Lens space index and global properties for 4d $\mathcal{N} = 2$ models

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ABSTRACT: The additional data necessary to univocally fix the gauge group for a given algebra are represented by the same charge lattices of mutually local Wilson and 't Hooft lines for both 4d $\mathcal{N} = 4$ SYM and $\mathcal{N} = 2$ elliptic models. Motivated by this equivalence in this paper we study the Lens space index of these $\mathcal{N} = 2$ elliptic models. The index is indeed sensitive to the global properties and in the $\mathcal{N} = 4$ case it is expected to coincide among S-dual models with different global properties, while it gives different results for models that lie in other S-duality orbits. Here by an explicit calculation we show that the same results hold for the $\mathcal{N} = 2$ elliptic models as well.

KEYWORDS: Gauge Symmetry, Supersymmetric Gauge Theory, Wilson, 't Hooft and Polyakov loops

ARXIV EPRINT: [1911.13264](https://arxiv.org/abs/1911.13264)

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1 Introduction

S-duality is an intriguing symmetry, originally conjectured by [1] for the case of 4d $\mathcal{N} = 4$ SYM. When the gauge group is $SU(N)$ S-duality acts by inversion on the holomorphic gauge coupling, $\tau \rightarrow -\frac{1}{\tau}$ providing a strong/weak duality. When this inversion is considered together with another symmetry acting on the gauge coupling, $\tau \rightarrow \tau + 1$, the S-duality group is $SL(2, \mathbb{Z})$. This provides an origin of this symmetry in terms of the invariance of the dilaton of 10 SUGRA, connecting the S-duality group of $\mathcal{N} = 4$ SYM to the S-duality of string theory. S-duality has been further extended to cases with lower supersymmetry (see e.g. [2]). For example, when considering elliptic models, it has been shown that the full S-duality group is generated by the two generators above and other shift and permutation symmetries. In this case the group becomes the mapping class group of the punctured Riemann surface that describes the M-theory construction of the model. Myriads of applications and connection with the geometry (e.g. the string embedding, lower dimensional dualities, etc...) made S-duality one of the more prominent tools in understanding non-perturbative aspects of supersymmetric QFTs in the last decades.

Here we focus on a result obtained in [3],¹ relating S-duality to the global properties of the gauge group of $\mathcal{N} = 4$ SYM. Concretely, when one considers a QFT with a gauge algebra $\mathfrak{g}$ one should provide also additional informations on the full gauge group $\mathbf{G}$. In absence of matter fields, being the vector multiplet in the adjoint representation of the gauge group, one can equally choose either the universal covering group $\tilde{\mathbf{G}}$ or some other group $\mathbf{G} = \tilde{\mathbf{G}}/\mathbf{H}$, where $\mathbf{H}$ is a subgroup of the center Γ . When matter fields are considered, some of the possible groups $\mathbf{G}$ are forbidden, depending on the representation of the matter fields under the gauge group. This issue does not arise for $\mathcal{N} = 4$ SYM because the matter

¹See also [4, 5].

fields are in the adjoint representation. In this last case additional data are necessary to specify the global structure of the gauge group. A necessary and sufficient condition, fixing completely such data, has been provided in [3]. It has been indeed shown that specifying the spectrum of charges of mutually local line operators, i.e. Wilson and 't Hooft lines [6], univocally fixes the gauge group $\mathbf{G}$. This result has a deep consequence on S-duality. This is because by acting with S-duality on the lattice that identifies the spectrum of charges of the line operators one can pass from a group $\mathbf{G}$ to a group $\mathbf{G}'$, showing that theories with different gauge groups are usually related by an S-duality transformation. However it is not always the case because there are lattices that are not connected by S-duality to the others, signaling the presence of orbits in the S-duality group. For example, in the case of $\mathcal{N} = 4$ $SU(N)$ SYM, these orbits are present whenever squares appear in the decomposition of N into prime factors (i.e. whenever N is not square free).

This mechanism has been extended to cases with $\mathcal{N} = 2$ supersymmetry [7]. Also in this case, by organizing the spectrum of mutually local line operators into lattices of electric and magnetic charges it has been possible to associate theories with different global properties to S-duality transformations. These lattices follow from the presence of bifundamental fields connecting the nodes of the quiver. In such cases the matter fields have a trivial charge under the diagonal subgroup of the center $\Gamma_{\text{diag}} \equiv \text{diag}(\prod_{i=1}^{n_G} \Gamma_i)$, where n_G represents the number of gauge nodes in the quiver.

In this paper we corroborate this result by studying and comparing the Lens space index of theories with a different global structure but mapped by an $SL(2, \mathbb{Z})$ transformation. The Lens space index was originally derived in [8]. It is a supersymmetric index computed on the product space $L(r, 1) \times S^1$ where $L(r, 1)$ corresponds to the Lens space. Being not simply connected it is sensitive to the global properties of the gauge groups. Indeed it takes contributions from two simultaneously conjugated commuting holonomies, one from the S^1 and one from the S^1 inside $L(r, 1)$. If the gauge group is the universal covering $\tilde{\mathbf{G}}$ then the only commuting holonomies contributing to the index, say g and h , respect a relation $[g, h] = 1$. If the gauge group is $\mathbf{G} = \tilde{\mathbf{G}}/\mathbf{H}$ the relation can be modified, by considering other elements of the center Γ instead of the identity. These contributions come from the so called almost commuting holonomies, i.e. holonomies that do not commute in $\tilde{\mathbf{G}}$ but commute in $\mathbf{G}$. In the case of $\mathcal{N} = 4$ SYM this index was computed in [9] and it was shown that it coincides among theories with different gauge group if these theories are mapped by an S-duality transformation. It was also observed that in the case of theories living in different orbits of $SL(2, \mathbb{Z})$ the index does not match as expected. Here we provide the extension of the result to the models with lower supersymmetry discussed above.

The paper is organized as follows. In section 2 we review the construction of [3] and the additional data necessary for distinguishing models with different gauge groups. In section 3 we review the construction of [9] for the Lens space index and its relations with the global properties of $\mathcal{N} = 4$ SYM. In section 4 we extend the construction to $\mathcal{N} = 2$ elliptic models, providing a general formula for the Lens space index in these cases. We apply this formalism to a simple model, namely an $\mathcal{N} = 2$ elliptic model with two $SU(N)$ gauge groups and we show that for $N = 2, 3, 4$ we recover the expected results, i.e. the index coincides for all the models in the same S-duality orbit and in the $N = 4$ case we

also find a different value of the index for the model corresponding to the new orbit. In section 5 we conclude. In appendix A we furnish the structure of the Lens space index for a generic $\mathcal{N} = 1$ quiver.

2 Global properties and S-duality

In a general QFT fixing a connected gauge group $\mathbf{G}$ from gauge Lie algebra $\mathfrak{g}$ requires to specify additional data, because there are different possible gauge groups corresponding to the same algebra. The gauge group is fixed as follows: consider the universal covering group $\tilde{\mathbf{G}}$ and mod it by its center Γ or a subgroup $\mathbf{H} \subset \Gamma$. The gauge group is $\mathbf{G} = \tilde{\mathbf{G}}/\mathbf{H}$. This group is commonly referred to as the electric gauge group. Indeed, by keeping in mind the electric-magnetic duality for Abelian theories, fixing the gauge theory requires also a similar discussion on the magnetic side. This corresponds to fixing the global properties of the GNO-dual [10] $\mathfrak{g}^*$ algebra. The magnetic group $\mathbf{G}^*$ is not completely fixed yet. Indeed for a given group $\tilde{\mathbf{G}}/\mathbf{H}$, there can be different possible choices of the GNO-dual group $\mathbf{G}^*$.

A possible way to fix the gauge group consists of choosing a (maximal) set of allowed line operators. These line operators are Wilson (W) and 't Hooft (H) lines. The W-lines are related to the electric aspects of the gauge theory while the H-lines are related to the magnetic ones.

Here we consider $\mathcal{N} = 4$ SYM with $\mathfrak{g} = su(N)$ and review how line operators are used to specify the global properties of the gauge group. Once we fix the group $\mathbf{G} = \tilde{\mathbf{G}}/\mathbf{H}$ the line operators have to be invariant under $\mathbf{H}$. Here W-lines are invariant under $\mathbf{H}$, i.e. they are labeled by representations of $\mathbf{G}$. Analogously the H-lines are labeled by representations of $\mathbf{G}^*$. The spectrum of the allowed operators can be put in 1-1 correspondence with the gauge group. However it is not necessary to specify all the possible representations but one can restrict to a subset of them, identified by their charge under the center Γ . In the rest of this section we review the relations between the charges of the line operators and the global properties of a gauge group.

When the gauge group is the universal cover $\tilde{\mathbf{G}}$ there is no restriction on the allowed representations of the Wilson lines. We can associate the representations to a lattice, the weight lattice Λ_w of $\mathfrak{g}$, modulo the Weyl group W . Modding $\tilde{\mathbf{G}}$ by Γ only the representations invariant under Γ survive. They are obtained by moving with the roots of $\mathfrak{g}$ on a sub-lattice specified by the adjoint representation. The weights of the adjoint are the roots of the algebra $\mathfrak{g}$ and this sub-lattice is denoted as the root lattice Λ_r . One can choose also a subgroup $\mathbf{H} \subset \Gamma$. In this case the representation lives in the co-character lattice $\Gamma_{\mathbf{G}^*}$. The three lattices are related as $\Lambda_r \subset \Gamma_{\mathbf{G}^*} \subset \Lambda_w$. The center $\mathbf{H}$ and the fundamental group $\pi_1(\mathbf{G})$ are

$$\mathbf{H} = \Gamma_{\mathbf{G}^*}/\Lambda_r, \quad \pi_1(\mathbf{G}) = \Lambda_w/\Gamma_{\mathbf{G}^*}. \quad (2.1)$$

A similar discussion can be replied for the GNO-dual algebra $\mathfrak{g}^*$. The center is $\Gamma^* = \Gamma$ and the Weyl group is $W^* = W$. The magnetic weight lattice Λ_{mw} corresponds to the dual of the root lattice of $\mathfrak{g}$. The magnetic root lattice is called Λ_{cr} , and it is usually called the co-root lattice. The dual of the co-character lattice is called the character lattice $\Gamma_{\mathbf{G}^*}$. The

inclusion here is $\Lambda_{cr} \subset \Gamma_{\mathbf{G}^*} \subset \Lambda_{mw}$. The center $\mathbf{H}$ and the fundamental group $\pi_1(\mathbf{G}^*)$ are

$$\mathbf{H}^* = \Lambda_{mw}/\Gamma_{\mathbf{G}^*}, \quad \pi_1(\mathbf{G}^*) = \Gamma_{\mathbf{G}^*}/\Lambda_{cr}. \quad (2.2)$$

In the case the magnetic set of lines are the 't Hooft lines. Fixing the gauge group requires to specify the allowed representations of the H-lines as well.

Observe that there are not only purely electric W-lines or purely magnetic H-lines, but we can allow also for dyonic (W,H)-lines. It turns out that specifying the set of allowed W-lines and H-lines corresponds to specify a sub-lattice $\Lambda_w \times \Lambda_{mw}$ modulo the action of the Weyl group.

Consider a generic dyonic line $(\lambda_e, \lambda_m) \in \Lambda_w \times \Lambda_{mw}$, it is identified to the line $(w\lambda_e, w\lambda_m)$, where w is an element of W . If the line (λ_e, λ_m) is allowed also the line $(-\lambda_e, -\lambda_m)$ is allowed. Moreover if two lines (λ_e, λ_m) and (λ'_e, λ'_m) are allowed also the line $(\lambda_e + \lambda'_e, \lambda_m + \lambda'_m)$ is allowed. For each choice of $\mathbf{G}$ there is always a pure electric line $(r_e, 0)$ with $r_e \in \Lambda_r$ and a pure magnetic line $(0, r_m)$ with $r_m \in \Lambda_{cr}$.

At this level of the discussion we are still considering the electric and magnetic sub-lattices of Λ_w and Λ_{mw} . We can project them to smaller sub-lattices using the following observation. We can consider a dyonic line (λ_e, λ_m) in the lattice which specifies a set of points. Then we can always add the line (pr_e, qr_m) with $p, q \in \mathbb{Z}$. In this way we reach all the points of the lattice with the same charge as (λ_e, λ_m) mod the dimension of the center. This explains why the lines can be organized in classes distinguished by their charge under the center. A generic point is of the form $(z_e, z_m) \in \Gamma \times \Gamma$. A theory is specified by a complete set of allowed charges. The lines are obtained from a condition of mutual locality, corresponding to a DSZ quantization condition on the set of charges. In the case of $\mathbf{g} = su(N)$, one has that $\Gamma = \mathbb{Z}_n$ and $\mathbf{H} = \mathbb{Z}_{k'}$ where $k'k = N$ and $k, k', N \in \mathbb{N}$. We can choose the representative of the equivalence class of charges by observing that the fundamental representation has unitary charge under the center. A generic point with charge (e, m) can be associated to the representation $(Sym^e_{\square}; Sym^m_{\square})$, i.e. the symmetric product of e and m fundamentals.

The DSZ quantization condition for two pairs of charges (e_1, m_1) and (e_2, m_2) is

$$e_1 m_2 - m_1 e_2 = 0 \text{ mod } N. \quad (2.3)$$

The gauge group is in general $SU(N)/\mathbb{Z}_k$. The electric lattice admits only charges $(e_\ell, 0)$ with $e_\ell = \ell k$ and $\ell \in \mathbb{Z}$. There is always the possibility to choose the charge $(0, m_\ell)$ with $m_\ell = \ell k'$ and k' integer. This is because $\pi_1(\mathbf{G}) = \mathbb{Z}_{k'}$. Other choices are anyway possible. They come from the fact that the lattices are generated by the DSZ quantization condition. In general the magnetic charge with the smallest m is $(e = \ell, m_{\min} = k')$. There are k different (inequivalent) choices, $\ell = 0, \dots, k-1$. Once this second point in the lattice of the charges is specified we can use the two vectors $(e_{\min} = k, 0)$ and $(\ell, m_{\min})$ to reconstruct the whole lattice. The reason is that we can combine linearly the lines and then their charges: if the lines with charge (e_1, m_1) and (e_2, m_2) are in the lattice then also the line with charges $(e_1 + e_2, m_1 + m_2)$ is in the lattice. The group associated to the lattice generated by these two vectors is denoted as $(SU(N)/\mathbb{Z}_k)_\ell$, $\ell = 0, \dots, k-1$. Theories identified by

different lattices have different global properties but they are connected by the action of the generators S and T of the $SL(2, \mathbb{Z})$ S-duality group. However if N is not square free there are theories that sit in different orbits of the S-duality group. i.e. self dual theories that cannot be reached by applying any $SL(2, \mathbb{Z})$ transformations.

3 Lens space index

The Lens space index is a generalization of the superconformal index of [11, 12], computed on $L(r, 1) \times S^1$. The Lens space $L(r, 1)$ is an orbifold of S^3 defined by the identification

$$(z_1, z_2) \sim (z_1 e^{\frac{2\pi i}{r}}, z_2 e^{-\frac{2\pi i}{r}}) \quad (3.1)$$

where z_i are complex and the three sphere is defined by the relation $|z_1|^2 + |z_2|^2 = 1$. The index of a chiral multiplet Φ of R -charge Δ is

$$\begin{aligned} I_{\Phi}^{(\Delta)}(z) &= \Gamma_e \left((pq)^{\frac{\Delta}{2}} z; q^r, pq \right) \Gamma_e \left((pq)^{\frac{\Delta}{2}} p^r z; p^r, pq \right) \\ &= \Gamma_e \left((pq)^{\frac{\Delta}{2}} q^r z; q^r, pq \right) \Gamma_e \left((pq)^{\frac{\Delta}{2}} z; p^r, pq \right), \end{aligned} \quad (3.2)$$

where Γ_e represents an elliptic Gamma function (see [13, 14] for a physical definition) corresponding to

$$\Gamma_e(z; p, q) \equiv \prod_{i,j=0}^{\infty} \frac{1 - p^{i+1} q^{j+1} / z}{1 - p^i q^j z}. \quad (3.3)$$

The fugacities p and q refer to the two $SU(2)$ of the original S^3 while z is a collective fugacity for the other internal symmetries. In presence of global internal symmetries the index can be refined. For each global symmetry one can consider a fugacity z and also turn on a non trivial holonomy around a non-contractible cycle in $L(r, 1)$. Going around such cycle the field acquires a phase $e^{\frac{2\pi i m}{r}}$. The net result for the index of Φ in presence of such an holonomy is

$$I_{\Phi}^{(\Delta)}(m, z) = I_0^{(\Delta)}(m, z) \Gamma_e \left((pq)^{\frac{\Delta}{2}} q^{r-m} z; q^r, pq \right) \Gamma_e \left((pq)^{\frac{\Delta}{2}} p^m z; p^r, pq \right), \quad (3.4)$$

where I_0 represents the vacuum energy, that, for non-anomalous symmetries (the only one that will be considered here) is

$$I_0^{(\Delta)}(m, z) = \left(\frac{(pq)^{1-\Delta}}{z^2} \right)^{\frac{m(r-m)}{4r}} \left(\frac{p}{q} \right)^{\frac{m(r-m)(r-2m)}{12r}}. \quad (3.5)$$

The contribution of an $\mathcal{N} = 1$ vector multiplet to the Lens index is

$$I_V(m, z) = \frac{I_0^V(m, z)}{(1 - 1/z)^{\delta_{m,0}} \Gamma_e(q^m/z; q^r, pq) \Gamma_e(p^{r-m}/z; p^r, pq)}, \quad (3.6)$$

where

$$I_0^V(m, z) = \left(\frac{pq}{z^2} \right)^{-\frac{m(r-m)}{4r}} \left(\frac{q}{p} \right)^{\frac{m(r-m)(r-2m)}{12r}}. \quad (3.7)$$

If there is also a gauge group one has to properly sum the contribution of the non-trivial holonomies on the matrix integral. The path integral over the gauge group $\mathbf{G}$ localizes indeed onto flat connections, labeled by the holonomies around the S^1 and the cycle inside $L(r, 1)$. These holonomies are elements of the gauge group and can be labeled by h and g , provided that they respect the conditions $g^r = 1$ and $ghg^{-1}h^{-1} = 1$. The holonomies must be considered up to simultaneous conjugation, leading to discrete set of choices for g . One can fix g to lie in the maximal torus. This is the end of the story for simply connected groups. In this case one obtain the Lens space index by summing over the inequivalent discrete choices of g , by integrating over the eigenvalues of h and by properly considering the Haar measure of the gauge group preserved by g in the matrix integral.

Let us consider the case of $\tilde{\mathbf{G}} = \text{SU}(N)$, even if the situation is more general. If one considers a non simply connected gauge group $\mathbf{G} = \text{SU}(N)/\mathbb{Z}_d$, obtained by modding the universal covering group $\tilde{\mathbf{G}} = \text{SU}(N)$ by a (sub)group $\mathbf{H} = \mathbb{Z}_d$ of the center $\Gamma = \mathbb{Z}_N$, the situation is more complicated. The holonomies in this case have to satisfy the following constraints

$$g^r = \mu = e^{\frac{2\pi i k}{N}} \quad \text{and} \quad ghg^{-1}h^{-1} = \nu = e^{\frac{2\pi i \ell}{N}}, \quad (3.8)$$

where k and ℓ are integer and $k \simeq k + \text{gcd}(N, r)$ and $\ell = \hat{\ell}N/\text{gcd}(N, r)$, such that k and $\hat{\ell}$ can be chosen to lie between 0 and $\text{gcd}(N, r) - 1$. In the rest of the discussion we fix $r = N$, that is the simplest case in which the global properties give different contributions to the integrand. These holonomies are referred as almost commuting and their contribution has to be computed separately. In such cases the integral over the gauge group is reduced, because, depending on the choices of μ and ν , some of the eigenvalues of h are fixed. In the index the contribution of the fields is usually obtained by decomposing them in the weight space. If g and h are almost commuting they do not lie in the maximal torus, and it implies that the action on the weight basis is not necessarily diagonal. In this case one has to find a basis that makes such an action diagonal and sum over the eigenvalues in this basis. In the next section we will review the choice of the basis for the case of the adjoint representation of $\text{SU}(N)$. The index becomes a sum over the sectors identified by different values of μ and ν . Each sector is weighted by a discrete theta angle, identified by a phase involving rational number $c(w_2)$ where w_2 represents the second Stiefel-Whitney class. The index is formally

$$I = \frac{1}{|\mathbf{H}|} \sum_{\mu, \nu} e^{ic(w_2(\mu, \nu))} Z_{\mu, \nu}. \quad (3.9)$$

3.1 $\mathcal{N} = 4$ SYM and global properties

Here we review the results of [9] for the derivation of the Lens space index for the case of $\text{SU}(N)$ $\mathcal{N} = 4$ SYM. This is necessary to set the notations for the derivation of the index in cases with lower supersymmetry.

Let us first review the results for the sector $\hat{\ell} = 0$ with a generic k . In this case the matrices $\tilde{g}$ and $\tilde{h}$ are both diagonal. The elements of $\tilde{g}$ are $e^{2\pi i/r(m_i + k/N)}$ with $m_1 + \dots +$

$m_N + k = 0 \bmod r$ and one can fix $r > m_1 \geq m_2 \cdots \geq m_N \geq 0$. Each $\mathcal{N} = 1$ adjoint chiral multiplet Φ_i contributes as

$$I_{\Phi}^{(\Delta)} = I_{\Phi}^{(\Delta)}(0, 1)^{N-1} \prod_{i \neq j} I_{\Phi}^{(\Delta)}([m_i - m_j], z_i z_j^{-1}) \quad (3.10)$$

where $[m] \equiv n | 0 \leq n \leq r-1, n = m \bmod r$. In the following we omit to indicate Δ , because in models with non-abelian R-symmetry we always have $\Delta = 2/3$. The $\mathcal{N} = 1$ vector multiplet contributes as

$$I_V = I_V(0, 1)^{N-1} \prod_{i \neq j} I_V([m_i - m_j], z_i z_j^{-1}) \quad (3.11)$$

and the measure factor is

$$\Delta_{\vec{m}}(z) = \frac{1}{|W_{\vec{m}, N}|} \prod_{i \neq j} (1 - z_i/z_j)^{\delta_{[m_i - m_j], 0}} \quad (3.12)$$

where $|W_{\vec{m}, N}|$ represents the size of the Weyl group left unbroken.

The next step consist of computing the contribution from the almost commuting holonomies, coming from the sectors with $\hat{\ell} \neq 0$, i.e. $\nu \neq 1$. This was done in [9] as follows. First one studies the case where N and ℓ are coprime. In this case the almost commuting holonomies can be represented by a pair of matrices denoted as “clock” and “shift” matrices, the first one corresponding to $C_{\ell, N} = \epsilon_N^{\ell} \text{diag}(e^{\frac{2\pi i \ell j}{N}})$ with $j = 0, \dots, N-1$ and the second one $S_N = \epsilon_N \delta_{i+1, i}$ (where ϵ_N is necessary for $\tilde{g}$ and $\tilde{h}$ to have unit determinant). These are the unique matrices, up to simultaneous conjugation and multiplication by a scalar, satisfying (3.8).

By identifying the clock and the shift matrices with the matrices $\tilde{g}$ and $\tilde{h}$, the contribution of from such holonomies can be computed. The computation requires a change of basis for the matter and vector fields in the adjoint representation, because the action of $\tilde{g}$ and $\tilde{h}$ is not diagonal in the weight basis. By denoting the weight basis as $E_{i,j} = e_i \otimes e_j^*$, where e_i represents base element of $\mathbb{C}^N$, the new basis is

$$F_{m,n} = \sum_j e^{2\pi i m j/N} E_{j, n-j}. \quad (3.13)$$

The traceless condition $\sum_i E_{i,i} = 0$ becomes $F_{0,0} = 0$ in this new basis. The adjoint action of $\tilde{g}$ and $\tilde{h}$ is diagonal in this basis, with eigenvalues given by

$$\tilde{g} \cdot F_{m,n} = e^{2\pi i n \ell/N} F_{m,n} \quad \text{and} \quad \tilde{h} \cdot F_{m,n} = e^{2\pi i m/N} F_{m,n}. \quad (3.14)$$

The index for an $\mathcal{N} = 1$ adjoint chiral fields is obtained by considering the allowed values of m and n . The contribution to the index of a vector multiplet and of an adjoint chiral field are

$$\prod_{m,n=0}^{N-1}{}' I_V\left(\frac{n\ell r}{N}, e^{2\pi i m/N}\right) \quad \text{and} \quad \prod_{m,n=0}^{N-1}{}' I_{\Phi}\left(\frac{n\ell r}{N}, e^{2\pi i m/N}\right) \quad (3.15)$$

respectively, where the apex denotes the traceless condition, i.e. we have to avoid to consider m and n vanishing simultaneously. In this case there is no integral to perform over the gauge group, because the matrix $\tilde{h}$ has been completely fixed.

In general one has to consider cases in which N and ℓ are not coprime. Let us define $d = \gcd(N, \ell)$. In this case the holonomies can be written in the form

$$\tilde{g} = \text{diag}\{e^{\frac{2\pi i}{r}(m_i + \tilde{k}/N)}\} \otimes C_{\ell/d, N/d} \quad \text{and} \quad \tilde{h} = \text{diag}\{z_i\} \otimes S_{N/d} \quad (3.16)$$

where $\tilde{k} = k$ for N/d odd and $\tilde{k} = k - rd/2$ for N/d even. The only pairs allowed satisfy the condition $\tilde{k}\ell = 0 \pmod{N}$. In this case one can turn to a diagonal basis by a tensor product of the weight basis and of the $F_{m,n}$ basis. This gives a basis $F_{m,n;i,j}$ with $m, n = 0, \dots, N/d - 1$ and $i, j = 1, \dots, d$. The eigenvalues are

$$\tilde{g} \cdot F_{m,n;i,j} = e^{2\pi i n \ell / N + (m_i - m_j)/r} F_{m,n;i,j} \quad \text{and} \quad \tilde{h} \cdot F_{m,n;i,j} = e^{2\pi i m / N} z_i / z_j F_{m,n;i,j} \quad (3.17)$$

and for each allowed value of m, n, i and j we have a contribution to the index of the form $I_\Phi([n\ell r/N + m_i - m_j], e^{2\pi i m d/N} z_i/z_j)$ where $[m] \equiv n|0 \leq m \leq r-1, n = m \pmod{r}$. Analogous contributions can be found from the vector multiplet and for the measure factor.

Eventually one can collect all the results reviewed above and write the general form of the index. The final form of the index given in (3.9) where the different terms $Z_{\mu,\nu}$ get multiplied by a phase in the partition function. This phase depends on the discrete theta angle and in this case, as explained in [9], we have

$$I = \frac{1}{d} \sum_{\hat{k}, \hat{\ell}=0}^{d-1} e^{2\pi i n \hat{k} \hat{\ell} / d} Z_{\hat{k}, \hat{\ell}} \quad (3.18)$$

where $\tilde{k} = \hat{k}N/d$ and $\ell = \hat{\ell}N/d$. Each sector contributes as

$$\begin{aligned} Z_{\hat{k}, \hat{\ell}} = & \sum_{m_i} \oint \prod_{i=1}^N \frac{dz_i}{2\pi i z_i} \Delta_{\vec{m}}^\ell(z) \\ & \times \prod_{m,n=0}^{N/d-1'} \prod_{i,j=1}^d \left(I_V \left(\left[\frac{n\ell r}{N} + m_{i,j} \right], e^{\frac{2\pi i m d}{N}} z_{i,j} \right) \prod_{I=1}^3 I_{\Phi_I}^{(\frac{2}{3})} \left(\left[\frac{n\ell r}{N} + m_{i,j} \right], e^{\frac{2\pi i m d}{N}} z_{i,j} \right) \right) \end{aligned} \quad (3.19)$$

where we defined $z_{i,j} \equiv z_i/z_j$ and $m_{i,j} \equiv m_i - m_j$ and the sums over m_i are restricted by $\sum_i m_i = -\tilde{k}d/N \pmod{rd/N}$. The notation $\oint$ refers to the constraint $\prod_{i=1}^N z_i = 1$. Observe that this index can be refined by adding two further fugacities that distinguish the different chiral adjoint fields, but here we omitted this contribution because such a refinement it is not necessary for our discussion.

4 The Lens space index for $\mathcal{N} = 2$ elliptic models

In this section we study the Lens space index for infinite families of quiver gauge theories arising from M-theory or type IIA. In M-theory these models describe a stack of N M5 branes wrapped on a genus $g = 1$ Riemann surface with n_G punctures. In type IIA the

model is related to a stack of N D4 branes extended along the directions 01236, where the direction x_6 is compact. There are also n_G parallel NS5 branes placed at different positions x_6^i along the circle. On the field theory side the model corresponds to a necklace $\mathcal{N} = 2$ quiver gauge theory with n_G gauge groups $SU(N)$ and one bifundamental hypermultiplet connecting each pair of consecutive nodes.

It was realized in [7] that the gauge group of the theory is actually determined once additional data on the (W,H) lines are specified. This is because the original gauge group is $U(N)_1 \times U(N)_2 \times \cdots \times U(N)_{n_G}$ and in the IR $n_G - 1$ combinations of the $U(1)$'s become baryonic and the remaining overall $U(1)$ is decoupled. The inequivalent ways in which such an $U(1)$ decouples give raise to different gauge groups identified by their global properties. By borrowing the notations adopted for $\mathcal{N} = 4$ SYM the gauge group here is specified by

$$G = \left(\left(\prod_{j=1}^{n_G} SU(N)_j \right) / \mathbb{Z}_k \right)_i. \quad (4.1)$$

The bifundamentals break the center $(\mathbb{Z}_N)^{n_G}$ to its diagonal subgroup $\mathbb{Z}_N^{\text{diag}}$, implying $\mathbb{Z}_k \subset \mathbb{Z}_N^{\text{diag}}$ in (4.1). One can connect theories with different global properties by the action of the $SL(2, \mathbb{Z})$ group, that represents a subgroup of the full S-duality group, identified with the mapping class group of the n_G -punctured Riemann surface [15]. Analogously to the case of $\mathcal{N} = 4$ SYM, if N is not square free there are orbits in the action of this S-duality subgroup. In the following we show how to compute the Lens space index for this family of theories, matching the index under S-duality for some low rank case and verifying the existence of the S-duality orbits.

We consider, as in the case of $\mathcal{N} = 4$ SYM, only the $\mathcal{N} = 1$ index of [8]. The corresponding $\mathcal{N} = 1$ R -current is generated by the combination of the abelian generators of the R -symmetry and it assigns R -charge $2/3$ to each matter field. Each $su(N)$ factor has center $\Gamma = \mathbb{Z}_N$, but in this case the matter fields break it to its diagonal subgroup. It modifies also the holonomies g and h contributing to the index.

In general we can re-organize the matrices g and h as $n_G N \times n_G N$ block matrices, each with n_G diagonal block of dimension $N \times N$

$$g = \begin{pmatrix} g_1 & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 \\ 0 & 0 & g_{n_G-1} & 0 \\ 0 & 0 & 0 & g_{n_G} \end{pmatrix} \quad h = \begin{pmatrix} h_1 & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 \\ 0 & 0 & h_{n_G-1} & 0 \\ 0 & 0 & 0 & h_{n_G} \end{pmatrix}. \quad (4.2)$$

The constraints (3.8) must be satisfied by the holonomies in each of these n_G sectors separately. At each block we have the relations

$$g_I^r = \mu = e^{\frac{2\pi i k}{N}}, \quad g_I h_I g_I^{-1} h_I^{-1} = \nu = e^{\frac{2\pi i \ell}{N}} \quad (4.3)$$

for $I = 1, \dots, n_G$. A crucial observation is that the $2n_G$ matrix equations (4.3) have to be satisfied for the same value of k and ℓ . This is because only the diagonal subgroup of the center survives in presence of the bifundamental hypermultiplets. This allows us to

organize the computation as above, by separating the contribution of the sectors with the integers k and ℓ .

Let us start by discussing the index for the $Z_{k,\ell}$ sector where $\ell = 0$. For each k the index is

$$Z_{k,0} = \sum_{m_i^I} (I_{V_I}(0,1) I_{\Phi_I}(0,1))^{(N-1)n_G} \oint \prod_{I=1}^{n_G} \prod_{i=1}^N \frac{dz_i^I}{2\pi i z_i^I} \prod_{I \rightarrow J} I_{\Phi_{I,J}}([m_{i,j}^{I,J}], z_{i,j}^{I,J}) \quad (4.4)$$

$$\times \prod_{I=1}^{n_G} \prod_{1 \leq i < j \leq N} \left(\frac{1}{2} (1 - z_{i,j}^{I,I})^{\pm 1} I_{V_I}([m_{i,j}^{I,I}], (z_{i,j}^{I,I})^{\pm 1}) I_{\Phi_I}([m_{i,j}^{I,I}], (z_{i,j}^{I,I})^{\pm 1}) \right)$$

where we defined $m_{i,j}^{I,J} \equiv m_i^I - m_j^J$ and $z_{i,j}^{I,J} \equiv z_i^I / z_j^J$. Furthermore the notation $I \rightarrow J$ refers to the bifundamentals connecting the nodes of the quiver (for example here the only possibilities are $J = I + 1$ and $J = I - 1$). We also used the notation $f(x^{\pm 1}) \equiv f(x)f(x^{-1})$. In addition we are not refining the index, i.e. all the chemical potentials for the global symmetries are set to zero.

We can now choose each g_I -block as done by [9], by constraining the sums over $\vec{m}^I$. Here we must consider each $\vec{m}^I$ such that $\sum_i m_i^I + k = 0 \bmod r$, where i runs from 1 to N and $I = 1, \dots, n_G$. The choice $\ell = 0$ fixes $h_I = \text{diag}\{z_i^I\}$, where z_i^I are defined on the circle and $\prod_i z_i^I = 1$ for each I .

The sector with $\ell \neq 0$ has to be analyzed by diagonalizing the holonomies in a non conventional basis. Let us consider separately the contribution of the adjoints and of the bifundamentals. First we analyze the case in which $d = 1$, where $d = \text{gcd}(N, \ell)$. In this case the matrices g and h have again two diagonal blocks of the type discussed in [9].

The novelty in this case is the presence of the bifundamentals. They are defined by the representations $E_{i,j}^{(I,J)} = e_i^I \otimes e_j^{J*}$ in the weight basis. A diagonal basis in this case is

$$F_{m,n}^{(I,J)} = \sum_j e^{2\pi i m j / N} E_{j, n-j}^{(I,J)}. \quad (4.5)$$

The eigenvalues are

$$g \cdot F_{m,n}^{(I,J)} = e^{2\pi i n \ell / N} F_{m,n}^{(I,J)} \quad \text{and} \quad h \cdot F_{m,n}^{(I,J)} = e^{2\pi i m / N} F_{m,n}^{(I,J)}. \quad (4.6)$$

The contribution to the index is

$$\prod_{m,n=0}^{N-1} I_{\phi_{I,I}}\left(\frac{n\ell r}{N}, e^{2\pi i m / N}\right) \quad (4.7)$$

for the adjoints and

$$\prod_{m,n=0}^{N-1} I_{\phi_{I,J}}\left(\frac{n\ell r}{N}, e^{2\pi i m / N}\right) \quad (4.8)$$

for the bifundamentals, where in the second case we did not exclude the value $n = m = 0$, because there is no traceless condition on these fields. The contribution of the vector multiplets can be computed straightforwardly in this basis, following [9].

The only step that is missing is the calculation of the contributions of the sectors with $d > 1$. Also in this case one can compute them following the strategy above and applying it to the results of [9].

The calculation is identical until formula (5.26) of [9]: in this case the basis is $F_{m,n;i,j}^{(I,J)}$ and equation (5.26) is modified as

$$g \cdot F_{m,n;i,j}^{(I,J)} = e^{2\pi i n \ell / N + (m_i^I - m_j^J)} F_{m,n;i,j}^{(I,J)} \quad \text{and} \quad h \cdot F_{m,n;i,j}^{(I,J)} = e^{2\pi i m d / N} z_i^I / z_j^J F_{m,n;i,j}^{(I,J)} \quad (4.9)$$

and the traces correspond to $\sum_i F_{0,0;i,i}^{(I,I)}$. Again the contribution of the adjoint requires to not consider $m = n = 0$ while for the bifundamentals such term contributes. By summing the various contribution together the generic expression for $Z_{k,\ell}$ of an elliptic model with n_G gauge groups is

$$Z_{k,\ell} = \sum_{m_i^I} \oint \prod_{I=1}^{n_G} \frac{dz_i^I}{2\pi i z_i^I} \Delta_{\vec{m}^I}^\ell(z^I) \prod_{m,n=0}^{N/d-1} \prod_{i,j=1}^d \prod_{I \rightarrow J} I_{\Phi_{I,J}} \left(\left[\frac{n\ell r}{N} + m_{i,j}^{I,J} \right], e^{\frac{2\pi i m d}{N}} z_{i,j}^{I,J} \right) \quad (4.10)$$

$$\times \prod_{m,n=0}^{N/d-1'} \prod_{i,j=1}^d \left(I_{V_I} \left(\left[\frac{n\ell r}{N} + m_{i,j}^{I,I} \right], e^{\frac{2\pi i m d}{N}} z_{i,j}^{I,I} \right) I_{\Phi_I} \left(\left[\frac{n\ell r}{N} + m_{i,j}^{I,I} \right], e^{\frac{2\pi i m d}{N}} z_{i,j}^{I,I} \right) \right)$$

where the sum over m_i^I is constrained by $\sum_i m_i^I = -\tilde{k}d/N \pmod{rd/N}$. The index is obtained by applying (3.18). Again we did not consider the presence of flavor fugacities, but they can be turned on straightforwardly.

In the following we focus on the example with $n_G = 2$. Despite the simplicity of this model its Lens space index contains all the necessary ingredients of the one expected for generic n_G . In this case we have two gauge groups $SU(N)_1 \times SU(N)_2$ and two bifundamental hypermultiplets. In $\mathcal{N} = 1$ notations there are two adjoint matter fields Φ_1 and Φ_2 , two bifundamentals a_1 and a_2 and two antifundamentals b_1 and b_2 . The superpotential is

$$W = a_1 \Phi_1 b_1 - a_2 \Phi_1 b_2 - a_1 \Phi_2 b_1 + a_2 \Phi_2 b_2. \quad (4.11)$$

By turning off the global symmetries the model has an $SU(2) \times U(1)$ R -symmetry. Here we only consider the $\mathcal{N} = 1$ R -symmetry that assigns charges $R = 2/3$ to the adjoints and the bifundamental fields. It corresponds to $R_{\mathcal{N}=1} = \frac{4}{3}J_3 + \frac{1}{3}R_{\mathcal{N}=2}$, where J_3 is the diagonal generator of $SU(2)$. We provide the explicit calculation of the Lens space index $I_{L(r,1) \times S^1}$ for this model. We focus on the case $r = N$ and explicitly compute the cases $N = 2, 3, 4$. In the first two case, $N = 2$ and $N = 3$ we found that the index coincides, signaling that the models are related by S-duality. In the last case we find that the Lens space index differs in the two orbits of the S-duality groups.

The $SU(2) \times SU(2)$ case

$\ell \backslash \tilde{k}$	0	1	$a = 2 + 26x^{4/3} - 12x^2 + 98x^{8/3} - 100x^{10/3} + 246x^4 + \dots$
0	a	b	$b = 1 + 2x^{2/3} + 17x^{4/3} + 4x^2 + 66x^{8/3} - 70x^{10/3} + 234x^4 + \dots$
1	c	0	$c = 1 - 2x^{2/3} + 9x^{4/3} - 16x^2 + 32x^{8/3} - 30x^{10/3} + 12x^4 + \dots$

(4.12)

$$\begin{aligned}
 I_{\text{SU}(2) \times \text{SU}(2)} &= a = 2 + 26x^{4/3} - 12x^2 + 98x^{8/3} - 100x^{10/3} + 246x^4 \\
 I_{((\text{SU}(2) \times \text{SU}(2))/\mathbb{Z}_2)_{0,1}} &= \frac{a+b+c}{2} = 2 + 26x^{4/3} - 12x^2 + 98x^{8/3} - 100x^{10/3} + 246x^4 + \dots
 \end{aligned} \tag{4.13}$$

As expected the index for the $\text{SU}(2) \times \text{SU}(2)$ case coincides with the one of the $(\text{SU}(2) \times \text{SU}(2))/\mathbb{Z}_2)_i$ cases, for both $i = 0$ and $i = 1$.

The $\text{SU}(3) \times \text{SU}(3)$ case

$\ell \backslash \tilde{k}$	0	1	2
0	a	b	b
1	c	0	0
2	c	0	0

$$\begin{aligned}
 a &= 4 + 4x^{2/3} + 42x^{4/3} + 86x^2 + 183x^{8/3} + \dots \\
 b &= 3 + x^{2/3} + 39x^{4/3} + 84x^2 + 192x^{8/3} + \dots \\
 c &= 1 - 2x^{2/3} + 3x^{4/3} + 2x^2 - 9x^{8/3} + \dots
 \end{aligned} \tag{4.14}$$

$$\begin{aligned}
 I_{\text{SU}(3) \times \text{SU}(3)} &= a = 4 + 4x^{2/3} + 42x^{4/3} + 86x^2 + 183x^{8/3} + \dots \\
 I_{((\text{SU}(3) \times \text{SU}(3))/\mathbb{Z}_3)_{0,1,2}} &= \frac{a+2b+2c}{3} = 4 + 4x^{2/3} + 42x^{4/3} + 86x^2 + 183x^{8/3} + \dots
 \end{aligned} \tag{4.15}$$

Also in this case we observe that the index for the $\text{SU}(3) \times \text{SU}(3)$ case coincides with the one of the $(\text{SU}(3) \times \text{SU}(3))/\mathbb{Z}_3)_i$ cases, for $i = 0, 1, 2$.

The $\text{SU}(4) \times \text{SU}(4)$ case

$\ell \backslash \tilde{k}$	0	1	2	3
0	a	b	c	b
1	d	0	0	0
2	e	0	f	0
3	d	0	0	0

$$\begin{aligned}
 a &= 10 + 20x^{2/3} + 162x^{4/3} + 320x^2 + \dots \\
 b &= 8 + 24x^{2/3} + 152x^{4/3} + 336x^2 + \dots \\
 c &= 9 + 22x^{2/3} + 159x^{4/3} + 324x^2 + \dots \\
 d &= 1 - 2x^{2/3} + 3x^{4/3} - 4x^2 + \dots \\
 e &= 2 - 4x^{2/3} + 10x^{4/3} - 16x^2 + \dots \\
 f &= 1 - 2x^{2/3} + 7x^{4/3} - 12x^2 + \dots
 \end{aligned} \tag{4.16}$$

$$\begin{aligned}
 I_{\text{SU}(4) \times \text{SU}(4)} &= a = 10 + 20x^{2/3} + 162x^{4/3} + 320x^2 + \dots \\
 I_{((\text{SU}(4) \times \text{SU}(4))/\mathbb{Z}_2)_0} &= \frac{a+c+e+f}{2} = 11 + 18x^{2/3} + 169x^{4/3} + 308x^2 + \dots \\
 I_{((\text{SU}(4) \times \text{SU}(4))/\mathbb{Z}_2)_1} &= \frac{a+c+e-f}{2} = 10 + 20x^{2/3} + 162x^{4/3} + 320x^2 + \dots \\
 I_{((\text{SU}(4) \times \text{SU}(4))/\mathbb{Z}_4)_i} &= \frac{a+2b+c+2d+e+f}{4} = 10 + 20x^{2/3} + 162x^{4/3} + 320x^2 + \dots
 \end{aligned} \tag{4.17}$$

where in the last line $i = 0, \dots, 3$. Here the situation is more interesting because not all the indices coincide. This is because the lattice of charges associated to the gauge group $\mathbf{G} = ((\text{SU}(4) \times \text{SU}(4))/\mathbb{Z}_2)_0$ is self dual under $\text{SL}(2, \mathbb{Z})$. This model represents an S-duality orbit. The other orbit includes all the other gauge groups, because their lattices are all related by $\text{SL}(2, \mathbb{Z})$ transformations. Here we have confirmed this expectation showing that the Lens space index assumes different values on the two orbits. This provides a non trivial check of S-duality in this model.

5 Conclusions

In this paper we have studied the Lens space index for a family of $\mathcal{N} = 2$ elliptic models with unitary gauge groups. We have computed explicitly such an index for the model with two gauge groups and for low ranks, verifying the action of the S-duality group.

One could try to extend the analysis to other quiver gauge theories, either with real gauge groups or with a reduced amount of supersymmetry or with tensorial matter. In each case the global properties are specified by the additional data, and the Lens space index should reproduce this structure. It may be useful to study the role of S-duality in these cases. Another extension of this work regards the study of other $\mathcal{N} = 2$ models as for example class S-theories [16]. It should be interesting apply the results of [17–19], like the relation between the twisted character of the chiral algebra and the Lens space index, in order to reproduce the lattices and the S-duality structure obtained in [20–28]. Also the results of [29], investigating the geometric origin of the global properties from the 6d $\mathcal{N} = (1, 0)$ perspective, may be useful for the study of $\mathcal{N} = 2$ models.

Acknowledgments

We are grateful to Domenico Orlando and Susanne Reffert for early collaboration on this project. This work has been supported in part by Italian Ministero dell’Istruzione, Università e Ricerca (MIUR), in part by Istituto Nazionale di Fisica Nucleare (INFN) through the “Gauge Theories, Strings, Supergravity” (GSS) research project and in part by MIUR-PRIN contract 2017CC72MK-003.

A $\mathcal{N} = 1$ theories

In this section we discuss the global properties of a family of 4d $\mathcal{N} = 1$ SCFTs. These theories are quiver gauge theories with bifundamental matter fields connecting the various nodes. Despite the fact that in presence of matter fields the center of the various gauge factors is (partially) broken, we show that the theories discussed here have the same structure for the global properties of $\mathcal{N} = 4$ SYM and of the $\mathcal{N} = 2$ elliptic models.

Let us first introduce the setup. We consider the gauge theories living on N D3 branes on toric CY-threefolds, corresponding to $\mathcal{N} = 1$ SCFTs that can be represented by a quiver. This quiver is a connected graph, in which the nodes represent the gauge groups and the oriented arrows are bifundamental fields connecting the I -th to the J -th node. A crucial property of such a quiver is that it can be represented as a planar graph on a torus. In this case each closed path represent a superpotential term. Having a connected graph on a torus is enough to argue the existence of pair of paths connecting each pair on nodes with opposite orientation. This has a consequence on the line operators. Let us define the charge of a generic line as $(\vec{e}; \vec{m}) = (e_1, \dots, e_{n_G}; m_1, \dots, m_{n_G})$. The matter fields impose that $m_I = m_J$ for each pair of integers I, J from 1 to n_G . On the other hand an electric line $(\vec{e}; \vec{0})$ can always be put in the form $(\sum_{I=1}^{n_G} e_I, 0, \dots, 0; \vec{0})$, where $\sum_{I=1}^{n_G} e_I$ must be considered mod N . Again this structure for the charges of the lines allows different

lattices, generated by the vectors $(k, 0)$ and (i, k') with $kk' = N$ and $0 \leq i < k$. This implies that also in the case of $\mathcal{N} = 1$ toric quiver gauge theories the gauge group is

$$G = \left(\prod_{i=1}^{n_G} \text{SU}(N)/\mathbb{Z}_k \right)_i. \quad (\text{A.1})$$

The $\text{SL}(2, \mathbb{Z})$ symmetry that allows to pass from one to another lattice acts on the diagonal holomorphic gauge coupling τ_{diag} . It can be identified with the dilaton ϕ of the $\text{AdS}_5 \times X_5$ string theory dual background, combined with the theta angle. This can be reformulated in geometric terms by considering the gauge theory from the perspective of the brane tiling. We have N D5₀₁₂₃₅₇, one NS5₀₁₂₃₄₅ and one NS5₀₁₂₃₆₇. The diagonal holomorphic coupling is identified by the combination $e^{C_{57}} + ie^{-\phi}$. This corresponds to a marginal deformation identified in [30], common to all the theories in this class. In particular it corresponds to the holomorphic gauge coupling if $X_5 = S^5$, i.e. if we consider $\mathcal{N} = 4$ SYM. It is then natural to expect that the $\text{SL}(2, \mathbb{Z})$ action allowing to connect the various lattices of the $\mathcal{N} = 4$ theory acts in a similar manner in the generalization to $\mathcal{N} = 1$. Observe that another reason for such similarity consists of the presence of an overall $\text{U}(1)$ in the quiver under which every field is uncharged. The different ways to decouple this $\text{U}(1)$ symmetry correspond to the different lattices on the field theory side [31]. An interesting consequence of the construction is the fact that also in the case of $\mathcal{N} = 1$ theories one expect orbits in the action of $\text{SL}(2, \mathbb{Z})$ on τ_{diag} .

Here, motivated by this expectation, we study the Lens space index for these $\mathcal{N} = 1$ quiver gauge theories. The derivation of the index follows the one discussed in the body of the paper and we will not repeat it here. The only differences are the fact that in this case the quiver is not necessarily vector like and that the R -charges of the bifundamentals are not necessarily $R_{\Phi_{IJ}} = 2/3$. For a generic $\mathcal{N} = 1$ quiver each sector specified by $\tilde{k}$ and ℓ contributes to the index as

$$\begin{aligned} Z_{\tilde{k}, \ell} = & \sum_{m_i^I} \oint \prod_{I=1}^{n_G} \frac{dz_i^I}{2\pi i z_i^I} \Delta_{\tilde{m}^I}^\ell(z^I) \prod_{m,n=0}^{N/d-1'} \prod_{i,j=1}^d I_{V_I} \left(\left[\frac{n\ell r}{N} + m_{i,j}^{I,I} \right], e^{\frac{2\pi i m d}{N}} z_{i,j}^{I,I} \right) \\ & \times \prod_{I \rightarrow J} I_{\Phi_{I,J}}^{(\Delta_{I,J})}(0, 1)^{-\delta_{I,J}} \prod_{m,n=0}^{N/d-1} \prod_{i,j=1}^d I_{\Phi_{I,J}}^{(\Delta_{I,J})} \left(\left[\frac{n\ell r}{N} + m_{i,j}^{I,J} \right], e^{\frac{2\pi i m d}{N}} \mu_{\{I,J\}} z_{i,j}^{I,J} \right) \quad (\text{A.2}) \end{aligned}$$

with the constraints $\sum_i m_i^I = -\tilde{k} \frac{d}{N} \pmod{r \frac{d}{N}}$. The R -charge of the field $\Phi_{I,J}$ has been denoted as $\Delta_{I,J}$ and we also turned on the chemical potentials $\mu_{\{I,J\}}$ that take into account the possible non R -global charges for each field.

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