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Entanglement spectrum of geometric states

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ABSTRACT: The reduced density matrix of a given subsystem, denoted by ρ_A , contains the information on subregion duality in a holographic theory. We may extract the information by using the spectrum (eigenvalue) of the matrix, called entanglement spectrum in this paper. We evaluate the density of eigenstates, one-point and two-point correlation functions in the microcanonical ensemble state $\rho_{A,m}$ associated with an eigenvalue λ for some examples, including a single interval and two intervals in vacuum state of 2D CFTs. We find there exists a microcanonical ensemble state with λ_0 which can be seen as an approximate state of ρ_A . The parameter λ_0 is obtained in the two examples. For a general geometric state, the approximate microcanonical ensemble state also exists. The parameter λ_0 is associated with the entanglement entropy of A and Rényi entropy in the limit $n \rightarrow \infty$. As an application of the above conclusion we reform the equality case of the Araki-Lieb inequality of the entanglement entropies of two intervals in vacuum state of 2D CFTs as conditions of Holevo information. We show the constraints on the eigenstates. Finally, we point out some unsolved problems and their significance on understanding the geometric states.

KEYWORDS: AdS-CFT Correspondence, Conformal Field Theory

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1 Introduction

For a quantum field theory with a gravity dual, there exists some states that can be effectively described by a classical geometry in the semi-classical limit $G \rightarrow 0$. We call these states geometric states. These states should show special properties, for example the probes, such as correlation functions, in geometric states should be well-defined in the limit $G \rightarrow 0$. Among all the probes there exists some special ones that can be associated with geometric objects in the bulk. We may call them geometric probes.

At present we still have no methods to judge whether a state is geometric or not. But we can use the geometric probes to detect the properties of geometric states. Quantum entanglement is a useful concept to understand the geometry. Among various measures of entanglement, entanglement entropy (EE) of a subsystem A is the most useful one since the discovery of the Ryu-Takayanagi (RT) formula [1]

$$S_A = \frac{\min(\text{Area}\{\gamma_A\})}{4G}, \quad (1.1)$$

where S_A is the EE of the subsystem A , γ_A is the bulk surface that is homologous to A . By using this simple relation one may obtain fruitful results which help us to catch the properties of the geometric states. Motivated by the holographic EE, in [2, 3] the author finds the secret relation between quantum entanglement and connectivity of spacetime. Another way to understand geometric states is using the analogy between tensor networks and the geometry of spatial slice of AdS/CFT [4]. The geometric states can also be constrained by using the inequality of entropy. In [5] the authors show the so-called tripartite information (a linear combinations of EEs of arbitrary three regions) in the geometric states should be non-negative, thus give strong constraints on the geometric states. Direct construction of geometric states as coherent state are studied in [6]–[8], see also [9]–[13]. On the field theory side we find a series of necessary conditions on expectation values of quasi-primary operators in AdS₃/CFT₂ by using the scaling behavior of large c limit [14].

Here we only list some attempts to characterize various aspects of geometric states. Many of them are associated with EE. But EE as a functional of the reduced density matrix ρ_A only contains limited information of ρ_A . In this paper one of our motivation is to study the spectrum of ρ_A for the geometric states. We will shortly call them entanglement spectrum following [15], in which it is used to identify topological order. In this paper we will also use the term “eigenvalue” instead of spectrum, though in general they may be different.

Generally, the density matrix ρ_A has the following spectrum decomposition

$$\rho_A = \sum_i \lambda_i |\lambda_i\rangle_A \langle \lambda_i|, \quad (1.2)$$

where $|\lambda_i\rangle_A$ are generally degenerate. The maximal eigenvalue of ρ_A is λ_m . We can parameterize the eigenvalue λ as $\lambda = \lambda_m e^{-t}$ with $t \in [0, +\infty)$. The distributions of entanglement spectrum can be obtained if one knows the Rényi entropy $S_A^{(n)}$ of all the index n . This is done in [16] for a single interval in the vacuum state of 2D CFTs. The results can be easily generalized to the cases that are studied in [17]. In this paper we will discuss more

non-trivial examples. Further, by using the method of inverse Laplace transformation similar with [18–20], we also study the correlation functions in the so-called microcanonical ensemble states $\rho_{A,m}$,

$$\rho_{A,m} := \frac{1}{\mathcal{P}(t)} \sum_i |\lambda_i\rangle_A \langle \lambda_i| \delta(t_i - t), \quad (1.3)$$

where $\mathcal{P}(t)$ is the density of eigenstate with respect to t . Our motivation is to explore the behavior of eigenvalues and eigenstates of ρ_A in the large c limit in 2D CFTs. Our examples include a single interval and two intervals in vacuum state. The examples show one could find a microcanonical ensemble state to be an approximate state of ρ_A in the large c limit, if the probes are located in a small region of A and far away from the boundary of A . We make the above conclusions by directly comparing the one-point and two-point functions in $\rho_{A,m}$ with the ones in ρ_A .

The results can be generalized to arbitrary geometric states by using the fact that $S_A^{(n)} \sim O(c)$ or $O(1/G)$, which follows the holographic Rényi entropy proposal in [21]. The microcanonical ensemble state are associated with the EE $S(\rho_A)$ and $S^\infty = \lim_{n \rightarrow \infty} S_A^{(n)}$. More precisely, the parameter t_0 of the special microcanonical ensemble state is given by

$$t_0 = S(\rho_A) - S^\infty. \quad (1.4)$$

Our results actually give a general feature of the entanglement spectrum of geometric states.

One of the interesting phenomenon that one can obtain from (1.1) is the phase transition of the EE of two disconnected subsystems. Without loss of generality we will focus on two intervals in the vacuum state of 2D CFTs. In this paper we will choose $A_1 = [-R, -T]$ and $A_2 = [T, R]$ with $R > T > 0$ and denote $A_3 = [-T, T]$. By using the RT formula (1.1) one can find the critical point at $T_0 = (3 - 2\sqrt{2})R$. For $T > T_0$ we have $S_{A_1 A_2} = S_{A_1} + S_{A_2}$ which means A_1 and A_2 lose correlations. For the case $T < T_0$ we have

$$S_{A_1 A_2} = S_{A_3} + S_{A_1 A_2 A_3}. \quad (1.5)$$

The equality is satisfied for the Araki-Lieb inequality $S_{AB} \geq S_A - S_B$ with $A = A_1 A_2$ and $B = A_3$.

In this paper we also study the implication of the equality condition of Araki-Lieb inequality on the spectrum of the reduced density matrices $\rho_{A_1 A_2 A_3}$ and ρ_{A_3} . This is based on reforming the above condition as Holevo information, that is

$$\chi(\rho_{A_1 A_2}) = H(\lambda_i) + O(c^0) \quad \text{and} \quad \chi(\rho_{A_3}) = 0 + O(c^0), \quad (1.6)$$

where $H(\lambda_i)$ is the EE of A , $\rho_{A_1 A_2} = \sum_i \lambda_i \rho_{i, A_1 A_2}$ and $\rho_{A_3} = \sum_i \lambda_i \rho_{i, A_3}$ with $\rho_{i, A_1 A_2} := \text{tr}_{A_3} |\lambda_i\rangle_A \langle \lambda_i|$ and $\rho_{i, A_3} := \text{tr}_{A_1 A_2} |\lambda_i\rangle_A \langle \lambda_i|$. This condition gives the constraints on measurement or quasi-primary operators of the vacuum conformal family in the single eigenstate.

Finally, we discuss the possible extensions based on present paper. One interesting question is how to find the critical point of the distinguishability and indistinguishability

of $\rho_{A,m}$ with $t = t_0$ from ρ_A . We also discuss the possibility of the Holevo information as a geometric probe. At last we compare the microcanonical ensemble state in this paper with the approximate state constructed by tensor networks and the fixed-area state in the quantum error-correcting code of AdS.

2 Entanglement spectrum and microcanonical ensemble state

2.1 Density of eigenstates

For a pure state $|\psi\rangle$ a Schmidt decomposition of the subsystem A and its complement $\bar{A}$ is given by

$$|\psi\rangle = \sum_i \sqrt{\lambda_i} |\lambda_i\rangle_A \otimes |\bar{\lambda}_i\rangle_{\bar{A}}, \quad (2.1)$$

where $|\lambda_i\rangle_{A(\bar{A})} \in \mathcal{H}_{A(\bar{A})}$ satisfy ${}_{A(\bar{A})}\langle\lambda_i|\lambda_j\rangle_{A(\bar{A})} = \delta_{ij}$. It is obvious that $|\lambda_i\rangle_{A(\bar{A})}$ are the eigenstates of $\rho_{A(\bar{A})}$. For QFTs the spectrum of ρ_A is continous, thus the sum of the decomposition should be replaced by integration. If $|\psi\rangle = |0\rangle$ and A is an interval, the eigenvalues and the corresponding eigenstates should only depend on the length of the interval by translation invariance. The reduced density matrix of A is $\rho_A = \sum_i \lambda_i |\lambda_i\rangle_A {}_A\langle\lambda_i|$, where λ_i is the eigenvalue of ρ_A . The EE of A is given by the Shannon entropy $H(\lambda_i) := -\sum_i \lambda_i \log \lambda_i$.

With the Rényi entropy $S_A^{(n)} := \frac{\log \text{tr} \rho_A^n}{1-n}$ of all the index n , one could construct density of eigenstates of the reduced density matrix ρ_A . The density of the states at the eigenvalues λ is defined as

$$P(\lambda) := \sum_k \delta(\lambda_k - \lambda), \quad (2.2)$$

which satisfies the normalization condition $\int_0^{\lambda_m} d\lambda \lambda P(\lambda) = 1$ with λ_m is the maximal eigenvalue of ρ_A . One may calculate $P(\lambda)$ of one interval in vacuum state of 2D CFTs by using the method in [16]. For our motivation we will use the inverse Laplace transformation method that is mentioned in the same reference. The Laplace transformation method is used in [22] to study the entanglement sepectrum of a sphere in the vacuum state. By the definition of the Rényi entropy we can find the maximal eigenvalue λ_m of ρ_A by

$$b := -\log \lambda_m = \lim_{n \rightarrow \infty} S_A^{(n)}. \quad (2.3)$$

We also have

$$\text{tr} \rho_A^n = \int_0^{\lambda_m} d\lambda \lambda^n P(\lambda) = e^{(1-n)S_A^{(n)}}. \quad (2.4)$$

Let's parameterize λ as $\lambda = \lambda_m e^{-t}$, the integral becomes the form of Laplace transformation,

$$\int_0^{+\infty} P(\lambda_m e^{-t}) e^{-(n+1)t} dt = \lambda_m^{-(n+1)} e^{(1-n)S_A^{(n)}}. \quad (2.5)$$

By inverse Laplace transform we have the density of eigenstates with respect to t

$$\mathcal{P}(t) := P(\lambda_m e^{-t}) \lambda_m e^{-t} = \mathcal{L}^{-1} \left[\lambda_m^{-n} e^{(1-n)S_A^{(n)}} \right] (t) := \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} dn e^{nt} e^{nb+(1-n)S_A^{(n)}}. \quad (2.6)$$

Using this one can get the density of eigenstates $P(\lambda)$

$$P(\lambda) = \frac{\lambda_m}{\lambda} \mathcal{L}^{-1} [\lambda_m^{-(n+1)} e^{(1-n)S_A^{(n)}}](t) \Big|_{t \rightarrow \log \frac{\lambda_m}{\lambda}}. \quad (2.7)$$

2.1.1 One-point functions

The one-point correlation functions are useful probes to study the properties of the eigenstates of ρ_A . For an operator $\mathcal{O}$ we define the one-point functions

$$P_{\mathcal{O}}(\lambda) := \sum_i \int_A \langle \lambda_i | \mathcal{O} | \lambda_i \rangle_A \delta(\lambda_i - \lambda). \quad (2.8)$$

The density of eigenstate $P(\lambda)$ is a special case with $\mathcal{O} = I$. We assume the operator $\mathcal{O}$ is located in the region A . So we have

$$\text{tr}(\rho_A \mathcal{O}) = \sum_j \int_A \langle \lambda_j | \mathcal{O} | \lambda_j \rangle_A = \int_0^{\lambda_m} d\lambda \lambda P_{\mathcal{O}}(\lambda), \quad (2.9)$$

and

$$\text{tr}(\rho_A^n \mathcal{O}) := \int_0^{\lambda_m} d\lambda \lambda^n P_{\mathcal{O}}(\lambda). \quad (2.10)$$

Taking $\lambda = \lambda_m e^{-t}$, the above integral becomes

$$\text{tr}(\rho_A^n \mathcal{O}) = \lambda_m^{n+1} \int_0^{+\infty} dt e^{-(n+1)t} P_{\mathcal{O}}(\lambda_m e^{-t}). \quad (2.11)$$

By an inverse Laplace transformation we get

$$\mathcal{P}_{\mathcal{O}}(t) := P_{\mathcal{O}}(\lambda_m e^{-t}) \lambda_m e^{-t} = \mathcal{L}^{-1} [\lambda_m^{-n} \text{tr}(\rho_A^n \mathcal{O})](t). \quad (2.12)$$

The one-point functions $P_{\mathcal{O}}(\lambda) = \mathcal{P}_{\mathcal{O}}(t)/\lambda|_{t \rightarrow \log(\lambda_m/\lambda)}$. To evaluate them one needs $\text{tr}(\rho_A^n \mathcal{O})$. In the path integral formalism $\text{tr}(\rho_A^n) = Z_n(A)/Z_1^n$, where $Z_n(A)$ is the path integral on the n -sheeted surface $\mathcal{R}_n$. By the definition of the one-point function of $\mathcal{O}$ on the surface $\mathcal{R}_n$ we have

$$\langle \mathcal{O} \rangle_{\mathcal{R}_n} := \frac{\int_{\mathcal{R}_n} D\phi \mathcal{O} e^{-I_E}}{\int_{\mathcal{R}_n} D\phi e^{-I_E}} = \frac{\text{tr}(\rho_A^n \mathcal{O})}{\text{tr} \rho_A^n}. \quad (2.13)$$

Thus we have

$$\begin{aligned} \mathcal{P}_{\mathcal{O}}(t) &= \mathcal{L}^{-1} \left[\lambda_m^{-n} e^{(1-n)S_A^{(n)}} \langle \mathcal{O} \rangle_{\mathcal{R}_n} \right](t) \\ &= \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} dn e^{nt} e^{nb+(1-n)S_A^{(n)}} \langle \mathcal{O} \rangle_{\mathcal{R}_n}. \end{aligned} \quad (2.14)$$

For a single interval in some special states of 2D CFTs one may calculate $\langle \mathcal{O} \rangle_{\mathcal{R}_n}$. We will show some examples later. In the limit $n \rightarrow \infty$ the domain contribution to $\text{tr}(\rho_A^n \mathcal{O})$ is given by

$$\lambda_m^n \sum_{m_i} \int_A \langle \lambda_{m_i} | \mathcal{O} | \lambda_{m_i} \rangle_A \delta(\lambda_{m_i} - \lambda_m). \quad (2.15)$$

By using (2.13) we find

$$\sum_{m_i} A \langle \lambda_{m_i} | \mathcal{O} | \lambda_{m_i} \rangle_A \delta(\lambda_{m_i} - \lambda_m) = \lim_{n \rightarrow \infty} \langle \mathcal{O} \rangle_{\mathcal{R}_n}. \quad (2.16)$$

To calculate the one-point functions in other eigenstates one should evaluate the inverse Laplace transformation. We define the average one-point functions

$$\bar{\mathcal{P}}_{\mathcal{O}}(t) := \frac{\mathcal{P}_{\mathcal{O}}(t)}{\mathcal{P}(t)}. \quad (2.17)$$

2.2 Example: one interval in the vacuum state of 2D CFTs

2.2.1 Density of eigstates

Once Rényi entropy of all the index n are known we could obtain the density of eigenstates by using (2.6). For 2D CFTs there are several known examples including a single interval with length l in the vacuum state or finite temperature and half infinite line in the regularized boundary states [17]. For all these examples the modular Hamiltonian H_A can be written as local integral over energy density in the subsystem A ,

$$H_A = \int_A dx f(x) T_{00}(x), \quad (2.18)$$

where $f(x)$ depends on the situation we are considering. For all the examples the Rényi entropy is like the form

$$S_A^{(n)} = \left(1 + \frac{1}{n}\right) b. \quad (2.19)$$

Let's consider some examples. $b = \frac{c}{6} \log \frac{l}{\epsilon}$ if the size of system is infinite, where ϵ is the UV cut-off. $b = \frac{c}{12} \log \frac{2l}{\epsilon}$ if A is the interval at the end of a semi-infinite line. $b = \frac{c}{6} \log \left[\frac{\beta}{\pi \epsilon} \sinh\left(\frac{\pi l}{\beta}\right) \right]$ if the system is in the canonical ensemble thermal state with inverse temperature β .

By using (2.6) and (2.19) we have

$$\mathcal{P}(t) = \delta(t) + \frac{\sqrt{b} I_1(2\sqrt{bt})}{\sqrt{t}} H(t), \quad (2.20)$$

where $H(t)$ is the Heaviside step function, $I_n(x)$ is the modified Bessel function of the first kind. One can check it satisfies the normalization condition

$$\text{tr} \rho_A = \int_0^{+\infty} \mathcal{P}(t) \lambda_m e^{-t} dt = 1, \quad (2.21)$$

and the EE

$$S_A = - \int_0^{+\infty} dt (\lambda_m e^{-t}) \log(\lambda_m e^{-t}) \mathcal{P}(t) = 2b. \quad (2.22)$$

In this paper we are interested in the behavior of $\mathcal{P}(t)$ in the large c limit. By using the fact $I_n(x) \simeq \frac{e^x}{\sqrt{2\pi x}}$ in the large x limit, $\mathcal{P}(t)$ can be approximated by

$$\mathcal{P}(t) \simeq \delta(t) + \frac{b e^{2\sqrt{bt}}}{\sqrt{4\pi} (bt)^{3/4}} H(t). \quad (2.23)$$

For $t = b$ or equally $\lambda = \lambda_0 := e^{-2b}$ we find $\mathcal{P}(t) \sim e^{2b}$. The entropy of the state $\rho_{A,m}$ is $S_{A,m} \simeq 2b + o(c)$. Here we define the microcanonical ensemble state

$$\rho_{A,m} = \frac{1}{\mathcal{P}(t)} \sum_i |\lambda_i\rangle_A \langle \lambda_i| \delta(t_i - t), \quad (2.24)$$

with $t_i = -\log(\lambda_i/\lambda_m)$. The above results show the EE of the microcanonical ensemble state with $t = b$ is equal to the EE of A at the leading order of c . This motivates us to propose the microcanonical ensemble state $\rho_{A,m}$ with $t = b$ can be an approximate state of ρ_A .

2.2.2 One-point functions of quasi-primary operators in the vacuum conformal family

The quasi-primary operators in the vacuum conformal family include the stress energy tensor $T(w)$ with conformal dimension $h_T = 2$, $\mathcal{A} := (TT) - \frac{3}{10}\partial^2 T$ with conformal dimension $h_{\mathcal{A}} = 4$ and so on. In this section for simplicity we only consider T and $\mathcal{A}$. One could refer to [23, 24] for more details on these operators.

By the conformal transformation $z = f(w) = \left(\frac{w+R}{w-R}\right)^{1/n}$ the n -sheet surface $\mathcal{R}_n$ is mapped to complex z -plane. One may get one-point function of T and $\mathcal{A}$ in $\mathcal{R}_n$

$$\langle T \rangle_{\mathcal{R}_n} = \frac{c}{6} \frac{n^2 - 1}{n^2} \frac{R^2}{(R^2 - w^2)^2}, \quad \text{and} \quad \langle \mathcal{A} \rangle_{\mathcal{R}_n} = \frac{c(5c + 22)}{180} \frac{(n^2 - 1)^2}{n^4} \frac{R^4}{(R^2 - w^2)^4} \quad (2.25)$$

Taking $\langle T \rangle_{\mathcal{R}_n}$ into (2.14) we get

$$\mathcal{P}_T(t) = \frac{cR^2}{6(R^2 - w^2)^2} \delta(t) + \frac{cR^2}{6(R^2 - w^2)^2} \frac{\sqrt{b}I_1(2\sqrt{bt})}{\sqrt{t}} \left(1 - \frac{t}{b}\right) H(t). \quad (2.26)$$

The first term gives the contribution from the maximal eigenvalue λ_m , which is consistent with (2.16). For $t \neq 0$ or equally $\lambda \neq \lambda_m$ the average one-point functions of T is

$$\bar{\mathcal{P}}_T(t) = \frac{cR^2}{6(R^2 - w^2)^2} \left(1 - \frac{t}{b}\right). \quad (2.27)$$

Similarly, we can derive

$$\mathcal{P}_{\mathcal{A}}(t) = \frac{c(5c + 22)R^4}{180(R^2 - w^2)^4} \delta(t) + \frac{c(5c + 22)R^4}{180(R^2 - w^2)^4} \frac{\sqrt{b}I_1(2\sqrt{bt})}{\sqrt{t}} \left[1 - 2\frac{t}{b} + \frac{t^2}{b^2} \frac{I_3(2\sqrt{bt})}{I_1(2\sqrt{bt})}\right] H(t). \quad (2.28)$$

For $t \neq 0$ the average one-point function is

$$\bar{\mathcal{P}}_{\mathcal{A}}(t) = \frac{c(5c + 22)R^4}{180(R^2 - w^2)^4} \left[1 - 2\frac{t}{b} + \frac{t^2}{b^2} \frac{I_3(2\sqrt{bt})}{I_1(2\sqrt{bt})}\right]. \quad (2.29)$$

Note that in the limit $c \rightarrow \infty$ the above expression can be approximated by

$$\bar{\mathcal{P}}_{\mathcal{A}}(t) \simeq \frac{c(5c + 22)R^4}{180(R^2 - w^2)^4} \left(1 - \frac{t}{b}\right)^2 \quad (2.30)$$

by using the fact $I_3(2\sqrt{bt})/I_1(2\sqrt{bt}) \rightarrow 1$.

Both of the two examples show the average one-point function $\bar{\mathcal{P}}(t)$ is divergent at R and $-R$, which are the boundary of the subsystem A . Actually this should be a general result for any operators. To obtain a well defined eigenstates and eigenvalues of H_A we should make some a UV cut-off ϵ near the ending point of subsystem A [17]. The divergence of one-point functions is closely related to the regularization. In the large c limit the one-point functions are vanishing in the eigenstates $t_0 = b$ or $\lambda_0 = \lambda_m^2$. This is consistent with our expectation that the microcanonical ensemble state $\rho_{A,m}$ (2.24) can approximate ρ_A since $\text{tr}(\rho_A T), \text{tr}(\rho_A \mathcal{A}) = 0$.

2.2.3 General operators

In appendix A we calculate more examples of the one-point functions of quasi-primary operator in vacuum conformal family. All these examples show $\mathcal{P}_{\mathcal{O}}(t)$ is vanishing for $t = b$ in the large c limit. In this section we would like to show the conclusion is true for any quasi-primary operators in vacuum family.

Under the conformal transformation $z = f(w)$ a general quasi-primary operator $\mathcal{X}(w)$ with conformal dimension $h_{\mathcal{X}}$ would transform as the following,

$$\mathcal{X}(w) = f'(w)^{h_{\mathcal{X}}} \mathcal{X}(z) + \sum_{\mathcal{Y},p} F_{\mathcal{Y},p}[f(w)] \partial^p \mathcal{Y}[f(w)] + S[f(w)], \quad (2.31)$$

where the sum over $\mathcal{Y}$ includes all the quasi-primary operators with conformal dimension $h_{\mathcal{Y}} \leq h_{\mathcal{X}}$, $F_{\mathcal{Y},p}[f(w)]$ and $S[f(w)]$ are functionals of $f(w)$ and its derivatives. By the conformal map $f(w) = \left(\frac{w+R}{w-R}\right)^{1/n}$ the image of n -sheeted surface $\mathcal{R}_n$ is complex z -plane $\mathcal{C}$. By symmetry we have $\langle \mathcal{Y}(z) \rangle_{\mathcal{C}} = 0$. This means $\langle \mathcal{X}(w) \rangle_{\mathcal{R}_n} = S[f(w)]$. Moreover, $S[f(w)]$ are composed by the Schwarzian derivative $s(w)$ and its derivatives, $s'(w), s''(w)$ and so on. By the definition of $s(w)$ we have

$$s(w) = \frac{(n^2 - 1)}{n^2} \frac{2R^2}{(R^2 - w^2)^2}. \quad (2.32)$$

By induction one can expand $S[f(w)]$ as

$$S[f(w)] = \sum_{I=1}^{h_{\mathcal{X}}/2} S_I \left(\frac{n^2 - 1}{n^2} \right)^I, \quad (2.33)$$

where S_I don't depend on n . The lower bound of the sum is not 0 since we expect $S[f(w)]$ is vanishing in the limit $n \rightarrow 1$. By using (2.12) we can derive the one-point functions of $\mathcal{X}$,

$$\mathcal{P}_{\mathcal{X}}(t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} dn e^{nt + \frac{b}{n}} S[f(w)]. \quad (2.34)$$

Since the inverse Laplace transformation is linear, to obtain the above result we only need to evaluate the I -th term

$$\mathcal{P}_{\mathcal{X},I}(t) := \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} dn e^{nt + \frac{b}{n}} \left(\frac{n^2 - 1}{n^2} \right)^I = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} dn e^{nt + \frac{b}{n} + I \log\left(\frac{n^2 - 1}{n^2}\right)}. \quad (2.35)$$

For a given I one could analytically calculate the integration by making derivatives on $\mathcal{P}(t)$ with respect to b . However, it is not easy to find a formula for any I . We assume $t \sim c$ which is very large. The above integral is dominated by a saddle point with

$$n \simeq \sqrt{\frac{b}{t}} + o(c^0). \quad (2.36)$$

So the saddle point approximation of the integral is

$$\mathcal{P}_{\mathcal{X},I}(t) \simeq \sqrt{\frac{b}{t}} e^{2\sqrt{bt}} \left(1 - \frac{t}{b}\right)^I. \quad (2.37)$$

We get the one-point function

$$\mathcal{P}_{\mathcal{X}}(t) = \sum_{I=1}^{h_{\mathcal{X}}/2} S_I \mathcal{P}_{\mathcal{X},I}(t). \quad (2.38)$$

For $t = b$ we conclude that the one-point functions $\mathcal{P}_{\mathcal{X}}(t)$ is vanishing for any quasi-primary operators in the vacuum conformal family.

2.2.4 Modular Hamiltonian

For the examples we consider the modular Hamiltonian can be written as a local integral over energy density in the subsystem A (2.18). One could get the function $f(x)$ by the method used in [17]. Let's consider A to be a single interval on infinite line. The modular Hamiltonian is given by

$$H_A = - \int_{-R}^R dw \frac{R^2 - w^2}{2R} T(w) - \int_{-R}^R d\bar{w} \frac{R^2 - \bar{w}^2}{2R} \bar{T}(\bar{w}). \quad (2.39)$$

We would like to calculate

$$\mathcal{P}_{H_A}(t) := \sum_i \langle \lambda_i | H_A | \lambda_i \rangle_A \delta(t_i - t), \quad (2.40)$$

by using (2.27). Taking (2.27) and (2.39) into the above equation, we find the integral is divergent near the boundary point of A . Therefore, we should make some regularization.

To avoid sharp bipartition of Hilbert space $\mathcal{H}$ into $\mathcal{H}_A$ and $\mathcal{H}_{\bar{A}}$, one should put a UV cut-off around the boundary of A and $\bar{A}$. In the Euclidean path integral expression this is done by introducing a slice around the boundary point of A and $\bar{A}$. The effect on the modular Hamiltonian is changing the integral region to be $[-R + \epsilon, R - \epsilon]$.

For $t \neq 0$ by using (2.27) and (2.39) we have

$$\mathcal{P}_{H_A}(t) = - \int_{-R+\epsilon}^{R-\epsilon} dw \frac{R^2 - w^2}{2R} \mathcal{P}_T(t) - \int_{-R+\epsilon}^{R-\epsilon} d\bar{w} \frac{R^2 - \bar{w}^2}{2R} \mathcal{P}_{\bar{T}}(t) = (t - b) \mathcal{P}(t), \quad (2.41)$$

where we have used $b = \frac{c}{6} \log \frac{2R}{\epsilon}$. For $t = 0$ by using (2.16) we have

$$\mathcal{P}_{H_A}(t = 0) = b. \quad (2.42)$$

With this we can calculate $\text{tr} e^{-H_A}$ by

$$\text{tr}(e^{-H_A}) = e^{-b} + \int_0^{+\infty} dt e^{b-t} \mathcal{P}(t) = e^{2b}, \quad (2.43)$$

where the first term comes from the contribution of $t = 0$. Therefore, the normalized reduced density matrix ρ_A is

$$\rho_A = e^{-H_A - 2b}. \quad (2.44)$$

Notice that the spectrum of $H_A + 2b$ is $t + b$, which is positive. So the operator $H_A + 2b$ is a positive operator.

One may construct other operators like the modular Hamiltonian, such as

$$Q_A = \int_{-R}^R dw \frac{(R^2 - w^2)^3}{R^3} \mathcal{A}(w) + \int_{-R}^R d\bar{w} \frac{(R^2 - \bar{w}^2)^3}{R^3} \bar{\mathcal{A}}(\bar{w}). \quad (2.45)$$

The expectation value of this operator in the microcanonical ensemble state (2.24) is

$$\mathcal{P}_{Q_A}(t) \simeq \frac{bc}{3} \left(1 - \frac{t}{b}\right)^2 \mathcal{P}(t), \quad (2.46)$$

where we have assumed the large c limit and used (2.30). It seems the operators Q_A and H_A don't have common eigenstates $|\lambda_i\rangle_A$. However, Q_A still show very similar behavior as the modular Hamiltonian. One also could construct other similar operators by $\mathcal{B}$, $\mathcal{D}$, etc. It is interesting to explore these kinds of operators in the future.

2.2.5 Two-point functions

The one-point functions of primary operators are vanishing for a single interval in infinite line because $\langle O \rangle_{\mathcal{R}_n} = 0$. For the interval at the end of a semi-infinite line or in the regularized boundary state the one-point functions are non-vanishing since the one point function in these two states can be associated with two-point functions on the complex plane. In this section we would like to discuss the two-point functions as probes to detect the microcanonical state $\rho_{A,m}$.

The two-point functions are defined as

$$\mathcal{P}_{\mathcal{O}\mathcal{O}}(t) := \sum_i \langle \lambda_i | \mathcal{O}(x) \mathcal{O}(y) | \lambda_i \rangle_A \delta(t_i - t). \quad (2.47)$$

The average two-point function is $\bar{\mathcal{P}}_{\mathcal{O}\mathcal{O}}(t) := \mathcal{P}_{\mathcal{O}\mathcal{O}}(t) / \mathcal{P}(t)$. Without loss of generality we choose $0 < x < R$ and $y = -x$. To calculate the two-point functions we need

$$\begin{aligned} \langle \mathcal{O}(x) \mathcal{O}(-x) \rangle_{\mathcal{R}_n} &= f'(x)^{h_{\mathcal{O}}} f'(-x)^{h_{\mathcal{O}}} (f(x) - f(-x))^{-2h_{\mathcal{O}}} \\ &= \left(\frac{2R}{n(R^2 - x^2)} \right)^{2h_{\mathcal{O}}} \left(\xi^{-\frac{1}{2n}} - \xi^{\frac{1}{2n}} \right)^{-2h_{\mathcal{O}}}, \end{aligned} \quad (2.48)$$

where $\xi := \left(\frac{R-x}{R+x} \right)^2$. By using (2.12) and (2.13) and the above result we can obtain

$$\mathcal{P}_{\mathcal{O}\mathcal{O}}(t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} dn e^{nt + \frac{b}{n}} \langle \mathcal{O}(x) \mathcal{O}(-x) \rangle_{\mathcal{R}_n}. \quad (2.49)$$

We cannot find an analytical result of the inverse Laplace transformation. If the distance between the two operators is small, i.e., $x \ll R$, we have $\xi \sim 1$ and

$$\langle \mathcal{O}(x)\mathcal{O}(-x) \rangle_{\mathcal{R}_n} = (2x)^{-2h_{\mathcal{O}}} + \frac{h_{\mathcal{O}}(n^2 - 1)(2x)^{2-2h_{\mathcal{O}}}}{3n^2 R^2} + o(x^{2-2h_{\mathcal{O}}}). \quad (2.50)$$

For $t \neq 0$ this gives

$$\mathcal{P}_{\mathcal{O}\mathcal{O}}(t) = \mathcal{P}(t)(2x)^{-2h_{\mathcal{O}}} \left(1 + \frac{h_{\mathcal{O}}(2x)^2}{3R^2} \left(1 - \frac{t}{b} \right) + o(x^2) \right). \quad (2.51)$$

Here we only list the results upto $O(x^2)$. In appendix B we calculate the two-point functions upto $O(x^6)$. An important feature of the results upto $O(x^6)$ is

$$\bar{\mathcal{P}}_{\mathcal{O}\mathcal{O}}(t = b) = (2x)^{-2h_{\mathcal{O}}} + o(x^6), \quad (2.52)$$

in the large c limit, which is consistent with the two point correlation function in vacuum state $\langle \mathcal{O}(x)\mathcal{O}(-x) \rangle = (2x)^{-2h_{\mathcal{O}}}$ upto $O(x^6)$. This further supports our expectation that the microcanonical state $\rho_{A,m}$ can approximate the reduced density state ρ_A in the large c limit.

If the distance between two operators is large, i.e., $x \sim R$, we have $\xi \sim 0$ and

$$\langle \mathcal{O}(x)\mathcal{O}(-x) \rangle_{\mathcal{R}_n} \simeq \left(\frac{2R}{n(R^2 - x^2)} \right)^{2h_{\mathcal{O}}} \xi^{\frac{h_{\mathcal{O}}}{n}}. \quad (2.53)$$

For $t \neq 0$ we get

$$\mathcal{P}_{\mathcal{O}\mathcal{O}}(t) = \left(\frac{2R}{R^2 - x^2} \right)^{2h_{\mathcal{O}}} \left(\frac{b + h_{\mathcal{O}} \log \frac{R-x}{R+x}}{t} \right)^{\frac{1-2h_{\mathcal{O}}}{2}} I_{2h-1} \left(2\sqrt{t \left(b + h_{\mathcal{O}} \log \frac{R-x}{R+x} \right)} \right). \quad (2.54)$$

Even if taking $t = b$ we find the two-point functions are not same as $\langle \mathcal{O}(x)\mathcal{O}(-x) \rangle$.

2.3 Indistinguishability of $\rho_{A,m}$ from ρ_A

In the previous sections we calculate the one-point functions and two-point functions in the microcanonical state $\rho_{A,m}$. If taking the large c limit, we find the one-point functions in the state $\rho_{A,m}$ with $t = b$ is vanishing. The two-point correlation functions of primary operators are equal to the ones in ρ_A upto $O(x^6)$, where $2x$ is the distance of the two operators. These are consistent with our expectation that $\rho_{A,m}$ with $t = b$ can be seen as an approximate state of ρ_A in the large c limit. However, if the distance between the two operators are large $x \sim R$, the two-point functions $\mathcal{P}_{\mathcal{O}\mathcal{O}}(t)$ are different from the ones in ρ_A . Therefore, a reasonable explanation of these results is $\rho_{A,m}$ is indistinguishable from ρ_A in large c limit only if the probes are located in a small region comparing with R , that is $T/R \ll 1$. One could define a reduced density matrix $\rho_{A_3,m} := \text{tr}_{A_1 A_2} \rho_{A,m}$. In the case $T/R \ll 1$ one could show that the distance between $\rho_{A_3,m}$ and ρ_{A_3} would approach to 0 in the large c limit, i.e.,

$$\lim_{c \rightarrow \infty} d(\rho_{A_3,m}, \rho_{A_3}) \rightarrow 0, \quad (2.55)$$

where $d(\rho, \sigma)$ are the quantities that characterize the distance between two density matrices, such as trace distance, relative entropy, etc. There are some studies on relative

entropy and trace distance in CFTs, see [25]–[32]. Here we will not discuss the details of the distance, but one could calculate the trace distance and relative entropy by using our results in previous sections.

However, in the case $T/R \sim 1$ the two states $\rho_{A,m}$ and ρ_A are distinguishable, for example one could use the two-point function with distance $2x$ as a probe. This phenomenon also appears in the discussion of the canonical ensemble thermal state and the microcanonical state. One could show the reduced density matrices of a small subsystem are indistinguishable for the two ensemble states (see the supplemental material of [33]). However, for large subsystem there exists many probes to distinguish them. One of them is the Rényi entropy [34, 35]. Here we can also use the Rényi entropy to distinguish $\rho_{A,m}$ from ρ_A . Specially, for the system A their Rényi entropies are different,

$$S^{(n)}(\rho_{A,m}) = 2b, \quad S^{(n)}(\rho_A) = \left(1 + \frac{1}{n}\right)b. \quad (2.56)$$

In a summary, $\rho_{A_3,m}$ and ρ_{A_3} are indistinguishable if the size of A_3 is small enough, but they are distinguishable if the size of A_3 is large.

3 Entanglement spectrum of geometric states

In this section we will generalize the results of the single interval example to arbitrary geometric states.

3.1 Two intervals in vacuum state

Before we discuss the general case, let's see a non-trivial example. Without loss of generality we could consider the two intervals A_1 and A_2 as mentioned in the introduction. Let's define the Rényi mutual information

$$I^{(n)}(\rho_{A_1 A_2}) := S^{(n)}(\rho_{A_1}) + S^{(n)}(\rho_{A_2}) - S^{(n)}(\rho_{A_1 A_2}). \quad (3.1)$$

Take $n \rightarrow 1$ we get the mutual information $I(\rho_{A_1 A_2})$. By conformal symmetry one can show the Rényi mutual information $I^{(n)}(\rho_{A_1 A_2})$ only depends on the cross ratio $\eta := \left(\frac{1-T/R}{1+T/R}\right)^2$. Let's denote $I^{(n)}(\rho_{A_1 A_2})$ to be $I_n(\eta)$. For $T > (3 - 2\sqrt{2})R$ we have $I(\rho_{A_1 A_2}) = O(c^0)$, which means

$$\rho_{A_1 A_2} = \rho_{A_1} \otimes \rho_{A_2} + \delta\rho, \quad (3.2)$$

where $\delta\rho$ is taken as a small perturbation in the large c limit. To derive the density of eigenstates of $\rho_{A_1 A_2}$ one needs the Rényi entropy for any index n as we have done for the single interval case. Unfortunately, we have no exact results of the Rényi entropy. But if the distance between the two interval A_1 and A_2 is large, one could perturbatively calculate the Rényi entropy by short interval method [36]–[38]. For $\eta \ll 1$ we have $S^{(n)}(\rho_{A_1 A_2}) = 2(1 + \frac{1}{n})b - I_n(\eta)$, where $b := \frac{c}{6} \log(R - T)/\epsilon$ and

$$I_n(\eta) = \frac{c(n-1)(n+1)^2\eta^2}{144n^3} + \frac{c(n-1)(n+1)^2\eta^3}{144n^3} + \frac{c(n-1)(n+1)^2(1309n^4 - 2n^2 - 11)\eta^4}{207360n^7} + o(\eta^4, c), \quad (3.3)$$

where we only keep the order of $O(c)$ results upto η^4 . One could refer to [23] for the higher order terms. Note that the above results only contain the contributions from the quasi-primary operators in the vacuum conformal family. In the theory with holographic dual we expect these operators gives the main contributions to the Rényi entropy.

We can formally write $\rho_{A_1 A_2}$ as

$$\rho_{A_1 A_2} = \sum_J \sqrt{\Lambda_J} |J\rangle_{A_1 A_2} \langle J|, \quad (3.4)$$

and define the density of eigenstates

$$P(\Lambda) = \sum_J \delta(\Lambda_J - \Lambda). \quad (3.5)$$

Using the expression (3.3) one can find the maximal eigenvalue Λ_m by

$$B = -\log \Lambda_m = \lim_{n \rightarrow \infty} S^{(n)}(\rho_{A_1 A_2}) = 2b + \delta b + o(\eta^4, c), \quad (3.6)$$

where $\delta b = -\frac{c\eta^2(1309\eta^2+1440\eta+1440)}{207360}$. By using (2.6) we obtain

$$\mathcal{P}(t) := P(\Lambda_m e^{-t}) \Lambda_m e^{-t} = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} dn e^{nt+nB+(1-n)S^{(n)}(\rho_{A_1 A_2})} \quad (3.7)$$

With some calculations the integral becomes

$$\mathcal{P}(t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} dn e^{nt+\frac{2b}{n}+n\delta b-(1-n)I_n(\eta)}. \quad (3.8)$$

One could evaluate this integral by the saddle point approximation. The approximation is safe if we assume $t \sim c$. We need to solve the equation

$$t - \frac{2b}{n^2} + \delta b - \frac{\partial}{\partial n}[(1-n)I_n(\eta)] = 0. \quad (3.9)$$

Since we take $\eta \ll 1$ the equation can be solved perturbatively. The solution is

$$\begin{aligned} n_0 = & \sqrt{\frac{2b}{t}} - \frac{c\eta^2(4b-3t)}{576\sqrt{2b^3t}} - \frac{c\eta^3(4b-3t)}{576\sqrt{2b^3t}} - \frac{c\eta^4}{6635520\sqrt{2b^7t}} \\ & \times \left[80b^2(524b+c) + t^2(225c-400b) - 72t(b(434b+5c)) + 154t^3 \right] + o(\eta^4). \end{aligned} \quad (3.10)$$

Taking the above equation back into (3.8), we have

$$\mathcal{P}(t) \simeq n_0 e^{s(t)}, \quad (3.11)$$

with

$$\begin{aligned} s(t) = & 2\sqrt{2bt} - \frac{c\eta^2(4b-t)\sqrt{bt}}{288\sqrt{2b^2}} - \frac{c\eta^3(4b-t)\sqrt{bt}}{288\sqrt{2b^2}} + \frac{c\eta^4\sqrt{t}}{3317760\sqrt{2b^{7/2}}} \\ & \times \left[80b^2(524b+c) + t^2(45c-80b) - 24t(b(434b+5c)) + 22t^3 \right] + o(\eta^4). \end{aligned} \quad (3.12)$$

Motivated by the discussions in section 2.1 we would like to find a microcanonical ensemble states similar with $\rho_{A,m}$ (2.24) with $t = t_0$ that can be taken as an approximate state of $\rho_{A_1 A_2}$. A necessary condition for this is the entropy of the microcanonical state $\log \mathcal{P}(t_0)$ is consistent with the EE of $\rho_{A_1 A_2}$. We find the non-trivial solution is

$$t_0 = 2b - \delta b + o(\eta^4). \quad (3.13)$$

Thus we have $s(t_0) = 4b + o(\eta^4)$ or

$$\mathcal{P}(t_0) \simeq n_0 e^{4b + o(\eta^4)}. \quad (3.14)$$

For $T < (3 - 2\sqrt{2})R$ we have $\eta \geq \frac{1}{2}$. Specially, for $T \ll (3 - 2\sqrt{2})R$ the perturbation results may broke down. However, we can use the following results [36],

$$I_n(\eta) = I_n(1 - \eta) + \frac{(n+1)c}{6n} \log \frac{\eta}{1 - \eta}. \quad (3.15)$$

This gives

$$S^{(n)}(\rho_{A_1 A_2}) = S^{(n)}(\rho_{A_1}) + S^{(n)}(\rho_{A_2}) - I_n(1 - \eta) - \frac{(n+1)c}{6n} \log \frac{\eta}{1 - \eta}. \quad (3.16)$$

Note that $S^{(n)}(\rho_{A_1}) = S^{(n)}(\rho_{A_2}) = \frac{c(n+1)}{6n} \log \frac{R-T}{\epsilon}$. The Rényi entropy can be written as

$$S^{(n)}(\rho_{A_1 A_2}) = \left(1 + \frac{1}{n}\right) b_1 + \left(1 + \frac{1}{n}\right) b_2 - I_n(1 - \eta), \quad (3.17)$$

with $b_1 = \frac{c}{6} \log \frac{2T}{\epsilon}$ and $b_2 = \frac{c}{6} \log \frac{2R}{\epsilon}$. The calculations are similar as we have done for the case $T > (3 - 2\sqrt{2})R$ by replacing $2b$ with $b_1 + b_2$ and η with $1 - \eta$. Therefore, t_0 is given by

$$t_0 = b_1 + b_2 - \delta b', \quad (3.18)$$

with $\delta b' = -\frac{c(1-\eta)^2(1309(1-\eta)^2 + 1440(1-\eta) + 1440)}{207360}$.

3.2 Arbitrary geometric states

Our previous discussions on entanglement spectrum on single interval or two intervals in vacuum states show interesting properties of the holographic theories. In this section we will generalize the discussions to arbitrary geometric states. To do that we need to know the scale behavior of Rényi entropy in the large c limit.

3.2.1 The gravity dual of Rényi entropy

Let's first review the proposal of holographic Rényi entropy in [21]. The holographic Rényi entropy also follow an area law like the RT formula. For a subsystem A it is given by

$$n^2 \partial_n \left(\frac{n-1}{n} S^{(n)}(\rho_A) \right) = \frac{\text{Area}(\mathcal{B}_n)}{4G}, \quad (3.19)$$

where $\mathcal{B}_n$ denotes a bulk codimension-2 cosmic brane homologous to the boundary region A . The tension T_n of $\mathcal{B}_n$ is associated with the Rényi index n by

$$T_n = \frac{n-1}{4nG}. \quad (3.20)$$

One may obtain the bulk geometry as well as the area of cosmic brane $\mathcal{B}_n$ by solving the Einstein equation with the Euclidean action $I = I_{\text{bulk}} + I_{\text{brane}}$, where I_{bulk} includes the Einstein-Hilbert action and the matter field, $I_{\text{brane}} = T_n \int d^{y-1} \sqrt{g}$. In this paper we will not pursue the solutions for special cases. What we need is that the metrics for any geometric states should be of order $O(G^0)$, since the action of the cosmic brane is of order $O(G^{-1})$ same as the bulk actions. Therefore, the area of $\mathcal{B}_n$ should be of order $O(G^0)$, which means Rényi entropy is of $O(G^{-1})$ or $O(c)$ by $c \sim 1/G$. The RT formula is a special case of the holographic Rényi entropy formula in the probe limit $n \rightarrow 1$.

In the limit $n \rightarrow \infty$ the tension $T_n \rightarrow T_\infty := \frac{1}{4G}$. By considering the back reaction of the cosmic brane with the tension T_∞ one could obtain the holographic Rényi entropy S^∞ . Near the probe limit $n \rightarrow 1$ one could perturbatively calculate the holographic Rényi entropy. We expect the following expansion,

$$S^{(n)}(\rho_A) = S(\rho_A) + \sum_{k=1} (1-n)^k b_k, \quad (3.21)$$

where $S(\rho_A)$ is the EE, b_k are constants of order $O(G^{-1})$. The single interval and two intervals examples both satisfy the ansatz (3.21). One could refer to [22, 39] for more supports on the ansatz. It is out of the scope of this paper to show the ansatz is true for any geometric states. To approach this one needs to study more details of the solutions of Einstein equation.

3.2.2 Density of eigenstates and microcanonical ensemble

The density of eigenstates can be obtained by using (2.6),

$$\mathcal{P}(t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} dn e^{s_n}, \quad (3.22)$$

with

$$s_n = nt + nS^\infty + (1-n) \left(S(\rho_A) + \sum_{k=1} (1-n)^k b_k \right), \quad (3.23)$$

where we have used the maximal eigenvalue $\lambda_m = e^{-S^\infty}$ and the ansatz (3.21). We will take the large c limit and assume $t \sim c$. Thus we could use the saddle point approximation to evaluate the inverse Laplace transformation. We can rewrite s_n as

$$s_n = nt + nS^\infty + (1-n)S(\rho_A) + (1-n)^2 \sum_{k=1} (1-n)^{k-1} b_k. \quad (3.24)$$

The saddle point approximation requires the solution of the equation

$$\partial_n s_n = t + S^\infty - S(\rho_A) - 2(1-n) \sum_{k=1} (1-n)^{k-1} b_k + (1-n)^2 \partial_n \left(\sum_{k=1} (1-n)^{k-1} b_k \right) = 0. \quad (3.25)$$

For general t it is hard to solve the above equation even if one knows the parameters b_k . However, a special case is at the point $t_0 = S(\rho_A) - S^\infty$, for which the solution is $n = 1$. Taking the solution back to s_n , the density of eigenstates is given by

$$\mathcal{P}(t_0) \simeq e^{S(\rho_A)}. \quad (3.26)$$

One could construct the microcanonical ensemble states $\rho_{A,m}$ (2.24) with $t = t_0$, the entropy of which is equal to the EE of A upto the leading order of c . This supports that the microcanonical ensemble state with can be taken as approximate state of ρ_A .

As a check of the general result, for $T < (3 - 2\sqrt{2})R$ in section 3.1 by using (3.17) we have $S^\infty = b_1 + b_2 + \delta b'$ and $S(\rho_{A_1 A_2}) = 2(b_1 + b_2)$. We obtain $t_0 = b_1 + b_2 - \delta b'$ by using the results in this section, which is consistent with (3.18).

3.2.3 Correlation functions in microcanonical ensemble state

Generally, we will consider $tr(\rho_{A,m}\mathcal{S})$ where $\mathcal{S}$ denotes the product of local operators, i.e.,

$$\mathcal{S} = \mathcal{O}(x_1)\mathcal{O}(x_2)\dots\mathcal{O}(x_n). \quad (3.27)$$

To get $tr(\rho_{A,m}\mathcal{S})$ one needs to know $\langle\mathcal{S}\rangle_{\mathcal{R}_n}$ by using (2.14). Let's denote the maximal distance between the boundary of A and the set $\{x_1, \dots, x_n\}$ to be D_m . And denoting x_m to be the maximal distance among the set $\{x_1, \dots, x_n\}$. If the operators of $\mathcal{S}$ is located in a small region in A and far away from the boundary, that is $x_m \ll D_m$, we expect the following expansion

$$\langle\mathcal{S}\rangle_{\mathcal{R}_n} = tr(\rho_A\mathcal{S}) + \sum_{k=1} (1-n)^k S_k, \quad (3.28)$$

where S_k are some functions depending on $x_1, \dots, x_n$. The expansion is consistent with the fact that $\lim_{n \rightarrow 1} \langle\mathcal{S}\rangle_{\mathcal{R}_n} = tr(\rho_A\mathcal{S})$. But we don't expect this expansion is still true for $x_m \sim D_m$. The correlation functions $\langle\mathcal{S}\rangle_{\mathcal{R}_n}$ on $\mathcal{R}_n$ is same as correlation functions with the inserting of the twist operators σ_n which is located at the boundary of A . For $x_m \sim D_m$ some operators in $\mathcal{S}$ are near the boundary of A . Therefore, the operator product expansion (OPE) of $\mathcal{S}$ and σ_n would give main contributions to the correlation functions. The expansion form (3.28) may broke down. Our example in section 2.2.5 can be taken as an non-trivial support on our argument. Specially, the expansion (3.28) is expected to be true for $\mathcal{S}$ being the quasi-primary operators $\mathcal{X}$ in the vacuum conformal family.

By using (2.12) and (2.13) we get the correlation functions in the microcanonical state

$$\mathcal{P}_{\mathcal{S}}(t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} dn e^{s_n} \langle\mathcal{S}\rangle_{\mathcal{R}_n}, \quad (3.29)$$

where s_n is given by (3.23). Since the saddle point approximation would give $n = 1$ if $t = t_0 = S(\rho_A) - S^\infty$, the only survive term in (3.28) is $tr(\rho_A\mathcal{S})$. This gives

$$\mathcal{P}_{\mathcal{S}}(t = t_0) \simeq \mathcal{P}(t_0) tr(\rho_A\mathcal{S}), \quad (3.30)$$

or

$$\bar{\mathcal{P}}_{\mathcal{S}}(t = t_0) := \frac{\mathcal{P}_{\mathcal{S}}(t = t_0)}{\mathcal{P}(t_0)} \simeq tr(\rho_A\mathcal{S}). \quad (3.31)$$

The physical meaning underlying the results is $\rho_{A,m}$ and ρ_A are indistinguishable if the probes are located in a small region $\tilde{A}$ and far away from the boundary of A .

We may define the reduced density matrix of the small region $\tilde{A}$ as $\rho_{\tilde{A},m} := tr_{A-\tilde{A}}\rho_{A,m}$ and $\rho_{\tilde{A}} := tr_{A-\tilde{A}}\rho_A$. Specially, for arbitrary states in 2D CFT one could show this statement by using the short interval expansion of relative entropy, which can be seen as a

measure of distance between two states $\rho_{\tilde{A},m}$ and $\rho_{\tilde{A}}$. The relative entropy can be expanded as powers of the length of interval ℓ [31, 32]. We can show any order of the expansion is associated with $\text{tr}(\rho_A \mathcal{X}) - \text{tr}(\rho_{A,m} \mathcal{X})$, where $\mathcal{X}$ denotes the operators contained in the theory. Our result (3.31) shows the relative entropy is vanishing to any order of ℓ in the order of $O(c)$. By using the Pinsker's inequality one can show the trace distance between the two states are also zero.

Of course, one can always find the probes to distinguish the two states, for example, the Rényi entropies of ρ_A and $\rho_{A,m}$ are different. This shows their difference will appear if the probes are not limited to $\tilde{A}$.

3.3 Higher dimensions

The set-up in previous sections can be easily generalized to higher dimensions case. Once the Rényi entropies $S_A^{(n)}$ are known, one could evaluate the density of eigenstates by using (2.6). For the holographic CFTs the Rényi entropy of a sphere are calculated in [22]. For d -dimensional CFTs dual to Einstein gravity the Rényi entropy of a sphere in the vacuum state is given by

$$S_A^{(n)}(d) = \frac{n}{8G(n-1)} V_\Sigma \left(2 - x_n^{d-2} - x_n^d \right), \quad (3.32)$$

where we take the radius of AdS $L = 1$, $x_n = \frac{1}{nd} \left(1 + \sqrt{1 - 2dn^2 + d^2n^2} \right)$, V_Σ is a universal term depending on dimension d

$$V_\Sigma(d) = \begin{cases} (-1)^{d/2-1} \frac{2}{\pi} \log(2R/\epsilon) & \text{for even } d, \\ (-1)^{(d-1)/2} & \text{for odd } d. \end{cases} \quad (3.33)$$

We have the EE $S(\rho_A) = \lim_{n \rightarrow 1} S_A^{(n)}(d) = \frac{1}{4G} V_\Sigma$. Taking the above result into (2.6) we get the density of eigenstates $\mathcal{P}(t)$. According to the discussions in section 3.2.2 we only need to show the Rényi entropies satisfy the general ansatz (3.21). In the region $n \sim 1$ we have the expansion

$$S_A^{(n)}(d) = \frac{1}{4G} V_\Sigma - \frac{n-1}{8G} + \frac{(4d^2 - 8d + 3)(n-1)^2}{24(d-1)^2 G} + O((n-1)^3), \quad (3.34)$$

which is consistent with the ansatz (3.21). Therefore, using the result in section 3.2.2 we have

$$\mathcal{P}(t = t_0) \simeq e^{S(\rho_A)}, \quad (3.35)$$

for $t_0 = S(\rho_A) - S^\infty$, where

$$S^\infty := \lim_{n \rightarrow \infty} S_A^{(n)}(d) = \frac{(1-d) \left(\frac{\sqrt{(d-2)d}}{d} \right)^d + d - 2}{4(d-2)G} V_\Sigma. \quad (3.36)$$

In the appendix C we show this result by direct calculation for $d = 4$.

To get the correlation functions of the local operators (3.27) in the microcanonical ensemble state with $t = t_0$ one needs to know the correlation functions on $\mathcal{R}_n$. In [40] the

authors studied the twist operators in higher dimensions. The one-point functions of stress energy tensor on $\mathcal{R}_n$ are evaluated in that paper. The results are

$$\langle T_{\mu\nu} \rangle_{\mathcal{R}_n} = \langle T_{\mu\nu} \sigma_n \rangle \propto h_n, \quad (3.37)$$

where σ_n is the higher dimensional twist operator, h_n is the dimension of the twist operator. An important result of [40] is

$$h_n = \sum_{k=1}^{\infty} \frac{1}{k!} H_k (n-1)^k, \quad (3.38)$$

where H_k are constants independent with n . Thus, the one-point functions $\langle T_{\mu\nu} \rangle_{\mathcal{R}_n}$ satisfy the ansatz (3.28) with $\langle T_{\mu\nu} \rangle = 0$ by symmetry of stress energy tensor on the infinite plane. Therefore, we will have

$$\bar{\mathcal{P}}_{T_{\mu\nu}}(t = t_0) := \langle T_{\mu\nu} \rangle = 0, \quad (3.39)$$

where $\bar{\mathcal{P}}_{T_{\mu\nu}}(t)$ is defined as (3.31) with $\mathcal{S} = T_{\mu\nu}$. This results support the idea that the microcanonical ensemble state with $t = t_0$ can be taken as an approximated state of ρ_A in the higher dimensions.

4 Applications of the microcanonical ensemble states

In previous sections we discuss the microcanonical ensemble state $\rho_{A,m}$. The number of the microstates in the ensemble is large which is of order $O(e^c)$. In this section we will come back to (1.5) and find its constraints on the eigenstates and eigenvalues of ρ_A .

4.1 Holevo information

Let's consider the subsystem $A = A_1 A_2 A_3$ in the vacuum state of 2D CFTs. We have the reduced density matrix $\rho_{A_1 A_2}$ and ρ_{A_3} ,

$$\rho_{A_1 A_2} = \sum_i \lambda_i \rho_{i, A_1 A_2}, \quad \rho_{A_3} = \sum_i \lambda_i \rho_{i, A_3}, \quad (4.1)$$

where $\rho_{i, A_1 A_2} := \text{tr}_{A_3} |\lambda_i\rangle_A \langle \lambda_i|$ and $\rho_{i, A_3} := \text{tr}_{A_1 A_2} |\lambda_i\rangle_A \langle \lambda_i|$. For an ensemble of a mixed state $\rho = \sum_i p_i \rho_i$, we may define the Holevo information,

$$\chi(\rho) := S(\rho) - \sum_i p_i S(\rho_i). \quad (4.2)$$

The Holevo information is an upper bound of the information that one can gain from the ensemble. It can also be used to characterize the distinguishability between the microstates ρ_i [41]. It is obvious that the Holevo information is non-negative, i.e., $\chi(\rho) \geq 0$. It can also be shown that $\chi(\rho) \leq H(p_i) = -\sum_i p_i \log p_i$. We are interested in the condition (1.5), which gives constraints on the spectrum and eigenstate of $\rho_{A_1 A_2 A_3}$. We may write (1.5) as

$$\chi(\rho_{A_1 A_2}) = \chi(\rho_{A_3}) + H(\lambda_i), \quad (4.3)$$

where $H(\lambda_i) = -\sum_i \lambda_i \log \lambda_i$, $\rho_{A_1 A_2}$ and ρ_{A_3} are given by the ensembles (4.1). In the above derivation we use the fact $S(\rho_{i, A_3}) = S(\rho_{i, A_1 A_2})$. Using the bound of the Holevo information we have

$$\chi(\rho_{A_1 A_2}) = H(\lambda_i) \quad \text{and} \quad \chi(\rho_{A_3}) = 0. \quad (4.4)$$

This means the microstates $\{\rho_{i, A_3}\}$ are indistinguishable from ρ_{A_3} , while $\{\rho_{i, A_1 A_2}\}$ can be perfectly distinguished.

The RT formula (1.1) and the relation (1.5) only count the leading order contribution in the limit $G \rightarrow 0$. In the CFT this corresponds to the limit $c \rightarrow \infty$ with $c \sim 1/G$. The EE should have quantum corrections which is of order $O(c^0)$. Therefore, the results should include higher order corrections of large c . More precisely, we have

$$\chi(\rho_{A_1 A_2}) = H(\lambda_i) + O(c^0) \quad \text{and} \quad \chi(\rho_{A_3}) = O(c^0). \quad (4.5)$$

4.2 Operational meaning of the Holevo information

We should stress that the formulas of Holevo information (4.5) is equal to (1.5). Holevo information has an operational meaning that is an upper bound of the accessible information from a given ensemble. The accessible information is associated with the operations or measurements on the ensemble. Generally, the measurement can be described by the so-called Positive Operator-Valued Measure (POVM) elements. The measurement includes the positive operators

$$E_k := M_k^\dagger M_k, \quad (4.6)$$

which satisfy the completeness condition $\sum_k E_k = I$. For the measurement on a state ρ the probability of outcome k is given by $p_k = \text{tr}(\rho E_k)$. The state after measurement is

$$\rho_k = \frac{M_k \rho M_k^\dagger}{\text{tr}(M_k^\dagger M_k \rho)}. \quad (4.7)$$

In the view of algebraic quantum field theories the POVM elements belong to the observable algebra $\mathcal{U}(A)$ associated with a region A .

For a given ensemble $\rho = \sum_{i \in X} p_i \rho_i$ we can take the index $X = \{1, 2, \dots\}$ as a random variable with the probability $\{p_1, p_2, \dots\}$. With the measurement $E_Y = \{E_1, E_2, \dots\}$ the outcomes Y denote the other random variable. We have the condition probability $p(y|x) = \text{tr}(\rho_x E_y)$ and the joint distribution $p_{x,y} = p_x p(y|x)$. The accessible information is given by the mutual information $I(X, Y) = H(p_x) + H(p_y) - H(p_{x,y})$ where $p_x := \sum_{j \in Y} p_{x,y}$ and $p_y := \sum_{i \in X} p_{x,y}$. The Holevo information is a bound of the accessible information, i.e., $I(X, Y) \leq \chi(\rho)$ with $\rho = \sum_{i \in X} p_i \rho_i$. With some calculations we have

$$H(X, Y) = \sum_{i \in X, j \in Y} p_i \text{tr}(\rho_i E_j) \log \frac{\text{tr}(\rho_i E_j)}{\text{tr}(\rho E_j)}. \quad (4.8)$$

Note that $H(X, Y) = 0$ if and only if $\text{tr}(\rho_i E_j) = \text{tr}(\rho E_j)$ for any $i \in X$ and $j \in Y$ that means any measurement cannot distinguish the states ρ_i ($i \in X$).

4.3 Holevo information $\chi(\rho_{A_3})$

Now let's turn to the reduced density matrix of a single interval in the vacuum state $\rho_A = \sum_i \lambda_i |i\rangle_A \langle i|$. Suppose the measurements are located in a small region, say A_3 with $T \ll R$, which means $E_j \in \mathcal{U}(A_3)$ ($j \in Y$). The mutual information $H(X, Y)$ in the state ρ_A is given by

$$H(X, Y)|_{\rho_A} = \sum_{i,j} \lambda_i \langle E_j \rangle_i \log \langle E_j \rangle_i - \sum_j \langle E_j \rangle_{\rho_A} \log \langle E_j \rangle_{\rho_A}, \quad (4.9)$$

where $\langle E_j \rangle_i := {}_A \langle i | E_j | i \rangle_A$ and $\langle E_j \rangle_{\rho_A} := \text{tr}(\rho_A E_j)$. $H(X, Y)|_{\rho_A}$ is equal to the one defined by the state $\rho_{A_3} = \sum_i \lambda_i \rho_{i, A_3}$ since ${}_A \langle i | E_j | i \rangle_A = \text{tr}(\rho_{i, A_3} E_j)$. The POVM elements E_j are bounded operators which are composed by the local operators $\mathcal{O}(x)$ with $x \in A_3$. In section 3.2.3 we have shown the expectation value of the products of the local operators $\mathcal{S}$ (3.27) in the microcanonical states with $t = t_0$ would approach to $\text{tr}(\rho_A \mathcal{S})$ in the large c limit. Therefore, if $T \ll R$ we expect $\text{tr}(\rho_{A, m} E_j) \rightarrow \text{tr}(\rho_A E_j)$ in the large c limit. The first term of (4.9) can be written as

$$\sum_{i,j} \lambda_i \langle E_j \rangle_i \log \langle E_j \rangle_i = \sum_j \int_0^{+\infty} dt \left(\mathcal{P}(t) \lambda_m e^{-t} \right) \bar{\mathcal{P}}_{E_j}, \quad (4.10)$$

where

$$\bar{\mathcal{P}}_{E_j} := \frac{\sum_i \langle E_j \rangle_i \log \langle E_j \rangle_i \delta(t_i - t)}{\mathcal{P}(t)}. \quad (4.11)$$

By using (2.23) for $t \neq 0$ we have

$$\mathcal{P}(t) \lambda_m e^{-t} = \frac{b e^{-(\sqrt{b} - \sqrt{t})^2}}{\sqrt{4\pi} (bt)^{3/4}} \rightarrow \delta(t - b) \quad (4.12)$$

in the limit $b \rightarrow \infty$ or $c \rightarrow \infty$. Therefore, we have

$$\sum_{i,j} \lambda_i \langle E_j \rangle_i \log \langle E_j \rangle_i \rightarrow \sum_j \frac{1}{\mathcal{P}(b)} \sum_i \langle E_j \rangle_i \log \langle E_j \rangle_i \delta(t_i - b). \quad (4.13)$$

Combination of the above results we find

$$H(X, Y)|_{\rho_A} \rightarrow H(X, Y)|_{\rho_{A, m}}, \quad (4.14)$$

in the large c or b limit. One could define the Holevo information of $\rho_{A_3, m} := \frac{1}{\mathcal{P}(b)} \sum_k \delta(t_i - b) \rho_{i, A_3}$ as $\chi(\rho_{A_3, m})$. We conclude if $T \ll R$

$$\chi(\rho_{A_3}) = \chi(\rho_{A_3, m}), \quad (4.15)$$

in the large c limit. This equality gives us a way to explain the almost vanishing Holevo information of $\chi(\rho_{A_3})$ in the case $T < (3 - 2\sqrt{2})R$. At present we only consider $T \ll R$. The mutual information $H(X, Y)|_{\rho_{A_3, m}} = H(X, Y)|_{\rho_{A, m}}$ is

$$H(X, Y)|_{\rho_{A_3, m}} = \frac{1}{\mathcal{P}(b)} \sum_{i,j} \langle E_j \rangle_i \log \frac{\langle E_j \rangle_i}{\langle E_j \rangle_{\rho_{A, m}}} \delta(t_i - b). \quad (4.16)$$

Let's denote $e_{ij} := \frac{\langle E_j \rangle_i}{\langle E_j \rangle_{\rho_{A,m}}}$. Since $\frac{1}{\mathcal{P}(b)} \sum_i e_{ij} \delta(t_i - b) = 1$ and $\langle E_j \rangle_i \geq 0$, it is obvious $e_{\max} = \max_{i,j} e_{ij} \geq 1$ and $e_{\min} = \min_{i,j} e_{ij} \leq 1$. The microstates in $\rho_{A_3,m}$ are eigenstates of H_A with the same eigenvalues. A natural assumption is that the measurements E_j cannot distinguish the microstates $|i\rangle_A$ at the leading order of c . For most of tensor e_{ij} ¹ we expect the expansion

$$\log e_{ij} = \tilde{e}_{ij} + O(1/c), \quad (4.17)$$

where $\tilde{e}_{ij}$ are of order $O(c^0)$. With the ansatz we have

$$H(X, Y)|_{\rho_{A_3,m}} \simeq \frac{1}{\mathcal{P}(b)} \sum_{i,j} \langle E_j \rangle_i \tilde{e}_{ij} \quad (4.18)$$

Support $\tilde{e}_{\max} := \max_{i,j} |\tilde{e}_{ij}|$. We have

$$H(X, Y)|_{\rho_{A_3,m}} \leq \tilde{e}_{\max}. \quad (4.19)$$

This result is consistent with $\chi(\rho_{A_3}) = O(c^0)$.

4.4 Constraints on one-point functions of quasi-primary operators

One could calculate the EE and Holevo information for arbitrary state $\rho = \sum_i \lambda_i \rho_i$ directly by using the short interval expansion method. In the appendix D we review the short expansion method. The Holevo information $\chi(\rho_A)$ is associated with the terms of the form

$$\langle \mathcal{X}_1 \rangle_\rho \dots \langle \mathcal{X}_m \rangle_\rho - \sum_i p_i \langle \mathcal{X}_1 \rangle_i \langle \mathcal{X}_m \rangle_i, \quad (4.20)$$

where $\langle \mathcal{X} \rangle_i := \text{tr}(\rho_i \mathcal{X})$. Support the length of the interval is ℓ . Let's consider the microcanonical ensemble state $\rho_{A,m}$. We have shown in section 2.2.3 the one-point function $\text{tr}(\rho_{A,m} \mathcal{X}) \rightarrow 0$ in the large c limit. Therefore, $\chi(\rho_{A,m})$ only depends on the $\langle \mathcal{X} \rangle_i := {}_A \langle \lambda_i | \mathcal{X} | \lambda_i \rangle_A$ with $\mathcal{X} = T, \mathcal{A}$ and their derivatives upto $O(\ell^{10})$.

We have calculated the average one-point functions of T and $\mathcal{A}$ in the microcanonical ensemble state $\rho_{A,m}$ in section 2.2.2. Let's define the functions

$$\begin{aligned} \mathbf{t}(w; \lambda_i) &:= {}_A \langle \lambda_i | T(w) | \lambda_i \rangle_A - \bar{\mathcal{P}}_T(t_i), \\ \mathbf{a}(w; \lambda_i) &:= {}_A \langle \lambda_i | \mathcal{A}(w) | \lambda_i \rangle_A - \bar{\mathcal{P}}_{\mathcal{A}}(t_i). \end{aligned} \quad (4.21)$$

By definition we have

$$\frac{1}{\mathcal{P}(t)} \sum_i \mathbf{t}(w; \lambda_i) \delta(t_i - t) = 0, \quad \frac{1}{\mathcal{P}(t)} \sum_i \mathbf{a}(w; \lambda_i) \delta(t_i - t) = 0. \quad (4.22)$$

By using the results (2.41) we have

$$\int_{-R+\epsilon}^{R-\epsilon} dw \frac{R^2 - w^2}{2R} {}_A \langle \lambda_i | T(w) | \lambda_i \rangle_A = \frac{1}{2}(b - t). \quad (4.23)$$

¹Here we don't need all of the tensor e_{ij} satisfy (4.17). Assume there exists some states with number N such that $e_{ij} \sim c$. As long as the number $N \ll \mathcal{P}(b) \sim O(e^c)$ their contributions to (4.16) are exponentially suppressed.

This gives

$$\int_{-R+\epsilon}^{R+\epsilon} dw \frac{R^2 - w^2}{2R} \mathfrak{t}(w; \lambda_i) = 0. \quad (4.24)$$

In general, we can expand $\mathfrak{t}(x; \lambda_i)$ by the Legendre Polynomials as follows²

$$\mathfrak{t}(w; \lambda_i) = \sum_{j \geq 3} c_{ij} P_j \left(\frac{w}{R} \right), \quad (4.25)$$

where c_{ij} are constants independent with w . By using the short interval expansion of Holevo information (D.3) and the condition $\chi(\rho_{A,m}) = O(c^0)$, we find the constraints on $\mathfrak{t}(w; \lambda_i)$ at order of $O(\ell^4)$,

$$\frac{1}{\mathcal{P}(b)} \sum_i \mathfrak{t}(w; \lambda_i)^2 \delta(t_i - b) \lesssim O(c), \quad (4.26)$$

where $\lesssim$ means the left hand side is at most of order $O(c)$. A natural assumption is that most of the functions $\mathfrak{t}(w; \lambda_i)$ are at most of order $O(\sqrt{c})$. One may define the maximal value of $\mathfrak{t}(w; \lambda_i)$ among these functions as $\mathfrak{t}(w)_{\max} := \max_{\lambda_i} |\mathfrak{t}(w; \lambda_i)|$. If $\mathfrak{t}(w)_{\max} \lesssim O(\sqrt{c})$, we would obtain (4.26).

Taking the large c limit, the term at order of $O(\ell^8)$ becomes

$$\frac{1}{630c^2} \frac{1}{\mathcal{P}(b)} \sum_i \left(\mathfrak{a}(w; \lambda_i) - \mathfrak{t}(w; \lambda_i)^2 \right)^2 \delta(t_i - b). \quad (4.27)$$

By the similar argument as above we conclude most of $\mathfrak{a}(w; \lambda_i)|_{\lambda_i=\lambda_0}$ should be at most of order $O(c)$. One may define $\mathfrak{a}(w)_{\max} := \max_{\lambda_i} |\mathfrak{a}(w; \lambda_i)|$. The assumption $\mathfrak{a}(w)_{\max} \lesssim O(c)$ ensures $\chi(\rho_{A,m}) \lesssim O(c^0)$ upto order $O(\ell^8)$.

One could check the term of Holevo information $\chi(\rho_{A,m}) \lesssim O(c^0)$ upto $O(\ell^{10})$ if $\mathfrak{t}(w)_{\max} \lesssim O(\sqrt{c})$ and $\mathfrak{a}(w)_{\max} \lesssim O(c)$ are satisfied.

5 Conclusions and discussions

In the context of AdS/CFT the geometric states should be the ones that are very special. They would show some well-defined and special properties in the semi-classical limit $G \rightarrow 0$ or $c \rightarrow \infty$. In this paper we focus on the entanglement spectrum of the geometric states, which contain more information of the reduced density matrix ρ_A than the EE. We can use the inverse Laplace transformation with respect to the index n of Rényi entropy.

A single interval A in the vacuum state of 2D CFTs is the example that one can exactly get the eigenstates and eigenvalues of ρ_A . By direct calculation we show there exists a microcanonical ensemble states $\rho_{A,m}$ with $t_0 = -\log \lambda_m$ can be taken as an approximate state of ρ_A in the large c limit if our probes are located in a small region of A and far away from the boundary. We get the conclusion by evaluating the one-point functions of primary and quasi-primary operators and two-point functions of primary operator in the microcanonical ensemble state. The one-point functions are always consistent with the

²We can ignore ϵ in the integration.

one in ρ_A , that is vanishing. The two-point functions are consistent only if the distance between two operators are small. We should stress that the results are only true in the semi-classical limit $c \rightarrow \infty$.

For the two intervals example we evaluate the microcanonical ensemble state with t_0 (3.18) by using saddle point approximation. The parameter b_0 is not only associated with the length of subsystem A_1 and A_2 , but also related to the cross ratio of the two intervals at the order of $O(c)$. Let's comment on the two point correlation functions

$$\mathcal{P}_{\mathcal{O}_{A_1}\mathcal{O}_{A_2}}(t=t_0) = \sum_J \langle \Lambda_J | \mathcal{O}_{A_1} \mathcal{O}_{A_2} | \Lambda_J \rangle \delta(t_J - t_0), \quad (5.1)$$

where $\mathcal{O}_{A_1(A_2)}$ are operators in the region $A_1(A_2)$. Or the normalized ones

$$\bar{\mathcal{P}}_{\mathcal{O}_{A_1}\mathcal{O}_{A_2}}(t=t_0) := \frac{\mathcal{P}_{\mathcal{O}_{A_1}\mathcal{O}_{A_2}}(t=t_0)}{\mathcal{P}(t=t_0)}, \quad (5.2)$$

where $\mathcal{P}(t=t_0)$ is given by (3.14). To get the two point functions one needs

$$\langle \mathcal{O}_{A_1} \mathcal{O}_{A_2} \rangle_{\mathcal{R}_n} = \langle \mathcal{O}_{A_1} \mathcal{O}_{A_2} \sigma_n(-R) \sigma_n(-T) \sigma_n(T) \sigma_n(R) \rangle, \quad (5.3)$$

where σ_n are the twist operators which are local in 2D CFTs, the correlator on r.h.s. is evaluated on n -copied CFTs. Let's consider the limit $\eta \ll 1$, which means the distance between A_1 and A_2 is far larger than their own sizes. By the OPE of CFTs we have

$$\langle \mathcal{O}_{A_1} \mathcal{O}_{A_2} \rangle_{\mathcal{R}_n} \simeq \langle \mathcal{O}_{A_1} \sigma_n(-R) \sigma_n(-T) \rangle \langle \mathcal{O}_{A_2} \sigma_n(T) \sigma_n(R) \rangle = \langle \mathcal{O}_{A_1} \rangle_{\mathcal{R}_n} \langle \mathcal{O}_{A_2} \rangle_{\mathcal{R}_n} \quad (5.4)$$

The above result doesn't satisfy the ansatz (3.28). Therefore, we expect the two point functions $\bar{\mathcal{P}}_{\mathcal{O}_{A_1}\mathcal{O}_{A_2}}(t=t_0)$ are different from $tr(\rho_{A_1 A_2} \mathcal{O}_{A_1} \mathcal{O}_{A_2}) = \langle \mathcal{O}_{A_1} \mathcal{O}_{A_2} \rangle$. This expectation is also consistent with our arguments in section 3.2.3. Here the distance between two operators $\mathcal{O}_{A_1}, \mathcal{O}_{A_2}$ are larger than their distances to the boundary.

By using the proposal of holographic Rényi entropy, we generalize the results to arbitrary geometric states. The key point is that the holographic Rényi entropy is that $S^{(n)} \sim O(c)$. This permits us to use the saddle point approximation to find the microcanonical ensemble state $\rho_{A,m}$ with the parameter t_0 . The solution of the saddle point approximation gives $n=1$. The parameter t_0 has a simple expression $t_0 = S(\rho_A) - S^\infty$, where $S(\rho_A)$ is the EE and S^∞ is $\lim_{n \rightarrow \infty} S^{(n)}$. The entropy of the microcanonical ensemble state $\rho_{A,m}$ with t_0 is equal to the EE of A . However, it cannot give the same Rényi entropy of A . The reason is that the microcanonical ensemble state can only be an approximate state of ρ_A only if the probes are located in a small region of A and far away from the boundary of A . This is consistent with the example of a single interval in the vacuum state. We check this by comparing $tr(\rho_{A,m} \mathcal{S})$ with $tr(\rho_A \mathcal{S})$, where $\mathcal{S}$ (3.27) is product of the local operators. A remarkable result is that $tr(\rho_{A,m} \mathcal{S}) \rightarrow tr(\rho_A \mathcal{S})$ if $\mathcal{S}$ permits the expansion as (3.28), which is expected to be true if the local operators are located in a small region in A and far away from the boundary.

Finally, we discuss the equality condition of the Araki-Lieb inequality (1.5). The condition can be reformed as the Holevo information, which can be taken as an upper bound

of information that one can gain by arbitrary measurements. To satisfy this condition we find the constraints on the expectation values of measurements and local operators. This constraints would help us to understand more on properties of geometric states.

In the following we will discuss some unsolved problems that are worth to explore in the future.

5.1 Transition between distinguishability and indistinguishability

As mentioned above we expect $\rho_{A,m}$ and ρ_A are indistinguishable at the leading order of c only if the probes are located in a small region. If using the Rényi entropy or two-point correlation functions with large distance, one would find the difference between the two states. However, we cannot find the critical point where the transition between distinguishability and indistinguishability happens. Let's see the two-point functions (2.51) in the microcanonical ensemble $\rho_{A,m}$. The average two-point function $\bar{P}_{\mathcal{OO}}(t) \rightarrow \text{tr}(\rho_A \mathcal{OO})$ if $t = b$ upto $O(x^6)$. But when the distance $2x$ between two operators is large enough, the perturbative expansion with respect to x may broke. The difference between ρ_A and $\rho_{A,m}$ with $t = b$ will appear.

An interesting question is whether the transition between distinguishability and indistinguishability is associated with the critical point $T = (3 - 2\sqrt{2})R$. For $T < (3 - 2\sqrt{2})R$ we have $\chi(\rho_{A_3}) = O(c^0)$. For $T > (3 - 2\sqrt{2})R$ we expect $\chi(\rho_{A_3}) \sim O(c)$. In section 4.3 we show the $\chi(\rho_{A_3}) \simeq \chi(\rho_{A_3,m})$ with $t=b$ if the length of A_3 is small enough. $\chi(\rho_{A_3,m}) = O(c^0)$ means the microstates $\rho_{A_3,i}$ of the microcanonical ensemble states are indistinguished at the leading order of c , that is the expression (4.17). In section 4.4 we directly calculate the Holevo information of $\rho_{A_3,m}$ by using short interval expansion. The almost vanishing $\chi(\rho_{A_3,m}) = O(c^0)$ gives (4.21) with $t(w; \lambda_i) \lesssim O(\sqrt{c})$ and $a(w; \lambda_i) \lesssim O(c)$. This means the difference of the expectation values of $\mathcal{X}$ in the microstates $|\lambda_i\rangle_A$ of $\rho_{A,m}$ is equal to $\text{tr}(\rho_{A,m} \mathcal{X})$ in the leading order of c , which are natural assumptions. Our opinion is that $\chi(\rho_{A_3,m}) = O(c^0)$ is a consequence of the indistinguishability of ρ_{A_3} from $\rho_{A_3,m}$. However, we cannot prove this or disprove this point at present. We will leave this to future works.

5.2 A possible geometric explanation of Holevo information

If a quantity can be associated with a geometric object in the bulk, we call it a geometric probe, for example the EE or Rényi entropy. To find these geometric probes are important since they would help us to understand more on the properties of geometric states. A natural requirement of the geometric quantity is that they should be order of $O(c)$. In section 4.1 we have shown the relation of holographic EE (1.5) in the case $T \leq (3 - 2\sqrt{2})R$ is equal to the conditions of Holevo information (4.5). $\chi(\rho_{A_3})$ is vanishing at the order of $O(c)$. However, if $T > (3 - 2\sqrt{2})R$, we expect $\chi(\rho_{A_3})$ should be the order of $O(c)$. For $T \sim R$ we will have $\chi(\rho_{A_3}) \simeq S(\rho_A) \sim O(c)$. As shown in [42] the Holevo information is monotonically increasing, that is if $\rho_X = \text{tr}_Y \rho_{XY}$ we have $\chi(\rho_X) \leq \chi(\rho_{XY})$. This leads to

$$\partial_T \chi(\rho_{A_3}) \geq 0. \quad (5.5)$$

Therefore, we expect $\chi(\rho_{A_3})$ should be a quantity that is order of $O(c)$ and monotonically increasing with T in the case $T > (3 - 2\sqrt{2})R$. We should note that $\chi(\rho_{A_3})$ is not only

dependent with the reduced density state ρ_{A_3} , but also on the spectrum decomposition of ρ_A . This is because the Holevo information depends on the ensemble. Generally, a mixed state ρ can be written as different ensembles, say $\rho = \sum_i p_i \rho_i$ and $\rho = \sum_j q_j \rho'_j$. The corresponding Holevo information $\chi(\sum_i p_i \rho_i)$ is different from $\chi(\sum_j q_j \rho'_j)$ in general. Our discussion is based on the fixed ensemble $\rho_A = \sum_i \lambda_i |\lambda_i\rangle_A \langle \lambda_i|$. Therefore, the Holevo information $\chi(\rho_{A_3})$ should depend on the eigenvalues and eigenstates of ρ_A . In some sense $\chi(\rho_{A_3})$ contain more information of the state ρ_A than the EE, which is only the trace of the eigenvalues. According to the so-called subregion/subregion duality in AdS/CFT [43]–[46], the bulk region surround by the RT surface γ_A and the boundary region A , named entanglement wedge, is expected to be dual to the reduced density matrix ρ_A . $\chi(\rho_{A_3})$ and $\chi(\rho_{A_1 A_2})$ do extract some information of ρ_A . If $\chi(\rho_{A_3})$ has the geometric dual, then the geometric object should be in the region inside the entanglement wedge.

However, to get the exact dual of one quantity one should be able to evaluate the quantity both in bulk and the boundary CFTs. One could also make a guess by comparing the properties on both sides like the one that is done in the proposal of holographic entanglement of purification [47, 48]. In the paper we are not trying to conjecture the duality of $\chi(\rho_{A_3})$.

Let's comment more on the ensemble dependence of $\chi(\rho_A)$. In [49] we show the $\chi(\rho_{A_3}) = O(c^0)$ (by the notation of the present paper) for the ensemble of ρ_{A_3} that seems not related to the one we used here. In that paper we consider the ensemble $\rho_A = \sum_k p_k |\psi\rangle_k \langle \psi|$ that would make the entropy $\sum_k p_k S(\tilde{\rho}_{k, A_3})$ minimal, where $\tilde{\rho}_{k, A_3} := \text{tr}_{A_1 A_2} |\psi\rangle_k \langle \psi|$. The entropy actually is defined as the entanglement of formation (EoF), which characterizes the correlation between A_1 and A_2 . By using the Koashi-Winter relation we find that $\chi(\sum_k p_k \tilde{\rho}_{k, A_3})$ is also the order of $O(c^0)$. It is interesting to study the relation between the ensemble $\rho_A = \sum_k p_k |\psi\rangle_k \langle \psi|$ and the spectrum decomposition (1.2) in the near future.

5.3 On construction of new geometric states

We only focus on the microcanonical ensemble state $\rho_{A, m}$ with $t = t_0 = S(\rho_A) - S^\infty$. We could construct a pure state

$$|\psi\rangle_0 = \frac{1}{\sqrt{\mathcal{P}(t_0)}} \sum_i |\lambda_i\rangle_A \otimes |\bar{\lambda}_i\rangle_{\bar{A}} \delta(t_i - t_0). \quad (5.6)$$

Of course, the state $|\psi\rangle_0$ is different from the vacuum state. In [50, 51] the authors constructed an approximate tensor networks for geometric states in AdS/CFT, which is generally given by

$$|\Psi\rangle = \sum_{I=0}^{e^{O(\sqrt{S})}} \sum_{i=0}^{e^{S-\sqrt{S}}} \sqrt{\lambda_i} |I, i\rangle_A \otimes |I, i\rangle_{\bar{A}}, \quad (5.7)$$

where $S := S(\rho_A)$ is the EE of A . It seems our state $|\psi\rangle_0$ is only part of the state $|\Psi\rangle$. So $|\psi\rangle_0$ and $\rho_{A, m}$ with $t = t_0$ only catch part information of the $|\Psi\rangle$ and ρ_A . This may be the reason why $\rho_{A, m}$ can be taken as the an approximate state ρ_A only if the probes are located in a small region of A and far away from the boundary. At present we still don't

know how to find the corrections of $|\psi\rangle_0$ and $\rho_{A,m}$ to make them to be better approximate states of $|\Psi\rangle$ and ρ_A .

At last, let's see the microcanonical ensemble state $\rho_{A,m}$ with $t \neq t_0$ in the single interval example. By using (2.37) we see that the one-point functions of quasi-primary operators $\mathcal{X}$ is non-vanishing for $t \neq b$. For example, the one-point function of T and $\mathcal{A}$ is given by (2.27) and (2.30). In [14] we find a series of conditions associated with the one-point functions of quasi-primary operators for geometric states ρ . The first condition is

$$\langle \mathcal{A} \rangle_\rho - \langle T \rangle_\rho^2 \sim O(c), \quad (5.8)$$

if $\langle T \rangle_\rho \sim O(c)$. One could check the state $\rho_{A,m}$ satisfies the above condition. By using the results in appendix A one can further show $\rho_{A,m}$ satisfies the conditions associated with higher order quasi-primary operators ($\mathcal{X} = \mathcal{B}, \mathcal{D}$). Therefore, this suggests the microcanonical ensemble state $\rho_{A,m}$ for any t may be taken as the reduced density matrix of A of some pure geometric state.

In the references [52, 53] (see also [54]) the authors discuss the so-called fixed-area states in the bulk side of AdS/CFT by using the quantum-error correcting code. A noticeable fact is the Rényi entropy in the fixed-area state is independent with n at the leading order of G^{-1} or $O(c)$. This property is similar with our microcanonical ensemble state $\rho_{A,m}$ constructed by field theory side. It is interesting to check whether $\rho_{A,m}$ has some relations to the fixed-area state.

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A More examples of one-point functions of quasi-primary operators

In section 2.2.2 we only calculate the one-point functions of T and $\mathcal{A}$. Here we will study more quasi-primary operators in the vacuum conformal family. There are two quasi-primary operators with conformal dimension 6, that is

$$\begin{aligned} \mathcal{B} &= (\partial T \partial T) - \frac{4}{5}(\partial^2 T T) - \frac{1}{42}\partial^4 T, \\ \mathcal{D} &= (T(TT)) - \frac{9}{10}(\partial^2 T T) - \frac{1}{28}\partial^4 T + \frac{93}{70c+29}\mathcal{B}. \end{aligned} \quad (A.1)$$

By using the same conformal map and the transformation law of $\mathcal{B}$ and $\mathcal{D}$ we have

$$\langle \mathcal{B} \rangle_{\mathcal{R}_n} = \frac{2c(n^2-1)^2(2(35c+61)n^2-93)}{1575n^6} \frac{R^6}{(R^2-w^2)^6}, \quad (A.2)$$

$$\langle \mathcal{D} \rangle_{\mathcal{R}_n} = \frac{c(2c-1)(5c+22)(7c+68)(n^2-1)^3}{216(70c+29)n^6} \frac{R^6}{(R^2-w^2)^6}. \quad (A.3)$$

With the above results we can calculate the one-point functions

$$\mathcal{P}_{\mathcal{B}}(t) = \frac{2c(70c+122)}{1575} \frac{R^6}{(R^2-w^2)^6} \delta(t) + \frac{2c}{1575b^3} \frac{R^6}{(R^2-w^2)^6} \frac{\sqrt{b}}{\sqrt{t}I_1(2\sqrt{bt})} \left[2b^3(35c+61) - b^2(140c+337)t + 14b(5c+22)t^2 \frac{I_3(2\sqrt{bt})}{I_1(2\sqrt{bt})} - 93t^3 \frac{I_5(2\sqrt{bt})}{I_1(2\sqrt{bt})} \right], \quad (\text{A.4})$$

and

$$\mathcal{P}_{\mathcal{D}}(t) = \frac{c(2c-1)(5c+22)(7c+68)}{216(70c+29)} \frac{R^6}{(R^2-w^2)^6} \delta(t) + \frac{c(2c-1)(5c+22)(7c+68)}{216(70c+29)} \frac{R^6}{(R^2-w^2)^6} \times \frac{\sqrt{b}I_1(2\sqrt{bt})}{\sqrt{t}} \left[\left(1 - \frac{3t}{b}\right) + \frac{3tI_3(2\sqrt{bt})}{bI_1(2\sqrt{bt})} - \frac{t^2I_5(2\sqrt{bt})}{b^2I_1(2\sqrt{bt})} \right]$$

In the large c or b limit one could find an approximate expression for $\mathcal{P}_{\mathcal{D}}(t)$ and $\mathcal{P}_{\mathcal{B}}(t)$,

$$\begin{aligned} \mathcal{P}_{\mathcal{B}}(t) &\simeq \frac{2c(70c+122)}{1575} \frac{R^6}{(R^2-w^2)^6} \delta(t) + \frac{2c}{1575} \frac{R^6}{(R^2-w^2)^6} \frac{\sqrt{b}I_1(2\sqrt{bt})}{\sqrt{t}} \left[2(35c+61) - 93\frac{t}{b} \right] \left(1 - \frac{t}{b}\right)^2, \\ \mathcal{P}_{\mathcal{D}}(t) &= \frac{c(2c-1)(5c+22)(7c+68)}{216(70c+29)} \frac{R^6}{(R^2-w^2)^6} \delta(t) + \frac{c(2c-1)(5c+22)(7c+68)}{216(70c+29)} \frac{R^6}{(R^2-w^2)^6} \\ &\times \frac{\sqrt{b}I_1(2\sqrt{bt})}{\sqrt{t}} \left(1 - \frac{t}{b}\right)^3. \end{aligned} \quad (\text{A.5})$$

$\mathcal{P}_{\mathcal{B}}(t)$ and $\mathcal{P}_{\mathcal{D}}(t)$ show similar properties as $\mathcal{P}_T(t)$ and $\mathcal{P}_{\mathcal{A}}(t)$. Specially, $\mathcal{P}_{\mathcal{B}}(t), \mathcal{P}_{\mathcal{D}}(t) \rightarrow 0$ when $t = b$ in the large c limit.

B Higher order corrections of two-point functions

In section 2.2.5 we calculate two-point functions $\mathcal{P}_{\mathcal{O}\mathcal{O}}(t)$ upto order $O(x^{2-2h_{\mathcal{O}}})$. Here we list more higher order terms, We have

$$\begin{aligned} \langle \mathcal{O}(x)\mathcal{O}(-x) \rangle_{\mathcal{R}_n} &= (2x)^{-2h_{\mathcal{O}}} \left(1 + \frac{h_{\mathcal{O}}(n^2-1)(2x)^2}{3n^2R^2} + \frac{h_{\mathcal{O}}(n^2-1)(5h_{\mathcal{O}}n^2+4n^2-5h_{\mathcal{O}}-1)(2x)^4}{90n^4R^4} \right. \\ &+ \frac{h_{\mathcal{O}}(n^2-1)(280h_{\mathcal{O}}^2n^4-560h_{\mathcal{O}}^2n^2+280h_{\mathcal{O}}^2+672h_{\mathcal{O}}n^4-840h_{\mathcal{O}}n^2+168h_{\mathcal{O}}+347n^4-136n^2+32)(2x)^6}{45360n^6R^6} \\ &\left. + o(x^6) \right). \end{aligned}$$

With the above results we can derive

$$\begin{aligned} \mathcal{P}_{\mathcal{O}\mathcal{O}}(t) &= \mathcal{P}(t)(2x)^{-2h_{\mathcal{O}}} \left\{ 1 + \frac{h_{\mathcal{O}}(2x)^2}{3R^2} \left(1 - \frac{t}{b}\right) + \left[5h_{\mathcal{O}}+4-5(2h_{\mathcal{O}}+1)\frac{t}{b} + (5h_{\mathcal{O}}+1)\frac{I_3(2\sqrt{bt})}{I_1(2\sqrt{bt})} \frac{t^2}{b^2} \right] \right. \\ &\times \frac{h_{\mathcal{O}}(2x)^4}{90R^4} + \left[280h_{\mathcal{O}}^2+672h_{\mathcal{O}}+347-21(40h_{\mathcal{O}}^2+72h_{\mathcal{O}}+23)\frac{t}{b} + 168(5h_{\mathcal{O}}^2+6h_{\mathcal{O}}+1) \right. \\ &\times \left. \left. \frac{I_3(2\sqrt{bt})}{I_1(2\sqrt{bt})} \left(\frac{t}{b}\right)^2 - 8(35h_{\mathcal{O}}^2+21h_{\mathcal{O}}+4)\frac{I_5(2\sqrt{bt})}{I_1(2\sqrt{bt})} \left(\frac{t}{b}\right)^3 \right] \frac{h_{\mathcal{O}}(2x)^6}{45360R^6} + o(x^6) \right\}. \end{aligned} \quad (\text{B.1})$$

If we take the large c limit of the above result, one would find

$$\begin{aligned} \mathcal{P}_{\mathcal{OO}}(t) = & \mathcal{P}(t)(2x)^{-2h_{\mathcal{O}}} + \mathcal{P}(t)(2x)^{-2h_{\mathcal{O}}} \left(1 - \frac{t}{b}\right) \left[\frac{h_{\mathcal{O}}(2x)^2}{3R^2} + \left((5h_{\mathcal{O}} + 4) - \frac{(5h_{\mathcal{O}} + 1)t}{b} \right) \frac{h_{\mathcal{O}}(2x)^4}{90R^4} \right. \\ & \left. + \left(347 + 672h_{\mathcal{O}} + 280h_{\mathcal{O}}^2 - 8(17 + 105h_{\mathcal{O}} + 70h_{\mathcal{O}}^2) \frac{t}{b} + 8(4 + 21h_{\mathcal{O}} + 35h_{\mathcal{O}}^2) \left(\frac{t}{b}\right)^2 \right) \frac{h_{\mathcal{O}}(2x)^6}{45360R^6} + o(x^6) \right]. \end{aligned}$$

The two-point functions are very simple upto $O(x^6)$ if $t = b$, that is

$$\mathcal{P}_{\mathcal{OO}}(t = b) = \mathcal{P}(b)(2x)^{-2h_{\mathcal{O}}} + o(x^6), \quad (\text{B.2})$$

or

$$\bar{\mathcal{P}}_{\mathcal{OO}}(t = b) = (2x)^{-2h_{\mathcal{O}}} + o(x^6). \quad (\text{B.3})$$

C Calculations for $d = 4$

In this section we will show the detial calculation of the density of eigenstates for dimension $d = 4$. The Rényi entropy is

$$S_A^{(n)}(d = 4) = \frac{40n^4 - 4 \left(2\sqrt{8n^2 + 1} + 3 \right) n^2 - \sqrt{8n^2 + 1} - 1}{128\pi G(n - 1)n^3} \log \left(\frac{2R}{\epsilon} \right). \quad (\text{C.1})$$

We have

$$b := S^\infty = \lim_{n \rightarrow \infty} S_A^{(n)}(d = 4) = \frac{5}{16\pi G} \log \frac{2R}{\epsilon}, \quad S(\rho_A) := \lim_{n \rightarrow 1} S_A^{(n)}(d = 4) = \frac{1}{2\pi G} \log \frac{2R}{\epsilon}. \quad (\text{C.2})$$

The density of eigenstates is given by

$$\mathcal{P}(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} dn e^{s_n}, \quad (\text{C.3})$$

where

$$s_n = nt + nS^\infty + (1 - n)S_A^{(n)}(d = 4), \quad (\text{C.4})$$

In the semi-classical limit $G \rightarrow 0$, we assume $t \sim O(G^{-1})$. The condition of the saddle point approximation $\partial_n s_n = 0$ gives

$$n = \sqrt{\frac{3b + \sqrt{15}\sqrt{bt}}{10t}}. \quad (\text{C.5})$$

Taking the above formula back to s_n (C.4) we have

$$\mathcal{P}(t) = e^{s_{n,\max}}, \quad (\text{C.6})$$

with

$$\begin{aligned} s_{n,\max} = & \sqrt{\frac{1}{10}t(\sqrt{15bt} + 3b)} + \frac{1}{2}\sqrt{\frac{5}{2}}b \left(\frac{t}{\sqrt{15bt} + 3b} \right)^{3/2} \\ & \times \left(\sqrt{\frac{4(\sqrt{15bt} + 3b)}{5t}} + 1 + \frac{2(\sqrt{15bt} + 3b) \left(2\sqrt{\frac{4(\sqrt{15bt} + 3b)}{5t}} + 1 + 3 \right)}{5t} + 1 \right). \end{aligned}$$

In the large t limit the density of eigenstates $\mathcal{P}(t) \sim e^{t^{3/4}}$ consistent with the result in [22]. We are interested in the case for which $\mathcal{P}(t_0) \simeq e^{S(\rho_A)}$. The non-trivial solution of $s_{n,\max} = S(\rho_A)$ is

$$t_0 = \frac{3b}{5}. \quad (\text{C.7})$$

Taking t_0 into (C.5) we find $n = 1$. Moreover, we also have $t_0 = S(\rho_A) - S^\infty$, which is consistent with our general results in section 3.2.2.

D Review of short interval expansion

In this section we briefly review the short interval expansion of the EE and Holevo information. One could refer to [14, 24, 33] for more details. If we only consider the contributions from the vacuum conformal family, the EE of A with length ℓ is given by

$$\begin{aligned} S_A = & \frac{c}{6} \log \frac{\ell}{\epsilon} + \ell^2 a_T \langle T \rangle_\rho + \ell^4 a_{TT} \langle T \rangle_\rho^2 + \ell^6 a_{TTT} \langle T \rangle_\rho^3 \\ & + \ell^8 (a_{\mathcal{A}\mathcal{A}} \langle A \rangle_\rho^2 + a_{TT\mathcal{A}} \langle T \rangle_\rho^2 \langle \mathcal{A} \rangle_\rho + a_{TTTT} \langle T \rangle_\rho^4) \\ & + \ell^{10} (a_{T\mathcal{A}\mathcal{A}} \langle T \rangle_\rho \langle \mathcal{A} \rangle_\rho^2 + a_{TTT\mathcal{A}} \langle T \rangle_\rho^3 \langle \mathcal{A} \rangle_\rho + a_{TTTTT} \langle T \rangle_\rho^5) + F(\partial) + O(\ell^{11}), \end{aligned} \quad (\text{D.1})$$

where the coefficients $a_{\mathcal{X}_1 \dots \mathcal{X}_m}$ are constants. We list them as follows

$$\begin{aligned} a_T = -\frac{1}{6}, \quad a_{TT} = -\frac{1}{30c}, \quad a_{TTT} = -\frac{4}{315c^2}, \quad a_{TT\mathcal{A}} = \frac{1}{315c^2}, \\ a_{\mathcal{A}\mathcal{A}} = -\frac{1}{126c(5c+22)}, \quad a_{TTTT} = -\frac{c+8}{630c^3}, \quad a_{T\mathcal{A}\mathcal{A}} = -\frac{16}{693c^2(5c+22)}, \\ a_{TTT\mathcal{A}} = \frac{32}{3465c^3}, \quad a_{TTTTT} = -\frac{16(c+5)}{3465c^4}. \end{aligned} \quad (\text{D.2})$$

$F(\partial)$ denotes the terms with derivatives $\partial^m \langle \mathcal{X} \rangle_\rho$. For example, at order of $O(\ell^3)$ the term is $\frac{a_T}{2} \partial \langle T \rangle_\rho$. These terms are vanishing for translation invariant states. Though in our discussion $F(\partial)$ is non-vanishing, they are not important for our results in section 4.4.

By the definition of Holevo information (4.2) one could calculate the Holevo information of $\rho = \sum_i p_i \rho_i$ by short interval method. The results are given by

$$\begin{aligned} \chi(\rho_A) = & \ell^4 a_{TT} (\langle T \rangle_\rho^2 - \sum_i p_i \langle T \rangle_i^2) + \ell^6 a_{TTT} (\langle T \rangle_\rho^3 - \sum_i p_i \langle T \rangle_i^3) \\ & + \ell^8 [a_{\mathcal{A}\mathcal{A}} (\langle A \rangle_\rho^2 - \sum_i p_i \langle A \rangle_i^2) + a_{TT\mathcal{A}} (\langle T \rangle_\rho^2 \langle \mathcal{A} \rangle_\rho - \sum_i p_i \langle T \rangle_i^2 \langle \mathcal{A} \rangle_i) \\ & + a_{TTTT} (\langle T \rangle_\rho^4 - \sum_i \langle T \rangle_i^4)] + \ell^{10} [a_{T\mathcal{A}\mathcal{A}} (\langle T \rangle_\rho \langle \mathcal{A} \rangle_\rho^2 - \sum_i p_i \langle T \rangle_i \langle \mathcal{A} \rangle_i^2) \\ & + a_{TTT\mathcal{A}} (\langle T \rangle_\rho^3 \langle \mathcal{A} \rangle_\rho - \sum_i p_i \langle T \rangle_i^3 \langle \mathcal{A} \rangle_i) + a_{TTTTT} (\langle T \rangle_\rho^5 - \sum_i p_i \langle T \rangle_i^5)] + G(\partial) + O(\ell^{11}). \end{aligned} \quad (\text{D.3})$$

where $G(\partial)$ denotes the terms associated with $\partial^m \langle \mathcal{X} \rangle_i$ and $\langle \mathcal{X} \rangle_i := \text{tr}(\mathcal{X} \rho_i)$ with $\mathcal{X} = T, \mathcal{A}$.

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