (\partial_z w(z) \partial_{\hat{z}} \hat{w}(\hat{z}))^h \left(\frac{1}{w(z) - \hat{w}(\hat{z})} \right)^{2h}.$$

In this expression we take the derivatives first and then $(z, \hat{z}) = (0_k, \hat{0}_k)$ which ensures that the $w(z) \rightarrow 0_k, \hat{w}(z) \rightarrow \hat{w}_k$ as can be seen from the maps (2.10) and (2.11). The strategy to

evaluate the derivatives with respect to z and $\hat{z}$ is to first expand the function $G(z, \hat{z}, n)$ around $n = 1$. This results in

$$G(z, \hat{z}, n) = \frac{1}{(1+z\hat{z})^{2h}} - (n-1) \times \frac{h}{(1+z\hat{z})^{2h}} \left\{ 2 + \log\left(\frac{z-u}{z-v}\right) + \log\left(\frac{1+u\hat{z}}{1+v\hat{z}}\right) \right. \\ \left. + \frac{2}{(u-v)(1+z\hat{z})} \left[(z-u)(1+v\hat{z}) \log\left(\frac{z-u}{z-v}\right) + (v-z)(1+u\hat{z}) \log\left(\frac{1+u\hat{z}}{1+v\hat{z}}\right) \right] \right\} \\ + O((n-1)^2). \quad (3.32)$$

Since we need to differentiate with respect to z and $\hat{z}$ equal number of times and then set $(z, \hat{z}) = (0_k, \hat{0}_k)$, we just need to examine terms in $G(z, \hat{z}, n)$ dependent on the product $z\hat{z}$. This is given by

$$G(z, \hat{z}, n)|_{z\hat{z}} = \frac{1}{(1+z\hat{z})^{2h}} - (n-1) \times \frac{h}{(1+z\hat{z})^{2h}} \left\{ 2 + \log\left(\frac{u}{v}\right) \right. \\ \left. + \frac{2}{(u-v)(1+z\hat{z})} \left[(-u+z\hat{z}v) \log\left(\frac{u}{v}\right) + 2z\hat{z}(v-u) \right] \right\} + O((n-1)^2). \quad (3.33)$$

Differentiating with respect to z and $\hat{z}$, l -times and then setting $(z, \hat{z}) = (0_k, \hat{0}_k)$. This differentiation is easy to carry out, we just have to group the terms proportional to $(z\hat{z})^l$. We obtain

$$\partial_z^l \partial_{\hat{z}}^l G(z, \hat{z}, n)|_{(z, \hat{z})=(0_k, \hat{0}_k)} = \frac{(-1)^l \Gamma(2h+l) l!}{\Gamma(2h)} [1 - 2(n-1)(h+l)(1+\pi x \cot \pi x)] \\ + O((n-1)^2), \quad (3.34)$$

where we have substituted the values of $u = e^{2\pi i x}$, $v = 1$ from (2.8). Note that prefactor $\frac{(-1)^l \Gamma(2h+l) l!}{\Gamma(2h)}$ is the norm of the global descendant $\partial^l \mathcal{O}$. Therefore the leading contribution to the entanglement entropy in the short distance limit is given by

$$S(\rho_{\partial^l \mathcal{O}}) = 2(h+l)(1-\pi x \cot \pi x) + \dots. \quad (3.35)$$

The result in (3.34) also implies that the two point function of the descendent is given by

$$\langle w \circ \partial^l \mathcal{O}(w_k) \hat{w} \circ \partial^l \mathcal{O}^*(\hat{w}_k) \rangle = \frac{(-1)^l \Gamma(2h+l) l!}{\Gamma(2h)} \left(\frac{\sin \pi x}{n \sin \frac{\pi x}{n}} \right)^{2(h+l)} (1 + O((n-1)^2)). \quad (3.36)$$

Here is it understood that the pre-factor should be expanded to $O(n-1)$ only. Thus the 2-point function of global descendants at level l of a primary of weight h in the same wedge on the uniformized plane is identical to that of a primary of weight $h+l$ to order $(n-1)$.

Virasoro descendants. A class of Virasoro descendants of the primary $\mathcal{O}$ can be defined by the OPE

$$T(z) \mathcal{O}(w) = \sum_{k \geq 0} \frac{\mathcal{O}^{(-k)}(w)}{(z-w)^{2-k}}. \quad (3.37)$$

Form this definition, we see that

$$\mathcal{O}^{(0)}(w) = h\mathcal{O}(w), \quad \mathcal{O}^{(-1)} = \partial_z \mathcal{O}. \quad (3.38)$$

Indeed another definition of these descendants is $\mathcal{O}^{-k} = L_{-k}\mathcal{O}$. From (3.37), the inverse relation which directly defines the descendants is given by

$$\mathcal{O}^{(-k)}(z) = \oint_z \frac{d\tilde{z}}{(\tilde{z} - z)^{k-1}} T(\tilde{z}) \mathcal{O}(z), \quad (3.39)$$

where the contour in $\tilde{z}$ is a small circle around z .

Let us now examine the descendants at level 2. We have 2 states, the global descendant $\partial^2 \mathcal{O} = L_{-1}^2 \mathcal{O}$ and the Virasoro descendant $\mathcal{O}^{-2} = L_{-2} \mathcal{O}$. We have already evaluated the two point function of global descendants of the same wedge, therefore we proceed to compute the two point function of the Virasoro descendant $\langle w \circ \mathcal{O}^{(-2)}(w_k) \hat{w} \circ \mathcal{O}^{(-2)}(\hat{w}_k) \rangle$. To compute this we would need the conformal transformation of this descendant, evaluated in (B.18). We see that the conformal transformation takes $\mathcal{O}^{-2}$ to a linear combination of $\mathcal{O}^{-2}$, $\mathcal{O}^{-1}$ and $\mathcal{O}$. This implies that we require the two point function between any 2 of these operators. These two point functions can be found in (B.29). There are 9 terms in all and finally we can expand the correlator to $O(n-1)^2$. We have performed this computation in Mathematica and we find

$$\langle w \circ \mathcal{O}^{(-2)}(w_k) \hat{w} \circ \mathcal{O}^{(-2)}(\hat{w}_k) \rangle = \left(\frac{c}{2} + 4h \right) \left(\frac{\sin \pi x}{n \sin \frac{\pi x}{n}} \right)^{2(h+2)} (1 + O((n-1)^2)). \quad (3.40)$$

The calculation is straight forward though tedious. To demonstrate a simple consistency check let us examine the term proportional to $O(c^2)$ in this correlator. From the conformal transformation of $\mathcal{O}^{(-2)}$ given in (B.18) we see that this term is given by

$$\begin{aligned} & \langle w \circ \mathcal{O}^{(-2)}(w_k) \hat{w} \circ \mathcal{O}^{(-2)}(\hat{w}_k) \rangle \Big|_{c^2} \\ &= c^2 \left(\frac{\sin \pi x}{n \sin \frac{\pi x}{n}} \right)^{2(h+2)} \left(-\frac{A^2}{2} + F \right) \left(-\frac{\hat{A}^2}{2} + \hat{F} \right) \frac{(w_k - \hat{w})^4}{(B_k \hat{B}_k)^2}, \\ &= c^2 \left(\frac{\sin \pi x}{n \sin \frac{\pi x}{n}} \right)^{2(h+2)} \left[16(n-1)^2 \sin^4(\pi x) + O((n-1)^3) \right]. \end{aligned} \quad (3.41)$$

Thus the term proportional to c^2 does not contribute at $O(n-1)$ which is consistent with (3.40). The equation (3.40) therefore implies that the leading contribution to the entanglement entropy in the short distance expansion is given by

$$S(\rho_{\mathcal{O}^{(-2)}}) = 2(h+2)(1 - \pi x \cot \pi x) + \dots. \quad (3.42)$$

Similarly let us evaluate the two point function of $\mathcal{O}^{(-3)}$ on the same wedge. For this we would use the conformal transformation of $\mathcal{O}^{(-3)}$ given in (B.20). This involves a linear combination of $\mathcal{O}^{(-3)}$, $\mathcal{O}^{(-2)}$, $\mathcal{O}^{(-1)}$, $\mathcal{O}$. We can evaluate the 16 two point functions that arise using the list given in (B.29). The end result is

$$\langle w \circ \mathcal{O}^{(-3)}(w_k) \hat{w} \circ \mathcal{O}^{(-3)}(\hat{w}_k) \rangle = -2(c+3h) \left(\frac{\sin \pi x}{n \sin \frac{\pi x}{n}} \right)^{2(h+3)} [1 + O((n-1)^2)]. \quad (3.43)$$

Again to perform a simple consistency check we can look at the term proportional to $O(c^2)$ in this correlator.

$$\begin{aligned}
 \langle w \circ \mathcal{O}^{(-3)}(w_k) \hat{w} \circ \mathcal{O}^{(-3)}(\hat{w}_k) \rangle \Big|_{c^2} & \\
 &= \left(\frac{c}{12}\right)^2 \left(\frac{\sin \pi x}{n \sin \frac{\pi x}{n}}\right)^{2(h+3)} (-AF + 2G)(-\hat{A}\hat{F} + 2\hat{G}) \frac{(w_k - \hat{w}_k)^6}{(B_k \hat{B}_k)^3}, \\
 &= \left(\frac{c}{12}\right)^2 \left(\frac{\sin \pi x}{n \sin \frac{\pi x}{n}}\right)^{2(h+3)} \left[-256 \cos^2 \pi x \sin^4 \pi (n-1)^2 + O((n-1)^3) \right].
 \end{aligned} \tag{3.44}$$

The result in (3.43) implies that the leading contribution to the entanglement entropy is given by

$$\hat{S}(\rho_{\mathcal{O}^{(-3)}}) = 2(h+3)(1 - \pi x \cot \pi x) + \dots \tag{3.45}$$

We have also repeated this calculation for the descendant $\partial \mathcal{O}^{(-2)}$ and have seen that again the leading contribution just depends on the level of the descendant.

$$\hat{S}(\rho_{\partial \mathcal{O}^{(-2)}}) = 2(h+3)(1 - \pi x \cot \pi x) + \dots \tag{3.46}$$

Based on the explicit calculations with the global descendants and all descendants till level 3 we expect the two point function for descendants at level l on same wedge in the uniformized plane is given by

$$\langle w \circ \mathcal{O}^{[l]}(w_k) \hat{w} \circ \mathcal{O}^{[l]}(\hat{w}_k) \rangle = (-1)^l \mathcal{N}_{(l)} \left(\frac{\sin \pi x}{n \sin \frac{\pi x}{n}} \right)^{2(h+l)} \left[1 + O((n-1)^2) \right]. \tag{3.47}$$

Here the superscript l refers to any descendant at level l and $\mathcal{N}_{(l)}$ is its norm. It will be interesting to provide a proof of the observation that the two point function on the same wedge of the uniformized plane of descendants of a primary of weight h at level l is same as that of a primary of weight $h+l$ to order $O(n-1)$.

3.3 Descendants: sub-leading contribution to entanglement entropy

We now proceed to evaluate the sub-leading contribution to the short interval expansion of the entanglement entropy of descendants. Just as we have discussed for the primaries, the leading and the sub-leading contribution to the short interval expansion of the $2n$ -point function on the uniformized plane is given by (3.4) where now $\mathcal{O}$ is a descendant of a primary of weight h . We have seen in section 3.2 that the two point function of descendant at level l on the same wedge behave as a primary of weight $h+l$. Therefore we get

$$\begin{aligned}
 \mathcal{C}_{2n} &= \left\langle \prod_{k=0}^{n-1} (w \circ \mathcal{O}^{[l]}(w_k) w \circ \mathcal{O}^{*[l]}(\hat{w}_k)) \right\rangle, \\
 &\simeq \left(\frac{\sin \pi x}{n \sin \frac{\pi x}{n}} \right)^{2n(h+l)} \\
 &\quad + \left(\frac{\sin \pi x}{n \sin \frac{\pi x}{n}} \right)^{2(n-2)(h+l)} \times \sum_{i,j=0, i \neq j}^{n-1} \left\langle w \circ \mathcal{O}^{[l]}(w_j) \hat{w} \circ \mathcal{O}^{*[l]}(\hat{w}_j) w \circ \mathcal{O}^{[l]}(w_k) \hat{w} \circ \mathcal{O}^{*[l]}(\hat{w}_k) \right\rangle_c \\
 &\quad + O((n-1)^2) + \dots
 \end{aligned} \tag{3.48}$$

Here $\mathcal{O}^{[l]}$ refers to a descendant of level l . Note that this relation is true only to $O((n-1)^2)$ since we have replaced the two point function of descendants at level one on the same wedge with that of the primary of weight $h+l$. Here are working with the normalised descendant at level l . What remains now is to evaluate the connected 4-point function on pairs of wedges in the conformal block approximation. We need the leading term in the short distance approximation at $O(n-1)$.

3.3.1 Level 1

We discuss the level one descendant $\partial\mathcal{O}$ in detail. This will serve to illustrate the methods and approximations involved in obtain the leading contribution from the 4-point function from the operators on pairs of wedges. The relevant correlator is given by

$$\begin{aligned}\mathcal{F}_4 &= \frac{1}{(-2h)^2} \left\langle w \circ \partial\mathcal{O}(w_j) \hat{w} \circ \partial\mathcal{O}^*(\hat{w}_j) w \circ \partial\mathcal{O}(w_k) \hat{w} \circ \partial\mathcal{O}^*(\hat{w}_k) \right\rangle_c, \\ &= \frac{1}{(2h)^2} (B_j \hat{B}_j B_k \hat{B}_k)^h \left(hA + B_j \partial_{w_j} \right) \left(h\hat{A} + \hat{B}_j \partial_{\hat{w}_j} \right) \left(hA + B_k \partial_{w_k} \right) \left(h\hat{A} + \hat{B}_k \partial_{\hat{w}_k} \right) \\ &\quad \times \left[\chi^2 \frac{(w_j - w_k)^{-2} (\hat{w}_j - \hat{w}_k)^{-2}}{(w_j - \hat{w}_j)^{2h-2} (w_k - \hat{w}_k)^{2h-2}} + C_{\mathcal{O}\mathcal{O}\mathcal{O}_p} C_{\mathcal{O}\mathcal{O}}^{\mathcal{O}_p} \frac{(w_j - w_k)^{-h_p} (\hat{w}_j - \hat{w}_k)^{-h_p}}{(w_j - \hat{w}_j)^{2h-h_p} (w_k - \hat{w}_k)^{2h-h_p}} \right].\end{aligned}\tag{3.49}$$

Here we have substituted for the leading contribution to the connected 4-point function from (3.9), used the definition of the cross ratio in (3.6) and used the action of the conformal transformation (2.10) and (2.11) on the level-1 descendants. We have also normalized the descendant operator. Expanding the action of the 4 terms of the kind $(hA + B\partial_w)$ one obtains 16 terms. However it is reasonably easy to isolate among these the leading terms which contribute in the short distance approximation. From (2.22) note that B 's vanish when $x \rightarrow 0$ as

$$B_k \sim \sin \pi x, \quad \hat{B}_k \sim -\sin \pi x \quad \text{as } x \rightarrow 0,\tag{3.50}$$

while A 's defined in (2.34) and (2.36) remain finite as $x \rightarrow 0$.

$$A \sim 2, \quad \hat{A} \sim -2, \quad \text{as } x \rightarrow 0.\tag{3.51}$$

This implies that for instance in $(hA + B_j \partial_{w_j})$, the term involving $B\partial_{w_j}$ is generally suppressed compared to hA in the short distance limit, unless the derivative acts on $(w_j - \hat{w}_j)^{-(2h-h_p)}$ since from (2.12) we see that

$$(w_j - \hat{w}_j) \sim \sin \frac{\pi x}{n}, \quad \text{as } x \rightarrow 0.\tag{3.52}$$

Thus the action of $B_j \partial_{w_j}$ or the multiplication of A on $(w_j - \hat{w}_j)^{-(2h-h_p)}$ have identical behaviour as $x \rightarrow 0$ limit. Similar arguments can be applied for all the terms of the form $(hA + B\partial_w)$. This leads us to conclude that the leading term in the short distance limit of

the 4-point function in (3.49) is given by

$$\begin{aligned}
 \mathcal{F}_4 = & \frac{1}{(-2h)^2} (B_j \hat{B}_j B_k \hat{B}_k)^h \\
 & \times \left[\chi_2(w_j - w_k)^{-2} (\hat{w}_j - \hat{w}_k)^{-2} (hA + B_j \partial_{w_j}) (h\hat{A} + \hat{B}_j \partial_{\hat{w}_j}) \frac{1}{(w_j - \hat{w}_j)^{2h-2}} \right. \\
 & \quad \times (hA + B_k \partial_{w_k}) (h\hat{A} + \hat{B}_k \partial_{\hat{w}_k}) \frac{1}{(w_k - \hat{w}_k)^{2h-2}} + O(x^5) \\
 & \quad + C_{\mathcal{O}\mathcal{O}\mathcal{O}_p} C_{\mathcal{O}\mathcal{O}}^{\mathcal{O}_p} (w_j - w_k)^{-h_p} (\hat{w}_j - \hat{w}_k)^{-h_p} (hA + B_j \partial_{w_j}) (h\hat{A} + \hat{B}_j \partial_{\hat{w}_j}) \frac{1}{(w_j - \hat{w}_j)^{2h-2h_p}} \\
 & \quad \left. \times (hA + B_k \partial_{w_k}) (h\hat{A} + \hat{B}_k \partial_{\hat{w}_k}) \frac{1}{(w_k - \hat{w}_k)^{2h-2h_p}} + O(x^{2h_p+1}) \right].
 \end{aligned} \tag{3.53}$$

To illustrate a further simplification, let us consider the following action of the derivatives

$$\begin{aligned}
 (hA + B_j \partial_{w_j}) (h\hat{A} + \hat{B}_j \partial_{\hat{w}_j}) \frac{1}{(w_j - \hat{w}_j)^{2h-h_p}} &= \frac{B_j \hat{B}_j}{(w_j - \hat{w}_j)^{2(h+1)-h_p}} \times \\
 & \left(h^2 A \hat{A} \frac{(w_j - \hat{w}_j)^2}{B_j \hat{B}_j} - (2h - h_p) h \hat{A} \frac{w_j - \hat{w}_j}{\hat{B}_j} + (2h - h_p) h A \frac{w_j - \hat{w}_j}{B_j} - (2h - q)(2h - q + 1) \right).
 \end{aligned} \tag{3.54}$$

Substituting this result in (3.53), we see that the term proportional to $C_{\mathcal{O}\mathcal{O}\mathcal{O}_p}$ reduces to

$$\begin{aligned}
 \mathcal{F}_4|_{C_{\mathcal{O}\mathcal{O}\mathcal{O}_p}} &= \frac{C_{\mathcal{O}\mathcal{O}\mathcal{O}_p} C_{\mathcal{O}\mathcal{O}}^{\mathcal{O}_p}}{(-2h)^2} \frac{(B_j \hat{B}_j B_k \hat{B}_k)^{h+1}}{(w_j - \hat{w}_j)^{2(h+1)} (w_k - \hat{w}_k)^{2(h+1)}} \times w^{h_p} \times \\
 & \left[h^2 A \hat{A} \frac{(w_j - \hat{w}_j)^2}{B_j \hat{B}_j} - (2h - h_p) h \hat{A} \frac{(w_j - \hat{w}_j)}{\hat{B}_j} + (2h - h_p) h A \frac{(w_j - \hat{w}_j)}{B_j} - (2h - q)(2h - q + 1) \right] \\
 & \times [j \rightarrow k].
 \end{aligned} \tag{3.55}$$

Here we have reassembled the dependences in w 's back to the cross ratio defined in (3.6). Substituting for the w 's from (2.12) and the B 's from (2.22) we obtain

$$\begin{aligned}
 \mathcal{F}_4|_{C_{\mathcal{O}\mathcal{O}\mathcal{O}_p}} &= \frac{1}{(-2h)^2} \left(\frac{\sin \pi x}{n \sin \frac{\pi x}{n}} \right)^{4(h+1)} \times \left(\frac{\sin \frac{\pi x}{n}}{\frac{\pi}{n}(j-k)} \right)^{2h_p} \times \\
 & \left[h^2 A \hat{A} \frac{(w_j - \hat{w}_j)^2}{B_j \hat{B}_j} - (2h - h_p) h \hat{A} \frac{(w_j - \hat{w}_j)}{\hat{B}_j} + (2h - h_p) h A \frac{(w_j - \hat{w}_j)}{B_j} - (2h - q)(2h - q + 1) \right] \\
 & \times [j \rightarrow k].
 \end{aligned} \tag{3.56}$$

From the definition of $w_j, \hat{w}_j$ in (2.12) and B_j from (2.22) we see that the terms in bracket of the second line or the third line are independent of j and k respectively. Let us define

$$\begin{aligned}
 D(h, h_p, n) &= \frac{1}{(-2h)} \times \\
 & \left(h^2 A \hat{A} \frac{(w_j - \hat{w}_j)^2}{B_j \hat{B}_j} - (2h - h_p) h \hat{A} \frac{(w_j - \hat{w}_j)}{\hat{B}_j} \right. \\
 & \quad \left. + (2h - h_p) h A \frac{(w_j - \hat{w}_j)}{B_j} - (2h - q)(2h - q + 1) \right) \Bigg|_{x \rightarrow 0}.
 \end{aligned} \tag{3.57}$$

The quantity $D(h, h_p, n)$ also depends on n , but is independent of j . We have taken the $x \rightarrow 0$ limit in this quantity since all higher orders in x contribute as $O(x^{2h_p+1})$. A equation similar to (3.56) results on performing the same analysis for the term proportional to χ_2 in (3.53) with $h_p \rightarrow 2$. Therefore the 4-point function can be written as

$$\mathcal{F}_4 = \left(\frac{\sin \pi x}{n \sin \frac{\pi x}{n}} \right)^{2(h+1)} \left[\chi_2 [D(h, 2, n)]^2 \times \left(\frac{\sin \frac{\pi x}{n}}{\frac{\pi}{n}(j-k)} \right)^4 + O(x^5) \right. \\ \left. + C_{\mathcal{O}\mathcal{O}\mathcal{O}_p} C_{\mathcal{O}\mathcal{O}}^{\mathcal{O}_p} (D(h, h_p, n))^2 \left(\frac{\sin \frac{\pi x}{n}}{\frac{\pi}{n}(j-k)} \right)^{2h_p} + O(x^{2h_p+1}) \right]. \quad (3.58)$$

Substituting this result for the 4-point function into (3.48) we obtain

$$C_{2n} = \left(\frac{\sin \pi x}{n \sin \frac{\pi x}{n}} \right)^{2(h+1)} \times \left[1 + \right. \\ \left. (n-1) \left\{ \frac{8h^2}{15c} [D(h, 2, n)]^2 \left(\sin \frac{\pi x}{n} \right)^4 + C_{\mathcal{O}\mathcal{O}\mathcal{O}_p} C_{\mathcal{O}\mathcal{O}}^{\mathcal{O}_p} (D(h, h_p, n))^2 \frac{\Gamma(\frac{3}{2})\Gamma(h_p+1)}{\Gamma(h_p+\frac{3}{2})} \right\} \right] \\ + ((n-1)^2). \quad (3.59)$$

Here we have performed the sums over pairs of wedges using (3.13). Since we are interested in the entanglement entropy we can take the $n \rightarrow 1$ limit on the second term in the curly bracket. Thus the factor $D(h, 2, n)$ can be evaluated in the $n \rightarrow 1$ limit which results in

$$D_{\partial\mathcal{O}}(h, h_p) = \lim_{n \rightarrow 1} D(h, h_p, n), \quad (3.60) \\ = \frac{2h - h_p + h_p^2}{2h}.$$

Substituting this result and evaluating the entanglement entropy we obtain

$$\hat{S}(\rho_{\partial\mathcal{O}}) = 2(h+1)(1 - \pi x \cot \pi x) - \frac{8(h+1)^2}{15c} (\sin \pi x)^4 \quad (3.61) \\ - C_{\mathcal{O}\mathcal{O}\mathcal{O}_p} C_{\mathcal{O}\mathcal{O}}^{\mathcal{O}_p} \left(\frac{2h - h_p + h_p^2}{2h} \right)^2 \frac{\Gamma(\frac{3}{2})\Gamma(h_p+1)}{2\Gamma(h_p+\frac{3}{2})} (\sin \pi x)^{2h_p} + \dots$$

Note that $D_{\partial\mathcal{O}}(h, 2) = \frac{h+1}{h}$ results in the enhancement of the contribution due to the stress tensor exchange. This factor is also expected, since the structure constant for the vacuum block should be proportional to the square of the weight of the external states. Comparing the sub-leading corrections to that of the primary in (3.14) we see that the position dependence of these corrections remain the same, but the pre-factors are modified by a dressing factor (3.60) which depends only on the weight of the primary and the dimension of the lowest lying primary.

The derivation of the sub-leading corrections in the short interval expansion of the entanglement entropy to the level 1 descendent makes it clear that the dressing factor is the ratio of the deformation of the norm of the level one primary to that of the original norm.

This is evident from (3.54) from which the dressing factor emerges. The deformation is in which the two point function of the primary is changed to that of an operator of weight $h - h_p/2$ while the conformal transformation of the descendant is that of an operator of weight h as it is clear from the left hand side of (3.54). Note that the terms in the brackets are finally evaluated in the $n \rightarrow 1$ limit and observe that when $h_p = 0$ one is essentially evaluating the two point function of the descendants. These observations are useful in evaluating the dressing factors at higher levels.

Let us perform a check of the result in (3.61) using the exact result for the entanglement entropy of the descendant ∂e^{ilX} in (2.65). Substituting $h = \frac{l^2}{2}$, $h_p = 1$, $C_{\mathcal{O}\mathcal{O}\mathcal{O}_p} C_{\mathcal{O}\mathcal{O}}^{\mathcal{O}_p} = l^2$ in (3.61) we obtain

$$\hat{S}(\rho_{\partial e^{ilX}}) = \frac{2}{3}\pi^2 x^2 + O(x^4), \quad (3.62)$$

which precisely agrees with the leading term in the expansion of (2.65).

3.3.2 Level 2

At level 2 we have two descendants $\partial^2 \mathcal{O}$ and $\mathcal{O}^{(-2)}$. The analysis for $\partial^2 \mathcal{O}$ proceeds identically to that discussed for the level one descendant. The final result for the entanglement entropy is given by

$$\begin{aligned} \hat{S}(\rho_{\partial^2 \mathcal{O}}) &= 2(h+2)(1 - \pi x \cot \pi x) - \frac{8(h+2)^2}{15c} (\sin \pi x)^4 \\ &\quad - C_{\mathcal{O}\mathcal{O}\mathcal{O}_p} C_{\mathcal{O}\mathcal{O}}^{\mathcal{O}_p} [D_{\partial^2 \mathcal{O}}(h, h_p)]^2 \frac{\Gamma(3/2)\Gamma(h_p+1)}{2\Gamma(h_p+3/2)} (\sin \pi x)^{2h_p} + \dots, \end{aligned} \quad (3.63)$$

$$\begin{aligned} D_{\partial^2 \mathcal{O}}(h, h_p) &= \frac{8h^2 + h(8h_p^2 - 8h_p + 4) + h_p(h_p - 1)(h_p^2 - h_p + 2)}{4h(2h+1)} \\ &= \frac{2!\Gamma(2h)}{\Gamma(2h-h_p)\Gamma(2h+2)} \sum_{k=0}^2 \frac{\Gamma(2h-h_p+k)}{k!} \left(\frac{\Gamma(h_p+2-k)}{(2-k)!\Gamma(h_p)} \right)^2. \end{aligned} \quad (3.64)$$

As a consistency check note that

$$D_{\partial^2 \mathcal{O}}(h, 0) = 1, \quad D_{\partial^2 \mathcal{O}}(h, 2) = \frac{h+2}{h}. \quad (3.65)$$

The first equation in (3.65) results due to the fact explained in the level 1 case. The dressing factor is a ratio of the deformed norm of the descendant to the undeformed norm and it should reduce to unity when $h_p = 0$. The second equation ensures that the stress tensor contribution in (3.63) is proportional to $(h+2)^2$.

We will discuss the Virasoro descendant $\mathcal{O}^{(-2)}$ in detail as it involves using conformal Ward identities to evaluate the relevant correlator. From (3.48) we see that we need the 4-point function

$$\mathcal{F}_4 = \frac{1}{(\frac{c}{2} + 4h)^2} \left\langle w \circ \mathcal{O}^{(-2)} \hat{w} \circ \mathcal{O}^{*(-2)}(\hat{w}_j) w \circ \mathcal{O}^{(-2)}(w_k) \hat{w} \circ \mathcal{O}^{*(-2)}(\hat{w}_k) \right\rangle_c. \quad (3.66)$$

The conformal transformation of $\mathcal{O}^{(-2)}$ is given by

$$w \circ \mathcal{O}^{(-2)}(w_k) = B_k^h \left[B_k^2 \mathcal{O}^{(-2)}(w_k) + \frac{3AB_k}{2} \mathcal{O}^{(-1)}(w_k) \right. \\ \left. + \frac{1}{12} \left(\left(5h - \frac{c}{2} \right) A^2 + 2 \left(4h + \frac{c}{2} \right) F \right) \mathcal{O}(w_k) \right], \quad (3.67)$$

$$\hat{w} \circ \mathcal{O}^{(-2)}(\hat{w}_k) = \hat{B}_k^h \left[\hat{B}_k^2 \mathcal{O}^{(-2)}(\hat{w}_k) + \frac{3\hat{A}\hat{B}_k}{2} \mathcal{O}^{(-1)}(\hat{w}_k) \right. \\ \left. + \frac{1}{12} \left(\left(5h - \frac{c}{2} \right) \hat{A}^2 + 2 \left(4h + \frac{c}{2} \right) \hat{F} \right) \mathcal{O}(\hat{w}_k) \right]. \quad (3.68)$$

Substituting the conformal transformation of $\mathcal{O}^{(-2)}$ we see that we need to evaluate 81 correlators involving $\mathcal{O}^{(-2)}, \partial\mathcal{O}, \mathcal{O}$. However as we have seen earlier for the level 1 descendant, we can simplify the calculation since we are interested in obtaining only the leading correction at short distance and also the $O((n-1))$ term. To illustrate the simplification consider the following term which occurs in the evaluation of $\mathcal{F}_4$ in (3.66).

$$\mathcal{F}_4^{(1)} = (B_j \hat{B}_j B_k \hat{B}_k)^h \langle B_k^2 \mathcal{O}^{(-2)}(w_j) \mathcal{O}^*(w_j) \mathcal{O}(w_k) \mathcal{O}^*(w_k) \rangle \\ \times \frac{1}{12} \left(\left(5h - \frac{c}{2} \right) A^2 + 2 \left(4h + \frac{c}{2} \right) F \right) \times \left[\frac{1}{12} \left(5h - \frac{c}{2} \right) \hat{A}^2 + 2 \left(4h + \frac{c}{2} \right) \hat{F} \right]^2. \quad (3.69)$$

The superscript in $\mathcal{F}_4^{(1)}$ just refers to the fact that we are looking at one term among the 81 terms. Due to the presence of B_k^2 along with $\mathcal{O}^{(-2)}$ which vanishes as $x \rightarrow 0$ limit we need to look only for the contributions which remain finite in this limit. The correlator we need can be evaluated using conformal Ward identities and the definition of $\mathcal{O}^{(-2)}$ in (3.39).

$$\langle \mathcal{O}^{(-2)}(w_j) \mathcal{O}^*(\hat{w}_j) \mathcal{O}(w_k) \mathcal{O}^*(\hat{w}_k) \rangle_c = \left[\frac{h}{(w_j - \hat{w}_j)^2} + \frac{1}{(w_j - \hat{w}_j)} \partial_{\hat{w}_j} + \frac{h}{(w_j - w_k)^2} \right. \\ \left. + \frac{1}{(w_j - w_k)} \partial_{\hat{w}_k} + \frac{h}{(w_j - \hat{w}_k)^2} + \frac{1}{(w_j - \hat{w}_k)} \partial_{\hat{w}_k} \right] \langle \mathcal{O}(w_j) \mathcal{O}^*(\hat{w}_j) \mathcal{O}(w_k) \mathcal{O}^*(\hat{w}_k) \rangle_c. \quad (3.70)$$

From (3.69) we see that the correlator in (3.70) occurs with a factor of B_j^2 , therefore $x \rightarrow 0$ limit only from the first two terms contribute at the leading order. The denominators in the second line of (3.70) are finite, therefore the factor $B_k^2 \sim x^2$ will render them sub-leading compared to the first two terms which come with an additional $1/x^2$. In conclusion, the leading terms can be obtained by restricting the Ward identity to

$$\langle \mathcal{O}^{(-2)}(w_j) \mathcal{O}^*(\hat{w}_j) \mathcal{O}(w_k) \mathcal{O}^*(\hat{w}_k) \rangle_c \simeq \\ \left[\frac{h}{(w_j - \hat{w}_j)^2} + \frac{1}{(w_j - \hat{w}_j)} \partial_{\hat{w}_j} \right] \langle \mathcal{O}(w_j) \mathcal{O}^*(\hat{w}_j) \mathcal{O}(w_k) \mathcal{O}^*(\hat{w}_k) \rangle_c. \quad (3.71)$$

This conclusion is true for all of the 81 correlators in (3.66). The leading contributions can be obtained from terms contain only powers of $(w_j - \hat{w}_j)$ or $(w_k - \hat{w}_k)$ in the denominator. We illustrate this with one more correlator whose leading terms are given by

$$\begin{aligned} & \langle \mathcal{O}(w_j) \mathcal{O}^*(\hat{w}_j) \mathcal{O}^{(-2)}(w_k) \mathcal{O}^{*(-2)}(\hat{w}_k) \rangle \\ & \simeq \left(\frac{\frac{c}{2} + 4h}{(w_k - \hat{w}_k)^4} + 3 \frac{1}{(w_k - \hat{w}_k)^3} \partial_{\hat{w}_k} \right) \langle \mathcal{O}(w_j) \mathcal{O}^*(\hat{w}_j) \mathcal{O}(w_k) \mathcal{O}^*(\hat{w}_k) \rangle \\ & \quad + \left(\frac{h+2}{(w_k - \hat{w}_k)^2} + \frac{1}{(w_k - \hat{w}_k)} \partial_{\hat{w}_k} \right) \langle \mathcal{O}(w_j) \mathcal{O}^*(\hat{w}_j) \mathcal{O}(w_k) \mathcal{O}^{*(-2)}(\hat{w}_k) \rangle. \end{aligned} \quad (3.72)$$

Here we have used the OPE in (B.23) and kept only the leading terms in the Ward identity.

Let us proceed with the approximation (3.71) for the Ward identity in the conformal block expansion of the four point function given in (3.9). The term proportional to the OPE coefficient in the conformal block expansion is given by

$$\begin{aligned} & \langle \mathcal{O}^{(-2)}(w_j) \mathcal{O}^*(\hat{w}_j) \mathcal{O}(w_k) \mathcal{O}^*(\hat{w}_k) \rangle_c |_{C_{\mathcal{O}\mathcal{O}\mathcal{O}_p}} \\ & \simeq \frac{C_{\mathcal{O}\mathcal{O}\mathcal{O}_p} C_{\mathcal{O}\mathcal{O}}^{\mathcal{O}_p}}{(\frac{c}{2} + 4h)^2} \left[\frac{h}{(w_j - \hat{w}_j)^2} + \frac{1}{(w_j - \hat{w}_j)} \partial_{\hat{w}_j} \right] \frac{1}{(w_j - \hat{w}_j)^{2h-h_p}} \times \frac{1}{(w_k - \hat{w}_k)^{2h-h_p}} \\ & \simeq \frac{C_{\mathcal{O}\mathcal{O}\mathcal{O}_p} C_{\mathcal{O}\mathcal{O}}^{\mathcal{O}_p}}{(\frac{c}{2} + 4h)^2} \frac{(3h - h_p)}{(w_j - \hat{w}_j)^{2h-h_p}} \frac{1}{(w_k - \hat{w}_k)^{2h-h_p}}. \end{aligned} \quad (3.73)$$

Substituting this approximation in (3.66) and re-grouping the cross ratio we obtain

$$\begin{aligned} \mathcal{F}_4^{(1)} |_{C_{\mathcal{O}\mathcal{O}\mathcal{O}_p}} &= \frac{C_{\mathcal{O}\mathcal{O}\mathcal{O}_p} C_{\mathcal{O}\mathcal{O}}^{\mathcal{O}_p}}{(\frac{c}{2} + 4h)^2} \frac{(B_j \hat{B}_j B_k \hat{B}_k)^{h+1}}{(w_j - \hat{w}_j)^{2(h+2)} (w_k - \hat{w}_k)^{2(h+2)}} \times w^{h_p} \\ & \times (3h - h_p) \frac{1}{12} \left(\left(5h - \frac{c}{2} \right) A^2 + 2 \left(4h + \frac{c}{2} \right) F \right) \left[\frac{1}{12} \left(\left(5h - \frac{c}{2} \right) \hat{A}^2 + 2 \left(4h + \frac{c}{2} \right) \hat{F} \right) \right]^2 \\ & \times \frac{(w_j - \hat{w}_j)^2 (w_k - \hat{w}_k)^4}{\hat{B}_j^2 \hat{B}_k^4}. \end{aligned} \quad (3.74)$$

Substituting for the w 's using (2.12) in the first line we obtain for the leading term

$$\begin{aligned} \mathcal{F}_4^{(1)} |_{C_{\mathcal{O}\mathcal{O}\mathcal{O}_p}} &= \frac{C_{\mathcal{O}\mathcal{O}\mathcal{O}_p} C_{\mathcal{O}\mathcal{O}}^{\mathcal{O}_p}}{(\frac{c}{2} + 4h)^2} \left(\frac{\sin \pi x}{n \sin \frac{\pi x}{n}} \right)^{2(h+2)} \times \left(\frac{\sin \frac{\pi x}{n}}{\sin \frac{\pi}{n} (j-k)} \right)^{2h_p} \\ & \times (3h - h_p) \frac{1}{12} \left(\left(5h - \frac{c}{2} \right) A^2 + 2 \left(4h + \frac{c}{2} \right) F \right) \left[\frac{1}{12} \left(\left(5h - \frac{c}{2} \right) \hat{A}^2 + 2 \left(4h + \frac{c}{2} \right) \hat{F} \right) \right]^2 \Big|_{x \rightarrow 0} \\ & \times \left(\frac{(w_j - \hat{w}_j)^2 (w_k - \hat{w}_k)^4}{\hat{B}_j^2 \hat{B}_k^4} \right) \Big|_{x \rightarrow 0} + O(x^{2h_p+1}). \end{aligned} \quad (3.75)$$

The last two lines of (3.75) are independent of j, k and reduce to constants in $x \rightarrow 0$ limit. Now from the previous analysis for the level 1 descendent we know that since we have the factor $[\sin \frac{\pi}{n} (j-k)]^{-(2h_p)}$ and we need to sum over j, k , the result is proportional to $(n-1)$.

Therefore we can set $n \rightarrow 1$ in the last 2 lines of (3.75).⁷ These manipulations lead us to

$$\begin{aligned} \mathcal{F}_4^{(1)}|_{C_{\mathcal{O}\mathcal{O}\mathcal{O}_p}} &= \frac{C_{\mathcal{O}\mathcal{O}\mathcal{O}_p} C_{\mathcal{O}\mathcal{O}}^{\mathcal{O}_p}}{(\frac{c}{2} + 4h)^2} \left(\frac{\sin \pi x}{n \sin \frac{\pi x}{n}} \right)^{2(h+2)} \times \left(\frac{\sin \frac{\pi x}{n}}{\sin \frac{\pi}{n}(j-k)} \right)^{h_p} \\ &\quad \times (3h - h_p) \left(\frac{28h - c}{12} \right)^3. \end{aligned} \quad (3.76)$$

These simplifications illustrate that the Ward identities involved in evaluating each of the 81 correlators in (3.66) factorize. We also note that it is sufficient to evaluate the coefficients involved in the conformal transformation by grouping them as in last 2 lines of (3.75), take $x \rightarrow 0$ limit and then take $n \rightarrow 1$ limit. These observations help us to evaluate the leading contribution to the 4-point function without much difficulty. This is given by

$$\begin{aligned} \mathcal{F}_4 &= \left(\frac{\sin \pi x}{n \sin \frac{\pi x}{n}} \right)^{2(h+2)} \left[\frac{2h^2}{c} [D_{\mathcal{O}(-2)}(h, 2)]^2 \left(\frac{\sin \frac{\pi x}{n}}{\sin \frac{\pi(j-k)}{n}} \right)^4 + O(x^5, (n-1)^2) \right. \\ &\quad \left. = C_{\mathcal{O}\mathcal{O}\mathcal{O}_p} C_{\mathcal{O}\mathcal{O}}^{\mathcal{O}_p} [D_{\mathcal{O}(-2)}(h, h_p)]^2 \left(\frac{\sin \frac{\pi x}{n}}{\sin \frac{\pi}{n}(j-k)} \right)^{2h_p} + O(x^{2h_p+1}, (n-1)^2) \right], \end{aligned} \quad (3.77)$$

where

$$D_{\mathcal{O}(-2)}(h, h_p) = \frac{\frac{c}{2} + 4(h + h_p(h_p - 1))}{\frac{c}{2} + 4h}, \quad D_{\mathcal{O}(-2)}(h, 2) = 1 + \frac{16}{c + 8h}. \quad (3.78)$$

This approximation of the 4-point function is sufficient to obtain the leading short distance corrections to the entanglement entropy which is given by

$$\begin{aligned} \hat{S}(\rho_{\mathcal{O}(-2)}) &= 2(h+2)(1 - \pi x \cot \pi x) - \frac{8h^2}{15c} D_{\mathcal{O}(-2)}(h, 2)^2 (\sin \pi x)^4 \\ &\quad - C_{\mathcal{O}\mathcal{O}\mathcal{O}_p} C_{\mathcal{O}\mathcal{O}}^{\mathcal{O}_p} D_{\mathcal{O}(-2)}(h, h_p)^2 \frac{\Gamma(\frac{3}{2})\Gamma(h_p+1)}{2\Gamma(h_p+\frac{3}{2})} (\sin \pi x)^{2h_p} + \dots. \end{aligned} \quad (3.79)$$

The dressing factor $D_{\mathcal{O}(-2)}(h, h_p)$ reduces to unity also when $c \gg h, h_p$.

3.3.3 Level 3

Using the same techniques we can evaluate the entanglement entropy of the states at level 3. We obtain the following results.

$$\begin{aligned} S(\rho_{\partial^3 \mathcal{O}}) &= 2(h+3)(1 - \pi x \cot \pi x) - \frac{8(h+3)^2}{15c} D_{\partial^3 \mathcal{O}}(h, 2)^2 (\sin \pi x)^4 \\ &\quad - C_{\mathcal{O}\mathcal{O}\mathcal{O}_p} C_{\mathcal{O}\mathcal{O}}^{\mathcal{O}_p} [D_{\partial^3 \mathcal{O}}(h, h_p)]^2 \frac{\Gamma(\frac{3}{2})\Gamma(h_p+1)}{2\Gamma(h_p+\frac{3}{2})} (\sin \pi x)^{2h_p} + \dots, \end{aligned} \quad (3.80)$$

⁷We can set $n = 1$ in the factor $\left(\frac{\sin \pi x}{n \sin \frac{\pi x}{n}} \right)^{2(h+2)} (\sin \frac{\pi x}{n})^{h_p}$ but it is convenient to retain this till the last step.

where

$$D_{\partial^3 \mathcal{O}}(h, h_p) = \frac{3! \Gamma(2h)}{\Gamma(2h - h_p) \Gamma(2h + 3)} \sum_{k=0}^3 \frac{\Gamma(2h - h_p + k)}{k!} \left(\frac{\Gamma(h_p + 3 - k)}{\Gamma(h_p)(3 - k)!} \right)^2. \quad (3.81)$$

This dressing factor reduces to the following as expected for the stress tensor exchange

$$D_{\partial^3 \mathcal{O}}(h, 2) = \frac{h + 3}{h}. \quad (3.82)$$

Another consistency check is that for $h_p = 0$, the dressing factor reduces to unity. This is expected since from the evaluation of the dressing factor one can see that it is the ratio of a deformed norm, that is for which the conformal transformation acts as though the state is a descendant of weight h while its two point function of the primary is deformed as though it has weight $h \rightarrow h - h_p/2$. We saw this clearly for the state $\partial \mathcal{O}$ as well as for the state $\mathcal{O}^{(-2)}$.

The entanglement entropy for the other states at level 3 obey the same expression as in (3.79), but with different dressing factors. Here are the dressing factors for the remaining states at level 3

$$\begin{aligned} D_{\mathcal{O}^{(-3)}}(h, h_p) &= \frac{c + 4h_p(h_p + 4) + h(4h_p + 3)}{c + 3h}, \\ D_{\mathcal{O}^{(-3)}}(h, 2) &= \frac{c + 11h + 48}{c + 3h}, \\ D_{\partial \mathcal{O}^{(-2)}}(h, h_p) &= \frac{1}{2[c(h + 2) + 2h(17 + 4h)]} \times \\ &\quad \left[16h^2 + c(4 + 2h + h_p(h_p - 1)) + 4h(17 + 6h_p(h_p - 1)) + 2h_p(h_p - 1)(19 + 4h_p(2 + h_p)) \right], \\ D_{\partial \mathcal{O}^{(-2)}}(h, 2) &= \frac{(h + 3)(8h + c + 34)}{c(h + 2) + 2h(17 + 4h)}. \end{aligned} \quad (3.83)$$

Note $D_{\mathcal{O}^{(-3)}}(h, 0) = 1$ and we observe as in the case of the Virasoro descendant $\mathcal{O}^{-2}$, the dressing factor reduces to unity when $c \gg h, h_p$.

3.3.4 Global descendants $\partial^l \mathcal{O}$

From the explicit calculations till level 3 we saw that the dressing factor $D_{\partial^l \mathcal{O}}(h, h_p)$ can be obtained by evaluating the deformed norm. That is we think of the $\mathcal{O}$ transforming as a primary of weight h , but its two point function is given a deformation in which the weight is shifted to $h \rightarrow h - h_p/2$. To evaluate this deformed norm, we can consider the $n = 1$ limit of the transformations (2.10) and (2.11) and choose $u = 0, v = 1$ for convenience. This results in the following $Sl(2, C)$ transformations

$$s(z) = \frac{z}{z - 1}, \quad \hat{s}(\hat{z}) = \frac{1}{1 + \hat{z}}. \quad (3.84)$$

We then evaluate the ‘deformed’ two point function to obtain the norm.

$$\begin{aligned} D_{\partial^l \mathcal{O}}(h, h_p) &= \frac{(-1)^l \Gamma(2h)}{l! \Gamma(2h + l)} \times \partial_z^l \partial_{\hat{z}}^l H(z, \hat{z})|_{(z, \hat{z}) = (z, 0)}, \\ H(z, \hat{z}) &= (\partial_z s(z) \partial_{\hat{z}} s(\hat{z}))^h \left(\frac{1}{s(z) - \hat{s}(\hat{z})} \right)^{2h - h_p}. \end{aligned} \quad (3.85)$$

Here we have implemented the fact that the primary transforms with the weight h , while the correlator is deformed as though it is obtained as a two point function of an operator of weight $h \rightarrow h - h_p/2$. Also it is easy to see by construction that $D_{\partial^l \mathcal{O}}(h, 0) = 0$ since we have taken the ratio of the deformed norm by the usual norm of the state $\partial^l \mathcal{O}$. Simplifying $H(z, \hat{z})$ we obtain

$$H(z, \hat{z}) = \frac{1}{(1 + z\hat{z})^{2h-h_p}(1-z)^{h_p}(1+\hat{z})^{h_p}}. \quad (3.86)$$

Now we can evaluate the l derivatives with respect to both z and $\hat{z}$ and then set $(z, \hat{z}) = (0, 0)$. This results in the following general expression for the dressing factor for global descendants.

$$D_{\partial^l \mathcal{O}}(h, h_p) = \frac{l! \Gamma(2h)}{\Gamma(2h - h_p) \Gamma(2h + l)} \sum_{k=0}^l \frac{\Gamma(2h - h_p + k)}{k!} \left[\frac{\Gamma(h_p + l - k)}{(l - k)! \Gamma(h_p)} \right]^2. \quad (3.87)$$

We can show that as expected

$$D_{\partial^l \mathcal{O}}(h, 2) = \frac{h + l}{h}, \quad D_{\partial^l \mathcal{O}}(h, 0) = 1. \quad (3.88)$$

3.4 Relative entropy between descendants

In this section we apply the results of the previous section to evaluate the relative entropy between two states not necessary only primaries. Consider $\mathcal{O}^{[-l]}$ and $\mathcal{O}'^{[-l']}$ be descendants of primaries of weight h, h' and levels l, l' respectively. Recall the superscript refers $[l]$ refers to the level of the descendant, it can include both Virasoro or global descendants. To obtain the relative entropy, from (2.19), we see that we would need to evaluate the following correlator in the short distance expansion

$$\hat{\mathcal{C}}_{2n} = \left\langle w \circ \mathcal{O}^{[-l]}(w_0) \hat{w} \circ \mathcal{O}^{*[-l]}(\hat{w}_0) \prod_{k=1}^{n-1} \left[w \circ \mathcal{O}'^{[-l']}(w_k) \hat{w} \circ \mathcal{O}'^{*[-l']}(w_k) \right] \right\rangle. \quad (3.89)$$

Using the results of the previous sections, the leading corrections to this correlator in the short interval arises from the following terms

$$\begin{aligned} \hat{\mathcal{C}}_{2n} \simeq & \left(\frac{\sin \pi x}{n \sin \frac{\pi x}{n}} \right)^{2(h+l+(n-1)(h'+l'))} + O((n-1)^2) \\ & + \left(\frac{\sin \pi x}{n \sin \frac{\pi x}{n}} \right)^{2(h+l+(n-3)(h'+l'))} \sum_{j,k=1, j \neq k}^{n-1} \left\langle w \circ \mathcal{O}'^{[-l']}(w_j) \hat{w} \circ \mathcal{O}'^{*[-l']}(w_j) w \circ \mathcal{O}'^{[-l']}(w_k) \hat{w} \circ \mathcal{O}'^{*[-l']}(w_k) \right\rangle_c \\ & \left(\frac{\sin \pi x}{n \sin \frac{\pi x}{n}} \right)^{2(n-2)(h'+l')} \sum_{j=1}^{n-1} \left\langle w \circ \mathcal{O}^{[-l]}(w_0) \hat{w} \circ \mathcal{O}^{*[-l]}(\hat{w}_0) w \circ \mathcal{O}'^{[-l']}(w_j) \hat{w} \circ \mathcal{O}'^{*[-l']}(w_j) \right\rangle_c + \dots \end{aligned} \quad (3.90)$$

Where the first line contains the leading contribution which arises from the factorisation of the correlator to $2n$ operators on the n wedges of the uniformized plane. Here we have used the result in section 3.2 which showed that the correlator to order $O(n-1)$ is equivalent to that of two point functions of operators $h+l$ and $h'+l'$. The second and third line contains the terms which contribute at the sub-leading order similar to that in equation (3.23) for the case of the primaries.

We can now use the methods developed in section 3.3 to conclude that the leading corrections to the four point functions of interest are given by

$$\begin{aligned} \langle w \circ \mathcal{O}'^{[-l]}(w_j) \hat{w} \circ \mathcal{O}^{*[-l]}(\hat{w}_j) w \circ \mathcal{O}'^{[-l]}(w_k) \hat{w} \circ \mathcal{O}^{*[-l]}(\hat{w}_k) \rangle_c = \\ \left(\frac{\sin \pi x}{n \sin \frac{\pi x}{n}} \right)^{4(h'+l')} \times \left[\frac{2h'^2}{c} [D_{\mathcal{O}'^{[-l']}}(h, 2)]^2 \left(\frac{\sin \frac{\pi x}{n}}{\sin \frac{\pi}{n}(j-k)} \right)^4 \right. \\ \left. + C_{\mathcal{O}'\mathcal{O}'\mathcal{O}_p} C_{\mathcal{O}'\mathcal{O}'}^{\mathcal{O}_p} [D_{\mathcal{O}'^{[-l']}}(h, h_p)]^2 \left(\frac{\sin \frac{\pi x}{n}}{\sin \frac{\pi}{n}(j-k)} \right)^{2h_p} \right] + \dots \end{aligned} \quad (3.91)$$

The analysis leading upto this equation is identical to that of say of (3.77) with the same approximations involved. Similarly the leading contributions from the four point function involving a pair of $\mathcal{O}^{[-l]}$ and a pair of $\mathcal{O}'^{[-l']}$ is given by

$$\begin{aligned} \langle w \circ \mathcal{O}^{[-l]}(w_0) \hat{w} \circ \mathcal{O}^{*[-l]}(\hat{w}_0) w \circ \mathcal{O}'^{[-l]}(w_j) \hat{w} \circ \mathcal{O}^{*[-l]}(\hat{w}_j) \rangle_c = \\ \left(\frac{\sin \pi x}{n \sin \frac{\pi x}{n}} \right)^{2(h+h'+l+l')} \times \left[\frac{2hh'}{c} D_{\mathcal{O}^{[-l]}}(h, 2) D_{\mathcal{O}'^{[-l']}}(h, 2) \left(\frac{\sin \frac{\pi x}{n}}{\sin \frac{\pi j}{n}} \right)^4 \right. \\ \left. + C_{\mathcal{O}\mathcal{O}\mathcal{O}_p} C_{\mathcal{O}'\mathcal{O}'}^{\mathcal{O}_p} D_{\mathcal{O}^{[-l]}}(h, h_p) D_{\mathcal{O}'^{[-l']}}(h, h_p) \left(\frac{\sin \frac{\pi x}{n}}{\sin \frac{\pi j}{n}} \right)^{2h_p} \right] + \dots \end{aligned} \quad (3.92)$$

A simple way to see that the above result is true, is to realise that the dressing factors of descendants in the conformal block approximation to $O((n-1))$ at the leading order in the short interval approximation can be evaluated by thinking of them as a generalized norm as we have discussed earlier. The generalized norm is one that we need to compute for the operators at $(0, \hat{0})$ times that for the operators at $(j, \hat{j})$. This leads to the expression in (3.92). We have also evaluated these correlators explicitly for various descendants till level 3 and seen that the result agrees with (3.92). We have assumed that the lowest weight of the primary that appears in the OPE of $\mathcal{O}$ as well as $\mathcal{O}'$ is the same operator $\mathcal{O}_p$ with weight h_p .

Now we substitute the expressions for the four point functions (3.91) and (3.92) into (3.90) to obtain and use the same steps as in equations (3.25) to (3.27). This leads to the following result for the relative entropy of descendants

$$\begin{aligned} S(\rho_{\mathcal{O}^{[-l]}} | \rho_{\mathcal{O}'^{[-l']}}) = \frac{8}{15c} \left[h D_{\mathcal{O}^{[-l]}}(h, 2) - h' D_{\mathcal{O}'^{[-l']}}(h, 2) \right]^2 (\sin \pi x)^4 \\ + \left\| (C_{\mathcal{O}\mathcal{O}\mathcal{O}_p} D_{\mathcal{O}^{[-l]}}(h, h_p) - C_{\mathcal{O}'\mathcal{O}'\mathcal{O}_p} D_{\mathcal{O}'^{[-l']}}(h, h_p)) \right\|^2 \frac{2\Gamma(\frac{3}{2})\Gamma(h_p+1)}{2\Gamma(h_p+\frac{3}{2})} (\sin \pi x)^{2h_p} + \dots \end{aligned} \quad (3.93)$$

Here $\|\cdot\|^2$ refers to the fact that we need to take the square of its argument together with the index $\mathcal{O}_p$ in one of the term raised by the Zamolodchikov metric. It is useful to write down the relative entropy of descendants of the same primary

$$\begin{aligned} S(\rho_{\mathcal{O}^{[-l]}} | \rho_{\mathcal{O}^{[-l']}}) = \frac{8h^2}{15c} \left[D_{\mathcal{O}^{[-l]}}(h, 2) - D_{\mathcal{O}^{[-l']}}(h, 2) \right]^2 (\sin \pi x)^4 \\ + C_{\mathcal{O}\mathcal{O}\mathcal{O}_p} C_{\mathcal{O}\mathcal{O}}^{\mathcal{O}_p} \left[D_{\mathcal{O}^{[-l]}}(h, h_p) - D_{\mathcal{O}^{[-l']}}(h, h_p) \right]^2 \frac{2\Gamma(\frac{3}{2})\Gamma(h_p+1)}{2\Gamma(h_p+\frac{3}{2})} (\sin \pi x)^{2h_p} + \dots \end{aligned} \quad (3.94)$$

Here $D_{\mathcal{O}[-l]}, D_{\mathcal{O}[-l']}$ refer to the dressing factors between descendants of the primary $\mathcal{O}$. The level can in fact be same but the descendant distinct for example $\partial^2 \mathcal{O}$ and $\mathcal{O}^{(-2)}$. Thus it is only the dressing factor which determines the distance between two descendants in this situation.

At this point we would like to mention that the methods of [21] can be generalised to evaluate the universal contributions to the Rényi entropy of descendants of primaries in the short distance expansion. It would be interesting to carry this out and compare with the results of this section.

4 Applications

In this section we apply the results of entanglement entropy of descendants to generalised free fields and descendants of the vacuum. The reason we focus on these applications is that they can in principle be verified using holography. Generalised free fields are dual to minimally coupled scalars in holographic conformal field theories. Entanglement entropy of descendants of the vacuum are universal to all conformal field theories and in particular for holographic conformal field theories. It should be possible to use the methods developed in [16, 25, 26] to verify the results of this section.

4.1 Generalised free fields

Generalised free fields are operators whose correlation functions can be evaluated by Wick contractions

$$\langle \prod_{i=0}^{n-1} \mathcal{O}(w_i) \mathcal{O}^*(\hat{w}_i) \rangle = \prod_{i=0}^{n-1} \langle \mathcal{O}(w_i) \mathcal{O}^*(\hat{w}_i) \rangle + \text{all distinct permutations.} \quad (4.1)$$

For example the four point function of generalised free fields of weight h is given by

$$\begin{aligned} \langle \mathcal{O}(w_j) \mathcal{O}^*(\hat{w}_j) \mathcal{O}(w_k) \mathcal{O}^*(\hat{w}_k) \rangle &= \frac{1}{(w_j - \hat{w}_j)^{2h} (w_k - \hat{w}_k)^{2h}} + \frac{1}{(w_j - w_k)^{2h} (\hat{w}_j - \hat{w}_k)^{2h}} \\ &+ \frac{1}{(w_j - \hat{w}_k)^{2h} (\hat{w}_j - w_k)^{2h}}. \end{aligned} \quad (4.2)$$

Such behaviour of correlators is expected in large c holographic CFT's. Indeed minimally coupled scalar scalars in AdS_3 are to dual generalised free fields [16]. As shown recently, these correlation functions get corrected at order $O(1/c)$ [26]. Here, we will restrict our attention to the leading order in c . The connected part of the four point function in (4.2) is given by

$$\begin{aligned} \langle \mathcal{O}(w_j) \mathcal{O}^*(\hat{w}_j) \mathcal{O}(w_k) \mathcal{O}^*(\hat{w}_k) \rangle_c &= \frac{1}{(w_j - \hat{w}_j)^{2h} (w_k - \hat{w}_k)^{2h}} \left(w^{2h} + \frac{w^{2h}}{(1-w)^{2h}} \right), \\ &= \frac{1}{(w_j - \hat{w}_j)^{2h} (w_k - \hat{w}_k)^{2h}} (2w^{2h} + \dots), \end{aligned} \quad (4.3)$$

where the cross ratio w is defined in (3.6). From the second line of the above equation, we can read out the lowest lying primary that appears in the OPE has dimension $h_p = 2h$ with

structure constant $C_{\mathcal{O}\mathcal{O}\mathcal{O}_p}C^{\mathcal{O}\mathcal{O}\mathcal{O}_p} = 2$. This primary corresponds to the composite $[\mathcal{O}\mathcal{O}]$ which is usually referred to as a double trace operator in holography. With this information of the four point function, it is easy to write down the entanglement entropy of a generalised free field in the short distance approximation. Using (3.14) we obtain

$$\hat{S}(\rho_{\mathcal{O}}) = 2h(1 - \pi x \cot \pi x) - (\sin \pi x)^{4h} \frac{\Gamma(\frac{3}{2})\Gamma(2h+1)}{\Gamma(2h+\frac{3}{2})}. \quad (4.4)$$

There is no contribution from the stress tensor exchange term which is proportional to $1/c$, this is expected since the Wick contraction rules for generalized free fields in (4.1) is true only to the leading order in c .

Let us use the result for the dressing factor for the descendants $\partial^l \mathcal{O}$ on generalised free fields. Since $h_p = 2h$, from (3.87) we see it is only the $k = 0$ term in the sum contributes. This results in the following dressing factor for the descendants of generalised free fields.

$$D_{\partial^l \mathcal{O}}(h, h_p) = \frac{\Gamma(2h+l)}{l!\Gamma(2h)} = \frac{\mathcal{N}_l}{(l!)^2}, \quad \mathcal{N}_l = \frac{l!\Gamma(2h+l)}{\Gamma(2h)}. \quad (4.5)$$

Here $\mathcal{N}_l$ is the norm of a descendant at level l . The dressing factor just depends on the conformal dimension and the level and it is related to the norm in an interesting way. Thus the entanglement entropy of the global descendants of generalised free fields is given by

$$\hat{S}(\rho_{\partial^l \mathcal{O}}) = 2(h+l)(1 - \pi x \cot \pi x) - (\sin \pi x)^{4h} \frac{\Gamma(\frac{3}{2})\Gamma(2h+1)}{\Gamma(2h+\frac{3}{2})} \times \left(\frac{\Gamma(2h+l)}{l!\Gamma(2h)} \right)^2. \quad (4.6)$$

As a cross check of the above result, we have directly used the four point function of generalized free fields in (4.2) and then evaluated the corresponding four point function for the global descendents. This just involves the knowledge of the two point function of the descendants since all generalized free field correlators are obtained from two point function. Evaluating the four point function of global descendants in this manner for the sub-leading corrections to the entanglement entropy also results in (4.6).

As we have seen the leading contributions to the entanglement entropy of generalised free fields and their descendants appear due to the exchange of the composite $[\mathcal{O}\mathcal{O}]$. The short distance expansion developed in [21] is not sensitive to the OPE coefficients of the twist fields with this composite and therefore cannot capture the second term of equation (4.6) in the short distance expansion in generalised free field theory.

Global descendants are obtained from primaries by the action of L_{-1} which corresponds to an isometry in AdS_3 . Therefore, this dressing factor can in principle be seen in holography using the methods of [16]. It will be interesting to perform this check. In [16] to obtain the entanglement entropy of generalised fields (4.4) from the bulk, both the Ryu-Takayanagi minimal surface as well as the bulk entanglement entropy across this surface contributed to the sub-leading term. Thus deriving (4.6) from the bulk will teach us how the action of symmetry L_{-1} will affect the Ryu-Takayanagi minimal surface as well as the bulk entanglement entropy across this surface.

There is one more observation regarding dressing factor of descendants of generalised free fields worth pointing out. In (3.78) and (3.83) we evaluated the dressing factors of

the Virasoro descendants $\mathcal{O}^{(-2)}, \mathcal{O}^{(-3)}$. All these go to unity as $c \rightarrow \infty$. Then from the expression of relative entropy in (3.94) we see that for generalised free fields, relative entropy among Virasoro descendants as well as primaries vanish at the leading order.

4.2 Descendants of the vacuum

Evaluating entanglement entropy of descendants of vacuum is interesting. Since stress tensor correlators depend only on the central charge the results are universal and depend only the central charge c of the conformal field theory. Consider the state $L_{-2}|0\rangle$ which corresponds to the operator $T(z)$, the stress tensor. To evaluate the leading contributions to the short interval entanglement entropy of this state, we first evaluate the two point function of the stress tensor on a wedge of the uniformized surface.

$$\langle w \circ T(w_j) \hat{w} T(\hat{w}_j) \rangle = \frac{2}{c} \left\langle (B_j^2 T(w_j) + \frac{c}{12} S(w_j, z)|_{z \rightarrow 0}) (\hat{B}_j^2 T(\hat{w}_j) + \frac{c}{12} S(\hat{w}_j, \hat{z})|_{\hat{z} \rightarrow 0}) \right\rangle, \quad (4.7)$$

where the Schwarzian is given by

$$S(w, z) = \frac{w'''}{w'} - \frac{3}{2} \left(\frac{w''}{w'} \right)^2. \quad (4.8)$$

In (4.7) we have used the fact that the stress tensor is a quasi-primary and transforms as

$$\langle w \circ T(w(z)) = (w')^2 T(w(z)) + S(w, z), \quad (4.9)$$

as well as the fact that its expectation value on the plane vanishes by translational invariance. The Schwarzian evaluated for the uniformization map is independent of the position j and is given by

$$S(w_j, z)|_{z \rightarrow 0} = 2e^{-2\pi i x} \frac{n^2 - 1}{n} (\sin \pi x)^2, \quad S(\hat{w}_j, \hat{z})|_{\hat{z} \rightarrow 0} = 2e^{2\pi i x} \frac{n^2 - 1}{n} (\sin \pi x)^2. \quad (4.10)$$

As expected the Schwarzian is proportional to $(n - 1)$. Using this result for the Schwarzian we obtain

$$\begin{aligned} \langle w \circ T(w_j) \hat{w} \circ T(\hat{w}_j) \rangle &= \left(\frac{\sin \pi x}{n \sin \frac{\pi x}{n}} \right)^4 + \frac{c(n^2 - 1)^2}{18n^4} (\sin \pi x)^4, \\ &= \left(\frac{\sin \pi x}{n \sin \frac{\pi x}{n}} \right)^4 + O((n - 1)^2). \end{aligned} \quad (4.11)$$

Thus the two point function of the stress tensor on the wedge of the uniformization plane is that of a primary field of weight 2 to $O(n - 1)$. This is the similar property satisfied by the conformal descendants seen in section 3.2

Now let us examine the 4-point function of the stress tensor on the uniformized plane in which a pair of operators are inserted on two distinct wedges. This correlator is given by

$$\mathcal{F}_4 = \frac{4}{c^2} \left\langle w \circ T(w_j) \hat{w} \circ T(\hat{w}_j) w \circ T(w_k) \hat{w} \circ T(\hat{w}_j) \right\rangle_c, \quad (4.12)$$

where we have divided by the norm of the stress tensor. The conformal transformation of the stress tensor (4.9) involves the Schwarzian. However since the Schwarzian is proportional to $(n-1)$ and we need the correlator in (4.12) to $O(n-1)$, these contributions arise only from terms at the linear order in the Schwarzian.

$$\begin{aligned}\mathcal{F}_4 &= \frac{4}{c^2} (B_j \hat{B}_j B_k \hat{B}_k)^2 \langle T(w_j) T(\hat{w}_j) T(w_k) T(\hat{w}_k) \rangle_c \\ &\quad + \frac{1}{3c} S(w_j, z)|_{z \rightarrow 0} \hat{B}_j B_k \hat{B}_k \langle T(\hat{w}_j) T(w_k) T(\hat{w}_k) \rangle + (3 \text{ permutations}) \\ &\quad + O((n-1)^2).\end{aligned}\tag{4.13}$$

The second term arises from the single insertion of the Schwarzian. Substituting for the 4 point and 3 point function of the stress tensors we obtain

$$\begin{aligned}\mathcal{F}_4 &= \left(\frac{\sin \pi x}{n \sin \frac{\pi x}{n}} \right)^4 \left[w^4 + \frac{w^4}{(1-w)^4} + \frac{8}{c} w^2 \frac{1-w-w^2}{(1-w)^2} \right] \\ &\quad - \frac{2(n^2-1)}{3n^8} \frac{\left(2 - \cos \frac{2(j-k+x)\pi}{n} - \cos \frac{2(j-k-2x)\pi}{n} \right)}{\left(\sin \frac{(j-k)\pi}{n} \sin \frac{(j-k+x)\pi}{n} \sin \frac{(j-k-2x)\pi}{n} \right)^2} \times \frac{(\sin \pi x)^8}{(\sin \frac{\pi x}{n})^2} \\ &\quad + O((n-1)^2)\end{aligned}\tag{4.14}$$

where w is the cross ratio (3.6). We have removed the 1 in the four point function of the stress tensor since this term is already accounted for in the pairwise contraction in each wedge. Though the second term is manifestly proportional to $n-1$, the sum over the wedges which is necessary to evaluate the entanglement entropy renders it to $O(n-1)^2$. To see this expand the second term in x

$$\begin{aligned}\mathcal{F}_4 &= \left(\frac{\sin \pi x}{n \sin \frac{\pi x}{n}} \right)^4 \left[w^4 + \frac{w^4}{(1-w)^4} + \frac{8}{c} w^2 \frac{1-w-w^2}{(1-w)^2} \right] \\ &\quad - \frac{8}{3} (n^2-1) \frac{(\sin \pi x)^8}{(\sin \frac{\pi x}{n})^2 (\sin \frac{(j-k)\pi}{n})^4} \left[1 + \left(\frac{3}{(\sin \frac{(j-k)\pi}{n})^2} - 2 \right) \frac{\pi^2 x^2}{n^2} + O(x^4) \right] \\ &\quad + O((n-1)^2).\end{aligned}\tag{4.15}$$

From (3.13), we see that the sum over the wedges j, k will result in another factor of $(n-1)$ and therefore the contribution of the second term can also be neglected for the purposes of evaluating the entanglement entropy. In the first line there are terms to the order c^0 , on comparison with (4.3) we see that this part of the correlator behaves as a generalized free field, while the terms at order c^{-1} start at w^2 as expected for the stress tensor exchange. Now keeping the leading terms in the short interval expansion and substituting for the cross ratio we obtain

$$\begin{aligned}\mathcal{F}_4 &= \left(\frac{\sin \pi x}{n \sin \frac{\pi x}{n}} \right)^4 \left[\frac{8}{c} \left(\frac{\sin \frac{\pi x}{n}}{\sin \frac{(j-k)\pi}{n}} \right)^4 + O(x^5) \right. \\ &\quad \left. + 2 \left(\frac{\sin \frac{\pi x}{n}}{\sin \frac{(j-k)\pi}{n}} \right)^8 + O(x^9) \right] + O((n-1)^2).\end{aligned}\tag{4.16}$$

Thus the first term arises due to the stress tensor exchange and the second term is similar to that of a generalized free field with $C_{TT\mathcal{O}_p}C_{TT}^{\mathcal{O}_p} = 2$ and $h_p = 2h = 4$. The composite operator involved in this exchange is bilinear in the stress tensor, $\mathcal{O}_p = [TT]$. Using the two point function on the wedge in (4.11) and the four point function (4.16) we use (3.48) to obtain the leading contributions to the single interval entanglement entropy of the state $L_{-2}|0\rangle$

$$\hat{S}(\rho_{L_{-2}}) = 4(1 - \pi x \cot \pi x) - \frac{32}{15c} \sin^4 \pi x - \frac{128}{315} \sin^8 \pi x + \dots \quad (4.17)$$

In large c holographic theories we expect to see only the 3rd term as the sub-leading correction. It will be interesting to verify this using the methods of [16]. This will involve the study of entanglement entropy of gravitons which has largely been unexplored.

Our derivation shows that for the purposes of entanglement entropy the stress tensor behaves as a primary of weight h and the lightest state $h_p = 4$ corresponding to the composite $\mathcal{O}_p = [TT]$ with $C_{TT\mathcal{O}_p}C_{TT}^{\mathcal{O}_p} = 2$. Therefore from (3.87) we can evaluate the dressing factors for the descendant $L_{-(l+2)}|0\rangle$ which corresponds to the operator $\partial^l T$.

$$D_{\partial^l T}(2, 4) = \frac{(l+3)(l+2)(l+1)}{3!}, \quad D_{\partial^l T}(2, 2) = \frac{l+2}{2}. \quad (4.18)$$

Using these expressions for the dressing factors we obtain the leading entanglement entropy of the descendants of the vacuum $L_{-(l+2)}|0\rangle$

$$\begin{aligned} \hat{S}(\rho_{L_{-(l+2)}|0\rangle}) &= 2(l+2)(1 - \pi x \cot \pi x) - \frac{8(l+2)^2}{15c} \sin^4 \pi x \\ &\quad - \left(\frac{(l+3)(l+2)(l+1)}{3!} \right)^2 \frac{128}{315} \sin^8 \pi x + \dots \end{aligned} \quad (4.19)$$

Again for holographic conformal field theories it is only the 3rd term that contributes at the sub-leading order.

It is again interesting to compare with the results of [21] which has evaluated the Rényi entropy of the descendants $L_{-2}|0\rangle, L_{-3}|0\rangle, L_{-4}|0\rangle$. These results can be read out in equations (4.5), (4.7), (4.8) respectively. The contribution S_n^{NL} in [21] has been evaluated to the order $(\frac{\ell}{L})^6$, all these terms in equation (4.5), (4.7), (4.8) with $n = 1$ agree precisely with the c^0 contribution of (4.19) at $l = 0, 1, 2$ to order x^6 . The expression in (4.19) is however valid to order x^8 . For S_n^{NNL} in [21], the leading term at order $(\frac{\ell}{L})^4$ in equations (4.5), (4.7), (4.8) with $n = 1$ agree with the coefficient of $\frac{x^4}{c}$ in (4.19) at $l = 0, 1, 2$. The reference [21] also evaluates the leading contribution to Rényi entropy at order $\frac{1}{c^2}$, while the expression in (4.19) gives the entanglement entropy to descendants $L_{-(2+l)}$ for arbitrary l to order $(c^0 x^8)$ and $\frac{x^4}{c}$.

Since the dual to the stress tensor is the graviton, it should be possible to verify equations (4.17), (4.19) by generalizing the methods of [16, 25] to the graviton in the bulk. This is interesting since it will give us some handle on defining entanglement entropy for gravitons.

5 Conclusions

In this paper we have studied the single interval entanglement entropy of descendants in 2 dimensional conformal field theories. We have found simplifications in the evaluation of contributions to the short interval expansion of entanglement entropy and relative entropies of descendants. One of the observations in the paper is that the leading contribution to the short interval expansion of a descendant at level l is identical to that of a primary with weight $h + l$. This came about since the two point function of the descendant on the same wedge in the uniformized plane behaved as that of a primary of weight $h + l$ to $O((n - 1))$ and deviated only at $O((n - 1)^2)$ where n is the replica index. It will be interesting to establish this property to all descendants as well as provide an algorithm to evaluate the dressing factor to all descendants using the methods of [27]. We have seen that the dressing factor can be thought of a ratio of generalised norm to the usual norm of the descendant. The operator methods developed in [27] to efficiently obtain conformal transformation of descendants would be useful to obtain this deformed norm.

The general question that motivated this study is how entanglement or relative entropy is affected by the action of symmetries of a theory and how can this effect be seen in holography. When the conformal field theory is deformed by a symmetry, this question has been studied earlier in [28–31]. Indeed when the theory is deformed by introducing higher spin fields, the leading corrections to entanglement entropy proved to be universal [29] and provided a test for the Wilson line proposal of [32, 33]. In this paper we studied the action of the Virasoro symmetry on the entanglement of low lying states of the conformal field theory. For global descendants of generalised free fields and descendants of the vacuum the modification to the leading contributions to the single interval entanglement entropy are universal and are given by (4.6), (4.19). It should be possible to verify these results by generalizing the methods of [16, 25, 26] in the bulk. This would show how the Ryu-Takayanagi surface as well as bulk entanglement entropy across this surface are modified by the action of the isometries. It will also help us understand how to evaluate entanglement entropies of gravitons.

Acknowledgments

We wish to thank Shouvik Datta for discussions during the initial phase of this project.

A 2n-point function of ∂e^{ilX} on the uniformised plane

We are interested in the correlator (2.37) given by

$$\begin{aligned} \mathcal{C}_{2n} &= \left\langle \prod_{k=0}^{n-1} w \circ (\partial e^{ilX(w_k)}) \hat{w} \circ (\partial e^{-ilX(w_k)}) \right\rangle, \\ &= \prod_{k=0}^{n-1} (B_k \hat{B}_k)^{\frac{l^2}{2}} \left[\frac{l^2}{2} A + B_k \partial_{u_k} \right] \left[\frac{l^2}{2} \hat{A} + \hat{B}_k \partial_{\hat{u}_k} \right] f(u, \hat{u}) \Big|_{(u, \hat{u})=(w, \hat{w})}, \end{aligned} \quad (\text{A.1})$$

where $f(u, \hat{u})$ is defined in (2.38), $B_k, \hat{B}_k$ defined in (2.22), and $A, \hat{A}$ are defined in (2.34) and (2.36). To un-clutter the expressions let us define the derivatives

$$D_k = B_k \partial_{u_k}, \quad \hat{D}_k = \hat{B}_k \partial_{\hat{u}_k}. \quad (\text{A.2})$$

Note that here is no summation over k . Opening out the expression for the correlator in (A.1), we obtain

$$\begin{aligned} \mathcal{C}_{2n} &= \left(\prod_{k=0}^{n-1} (B_k \hat{B}_k)^{\frac{l^2}{2}} \right) \\ &\times \sum_{p=0}^{2n} \sum_{\substack{i=0, i \leq n \\ 0 \leq p-i \leq n}}^p \left(\frac{l^2}{2} \right)^{2n-p} A^{n-i} \hat{A}^{n-p+i} \sum_{\substack{\{k_1, \dots, k_i\} \\ \{\hat{k}_1, \dots, \hat{k}_{p-i}\}}} [D_{k_1} \cdots D_{k_i}] [\hat{D}_{\hat{k}_1} \cdots \hat{D}_{\hat{k}_{p-i}}] f(u, \hat{u}). \end{aligned} \quad (\text{A.3})$$

The sum over $\{k_1, \dots, k_i\}$ runs over i integers in (A.3). None of these integers are equal and they all take values from 0 to $n-1$. Similarly for the set $\{\hat{k}_1, \dots, \hat{k}_{p-i}\}$. Also note that we can have maximum $2n$ derivatives. However, the intermediate i cannot exceed n , and $(p-i)$ cannot exceed n .

We now need to evaluate the derivatives acting on the function $f(u, \hat{u})$. To write down a general formula, it is first convenient to defined the notion of ‘contraction’ between the functions g, h which arise due to action the derivatives of the function $f(u, \hat{u})$ as seen in (2.45)

$$\begin{aligned} g_k(u, \hat{u}) &= l^2 \left[\sum_{j=0, j \neq k}^{n-1} \frac{1}{u_k - u_j} - \sum_{j=0}^{n-1} \frac{1}{u_k - \hat{u}_j} \right], \\ h_k(u, \hat{u}) &= l^2 \left[\sum_{j=0, j \neq k}^{n-1} \frac{1}{\hat{u}_k - \hat{u}_j} - \sum_{j=0}^{n-1} \frac{1}{\hat{u}_k - u_j} \right]. \end{aligned} \quad (\text{A.4})$$

Consider the product of $g_{k_1} g_{k_2}$, we suppress the dependence on $(u, \hat{u})$ to un-clutter the expression. Let us define the ‘contraction’ as follows

$$\overline{\square}_{g_{k_1} g_{k_2}} = \frac{l^2}{(u_{k_1} - u_{k_2})^2}. \quad (\text{A.5})$$

Here $k_1 \neq k_2$. Using this definition we see that

$$\partial_{u_{k_2}} g_{k_1} = \partial_{u_{k_1}} g_{k_2} = \frac{l^2}{(\hat{u}_{k_1} - \hat{u}_{k_2})^2} = \overline{g_{k_1} g_{k_2}}, \quad k_1 \neq k_2. \quad (\text{A.6})$$

Similarly let us define the contraction between two h 's as

$$\overline{h_{k_1} h_{k_2}} = \frac{l^2}{(\hat{u}_{k_1} - \hat{u}_{k_2})^2}. \quad (\text{A.7})$$

We see that

$$\partial_{\hat{u}_{k_2}} h_{k_1} = \partial_{\hat{u}_{k_1}} g_{k_2} = \frac{l^2}{(\hat{u}_{k_1} - \hat{u}_{k_2})^2} = \overline{h_{k_1} h_{k_2}}, \quad k_1 \neq k_2. \quad (\text{A.8})$$

Finally the contraction between g and h is defined as

$$\overline{g_{k_1} h_{k_2}} = -\frac{l^2}{(u_{k_1} - \hat{u}_{k_2})^2}. \quad (\text{A.9})$$

This definition of the contraction is related to the derivative

$$\partial_{\hat{u}_{k_2}} g_{k_1} = \partial_{u_{k_1}} h_{k_2} = -\frac{l^2}{(u_{k_1} - \hat{u}_{k_2})^2} = \overline{g_{k_1} h_{k_2}}, \quad k_1 \neq k_2. \quad (\text{A.10})$$

Note that higher order derivatives with respect to u_{k_3} or $\hat{u}_{k_3}$ with $k_3 \neq k_1, k_2$ on g, h vanish. The rules for these contractions are similar to Wick contractions, for instance if one has a single contraction on the product

$$[g_{k_1} g_{k_2} h_{k_3}]_{(1)} = \overline{g_{k_1} g_{k_2}} h_{k_3} + \overline{g_{k_1} g_{k_2} g_{k_3}} + \overline{g_{k_1} g_{k_2} h_{k_3}}. \quad (\text{A.11})$$

The subscript in the square bracket of the l.h.s. of the equation indicates the number of contractions performed. One can consider more contractions, for example the terms involving 2 contractions on the product

$$[g_{k_1} g_{k_2} h_{k_3} h_{k_4}]_{(2)} = \overline{g_{k_1} g_{k_2}} \overline{h_{k_3} h_{k_4}} + \overline{g_{k_1} g_{k_2} h_{k_3}} h_{k_4} + \overline{g_{k_1} g_{k_2} h_{k_3} h_{k_4}}. \quad (\text{A.12})$$

Using this notation the derivatives on the function f can be written as

$$\begin{aligned} \partial_{u_{k_1}} \cdots \partial_{u_{k_i}} \partial_{\hat{u}_{\hat{k}_1}} \cdots \partial_{\hat{u}_{\hat{k}_{p-i}}} f &= f g_{k_1} \cdots g_{k_i} h_{\hat{k}_1} \cdots h_{\hat{k}_{p-i}} \\ &+ f \sum_{c=1}^{\lfloor \frac{p}{2} \rfloor} [g_{k_1} \cdots g_{k_i} h_{\hat{k}_1} \cdots h_{\hat{k}_{p-i}}]_{(c)}, \end{aligned} \quad (\text{A.13})$$

where $[\frac{p}{2}]$ is the greatest integer less than or equal to $\frac{p}{2}$. This expression for the derivative can be seen to be true by using the Leibnitz property of derivatives together with the fact none of the integers in the set $\{k_1 \cdots k_i\}$ and $\{\hat{k}_1 \cdots \hat{k}_{p-i}\}$ are identical. As a simple check we can consider 3 derivatives acting on f .

$$\begin{aligned}
 \partial_{u_{k_1}} \partial_{u_{k_2}} \partial_{\hat{u}_{\hat{k}_1}} f &= f g_{k_1} g_{k_2} h_{\hat{k}_1} + f \overbrace{g_{k_1} g_{k_2}} h_{\hat{k}_1} + f \overbrace{g_{k_1} g_{k_2}} h_{\hat{k}_1} + f g_{k_1} g_{k_2} h_{\hat{k}_1}, \\
 &= f g_{k_1} g_{k_2} h_{\hat{k}_1} + f h_{\hat{k}_1} \frac{l^2}{(u_{k_1} - u_{k_2})^2} - f g_{k_2} \frac{l^2}{(u_{k_1} - \hat{u}_{\hat{k}_1})^2} - f g_{k_1} \frac{l^2}{(u_{k_2} - \hat{u}_{\hat{k}_1})^2}, \\
 \partial_{u_{k_1}} \partial_{\hat{u}_{\hat{k}_2}} \partial_{\hat{u}_{\hat{k}_2}} f &= f g_{k_1} h_{\hat{k}_1} h_{\hat{k}_2} + f \overbrace{g_{k_1} h_{\hat{k}_1}} h_{\hat{k}_2} + f \overbrace{g_{k_1} h_{\hat{k}_1}} h_{\hat{k}_2} + f g_{k_1} \overbrace{h_{\hat{k}_1} h_{\hat{k}_2}}, \\
 &= f g_{k_1} h_{\hat{k}_1} h_{\hat{k}_2} - f h_{\hat{k}_2} \frac{l^2}{(u_{k_1} - \hat{u}_{\hat{k}_1})^2} - f h_{\hat{k}_1} \frac{l^2}{(u_{k_1} - \hat{u}_{\hat{k}_2})^2} + f g_{k_1} \frac{l^2}{(\hat{u}_{\hat{k}_1} - \hat{u}_{\hat{k}_2})^2}.
 \end{aligned} \tag{A.14}$$

Similarly the action of 4 derivatives is given by

$$\begin{aligned}
 \partial_{u_{k_1}} \partial_{u_{k_2}} \partial_{\hat{u}_{\hat{k}_1}} \partial_{\hat{u}_{\hat{k}_2}} f &= f g_{k_1} g_{k_2} h_{\hat{k}_1} h_{\hat{k}_2} + f [g_{k_1} g_{k_2} h_{\hat{k}_1} h_{\hat{k}_2}]_{(1)} + f [g_{k_1} g_{k_2} h_{\hat{k}_1} h_{\hat{k}_2}]_{(2)}, \\
 &= f g_{k_1} g_{k_2} h_{\hat{k}_1} h_{\hat{k}_2} \\
 &\quad + f \overbrace{g_{k_1} g_{k_2}} h_{\hat{k}_1} h_{\hat{k}_2} + f \overbrace{g_{k_1} g_{k_2}} h_{\hat{k}_1} h_{\hat{k}_2} + f \overbrace{g_{k_1} g_{k_2}} h_{\hat{k}_1} h_{\hat{k}_2} \\
 &\quad + f g_{k_1} \overbrace{g_{k_2} h_{\hat{k}_1}} h_{\hat{k}_2} + f g_{k_1} \overbrace{g_{k_2} h_{\hat{k}_1}} h_{\hat{k}_2} + f g_{k_1} g_{k_2} \overbrace{h_{\hat{k}_1} h_{\hat{k}_2}} \\
 &\quad + f \overbrace{g_{k_1} g_{k_2}} h_{\hat{k}_1} h_{\hat{k}_2} + f \overbrace{g_{k_1} g_{k_2}} h_{\hat{k}_1} h_{\hat{k}_2} + f \overbrace{g_{k_1} g_{k_2}} h_{\hat{k}_1} h_{\hat{k}_2}.
 \end{aligned} \tag{A.15}$$

Substituting the expressions for the contraction in (A.5), (A.7), (A.9) we obtain

$$\begin{aligned}
 \partial_{u_{k_1}} \partial_{u_{k_2}} \partial_{\hat{u}_{\hat{k}_1}} \partial_{\hat{u}_{\hat{k}_2}} f &= g_{k_1} g_{k_2} h_{\hat{k}_1} h_{\hat{k}_2} f, \\
 &+ f \frac{l^2}{(u_{k_1} - u_{k_2})^2} h_{\hat{k}_1} h_{\hat{k}_2} - f \frac{l^2}{(u_{k_1} - \hat{u}_{\hat{k}_1})^2} g_{k_2} h_{\hat{k}_2} - f \frac{l^2}{(u_{k_1} - \hat{u}_{\hat{k}_2})^2} g_{k_2} h_{\hat{k}_1} \\
 &- f \frac{l^2}{(u_{k_2} - \hat{u}_{\hat{k}_1})^2} g_{k_1} h_{\hat{k}_2} - f \frac{l^2}{(u_{k_2} - \hat{u}_{\hat{k}_2})^2} g_{k_1} h_{\hat{k}_1} + f \frac{l^2}{(\hat{u}_{\hat{k}_1} - \hat{u}_{\hat{k}_2})^2} g_{k_1} g_{k_2} \\
 &+ f \frac{l^2}{(u_{k_1} - u_{k_2})^2} \frac{l^2}{(\hat{u}_{\hat{k}_1} - \hat{u}_{\hat{k}_2})^2} + f \frac{l^2}{(u_{k_1} - \hat{u}_{\hat{k}_1})^2} \frac{l^2}{(u_{k_2} - \hat{u}_{\hat{k}_2})^2} + f \frac{l^2}{(u_{k_1} - \hat{u}_{\hat{k}_2})^2} \frac{l^2}{(u_{k_2} - \hat{u}_{\hat{k}_1})^2}.
 \end{aligned} \tag{A.16}$$

The derivatives in u_k and $\hat{u}_{\hat{k}}$ that occur in (A.3) are always paired with the corresponding B_k and $\hat{B}_{\hat{k}}$ respectively. These can be simplified with the help of the identities (2.47) and (2.48), which we write down here again

$$\begin{aligned}
 B_k \partial_{u_k} f(u, \hat{u})|_{(u, \hat{u})=(w, \hat{w})} &= g_k(u, \hat{u}) f(u, \hat{u})|_{(u, \hat{u})=(w, \hat{w})}, \\
 &= -\frac{l^2}{2} f(w, \hat{w}) A, \\
 \hat{B}_{\hat{k}} \partial_{\hat{u}_{\hat{k}}} f(u, \hat{u})|_{(u, \hat{u})=(w, \hat{w})} &= g_k(u, \hat{u}) f(u, \hat{u})|_{(u, \hat{u})=(w, \hat{w})}, \\
 &= -\frac{l^2}{2} f(w, \hat{w}) \hat{A}.
 \end{aligned} \tag{A.17}$$

Note that the above equations do not involve a sum over k . This relation helps us simplify the various terms which involve the derivatives.

Let us simplify the derivatives that occur in (A.3), we have

$$D_{u_{k_1}} \cdots D_{u_{k_i}} \hat{D}_{\hat{u}_{\hat{k}_1}} \cdots D_{\hat{u}_{\hat{k}_{p-i}}} f|_{(u,\hat{u})=(w,\hat{w})} = (-1)^p \left(\frac{l^2}{2}\right)^p A^i \hat{A}^{p-i} f, \quad (\text{A.18})$$

$$+ \sum_{c=1}^{\lfloor \frac{p}{2} \rfloor} T_{(c)}(k_1, \cdots k_i; \hat{k}_1, \cdots \hat{k}_{p-i}).$$

Here we have used the relations in (A.17) to simplify the first term on the l.h.s. of (A.13) and written it in terms of only A and $\hat{A}$'s. $T_{(c)}$ is given by

$$T_{(c)}(k_1, \cdots k_i; \hat{k}_1, \cdots \hat{k}_{p-i}) = f B_{k_1} \cdots B_{k_i} \hat{B}_{\hat{k}_1} \cdots \hat{B}_{\hat{k}_{p-i}} [g_{k_1} \cdots g_{k_i} h_{\hat{k}_1} \cdots h_{\hat{k}_{p-i}}]_{(c)}. \quad (\text{A.19})$$

Now again using the identities (A.17) we see that the B_k and $\hat{B}_{\hat{k}}$ which are not associated with the contractions among g_k, h_k can be reduced to A and $\hat{A}$ respectively. What remains are the $B_k, \hat{B}_{\hat{k}}$ associated with the contractions. For example

$$T_{(1)}(k_1, k_2; \hat{k}_1) = -\frac{l^2}{2} \hat{A} \frac{B_{k_1} B_{k_2} l^2}{(u_{k_1} - u_{k_2})^2} f - \frac{l^2}{2} A \left(-\frac{B_{k_1} \hat{B}_{\hat{k}_1} l^2}{(u_{k_1} - \hat{u}_{\hat{k}_1})^2} - \frac{B_{k_2} \hat{B}_{\hat{k}_1} l^2}{(u_{k_2} - \hat{u}_{\hat{k}_1})^2} \right) f. \quad (\text{A.20})$$

Similarly

$$T_{(2)}(k_1, k_2; \hat{k}_1, \hat{k}_2) = \quad (\text{A.21})$$

$$f \left(\frac{B_{k_1} B_{k_2} l^2}{(u_{k_1} - u_{k_2})^2} \frac{B_{\hat{k}_1} B_{\hat{k}_2} l^2}{(\hat{u}_{\hat{k}_1} - \hat{u}_{\hat{k}_2})^2} + \frac{B_{k_1} \hat{B}_{\hat{k}_1} l^2}{(u_{k_1} - \hat{u}_{\hat{k}_1})^2} \frac{B_{k_2} \hat{B}_{\hat{k}_2} l^2}{(u_{k_2} - \hat{u}_{\hat{k}_2})^2} + \frac{B_{k_1} \hat{B}_{\hat{k}_2} l^2}{(u_{k_1} - \hat{u}_{\hat{k}_2})^2} \frac{B_{k_2} \hat{B}_{\hat{k}_1} l^2}{(u_{k_2} - \hat{u}_{\hat{k}_1})^2} \right).$$

On examining the terms that occur in (A.18), we see that they are organised in terms of powers of l^2 . We see that a pair of B 's are associated with l^2 , while a single A or $\hat{A}$ is associated with $\frac{l^2}{2}$. Note also the presence of (-1) associated with A or $\hat{A}$, that arises due to the use of the identity (A.17).

With all these observations, let us label the terms which occur in the correlator according to the pair of $B_k, \hat{B}_{\hat{k}}$, that occur. First consider the term with no B or $\hat{B}$. These terms arise from the first term in the expansion of the derivative in (A.18). Note that from (A.3)

and (A.18), they all occur with the coefficient $\left(\frac{l^2}{2}\right)^{2n}$.

$$\begin{aligned}
 \mathcal{C}_{2n}|_{\left(\frac{l^2}{2}\right)^{2n}} &= \left(\prod_{k=0}^{n-1} (B_k \hat{B}_k)^{\frac{l^2}{2}}\right) f(u, \hat{u})|_{(w, \hat{w})} (A^n \hat{A}^n) \left(\frac{l^2}{2}\right)^{2n} \\
 &\times \sum_{p=0}^{2n} \sum_{\substack{i=0, i \leq n \\ 0 \leq p-i \leq n}}^p \sum_{\substack{\{k_1, \dots, k_i\} \\ \{\hat{k}_1, \dots, \hat{k}_{p-i}\}}} (-1)^p, \\
 &= (A^n \hat{A}^n) \left(\frac{l^2}{2}\right)^{2n} \sum_{p=0}^{2n} \sum_{\substack{i=0, i \leq n \\ 0 \leq p-i \leq n}}^p (-1)^p \binom{n}{i} \binom{n}{p-i}, \\
 &= (A^n \hat{A}^n) \left(\frac{l^2}{2}\right)^{2n} \sum_{p=0}^{2n} \binom{2n}{p} (-1)^p = 0.
 \end{aligned} \tag{A.22}$$

In the first step we have used (2.29), (A.17). Then we have replaced the sums over $\{k_1, \dots, k_i\}, \{\hat{k}_1, \dots, \hat{k}_{p-i}\}$ by the number of terms. To obtain the last step we have used the combinatorial identity

$$\sum_{\substack{i=0, i \leq n \\ 0 \leq p-i \leq n}}^p \binom{n}{i} \binom{n}{p-i} = \binom{2n}{p}. \tag{A.23}$$

This same set of steps can be used to demonstrate that any term containing A or $\hat{A}$ vanishes. Before going to the general case, let us illustrate it for the term containing a single contraction $B_0 B_1 l^2$ in the correlator which is proportional to $\left(\frac{l^2}{2}\right)^{2n-2} l^2$. This term in the correlator is given by

$$\begin{aligned}
 \mathcal{C}_{2n}|_{\left(\frac{l^2}{2}\right)^{2n-2} l^2} &= \left(\prod_{k=0}^{n-1} (B_k \hat{B}_k)^{\frac{l^2}{2}}\right) f(u, \hat{u}) \frac{B_0 B_1 l^2}{(u_0 - u_1)^2} \times \\
 &\sum_{p=0}^{2n-2} \sum_{\substack{i=0, i \leq n-2 \\ 0 \leq p-i \leq n}}^p \left(\frac{l^2}{2}\right)^{2n-p} A^{n-i} \hat{A}^{n-p+i} \sum_{\substack{\{k_1, \dots, k_i\}, k \neq 0, 1 \\ \{\hat{k}_1, \dots, \hat{k}_{p-i}\}}} B_{k_1} g_{k_1} \dots B_{k_i} g_{k_i} \hat{B}_{k_1} h_{\hat{k}_1} \dots \hat{B}_{\hat{k}_{p-i}} \hat{h}_{\hat{k}_{p-i}} \Big|_{(u, \hat{u})=(w, \hat{w})}, \\
 &= \frac{B_0 B_1 l^2}{(w_0 - w_1)^2} \left(\frac{l^2}{2}\right)^{2n-2} A^{n-2} \hat{A}^n \sum_{\substack{\{k_1, \dots, k_i\}, k \neq 0, 1 \\ \{\hat{k}_1, \dots, \hat{k}_{p-i}\}}} (-1)^p, \\
 &= \frac{B_0 B_1 l^2}{(w_0 - w_1)^2} \left(\frac{l^2}{2}\right)^{2n-2} A^{n-2} \hat{A}^n \sum_{p=0}^{2n-2} \sum_{\substack{i=0, i \leq n-2 \\ 0 \leq p-i \leq n}}^p (-1)^p \binom{n-2}{i} \binom{n}{p-i}, \\
 &= \frac{B_0 B_1 l^2}{(w_0 - w_1)^2} \left(\frac{l^2}{2}\right)^{2n-2} A^{n-2} \hat{A}^n \sum_{p=0}^{2n-2} (-1)^p \binom{2n-2}{p}, \\
 &= 0.
 \end{aligned} \tag{A.24}$$

The first two lines of the above equation have all the terms containing $B_0 B_1$ that occur from summing the various derivatives in (A.3). Note that this specific single contraction

terms are present in $T_{(1)}$ as defined in (A.19). We then use (2.29), (A.17) and follow the same steps as before. When one sums over the set of integers $\{k_1, \dots, k_i\}$, none of these integers can contain $\{0, 1\}$ since g_0, g_1 are contracted. This is denoted in the subscript of the sum in the third line as a result, the number of these set of integers are given by $\binom{n-i}{i}$.

Now let us move to the more general case. Consider the term which arises from a pairs of contractions involving g 's, b pair of contractions involving h 's and c contractions between g and h . We also have $2(a + b + c) < 2n$. Such a contraction will be present in $T_{(a+b+c)}$. Among this type of contractions we focus on the term which contains the following products of $B, \hat{B}$.

$$\begin{aligned} \mathcal{B}(u, \bar{u}) = & \left[\frac{B_0 B_1 l^2}{(u_0 - u_1)^2} \cdots \frac{B_{2a-2} B_{2a-1} l^2}{(u_{2a-2} - u_{2a-1})^2} \right] \times \left[\frac{\hat{B}_0 \hat{B}_1 l^2}{(u_0 - u_1)^2} \cdots \frac{\hat{B}_{2b-2} \hat{B}_{2b-1} l^2}{(\hat{u}_{2b-2} - \hat{u}_{2b-1})^2} \right] \\ & \times \left[\frac{-B_{2a} \hat{B}_{2b} l^2}{(u_{2a} - u_{2b})^2} \cdots \frac{-\hat{B}_{2a+c-1} \hat{B}_{2b+c-1} l^2}{(\hat{u}_{2a+c-1} - \hat{u}_{2b+c-1})^2} \right]. \end{aligned} \quad (\text{A.25})$$

Let us now collect all the terms in the derivatives of the correlator (A.3) which contains this specific contraction.

$$\begin{aligned} \mathcal{C}_{2n} \Big|_{\left(\frac{l^2}{2}\right)^{2(n-a-b-c)} l^{2(a+b+c)}} &= \left(\prod_{k=0}^{n-1} (B_k \hat{B}_k)^{\frac{l^2}{2}} \right) f(u, \hat{u}) \mathcal{B}(u, \bar{u}) \\ &\times \sum_{p=0}^{2(n-a-b-c)} \sum_{\substack{i=0, i \leq n-2a-c \\ 0 \leq p-i \leq n-2b-c}}^p \left(\frac{l^2}{2} \right)^{2n-p} A^{n-i} \hat{A}^{n-p+i} \sum_{\substack{\{k_1, \dots, k_i\}, k \notin \{0, 1, \dots, 2a+c-1\}, \\ \{\hat{k}_1, \dots, \hat{k}_{p-i}\}, k \notin \{0, 1, \dots, 2b+c-1\}}} \left(\right. \\ &\quad \left. B_{k_1} g_{k_1} \cdots B_{k_i} g_{k_i} \hat{B}_{\hat{k}_1} h_{\hat{k}_1} \cdots \hat{B}_{\hat{k}_{p-i}} h_{\hat{k}_{p-i}} \right) \Big|_{(u, \hat{u})=(w, \hat{w})}, \\ &= \mathcal{B}(w, \hat{w}) \left(\frac{l^2}{2} \right)^{2(n-a-b-c)} A^{n-2a-c} \hat{A}^{n-2b-c} \sum_{\substack{\{k_1, \dots, k_i\}, k \notin \{0, 1, \dots, 2a+c-1\}, \\ \{\hat{k}_1, \dots, \hat{k}_{p-i}\}, k \notin \{0, 1, \dots, 2b+c-1\}}} (-1)^p, \\ &= \mathcal{B}(w, \hat{w}) \left(\frac{l^2}{2} \right)^{2(n-a-b-c)} A^{n-2a-c} \hat{A}^{n-2b-c} \sum_{p=0}^{2(n-a-b-c)} \sum_{\substack{i=0, i \leq n-2a-c \\ 0 \leq p-i \leq n-2b-c}} (-1)^p \binom{n-2a-c}{i} \binom{n-2b-c}{p-i}, \\ &= \mathcal{B}(w, \hat{w}) \left(\frac{l^2}{2} \right)^{2(n-a-b-c)} A^{n-2a-c} \hat{A}^{n-2b-c} \sum_{p=0}^{2(n-a-b-c)} \binom{2(n-a-b-c)}{p} (-1)^p, \\ &= 0. \end{aligned} \quad (\text{A.26})$$

The steps are as discussed before for the single contraction. Note that it is clear from this derivation that for any contraction of this form, not necessary the special one chosen in (A.25) vanishes as long as $2(n - a - b - c) > 0$. Thus the only non-zero element in the correlator is given in which all the n g 's and n h 's are contracted. Such a term occurs only once in (A.3) and it occurs in the derivative of order $2n$. Therefore the correlator of interest

is given by

$$\mathcal{C}_{2n} = l^{2n} (-1)^n \left(\prod_{k=0}^{n-1} (B_k \hat{B}_k) \right) \mathcal{J}, \quad (\text{A.27})$$

$$\mathcal{J} = \left(\prod_{k=0}^{n-1} \frac{1}{(w_k - \hat{w}_k)} + \text{distinct permutations} \right).$$

The $(-1)^n$ arises due to the contractions between g and h (A.9). $\mathcal{J}$ is identical to the correlator that one would obtain by considering the $2n$ point functions of $U(1)$ currents on the uniformised plane and the $B, \hat{B}$'s are slopes of the conformal transformations (2.10) and (2.11). Note that it is clear from the steps leading to this result, the simplification in the $2n$ point function occurs only when the descendants are placed on the uniformized plane. This concludes the derivation of the $2n$ point function of the descendant ∂e^{ilX} on the uniformized plane.

B Properties of descendants

In this appendix we summarise as well as derive the properties of conformal descendants till level 3.

B.1 Conformal transformations of descendants

Transformations of descendants obtained by the action of derivatives on primaries are easily derived. Here is the list till level-3

Level 1.

$$w \circ \partial \mathcal{O}(w(z)) = \left(\frac{\partial w}{\partial z} \right)^{h+1} \left[h \left(\frac{\partial w}{\partial z} \right)^{-2} \left(\frac{\partial^2 w}{\partial z^2} \right) + \partial_w \right] \mathcal{O}(w(z)). \quad (\text{B.1})$$

For the purposes of this paper we would need the transformations under the conformal maps (2.10), (2.11) where these maps take $z \rightarrow w_k, \hat{z} \rightarrow \hat{w}_k$. Using the definition of the slopes in (2.22) and the second derivatives of these transformations in (2.33) and (2.35), we can write (B.1) as

$$w \circ \partial O(w_k) = B_k^{h+1} \left(h \frac{A}{B_k} + \partial_{w_k} \right) O(w_k), \quad (\text{B.2})$$

$$\hat{w} \circ \partial O(\hat{w}_k) = \hat{B}_k^{h+1} \left(h \frac{\hat{A}}{\hat{B}_k} + \partial_{\hat{w}_k} \right) O(\hat{w}_k). \quad (\text{B.3})$$

Level 2. The second derivative of the operator transforms as

$$w \circ \partial^2 O(w(z)) = w'^{h+2} \left[h \left((h-1) \frac{w''^2}{w'^4} + \frac{w'''}{w'^3} \right) + (2h+1) \frac{w''}{w'^2} \partial_w + \partial_w^2 \right] O(w). \quad (\text{B.4})$$

Let us introduce the third derivatives of the conformal maps to the uniformized plane by

$$\left. \frac{d^3 w}{dz^3} \right|_{w_k} = B_k (A^2 + F), \quad \left. \frac{d^3 \hat{w}}{d\hat{z}^3} \right|_{\hat{w}_k} = \hat{B}_k (\hat{A}^2 + \hat{F}), \quad (\text{B.5})$$

$$F = (1 + e^{-4\pi i x}) + \frac{1}{n} (1 - e^{-4\pi i x}), \quad \hat{F} = (1 + e^{4\pi i x}) + \frac{1}{n} (1 - e^{4\pi i x}). \quad (\text{B.6})$$

Then we can write the conformal transformations as

$$\begin{aligned} w \circ \partial^2 \mathcal{O}(w_k) &= B_k^{h+2} \left[h \left((h-1) \frac{A^2}{B_k^2} + \frac{A^2 + F}{B_k^2} \right) + (2h+1) \frac{A}{B_k} \partial_{w_k} + \partial_{w_k^2} \right] \mathcal{O}(w_k), \\ \hat{w} \circ \partial^2 \mathcal{O}(\hat{w}_k) &= \hat{B}_k^{h+2} \left[h \left((h-1) \frac{\hat{A}^2}{\hat{B}_k^2} + \frac{\hat{A}^2 + \hat{F}}{\hat{B}_k^2} \right) + (2h+1) \frac{\hat{A}}{\hat{B}_k} \partial_{\hat{w}_k} + \partial_{\hat{w}_k^2} \right] \mathcal{O}(\hat{w}_k). \end{aligned} \quad (\text{B.7})$$

Level 3. The conformal transformation of the third derivative of the primary is given by

$$\begin{aligned} \partial_z^3 \mathcal{O}(z) &= w'^{h+3} \left[\partial_w^3 + \frac{3(h+1)w''}{(w')^2} \partial_w^2 + \frac{3h^2(w'')^2 + (1+3h)w'w'''}{(w')^4} \partial_w \right. \\ &\quad \left. + \frac{h((h-1)w''((h-2)(w'')^2 + 3w'w''') + (w')^2w''')}{(w')^6} \right] \mathcal{O}(w). \end{aligned} \quad (\text{B.8})$$

The fourth derivatives to the maps (2.10), (2.11) to the uniformized plane is given by

$$\left. \frac{d^4 w}{dz^4} \right|_{w_k} = B_k(A^3 + 3AF + 2G), \quad \left. \frac{d^3 \hat{w}}{dz^3} \right|_{\hat{w}_k} = \hat{B}_k(\hat{A}^3 + 3\hat{A}\hat{F} + 2\hat{G}), \quad (\text{B.9})$$

$$G = (1 + e^{-6\pi i x}) + \frac{1}{n}(1 - e^{-6\pi i x}), \quad \hat{G} = -(1 + e^{6\pi i x}) - \frac{1}{n}(1 - e^{6\pi i x}). \quad (\text{B.10})$$

We use these expression to write the conformal transformation as

$$\begin{aligned} w \circ \partial^3 \mathcal{O}(w_k) &= B_k^{h+3} \left[\frac{h(2G + hA(3F + hA^2))}{B_k^3} + \frac{(3h+1)F + (1+3h+3h^2)A^2}{B_k^2} \partial_{w_k} \right. \\ &\quad \left. + \frac{3(h+1)A}{B_k} \partial_{w_k}^2 + \partial_{w_k}^3 \right] \mathcal{O}(w_k), \\ \hat{w} \circ \partial^3 \mathcal{O}(\infty_k) &= \hat{B}_k^{h+3} \left[\frac{h(2\hat{G} + h\hat{A}(3\hat{F} + h\hat{A}^2))}{\hat{B}_k^3} + \frac{(3h+1)\hat{F} + (1+3h+3h^2)\hat{A}^2}{\hat{B}_k^2} \partial_{\hat{w}_k} \right. \\ &\quad \left. + \frac{3(h+1)\hat{A}}{\hat{B}_k} \partial_{\hat{w}_k}^2 + \partial_{\hat{w}_k}^3 \right] \mathcal{O}(\hat{w}_k). \end{aligned} \quad (\text{B.11})$$

Let us now obtain the conformal transformation of Virasoro descendants defined by

$$\mathcal{O}^{(-k)}(z) = \oint_z \frac{d\tilde{z}}{(\tilde{z} - z)^{k-1}} T(\tilde{z}) \mathcal{O}(z). \quad (\text{B.12})$$

Here the contour is around z . Consider the conformal transformation $w(z)$, we can use the conformal transformation of the stress tensor and the primary to write the above equation as

$$\mathcal{O}^{(-k)}(z) = \oint_e \frac{d\tilde{w}}{(\tilde{z} - z)^{k-1}} \frac{d\tilde{z}}{d\tilde{w}} \left(\left(\frac{d\tilde{w}}{d\tilde{z}} \right)^2 T(\tilde{w}) + \frac{c}{12} S[\tilde{w}, \tilde{z}] \right) \left(\frac{dw}{dz} \right)^h \mathcal{O}(w) \quad (\text{B.13})$$

where $\tilde{w} = w(\tilde{z})$ and the contour is around w . The Schwarzian is defined as

$$S[w, z] = \frac{2w'''w' - 3(w'')^2}{2(w')^2}, \quad w' = \frac{dw}{dz}. \quad (\text{B.14})$$

The next step is to write the denominator $\tilde{z} - z$ in terms of $\tilde{w} - w$ and also write the slope $\frac{d\tilde{z}}{d\tilde{w}}$ in terms of the slope $\frac{dw}{dz}$. This can be done by inverting the Taylor series of the function $\tilde{w}(\tilde{z})$ around z which is given by

$$\tilde{w} = w(z) + (\tilde{z} - z)w' + \frac{1}{2!}(\tilde{z} - z)^2w'' + \frac{1}{3!}(\tilde{z} - z)^3w''' + \frac{1}{4!}(\tilde{z} - z)^4w'''' + \dots \quad (\text{B.15})$$

Using this we can obtain

$$\begin{aligned} \tilde{z} - z &= \frac{1}{w'}(\tilde{w} - w) - \frac{1}{2} \frac{w''}{(w')^3}(\tilde{w} - w)^2 + \frac{1}{(w')^5} \left(\frac{(w'')^2}{2} - \frac{1}{3!}w'''w' \right) (\tilde{w} - w)^3 \\ &\quad - \frac{1}{(w')^7} \left(\frac{15(w'')^3 - 10w'w''w''' + (w')^2w''''}{4!} \right) (\tilde{w} - w)^4 + \dots, \\ \frac{d\tilde{z}}{d\tilde{w}} &= \frac{1}{w'} - \frac{w''}{(w')^3}(\tilde{w} - w) + \frac{3}{(w')^5} \left(\frac{(w'')^2}{2} - \frac{1}{3!}w'''w' \right) (\tilde{w} - w)^2 \\ &\quad - \frac{1}{6(w')^7} (15(w'')^3 - 10w'w''w''' + (w')^2w''') (\tilde{w} - w)^3 + \dots \end{aligned} \quad (\text{B.16})$$

Substituting these expressions in (B.13) and performing the contour integral with $k = 2, 3$ we can obtain the conformal transformation of the Virasoro descendants.

$\mathcal{O}^{(-2)}$. The conformal transformation of the level-2 Virasoro descendant is given by

$$\begin{aligned} w \circ \mathcal{O}^{(-2)}(w(z)) &= (w')^{h+2} \left[\mathcal{O}^{(-2)}(w) + \frac{3w''}{2(w')^2} \mathcal{O}^{(-1)}(w) + \left(\frac{3h(w'')^2}{4w'^4} + \left(4h + \frac{c}{2} \right) \frac{S[w, z]}{6(w')^2} \right) \mathcal{O}(w) \right], \\ &= (w')^{h+2} \left[\mathcal{O}^{(-2)}(w) + \frac{3w''}{2w'^2} \mathcal{O}^{(-1)}(w) \right. \\ &\quad \left. + \frac{1}{12(w')^4} \left(\left(-3h - \frac{3c}{2} \right) (w'')^2 + 2 \left(4h + \frac{c}{2} \right) w'''w' \right) \mathcal{O}(w) \right]. \end{aligned} \quad (\text{B.17})$$

This agrees with the transformation given in [8] which uses the operator approach of [27] to obtain conformal transformations of descendants. For the maps in (2.10) and (2.11), this transformation can be written as

$$\begin{aligned} w \circ \mathcal{O}^{(-2)}(w_k) &= B_k^{h+2} \left[\mathcal{O}^{(-2)}(w_k) + \frac{3A}{2B_k} \mathcal{O}^{(-1)}(w_k) + \frac{1}{12} \left(\left(5h - \frac{c}{2} \right) \frac{A^2}{B_k^2} + 2 \left(4h + \frac{c}{2} \right) \frac{F}{B_k^2} \right) \mathcal{O}(w_k) \right], \\ \hat{w} \circ \mathcal{O}^{(-2)}(\hat{w}_k) &= \hat{B}_k^{h+2} \left[\mathcal{O}^{(-2)}(\hat{w}_k) + \frac{3\hat{A}}{2\hat{B}_k} \mathcal{O}^{(-1)}(\hat{w}_k) + \frac{1}{12} \left(\left(5h - \frac{c}{2} \right) \frac{\hat{A}^2}{\hat{B}_k^2} + 2 \left(4h + \frac{c}{2} \right) \frac{\hat{F}}{\hat{B}_k^2} \right) \mathcal{O}(\hat{w}_k) \right]. \end{aligned} \quad (\text{B.18})$$

$\mathcal{O}^{(-3)}$. Similarly the transformation of the level 3 descendant is given by

$$w \circ \mathcal{O}^{(-3)}(w(z)) = (w')^{h+3} \left[\mathcal{O}^{(-3)}(w) + \frac{2w''}{(w')^2} \mathcal{O}^{(-2)}(w) + \left(\frac{(w'')^2}{4(w')^4} + \frac{5}{6} \frac{w'''}{(w')^3} \right) \mathcal{O}^{(-1)}(w) \right. \\ \left. + \left(\frac{3w'w'''' - 2w''w'''}{12(w')^5} - \frac{c}{12} \frac{6(w'')^3 - 6w'w''w''' + (w')^2w''''}{(w')^8} \right) \mathcal{O}(w) \right]. \quad (\text{B.19})$$

Restricting the conformal transformation to the uniformisation map we can write the above transformation as

$$w \circ \mathcal{O}^{(-3)}(w_k) = B_k^{(h+3)} \left[\mathcal{O}^{(-3)}(w_k) + 2 \frac{A}{B_k} \mathcal{O}^{(-2)}(w_k) + \frac{13A^2 + 10F}{12B_k^2} \mathcal{O}^{(-1)}(w_k) \right. \\ \left. + \left(\frac{h(A^3 + 7AF + 6G)}{12B_k^3} + \frac{c}{12} \frac{(-AF + 2G)}{B_k^3} \right) \mathcal{O}(w_k) \right], \quad (\text{B.20})$$

$$\hat{w} \circ \mathcal{O}^{(-3)}(\hat{w}_k) = \hat{B}_k^{(h+3)} \left[\mathcal{O}^{(-3)}(\hat{w}_k) + 2 \frac{\hat{A}}{\hat{B}_k} \mathcal{O}^{(-2)}(\hat{w}_k) + \frac{13\hat{A}^2 + 10\hat{F}}{12\hat{B}_k^2} \mathcal{O}^{(-1)}(\hat{w}_k) \right. \\ \left. + \left(\frac{h(\hat{A}^3 + 7\hat{A}\hat{F} + 6\hat{G})}{12\hat{B}_k^3} + \frac{c}{12} \frac{(-\hat{A}\hat{F} + 2\hat{G})}{\hat{B}_k^3} \right) \mathcal{O}(\hat{w}_k) \right].$$

The conformal transformations of the last state at level 3 is given by

$$w \circ \partial \mathcal{O}^{(-2)}(w_k) = B_k^{(h+3)} \left[\partial_{w_k} \mathcal{O}^{(-2)}(w_k) + \frac{A(h+2)}{B_k} \mathcal{O}^{(-2)}(w_k) \right. \\ \left. + \left\{ \frac{3A}{2B_k} \partial_{w_k}^2 + \frac{A^2(-c+46h+36) + 2F(c+8h+18)}{24B_k^2} \partial_{w_k} \right. \right. \\ \left. \left. + \frac{1}{24B_k^3} \left(2A^2(c+8h) + 2F(c+8h) - A^3(2(8-5h)h + c(2+h)) \right. \right. \right. \\ \left. \left. \left. + 2AF(c(-4+h) + 2h(-7+4h)) \right) \right\} \mathcal{O}(w_k) \right],$$

$$\hat{w} \circ \partial \mathcal{O}^{(-2)}(\hat{w}) = \hat{B}_k^{(h+3)} \left[\partial_{\hat{w}_k} \mathcal{O}^{(-2)}(\hat{w}_k) + \frac{\hat{A}(h+2)}{\hat{B}_k} \mathcal{O}^{(-2)}(\hat{w}_k) \right. \\ \left. + \left\{ \frac{3\hat{A}}{2\hat{B}_k} \partial_{\hat{w}_k}^2 + \frac{\hat{A}^2(-c+46h+36) + 2\hat{F}(c+8h+18)}{24\hat{B}_k^2} \partial_{\hat{w}_k} \right. \right. \\ \left. \left. + \frac{1}{24\hat{B}_k^3} \left(2\hat{A}^2(c+8h) + 2\hat{F}(c+8h) - \hat{A}^3(2(8-5h)h + c(2+h)) \right. \right. \right. \\ \left. \left. \left. + 2\hat{A}\hat{F}(c(-4+h) + 2h(-7+4h)) \right) \right\} \mathcal{O}(\hat{w}_k) \right]. \quad (\text{B.21})$$

B.2 Correlators with descendants

Correlators with derivatives of primaries are straightforward to obtain from the corresponding correlators of the primaries. To evaluate correlators involving Virasoro descendants we would

need the OPE with the stress tensor. To evaluate the OPE, we first use the definition (B.12) and use the TT OPE to obtain

$$\begin{aligned} T(w) \oint_z \frac{dz'}{(z' - z)^{k-1}} T(z') \mathcal{O}(z) \\ = \oint_z \frac{dz'}{(z' - z)^{k-1}} \left(\frac{2T(z')}{w - z'} + \frac{\partial_{z'} T(z')}{w - z'} + \frac{c}{2} \frac{1}{(w - z')^4} + :T(w)T(z'): \right) \mathcal{O}(z). \end{aligned} \quad (\text{B.22})$$

We can then use the OPE of the stress tensor with the operator $\mathcal{O}$. For $k = 2$ we obtain

$$T(w) \mathcal{O}^{-2}(z) \sim \frac{(\frac{c}{2} + 4h) \mathcal{O}(z)}{(w - z)^4} + \frac{3\partial_z \mathcal{O}(z)}{(w - z)^3} + \frac{(h + 2) \mathcal{O}^{(-2)}(z)}{(w - z)^2} + \frac{\partial_z \mathcal{O}^{(-2)}}{(w - z)}. \quad (\text{B.23})$$

Similarly for $k = 3$ we get

$$T(w) \mathcal{O}^{-3}(z) \sim \frac{2(c + 3h) \mathcal{O}(z)}{(w - z)^5} + \frac{5\partial_z \mathcal{O}(z)}{(w - z)^4} + \frac{4\mathcal{O}^{(-2)}(z)}{(w - z)^3} + \frac{(h + 3)\partial \mathcal{O}^{(-3)}}{(w - z)^2} + \frac{\partial_z \mathcal{O}^{(-3)}}{(w - z)}. \quad (\text{B.24})$$

Let us illustrate how to use these OPE's in a couple of correlators. Consider

$$\langle \mathcal{O}(z_1) \mathcal{O}^{(-2)}(z_2) \rangle = \langle \mathcal{O}(z_1) \oint_{z_2} \frac{dz'}{z' - z_2} T(z') \mathcal{O}(z_2) \rangle. \quad (\text{B.25})$$

Here by the definition of $\mathcal{O}^{(-2)}$ the contour is around z_2 . By deforming the contour around infinity we obtain

$$\langle \mathcal{O}(z_1) \mathcal{O}^{(-2)}(z_2) \rangle = - \langle \oint_{z_1} \frac{dz'}{z' - z_2} T(z') \mathcal{O}(z_1) \mathcal{O}(z_2) \rangle. \quad (\text{B.26})$$

Note that now the contour is around z_1 . Using the OPE of the stress tensor with $\mathcal{O}(z_1)$ we obtain

$$\begin{aligned} \langle \mathcal{O}(z_1) \mathcal{O}^{(-2)}(z_2) \rangle &= - \left\langle \oint_{z_1} \frac{dz'}{z' - z_2} \left(\frac{h \mathcal{O}(z_1)}{(z' - z_1)^2} + \frac{\partial_{z_1} \mathcal{O}(z_1)}{(z' - z_2)} \right) \mathcal{O}(z_2) \right\rangle, \\ &= \frac{h}{(z_1 - z_2)^2} \langle \mathcal{O}(z_1) \mathcal{O}(z_2) \rangle - \frac{1}{(z_1 - z_2)} \partial_{z_1} \langle \mathcal{O}(z_1) \mathcal{O}(z_2) \rangle, \\ &= \frac{3h}{(z_1 - z_2)^{2h+2}}. \end{aligned} \quad (\text{B.27})$$

Consider another example, the two point function of $\mathcal{O}^{(-2)}$ and $\mathcal{O}^{(-3)}$.

$$\begin{aligned} &\langle \mathcal{O}^{(-2)}(z_1) \mathcal{O}^{(-3)}(z_2) \rangle, \\ &= \langle \mathcal{O}^{(-2)}(z_1) \oint_{z_2} \frac{dz'}{(z' - z_2)^2} T(z') \mathcal{O}(z_2) \rangle, \\ &= - \left\langle \oint_{z_1} \frac{dz'}{(z' - z_2)^2} T(z') \mathcal{O}^{(-2)}(z_1) \mathcal{O}(z_2) \right\rangle, \\ &= - \left\langle \oint_{z_1} \frac{dz'}{(z' - z_2)^2} \left[\frac{(\frac{c}{2} + 4h) \mathcal{O}(z_1)}{(z' - z_1)^4} + \frac{3\partial_{z_1} \mathcal{O}(z_1)}{(z' - z_1)^3} + \frac{(h + 2) \mathcal{O}^{(-2)}(z_1)}{(z' - z_1)^2} + \frac{\partial_{z_1} \mathcal{O}^{(-2)}(z_1)}{(z' - z_1)} \right] \mathcal{O}(z_2) \right\rangle, \\ &= \frac{2c + 52h + 12h^2}{(z_1 - z_2)^{2h+5}}. \end{aligned} \quad (\text{B.28})$$

In the first step we used the definition of $\mathcal{O}^{(-3)}$, then deformed the contour so that it is around z_1 . We then applied the OPE (B.23), performed the contour integrals and also used the two point function (B.27).

Using similar methods we can obtain the following two point functions.

$$\begin{aligned}
\langle \mathcal{O}(z_1) \mathcal{O}^{-1}(z_2) \rangle &= \frac{2h}{(z_1 - z_2)^{2h+1}}, \\
\langle \mathcal{O}(z_1) \mathcal{O}^{-2}(z_2) \rangle &= \frac{3h}{(z_1 - z_2)^{2h+2}}, \\
\langle \mathcal{O}(z_1) \mathcal{O}^{-3}(z_2) \rangle &= \frac{4h}{(z_1 - z_2)^{2h+3}}, \\
\langle \mathcal{O}^{-2}(z_1) \mathcal{O}^{-2}(z_2) \rangle &= \frac{\frac{c}{2} + h(9h + 22)}{(z_1 - z_2)^{2h+4}}, \\
\langle \mathcal{O}^{-2}(z_1) \mathcal{O}^{-3}(z_2) \rangle &= \frac{2c + h(12h + 52)}{(z_1 - z_2)^{2h+5}}, \\
\langle \mathcal{O}^{-3}(z_1) \mathcal{O}^{-3}(z_2) \rangle &= \frac{-10c - 2h(71 + 8h)}{(z_1 - z_2)^{2h+6}}.
\end{aligned} \tag{B.29}$$

All the other 2-point functions required in this paper can be obtained by the action of derivatives on the above correlators.

Open Access. This article is distributed under the terms of the Creative Commons Attribution License ([CC-BY 4.0](https://creativecommons.org/licenses/by/4.0/)), which permits any use, distribution and reproduction in any medium, provided the original author(s) and source are credited.

References

- [1] S. Ryu and T. Takayanagi, *Holographic derivation of entanglement entropy from AdS/CFT*, *Phys. Rev. Lett.* **96** (2006) 181602 [[hep-th/0603001](#)] [[INSPIRE](#)].
- [2] V.E. Hubeny, M. Rangamani and T. Takayanagi, *A Covariant holographic entanglement entropy proposal*, *JHEP* **07** (2007) 062 [[arXiv:0705.0016](#)] [[INSPIRE](#)].
- [3] D.L. Jafferis, A. Lewkowycz, J. Maldacena and S.J. Suh, *Relative entropy equals bulk relative entropy*, *JHEP* **06** (2016) 004 [[arXiv:1512.06431](#)] [[INSPIRE](#)].
- [4] H. Casini, *Relative entropy and the Bekenstein bound*, *Class. Quant. Grav.* **25** (2008) 205021 [[arXiv:0804.2182](#)] [[INSPIRE](#)].
- [5] A.C. Wall, *A Proof of the generalized second law for rapidly-evolving Rindler horizons*, *Phys. Rev. D* **82** (2010) 124019 [[arXiv:1007.1493](#)] [[INSPIRE](#)].
- [6] A.C. Wall, *A proof of the generalized second law for rapidly changing fields and arbitrary horizon slices*, *Phys. Rev. D* **85** (2012) 104049 [Erratum *ibid.* **87** (2013) 069904] [[arXiv:1105.3445](#)] [[INSPIRE](#)].
- [7] T. Pálmai, *Excited state entanglement in one dimensional quantum critical systems: Extensivity and the role of microscopic details*, *Phys. Rev. B* **90** (2014) 161404 [[arXiv:1406.3182](#)] [[INSPIRE](#)].
- [8] P. Caputa and A. Veliz-Orsorio, *Entanglement constant for conformal families*, *Phys. Rev. D* **92** (2015) 065010 [[arXiv:1507.00582](#)] [[INSPIRE](#)].

- [9] L. Taddia, F. Ortolani and T. Pálmai, *Renyi entanglement entropies of descendant states in critical systems with boundaries: conformal field theory and spin chains*, *J. Stat. Mech.* **1609** (2016) 093104 [[arXiv:1606.02667](#)] [[INSPIRE](#)].
- [10] E.M. Brehm and M. Broccoli, *Correlation functions and quantum measures of descendant states*, *JHEP* **04** (2021) 227 [[arXiv:2012.11255](#)] [[INSPIRE](#)].
- [11] E. Witten, *A Mini-Introduction To Information Theory*, *Riv. Nuovo Cim.* **43** (2020) 187 [[arXiv:1805.11965](#)] [[INSPIRE](#)].
- [12] F.C. Alcaraz, M.I. Berganza and G. Sierra, *Entanglement of low-energy excitations in Conformal Field Theory*, *Phys. Rev. Lett.* **106** (2011) 201601 [[arXiv:1101.2881](#)] [[INSPIRE](#)].
- [13] M.I. Berganza, F.C. Alcaraz and G. Sierra, *Entanglement of excited states in critical spin chains*, *J. Stat. Mech.* **1201** (2012) P01016 [[arXiv:1109.5673](#)] [[INSPIRE](#)].
- [14] N. Lashkari, *Relative Entropies in Conformal Field Theory*, *Phys. Rev. Lett.* **113** (2014) 051602 [[arXiv:1404.3216](#)] [[INSPIRE](#)].
- [15] G. Sárosi and T. Ugajin, *Relative entropy of excited states in two dimensional conformal field theories*, *JHEP* **07** (2016) 114 [[arXiv:1603.03057](#)] [[INSPIRE](#)].
- [16] A. Belin, N. Iqbal and S.F. Lokhande, *Bulk entanglement entropy in perturbative excited states*, *SciPost Phys.* **5** (2018) 024 [[arXiv:1805.08782](#)] [[INSPIRE](#)].
- [17] P. Calabrese, F.H.L. Essler and A.M. Läuchli, *Entanglement entropies of the quarter filled hubbard model*, *J. Stat. Mech.* **2014** (2014) P09025.
- [18] P. Ruggiero and P. Calabrese, *Relative Entanglement Entropies in 1 + 1-dimensional conformal field theories*, *JHEP* **02** (2017) 039 [[arXiv:1612.00659](#)] [[INSPIRE](#)].
- [19] J. Bhattacharya, M. Nozaki, T. Takayanagi and T. Ugajin, *Thermodynamical Property of Entanglement Entropy for Excited States*, *Phys. Rev. Lett.* **110** (2013) 091602 [[arXiv:1212.1164](#)] [[INSPIRE](#)].
- [20] D.D. Blanco, H. Casini, L.-Y. Hung and R.C. Myers, *Relative Entropy and Holography*, *JHEP* **08** (2013) 060 [[arXiv:1305.3182](#)] [[INSPIRE](#)].
- [21] F.-L. Lin, H. Wang and J.-j. Zhang, *Thermalilty and excited state Rényi entropy in two-dimensional CFT*, *JHEP* **11** (2016) 116 [[arXiv:1610.01362](#)] [[INSPIRE](#)].
- [22] E. Perlmutter, *Virasoro conformal blocks in closed form*, *JHEP* **08** (2015) 088 [[arXiv:1502.07742](#)] [[INSPIRE](#)].
- [23] P. Calabrese, J. Cardy and E. Tonni, *Entanglement entropy of two disjoint intervals in conformal field theory II*, *J. Stat. Mech.* **1101** (2011) P01021 [[arXiv:1011.5482](#)] [[INSPIRE](#)].
- [24] C.T. Asplund, A. Bernamonti, F. Galli and T. Hartman, *Holographic Entanglement Entropy from 2d CFT: Heavy States and Local Quenches*, *JHEP* **02** (2015) 171 [[arXiv:1410.1392](#)] [[INSPIRE](#)].
- [25] A. Belin, N. Iqbal and J. Kruthoff, *Bulk entanglement entropy for photons and gravitons in AdS_3* , *SciPost Phys.* **8** (2020) 075 [[arXiv:1912.00024](#)] [[INSPIRE](#)].
- [26] A. Belin and S. Colin-Ellerin, *Bootstrapping quantum extremal surfaces. Part I. The area operator*, *JHEP* **11** (2021) 021 [[arXiv:2107.07516](#)] [[INSPIRE](#)].
- [27] M. Gaberdiel, *A General transformation formula for conformal fields*, *Phys. Lett. B* **325** (1994) 366 [[hep-th/9401166](#)] [[INSPIRE](#)].

- [28] S. Datta, J.R. David, M. Ferlino and S.P. Kumar, *Higher spin entanglement entropy from CFT*, *JHEP* **06** (2014) 096 [[arXiv:1402.0007](#)] [[INSPIRE](#)].
- [29] S. Datta, J.R. David, M. Ferlino and S.P. Kumar, *Universal correction to higher spin entanglement entropy*, *Phys. Rev. D* **90** (2014) 041903 [[arXiv:1405.0015](#)] [[INSPIRE](#)].
- [30] S. Datta, J.R. David and S.P. Kumar, *Conformal perturbation theory and higher spin entanglement entropy on the torus*, *JHEP* **04** (2015) 041 [[arXiv:1412.3946](#)] [[INSPIRE](#)].
- [31] B.G. Chowdhury, S. Datta and J.R. David, *Rényi divergences from Euclidean quenches*, *JHEP* **04** (2020) 094 [[arXiv:1912.07210](#)] [[INSPIRE](#)].
- [32] J. de Boer and J.I. Jottar, *Thermodynamics of higher spin black holes in AdS_3* , *JHEP* **01** (2014) 023 [[arXiv:1302.0816](#)] [[INSPIRE](#)].
- [33] M. Ammon, A. Castro and N. Iqbal, *Wilson Lines and Entanglement Entropy in Higher Spin Gravity*, *JHEP* **10** (2013) 110 [[arXiv:1306.4338](#)] [[INSPIRE](#)].