Neutrino oscillations in matter: from microscopic to macroscopic description

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ABSTRACT: Neutrino flavour transmutations in nonuniform matter are described by a Schrödinger-like evolution equation with coordinate-dependent potential. In all the derivations of this equation it is assumed that the potential, which is due to coherent forward scattering of neutrinos on matter constituents, is a continuous function of coordinate that changes slowly over the distances of the order of the neutrino de Broglie wavelength. This tacitly assumes that some averaging of the microscopic potential (which takes into account the discrete nature of the scatterers) has been performed. The averaging, however, must be applied to the microscopic evolution equation as a whole and not just to the potential. Such an averaging has never been explicitly carried out. We fill this gap by considering the transition from the microscopic to macroscopic neutrino evolution equation through a proper averaging procedure. We discuss some subtleties related to this procedure and establish the applicability domain of the standard macroscopic evolution equation. This, in particular, allows us to answer the question of when neutrino propagation in rarefied media (such as e.g. low-density gases or interstellar or intergalactic media) can be considered within the standard theory of neutrino flavour evolution in matter.

KEYWORDS: Neutrino Physics, Solar and Atmospheric Neutrinos

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1 Introduction

Neutrino flavour transformations in nonuniform matter are described by a Schrödinger-like evolution equation with coordinate-dependent potential. It had been first suggested by Wolfenstein [1] basing on heuristic considerations and was subsequently derived more rigorously within the relativistic quantum mechanics and quantum field theory frameworks [2–8]. This evolution equation has been employed in virtually all studies of the neutrino flavour transition effects in nonuniform matter, including the explorations of the Mikheyev-Smirnov-Wolfenstein effect [1, 9] and of the parametric resonance of neutrino oscillations [10–12] (see ref. [13] for a review).

In all the derivations of the neutrino evolution equation it is assumed that the neutrino potential, which is due to coherent forward scattering of neutrinos on matter constituents, is a continuous function of coordinate that changes slowly over the distances of the order of the neutrino de Broglie wavelength $\lambda_D = 1/p$. This means that some averaging of the microscopic potential (which takes into account the discrete nature of the scatterers) is tacitly assumed.

Indeed, even for neutrinos of energy as small as $E \sim 1 \text{ MeV}$ (which is close to the lower end of the spectra of detectable solar neutrinos and reactor antineutrinos) the de Broglie wavelength is on the order of 10^{-11} cm , which is much smaller than the interatomic distance $\sim 10^{-9}\text{--}10^{-8} \text{ cm}$. Thus, the average number of scatterers inside a volume of linear size $\sim \lambda_D$ is much less than one, and the matter-induced neutrino potential cannot even approximately be considered as a smooth function on such length scales. This means that some coarse-graining (averaging) must be performed to justify the standard neutrino evolution equation. It is important to note that such an averaging must be applied to the microscopic evolution equation as a whole and not just to the potential. No such averaging has been explicitly carried out so far.

In the present paper we fill this gap by considering the transition from the microscopic to macroscopic neutrino evolution equation through a proper averaging procedure. We perform a coarse graining — a coordinate-space averaging over macroscopic volumes v_0 that contain large numbers of particles of the medium and at the same time are small enough, so that the macroscopic characteristics of the medium (such as density) are nearly constant within v_0 .¹ Such an averaging is actually necessary because of the very large number of the scatterers, which makes a microscopic description of neutrino flavour evolution practically impossible; it is also sufficient, as we only need the coarse-grained neutrino wave functions to predict the outcomes of neutrino detection experiments.

The averaging procedure we consider is similar to the one employed in classical electrodynamics of continuous media in going from microscopic to macroscopic Maxwell equations. There are, however, important differences between these two procedures. In electrodynamics, each term of the microscopic Maxwell equations contains either derivative of electric or magnetic field, or charge/current density of the particles of the medium, but not the products of the two. This makes the averaging of the microscopic Maxwell equations technically simple. In contrast to this, there are terms in the microscopic neutrino evolution equation in medium that contain products of the neutrino wave function and the matter-induced potential of neutrinos (or the gradient of this potential). As the average of the product of two functions is in general different from the product of their averages, this is a nontrivial issue requiring special consideration.

In the present paper we discuss the subtleties related to the averaging of the microscopic neutrino evolution equation in matter and establish the applicability domain of the standard macroscopic evolution equation. This, in particular, allows us to answer the question of when neutrino propagation in rarefied media (such as e.g. low-density gases or interstellar or intergalactic media) can be considered within the standard theory of neutrino flavour evolution in matter.

2 Microscopic neutrino evolution equation in matter

Let us first consider the case of Dirac neutrinos; generalization to the Majorana neutrino case will be discussed in section 7. The effective Lagrangian for neutrinos propagating in

¹In some situations also the averaging over the velocities and spins of the particles of the medium has to be done. This is discussed in section 6.

matter can be written as [2]²

$$\mathcal{L} = \bar{\nu}_L[i\cancel{\partial} - \cancel{V}(x)]\nu_L + \bar{\nu}_R i\cancel{\partial}\nu_R - \bar{\nu}_L M \nu_R - \bar{\nu}_R M^\dagger \nu_L, \quad (2.1)$$

where ν_L and ν_R are the left-handed (LH) and right-handed (RH) neutrino fields, $V^\mu(x)$ is the 4-vector potential induced by coherent neutrino forward scattering on matter constituents and M is the neutrino mass matrix. Note that $V^\mu(x)$ and M are matrices in flavour space, whereas ν_L and ν_R are flavour vectors. For definiteness, we shall speak about neutrino forward scattering on electrons, but our results will also apply to neutrino scattering on other matter constituents. The equations of motion for ν_L and ν_R following from the Lagrangian (2.1) are

$$i\cancel{\partial}\nu_L - M\nu_R = \cancel{V}\nu_L, \quad (2.2)$$

$$i\cancel{\partial}\nu_R - M^\dagger\nu_L = 0. \quad (2.3)$$

It will be convenient for us to use the chiral (Weyl) representation for the γ -matrices, in which γ_5 is diagonal and

$$\nu_L = \begin{pmatrix} \phi \\ 0 \end{pmatrix}, \quad \nu_R = \begin{pmatrix} 0 \\ \chi \end{pmatrix}, \quad (2.4)$$

with ϕ and χ being the 2-component LH and RH neutrino fields, respectively. Eqs. (2.2) and (2.3) then can be written as

$$(i\partial^0 - i\vec{\sigma}\vec{\nabla})\phi - M\chi = (V^0 + \vec{\sigma}\vec{V})\phi, \quad (2.5)$$

$$(i\partial^0 + i\vec{\sigma}\vec{\nabla})\chi - M^\dagger\phi = 0. \quad (2.6)$$

We shall be assuming that the 4-vector of matter-induced potentials $V^\mu(x)$ depends on coordinate but does not change with time:

$$V^\mu(x) = V^\mu(\vec{x}). \quad (2.7)$$

One can then look for the solutions of eqs. (2.5)–(2.6) in the form $\phi(x) = e^{-iEt}\phi(\vec{x})$, $\chi(x) = e^{-iEt}\chi(\vec{x})$.³ Substituting this into eqs. (2.5) and (2.6), we obtain

$$(E - i\vec{\sigma}\vec{\nabla})\phi - M\chi = (V^0 + \vec{\sigma}\vec{V})\phi, \quad (2.8)$$

$$(E + i\vec{\sigma}\vec{\nabla})\chi - M^\dagger\phi = 0. \quad (2.9)$$

We shall be assuming that neutrinos are relativistic with $E \gg M, |V^\mu|$. Eliminating the RH neutrino field χ from eqs. (2.8) and (2.9), to lowest order in V^μ/E and $MM^\dagger/E^2$ we find

$$\left[\vec{\nabla}^2 + E^2 - MM^\dagger - 2E(V^0 - \vec{v}\vec{\nabla}) - i\vec{\sigma}\left[\vec{\nabla}(V^0 + \vec{\sigma}\vec{V})\right]\right]\phi = 0, \quad (2.10)$$

where $\vec{v}$ is the neutrino velocity (see appendix A for a detailed derivation). Note that, although this equation was derived for the neutrino field, the neutrino wave function

²We use the natural units $\hbar = c = 1$.

³These solutions can then be used as a basis for constructing neutrino wave packets with nonvanishing energy spread.

$\langle 0|\phi(\vec{x})|\nu\rangle$ satisfies the same evolution equation. In what follows we shall consider the flavour evolution of the neutrino wave function, for which for conciseness we will use the same notation $\phi(\vec{x})$ as was up to now used for the neutrino field.

The neutrino potential V^μ in general contains both polar-vector and axial-vector contributions: $V^\mu = V^{\mu V} + V^{\mu A}$ [1]. To simplify our discussion, we shall mainly concentrate on neutrino propagation in media consisting of nonrelativistic (or randomly moving) and unpolarized particles. This is the case in many applications of interest, such as oscillations of solar, atmospheric or accelerator neutrinos. The spatial component $\vec{V}$ of the neutrino potential as well as the axial-vector part V^{0A} of its time component can then be neglected. The case of non-vanishing $\vec{V}$ and V^{0A} will be briefly discussed in section 6.

With $\vec{V}$ set equal to zero, eq. (2.10) becomes

$$\left[\vec{\nabla}^2 + E^2 - MM^\dagger - 2EV^0 - i\left(\vec{\sigma}\vec{\nabla}V^0\right)\right]\phi = 0. \quad (2.11)$$

The standard approach is then to assume that the potential V^0 varies very slowly on the length scales of the order of the neutrino de Broglie wavelength $\sim 1/E$ and neglect the term containing the gradient of the potential. Eq. (2.10) then takes the form

$$\left[F + \vec{\nabla}^2\right]\phi = 0, \quad (2.12)$$

where

$$F = E^2 - MM^\dagger - 2EV^0. \quad (2.13)$$

For one-dimensional neutrino motion along the z -axis, eq. (2.12) can be factorized as [2]

$$\left[\sqrt{F} - i\frac{d}{dz}\right]\left[\sqrt{F} + i\frac{d}{dz}\right]\phi(z) = 0, \quad (2.14)$$

where, to lowest order in $MM^\dagger/E^2$ and V^0/E ,

$$\sqrt{F} = E - \frac{MM^\dagger}{2E} - V^0(z). \quad (2.15)$$

Note that the term E here does not affect neutrino flavour transitions and can be omitted. For neutrinos propagating in the positive direction of the z -axis, eq. (2.14) reduces to

$$\left[i\frac{d}{dz} + \sqrt{F}\right]\phi(z) = 0. \quad (2.16)$$

This equation, with $\sqrt{F}$ from eq. (2.15), coincides in form with the standard evolution equation for neutrino oscillations in matter, the difference being that eq. (2.16) is actually a microscopic equation, while the neutrino evolution equation is usually interpreted as the macroscopic one.

In refs. [7, 8] neutrino flavour evolution in nonuniform matter was studied by considering the in-medium neutrino propagator. No assumption of one-dimensional neutrino propagation was explicitly made; however, it was assumed that the condition $E|\vec{x}| \gg 1$ is satisfied, that is, the distance $|\vec{x}|$ from the neutrino production point is large compared

to the neutrino de Broglie wavelength. Under this condition the evolution is in fact one-dimensional. It was also assumed that the terms containing the gradient of the neutrino potential can be neglected. The obtained evolution equation coincides with eq. (2.16) with the potential V^0 in the expression for $\sqrt{F}$ taken at the point $\vec{x}$ and d/dz being the directional derivative along $\vec{x}$: $d/dz \equiv \vec{n}_{\vec{x}} \cdot \vec{\nabla}$, where $\vec{n}_{\vec{x}}$ is the unit vector in the direction of $\vec{x}$.

3 Averaging procedure and macroscopic evolution equation

In the above derivation of the neutrino evolution equation in matter we had to assume the potential $V^0(\vec{x})$ to be a slowly varying function of coordinate on the length scale of the order of the neutrino de Broglie wavelength $\lambda_D \sim 1/E$. As was discussed in the Introduction, this is in general not justified for microscopic neutrino potentials. We will therefore consider now the transition from microscopic to macroscopic description of neutrino flavour evolution in matter by averaging the microscopic evolution equation (2.11). To this end, we will integrate it over a small but macroscopic volume v_0 around each point $\vec{x}$. The volume v_0 should be sufficiently large to contain a large number of particles of the medium, but small enough such that the macroscopic characteristics of the medium be nearly constant within it.⁴ We will be using the “hat” notation for the coarse-grained (averaged) quantities, that is, for any integrable in v_0 function $g(\vec{x})$ we define

$$\hat{g}(\vec{x}) \equiv \frac{1}{v_0} \int_{v_0} d^3x' g(\vec{x} + \vec{x}'). \quad (3.1)$$

Obviously, differentiation and averaging operations commute, that is, the average of a derivative is equal to the derivative of the average. In particular,

$$\widehat{(\vec{\nabla} g)}(\vec{x}) = \vec{\nabla} \hat{g}(\vec{x}). \quad (3.2)$$

Thus, the averaging of all terms in eq. (2.11) except those containing the products of the neutrino wave function ϕ and the potential V^0 or its gradient is straightforward. In particular, upon the averaging,

$$(\vec{\nabla}^2 + E^2 - MM^\dagger) \phi(\vec{x}) \longrightarrow (\vec{\nabla}^2 + E^2 - MM^\dagger) \hat{\phi}(\vec{x}). \quad (3.3)$$

Before turning to the averaging of the remaining terms in eq. (2.11), let us consider the microscopic potential $V^0(\vec{x})$ and its averaging. We shall assume here the electrons of the medium to be pointlike particles with coordinates $\vec{x}_i$; the case when the electrons are described by atomic wave functions is discussed in appendix B.

For definiteness, we consider the potential due to coherent forward neutrino-electron scattering mediated by weak charged currents, though the exact nature of the underlying interaction is not important for our discussion. The microscopic neutrino potential is then

$$V^0(\vec{x}) = \sqrt{2} G_F v_F \sum_i \delta^3(\vec{x} - \vec{x}_i), \quad (3.4)$$

⁴By this we mean that if we divide v_0 arbitrarily into two or more sub-volumes that are still macroscopic, the intensive macroscopic characteristics of the medium (such as density or temperature) in each of them will to a high accuracy be the same.

where G_F is the Fermi constant and v_F is a coordinate-independent matrix characterizing the flavour structure of $V^0(\vec{x})$. In the flavour basis $(\nu_e, \nu_\mu, \nu_\tau)$ we have $v_F \equiv \text{diag}(1, 0, 0)$. Let the total number of electrons inside the averaging volume v_0 around the point $\vec{x}$ be $N_0(\vec{x}, v_0)$. As the volume v_0 is chosen to be sufficiently small to ensure that the matter density in it is essentially constant, $N_0(\vec{x}, v_0)$ is proportional to v_0 , i.e. the ratio

$$n_e(\vec{x}) \equiv \frac{N_0(\vec{x}, v_0)}{v_0} \quad (3.5)$$

is v_0 -independent. The quantity $n_e(\vec{x})$ is the macroscopic electron number density in the medium, which is a smooth function of coordinate. From eqs. (3.4) and (3.1), for the averaged potential $\hat{V}^0(\vec{x})$ we then find

$$\hat{V}^0(\vec{x}) = \sqrt{2}G_F v_F \frac{1}{v_0} \int_{v_0} d^3x' \sum_{i=1}^{N_0(\vec{x}, v_0)} \delta^3(\vec{x} + \vec{x}' - \vec{x}_i). \quad (3.6)$$

Here the sum is over all the electrons in v_0 . The integration is trivial, and we obtain

$$\hat{V}^0(\vec{x}) = \sqrt{2}G_F v_F n_e(\vec{x}). \quad (3.7)$$

This is the standard Wolfenstein potential employed in most studies of neutrino flavour transformations in matter.

We are now in a position to perform the averaging of the terms in eq. (2.11) that contain the potential $V^0(\vec{x})$ and its gradient. Consider first the term $2EV^0\phi(\vec{x})$. By definition

$$\widehat{(V^0\phi)}(\vec{x}) = \frac{1}{v_0} \int_{v_0} d^3x' V^0(\vec{x} + \vec{x}') \phi(\vec{x} + \vec{x}'). \quad (3.8)$$

Substituting here $V^0(\vec{x})$ from eq. (3.4) yields

$$\widehat{(V^0\phi)}(\vec{x}) = \hat{V}^0(\vec{x}) [\hat{\phi}(\vec{x})]_{\text{M.C.}}, \quad (3.9)$$

where we have used eq. (3.7) and denoted

$$[\hat{\phi}(\vec{x})]_{\text{M.C.}} \equiv \frac{1}{N_0(\vec{x}, v_0)} \sum_{i=1}^{N_0(\vec{x}, v_0)} \phi(\vec{x}_i). \quad (3.10)$$

We shall be assuming that the electrons of the medium are randomly distributed in the volume v_0 , i.e. that $\vec{r}_i$ are random coordinates in v_0 with the uniform probability distribution function.

The quantity $[\hat{\phi}(\vec{x})]_{\text{M.C.}}$ is then nothing but the basic Monte Carlo estimator for the integral defining $\hat{\phi}(\vec{x})$ according to eq. (3.1) [14].

Generally, Monte Carlo integration of a function $g(\vec{x})$ gives an accurate result when the number N_0 of the points $\vec{x}_i$ at which $g(\vec{x})$ is sampled is large, with the relative integration error being $\mathcal{O}(N_0^{-1/2})$ (see section 4 for a more detailed discussion). As the averaging volume v_0 contains a macroscopically large number of electrons, in all situations of practical

interest N_0 is very large and one can safely neglect the difference between $[\hat{\phi}(\vec{x})]_{\text{M.C.}}$ and $\hat{\phi}(\vec{x})$. Eq. (3.9) then becomes

$$\widehat{(V^0\phi)}(\vec{x}) = \hat{V}^0(\vec{x})\hat{\phi}(\vec{x}). \quad (3.11)$$

That is, even though the average of the product of $V^0(\vec{x})$ and $\phi(\vec{x})$ does not in general factorize into the product of their averages, such a factorization does take place with high accuracy under the conditions that the electrons of the medium are pointlike and are randomly distributed in the averaging volumes, and that the total number of electrons in each averaging volume v_0 is sufficiently large. In appendix B we show that the assumption of pointlike electrons can actually be lifted.

Next, we consider the averaging of the term $[\vec{\sigma}\vec{\nabla}V^0(\vec{x})]\phi(\vec{x})$ in (2.11). We have

$$\begin{aligned} [(\vec{\nabla}V^0)\phi](\vec{x}) &\equiv \frac{1}{v_0} \int_{v_0} d^3x' [\vec{\nabla}_{\vec{x}+\vec{x}'} V^0(\vec{x}+\vec{x}')] \phi(\vec{x}+\vec{x}') \\ &= \frac{1}{v_0} \left\{ \int_{v_0} d^3x' \vec{\nabla}_{\vec{x}+\vec{x}'} [V^0(\vec{x}+\vec{x}')\phi(\vec{x}+\vec{x}')] - \int_{v_0} d^3x' V^0(\vec{x}+\vec{x}') \vec{\nabla}_{\vec{x}+\vec{x}'} \phi(\vec{x}+\vec{x}') \right\}. \end{aligned} \quad (3.12)$$

Let us consider the two terms in the second line of eq. (3.12). For the first term we have

$$\frac{1}{v_0} \int_{v_0} d^3x' \vec{\nabla}_{\vec{x}+\vec{x}'} [V^0(\vec{x}+\vec{x}')\phi(\vec{x}+\vec{x}')] = \vec{\nabla} \frac{1}{v_0} \int_{v_0} d^3x' V^0(\vec{x}+\vec{x}')\phi(\vec{x}+\vec{x}') = \vec{\nabla} [\hat{V}^0(\vec{x})\hat{\phi}(\vec{x})], \quad (3.13)$$

where in the last equality we have used eq. (3.11). Consider now the last term in the second line in eq. (3.12). Repeating the arguments that led to eq. (3.11), we find

$$\frac{1}{v_0} \int_{v_0} d^3x' V^0(\vec{x}+\vec{x}') \vec{\nabla}_{\vec{x}+\vec{x}'} \phi(\vec{x}+\vec{x}') = \hat{V}^0(\vec{x}) \widehat{\vec{\nabla}\phi}(\vec{x}) = \hat{V}^0(\vec{x}) \vec{\nabla}\hat{\phi}(\vec{x}). \quad (3.14)$$

Finally, using eqs. (3.13) and (3.14) in (3.12) we obtain

$$[(\vec{\nabla}V^0)\phi](\vec{x}) = \vec{\nabla} [\hat{V}^0(\vec{x})\hat{\phi}(\vec{x})] - \hat{V}^0(\vec{x}) \vec{\nabla}\hat{\phi}(\vec{x}) = [\vec{\nabla}\hat{V}^0(\vec{x})] \hat{\phi}(\vec{x}). \quad (3.15)$$

We now have all the ingredients in order to perform the averaging of the microscopic equation (2.11). Combining eqs. (3.3), (3.11) and (3.15) yields

$$[\vec{\nabla}^2 + E^2 - 2E\hat{V}^0 - MM^\dagger - i(\vec{\sigma}\vec{\nabla}\hat{V}^0)] \hat{\phi}(\vec{x}) = 0. \quad (3.16)$$

As the coarse-grained potential $\hat{V}^0(\vec{x})$ is a continuous function of coordinate, one can now use the argument that the term containing the gradient of $\hat{V}^0(\vec{x})$ can be neglected if this potential changes slowly on the length scales of the order of the neutrino de Broglie wavelength. This condition is always satisfied in practice, and we therefore drop the last term in the square brackets in eq. (3.16). Following the same arguments that led from eq. (2.11) to (2.16) and dropping the irrelevant term E from $\sqrt{\hat{F}}$, we finally arrive at

$$i \frac{d}{dz} \hat{\phi}(z) = \left[\frac{MM^\dagger}{2E} + \hat{V}^0(z) \right] \hat{\phi}(z) = 0. \quad (3.17)$$

This is the standard neutrino evolution equation that was, though without proper justification, used in most studies of neutrino flavour transformations in matter.

4 Accuracy of Monte Carlo integration

Let us now discuss the accuracy of approximating the integrals involved in the averaging procedure by their basic Monte Carlo estimators, such as the one defined in eq. (3.10). We consider here the case of neutrino scattering on pointlike electrons studied in the previous section; the generalization to the case of neutrino scattering on electrons in atoms (or molecules) is given in appendix B.

In general, for random $\vec{x}_i$ uniformly distributed in v_0 the expected value of the quantity $[\hat{g}(\vec{x})]_{\text{M.C.}} \equiv (1/\mathcal{N}_0) \sum_i^{\mathcal{N}_0} g(\vec{x}_i)$ coincides with $\hat{g}(\vec{x})$, and its variance scales as $1/\mathcal{N}_0$. The error of the Monte Carlo estimation of $\hat{g}(\vec{x})$ therefore scales as $(\mathcal{N}_0)^{-1/2}$. The proof is very simple; we give it here for the particular instance of the coarse-grained neutrino wave function $\hat{\phi}(\vec{x})$ in the case of neutrino scattering on pointlike electrons.

For the expected value of the quantity $[\hat{\phi}(\vec{x})]_{\text{M.C.}}$ defined in eq. (3.10) we have

$$\text{E} \left([\hat{\phi}(\vec{x})]_{\text{M.C.}} \right) = \frac{1}{N_0(\vec{x}, v_0)} \sum_{i=1}^{N_0(\vec{x}, v_0)} \text{E}(\phi(\vec{x}_i)) = \frac{1}{N_0(\vec{x}, v_0)} \sum_{i=1}^{N_0(\vec{x}, v_0)} \int_{v_0} d^3x' \phi(\vec{x} + \vec{x}') \text{PDF}(\vec{x}'). \quad (4.1)$$

As the random variable $\vec{x}_i$ is uniformly distributed in v_0 , its probability distribution function $\text{PDF}(\vec{x}) = 1/v_0$, and we obtain

$$\text{E} \left([\hat{\phi}(\vec{x})]_{\text{M.C.}} \right) = \hat{\phi}(\vec{x}). \quad (4.2)$$

That is, $[\hat{\phi}(\vec{x})]_{\text{M.C.}}$ is an unbiased estimator of $\hat{\phi}(\vec{x})$. The expression for the variance of $[\hat{\phi}(\vec{x})]_{\text{M.C.}}$ can be found similarly. Direct calculation yields

$$\text{var} \left([\hat{\phi}(\vec{x})]_{\text{M.C.}} \right) = \frac{\sigma^2[\phi(\vec{x})]}{N_0(\vec{x}, v_0)}, \quad (4.3)$$

where $\sigma^2[\phi(\vec{x})]$ is defined as

$$\sigma^2[\phi(\vec{x})] \equiv \frac{1}{v_0} \int_{v_0} d^3x' |\phi(\vec{x} + \vec{x}')|^2 - \left| \frac{1}{v_0} \int_{v_0} d^3x' \phi(\vec{x} + \vec{x}') \right|^2 = |\widehat{\phi(\vec{x})}|^2 - |\hat{\phi}(\vec{x})|^2. \quad (4.4)$$

This quantity characterizes the speed of variation of $\phi(\vec{x})$ with coordinate in the averaging volume v_0 . In particular, it vanishes for $\phi(\vec{x}) = \text{const.}$

Eq. (4.3) has a simple meaning. The statistical error (standard deviation) introduced when replacing the coarse-grained neutrino wave function $\hat{\phi}(\vec{x})$ by its basic Monte Carlo estimator (3.10) is $\sigma[\phi(\vec{x})]/\sqrt{N_0(\vec{x}, v_0)}$. If the function $\phi(\vec{x})$ is nearly constant throughout the averaging volume v_0 , the quantity $\sigma[\phi(\vec{x})]$ is strongly suppressed; in this case the Monte Carlo integration of $\phi(\vec{x})$ with even a single sampling point should give an accurate value of $\hat{\phi}(\vec{x})$. On the other hand, if $\phi(\vec{x})$ is a fast varying function in v_0 , then $\sigma[\phi(\vec{x})]$ is of the order $|\hat{\phi}(\vec{x})|$ or even larger. In this case a large number N_0 of the sampling points is necessary to achieve a good accuracy of Monte Carlo integration. In the case we consider, the role of the sampling points is played by the coordinates of the electrons in the medium.

As we assume the averaging volumes v_0 to be macroscopic, N_0 is typically $\gtrsim 10^{12}$,⁵ and the approximation of replacing the averaging integrals by their basic Monte Carlo estimators is very accurate. A possible exception is the case of neutrino propagation in rarefied media, which will be discussed in the next section.

5 Neutrino flavour transitions in rarefied media

The averaging volumes v_0 that we use in our coarse-graining procedure have to satisfy several requirements. On the one hand, to make a statistical description possible, v_0 must be large enough to contain macroscopically large numbers N_0 of the particles of the medium. Very large N_0 also allowed us to replace, in the course of the coarse-graining, some averaging integrals by their basic Monte Carlo estimators. And finally, this allowed us to drop the term proportional to $\vec{\nabla}\hat{V}^0(\vec{x})$ from the macroscopic neutrino evolution equation and reduce it to the standard form (3.17).

On the other hand, v_0 must be small enough such that inside it one could consider the macroscopic characteristics of the medium (and, in particular, the number density of the particles) as nearly constant. There is, however, one more consideration that bounds v_0 from above. As detection processes do not allow exact determination of the coordinate of each neutrino detection event, the experiments yield the detection data averaged over the active volume of the detector or, in case the detector allows some position resolution, over the volume v_d of the corresponding detection region. The experiments therefore probe the flavour content of the incoming neutrino state with the same spatial resolution. The volume v_0 used in the averaging of the microscopic neutrino evolution equation must not exceed the volume of the detection region v_d , as otherwise the coarse-graining procedure would be too rough to allow an accurate prediction of the outcome of the experiment.

Let us discuss the consequences of this constraint. As before, we for definiteness consider the effects of coherent neutrino forward scattering on the electrons of the medium. Let n_e be a characteristic electron number density in the medium that affects the flavour transformations of neutrinos in the course of their propagation. Requiring that $N_0 = v_0 n_e$ be, say, of the order 10^{12} or larger, from $v_d > v_0$ we find $v_d n_e \gtrsim 10^{12}$. For the electron number density in the medium we therefore obtain the lower limit

$$n_e \gtrsim \frac{10^{12}}{v_d}. \quad (5.1)$$

The linear sizes of neutrino detectors are typically in the ~ 1 meter to 1 km range, but the position resolution for the neutrino events is usually much better. Taking as an example $v_d \sim 1 \text{ m}^3$, from eq. (5.1) we find the lower limit on the electron number density $n_e \gtrsim 10^6 \text{ cm}^{-3}$. If, however, the coordinate of the neutrino detection point is known with a cm accuracy, we find $n_e \gtrsim 10^{12} \text{ cm}^{-3}$. At the same time, the nuclear emulsion film

⁵We adopt the definition of macroscopic volumes as those of linear size $\gtrsim 1 \mu\text{m}$ (see, e.g., [15], p. 2). Noting that the electron number density can be written as $n_e = N_A \rho Y_e \text{ cm}^{-3}$ where N_A is the Avogadro constant, ρ is the matter density in g/cm^3 and Y_e is the number of electrons per nucleon, we find that for $\rho \sim 3 \text{ g/cm}^3$ and $Y_e \simeq 1/2$ the number of electrons in a volume $\sim (1 \mu\text{m})^3$ is of the order 10^{12} .

technology allows coordinate resolution at a μm level [16]; in this case eq. (5.1) yields the condition $n_e \gtrsim 10^{24} \text{ cm}^{-3}$. For comparison, the electron number density in dry air at sea level at 20°C is $\sim 3.6 \times 10^{20} \text{ cm}^{-3}$. Thus, one might conclude that neutrino oscillations in air may be considered within the standard approach based on the macroscopic evolution equation (3.17) provided that the position resolution of the detector is not better than $\sim 15 \mu\text{m}$.

It is natural to ask, however, whether taking matter effects into account for neutrinos propagating in air (or in any other low-density medium) makes any sense at all. There are two issues to be examined. First, matter effects on neutrino oscillations are typically important when the Wolfenstein potential $\hat{V}^0$ is at least of the same order as the neutrino kinetic energy difference $\Delta m^2/(2E)$.⁶ This means that low-density media are expected to affect flavour transitions of neutrinos of sufficiently high energy. The potential $\hat{V}^0$ can be written in convenient units as

$$\hat{V}^0 = \sqrt{2}G_F n_e \simeq 1.267 \times 10^{-37} \left(\frac{n_e}{\text{cm}^{-3}} \right) \text{ eV}. \quad (5.2)$$

Taking for the estimate Δm^2 to be the “solar” mass squared difference ($\simeq 7.5 \times 10^{-5} \text{ eV}^2$), we find that in air the condition $\hat{V}^0 \gtrsim \Delta m^2/(2E)$ is satisfied for neutrinos of energies $E \gtrsim 800 \text{ GeV}$. A small fraction of atmospheric neutrinos as well as high-energy neutrinos of astrophysical origin satisfy this condition.

However, for matter effects in neutrino oscillations to be noticeable, yet another condition has to be met: neutrinos must propagate sufficiently large distances l in matter [17, 18]. In practical terms, this so-called “minimal length condition” implies that l must at least exceed the refraction length l_0 defined as

$$l_0 \equiv \frac{2\pi}{\sqrt{2}G_F n_e} \simeq 9.8 \times 10^{32} \left(\frac{\text{cm}^{-3}}{n_e} \right) \text{ cm}. \quad (5.3)$$

For air we have $l_0 \sim 3 \times 10^7 \text{ km}$, and so the effects of the earth’s atmosphere on neutrino oscillations can obviously be neglected. Note that the minimum length condition $l \gtrsim l_0$ is quite universal; in particular, it has to be also satisfied for the parametric enhancement of neutrino oscillations, for which the matter-induced neutrino potential $\hat{V}^0$ may be much smaller than $\Delta m^2/(2E)$ [10–12, 17].

What about other low-density media? Consider, e.g., astrophysical neutrinos propagating in outer space. In the interstellar medium, the average electron number density is about 1 cm^{-3} . As follows from eq. (5.3), the minimum length condition then implies that for the effects of the medium on neutrino flavour transitions to be noticeable, the neutrinos should propagate at least distances $l \sim 10^{33} \text{ cm}$, which is about four orders of magnitude larger than the diameter of the observable Universe. Clearly, the effects of the interstellar medium on flavour evolution of astrophysical neutrinos can be safely neglected. More detailed discussion of the consequences of the minimum length condition on neutrino oscillations in various media can be found in [17, 18].

⁶A possible exception is the parametric resonance of neutrino oscillations in matter, see below.

Thus, although in some rarefied media the number density of particles may be too low to allow the transition from microscopic to macroscopic description of neutrino flavour evolution, quite often the minimum length condition is then also violated. In those cases one can simply neglect all matter effects and consider neutrino oscillations as occurring in vacuum. Obviously, no averaging is needed in such situations.

Let us now return to the lower bound (5.1) on the electron number density that follows from the coarse-graining conditions and is related to the coordinate resolution of the detector. When is it more restrictive than the constraint coming from the minimum length condition $l \gtrsim l_0$ and so has to be taken into account? Comparing eqs. (5.1) and (5.3), we find that this happens when

$$v_d^{1/3} \lesssim 10^{-7} \left(\frac{l}{1 \text{ cm}} \right)^{1/3} \text{ cm}. \quad (5.4)$$

As an example, for the coordinate resolution of the neutrino detection $v_d^{1/3} \sim 1 \mu\text{m}$ the condition in eq. (5.1) is more restrictive than the minimum length condition if the distance l traveled by neutrinos in matter exceeds about 10^4 km , which is of the same order as the diameter of the earth. Eq. (5.1) then requires $n_e \gtrsim 10^{24} \text{ cm}^{-3}$, that is, the average matter density should satisfy $\rho \gtrsim 3 \text{ g/cm}^3$, which is fulfilled for the matter of the earth.

6 Media with bulk currents and magnetization

As was pointed out in section 2, the spatial component $\vec{V}$ of the matter-induced neutrino potential and the axial-vector part of its time component, V^{0A} , have negligible effects on neutrino flavour transitions in media consisting of nonrelativistic or randomly moving particles with no spin polarization. They may, however, play an important role for neutrinos propagating in magnetized backgrounds or in media with bulk particle currents which may exist, e.g., at certain stages of supernova explosions. In this section we briefly discuss the averaging of the neutrino evolution equation with the $\vec{V}$ and V^{0A} contributions included.

Consider first neutrino propagation in a medium with unpolarized electrons; the axial-vector part of the neutrino potential then vanishes. We will assume, however, that there are macroscopic electron currents in the medium. In this case one should take into account the spatial component of the polar-vector part of the neutrino potential. For pointlike electrons we have

$$\vec{V}^V = \vec{V}^V(\vec{x}, \{\vec{v}_i\}) = \sqrt{2} G_F v_F \sum_i \vec{v}_i \delta^3(\vec{x} - \vec{x}_i), \quad (6.1)$$

where $\vec{v}_i$ is the velocity of the i th electron. For velocity-dependent quantities the averaging procedure of eq. (3.1) has to be modified: in addition to the spatial averaging, it should include the averaging over the electron velocities. Keeping the same “hat” notation for the averaged quantities as before, we now define the average of a function $g(\vec{x}, \vec{v})$ as

$$\hat{g}(\vec{x}) \equiv \frac{1}{v_0} \int_{v_0} d^3 x' \int d^3 v f_{\vec{x}}(\vec{v}) g(\vec{x} + \vec{x}', \vec{v}). \quad (6.2)$$

Here $f_{\vec{x}}(\vec{v})$ is the electron velocity distribution function, which we assume to be time-independent⁷ and normalized according to $\int d^3v f_{\vec{x}}(\vec{v}) = 1$. Note that it may be different in different parts of the system but, as any other intensive macroscopic characteristic of the medium, it is essentially position-independent over distances of the order of the linear size of v_0 .⁸ For this reason we use $f_{\vec{x}}(\vec{v})$ rather than $f_{\vec{x}+\vec{x}'}(\vec{v})$ in the integrand in eq. (6.2). The averaged quantities, however, in general depend on the position of the point $\vec{x}$ on which the volume v_0 is centered. In particular, for the average of the microscopic potential (6.1) we find

$$\hat{\vec{V}}^V(\vec{x}) = \sqrt{2} G_F v_F n_e(\vec{x}) \hat{v}_e(\vec{x}), \quad (6.3)$$

where

$$\hat{v}_e(\vec{x}) = \int d^3v f_{\vec{x}}(\vec{v}) \vec{v} \quad (6.4)$$

is the macroscopic local electron velocity.

Let us now discuss the averaging of the evolution equation (2.10). Obviously, for velocity-independent terms of this equation the averaging procedure (6.2) gives the same results as the one in eq. (3.1). Consider the averaging of the terms proportional to $\vec{V}^V \phi$. Straightforward calculation gives

$$\widehat{(\vec{V}^V \phi)}(\vec{x}) = \hat{\vec{V}}^V(\vec{x}) [\hat{\phi}(\vec{x})]_{\text{M.C.}} \quad (6.5)$$

with $[\hat{\phi}(\vec{x})]_{\text{M.C.}}$ defined in eq. (3.10). As was discussed in section 3, since the averaging volume v_0 contains a macroscopically large number of randomly distributed electrons, one can safely replace $[\hat{\phi}(\vec{x})]_{\text{M.C.}}$ by $\hat{\phi}(\vec{x})$, and eq. (6.5) becomes $\widehat{(\vec{V}^V \phi)}(\vec{x}) = \hat{\vec{V}}^V(\vec{x}) \hat{\phi}(\vec{x})$. The averaging of the terms in eq. (2.10) containing the spatial derivatives of $\vec{V}^V$ is then done similarly to the averaging of $(\vec{\nabla} V^0) \phi$ performed in section 3, and we obtain

$$\widehat{(\nabla^i \vec{V}^V)}(\vec{x}) = [\nabla^i \hat{\vec{V}}^V(\vec{x})] \hat{\phi}(\vec{x}). \quad (6.6)$$

Let us now turn to neutrino propagation in magnetized media. In this case the electrons of the medium have non-vanishing average spin polarization, which leads to a non-zero axial-vector contribution to the macroscopic neutrino potential $\hat{V}^\mu$. The averaging procedure of eq. (6.2) will then have to be modified in the following way:

- The electron velocity distribution function $f_{\vec{x}}(\vec{v})$ should be replaced by the electron velocity and spin distribution function $f_{\vec{x}}(\vec{v}, \vec{s})$, where $\vec{s} = \psi_e^\dagger \vec{\sigma}_e \psi_e$ is the electron polarization vector, ψ_e being the 2-component electron spinor.
- In addition to integrations over $\vec{x}'$ and $\vec{v}$, the summation over the electron polarization states should be performed.

⁷Actually, it will be sufficient for us to assume that the velocity distribution changes negligibly over the time intervals of the order of the time of neutrino passage through the averaging volume v_0 .

⁸The velocity distribution function $f_{\vec{x}}(\vec{v})$ can be considered as an average of the phase space density $\mathcal{F}(\vec{x}, \vec{v})$ over spatial volumes of the order of v_0 [19].

The averaging of the neutrino evolution equation (2.10) is then done along the same lines as discussed above. The resulting macroscopic evolution equation has the same form as eq. (2.10), but with the V^0 , $\vec{V}$ and ϕ replaced by $\hat{V}^0$, $\hat{\vec{V}}$ and $\hat{\phi}$ respectively. As discussed in section 3, the terms with spatial derivatives of the components of the macroscopic neutrino potential $\hat{V}^\mu$ can then be neglected, and the evolution equation can be reduced to a first order one. It coincides in form with that in eq. (3.17), except that the macroscopic potential $\hat{V}^0$ has to be replaced by $\hat{V} \equiv \hat{V}^0 - \vec{v}\hat{\vec{V}}$, where the coarse-graining procedure now in general includes, in addition to the spatial averaging, the averaging over the velocities and spin polarizations of the background electrons.

7 The case of Majorana neutrinos

In section 2, the microscopic evolution equation describing neutrino oscillations in matter was derived in the Dirac neutrino case from the equations of motion for the LH and RH neutrino fields, (2.8) and (2.9). For Majorana neutrinos, the LH and RH neutrino fields are not independent: they are related by $\chi = -i\sigma_2\phi^*$. Equations of motion (2.5) and (2.6) then have to be replaced by

$$(i\partial^0 - i\vec{\sigma}\vec{\nabla})\phi - M(-i\sigma_2\phi^*) = (V^0 + \vec{\sigma}\vec{V})\phi, \quad (7.1)$$

$$(i\partial^0 + i\vec{\sigma}\vec{\nabla})(-i\sigma_2\phi^*) - M^*\phi = (-V^0 + \vec{\sigma}\vec{V})(-i\sigma_2\phi^*). \quad (7.2)$$

Note that eqs. (7.1) and (7.2) are actually equivalent. Unlike its Dirac-case analogue (2.6), eq. (7.2) depends on the neutrino potentials ($V^0, \vec{V}$); this is because, in contrast to the Dirac neutrino case, for Majorana neutrinos the RH neutrino states are not sterile.

As in section 2, we look for the solution of the equations of motion in the form $\phi(x) = e^{-iEt}\phi(\vec{x})$. Eliminating ϕ^* from eqs. (7.1) and (7.2), to lowest order in V^0/E and $MM^\dagger/E^2$ we again obtain eq. (2.10). Its averaging and the transition to the first-order neutrino evolution equation are then carried out exactly as in the Dirac neutrino case, leading to the same macroscopic evolution equation. Thus, under our assumption that neutrinos are relativistic with $E \gg M, |V^\mu|$, flavour transitions of Dirac and Majorana neutrinos in matter are described by the same evolution equation.

8 Summary and discussion

In this paper we considered a transition from microscopic (fine-grained) to macroscopic (coarse-grained) description of neutrino flavour transitions in matter through a proper averaging in coordinate space. Our primary motivation was to justify neglecting the term proportional to the gradient of the potential in the neutrino evolution equation, which allows one to reduce this equation to the standard form (3.17). However, the transition to a statistical (macroscopic) description is also necessary on more general grounds: it is not possible in practice to solve the microscopic evolution equation, not even to mention that this would require the knowledge of the coordinates of all the matter constituents. As neutrino experiments yield the detection data averaged over macroscopic volumes determined

by the coordinate resolution of the detectors, coarse-grained neutrino wave functions are adequate for the practical purposes of predicting the expected outcomes of the experiments or interpreting the obtained data.

In all the previous studies of neutrino flavour transitions in matter it was implicitly assumed that some averaging of the microscopic neutrino potential has already been done. A consistent approach, however, would require to average the microscopic evolution equation as a whole and not just the neutrino potential. To the best of our knowledge, no such averaging has been carried out in the past.

In the present paper we performed the averaging of the microscopic neutrino evolution equation by integrating it over a small but macroscopic volume v_0 around each point in coordinate space. The choice of the averaging volume was dictated by a number of factors. On the one hand, it must be large enough to contain macroscopically large numbers N_0 of the particles of the medium. This allows a statistical description of the medium. On the other hand, v_0 must be small enough so that inside it one could consider the intensive macroscopic characteristics of the medium as nearly constant. Another upper limit on the averaging volume v_0 comes from experimental considerations: it should not exceed the volume v_d of the detection region that is determined by the spatial resolution of the detector. The same consideration, together with the requirement that the number of particles in the volume v_0 be macroscopically large, puts a lower limit on the number density of the particles in the medium.

In the course of the averaging of the microscopic neutrino evolution equation in matter one encounters a difficulty related to the presence of the terms containing the products of the neutrino potential or its gradient and the neutrino wave function. As the average of the product of two functions is in general different from the product of their averages, such terms require special consideration. We have demonstrated that for the product terms in the neutrino evolution equation the factorization does take place with very high accuracy provided that the electrons of the medium are randomly distributed in the averaging volumes and that the total number N_0 of electrons in each averaging volume v_0 is macroscopically large. Our key observation was that under these conditions one can replace the integral over the averaging volume v_0 of the neutrino wave function by its basic Monte Carlo estimator, which immediately leads to the desired factorization.

We have also established a lower bound on the number density of the particles of the medium that has to be satisfied in order for the coarse-graining procedure to be adequate to experiments with a given spatial resolution of the neutrino detection events. This bound, in principle, establishes under what conditions neutrino oscillations in low-density media can be described by the standard macroscopic evolution equation. This condition, however, becomes irrelevant if the matter density is so small that the distance neutrinos propagate in it is small compared to the refraction length l_0 defined in eq. (5.3). This is because in this case matter effects on neutrino oscillations can be safely neglected.

To conclude, by performing a coarse-graining of the microscopic neutrino evolution equation in matter we have derived and justified the standard macroscopic evolution equation (3.17) which was previously used without proper justification. We have also found the validity conditions for this equation. In addition to the usual requirement that the neutri-

nos must be relativistic with $E \gg M, |V^\mu|$, we had to assume that the averaging volumes v_0 contain macroscopically large numbers of electrons which are distributed randomly within v_0 . Our treatment is therefore not applicable to the case of neutrino propagation in media that are ordered at the subatomic level, such as crystals.⁹ It is, however, still valid for the macroscopically ordered media, such as e.g. periodic structures with macroscopic periods of density modulation, including those that lead to parametric enhancement of neutrino oscillations in matter.

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A Derivation of eq. (2.10)

Acting on eq. (2.8) with $(E + i\vec{\sigma}\vec{\nabla})$ and making use of eq. (2.9), we obtain

$$\left[\vec{\nabla}^2 + E^2 - MM^\dagger\right]\phi = \left(E + i\vec{\sigma}\vec{\nabla}\right)\left[(V^0 + \vec{\sigma}\vec{V})\phi\right]. \quad (\text{A.1})$$

For the right-hand side of this equation we have

$$\begin{aligned} \text{r.h.s.} &= E(V^0 + \vec{\sigma}\vec{V})\phi + i(\vec{\sigma}\vec{\nabla}V^0)\phi + V^0(i\vec{\sigma}\vec{\nabla}\phi) + i\vec{\sigma}\vec{\nabla}(\vec{\sigma}\vec{V}\phi) \\ &= E(V^0 + \vec{\sigma}\vec{V})\phi + i(\vec{\sigma}\vec{\nabla}V^0)\phi + V^0(i\vec{\sigma}\vec{\nabla}\phi) + i[\vec{\sigma}\vec{\nabla}(\vec{\sigma}\vec{V})]\phi + i\sigma^i\sigma^k V^k \nabla^i \phi \\ &= E(V^0 + \vec{\sigma}\vec{V})\phi + i[\vec{\sigma}\vec{\nabla}(V^0 + \vec{\sigma}\vec{V})]\phi + V^0(i\vec{\sigma}\vec{\nabla}\phi) + i[-\sigma^k\sigma^i + 2\delta^{ik}]V^k \nabla^i \phi \\ &= E(V^0 + \vec{\sigma}\vec{V})\phi + i[\vec{\sigma}\vec{\nabla}(V^0 + \vec{\sigma}\vec{V})]\phi + (V^0 - \vec{\sigma}\vec{V})(i\vec{\sigma}\vec{\nabla}\phi) + 2\vec{V}(i\vec{\nabla}\phi). \end{aligned} \quad (\text{A.2})$$

We assume neutrinos to be relativistic with $E \gg M, |V^\mu|$ and evaluate r.h.s. to leading order in $MM^\dagger/E^2$ and $|V^\mu|/E$. As follows from eqs. (2.8) and (2.9), the expression $i\vec{\sigma}\vec{\nabla}\phi$, which enters in eq. (A.2) multiplied by the components of the neutrino potential V^μ , can then be replaced there by $E\phi$. In addition, one can replace in eq. (A.2) the quantity $i\vec{\nabla}\phi$ by $-\vec{p}\phi = -E\vec{v}\phi$, where $\vec{p}$ is the neutrino momentum and $\vec{v}$ is its velocity.¹⁰ Taking this into account, from the last line in eq. (A.2) we find

$$\text{r.h.s.} = 2E(V^0 - \vec{v}\vec{V})\phi + i[\vec{\sigma}\vec{\nabla}(V^0 + \vec{\sigma}\vec{V})]\phi. \quad (\text{A.3})$$

Substituting this for the right-hand side of eq. (A.1), we arrive at eq. (2.10).

⁹Note that for such media it is sometimes possible to solve microscopic neutrino evolution equations [20].

¹⁰Strictly speaking, propagating neutrinos are described not by plane waves of momentum $\vec{p}$, but by wave packets with mean momentum $\vec{p}$ and momentum uncertainty σ_p . Therefore, $i\vec{\nabla}\phi = [-\vec{p} + \mathcal{O}(\sigma_p)]\phi$. However, because $\sigma_p \ll |\vec{p}|$ and the expression $i\vec{\nabla}\phi$ enters in eq. (A.2) multiplied by $\vec{V}$, the term proportional to σ_p is subleading and can to the adopted accuracy be neglected.

B Neutrino coherent forward scattering on electrons in atoms and molecules

We generalize here the results of sections 3 and 4 to the case of neutrino forward scattering on atomic and molecular electrons. The results of section 6 can be generalized quite similarly.

B.1 Approximation of pointlike atoms

Consider first the idealized situation when one can neglect the size of the atoms and treat them as pointlike objects. This case is similar to the one considered in sections 3 and 4, the difference being that the scatterers may now have electron numbers different from one.

Let the medium consist of K types of pointlike objects (scatterers) with the electron numbers Z_k ($k = 1, \dots, K$), and let the total numbers of the scatterers of the k th type in the medium be N_k . The microscopic neutrino potential is in this case

$$V^0(\vec{x}) = \sqrt{2}G_F v_F \sum_{k=1}^K Z_k \sum_{i=1}^{N_k} \delta^3(\vec{x} - \vec{x}_i). \quad (\text{B.1})$$

Its averaging leads to the standard Wolfenstein potential (3.7) with the macroscopic electron number density

$$n_e(\vec{x}) = \frac{1}{v_0} \sum_{k=1}^K Z_k N_{0k}(\vec{x}, v_0). \quad (\text{B.2})$$

Here $N_{0k}(\vec{x}, v_0)$ is the number of the scatterers of the k th type in the averaging volume v_0 around the point $\vec{x}$, so that the sum in (B.2) is just the total number of electrons in v_0 . Substituting the expression for $V^0(\vec{x})$ from eq. (B.1) into eq. (3.8) yields eq. (3.9), where $[\hat{\phi}(\vec{x})]_{\text{M.C.}}$ is now given by

$$[\hat{\phi}(\vec{x})]_{\text{M.C.}} \equiv \frac{\sum_{k=1}^K Z_k \sum_{i=1}^{N_{0k}(\vec{x}, v_0)} \phi(\vec{x}_i)}{\sum_{k=1}^K Z_k N_{0k}(\vec{x}, v_0)}. \quad (\text{B.3})$$

As before, we assume that all the scatterers are randomly and uniformly distributed in the averaging volumes v_0 ; the obtained results then essentially coincide with those of section 3 and 4. In particular, the macroscopic neutrino evolution equation is again given by eq. (3.17); it is also easy to show that the expected value of $[\hat{\phi}(\vec{x})]_{\text{M.C.}}$ coincides with $\hat{\phi}(\vec{x})$. For the variance of $[\hat{\phi}(\vec{x})]_{\text{M.C.}}$ we find

$$\text{var} \left([\hat{\phi}(\vec{x})]_{\text{M.C.}} \right) = \frac{\sum_{k=1}^K Z_k^2 N_{0k}(\vec{x}, v_0)}{\left[\sum_{k=1}^K Z_k N_{0k}(\vec{x}, v_0) \right]^2} \sigma^2 [\phi(\vec{x})]. \quad (\text{B.4})$$

This quantity typically scales as $\sim 1/N_{0j}$,¹¹ where N_{0j} is the number of the scatterers with the largest $Z_j N_{0j}$ contained in the averaging volume v_0 .

¹¹Except when $Z_j^2 N_{0j} \ll Z_k^2 N_{0k}$ for some $k \neq j$; in that case $\text{var}([\hat{\phi}(\vec{x})]_{\text{M.C.}})$ scales as $(Z_k^2/Z_j^2)(N_{0k}/N_{0j}^2)$.

B.2 Neutrino forward scattering on atoms and molecules of finite size

Let us now lift our assumption of pointlike atoms and consider a medium consisting of atoms of finite size described by atomic wave functions. As in the previous subsection, we shall consider a medium containing K types of atoms with atomic numbers Z_k ($k = 1, \dots, K$). Let the atomic wave functions be $\Psi_k(\vec{x}_1, \dots, \vec{x}_{Z_k}; \vec{x}_0)$, where $\vec{x}_0$ is the coordinate of the center of the atom. For an atom with Z_k electrons and the center at $\vec{x}_0$, the electron number density is

$$\rho_k(\vec{x}, \vec{x}_0) = \sum_{a=1}^{Z_k} \int |\Psi_k(\vec{x}_1, \dots, \vec{x}_{Z_k}; \vec{x}_0)|^2 \delta^3(\vec{x} - \vec{x}_a) d^3x_1 \dots d^3x_{Z_k}. \quad (\text{B.5})$$

We adopt the standard normalization convention in which the integrals of the squared moduli of the atomic wave functions are normalized to unity; $\rho_k(\vec{x}, \vec{x}_0)$ then satisfies

$$\int d^3x \rho_k(\vec{x}, \vec{x}_0) = Z_k. \quad (\text{B.6})$$

The microscopic neutrino potential is in this case

$$V^0(\vec{x}) = \sqrt{2} G_F v_F \sum_{k=1}^K \sum_{i=1}^{N_k} \rho_k(\vec{x}, \vec{x}_i). \quad (\text{B.7})$$

The coarse-grained neutrino potential $\hat{V}^0(\vec{x})$ takes the standard form (3.7) with the macroscopic electron density $n_e(\vec{x})$ given by eq. (B.2), as in the case of neutrino forward scattering on pointlike atoms. The average of the product $V^0(\vec{x})\phi(\vec{x})$ has the same form as in eq. (3.9), but with $[\hat{\phi}(\vec{x})]_{\text{M.C.}}$ defined as

$$[\hat{\phi}(\vec{x})]_{\text{M.C.}} \equiv \frac{\sum_{k=1}^K \sum_{i=1}^{N_{0k}(\vec{x}, v_0)} \Phi_k(\vec{x}, \vec{x}_i)}{\sum_{k=1}^K Z_k N_{0k}(\vec{x}, v_0)}, \quad (\text{B.8})$$

where

$$\Phi_k(\vec{x}, \vec{x}_i) \equiv \int_{v_0} d^3x' \rho_k(\vec{x} + \vec{x}', \vec{x}_i) \phi(\vec{x} + \vec{x}'). \quad (\text{B.9})$$

It is easy to show that $[\hat{\phi}(\vec{x})]_{\text{M.C.}}$ is an unbiased estimator of $\hat{\phi}(\vec{x})$, i.e. $E([\hat{\phi}(\vec{x})]_{\text{M.C.}}) = \hat{\phi}(\vec{x})$. To prove this, let us first notice that due to translational invariance $\rho_k(\vec{x}, \vec{x}_0) = \rho_k(\vec{x} - \vec{x}_0)$, so that in the normalization condition (B.6) one can replace the integration over $\vec{x}$ by that over $\vec{x}_0$. Therefore,

$$E(\Phi_k(\vec{x}, \vec{x}_i)) = \int d^3x_i \Phi_k(\vec{x}, \vec{x}_i) \text{PDF}(\vec{x}_i) = Z_k \hat{\phi}(\vec{x}), \quad (\text{B.10})$$

where, as usual, we assumed that the scattering centers are randomly distributed in v_0 with the uniform probability distribution function. Together with the definition of $[\hat{\phi}(\vec{x})]_{\text{M.C.}}$ in eq. (B.8), this gives $E([\hat{\phi}(\vec{x})]_{\text{M.C.}}) = \hat{\phi}(\vec{x})$.

For the variance of $[\hat{\phi}(\vec{x})]_{\text{M.C.}}$ we find

$$\text{var}([\hat{\phi}(\vec{x})]_{\text{M.C.}}) = \frac{\sum_{k=1}^K Z_k^2 N_{0k}(\vec{x}, v_0) \sigma^2[\Phi_k(\vec{x})]}{[\sum_{k=1}^K Z_k N_{0k}(\vec{x}, v_0)]^2}, \quad (\text{B.11})$$

where

$$\sigma^2[\Phi_k(\vec{x})] \equiv \frac{1}{Z_k^2 v_0} \int_{v_0} d^3 x_i |\Phi_k(\vec{x}, \vec{x}_i)|^2 - |\hat{\phi}(\vec{x})|^2. \quad (\text{B.12})$$

This quantity depends on the atomic wave function $\Psi_k(\vec{x}_1, \dots, \vec{x}_{Z_k}; \vec{x}_i)$ and, unlike $\sigma[\phi(\vec{x})]$, it cannot be expressed solely through $\phi(\vec{x})$. However, it shares with $\sigma[\phi(\vec{x})]$ the property of being strongly suppressed for nearly constant $\phi(\vec{x})$ and also vanishes when $\phi(\vec{x})$ is constant. Just like in the case of neutrino forward scattering on pointlike atoms, $\text{var}([\hat{\phi}(\vec{x})]_{\text{M.C.}})$ typically scales as $\sim 1/N_{0j}$, where N_{0j} is the number of the scatterers with the largest $Z_j N_{0j}$ contained in the averaging volume v_0 .

As the averaging volumes v_0 are assumed to contain macroscopically large numbers of atoms, it is justified to replace the averaged quantities $\hat{\phi}(\vec{x})$ and $\vec{\nabla}\hat{\phi}(\vec{x})$ by their Monte Carlo estimators. Thus, for the case of neutrino forward scattering on atomic electrons the results of the averaging of the microscopic neutrino evolution equation coincide with those obtained in section 3. The difference is that for the Monte Carlo estimator of the macroscopic neutrino field $\hat{\phi}(\vec{x})$ we actually take, instead of a linear combination of the values of ϕ at random coordinates $\vec{x}_i$ in v_0 , a linear combination of the values of ϕ averaged over small (atomic size) volumes around the random coordinates of the centers of the atoms inside v_0 , and similarly for $\vec{\nabla}\hat{\phi}(\vec{x})$.

We were assuming here that all the electrons of the medium are contained in atoms; it is easy to see, however, that our results are also directly applicable to the case of media consisting of arbitrary mixture of molecules, neutral atoms, ions and free electrons.

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