The chiral algebra of genus two class $\mathcal{S}$ theory

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ABSTRACT: We construct the chiral algebra associated with the A_1 -type class $\mathcal{S}$ theory for the genus two Riemann surface without punctures. By solving the BRST cohomology problem corresponding to a marginal gauging in four dimensions, we find a set of chiral algebra generators that form closed OPEs. Given the fact that they reproduce the spectrum of chiral algebra operators up to large dimensions, we conjecture that they are the complete set of generators. Remarkably, their OPEs are invariant under an action of $SU(2)$ which is not associated with any conserved one-form current in four dimensions. We find that this novel $SU(2)$ strongly constrains the OPEs of non-scalar Schur operators. For completeness, we also check the equivalence of Schur indices computed in two S-dual descriptions with a non-vanishing flavor fugacity turned on.

KEYWORDS: Conformal Field Theory, Extended Supersymmetry

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1 Introduction

One of the most interesting recent developments in the study of four-dimensional $\mathcal{N} = 2$ superconformal field theories (SCFTs) is the new duality proposed by [1],

$$\chi : 4d \mathcal{N} = 2 \text{ SCFT} \rightarrow 2d \text{ chiral algebra}, \quad (1.1)$$

which associates a 2d chiral algebra to every $\mathcal{N} = 2$ SCFT in four dimensions.¹ This map can be applied to arbitrary four-dimensional $\mathcal{N} = 2$ SCFT, regardless of the existence of a weakly-coupled Lagrangian description. The associated chiral algebra constructed in this way captures the OPEs of protected BPS operators called “Schur operators”, which are operators contributing to the Schur limit of the superconformal index [2, 3]. While the scalar Schur operators are known as Higgs branch operators and well-studied, the nature of their non-scalar cousins is not generally understood.

Associated chiral algebras have revealed various general properties of 4d $\mathcal{N} = 2$ SCFTs. Indeed, in [1, 4–7], many new unitarity bounds were found by using this 2d/4d correspondence. It was also shown in [8] that the Higgs branch of the 4d theory is reconstructed as the associated variety [9] of the corresponding chiral algebra, which was further studied in [10–14]. Since the map (1.1) preserves the non-perturbative structure of OPEs among the Schur operators, the associated chiral algebra is also a powerful tool to analyze strongly coupled SCFTs, such as Argyres-Douglas type theories [15–18] and T_N theories [19]. See [20–59] for recent works in this and other directions.

After these works on the associated chiral algebras, one important thing that is still to be understood is the chiral algebra nature of non-scalar Schur operators. In particular, chiral algebra generators corresponding to non-scalar Schur operators in the semi-short multiplet, $\hat{\mathcal{C}}_{R(j,\bar{j})}$, are not well-understood, except for the stress tensor multiplet $\hat{\mathcal{C}}_{0(0,0)}$.² Most of the earlier works focused on the case in which the chiral algebra is generated by the stress tensor and those corresponding to Higgs branch operators (i.e., scalar Schur operators), possibly with their partners created by extra supersymmetries.³ In such cases, the OPEs of the associated chiral algebras are often fixed by the 4d Higgs branch chiral ring and 2d Jacobi identities. However, there are SCFTs whose chiral algebra contains a generator corresponding to generic $\hat{\mathcal{C}}_{R(j,\bar{j})}$.⁴ In this case, 2d OPEs are not simply reconstructed from the 4d Higgs branch.

One simple such SCFT is the A_1 -type class $\mathcal{S}$ theory for a genus two Riemann surface without punctures. Here, A_1 -type class $\mathcal{S}$ theories are defined as the IR limit of the six-dimensional $\mathcal{N} = (2, 0)$ A_1 SCFT compactified on a punctured Riemann surface, $\mathcal{C}_{g,s}$, where g is genus and s is the number of punctures.⁵ We denote this theory by $\mathcal{T}_{\mathcal{C}_{g,s}}$, especially focusing on $\mathcal{T}_{\mathcal{C}_{2,0}}$. The flavor $U(1)_f$ symmetry of $\mathcal{T}_{\mathcal{C}_{2,0}}$ is emergent and not associated with

¹In mathematical literature, chiral algebras are also called vertex operator algebras (VOAs).

²Here, we use the convention of [60].

³In some cases, the 2d stress tensor is a composite operator and therefore not an independent generator.

⁴See [21, 29] for examples of such chiral algebras.

⁵When the six-dimensional theory is $\mathcal{N} = (2, 0)$ $\mathfrak{g}$ SCFT for general $\mathfrak{g} \in \{A_n, D_n, E_n\}$, the class $\mathcal{S}$ theory is also characterized by embeddings $\mathfrak{su}(2) \hookrightarrow \mathfrak{g}$ at each puncture on $\mathcal{C}_{g,s}$. In the case of $\mathfrak{g} = A_1$, there is only one such embedding in addition to the trivial one.

any puncture on the Riemann surface. This is an example of flavor enhancement of the class $\mathcal{S}$ theories. While $\mathcal{T}_{C_{2,0}}$ has a Lagrangian description, its associated chiral algebra has not been identified.⁶ Moreover, it was proven in [1] that its chiral algebra must contain generators corresponding to non-scalar Schur operators in $\widehat{\mathcal{C}}_{1(0,0)}$. These operators are neither related to the stress tensor multiplet nor to Higgs branch operators, and still to be understood. For example, it is not known whether these generators are Virasoro primary operators.

In this paper, we identify the chiral algebra $\chi[\mathcal{T}_{C_{2,0}}]$ associated with $\mathcal{T}_{C_{2,0}}$. Our strategy is to start with a weakly-coupled description of the theory and apply to it the 2d interpretation of marginal gaugings proposed in [1]. This 2d interpretation involves a BRST reduction associated with a 4d marginal gauging. We evaluate this BRST reduction and find a set of chiral algebra generators equipped with closed OPEs. These generators are of dimension less than or equal to three, and reproduce the correct operator spectrum by normal-ordered products at least up to dimension six. Given this fact and their closed OPEs, we conjecture that they are the complete set of generators. As a further consistency check, we compute the character of $\chi[\mathcal{T}_{C_{2,0}}]$ based on our conjecture, and find the result in perfect agreement with the Schur index of $\mathcal{T}_{C_{2,0}}$ up to $\mathcal{O}(q^9)$. The 4d superconformal multiplets corresponding to these generators are also identified. We particularly show that the generators corresponding to $\widehat{\mathcal{C}}_{1(0,0)}$ are all Virasoro primaries.

One remarkable consequence of our OPEs is that there exists an unexpected $SU(2)$ acting on $\chi[\mathcal{T}_{C_{2,0}}]$ as an automorphism. Since $\chi[\mathcal{T}_{C_{2,0}}]$ has no $\widehat{\mathfrak{su}(2)}$ current, this $SU(2)$ symmetry is not associated with any 2d conserved current. Moreover, one can show that there is no 4d conserved one-form current corresponding to this $SU(2)$.⁷ Therefore, this either corresponds to a 4d symmetry without conserved current, or is an accidental symmetry in two dimensions. We find that the action of this $SU(2)$ is trivial on the space of 2d operators corresponding to scalar Schur operators. Therefore, it is a symmetry characterizing the OPEs of non-scalar Schur operators. Indeed, the invariance under this $SU(2)$ forbids various otherwise-possible terms in the OPEs of these operators.⁸ It would be an interesting open problem to see if this new symmetry corresponds to a 4d global symmetry without conserved current.

The organization of this paper is as follows. In section 2, we will briefly review the 4d/2d duality and the BRST interpretation of 4d marginal gaugings, following [1]. In section 3, we will collect aspects of the genus two theory $\mathcal{T}_{C_{2,0}}$, and describe our strategy of identifying the corresponding chiral algebra $\chi[\mathcal{T}_{C_{2,0}}]$. Sections 4 and 5 include our main results. In section 4, we identify the chiral algebra generators and their OPEs, and in section 5 we discuss automorphisms of $\chi[\mathcal{T}_{C_{2,0}}]$ including a new $SU(2)$ symmetry that is

⁶Note that, while chiral algebras of class $\mathcal{S}$ at genus zero are well-studied in [13] even for a general Lie algebra of the 6d (2,0) theory, its generalization to higher genus cases is still to be understood. See [61] for the derived extension of [13], which will be helpful for understanding the chiral algebra interpretation of gluings that increase the genus of the Riemann surface.

⁷By “conserved one-form current”, we mean $j = j_\mu dx^\mu$ such that $\partial_\mu j^\mu = 0$, as in [62]. The corresponding Noether charge is given by $\int *j$.

⁸Such a symmetry is also present in the chiral algebra studied in [29].

Multiplet	h	spin	$U(1)_r$
$\hat{\mathcal{B}}_R$	R	$(0, 0)$	0
$\mathcal{D}_{R(0, \bar{j})}$	$R + \bar{j} + 1$	$(0, \bar{j} + \frac{1}{2})$	$\bar{j} + \frac{1}{2}$
$\bar{\mathcal{D}}_{R(j, 0)}$	$R + j + 1$	$(j + \frac{1}{2}, 0)$	$-j - \frac{1}{2}$
$\hat{\mathcal{C}}_{R(j, \bar{j})}$	$R + j + \bar{j} + 2$	$(j + \frac{1}{2}, \bar{j} + \frac{1}{2})$	$\bar{j} - j$

Table 1. The list of Schur multiplets in [60] notation. h and r are a 2d holomorphic dimension and $U(1)_r$ charge of Schur operators, respectively.

not associated with any 4d conserved one-form current. The final section 6 is devoted to the conclusion and discussions about future works. While the genus two theory has two independent weak coupling descriptions, we focus on one of them. The S-dual equivalence between them is checked via the superconformal index with its flavor fugacity turned off [2]. In appendix A, we provide its extension to the case of non-vanishing flavor fugacity. In appendix B, we list the null operators in $\chi[\mathcal{T}_{C_{2,0}}]$ up to dimension six.

2 Brief review of the chiral algebra conjecture

In this section, we will review the duality between four-dimensional $\mathcal{N} = 2$ SCFTs and 2d chiral algebras, following [1] (see also [63] for a lecture note). Readers familiar with this 2d/4d correspondence can skip this section. We follow the convention of [1] unless otherwise stated.

2.1 General properties of associated chiral algebras

The map (1.1) from 4d $\mathcal{N} = 2$ SCFTs to chiral algebras was defined in [1] by considering cohomology classes of local operators with respect to a particular linear combination of Poincaré and conformal supercharges. At the origin, each cohomology classes is represented by a local operator annihilated by $\mathcal{Q}_-^1$, $\tilde{\mathcal{Q}}_{2-}$, $\mathcal{S}_1^-$ and $\tilde{\mathcal{S}}^{2-}$, which is called a Schur operator. The 4d unitarity implies that the quantum numbers of Schur operators satisfy

$$E = j + \bar{j} + 2R, \quad r = \bar{j} - j, \quad (2.1)$$

where E is the conformal dimension, $(j, \bar{j})$ is the Lorentz spin, R is the $SU(2)_R$ charge, and r is the $U(1)_r$ charge.

Schur operators are classified by superconformal multiplets containing them. We call an $\mathcal{N} = 2$ superconformal multiplet containing a Schur operator, a ‘‘Schur multiplet’’. In the language of [60], the only Schur multiplets are $\hat{\mathcal{B}}_R$, $\mathcal{D}_{R(0, \bar{j})}$, $\bar{\mathcal{D}}_{R(j, 0)}$ and $\hat{\mathcal{C}}_{R(j, \bar{j})}$, where $R, j, \bar{j}$ are quantum numbers of the bottom component in each multiplet. The Lorentz spin and $U(1)_r$ charge of a Schur operator in these multiplets are shown in table 1, together with the holomorphic dimensions of the corresponding 2d operators. Note that scalar Schur operators are only in $\hat{\mathcal{B}}_R$.

The OPEs of the chiral algebra $\chi[\mathcal{T}]$ associated with a 4d $\mathcal{N} = 2$ SCFT $\mathcal{T}$ is determined by OPEs of Schur operators in $\mathcal{T}$. Since the map from 4d OPEs to 2d ones involves twisting

the translation with $SU(2)_R$ rotation, the $SU(2)_R$ symmetry of $\mathcal{T}$ is not preserved by the OPEs of $\chi[\mathcal{T}]$. On the other hand, the flavor and $U(1)_r$ symmetries of $\mathcal{T}$ give rise to a conserved global symmetry of $\chi[\mathcal{T}]$.

The associated chiral algebra $\chi[\mathcal{T}]$ contains special sub-algebras uniquely fixed by the 4d symmetry of $\mathcal{T}$. In particular, the Virasoro sub-algebra of $\chi[\mathcal{T}]$ arises from the self-OPEs of the $SU(2)_R$ current in the stress tensor multiplet, $\hat{\mathcal{C}}_{0(0,0)}$. It is for this reason that the 2d Virasoro central charge, c_{2d} , is fixed by a 4d Weyl anomaly coefficient, c_{4d} , as⁹

$$c_{2d} = -12c_{4d}. \quad (2.2)$$

When $\mathcal{T}$ has a flavor symmetry G , the flavor current multiplet $\hat{\mathcal{B}}_1$ gives rise to the following affine Kac-Moody G sub-algebra in $\chi[\mathcal{T}]$:

$$J^A(z)J^B(0) \sim \frac{k_{2d}\delta^{AB}}{z^2} + \frac{if^{AB}{}_C J^C(0)}{z}, \quad (2.3)$$

where $f^{AB}{}_C$ is the structure constant of G , and $A, B, C = 1, \dots, \dim G$. The level, k_{2d} , of the affine Kac-Moody algebra is fixed by the 4d flavor central charge, k_{4d} , as¹⁰

$$k_{2d} = -\frac{1}{2}k_{4d}. \quad (2.4)$$

This implies that every 4d global symmetry that commutes with the $\mathcal{N} = 2$ superconformal symmetry gives rise to a corresponding affine current in $\chi[\mathcal{T}]$ unless the 4d symmetry has no conserved current.

The OPEs of the other 2d operators are more non-trivial, reflecting the 4d OPEs of Schur operators that are not fixed by the global symmetry of $\mathcal{T}$. Among these operators, those corresponding to $\hat{\mathcal{B}}_R$ are relatively well-understood, since the corresponding Schur operators are Higgs branch operators whose vacuum expectation values parameterize the Higgs branch of $\mathcal{T}$. Indeed, the OPEs of 2d operators corresponding to $\hat{\mathcal{B}}_R$ are often reconstructed from the Higgs branch chiral ring of $\mathcal{T}$. The Hall-Littlewood (HL) chiral ring is an extension of the Higgs branch chiral ring by Schur operators in $\bar{\mathcal{D}}_{R(j,0)}$.¹¹ Similarly, HL anti-chiral ring is defined for $\mathcal{D}_{R(0,\bar{j})}$ and $\hat{\mathcal{B}}_R$. A common feature of these multiplets, $\bar{\mathcal{D}}_{R(j,0)}$, $\mathcal{D}_{R(0,\bar{j})}$ and $\hat{\mathcal{B}}_R$, is that the corresponding 2d operators are guaranteed to be Virasoro primary operators.

When the 4d theory $\mathcal{T}$ has a Lagrangian description, Schur operators in the $\mathcal{D}_{R(0,\bar{j})}$ and $\bar{\mathcal{D}}_{R(j,0)}$ multiplets are gauge invariant composite operators involving at least one gaugino in a vector multiplet. Therefore, for Lagrangian theories, 2d operators corresponding to $\bar{\mathcal{D}}_{R(j,0)}$ or $\mathcal{D}_{R(0,\bar{j})}$ are always nilpotent with respect to the normal ordered product. The simplest example of $\mathcal{D}_{R(0,\bar{j})}, \bar{\mathcal{D}}_{R(j,0)}$ type multiplets is an $\mathcal{N} = 2$ free vector multiplet, $\mathcal{D}_{0(0,0)} \oplus \bar{\mathcal{D}}_{0(0,0)}$, whose associated chiral algebra is the small (b, c) -ghost system.¹² Its OPE

⁹Here, we normalized c_{4d} such that a free hypermultiplet has $c_{4d} = \frac{1}{12}$.

¹⁰We use the convention that the $k_{4d} = 2$ for a hypermultiplet in the fundamental representation of G .

¹¹These operators are those contributing to the HL limit of the superconformal index [3].

¹²The “small” means there are no c_0 mode in the spectrum, and on the other hands, full (b, c) -ghost system contain c_0 .

is expressed in terms of $\lambda := b, \tilde{\lambda} := \partial c$ as

$$\lambda(z)\tilde{\lambda}(0) \sim \frac{1}{z^2}. \quad (2.5)$$

It is also known that, in A_1 type class $\mathcal{S}$ theories, $\bar{\mathcal{D}}_{R(j,0)}$ and $\bar{\mathcal{D}}_{R(0,\bar{j})}$ exist only if the genus of the Riemann surface is non-zero.

The last type of Schur multiplet, $\hat{\mathcal{C}}_{R(j,\bar{j})}$, is not well-understood, except for the stress tensor multiplet $\hat{\mathcal{C}}_{0(0,0)}$. As mentioned above, the Schur operator in $\hat{\mathcal{C}}_{0(0,0)}$ is the highest weight component of the $SU(2)_R$ conserved current, and corresponds to the 2d stress tensor in the associated chiral algebra. The chiral algebra nature of the other $\hat{\mathcal{C}}$ -type multiplets are, however, not very clear. Indeed, unlike the other Schur multiplets, a chiral algebra generator corresponding to $\hat{\mathcal{C}}_{R(j,\bar{j})}$ is not guaranteed to be a Virasoro primary operator.

2.2 BRST reduction corresponding to gauging

Let $\mathcal{T}$ be a 4d $\mathcal{N} = 2$ SCFT with a flavor symmetry G whose flavor central charge is $k_{4d} = 4h^\vee$, where $h^\vee$ is the dual Coxeter number of G . In this case, we can construct a new $\mathcal{N} = 2$ SCFT, $\mathcal{T}_G$, by marginally gauging the flavor G symmetry of $\mathcal{T}$. Indeed, $k_{4d} = 4h^\vee$ precisely coincides with the condition for a vanishing β -function.

On the associated chiral algebra side, this procedure corresponds to considering a BRST cohomology of the tensor product of $\chi[\mathcal{T}]$ and the small (b, c) -ghost system in the adjoint representation of G . To describe this, let us consider the following BRST charge

$$Q_{\text{BRST}} := \oint \frac{dz}{2\pi i} J_{\text{BRST}}, \quad J_{\text{BRST}} := c_A \left(J^A + \frac{1}{2} J_{\text{gh}}^A \right), \quad (2.6)$$

where J^A for $A = 1, \dots, \dim G$ is the affine G currents in $\chi[\mathcal{T}]$, and

$$J_{\text{gh}}^A := -i f^{ABC} b^B c^C \quad (2.7)$$

is the affine G currents in the (b, c) -ghost sector. Note that $k_{4d} = 4h^\vee$ implies the affine currents J^A have level $k_{2d} = -2h^\vee$, which is precisely the condition that $Q_{\text{BRST}}^2 = 0$. When the gauge coupling is turned off, the chiral algebra of the $\mathcal{T}_G$ is given by

$$\left\{ |\psi\rangle \in \chi[\mathcal{T}] \otimes (b^A, \partial c_A) \mid J_{\text{tot},0}^A |\psi\rangle = 0 \right\}, \quad (2.8)$$

where $J_{\text{tot},0}^A$ is the zero-mode of $J_{\text{tot}}^A := J^A + J_{\text{gh}}^A$, and $(b^A, \partial c_A)$ stands for the small (b, c) -ghost system associated with G . The constraint $J_{\text{tot},0}^A |\psi\rangle = 0$ corresponds to the Gauss law constraint that is present even for the zero gauge coupling. When the gauge coupling is turned on, some Schur operators are lifted to non-Schur operators, giving rise to a reduced chiral algebra. This reduced algebra was conjectured in [1] to be given by the Q_{BRST} -cohomology

$$\chi[\mathcal{T}_G] = H_{Q_{\text{BRST}}} \left(\left\{ |\psi\rangle \in \chi[\mathcal{T}] \otimes (b^A, \partial c_A) \mid J_{\text{tot},0}^A |\psi\rangle = 0 \right\} \right). \quad (2.9)$$

As shown in [1], the associated chiral algebra is independent of marginal couplings of the four-dimensional theory. Therefore (2.9) is identified as the chiral algebra of $\mathcal{T}_G$ at any generic value of the gauge coupling.

In section 4, we will use the above prescription to identify the chiral algebra $\chi[\mathcal{T}_{C_{2,0}}]$ associated with genus two class $\mathcal{S}$ theory.

Figure 1. Two weak-coupling descriptions of the theory $\mathcal{T}_{C_{2,0}}$, which are S-dual to each other. Each circle node stands for an $SU(2)$ gauge group, while each triangle node stands for the T_2 theory, i.e., the theory of four free hypermultiplets. Each corner of the triangles stands for an $SU(2)$ flavor subgroup of T_2 . When a corner of a triangle is connected to a circle, the corresponding $SU(2)$ flavor subgroup of T_2 is coupled to the gauge group associated with the circle.

3 Genus two theory

The main purpose of this paper is to identify the chiral algebra of the genus two class $\mathcal{S}$ theory, $\mathcal{T}_{C_{2,0}}$, obtained by compactifying the six-dimensional $\mathcal{N} = (2, 0), A_1$ theory on a genus two Riemann surface $C_{2,0}$. Here we collect known facts about this theory, mainly following section 5.4 of [1], so that we can use them in identifying its chiral algebra in the next section.

3.1 Weak coupling descriptions

The $\mathcal{T}_{C_{2,0}}$ theory has two different weak coupling descriptions, as shown in quiver gauge theories in figure 1, corresponding to two different pants decompositions of $C_{2,0}$. These two descriptions are expected to be S-dual to each other. In figure 1, each circle node stands for an $SU(2)$ gauge group, while each triangle node stands for the T_2 theory, i.e., the theory of four free hypermultiplets. The T_2 theory has $Sp(4)$ flavor symmetry, whose $SU(2)^3$ subgroup is manifest in figure 1; each corner of the triangles stands for one $SU(2)$ flavor subgroup. When a corner of a triangle attaches to a circle, the corresponding $SU(2)$ flavor subgroup is gauged by the gauge group associated with the circle.

The above $SU(2)^3$ flavor subgroup of the T_2 theory can be explicitly seen as follows. In the $\mathcal{N} = 1$ language, the T_2 theory is composed of eight chiral multiplets, which we denote by Q_{ab} and $\tilde{Q}^{ab}$ for $a, b = 1, 2$. There is an obvious $\mathcal{N} = 2$ flavor $SU(2)^2$ symmetry under which Q_{ab} and $\tilde{Q}^{ab}$ transform respectively as $\mathbf{2} \otimes \mathbf{2}$ and $\bar{\mathbf{2}} \otimes \bar{\mathbf{2}}$.¹³ In addition, there exists an extra flavor $SU(2)$ symmetry under which (Q_{1bc}, Q_{2bc}) defined by

$$Q_{1bc} := Q_{bc}, \quad Q_{2bc} := \epsilon_{bd}\epsilon_{ce}\tilde{Q}^{de}, \quad (3.1)$$

transforms as a fundamental representation, where ϵ_{ab} is the anti-symmetric tensor such that $\epsilon^{12} = -\epsilon_{12} = 1$. Therefore, Q_{abc} can be regarded as a “half-hypermultiplet” in the trifundamental representation, $\mathbf{2} \otimes \mathbf{2} \otimes \mathbf{2}$, of $SU(2)^3$ [19].

As mentioned in section 1, the genus two theory $\mathcal{T}_{C_{2,0}}$ has an accidental flavor $U(1)_f$ symmetry that is not visible in its class $\mathcal{S}$ construction (see [64] for similar accidental

¹³The indices, a and b , express components of the fundamental or anti-fundamental representation of $SU(2)$ here.

enhancements of flavor symmetries of A_1 -type class $\mathcal{S}$ theories). As a result, the most general expression for its superconformal index contains a fugacity for this flavor $U(1)_f$ symmetry. The explicit action of this $U(1)_f$ will be shown in sub-section 3.2.

The S-duality equivalence between the two quiver descriptions in figure 1 has been checked in terms of the Schur limit of their superconformal indices with the flavor fugacity turned off [2]. The extension of their proof to the case with a non-vanishing flavor fugacity is discussed in appendix A of this paper. Given these results, we assume the S-duality equivalence between the two quiver descriptions. In the rest of this paper, we mainly focus on the left description in figure 1, which we call the “dumbbell quiver”.

3.2 Higgs branch and flavor symmetry

In this sub-section, we describe the $U(1)_f$ flavor symmetry and a chiral ring relation of the genus two theory $\mathcal{T}_{C_{2,0}}$. We focus on the dumbbell quiver description of the theory (i.e., the left quiver in figure 1), and denote by Q_{abc} and S_{def} two trifundamental half-hypermultiplets of $SU(2)^3$. Let $SU(2)_1$, $SU(2)_2$ and $SU(2)_3$ be the gauge groups corresponding to the left, middle and right circles in the quiver, respectively. Without loss of generality, we say $SU(2)_1$ is diagonally gauging the $SU(2) \times SU(2)$ symmetry corresponding to b and c of Q_{abc} , and $SU(2)_3$ is diagonally gauging the $SU(2) \times SU(2)$ symmetry corresponding to b and c of S_{abc} . The remaining gauge group $SU(2)_2$ is diagonally gauging $SU(2)$ corresponding to a of Q_{abc} and S_{abc} . Then we see that, under $SU(2)_1$, the left half-hypermultiplet Q_{abc} is decomposed into $\mathbf{1} \oplus \mathbf{3}$ as¹⁴

$$Q_a := \frac{1}{\sqrt{2}} \epsilon^{bc} Q_{abc}, \quad Q_a^A := \sigma^{Abc} Q_{abc}. \quad (3.2)$$

Similarly, S_{abc} also decomposes into $\mathbf{1} \oplus \mathbf{3}$ under $SU(2)_3$ as

$$S_a := \frac{1}{\sqrt{2}} \epsilon^{bc} S_{abc}, \quad S_a^A := \sigma^{Bbc} S_{abc}. \quad (3.3)$$

Note that these are all in the fundamental representation of $SU(2)_2$. It is also useful for us below to define

$$\phi_a := \frac{1}{\sqrt{2}} (Q_a + iS_a), \quad \bar{\phi}_a := \frac{1}{\sqrt{2}} (Q_a - iS_a). \quad (3.4)$$

The superpotential of the dumbbell quiver is then written as

$$\begin{aligned} W_{C_{2,0}} = & 4if_{AB}{}^C Q_a^A Q_b^B \Phi_C^{(1)} \epsilon^{ab} + 4if_{DE}{}^F S_d^D S_e^E \Phi_F^{(3)} \epsilon^{de} \\ & + 2(Q_a^A Q_{Ab} + S_a^D S_{Db} + \phi_a \bar{\phi}_b) \sigma^{Aab} \Phi_A^{(2)}, \end{aligned} \quad (3.5)$$

where $\Phi_A^{(i)}$ is the chiral superfield arising from the vector multiplet associated with $SU(2)_i$.

From the above superpotential, we see that there is a flavor $U(1)_f$ symmetry under which ϕ_a and $\bar{\phi}_a$ have charge +1 and −1 while all the other fields are neutral. The

¹⁴The σ matrices and antisymmetric tensor, ϵ , are in the same convention with [65] and $(\sigma^A)_{ab} := \epsilon_{ac}(\sigma^A)^c{}_b$, $(\sigma^A)^{ab} := \epsilon^{bc}(\sigma^A)^a{}_c$. $(\sigma^A)_{ab}$ are symmetric tensor.

generators of the Higgs branch chiral ring can also be read off from the above superpotential. Indeed, it is generated by the moment map of $U(1)_f$ ¹⁵

$$M := -\phi_a \bar{\phi}^a, \quad (3.6)$$

and the following Schur operators in the $\widehat{\mathcal{B}}_2$ multiplet:

$$\mathcal{O}_1 := 2\phi_a \phi_b Q_A^a Q^{Ab}, \quad \mathcal{O}_2 := 2\bar{\phi}_a \bar{\phi}_b Q_A^a Q^{Ab}, \quad (3.7)$$

where $\bar{\phi}^a := \epsilon^{ab} \bar{\phi}_b$ and $Q_A^a := \epsilon^{ab} Q_{Ab}$. These Higgs branch operators satisfy the chiral ring relation

$$\mathcal{O}_1 \mathcal{O}_2 = M^4, \quad (3.8)$$

as seen from the superpotential [66].

Note that, when taking the decoupling limit of the middle gauge group $SU(2)_2$, the second line of (3.5) vanishes. Then the theory splits into the following three sectors:

- $\mathcal{N} = 4$ $SU(2)$ super Yang-Mills (SYM) theory described by $(Q_a^A, \Phi_C^{(1)})$.
- $\mathcal{N} = 4$ $SU(2)$ SYM theory described by $(S_a^A, \Phi_C^{(3)})$.
- a fundamental hypermultiplet of $SU(2)$ described by $(\phi_a, \bar{\phi}_a)$.

This means that the genus two theory $\mathcal{T}_{C_{2,0}}$ can be regarded as a theory obtained by marginally gauging a diagonal $SU(2)_2$ global symmetry of the above three sectors.¹⁶ We will use this construction of $\mathcal{T}_{C_{2,0}}$ in identifying its associated chiral algebra in the next section.

4 Chiral algebra of genus two theory

In this section, we construct the chiral algebra $\chi[\mathcal{T}_{C_{2,0}}]$ associated with the genus two theory $\mathcal{T}_{C_{2,0}}$. We regard $\mathcal{T}_{C_{2,0}}$ as a theory obtained by marginally gauging the three sectors discussed at the end of the previous section. Then $\chi[\mathcal{T}_{C_{2,0}}]$ is constructed via the BRST reduction corresponding to this marginal gauging.

4.1 Chiral algebras of the three sectors

The chiral algebras of the three sectors discussed at the end of sub-section 3.2 are already identified.

The chiral algebra of a fundamental hypermultiplet of $SU(2)$ is the symplectic boson algebra generated by ϕ^{ai} for $a = 1, 2$ and $i = \pm$ such that

$$\phi^{a\pm}(z) \phi^{b\mp}(0) \sim \frac{\epsilon^{ab}}{z}. \quad (4.1)$$

¹⁵The flavor moment map is a Schur operator in the flavor current multiplet $\widehat{\mathcal{B}}_1$.

¹⁶Note that $\mathcal{N} = 4$ SYM theory has an $\mathcal{N} = 2$ flavor $SU(2)$ symmetry.

Here, the 2d operators ϕ^{a+} and ϕ^{a-} correspond respectively to the 4d Schur operators ϕ^a and $\bar{\phi}^a$ described in (3.4).¹⁷ Recall that the 4d $U(1)_r$ symmetry is generally preserved by 2d OPEs. Since the 4d Schur operators ϕ^a and $\bar{\phi}^a$ are neutral under $U(1)_r$, so are $\phi^{a\pm}$. This chiral algebra has $\widehat{\mathfrak{sp}(4)}$ currents, which contains as sub-algebras the $\widehat{\mathfrak{su}(2)}$ currents

$$J_{\text{matter}}^A := -\frac{1}{2}\sigma_{ab}^A \phi^{a+} \phi^{b-}, \quad (4.2)$$

and the $\widehat{\mathfrak{u}(1)}$ current

$$J := \epsilon_{ab} \phi^{a+} \phi^{b-}. \quad (4.3)$$

In the BRST reduction discussed below, J_{matter}^A will be involved in the BRST current and therefore disappears from the spectrum. On the other hand, J will give rise to a non-trivial BRST cohomology class corresponding to the flavor $U(1)_f$ symmetry of $\mathcal{T}_{C_{2,0}}$.

Let us now turn to the other sectors, i.e., $\mathcal{N} = 4$ $SU(2)$ SYM theories. The chiral algebra associated with the $\mathcal{N} = 4$ $SU(2)$ SYM was conjectured in [1] to be the small $\mathcal{N} = 4$ super Virasoro algebra at the Virasoro central charge $c = -9$. This algebra is generated by $\widehat{\mathfrak{su}(2)}$ currents J^A at level $-3/2$, and $\mathcal{N} = 4$ supercurrents $G^a, \bar{G}^a$. Their non-trivial OPEs are given by

$$J^A(z)J^B(0) \sim -\frac{3}{4}\frac{\delta^{AB}}{z^2} + \frac{if^{ABC}J_C}{z}, \quad (4.4)$$

$$J^A(z)G^a(0) \sim -\frac{1}{2}\frac{(\sigma^A)^a_b G^b}{z}, \quad (4.5)$$

$$J^A(z)\bar{G}^a(0) \sim -\frac{1}{2}\frac{(\sigma^A)^a_b \bar{G}^b}{z}, \quad (4.6)$$

$$G^a(z)\bar{G}^b(0) \sim -\frac{6\epsilon^{ab}}{z^3} + \frac{4(\sigma^A)^{ab}J_A}{z^2} + \frac{2(\sigma^A)^{ab}\partial J_A + 2\epsilon^{ab}T_{\text{sug}}}{z}, \quad (4.7)$$

where $f^{ABC} = \epsilon^{ABC}$ is the structure constant of $SU(2)$, and $T_{\text{sug}} := 2J_A J_A$ is the Sugawara stress tensor. The $U(1)_r$ charges of J^A, G^a and $\bar{G}^a$ are respectively 0, $\frac{1}{2}$ and $-\frac{1}{2}$.¹⁸ Note that the above OPEs preserve this $U(1)_r$ symmetry as expected.

The small $\mathcal{N} = 4$ Virasoro algebra at $c = -9$ has various null operators, which we need to remove from the spectrum. Fortunately, there is a special free field realization of this algebra which makes all these null operators automatically vanishing [67]. As shown in [12], this free field realization is concisely expressed in terms of a $\beta\gamma bc$ ghost system. To describe it, let us change variables as $J^\pm := -i(J^1 \pm iJ^2)$, $J^0 := 2J^3$, $G^+ := -\frac{1+i}{2}G^2$, $G^- := \frac{1-i}{2}G^1$, $\bar{G}^+ := -\frac{1+i}{2}\bar{G}^2$ and $\bar{G}^- := \frac{1-i}{2}\bar{G}^1$. Then the free field realization is given by

$$J^+ = \beta, \quad J^0 = bc + 2\beta\gamma, \quad J^- = \beta\gamma\gamma + \gamma bc - \frac{3}{2}\partial\gamma,$$

¹⁷The 2d operators $q^{a,1} := \frac{1}{\sqrt{2}}(\phi^{a+} + \phi^{a-})$ and $q^{a,2} := \frac{1}{\sqrt{2}i}(\phi^{a+} - \phi^{a-})$ correspond to the 4d Schur operators Q^a and S^a , respectively. Their 2d OPEs are given by $q^{a,i}(z)q^{b,j}(0) \sim \epsilon^{ab}\delta^{ij}/z$.

¹⁸The Schur operators corresponding to G^a and $\bar{G}^a$ are in $\mathcal{D}_{\frac{1}{2}(0,0)}$ and $\bar{\mathcal{D}}_{\frac{1}{2}(0,0)}$, respectively. Their $U(1)_r$ charges can be seen from table 1.

$$G^+ = b^i, \quad G^- = b\gamma, \quad \bar{G}^+ = c\partial\beta + 2(\partial c)\beta, \quad \bar{G}^- = -b(\partial c)c + 2\beta\gamma\partial c + \partial\beta\gamma c - \frac{3}{2}\partial^2 c.$$

Here the OPEs of the $\beta\gamma bc$ system are given by

$$\beta(z)\gamma(0) \sim -\frac{1}{z}, \quad b(z)c(0) \sim \frac{1}{z}. \quad (4.8)$$

In the next sub-section, we use this free field realization for each of the two $\mathcal{N} = 4$ SYM sectors. Note that b and c that appear in the above free field realization have nothing to do with the bc ghost system arising in the BRST reduction.

4.2 BRST reduction

Let us now consider marginally gauging the three sectors to obtain the genus two theory $\mathcal{T}_{C_{2,0}}$. As reviewed in sub-section 2.2, on the chiral algebra side, this corresponds to a BRST reduction of the tensor product of chiral algebras associated with these three sectors.

Recall that two of the three sectors are described by $\mathcal{N} = 4$ $SU(2)$ SYM. For $k \in \{1, 2\}$, let $J_{(k)}^A, G_{(k)}^a$ and $\bar{G}_{(k)}^a$ be the generators of the small $\mathcal{N} = 4$ super Virasoro algebra (at $c = -9$) associated with the k -th $\mathcal{N} = 4$ SYM sector. The remaining sector is the theory of a fundamental hypermultiplet of $SU(2)$, whose chiral algebra is generated by the symplectic bosons $\phi^{a\pm}$ such that (4.1).

In the tensor product of these three chiral algebras,

$$J^A := J_{(1)}^A + J_{(2)}^A + J_{\text{matter}}^A \quad (4.9)$$

is the $\widehat{\mathfrak{su}(2)}$ current corresponding to the 4d $SU(2)$ symmetry that we are gauging, where J_{matter}^A is given by (4.2). With this J^A , the relevant BRST current is given by

$$J_{\text{BRST}} = c_A \left(J^A + \frac{1}{2} J_{\text{gh}}^A \right), \quad (4.10)$$

where J_{gh}^A is defined in (2.7). We see that $Q_{\text{BRST}} = \oint \frac{dz}{2\pi i} J_{\text{BRST}}$ is nilpotent, which reflects the fact that the corresponding $SU(2)$ gauging is exactly marginal in four dimensions.

According to the gauging prescription reviewed in sub-section 2.2, the chiral algebra $\chi[\mathcal{T}_{C_{2,0}}]$ is identified as the following BRST cohomology:

$$\chi[\mathcal{T}_{C_{2,0}}] = H_{Q_{\text{BRST}}} \left(\left\{ |\psi\rangle \in \left(\bigotimes_{k=1}^2 (J_{(k)}^A, G_{(k)}^a, \bar{G}_{(k)}^a) \right) \otimes (\phi^{a\pm}) \otimes (b^A, \partial c^A) \middle| J_{\text{tot},0}^A |\psi\rangle = 0 \right\} \right), \quad (4.11)$$

where $(x, y, \dots)$ stands for the chiral algebra generated by $x, y, \dots$, and $J_{\text{tot}}^A := J^A + J_{\text{gh}}^A$. We evaluate this cohomology explicitly, using the Mathematica package `OPedefs` developed by [68, 69]. While this computation is rather involved, one can simplify it by using the free field realization of $(J_{(k)}^A, G_{(k)}^a, \bar{G}_{(k)}^a)$ reviewed in the previous sub-section.¹⁹ Evaluating the cohomology classes in (4.11) up to a high order of holomorphic dimensions of operators, we find that there exists a set, $\mathbb{S}$, of operators in $\chi[\mathcal{T}_{C_{2,0}}]$ with the following properties.

¹⁹One thing that makes this computation involved is the presence of various null operators in $(J_{(k)}^A, G_{(k)}^a, \bar{G}_{(k)}^a)$ that we need to remove from the spectrum. Using the free field realization, all these null operators are automatically zero, which simplifies the computation with Mathematica.

2d generator	h	f	r	4d multiplet
J	1	0	0	$\hat{\mathcal{B}}_1$
B^+	2	2	0	$\hat{\mathcal{B}}_2$
B^-	2	-2	0	$\hat{\mathcal{B}}_2$
D^{+I}	2	1	$\frac{1}{2}$	$\mathcal{D}_{1(0,0)}$
$\bar{D}_I^+$	2	1	$-\frac{1}{2}$	$\bar{\mathcal{D}}_{1(0,0)}$
D^{-I}	2	-1	$\frac{1}{2}$	$\mathcal{D}_{1(0,0)}$
$\bar{D}_I^-$	2	-1	$-\frac{1}{2}$	$\bar{\mathcal{D}}_{1(0,0)}$
T	2	0	0	$\hat{\mathcal{C}}_{0(0,0)}$
X	3	0	1	$\mathcal{D}_{\frac{3}{2}(0,\frac{1}{2})}$
$\bar{X}$	3	0	-1	$\bar{\mathcal{D}}_{\frac{3}{2}(\frac{1}{2},0)}$
C_A	3	0	0	$\hat{\mathcal{C}}_{1(0,0)}$

Table 2. The list of generators of genus two chiral algebra that we found. Here, h is the holomorphic dimension of operators, f is the global $U(1)$ charge associated with the $\widehat{\mathfrak{u}(1)}$ current J , and r is the $U(1)_r$ charge. We conjecture that this is the complete set of generators. Note that the indices $I = \uparrow, \downarrow$ and $A = 1, 2, 3$ have nothing to do with the global $U(1)$ symmetry. They play an important role in the study of automorphisms of $\chi[\mathcal{T}_{C_{2,0}}]$ in section 5.2.

- The operators in $\mathbf{S}$ are of dimension less than or equal to three.
- Every Q_{BRST} -cohomology class of dimension less than or equal to six is either an operator in $\mathbf{S}$, the derivative of an operator in $\mathbf{S}$, or a normal ordered product of these operators.
- The operators in $\mathbf{S}$ form closed OPEs.

The list of operators in $\mathbf{S}$ is shown in table 2 with their quantum numbers and corresponding 4d superconformal multiplets.

Given these facts, we *conjecture* that the above $\mathbf{S}$ is the complete set of generators of the chiral algebra $\chi[\mathcal{T}_{C_{2,0}}]$. Similar conjectures have been made for various theories [1, 28, 40], leading to consistent results. We will perform a further consistency check of our conjecture in sub-section 4.5. In the next sub-section, we describe which Q_{BRST} -cohomology classes correspond to the operators in $\mathbf{S}$.

Note that, as shown in [1], the Macdonald index of $\mathcal{T}_{C_{2,0}}$ implies that there are *at least* those generators listed in table 2. Although the index computation does not imply the absence of extra generators, we have shown here that there is no extra generators up to dimension six. Combined with the fact that they form closed OPEs, this is a strong evidence for the above conjecture.

4.3 List of generators of $\chi[\mathcal{T}_{C_{2,0}}]$

We here describe the Q_{BRST} -cohomology classes corresponding to the generators listed in table 2, in terms of their representatives. Note that these cohomology classes are labeled by three quantum numbers (h, r, f) , where h is the holomorphic dimension, r is $U(1)_r$ charge, and f is flavor $U(1)_f$ charge.

First, at dimension one, the only generator is the $\widehat{\mathfrak{u}(1)}$ current

$$J = \epsilon_{ab} \phi^{a+} \phi^{b-}. \quad (4.12)$$

At dimension two, there are three independent bosonic generators. The one with vanishing global $U(1)_f$ charge is the stress tensor²⁰

$$T = \sum_{i=1}^2 T_{\text{sug}(i)} - \frac{1}{2} \epsilon_{ab} (\phi^{a+} \partial \phi^{b-} + \phi^{a-} \partial \phi^{b+}) - b^A \partial c_A, \quad (4.13)$$

where $T_{\text{sug}(i)} := J_{(i)}^A J_{(i)A}$. Since the genus two theory $\mathcal{T}_{C_{2,0}}$ has $c_{4d} = 13/6$, the Virasoro central charge is $c_{2d} = -26$. The other two bosonic operators are written as

$$B^\pm = -2 \left(J_{(1)}^A - J_{(2)}^A \right) \sigma_{Aab} \phi^{a\pm} \phi^{b\pm}. \quad (4.14)$$

These operators have $U(1)_f$ charge ± 2 , and correspond to the 4d Higgs branch operators, $\mathcal{O}_1$ and $\mathcal{O}_2$, shown in (3.7).

At dimension two, there are also the following fermionic generators:

$$D^{\pm\uparrow} = -\epsilon_{ab} \phi^{a\pm} \left(iG_{(1)}^b \pm G_{(2)}^b \right), \quad D^{\pm\downarrow} = -\epsilon_{ab} \phi^{a\pm} \left(iG_{(1)}^b \mp G_{(2)}^b \right), \quad (4.15)$$

$$\bar{D}_\uparrow^\pm = -\epsilon_{ab} \phi^{a\pm} \left(i\bar{G}_{(1)}^b \pm \bar{G}_{(2)}^b \right), \quad \bar{D}_\downarrow^\pm = -\epsilon_{ab} \phi^{a\pm} \left(i\bar{G}_{(1)}^b \mp \bar{G}_{(2)}^b \right). \quad (4.16)$$

The $U(1)_f$ charge of $D^{\pm I}$ and $\bar{D}_I^\pm$ are both ± 1 . Here, one could instead take $\epsilon_{ab} \phi^{a\pm} G_{(i)}^b$ and $\epsilon_{ab} \phi^{a\pm} \bar{G}_{(i)}^b$ as independent operators, but we will see below that the above linear combinations make automorphisms of $\chi[\mathcal{T}_{C_{2,0}}]$ more transparent. From their $U(1)_r$ charges, we see that the corresponding Schur multiplets of $D^{\pm I}$ and $\bar{D}_I^\pm$ are as in table 2.

At dimension three, there are five generators, which are all neutral under the global $U(1)_f$ symmetry. We split them into two parts;

$$X = i\epsilon_{ab} G_{(1)}^a G_{(2)}^b, \quad \bar{X} = -i\epsilon_{ab} \bar{G}_{(1)}^a \bar{G}_{(2)}^b, \quad (4.17)$$

and

$$C_1 = 4J_{(1)}^A J_{(1)A} \epsilon_{ab} \phi^{a+} \phi^{b-} + 2J_{(1)}^A \sigma_{Aab} \left(\phi^{a+} \partial \phi^{b-} - \phi^{a-} \partial \phi^{b+} \right) - (1 \leftrightarrow 2), \quad (4.18)$$

$$C_2 = \epsilon_{ab} (G_{(1)}^a \bar{G}_{(2)}^b + G_{(2)}^a \bar{G}_{(1)}^b), \quad (4.19)$$

$$C_3 = i\epsilon_{ab} (G_{(1)}^a \bar{G}_{(2)}^b - G_{(2)}^a \bar{G}_{(1)}^b). \quad (4.20)$$

From their $U(1)_r$ charges, we see that X and $\bar{X}$ correspond to Schur operators in $\mathcal{D}_{\frac{3}{2}(0, \frac{1}{2})}$ and $\bar{\mathcal{D}}_{\frac{3}{2}(\frac{1}{2}, 0)}$, and similarly C_A correspond to $\hat{\mathcal{C}}_{1(0,0)}$.

²⁰Note that T cannot be the Sugawara-type stress tensor associated with J since $c_{2d} = -26$.

4.4 OPEs of generators

Here, we describe the OPEs of generators listed in table 2. Note that these are uniquely fixed by the BRST cohomology computation. First, the OPEs of T and J are given by

$$T(z)T(0) \sim -\frac{13}{z^4} + \frac{2T}{z^2} + \frac{\partial T}{z}, \quad J(z)J(0) \sim -\frac{2}{z^2}, \quad T(z)J(0) \sim \frac{J}{z^2} + \frac{\partial J}{z}. \quad (4.21)$$

Recall here that the Virasoro central charge $c_{2d} = -26$ follows from the 4d central charge $c_{4d} = 13/6$.

Next, we find that all the other generators in table 2 are primary operators in the sense of the Virasoro sub-algebra generated by T and the $\widehat{\mathfrak{u}(1)}$ sub-algebra generated by J . Note that, while operators corresponding to $\widehat{B}$, $\mathcal{D}$ and $\bar{\mathcal{D}}$ -type multiplets are guaranteed to be Virasoro primaries [1], those corresponding to $\hat{\mathcal{C}}$ -type multiplets are not. Therefore, this result is already non-trivial. The fact that they are primary operators implies that their non-vanishing OPEs with J and T are the following:

$$J(z)B^\pm(0) \sim \pm \frac{2B^\pm}{z}, \quad T(z)B^\pm(0) \sim \frac{2B^\pm}{z^2} + \frac{\partial B^\pm}{z}, \quad (4.22)$$

$$J(z)D^{\pm I}(0) \sim \pm \frac{D^{\pm I}}{z}, \quad T(z)D^{\pm I}(0) \sim \frac{2D^{\pm I}}{z^2} + \frac{\partial D^{\pm I}}{z}, \quad (4.23)$$

$$J(z)\bar{D}_I^\pm(0) \sim \pm \frac{\bar{D}_I^\pm}{z}, \quad T(z)\bar{D}_I^\pm(0) \sim \frac{2\bar{D}_I^\pm}{z^2} + \frac{\partial \bar{D}_I^\pm}{z}, \quad (4.24)$$

$$T(z)X(0) \sim \frac{3X}{z^2} + \frac{\partial X}{z}, \quad T(z)\bar{X}(0) \sim \frac{3\bar{X}}{z^2} + \frac{\partial \bar{X}}{z}, \quad T(z)C_A(0) \sim \frac{3C_A}{z^2} + \frac{\partial C_A}{z}. \quad (4.25)$$

The OPEs of $B^\pm, D^{\pm I}, \bar{D}_I^\pm, X, \bar{X}$ and C_A are more non-trivial. We evaluate them to find that the only non-vanishing OPEs of these operators are the following. First, the OPEs of dimension-two operators are

$$B^+(z)B^-(0) \sim \frac{72}{z^4} - \frac{72J}{z^3} + \frac{-16T - 36\partial J + 32J^2}{z^2} + \frac{-20\partial^2 J - 8J^3 + 16\partial J^2 - 8\partial T + 16TJ}{z}, \quad (4.26)$$

$$B^\pm(z)D^{\mp I}(0) \sim \frac{6D^{\pm I}}{z^2} \mp \frac{4D^{\pm I}J}{z}, \quad B^\pm(z)\bar{D}_I^\mp(0) \sim \frac{6\bar{D}_I^\pm}{z^2} \mp \frac{4\bar{D}_I^\pm J}{z}, \quad (4.27)$$

$$D^{+I}(z)D^{-J}(0) \sim \frac{-2\epsilon^{IJ}X}{z}, \quad \bar{D}_I^+(z)\bar{D}_J^-(0) \sim \frac{-2\epsilon_{IJ}\bar{X}}{z}, \quad (4.28)$$

$$D^{\pm I}(z)\bar{D}_J^\pm(0) \sim \delta_J^I \left(\frac{2B^\pm}{z^2} + \frac{\partial B^\pm}{z} \right), \quad (4.29)$$

$$D^{\pm I}(z)\bar{D}_J^\mp(0) \sim \delta_J^I \left(\frac{24}{z^4} \mp \frac{12J}{z^3} + \frac{(2J^2 - 4T \mp 6\partial J)}{z^2} \right) + \delta_J^I \frac{\partial(J^2 - 2T) \pm (2TJ - 3\partial^2 J)}{z} \pm \frac{(\sigma^A)^I{}_J C_A}{z}, \quad (4.30)$$

where ϵ_{IJ} and ϵ^{IJ} are anti-symmetric tensors such that $\epsilon^{\uparrow\downarrow} = \epsilon_{\downarrow\uparrow} = 1$, and $(\sigma^A)^I{}_J$ is the (I, J) -element of the Pauli matrix σ^A .

The OPEs of dimension-two and three operators are

$$B^\pm(z)X(0) \sim \frac{\pm\epsilon_{IJ}D^{\pm I}D^{\pm J}}{z}, \quad B^\pm(z)\bar{X}(0) \sim \frac{\pm\epsilon^{IJ}\bar{D}_I^\pm\bar{D}_J^\pm}{z} \quad (4.31)$$

$$B^\pm(z)C^A(0) \sim \frac{\pm 2(\sigma^A)^I{}_J D^{\pm J}\bar{D}_I^\pm}{z}, \quad (4.32)$$

$$X(z)\bar{D}_I^\pm(0) \sim \epsilon_{IK} \left(\frac{\pm 6D^{\pm K}}{z^3} + \frac{2(JD^{\pm K} \pm 2\partial D^{\pm K})}{z^2} + \frac{2(J\partial D^{\pm K} \pm (\partial^2 D^{\pm K} - TD^{\pm K}))}{z} \right) \quad (4.33)$$

$$\bar{X}(z)D^{\pm I}(0) \sim -\epsilon^{IK} \left(\frac{\pm 6\bar{D}_K^\pm}{z^3} + \frac{2(J\bar{D}_K^\pm \pm 2\partial\bar{D}_K^\pm)}{z^2} + \frac{2(J\partial\bar{D}_K^\pm \pm (\partial^2\bar{D}_K^\pm - T\bar{D}_K^\pm))}{z} \right) \quad (4.34)$$

$$C^A(z)D^{\pm I}(0) \sim (\sigma^A)^I{}_K \left(\frac{\pm 6D^{\pm K}}{z^3} + \frac{2(JD^{\pm K} \pm 2\partial D^{\pm K})}{z^2} + \frac{2(J\partial D^{\pm K} \pm (\partial^2 D^{\pm K} - TD^{\pm K}))}{z} \right), \quad (4.35)$$

$$C^A(z)\bar{D}_I^\pm(0) \sim (\sigma^A)^K{}_I \left(\frac{\pm 6\bar{D}_K^\pm}{z^3} + \frac{2(J\bar{D}_K^\pm \pm 2\partial\bar{D}_K^\pm)}{z^2} + \frac{2(J\partial\bar{D}_K^\pm \pm (\partial^2\bar{D}_K^\pm - T\bar{D}_K^\pm))}{z} \right). \quad (4.36)$$

Finally, the OPEs of dimension-three operators are

$$\begin{aligned} X(z)\bar{X}(0) &\sim -\frac{72}{z^6} + \frac{4(J^2 + 4T)}{z^4} + \frac{2\partial(J^2 + 4T)}{z^3} \\ &\quad - \frac{2}{z^2} \left(2T^2 + 2\partial(JT) - \partial^3 J - 3\partial^2 T + J\partial^2 J - D^{+I}\bar{D}_I^- \right) \\ &\quad - \frac{1}{z} \left(2T^2 + 2\partial(JT) - \partial^3 J - \frac{7}{3}\partial^2 T + \frac{1}{3}(\partial J)^2 + \frac{4}{3}J\partial^2 J \right) \\ &\quad + \frac{1}{z} \left(\partial D^{+I}\bar{D}_I^- - \bar{D}_I^+ \partial D^{-I} \right), \end{aligned} \quad (4.37)$$

$$C^A(z)X(0) \sim -\frac{2(\sigma^A)_{IJ}D^{+I}\partial D^{-J}}{z}, \quad C^A(z)\bar{X}(0) \sim -\frac{2(\sigma^A)^{IJ}\bar{D}_I^+\partial\bar{D}_J^-}{z} \quad (4.38)$$

$$\begin{aligned} C^A(z)C^B(0) &\sim \delta^{AB} \left\{ \frac{144}{z^6} - \frac{8(J^2 + 4T)}{z^4} - \frac{4\partial(J^2 + 4T)}{z^3} \right. \\ &\quad + \frac{4}{z^2} \left(2T^2 + 2\partial(JT) - \partial^3 J - 3\partial^2 T + J\partial^2 J - D^{+I}\bar{D}_I^- \right) \\ &\quad + \frac{2}{z} \left(2T^2 + 2\partial(JT) - \partial^3 J - \frac{7}{3}\partial^2 T + \frac{1}{3}(\partial J)^2 + \frac{4}{3}J\partial^2 J - D^{+I}\bar{D}_I^- \right) \Big\} \\ &\quad - \frac{2if^{AB}{}_C(\sigma^C)^I{}_J \left(D^{+J}\partial\bar{D}_I^- + \bar{D}_I^+\partial D^{-J} \right)}{z}. \end{aligned} \quad (4.39)$$

Note that the OPEs of generators that are not listed above all vanish. In particular, we find $B^\pm(z)D^{\pm I}(0) \sim 0$, $D^{\pm I}(z)D^{\pm J}(0) \sim 0$ and $\bar{D}_I^\pm(z)\bar{D}_J^\pm(0) \sim 0$. The vanishing of these OPEs

are necessary for our conjecture that those in table 2 is the complete set of generators, since no operator composed of those in table 2 can appear in these OPEs for dimensional and symmetry reasons.

4.5 Consistency check with Schur index

In sub-section 4.2, we conjecture that $\mathbf{S}$ is the complete set of generators of the chiral algebra $\chi[\mathcal{T}_{C_{2,0}}]$ associated with the genus two theory $\mathcal{T}_{C_{2,0}}$. Operators in $\mathbf{S}$ are listed in table 2. As discussed in sub-section 4.2, we have checked that every Q_{BRST} -cohomology class of dimension less than or equal to six is generated by those in $\mathbf{S}$. Here, we perform a further non-trivial check of this conjecture by computing the character of $\chi[\mathcal{T}_{C_{2,0}}]$ up to higher dimensions.

The (normalized) vacuum character of $\chi[\mathcal{T}_{C_{2,0}}]$ is defined by

$$\mathcal{I}(q, a) := \text{Tr}_{\chi[\mathcal{T}_{C_{2,0}}]} (-1)^F q^{L_0} a^{J_0}, \quad (4.40)$$

where L_0 and J_0 are the zero modes of T and J respectively, and $q, a \in \mathbb{C}$ such that $|q| < 1$ and $|a| = 1$. This character can be computed order by order in q as follows. First, if $\chi[\mathcal{T}_{C_{2,0}}]$ has no null operator, $\mathcal{I}(q, a)$ is given by

$$P.E. \left[\frac{1}{1-q} \left(q + (a^2 + 1 + a^{-2})q^2 - (4a + 4a^{-1})q^2 + 5q^3 \right) \right], \quad (4.41)$$

where $P.E.[g(q, a)] := \exp \left[\sum_{n=1}^{\infty} \frac{1}{n} g(q^n, a^n) \right]$ for any function g . Indeed, the above expression counts all composite operators built out of $\partial^k \mathcal{O}$ for $\mathcal{O} \in \mathbf{S}$ and $k \geq 0$. However, $\chi[\mathcal{T}_{C_{2,0}}]$ contains many null operators, corresponding to 4d operator relations in $\mathcal{T}_{C_{2,0}}$. These null operators must be removed from the spectrum. Therefore, the vacuum character $\mathcal{I}(q, a)$ of $\chi[\mathcal{T}_{C_{2,0}}]$ is obtained by subtracting the contributions of these null operators from (4.41). This can be done by identifying the null operators order by order in q . We list all the null operators up to dimension nine in appendix B.

On the 4d side, the vacuum character $\mathcal{I}(q, a)$ is identified with the Schur index defined by

$$\mathcal{I}_{\text{Schur}}(q, a) := \text{Tr}_{\mathcal{H}} (-1)^F q^{E-R} a^f, \quad (4.42)$$

where $\mathcal{H}$ is the space of local operators in $\mathcal{T}_{C_{2,0}}$, and E, R and f are the dimension, $SU(2)_R$ charge and $U(1)_f$ charge of operators. Since $\mathcal{T}_{C_{2,0}}$ has a Lagrangian description, one can compute it via an integral formula [70, 71].

We have checked that (4.42) agrees with (4.40) up to $\mathcal{O}(q^9)$. Combined with our check of the BRST cohomology classes up to dimension six, this gives a strong evidence for our conjecture that $\mathbf{S}$ is the complete set of generators of $\chi[\mathcal{T}_{C_{2,0}}]$.

5 Automorphisms and new $SU(2)$

In this section, we discuss automorphisms of $\chi[\mathcal{T}_{C_{2,0}}]$, based on its OPEs we identified in the previous section. We will see that there exists an unexpected $SU(2)$ automorphism sub-group in addition to those associated with 4d flavor and $U(1)_r$ symmetries.

5.1 Expected automorphisms

Let us first recall that 4d flavor $U(1)_f$ symmetry gives rise to an $\widehat{\mathfrak{u}(1)}$ current, whose zero-mode J_0 gives a $U(1)$ automorphism sub-group of $\chi[\mathcal{T}_{C_{2,0}}]$. The $U(1)_f$ charges of the generators are shown in table 2.

The charge conjugate for this $U(1)_f$ leads to a $\mathbb{Z}_2$ automorphism under which

$$\begin{aligned} J &\rightarrow -J, & B^\pm &\rightarrow -B^\mp, & D^{\pm\uparrow} &\rightarrow \pm i D^{\mp\downarrow}, & D^{\pm\downarrow} &\rightarrow \pm i D^{\mp\uparrow}, \\ \bar{D}_\uparrow^\pm &\rightarrow \pm i \bar{D}_\downarrow^\mp, & \bar{D}_\downarrow^\pm &\rightarrow \pm i \bar{D}_\uparrow^\mp, & C_1 &\rightarrow -C_1, \end{aligned} \quad (5.1)$$

with the other generators kept fixed. This $\mathbb{Z}_2$ corresponds to $\phi^{a\pm} \rightarrow \pm i \phi^{a\mp}$, which preserves (4.1) and Q_{BRST} , and therefore is an automorphism of $\chi[\mathcal{T}_{C_{2,0}}]$.

There is also another $\mathbb{Z}_2$ automorphism corresponding to exchanging two small $\mathcal{N} = 4$ super Virasoro algebras. Under this $\mathbb{Z}_2$, the generators of $\chi[\mathcal{T}_{C_{2,0}}]$ transform as

$$B^\pm \rightarrow -B^\pm, \quad D^{\pm\uparrow} \rightarrow \mp i D^{\pm\downarrow}, \quad D^{\pm\downarrow} \rightarrow \pm i D^{\pm\uparrow}, \quad (5.2)$$

$$\bar{D}_\uparrow^\pm \rightarrow \mp i \bar{D}_\downarrow^\pm, \quad \bar{D}_\downarrow^\pm \rightarrow \pm i \bar{D}_\uparrow^\pm, \quad C_1 \rightarrow -C_1, \quad C_3 \rightarrow -C_3, \quad (5.3)$$

with the other generators kept fixed. The combined action of the above two $\mathbb{Z}_2$ corresponds to exchanging the left and right sides of the left quiver in figure 1. Note that $D^{\pm I}$ and $\bar{D}_I^\pm$ are eigenstates of this combined $\mathbb{Z}_2$ -action, where $D^{\pm\uparrow}$ and $\bar{D}_\uparrow^\pm$ have eigenvalue +1 while $D^{\pm\downarrow}$ and $\bar{D}_\downarrow^\pm$ have eigenvalue -1.

As reviewed in sub-section 2.1, the 4d $U(1)_r$ symmetry of $\mathcal{T}_{C_{2,0}}$ also gives rise to a $U(1)$ automorphism of $\chi[\mathcal{T}_{C_{2,0}}]$. The $U(1)_r$ charges of the chiral algebra generators are shown in table 2. It is straightforward to see that the OPEs shown in sub-section 4.4 are all invariant under $U(1)_r$.

5.2 New $SU(2)$ symmetry

In addition to the above $U(1)_f \times U(1)_r \times (\mathbb{Z}_2)^2$ automorphism sub-group expected from the construction, the chiral algebra $\chi[\mathcal{T}_{C_{2,0}}]$ turns out to have an unexpected action of $SU(2)$ under which

1. $D^{\pm I}$ and $\bar{D}_{\pm I}$ transform as doublets, with the upper and lower indices I being the fundamental and anti-fundamental indices, respectively,
2. C^A transform as a triplet with A being the adjoint index,
3. and all the other generators transform as singlets.

Indeed, it is straightforward to see that the OPEs shown in sub-section 4.4 are all covariant under this action of $SU(2)$.

In contrast to $U(1)_f$ and $U(1)_r$, the above $SU(2)$ symmetry does not correspond to any conserved one-form current in four dimensions. This follows from the fact that every conserved one-form current in a 4d $\mathcal{N} = 2$ SCFT is either a flavor current or an R-symmetry

current.²¹ It is clear that our new $SU(2)$ is not associated with any 4d R-current.²² It is also clear that this $SU(2)$ does not correspond to a 4d flavor current, since the genus two theory $\mathcal{T}_{C_{2,0}}$ has only one flavor current multiplet, $\hat{\mathcal{B}}_1$, corresponding to $U(1)_f$. Therefore, the above $SU(2)$ symmetry either corresponds to a 4d global symmetry without conserved currents, or is an accidental symmetry in two dimensions.²³ It would be interesting to study each of these possibilities further.

While its four-dimensional origin is still to be understood, we see that this $SU(2)$ symmetry strongly constrains the OPEs of $\chi[\mathcal{T}_{C_{2,0}}]$. For instance, the $U(1)_f \times U(1)_r \times (\mathbb{Z}_2)^2$ symmetry does not forbid $C_2(0)/z^3$ to arise in the OPE of $X(z)\bar{X}(0)$, which is however prohibited by the $SU(2)$ symmetry; an $SU(2)$ triplet just cannot appear in the tensor product of singlets. There are indeed various terms in the OPEs of generators that are forbidden by the $SU(2)$ symmetry.

Note that, since 2d OPEs are determined by 4d OPEs, this $SU(2)$ symmetry also constrains the OPEs of Schur operators in the genus two theory $\mathcal{T}_{C_{2,0}}$. In particular, every 4d operator relation involving Schur operators gives rise to a null operator in $\chi[\mathcal{T}_{C_{2,0}}]$, and therefore must be in some representation of $SU(2)$. Since J and $B^\pm$ are neutral under it, this $SU(2)$ acts trivially on the Higgs branch. To see the effects of this $SU(2)$, one needs to look at Schur operators with non-vanishing spins.²⁴ This means that the above unexpected $SU(2)$ symmetry of $\chi[\mathcal{T}_{C_{2,0}}]$ constrains the OPEs of *non-scalar* Schur operators in $\mathcal{T}_{C_{2,0}}$. We leave a detailed study of these non-trivial constraints for future work.

6 Conclusions and discussions

In this paper, we studied the chiral algebra $\chi[\mathcal{T}_{C_{2,0}}]$ associated with the A_1 -type genus two class $\mathcal{S}$ theory $\mathcal{T}_{C_{2,0}}$. We focus on the weak coupling description of the theory corresponding to the left quiver in figure 1, and apply the BRST reduction reviewed in sub-section 2.2. We found that (1) BRST cohomology classes up to dimension six are all built out of generators listed in sub-section 4.3, and (2) these generators form closed OPEs. Given these facts, we conjecture that these are the complete set of generators of the chiral algebra $\chi[\mathcal{T}_{C_{2,0}}]$. As mentioned at the end of sub-section 4.2, similar conjectures were made for various other theories, leading to consistent results. We also check that our conjecture is perfectly consistent with the Schur index of $\mathcal{T}_{C_{2,0}}$ up to $\mathcal{O}(q^9)$. The OPEs of these generators are shown in sub-section 4.4.

One important and remarkable consequence of our OPEs is that there exists an unexpected $SU(2)$ automorphism sub-group of $\chi[\mathcal{T}_{C_{2,0}}]$. As discussed in section 5, this $SU(2)$

²¹Here, the flavor symmetry of a 4d $\mathcal{N} = 2$ SCFT is defined as a global symmetry that commutes with $\mathcal{N} = 2$ superconformal symmetry.

²²The only $SU(2)$ sub-group of the 4d R-symmetry group is $SU(2)_R$, which is broken in the 4d/2d correspondence.

²³In the latter case, it can be an accidental enhancement of a smaller 4d global symmetry. It would particularly interesting to see if this $SU(2)$ is an accidental enhancement of an abelian higher-form symmetry [62] in four dimensions. Note that the above $SU(2)$ symmetry cannot be interpreted as coming from an “ $SU(2)$ higher form symmetry” in four dimensions since every higher form symmetry is abelian.

²⁴Recall that Schur operators are in $\hat{\mathcal{B}}_R$, $\mathcal{D}_{R(0,j_2)}$, $\bar{\mathcal{D}}_{R(j_1,0)}$ or $\hat{\mathcal{C}}_{R(j_1,j_2)}$. As seen from table 1, the only scalar Schur operators are those in $\hat{\mathcal{B}}_R$, i.e., Higgs branch operators.

symmetry is not related to any conserved one-form current in four dimensions, and therefore either corresponds to a 4d symmetry without conserved current or is an accidental symmetry in two dimensions. We found that this $SU(2)$ acts trivially on 2d operators corresponding to 4d Higgs branch operators. Therefore, this $SU(2)$ is a symmetry characterizing the OPEs of non-scalar Schur operators.

While there has been various progress about the associated chiral algebra of class $\mathcal{S}$, our work is the first step to understand the chiral algebras of class $\mathcal{S}$ at higher genera. There are indeed many open problems related to this work in this direction:

- What is the four-dimensional origin of the new $SU(2)$ symmetry that we discussed in sub-section 5.2? As mentioned already, there is no 4d conserved one-form current corresponding to it. One possible way to understand it is to see how this $SU(2)$ symmetry emerges in the localization computation studied in [42, 52].
- What is the chiral algebra of class $\mathcal{S}$ associated with a Riemann surface of higher genus? It would be particularly interesting to see if there is a non-abelian automorphism sub-group that does not correspond to a 4d conserved one-form current. It would also be interesting to consider a generalization to genus two theories arising from higher rank 6d (2,0) theories.
- Is there any free field realization of $\chi[\mathcal{T}_{C_{2,0}}]$? As shown in [12, 14, 57], the chiral algebras of a class of 4d $\mathcal{N} = 2$ SCFTs have a beautiful free field realization that makes all the null operators trivially vanishing. It would be interesting to search for a similar realization of $\chi[\mathcal{T}_{C_{2,0}}]$, which would also be useful for solving the problem discussed in the previous bullet.
- As shown in appendix D of [8], the (normalized) Schur index of the genus two theory $\mathcal{T}_{C_{2,0}}$ satisfies a sixth-order modular linear differential equation. This suggests that $\chi[\mathcal{T}_{C_{2,0}}]$ has a null operator of dimension twelve, in addition to those listed in appendix B. One can perform a further consistency check of our conjectured chiral algebra by identifying this null operator in it.

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A S-duality equivalence of Schur index

As mentioned in section 3, $\mathcal{T}_{C_{2,0}}$ has two weak coupling Lagrangian descriptions corresponding to the quiver diagrams shown in figure 1. These two quiver descriptions are expected to be related by S-duality, which implies the Schur indices computed via these quivers are

equivalent. This equivalence has been checked in the case of vanishing flavor fugacity in [2]. In this appendix, we will extend it to the case of non-vanishing flavor fugacity. Note that, since the flavor $U(1)_f$ symmetry of this theory is not visible in its class $\mathcal{S}$ construction, this extension does not immediately follow from the class $\mathcal{S}$ interpretation of S-duality.

First, we focus on the left quiver diagram in figure 1. Let a be a $U(1)_f$ flavor fugacity. The Schur index of $\mathcal{T}_{C_{2,0}}$ is evaluated as

$$\mathcal{I}_1(q; a) = \oint_{|x_i|=1} \left(\prod_{k=1}^3 \frac{dx_k}{2\pi i x_k} \Delta(x_k) \mathcal{I}_{\text{vect}}(q; x_k) \right) \mathcal{I}_{\text{fund}}(q; x_2, a) \mathcal{I}_{\text{adj}}(q; x_1, x_2) \mathcal{I}_{\text{adj}}(q; x_3, x_2), \quad (\text{A.1})$$

where the contour integrations are taken over $|x_i| = 1$, $\Delta(x_k) := \frac{1}{2}(1 - x_k^2)(1 - x_k^{-2})$ is the factor arising from the Harr measure of $SU(2)_k$ gauge group, and

$$\mathcal{I}_{\text{vect}}(q; x) := P.E. \left[\frac{-2q}{1-q} (x^2 + 1 + x^{-2}) \right], \quad (\text{A.2})$$

$$\mathcal{I}_{\text{fund}}(q; x, a) := P.E. \left[\frac{q^{\frac{1}{2}}}{1-q} (x + x^{-1})(a + a^{-1}) \right], \quad (\text{A.3})$$

$$\mathcal{I}_{\text{adj}}(q; x, a) := P.E. \left[\frac{q^{\frac{1}{2}}}{1-q} (x^2 + 1 + x^{-2})(a + a^{-1}) \right], \quad (\text{A.4})$$

are respectively the index contributions from an $SU(2)$ vector multiplet, a fundamental hypermultiplet, and an adjoint hypermultiplet. Here, we used the plethystic exponential defined by

$$P.E.[g(q; x_1, \dots, x_k)] := \exp \left[\sum_{n=1}^{\infty} \frac{1}{n} g(q^n; x_1^n, \dots, x_k^n) \right], \quad (\text{A.5})$$

for arbitrary function g of fugacities.

On the other hands, the right quiver diagram in figure 1 indicates the index of $\mathcal{T}_{C_{2,0}}$ is evaluated as

$$\mathcal{I}_2(q; b) = \oint_{|x_i|=1} \left(\prod_{k=1}^3 \frac{dx_k}{2\pi i x_k} \Delta(x_k) \mathcal{I}_{\text{vect}}(q; x_k) \right) \prod_{s=\pm 1} \mathcal{I}_{\text{tri-fund}}^{\text{half}}(q; x_1, x_2, x_3, b^s), \quad (\text{A.6})$$

where b is a fugacity for the $U(1)_f$ flavor symmetry, and

$$\mathcal{I}_{\text{tri-fund}}^{\text{half}}(q; x_1, x_2, x_3, b) := P.E. \left[\frac{b q^{\frac{1}{2}}}{1-q} \prod_{k=1}^3 (x_k + x_k^{-1}) \right], \quad (\text{A.7})$$

is the index contributions from trifundamental hypermultiplet. The relation between a and b will be clear below.

When the flavor fugacity is turned off, the equivalence of (A.1) and (A.6) was shown in [2]. Indeed, it was shown in [2] that

$$\mathcal{I}_{\text{tri-fund}}^{\text{half}}(q; x_1, x_2, x_3, 1) = (q^2; q) \left[\prod_{k=1}^3 \mathcal{I}_{\text{vect}}(q; x_k) \right]^{-\frac{1}{2}} \sum_{R: \text{irreps of } \mathfrak{su}(2)} \frac{\chi_R(x_1) \chi_R(x_2) \chi_R(x_3)}{[\dim R]_q}, \quad (\text{A.8})$$

where R runs over irreducible representations of $\mathfrak{su}(2)$, $(x; q) := \prod_{k=0}^{\infty} (1 - xq^k)$, $[n]_q := \frac{(q^{\frac{n}{2}} - q^{-\frac{n}{2}})}{(q^{\frac{1}{2}} - q^{-\frac{1}{2}})}$, and $\chi_R(x) := (x^{\dim R} - x^{-\dim R}) / (x - x^{-1})$. Using this expression together with

$$\mathcal{I}_{\text{fund}}(q; x_2, 1) \mathcal{I}_{\text{adj}}(q; x_1, x_2) \mathcal{I}_{\text{adj}}(q; x_3, x_2) = \mathcal{I}_{\text{tri-fund}}^{\text{half}}(q; x_1, x_1, x_2, 1) \mathcal{I}_{\text{tri-fund}}^{\text{half}}(q; x_3, x_3, x_2, 1), \quad (\text{A.9})$$

and

$$\int_{|x|=1} \frac{dx}{2\pi i x} \Delta(x) \chi_{R_1}(x) \chi_{R_2}(x) = \delta_{R_1 R_2}, \quad (\text{A.10})$$

one can show that the equivalence

$$\mathcal{I}_1(q; 1) = \mathcal{I}_2(q; 1) = \sum_R \frac{(q^2; q)}{([\dim R]_q)^2}, \quad (\text{A.11})$$

in the case of $a = b = 1$.

Our aim in this appendix is to generalize the above proof to the case of $a, b \neq 1$. First, by comparing the first few terms of $\mathcal{I}_1(q; a)$ and $\mathcal{I}_2(q; b)$, we see that the correct identification of the flavor fugacity is

$$a = b^2. \quad (\text{A.12})$$

Below, we rewrite $\mathcal{I}_1$ and $\mathcal{I}_2$ to show that $\mathcal{I}_1(q; b^2) = \mathcal{I}_2(q; b)$.

A.1 Rewriting $\mathcal{I}_1$

From the identity

$$\mathcal{I}_{\text{adj}}(q; x, a) = P.E. \left[\frac{-q^{\frac{1}{2}}}{1 - q} (a + a^{-1}) \right] \times \mathcal{I}_{\text{tri-fund}}^{\text{half}}(q; x, x, a, 1), \quad (\text{A.13})$$

and (A.3), we see that

$$\begin{aligned} \mathcal{I}_1(q; b^2) &= \oint_{|x_i|=1} \left(\prod_{k=1}^3 \frac{dx_k}{2\pi i x_k} \Delta(x_k) \mathcal{I}_{\text{vect}}(q; x_k) \right) P.E. \left[\frac{q^{\frac{1}{2}}}{1 - q} (x_2 + x_2^{-1})(b^2 + b^{-2} - 2) \right] \\ &\quad \times \mathcal{I}_{\text{tri-fund}}^{\text{half}}(q; x_1, x_1, x_2) \mathcal{I}_{\text{tri-fund}}^{\text{half}}(q; x_3, x_3, x_2). \end{aligned} \quad (\text{A.14})$$

Using the identity (A.10), we can also rewrite it as²⁵

$$\begin{aligned} \mathcal{I}_1(q; b^2) &= (q^2; q)^2 \oint_{|x|=1} \frac{dx}{2\pi i x} \Delta(x) P.E. \left[\frac{q^{\frac{1}{2}}}{1 - q} (b^2 + b^{-2} - 2)(x + x^{-1}) \right] \left(\sum_{n=1} \frac{\chi_n(x)}{[\dim n]_q} \right)^2 \\ &= \oint_{|x|=1} \frac{dx}{2\pi i x} (1 - x^2) f(x) \left(\sum_{m=0} \frac{q^m}{(1 - q^{\frac{1}{2}+m} x)(1 - q^{\frac{1}{2}+m} x^{-1})} \right)^2, \end{aligned}$$

²⁵

$$\begin{aligned} \sum_{n=1} \frac{\chi_n(x)}{[\dim n]_q} &= \frac{q^{-\frac{1}{2}} - q^{\frac{1}{2}}}{x - x^{-1}} \sum_{n=1} \left(\frac{q^{\frac{n}{2}} (x^n - x^{-n})}{1 - q^n} \right) = \frac{1 - q}{x - x^{-1}} \sum_{m=0} \left(\frac{q^{\frac{1}{2}+m} x}{1 - q^{\frac{1}{2}+m} x} - \frac{q^{\frac{1}{2}+m} x^{-1}}{1 - q^{\frac{1}{2}+m} x^{-1}} \right) \\ &= \sum_{k=0} \frac{(1 - q) q^k}{(1 - q^{\frac{1}{2}+k} x)(1 - q^{\frac{1}{2}+k} x^{-1})} \end{aligned}$$

where

$$f(x) = (q; q)^2 P.E. \left[\frac{q^{\frac{1}{2}}}{1-q} (b^2 + b^{-2} - 2)(x + x^{-1}) \right] = \frac{(q; q)^2 (q^{\frac{1}{2}}x; q)^2 (q^{\frac{1}{2}}x^{-1}; q)^2}{\prod_{s_1, s_2 = \pm} (q^{\frac{1}{2}}b^{2s_1}x^{s_2}; q)}.$$

The last line are rewritten for symmetry $f(x^{-1}) = f(x)$.

We now evaluate the residues of the contour integral. The poles of the integrand are at $x = q^{\frac{1}{2}+k}b^{\pm 2}$ for $k \geq 0$.²⁶ First, we evaluate the residue at the pole at $x = q^{\frac{1}{2}+k}b^2$. The residue involves the following factor from the Pochhammer symbol

$$\frac{(x - q^{\frac{1}{2}+k}b^2)}{x(q^{\frac{1}{2}+k}b^2x^{-1}; q)} \Big|_{x=q^{\frac{1}{2}+k}b^2} = \prod_{t=0}^{k-1} \frac{1}{(1 - q^{t-k})} \prod_{t'=k+1}^{\infty} \frac{1}{(1 - q^{t'-m})} = \frac{(-1)^k q^{\frac{1}{2}k(k+1)} (q^{k+1}; q)}{(q; q)^2},$$

and therefore we find

$$\begin{aligned} \frac{(x - q^{\frac{1}{2}+k}b^2)}{x} f(x) \Big|_{x=q^{\frac{1}{2}+k}b^2} &= \frac{(-1)^k q^{\frac{1}{2}k(k+1)} (q^{k+1}b^2; q)^2 (q^{-k}b^{-2}; q)^2}{(q^{k+1}b^4; q)(q^{-k}b^{-4}; q)} \\ &= \frac{1 - b^{-2}}{1 + b^{-2}} \frac{(b^2q; q)^2 (b^{-2}q; q)^2}{(b^4q; q)(b^{-4}q; q)}. \end{aligned} \quad (\text{A.15})$$

In the last equality, we used the formula

$$(aq^{k+1}; q)(a^{-1}q^{-k}; q) = (-a^{-1})^k q^{-\frac{k(k+1)}{2}} (aq; q)(a^{-1}; q). \quad (\text{A.16})$$

From the above calculations, we see that the residue at $x = q^{\frac{1}{2}+k}b^2$ is

$$\frac{1 - b^{-2}}{(1 - b^4q^{2k+1})(1 + b^{-2})} \frac{(b^2q; q)^2 (b^{-2}q; q)^2}{(b^4q; q)(b^{-4}q; q)} \left(\sum_{m=0}^{\infty} \frac{(1 - b^4q^{2m+1})q^k}{(1 - q^{k+m+1}b^2)(1 - q^{m-k}b^{-2})} \right)^2. \quad (\text{A.17})$$

Note here that one can further rewrite the sum over m as

$$\begin{aligned} \sum_{m=0}^{\infty} \frac{(1 - b^4q^{2k+1})q^m}{(1 - b^2q^{k+m+1})(1 - b^{-2}q^{m-k})} &= \sum_{m=0}^{\infty} \left(\frac{-b^4q^{2k+m+1}}{1 - b^2q^{k+m+1}} - \frac{b^2q^k}{1 - b^2q^{k-m}} \right) \\ &= -b^2q^k \left(k + \frac{1}{1 - b^2} + \sum_{\ell=1}^{\infty} \frac{(b^{2\ell} - b^{-2\ell})q^{\ell}}{1 - q^{\ell}} \right). \end{aligned} \quad (\text{A.18})$$

Combining the above result with the residues at $x = q^{\frac{1}{2}+k}b^{-2}$, we finally get the following expression for $\mathcal{I}_1(q; b^2)$:

$$\begin{aligned} \mathcal{I}_1(q; b^2) &= \frac{1 - b^2}{1 + b^2} \frac{(qb^2; q)^2 (qb^{-2}; q)^2}{(qb^4; q)(qb^{-4}; q)} \\ &\quad \times \left(\sum_{m=0}^{\infty} \frac{-b^4q^{2m}}{(1 - b^4q^{2m+1})} \left(m + \frac{1}{1 - b^2} + \sum_{\ell=1}^{\infty} \frac{(b^{2\ell} - b^{-2\ell})q^{\ell}}{1 - q^{\ell}} \right)^2 - (b \rightarrow b^{-1}) \right). \end{aligned} \quad (\text{A.19})$$

²⁶There is no pole at $x = 0$, since $f(0)$ takes finite value and the summation part in the integrand is zero in the limit $x \rightarrow 0$.

A.2 Rewriting $\mathcal{I}_2$

Let us now turn to $\mathcal{I}_2(q; b)$. Using the identity

$$\begin{aligned} & \mathcal{I}_{\text{tri-fund}}^{\text{half}}(q; x_1, x_2, x_3, b) \mathcal{I}_{\text{tri-fund}}^{\text{half}}(q; x_1, x_2, x_3, b) \\ &= \mathcal{I}_{\text{tri-fund}}^{\text{half}}(q; x_1, bx_2, x_3, 1) \mathcal{I}_{\text{tri-fund}}^{\text{half}}(q; x_1, b^{-1}x_2, x_3, 1) \end{aligned} \quad (\text{A.20})$$

and (A.8), we can more simplify $\mathcal{I}_2(q; b)$ as

$$\begin{aligned} & \mathcal{I}_2(q; b) \\ &= (q^2; q)^2 \oint_{|x|=1} \frac{dx}{2\pi i x} (1-x^2) P.E. \left[\frac{q}{1-q} (b^2 + b^{-2} - 2)(x^2 + x^{-2}) \right] \sum_R \frac{\chi_R(bx) \chi_R(b^{-1}x)}{[\dim R]^2}. \end{aligned} \quad (\text{A.21})$$

Note that the sum over R can be rewritten as

$$\sum_R \frac{\chi_R(bx) \chi_R(b^{-1}x)}{([\dim R]_q)^2} = \frac{x^2(1-q)^2}{(1-b^2x^2)(1-b^{-2}x^2)} g(x), \quad (\text{A.22})$$

where

$$g(x) := \sum_{k=0} (k+1) q^k \left(\frac{x^2 + x^{-2} - 2q^{k+1}}{(1-x^2q^{k+1})(1-x^{-2}q^{k+1})} - \frac{b^2 + b^{-2} - 2q^{k+1}}{(1-b^2q^{k+1})(1-b^{-2}q^{k+1})} \right). \quad (\text{A.23})$$

From this and

$$P.E. \left[\frac{q}{1-q} (b^2 + b^{-2} - 2)(x^2 + x^{-2}) \right] = \frac{(q; q)^2 (qx^2; q)^2 (qx^{-2}; q)^2}{\prod_{s_1, s_2 = \pm 1} (qb^{2s_1} x^{2s_2})}, \quad (\text{A.24})$$

we see that the integrand of (A.21) has a pole at $x = \pm b^{\pm 1} q^{\frac{k+1}{2}}$ for all $k \in \mathbb{Z}_{\geq 0}$.²⁷

Note that the residue at $x = bq^{\frac{k+1}{2}}$ is evaluate as

$$\frac{-(1-b^2)(b^2; q)^2 (b^{-2}q; q)^2}{2(1+b^2)(b^4q; q)(b^{-4}q; q)} \frac{b^2q^{k+1}}{1-b^2q^{k+1}} g(bq^{\frac{k+1}{2}}). \quad (\text{A.25})$$

Combining this and its cousin obtained by $b \rightarrow -b^{-1}$, we find

$$\begin{aligned} \mathcal{I}_2(q; b) &= \frac{-(1-b^2)(b^2; q)^2 (b^{-2}q; q)^2}{(1+b^2)(b^4q; q)(b^{-4}q; q)} \sum_{k=0, m=0} \frac{(m+1)b^2q^{k+m+1}}{1-b^2q^{k+1}} \frac{b^2q^{k+1} + b^{-2}q^{-(k+1)} - 2q^{m+1}}{(1-b^2q^{k+m+2})(1-b^{-2}q^{k-m})} \\ &+ (b \rightarrow b^{-1}). \end{aligned} \quad (\text{A.26})$$

Since the sum over k and m is rewritten as²⁸

$$\begin{aligned} & \sum_{m, k=0} \frac{(m+1)b^2q^{k+m+1}}{1-b^2q^{k+1}} \left(\frac{-1}{q^{k+1} - b^2q^{m+1}} \right) + \sum_{\ell=1} \frac{b^{2\ell}q^{(m+2)\ell} - (b^{2\ell} + b^{-2\ell})q^\ell}{(1-q^\ell)^2} \\ &= \sum_{m=0} \frac{(m+1)b^2q^m}{1-b^2q^{m+1}} \left(\frac{1}{2}m + \frac{1}{1-b^2} + \sum_{\ell=1} \frac{(b^{2\ell} - b^{-2\ell})q^\ell}{1-q^\ell} \right), \end{aligned} \quad (\text{A.31})$$

²⁷For careful evaluation, we can see that $x = 0, \pm b^{\pm 1}$ are also not poles.

²⁸Here we use

$$\sum_{k=0} \frac{-(k+1)q^{k+1}}{q^{1+k} - b^2q^{m+1}} = \sum_{k=0}^m \frac{-(k+1)}{1-b^2q^{m-k}} + \sum_{k=0}^{\infty} \frac{(k+1+m+1)b^{-2}q^{k+1}}{1-b^{-2}q^{k+1}} \quad (\text{A.27})$$

the expression (A.26) is further rewritten as

$$\mathcal{I}_2(q; b) = \frac{1 - b^2}{1 + b^2} \frac{(qb^2; q)^2 (qb^{-2}; q)^2}{(qb^4; q)(qb^{-4}; q)} \sum_{m=0}^{\infty} \left(\frac{(m+1)b^2 q^m}{1 - b^2 q^{m+1}} \left(\frac{1}{2}m + \frac{1}{1 - b^2} + \sum_{\ell=1}^{\infty} \frac{(b^{2\ell} - b^{-2\ell})q^\ell}{1 - q^\ell} \right) - (b \rightarrow b^{-1}) \right). \quad (\text{A.32})$$

A.3 Proof of $\mathcal{I}_1 = \mathcal{I}_2$

From (A.19) and (A.32), we see that proving $\mathcal{I}_1 = \mathcal{I}_2$ is equivalent to proving

$$h_1(b) + h_2(b) = h_1(b^{-1}) + h_2(b^{-1}), \quad (\text{A.33})$$

where

$$h_1(b) := \sum_{m=0}^{\infty} \left(\frac{b^4 q^{2m}}{1 - b^4 q^{2m+1}} \left(m + \sum_{k=1}^{\infty} \frac{b^2 q^k}{1 - b^2 q^k} - \sum_{k=0}^{\infty} \frac{b^{-2} q^k}{1 - b^{-2} q^k} \right)^2 \right), \quad (\text{A.34})$$

$$h_2(b) := \sum_{m=0}^{\infty} \frac{(m+1)b^2 q^m}{1 - b^2 q^{m+1}} \left(\frac{m}{2} + \sum_{k=1}^{\infty} \frac{b^2 q^k}{1 - b^2 q^k} - \sum_{k=0}^{\infty} \frac{b^{-2} q^k}{1 - b^{-2} q^k} \right). \quad (\text{A.35})$$

In the rest of this appendix, we prove (A.33) by showing that both sides of (A.33) have the same poles and residues. Both sides of (A.33) have poles only at $b = \pm q^{\frac{\ell}{2}}$, $\ell \in \mathbb{Z}$. Note that potential poles of the l.h.s. at $b = \pm s q^{-\frac{\ell}{2} - \frac{1}{4}}$ for $s = 1, i$, and those of the r.h.s. at $b = \pm s q^{\frac{\ell}{2} + \frac{1}{4}}$ for $\ell \in \mathbb{Z}_{\geq 0}$ have vanishing residues. For instance, the residue of the expression in the most inner bracket in (A.34), at $b = s q^{-\frac{\ell}{2} - \frac{1}{4}}$, is

$$\begin{aligned} & \ell + \frac{1}{1 - s^2 q^{-\ell - \frac{1}{2}}} + \sum_{k=1}^{\infty} \left(\frac{s^2 q^{k - \ell - \frac{1}{2}}}{1 - s^2 q^{k - \ell - \frac{1}{2}}} - \frac{s^2 q^{k + \ell + \frac{1}{2}}}{1 - s^2 q^{k + \ell + \frac{1}{2}}} \right) \\ &= \ell - \frac{s^2 q^{\ell + \frac{1}{2}}}{1 - s^2 q^{\ell + \frac{1}{2}}} + \sum_{k=1}^{2\ell} \frac{s^2 q^{k - \ell - \frac{1}{2}}}{1 - s^2 q^{k - \ell - \frac{1}{2}}} + \frac{s^2 q^{\ell + \frac{1}{2}}}{1 - s^2 q^{\ell + \frac{1}{2}}} + \sum_{k=1}^{\infty} \left(\frac{s^2 q^{k + \ell + \frac{1}{2}}}{1 - s^2 q^{k + \ell + \frac{1}{2}}} - \frac{s^2 q^{k + \ell + \frac{1}{2}}}{1 - s^2 q^{k + \ell + \frac{1}{2}}} \right) \\ &= 0. \end{aligned}$$

Therefore the expression in the most inner bracket is regular at $b = s q^{-\frac{\ell}{2} - \frac{1}{4}}$. This means that $h_1(b)$ has no pole at this point. Similarly, one can show that both sides of (A.33) have no poles at $b = \pm s q^{-\frac{\ell}{2} - \frac{1}{4}}$ or $b = \pm s q^{\frac{\ell}{2} + \frac{1}{4}}$. Below, we show that the residues of both sides of (A.33) at $b = \pm q^{\frac{\ell}{2}}$ are identical, for $\ell \in \mathbb{Z}$.

$$= \sum_{k=0}^m \frac{k - (m+1)}{1 - b^2 q^k} + (m+1) \sum_{\ell=1}^{\infty} \frac{b^{-2\ell} q^\ell}{1 - q^\ell} + \sum_{\ell=1}^{\infty} \frac{b^{-2\ell} q^\ell}{(1 - q^\ell)^2} \quad (\text{A.28})$$

and

$$\sum_{k=0}^m \frac{1}{1 - b^2 q^k} = \frac{1}{1 - b^2} + m + \sum_{\ell=1}^{\infty} \frac{b^{2\ell} q^\ell (1 - q^{m\ell})}{1 - q^\ell} \quad (\text{A.29})$$

$$\sum_{k=1}^m \frac{k}{1 - b^2 q^k} = \frac{1}{2} m(m+1) + \sum_{\ell=1}^{\infty} \frac{b^{2\ell} (q^\ell - (m+1)q^{\ell(m+1)} + m q^{\ell(m+2)})}{(1 - q^\ell)^2}. \quad (\text{A.30})$$

A.3.1 Poles and residues on both sides

Let us focus on the poles at $b = q^{\frac{\ell}{2}}$ for $\ell > 0$. Its generalization to the other poles is straightforward. Note that these poles are second order poles. One can see that the coefficients of the most singular terms on both sides of (A.33) are identical. Indeed, when $b \sim q^{\frac{\ell}{2}}$, the l.h.s. behaves as

$$h_1(b) + h_2(b) \sim \frac{1}{1 - b^{-2}q^\ell} \sum_{m=0}^{\infty} \frac{q^{3\ell+2m}}{4(1 - q^{2m+2\ell+1})}. \quad (\text{A.36})$$

On the other hand, the r.h.s. behaves as

$$h_1(b^{-1}) + h_2(b^{-1}) \sim \frac{1}{1 - b^{-2}q^\ell} \left(\sum_{m=0}^{\infty} \frac{q^{2m-\ell}}{4(1 - q^{2(m-\ell)+1})} + \frac{\ell}{4} q^{\ell-1} \right) \quad (\text{A.37})$$

$$= \frac{1}{1 - b^{-2}q^\ell} \sum_{m=0}^{\infty} \frac{q^{3\ell+2m}}{4(1 - q^{2(m+\ell)+1})}. \quad (\text{A.38})$$

Thus, the most singular terms on both sides agree.

Let us next evaluate the residues on both sides. First, we compute the residue on the l.h.s. . The residue coming from $h_1(b)$ is evaluated as

$$\begin{aligned} & \sum_{m=0}^{\infty} \left[\frac{-q^{2(m+\ell)}q^{\frac{\ell}{2}}}{1 - q^{2(m+\ell)+1}} \left(m + \sum_{k=1}^{\infty} \frac{q^{k+\ell}}{1 - q^{k+\ell}} - \sum_{\substack{k=0 \\ (k \neq \ell)}}^{\infty} \frac{q^{k-\ell}}{1 - q^{k-\ell}} \right) + \frac{d}{db} \left(\frac{b^4 q^{2m}}{1 - b^4 q^{2m+1}} \frac{q^{2\ell}}{(b + q^{\frac{\ell}{2}})^2} \right) \right]_{b=q^{\frac{\ell}{2}}} \\ &= q^{\frac{\ell}{2}} \sum_{m=0}^{\infty} \left(\frac{-mq^{2m}}{1 - q^{2m+1}} + \frac{q^{2m}(3 + q^{2m+1})}{4(1 - q^{2m+1})^2} \right) + q^{\frac{\ell}{2}} \sum_{m=0}^{\ell-1} \left(\frac{mq^{2m}}{1 - q^{2m+1}} - \frac{q^{2m}(3 + q^{2m+1})}{4(1 - q^{2m+1})^2} \right), \end{aligned}$$

while, the residue coming from $h_2(b)$ is evaluated as

$$q^{\frac{\ell}{2}} \sum_{m=0}^{\infty} \frac{-(m+1)q^{m+\ell}}{2(1 - q^{m+\ell+1})} = q^{\frac{\ell}{2}} \sum_{m=0}^{\infty} \frac{-(m-\ell+1)q^m}{2(1 - q^{m+1})} + q^{\frac{\ell}{2}} \sum_{m=0}^{\ell-1} \frac{(m-\ell+1)q^m}{2(1 - q^{m+1})}. \quad (\text{A.39})$$

Combining the above two, we see that the residue of the l.h.s. of (A.33) is

$$\begin{aligned} & q^{\frac{\ell}{2}} \sum_{m=0}^{\infty} \left(\frac{-mq^{2m}}{1 - q^{2m+1}} + \frac{q^{2m}(3 + q^{2m+1})}{4(1 - q^{2m+1})^2} - \frac{(m+1-\ell)q^m}{2(1 - q^{m+1})} \right) \\ & + q^{\frac{\ell}{2}} \sum_{m=0}^{\ell-1} \left(\frac{mq^{2m}}{1 - q^{2m+1}} - \frac{q^{2m}(3 + q^{2m+1})}{4(1 - q^{2m+1})^2} + \frac{(m+1-\ell)q^m}{2(1 - q^{m+1})} \right). \end{aligned} \quad (\text{A.40})$$

We next turn to the r.h.s. of (A.33). The residue coming from $h_1(b^{-1})$ is

$$\sum_{m=0}^{\infty} \frac{q^{2(m-\ell)}q^{\frac{\ell}{2}}}{1 - q^{2(m-\ell)+1}} \left(m + \sum_{k=1, k \neq \ell}^{2\ell-1} \frac{q^{k-\ell}}{1 - q^{k-\ell}} \right) + \frac{d}{db} \frac{q^{2(m+\ell)}}{(b^4 - q^{2m+1})(b + q^{\frac{\ell}{2}})^2} \Big|_{b=q^{\frac{\ell}{2}}} \quad (\text{A.41})$$

$$= q^{\frac{\ell}{2}} \sum_{m=0}^{\infty} \left(\frac{mq^{2m}}{1 - q^{2m+1}} - \frac{q^{2m}(1 + 3q^{2m+1})}{4(1 - q^{2m+1})^2} \right) + \sum_{m=0}^{\ell-1} \left(\frac{m}{q(1 - q^{2m+1})} + \frac{(1 - 5q^{2m+1})}{4q(1 - q^{2m+1})^2} \right). \quad (\text{A.42})$$

and that arising from $h_2(b^{-1})$ is

$$\begin{aligned} & \sum_{m+1 \neq \ell} \frac{(m+1)q^{m-\frac{\ell}{2}}}{2(1-q^{m-\ell+1})} + \frac{l}{2}q^{\frac{\ell}{2}-1} \left(\frac{l-1}{2} + \sum_{k=1, k \neq \ell}^{2\ell-1} \frac{q^{k-\ell}}{1-q^{k-\ell}} \right) + \frac{d}{db} \left(\frac{lq^{2\ell-1}}{(b+q^{\frac{\ell}{2}})^2} \right) \Big|_{b=q^{\frac{\ell}{2}}} \\ &= q^{\frac{\ell}{2}} \sum_{m=0}^{\ell-1} \frac{(m+1-\ell)}{2q(1-q^{m+1})} + q^{\frac{\ell}{2}} \sum_{m=0}^{\ell-1} \frac{(m+\ell+1)q^m}{2(1-q^{m+1})} - \frac{\ell^2}{4}q^{\frac{\ell}{2}-1}. \end{aligned}$$

Therefore, the residue of the r.h.s. of (A.33) at $b = q^{\frac{\ell}{2}}$ is evaluated as

$$\begin{aligned} & q^{\frac{\ell}{2}} \sum_{m=0}^{\ell-1} \left(\frac{mq^{2m}}{1-q^{2m+1}} - \frac{q^{2m}(1+3q^{2m+1})}{4(1-q^{2m+1})^2} + \frac{(m+1+\ell)q^m}{2(1-q^{m+1})} \right) \\ &+ q^{\frac{\ell}{2}} \sum_{m=0}^{\ell-1} \left(\frac{m}{q(1-q^{2m+1})} + \frac{(1-5q^{2m+1})}{4q(1-q^{2m+1})^2} + \frac{(m+1-\ell)}{2q(1-q^{m+1})} \right) - \frac{\ell^2}{4}q^{\frac{\ell}{2}-1}. \quad (\text{A.43}) \end{aligned}$$

A.3.2 Coincidence of the residues

We here show that the residue (A.40) of the l.h.s. of (A.33) agrees with the residue (A.43) of the r.h.s. .

Note first that the second line of (A.40) minus that of (A.43) is simplified as

$$-q^{\frac{\ell}{2}} \sum_{m=0}^{\ell-1} \frac{(6m-2\ell+3)}{4q} + \frac{\ell^2}{4}q^{\frac{\ell}{2}-1} = 0. \quad (\text{A.44})$$

Therefore, all we need to show is the equivalence of the first lines of (A.40) and (A.43). Note that this is equivalent to proving the identity

$$\sum_{m=0}^{\infty} \left(\frac{2mq^{2m}}{1-q^{2m+1}} + \frac{(m+1)q^m}{1-q^{m+1}} \right) = \sum_{m=0}^{\infty} \frac{q^{2m}(1+q^{2m+1})}{(1-q^{2m+1})^2}. \quad (\text{A.45})$$

Using the identities

$$\sum_{m=0}^{\infty} \frac{2mq^{2m}}{1-q^{2m+1}} = \sum_{m=0}^{\infty} \left(\frac{2q^m}{(1-q^{2(m+1)})^2} + \frac{-2q^{2m}}{1-q^{2m+1}} \right), \quad (\text{A.46})$$

$$\sum_{m=0}^{\infty} \frac{(m+1)q^m}{1-q^{m+1}} = \sum_{m=0}^{\infty} \left(\frac{q^{2m}}{(1-q^{2(m+1)})^2} + \frac{q^{2m+1}}{(1-q^{2(m+1)})^2} \right), \quad (\text{A.47})$$

we see that proving (A.45) is equivalent to proving

$$\sum_{m=0}^{\infty} \left(\frac{q^m(2+q^{m+1})}{(1-q^{2(m+1)})^2} - \frac{2q^{2m}}{(1-q^{2m+1})^2} + \frac{q^{4m+1}}{(1-q^{2m+1})^2} \right) = 0. \quad (\text{A.48})$$

Below, we show that (A.48) indeed holds. To that end, first note that

$$\sum_{m=0}^{\infty} \frac{2q^{2m}}{(1-q^{2m+1})^2} = \sum_{m=0}^{\infty} \left(\frac{2q^m}{(1-q^{m+1})^2} - \frac{2q^{2m+1}}{(1-q^{2(m+1)})^2} \right), \quad (\text{A.49})$$

$$\sum_{m=0}^{\infty} \frac{q^{4m+1}}{(1-q^{2m+1})^2} = \sum_{m=0}^{\infty} \left(\frac{q^{2m+1}}{(1-q^{m+1})^2} - \frac{q^{4m+3}}{(1-q^{2(m+1)})^2} \right). \quad (\text{A.50})$$

Using these identities, the l.h.s. of (A.48) is rewritten as

$$\sum_{m=0}^{\infty} \left(\frac{-q^{2m+1}(1+2q^{m+1})}{(1-q^{2(m+1)})^2} + \frac{q^{2m+1}}{(1-q^{m+1})^2} - \frac{q^{4m+3}}{(1-q^{2(m+1)})^2} \right), \quad (\text{A.51})$$

which can be shown to vanish by a straightforward calculation. Therefore, (A.48) is an identity, which completes our proof of the equivalence between (A.40) and (A.43).

B Null operators of genus two chiral algebra

In this appendix, we list the null operators in the chiral algebra $\chi[\mathcal{T}_{C_{2,0}}]$ whose holomorphic dimension (which we denote by h) is less than or equal to nine. Interestingly, such null operators only exist at $h = 4, 5$ and 6 , up to composite operators involving them or their derivatives. Note that the absence of independent null operators at $h = 7, 8$ and 9 does not mean the absence of such operators at $h \geq 10$. Indeed, the modular linear differential equation studied in [8] suggests an independent null operator involving T^6 at $h = 12$. It would be interesting to extend our results here to higher dimensions.

Below, we list these null operators as operator relations. We also classify these operator relations in terms of the $U(1)_f$ charge, f , and the $U(1)_r$ charge r . Note that every pair of generators, X and Y , satisfies the following trivial operator relation

$$YX - (-1)^{|X||Y|}XY = \sum_{n \geq 1} \frac{(-1)^n}{n!} \partial^n [XY]_n, \quad (\text{B.1})$$

where $|\mathcal{O}| = 0$ or $|\mathcal{O}| = 1$ when $\mathcal{O}$ is bosonic or fermionic respectively, and $[XY]_n$ are operators such that $X(z)Y(0) \sim \sum_{n \geq 1} [XY]_n(0)/z^n$ [69]. Since these relations just follow from the definition of the normal ordered product, we do not list them below.²⁹

B.1 Dim 4

The null operators at dimension four are as follows.

Nulls with $f = r = 0$. First, null operator relations with $f = r = 0$ are

$$D^{+I} \bar{D}_J^- + \bar{D}_J^+ D^{-I} = 0, \quad (\text{B.2})$$

$$\partial C^A = J C^A + (\sigma^A)^I_J D^{+J} \bar{D}_I^-, \quad (\text{B.3})$$

$$\begin{aligned} (B^+ B^- - J^4) + 2(D^{+I} \bar{D}_I^-) &= 4T^2 - 6\partial^2 T - 8J^2 T + 12\partial(JT) - 4\partial J^3 \\ &\quad + 9(\partial J)^2 + 14J\partial^2 J - 5\partial^3 J. \end{aligned} \quad (\text{B.4})$$

We see that, when we omit derivatives and generators corresponding to $\mathcal{D}_{R(0,\bar{j})}$, $\bar{\mathcal{D}}_{R(j,0)}$ and $\hat{\mathcal{C}}_{R(j,\bar{j})}$, the last operator relation (B.4) is identical to the Higgs branch chiral ring relation (3.8) in four dimensions. The mixing with derivative operators and those arising from $\mathcal{D}_{R(0,\bar{j})}$, $\bar{\mathcal{D}}_{R(j,0)}$ or $\hat{\mathcal{C}}_{R(j,\bar{j})}$ is a common feature of associated chiral algebras.

²⁹In our computation of the character in section 4.5, these operator relations are taken into account by counting only one of XY and YX . Indeed, the expression (4.41) assumes that XY and YX are linearly dependent and therefore counts only one of them as an independent operator.

Nulls with $f = \pm 1$ and $r = \pm \frac{1}{2}$. Second, the null operator relations for $f = \pm 1$ and $r = \pm \frac{1}{2}$ are

$$\begin{aligned}\partial^2 D^{\pm I} &= T D^{\pm I} \pm \partial(J D^{\pm I}) + \frac{1}{2} \left(B^{\pm} D^{\mp I} - J^2 D^{\pm I} \pm \partial J D^{\pm I} \right), \\ \partial^2 \bar{D}_I^{\pm} &= T \bar{D}_I^{\pm} \pm \partial(J \bar{D}_I^{\pm}) + \frac{1}{2} \left(B^{\pm} \bar{D}_I^{\mp} - J^2 \bar{D}_I^{\pm} \pm \partial J \bar{D}_I^{\pm} \right).\end{aligned}\quad (\text{B.5})$$

Since these relations involve only operators corresponding to non-scalar Schur operators in four dimensions, they captures 4d operator relations that are not visible in the Higgs branch chiral ring.

Nulls with $(f, r) = (0, \pm 1)$ and $(\pm 2, 0)$. Similarly, the null operator relations with $f = 0$ and $r = \pm 1$ are

$$D^{+I} D^{-J} + D^{+J} D^{-I} = 0, \quad \bar{D}_I^+ \bar{D}_J^- + \bar{D}_J^+ \bar{D}_I^- = 0, \quad (\text{B.6})$$

$$\partial X = JX + \frac{1}{2} D^{+I} D_I^-, \quad \partial \bar{X} = J\bar{X} + \frac{1}{2} \bar{D}_I^+ \bar{D}^{-I}, \quad (\text{B.7})$$

and those with $f = \pm 2$ and $r = 0$ are

$$\partial^2 B^{\pm} = 2(T \pm \partial J) B^{\pm} \mp J \partial B^{\pm} + D^{\pm I} \bar{D}_I^{\pm}. \quad (\text{B.8})$$

While (B.8) involves $B^{\pm}$ arising corresponding to Higgs branch operators, this operator relation is not captured by 4d Higgs branch chiral ring relation, since every derivative operator is trivial in the Higgs branch chiral ring. Therefore, (B.8) captures more refined data of the OPEs of Higgs branch operators.

B.2 Dim 5

Let us move on to the null operators at dimension five. Note that there are such operators that are composite operators involving a null of dimension four, or the derivatives of lower-dimensional nulls. Below, we omit all such null operators that trivially follow from the lower-dimensional ones.

Nulls with $f = r = 0$. The null operators with $f = r = 0$ is

$$0 = 2TC^A + (\sigma^A)^I{}_J \left(\bar{D}_I^+ \partial D^{-J} + \partial D^{+J} \bar{D}_I^- \right), \quad (\text{B.9})$$

$$\begin{aligned}0 &= J^3 T + 3J(\partial J)^2 - 2JT^2 - J\partial(JT) + \frac{1}{2} \left(JD^{+I} \bar{D}_I^- + \partial D^{-I} \bar{D}_I^+ - \partial D^{+I} \bar{D}_I^- \right) \\ &\quad + J\partial^2 T - \partial J \partial T - \frac{5}{6} J \partial^3 J - 4\partial J \partial^2 J + \frac{1}{4} (B^+ \partial B^{-1} - \partial B^{+1} B^{-1}) + \frac{1}{3} \partial^3 T + \frac{1}{4} \partial^4 J.\end{aligned}\quad (\text{B.10})$$

Nulls with $f = 0$ and $r = \pm 1$. The null operators of $f = 0$ and $r = \pm 1$ are

$$TX + \frac{1}{4} \left(\partial D^{+I} D_I^- - D^{+I} \partial D_I^- \right) = 0 \quad T\bar{X} + \frac{1}{4} \left(\partial \bar{D}_I^+ \bar{D}^{-I} - \bar{D}_I^+ \partial \bar{D}^{-I} \right) = 0. \quad (\text{B.11})$$

Nulls with $f = \pm 1$ and $r = \pm \frac{1}{2}$. The null operator relations of $f = \pm 1$ and $r = \pm \frac{1}{2}$ are

$$2JTD^{\pm I} - \partial^2 JD^{\pm I} - 2\partial(J\partial D^{\pm I}) + X\bar{D}^{\pm I} \pm (2J\partial JD^{\pm I} - \partial B^{\pm} D^{\mp I}) = 0, \quad (\text{B.12})$$

$$2JT\bar{D}_I^{\pm} - \partial^2 J\bar{D}_I^{\pm} - 2\partial(J\partial\bar{D}_I^{\pm}) - \bar{X}D_I^{\pm} \pm (2J\partial J\bar{D}_I^{\pm} - \partial B^{\pm} \bar{D}_I^{\mp}) = 0, \quad (\text{B.13})$$

$$C^A D^{\pm I} + (\sigma^A)^{IJ} X \bar{D}_J^{\pm} = 0, \quad (\text{B.14})$$

$$C^A \bar{D}_I^{\pm} - (\sigma^A)_{IJ} \bar{X} D^{\pm J} = 0. \quad (\text{B.15})$$

Nulls with $f = \pm 1$ and $r = \pm \frac{3}{2}$. The null operators for $f = \pm 1$ and $r = \pm \frac{3}{2}$ are

$$XD^{\pm I} = 0, \quad \bar{X}\bar{D}^{\pm I} = 0. \quad (\text{B.16})$$

Nulls with $f = \pm 2$ and $r = 0$. The null operators with $(f, r) = (\pm 2, 0)$ are

$$B^{\pm} C^A + (\sigma^A)^J{}_I (JD^{\pm I} \bar{D}_J^{\pm} \mp \partial(D^{\pm I} \bar{D}_J^{\pm})) = 0. \quad (\text{B.17})$$

Nulls with $f = \pm 2$ and $r = \pm 1$. The null operators for $f = \pm 2$ and $r = \pm 1$ are

$$B^{\pm} X = \pm D^{\pm I} \partial D_I^{\pm} - \frac{1}{2} J D^{\pm I} D_I^{\pm}, \quad (\text{B.18})$$

$$B^{\pm} \bar{X} = \pm \bar{D}_I^{\pm} \partial \bar{D}^{\pm I} - \frac{1}{2} J \bar{D}_I^{\pm} \bar{D}^{\pm I}. \quad (\text{B.19})$$

B.3 Dim 6

We here list independent null operators of dimension six. We again omit all such operators that trivially follow from lower-dimensional nulls.

Nulls with $f = r = 0$. There are seven null operators with $f = r = 0$:

$$\begin{aligned} 0 = & T^3 - \frac{3}{4} (TD^{+I} \bar{D}_I^- + \partial D^{+I} \partial \bar{D}_I^- + X \bar{X}) + \frac{1}{2} \partial^2 (D^{+I} \bar{D}_I^-) \\ & - \frac{3}{8} (D^{+I} \partial^2 \bar{D}_I^- + \bar{D}_I^+ \partial^2 D^{-I} + JD^{+I} \partial \bar{D}_I^- + J \partial \bar{D}_I^+ D^{-I}) \\ & + \frac{1}{4} J^2 (\partial^2 T + \partial^3 J) + \frac{3}{2} (JT \partial T + \partial J T^2) + J \partial J \partial T + \frac{5}{4} (J \partial^2 J T - \partial J \partial^3 J) \\ & + \frac{3}{4} J \partial J \partial^2 J - J \partial^3 T - \frac{9}{4} \partial J \partial^2 T - \partial^3 J T - \frac{1}{2} (J \partial^4 J + (\partial^2 J)^2) \\ & - \frac{5}{2} T \partial^2 T - (\partial T)^2 + \frac{1}{2} (\partial J)^2 T - \frac{3}{2} \partial^2 J \partial T + \frac{5}{6} \partial^4 T + \frac{1}{4} \partial^5 J, \end{aligned} \quad (\text{B.20})$$

$$0 = C^A C_A + 6X \bar{X} + 3 \partial (D^{+I} \partial \bar{D}_I^- + \bar{D}_I^+ \partial D^{-I}), \quad (\text{B.21})$$

$$0 = C^{\{A} C^{B\}} - \frac{1}{3} \delta^{AB} C^D C_D. \quad (\text{B.22})$$

where $C^{\{A} C^{B\}} := \frac{1}{2} (C^A C^B + C^B C^A)$.³⁰ Note that the last equation implies five non-trivial operator relations.

³⁰For completeness, we here comment that there are also operator relations of the form $0 = C^{[A} C^{B]} + i f^{ABC} (\sigma^C)^J{}_I \partial (D^{+I} \partial \bar{D}_J^- + \bar{D}_J^+ \partial D^{-I})$. These are, however, relations trivially following from the OPE of $C^A(z) C^B(0)$. We do not list such operator relations as mentioned at the beginning of this appendix.

Nulls with $f = 0$ and $r = \pm 1$. The null operators with $(f, r) = (0, \pm 1)$ are

$$XC^A = (\sigma^A)_{IJ} \partial(D^{+I} \partial D^{+J}), \quad \bar{X}C^A = (\sigma^A)^{IJ} \partial(\bar{D}^{+I} \partial \bar{D}^{+J}). \quad (\text{B.23})$$

Nulls with $f = 0$ and $r = \pm 2$. Finally, it turns out that dimension six operators with $r = \pm 2$ are all null:

$$X^2 = 0, \quad \bar{X}^2 = 0, \quad (\text{B.24})$$

In general, $\mathcal{D}$ and $\bar{\mathcal{D}}$ type Schur operators in the Lagrangian theory are nilpotent since they involve gauginos [1, 8]. The above relations reflect this property of $\mathcal{D}$ and $\bar{\mathcal{D}}$ type operators in four dimensions.

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