

Constructing $SU(N)$ fractional instantons

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ABSTRACT: We study self-dual $SU(N)$ gauge field configurations on the 4 torus with twisted boundary conditions, known as fractional instantons. Focusing on the minimum non-zero action case, we generalize the constant field strength solutions discovered by 't Hooft and valid for certain geometries. For the general case, we construct the vector potential and field strength in a power series expansion in a deformation parameter of the metric. The next to leading term is explicitly computed. The methodology is an extension of that used by the author for $SU(2)$ fractional instantons and for vortices in two-dimensional Abelian Higgs models. Obviously, these solutions can also be seen as self-dual configurations in \mathbb{R}^4 having a crystal structure, where each node of the crystal carries a topological charge of $1/N$.

KEYWORDS: Solitons Monopoles and Instantons, $1/N$ Expansion, Nonperturbative Effects

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1 Introduction

The study of solutions of the classical equations of motions from the perspective of quantum field theory studies has a long history (see here [1, 2] some books written on the subject). For the case of gauge theories a pioneering role has been played by the BPST instanton solution [3] of the Yang-Mills euclidean equations of motion. Its interpretation and relevance in the quantum field theory setting as dominating tunneling trajectories was clarified by Polyakov [4]. As in all solutions to partial differential equations, boundary conditions matter. In the case of instantons the condition of finite action is equivalent to the compactification of \mathbb{R}^4 into S_4 . This automatically brings in the study of the topology of the bundles on compact manifolds of great interest to mathematicians [5]. These bundles can be classified according to Chern classes. In particular, the stability of the instanton solution is a consequence of it possessing a non-trivial second Chern-number, also known as instanton number. A similar phenomenon happens in two-dimensional abelian gauge theories with the vortex solution [6, 7] and its connection with the first Chern number. The history of these topics is beautiful and very rich but unfortunately we cannot review it here. This paragraph serves to put our work in context but we will now focus on aspects more directly related to our work.

The study of gauge fields on the four-dimensional torus within the Physics literature was initiated by 't Hooft. The torus has many advantages as a compact manifold since it is compatible with a flat metric and allows to respect a group of translations of practical interest. Furthermore, numerical studies of gauge theories are almost always performed on the torus. What 't Hooft put forward is a new class of topological sectors of $SU(N)$ gauge bundles on the torus, which he called twisted boundary conditions [8–10]. These sectors

are characterized by an antisymmetric tensor of integers modulo N . These integers can be interpreted as abelian fluxes through the faces of the torus. They are a remnant of the first Chern numbers of a $U(N)$ gauge theory when projecting onto $SU(N)$, or as mathematicians see it, as obstructions to lifting an $SU(N)/Z_N$ bundle to a $SU(N)$ one. These questions were clarified by van Baal in his thesis and early papers [11, 12]. One of the interesting aspects discovered by 't Hooft was the connection of the twist fluxes with the instanton number. It turns out that for certain twist tensors the instanton number is no longer an integer. All this is perfectly understood in mathematical terms as a result of the work of van Baal and others [13, 14]. An explicit construction of the bundles implementing all possible values of the instanton number and twist tensor can be seen in my lectures given several years ago [15].

The search of new classes of Yang-Mills classical solutions on the torus having fractional topological charge was initiated by 't Hooft [16]. We will refer to all self-dual solutions as *fractional instantons*, although the reader is warned about the use of different names in the literature for the same objects. 't Hooft analytic solutions are constant field strength solutions which become self-dual (and hence stable) only when the ratio of certain areas of the faces of the torus become certain rational numbers. The solutions have a somewhat abelian character since the electric and magnetic fields are aligned along a single direction in Lie Algebra space (and hence commute among themselves). However, using numerical methods one can obtain fractional instantons for a wide range of torus sizes including the Hamiltonian $T_3 \times \mathbb{R}$ geometry [17, 18]. These instantons are lumpy structures having a center in space and time and non-commuting non-constant electric and magnetic fields. The numerical technique reflects the methods used in mathematical approaches to the subject: gradient flows. Starting from non self-dual configurations on the twisted bundle, the flow converges in some cases to self-dual non-singular configurations.

In commenting about the relevance of these fractional instantons to the dynamics of Yang-Mills fields we should clarify a few points that unfortunately some fraction of the scientists seem to ignore. Boundary conditions are necessary only to stabilize the solutions. The classical equations of motion are *local* equations satisfied at each space-time point and this makes these structures relevant even if we modify the boundary conditions. Let us put two very simple examples to explain this point. Consider first the one-dimensional scalar field theory with a double well potential. If we put antiperiodic boundary conditions in time, there is a stable classical solution known as the kink. If we change the boundary conditions to periodic, the kink is no longer a stable solution, but configurations with kinks and antikinks provide unstable classical configurations of great dynamical relevance. Another quite different example is that of $Q = 1$ instanton on the torus without twist. This configuration is unstable [19]. The configuration will shrink in size with gradient flow tending towards a singular solution. Does this mean that compactifying space-time on a torus will make instantons absent in the Yang-Mills vacuum? Obviously not. Quantum fluctuations will produce instantons since boundary conditions only affect the total action through a term proportional to the boundary, which is subdominant with respect to the action or entropy of the bulk. Coming back to fractional instantons, twist only plays an stabilizing role. Since twist is a kind of flux modulo N , if we glue together several copies

of the torus in various directions we produce a larger torus with no twist. The resulting configuration is still a self-dual solution (hence minimum action) but the topological charge (which is not modulo N) is larger and hence the moduli space grows accordingly. This means that there are deformations that cost no action that destroy the periodicity under the small period. It is obvious that for small enough deformations the configurations will still look like a collection of lumps carrying fractional topological charge. Curiously, the moduli space of all self-dual solutions has precisely the same dimensionality as a four-dimensional gas of fractional instantons of minimum action ($8\pi^2/N$). This is the same as in abelian gauge theories in two dimensions, in which the moduli space is precisely given by the configuration space of a two-dimensional gas of vortices [20].

Do these fractional instantons play a dynamical role in the Yang-Mills vacuum? Instantons are important in solving the $U(1)$ problem [21] and producing a non-zero topological charge density, but cannot explain other phenomena such as confinement. Very early on some researchers proposed that instantons can dissociate into some constituents in a dense scenario which could be responsible for Confinement [22]. Unfortunately at the time the only possible candidates were some singular configurations known as merons [23]. Our proposed confinement scenario [24, 25] claims that the idea is basically correct but replacing the old singular meron solutions by the regular, smooth, self-dual fractional instantons. The idea arose quite naturally when pursuing a program initiated by Luscher [26] of trying to use the spatial volume as an interpolating parameter between the perturbative femto-world and the large volume confinement regime. Fractional instantons appear naturally as non-perturbative weak coupling solutions whose effect is to approach the system towards the confinement regime [27]. Further studies done by our group in Madrid hinted on the presence of fractional instantons on the large volume lattice configurations [28] and showed that an artificially created gas of fractional instantons leads to Wilson loops satisfying the area law [29]. At about the same time Zhitnitsky [30, 31] advocated the existence and relevance of fractional topological charge objects to explain the N dependence of the free energy in the presence of a θ parameter.

Coming back to fractional instantons at the classical level, one important difficulty is that there are no analytic formulas for the vector potentials or the field strengths beyond the special solutions found by 't Hooft. Of course, the same happens for the vortex and multivortex solutions in abelian two dimensional field theory. Thus, it became a wonderful surprise when analytic formulas were obtained for $Q = 1$ non-trivial holonomy calorons in which the dissociation mechanism is explicit [32–35]. The moduli space of these solutions nicely interpolates smoothly between a single ordinary instanton lump and a set of N local lumpy structures carrying fractional topological charge. These caloron constituents are intimately connected to fractional instantons. Indeed, for large separations these $S_1 \times \mathbb{R}^3$ fractional topological charge caloron components can be seen as a one-dimensional periodic array of minimum action fractional instantons. These fractional instanton constituents can also be arranged into two dimensional doubly periodic sheets [36–38] that make up solutions in $T_2 \times \mathbb{R}^2$, as in the three-dimensional $T_3 \times \mathbb{R}$ solutions mentioned earlier. All this makes a unified picture of these arrays of fractional instantons [39, 40], connected to each other by Nahm transformations [19, 41–44].

In search for analytical expressions of other types of general fractional instantons we developed a strategy based on deforming away from 't Hooft constant field strength solutions [45]. A perturbative expansion on the deformation parameter arises naturally and we were able to show that the equations can be solved order by order in a sequential fashion and to compute the leading orders in the expansion. Our analytical formulas reproduced the numerical solutions obtained for small deformations, confirming the validity of our results. Going beyond the first few orders is necessary if one wants to obtain a good approximation to general torii. Investigating this matter we realized that the deformation idea is very general. Indeed, something very similar happens for two-dimensional abelian gauge theories on the torus. A constant field strength solution exists for a particular value of the area. Deforming the area one obtains a hierarchy of equations that can be solved order by order to provide analytical formulas for the critical vortices on the torus [46]. Indeed, in this simpler case we were able to compute up to order 51 in the deformation parameter, which allows to reproduce nicely the critical vortex in \mathbb{R}^2 (infinite area). Even more so, the procedure allows the computation of multivortex solutions at all points in the moduli space. This leads to an analytic control of features such as vortex-vortex scattering [47] or quantum corrections to the multivortex energies [48]. The existence of a critical area is completely general in these two-dimensional abelian-Higgs systems as discovered in the thesis of mathematician Steven Bradlow [49]. Thus we named the expansion in the deformation parameter as Bradlow parameter expansion.

Many of the previous works, including our deformation perturbative approach [45] focused on $SU(2)$. This was mostly driven by simplicity and/or computational resources. However, some numerical work was done for $SU(N)$ fractional instantons [38, 50, 51] showing that the ideas and results extend to all values of N . Indeed, large N was always in the origin of our interest in twisted boundary conditions. Fractional instantons have free energies that survive the large N limit. Twist also plays a major role [52, 53] in preserving enough center symmetry at weak coupling, a necessary ingredient for the validity of volume reduction at large N [54]. Thus, we felt the necessity of extending our previous analytic expansion to $SU(N)$. This is indeed the main purpose of this paper.

The motivation for the extension has emerged from a recent interest in fractional instanton solutions emerging from a different perspective. This comes in the spirit of the ideas of resummation of the perturbative expansion and the proposed idea of Resurgence (See ref. [55] for a recent review). The claim is that even in the case of non-Borel summable expansions one can use the perturbative expansion to reconstruct all non-perturbative phenomena. A huge literature has emerged which we cannot review here. The main connection with our program is that in certain simpler systems fractional soliton solutions of various kinds have been found to be relevant in interpreting the singularity structure in the Borel plane [56]. The natural candidate to extend this phenomenon to four-dimensional gauge theories are precisely the fractional instantons. Hence, we found the courage to extend the construction to $SU(N)$, hoping our formulas will be of some help to other researchers. In so doing we have also generalized some of the steps that were previously carried only for special cases.

The lay-out of the paper is as follows. In the next section we collect several results about fractional instantons. Some of the concepts mentioned in this introduction will be put

in mathematical terms. The following section is devoted to constant field strength solution. We actually generalize the construction done by 't Hooft both in the group structure as in the geometrical setting. We will focus only on minimal action instantons given their unique character. The following section explains the philosophy of the deformation technique and show that it leads to hierarchy of equations. Section 5 is devoted to the computation of fractional configuration to first order in the deformation parameter. In section 6 we show the basic ingredients to extend the calculation to higher orders. Finally, in the last section we present our conclusions and explain how our results can be extended and/or used to compute other interesting quantities such as fermion zero modes.

2 Fractional instantons

In this section we recall some general facts about gauge fields on a four-dimensional torus. The torus \mathcal{T} is given as the quotient space \mathbb{R}^4/Λ where Λ is a discrete group of translations generated by 4 linearly independent vectors e_α for $\alpha \in \{0, 1, 2, 3\}$. In our favourite presentation we introduce $SU(N)$ gauge fields as connections in an $SU(N)$ vector bundle. The bundle itself is defined by its transition functions, which is given by a homomorphism from Λ to the space of gauge transformations. In this way we guarantee that all gauge-invariant quantities are well-defined on the torus. In an specific trivialization all we need to do is to associate an $SU(N)$ matrix to each generator

$$e_\alpha \longrightarrow \Omega_\alpha(x) \tag{2.1}$$

so that each section of the bundle $\Psi(x)$ transforms as

$$\Psi(x + e_\alpha) = \Omega_\alpha(x)\Psi(x) \tag{2.2}$$

Consistency then demands that

$$\Omega_\alpha(x + e_\beta)\Omega_\beta(x) = \Omega_\beta(x + e_\alpha)\Omega_\alpha(x) \tag{2.3}$$

The space of bundles can be classified into topologically inequivalent sectors by means of the Chern classes. The first Chern class integrated over non-trivial 2-cycles gives the first Chern numbers. Thus, to each face of the torus we can associate a number which can be interpreted as the flux through that face. However, for $SU(N)$, these numbers are all zero. We then have the second Chern class, which when integrated over the full space gives the second Chern number, instanton number or topological charge Q . This number is known to be an integer. The best way to compute this number is by introducing a connection $A = A_\mu(x)dx^\mu$ on the bundle with its corresponding curvature 2-form F . The instanton number is given by

$$Q = \frac{1}{8\pi^2} \int_{\mathcal{T}} \text{Tr}(F \wedge F) \tag{2.4}$$

Notice, however, that the number is a property of the bundle encoded in its transition matrices.

't Hooft realized that in pure gluodynamics the consistency conditions can be relaxed to the form [8, 9]

$$\Omega_\alpha(x + e_\beta)\Omega_\beta(x) = z_{\alpha\beta}\Omega_\beta(x + e_\alpha)\Omega_\alpha(x) \tag{2.5}$$

where $z_{\alpha\beta} = \exp\{2\pi i n_{\alpha\beta}/N\}$ and $n_{\alpha\beta}$ is an antisymmetric tensor of integers modulo N . The connection is still well-defined because it is insensitive to a transformation by an element of the center Z_N . These modified consistency conditions were called twisted boundary conditions by 't Hooft. The 6 independent integers of the twist tensor $n_{\alpha\beta}$, can be written as 2 integer 3-vectors \vec{k} and \vec{m} ($k_i = n_{0i}$ and $m_i = \epsilon_{ijk}n_{jk}/2$) defined modulo N . Their integer character shows that they characterize topologically inequivalent bundles. 't Hooft also realized that the instanton number is related to these vectors as follows

$$Q = -\frac{\vec{k}\vec{m}}{N} + \mathbb{Z} \tag{2.6}$$

We then see that for non-orthogonal twists ($\vec{k}\vec{m} \neq 0 \pmod N$) the instanton number becomes fractional. This apparent puzzle was clarified by Pierre van Baal in his thesis [11, 12]. In reality we are constructing an $SU(N)/Z_N$ bundle, and one should write the transition matrices in a center-blind representation as the adjoint. Twist becomes then an obstruction to lifting the bundle to one in $SU(N)$. Another way to look at twist is by starting with a $U(N)$ bundle and projecting it down to $SU(N)$ [57]. It is then clear how the first Chern number of the original bundle relates to the twist of the $SU(N)$. This clarifies the interpretation of twist as flux modulo N . For a more extensive description of the preceding, the reader can also consult the author's lectures [15], which includes an explicit construction of twist matrices for all values of the twist and instanton number for $N > 2$.

To generate the dynamics of gauge fields one introduces the Yang-Mills action functional

$$S = \frac{1}{2g^2} \int_{\mathcal{T}} dx \text{Tr}(F_{\mu\nu}F^{\mu\nu}) \tag{2.7}$$

Implicitly this demands the introduction of a metric on the torus (dx stands for the corresponding volume form), although one frequently takes it to be the euclidean metric. Here, we will stick to this case. However, as we will see, the expression of the constant metric tensor in a given coordinate system will play a fundamental role in what follows.

The action is bounded from below by a multiple of the absolute value of the topological charge [58]

$$S \geq \frac{8\pi^2}{g^2} |Q| \tag{2.8}$$

This Bogomolny bound is saturated by self-dual or anti-self-dual configurations. These configurations are called instantons (for the self-dual $Q > 0$ case), whose first representative is the celebrated BPST instanton [3] having $Q = 1$ on S_4 or \mathbb{R}^4 . For the torus case the possible solutions for non-integer Q are called fractional instantons. Their existence has been established mathematically in some cases [13]. One can start by a configuration in each sector (which is known to exist) and then apply a gradient flow to minimise the action. The limit does not necessarily exist (as happens for $Q = 1$ and $\vec{k} = \vec{m} = 0$) because the limiting configuration can be singular. The method has been used successfully to obtain

numerically precise approximations to the fractional instanton configurations for certain geometries [17, 18, 51].

The fractional instanton configurations are not unique, but depend on $4|Q|N$ real parameters as dictated by the index theorem. Particular interest is attributed to the lowest action fractional instanton having topological charge $|Q| = 1/N$. Apart from acting as building block for the higher topological charge solutions, it is essentially unique, since its 4 moduli parameters are associated to space-time translations. Notice that in this case the action remains finite in the large N limit and given by $8\pi^2/(g^2N)$.

Before describing the analytic construction of these solutions we should mention that the torus and the twisted boundary conditions are only auxiliary tools in their identification. The configurations can be seen as configurations in \mathbb{R}^4 satisfying certain periodicity conditions. They are still solutions of the classical equations of motion (with euclidean signature) although with infinite action (finite action over each cell). It is also important to realize that since twist fluxes are additive modulo N , fractional instantons also give rise to classical solutions on the torus with vanishing twist $\vec{k} = \vec{m} = 0$ and integer topological charge. These configurations look very different to a collection of $Q = 1$ instantons.

In the next section we will present all constant field strength fractional instanton solutions, which are valid for specific torus sizes, thus generalizing 't Hooft construction [16].

3 Constant field strength fractional instantons

't Hooft succeeded in obtaining analytical solutions for some fractional instantons [16]. A good deal of importance comes from choosing the transition matrices. He used a hybrid between the abelian and the twist-eating matrices [59, 60]:

$$\Omega_\alpha(x) = e^{i\pi\hat{\omega}(e_\alpha,x)T} \begin{pmatrix} \Gamma_\alpha^{(1)} & 0 \\ 0 & \Gamma_\alpha^{(2)} \end{pmatrix} \tag{3.1}$$

where $\Gamma_\alpha^{(a)}$ are constant $SU(N_a)$ matrices satisfying

$$\Gamma_\alpha^{(a)}\Gamma_\beta^{(a)} = e^{2\pi i n_{\alpha\beta}^{(a)}/N_a}\Gamma_\beta^{(a)}\Gamma_\alpha^{(a)} \tag{3.2}$$

and T is a hermitian traceless matrix commuting with all the $\Omega_\alpha(x)$. Explicitly we have

$$T = \begin{pmatrix} \frac{\mathbb{I}_1}{N_1} & 0 \\ 0 & -\frac{\mathbb{I}_2}{N_2} \end{pmatrix} \tag{3.3}$$

with \mathbb{I}_a the $N_a \times N_a$ identity matrix. We have split the space into two blocks such that $N_1 + N_2 = N$. Finally, $\hat{\omega}(x, y)$ is an antisymmetric bilinear form. Imposing the twisted boundary conditions one concludes that

$$n_{\mu\nu} = n_{\mu\nu}^{(1)} + n_{\mu\nu}^{(2)} \quad \Leftrightarrow \quad \vec{k} = \vec{k}^{(1)} + \vec{k}^{(2)} \quad ; \quad \vec{m} = \vec{m}^{(1)} + \vec{m}^{(2)} \tag{3.4}$$

and

$$\hat{\omega}(e_\alpha, e_\beta) = \frac{\Delta_{\alpha\beta}}{N} \equiv \frac{n_{\alpha\beta}^{(2)}N_1 - n_{\alpha\beta}^{(1)}N_2}{N} \tag{3.5}$$

We have used the freedom to redefine $n_{\alpha\beta}^{(a)}$ modulo N_a , to write these equations as exact and not modulo integers. We recall (see [15] and references therein) that the existence of solutions to eqs. (3.2) implies

$$\vec{k}^{(a)}\vec{m}^{(a)} = 0 \text{ mod } N_a \tag{3.6}$$

Associated to the aforementioned twisted transition matrices there is a natural constant field strength connection. The vector potential one-form is given by $\hat{A} = \pi\hat{\omega}(x, dx)T$ with field strength $\hat{F} = 2\pi\hat{\omega}(dx, dx)T$. We can use this connection to compute the topological charge

$$Q = \frac{\epsilon_{\mu\nu\rho\sigma}\Delta_{\mu\nu}\Delta_{\rho\sigma}}{8NN_1N_2} = \frac{\text{Pf}(\Delta)}{NN_1N_2} \tag{3.7}$$

where $\text{Pf}(\Delta)$ is the Pfaffian of the antisymmetric matrix $\Delta_{\alpha\beta}$. Although computed with the use of the connection \hat{A} the topological charge only depends on the transition matrices Ω_α .

Notice that the constant field strength connection \hat{A} , being proportional to the single Lie algebra generator T , is essentially abelian, but the bundle is non-abelian. These constant abelian gauge fields are solutions of the classical equations of motion (for constant metric tensor), but are in general unstable. Self-dual solutions are obviously stable.

We can now use symmetries of the system to write the solution in a simpler form. First of all we use the freedom to redefine the basis of the lattice Λ . This can be done by means of $SL(4, \mathbb{Z})$ transformations. By well-known properties [15] we can find an appropriate basis such that only $\Delta_{03} = -\Delta_{30} \equiv \Delta_A$ and $\Delta_{12} = -\Delta_{21} \equiv \Delta_B$ are non-zero. The topological charge is now simplified to

$$Q = \frac{\Delta_A\Delta_B}{NN_1N_2} \tag{3.8}$$

It is easy to show using eqs. (3.6) that $\Delta_A\Delta_B$ is proportional to N_1N_2 and the topological charge has the form put forward by 't Hooft.

If we want to find solutions having minimum non-zero action we should take $\Delta_A\Delta_B = N_1N_2$. Thus, a general solution is provided by introducing 4 positive integers M_{A1}, M_{A2}, M_{B1} and M_{B2} and writing

$$\Delta_A = M_{A1}M_{A2}; \quad \Delta_B = M_{B1}M_{B2}; \quad N_1 = M_{A1}M_{B1}; \quad N_2 = M_{A2}M_{B2} \tag{3.9}$$

't Hooft made the special choice $M_{A2} = M_{B1} = 1$.

Now we will enforce self-duality. The explicit formulas do depend on the metric. Essentially, the relevant piece of information needed is the value of the scalar products of the basis vectors of our lattice Λ :

$$\hat{g}_{\alpha\beta} = (e_\alpha, e_\beta) \tag{3.10}$$

where we use the notation (\cdot, \cdot) for the scalar product. This information translates into the lengths of e_α , the areas of the $\alpha - \beta$ faces, the total volume of the torus, etc. In retrospective, we can say that the idea of 't Hooft was to choose the metric $g = \hat{g}$ in such a way as to enforce the self-duality condition for the constant field-strength connection. We will explain this better in the next paragraphs.

Although, we have restricted ourselves to a flat metric, we will still need to use different sets of coordinates related by linear transformations. In a given set of coordinates, the

expression of the metric $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ defines a specific constant matrix $g_{\mu\nu}$. The self-duality condition expressed in this coordinate system is given by

$$F_{\mu\nu} = \frac{\sqrt{\det(g)}}{2} \epsilon_{\alpha\beta\mu\nu} F^{\alpha\beta} \equiv \tilde{F}_{\mu\nu} \quad (3.11)$$

where the metric tensor $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$ are used for lowering and rising indices in the standard way. In our particular problem there are two natural systems of coordinates which will be useful. The first one is that in which the coordinates are aligned along the basis vector of the lattice: $x = \sum_\alpha e_\alpha y^\alpha$. These coordinates (unit-period coordinates) have the advantage that the torus has periods of 1 in each direction ($y^\mu \rightarrow y^\mu + 1$). It is also in this coordinate system in which we can write down easily the form of the constant field strength tensor:

$$\hat{F} = \frac{2\pi}{N} (\Delta_A dy^0 \wedge dy^3 + \Delta_B dy^1 \wedge dy^2) T \equiv \frac{N_1 N_2}{2N} T f_{\alpha\beta} dy^\alpha \wedge dy^\beta \quad (3.12)$$

The metric in these coordinates can be written as $ds^2 = \hat{g}_{\alpha\beta} dy^\alpha dy^\beta$. This gives the lengths of the basis vectors $\|e_\alpha\| = \sqrt{\hat{g}_{\alpha\alpha}}$ and the volume of the torus $\mathcal{V} = \sqrt{\det(\hat{g})}$.

The other quite natural coordinate system is the one in which the metric tensor is the unit matrix. We label the corresponding coordinates by z^a . The change of variables is produced by the vierbein V_α^a . (which in our case is just a constant matrix):

$$z^a = V_\alpha^a y^\alpha ; \quad y^\alpha = W_a^\alpha z^a \quad (3.13)$$

where

$$\sum_a V_\alpha^a V_\beta^a = \hat{g}_{\alpha\beta} ; \quad \sum_a V_\mu^a W_a^\nu = \delta_{\mu\nu} \quad (3.14)$$

These conditions do not specify the z^a coordinates uniquely. We are still free to perform orthogonal transformations in the z^a variables. This freedom can be used to adopt a canonical form for the field strength in these coordinates $\hat{F} = \frac{1}{2} T \bar{F}_{ab} dz^a \wedge dz^b$ with

$$\bar{F} = \frac{2\pi}{N} \begin{pmatrix} 0 & 0 & 0 & \bar{f}_A \\ 0 & 0 & \bar{f}_B & 0 \\ 0 & -\bar{f}_B & 0 & 0 \\ -\bar{f}_A & 0 & 0 & 0 \end{pmatrix} \quad (3.15)$$

with $\bar{f}_A \geq \bar{f}_B > 0$. The quantities \bar{f}_A and \bar{f}_B can be expressed in terms of coordinate invariant quantities. In particular, we can take $\rho_S \equiv \frac{1}{4} \text{Tr}(F_{\mu\nu} F^{\mu\nu})$ and $\rho_Q \equiv \frac{1}{4} \text{Tr}(F_{\mu\nu} \tilde{F}^{\mu\nu})$. The formula is

$$\bar{f}_A = \frac{\sqrt{N N_1 N_2}}{2\pi} \sqrt{\rho_S + \sqrt{\rho_S^2 - \rho_Q^2}} \quad (3.16)$$

$$\bar{f}_B = \frac{\sqrt{N N_1 N_2}}{2\pi} \sqrt{\rho_S - \sqrt{\rho_S^2 - \rho_Q^2}} \quad (3.17)$$

The self-duality condition is then simply given by $\bar{f}_A = \bar{f}_B$. Notice that $\rho_Q = 4\pi^2 Q/\mathcal{V} = 4\pi^2 \bar{f}_A \bar{f}_B / (N N_1 N_2) = 4\pi^2 \Delta_A \Delta_B / (N N_1 N_2 \mathcal{V})$, where \mathcal{V} is the volume of the torus (another invariant). Thus, in the self-dual case $\bar{f}_A = \bar{f}_B = \sqrt{N_1 N_2 / \mathcal{V}}$.

There is a whole family of metrics $g = \hat{g}$ for which the constant field strength connection is self-dual. For any of these cases we have a minimum action fractional instanton with constant field strength. A particularly simple case is the one chosen by 't Hooft, in which the metric tensor is diagonal in the unit-period coordinates:

$$ds^2 = \sum_{\mu} l_{\mu}^2 dy^{\mu} dy^{\mu} \tag{3.18}$$

This amounts to assuming that the generators of the lattice Λ are orthogonal and have length $\|e_{\alpha}\| = l_{\alpha}$. Then we have

$$\rho_Q = \frac{4\pi^2}{NN_1N_2} \frac{\Delta_A \Delta_B}{l_0 l_1 l_2 l_3}; \quad \rho_S = \frac{4\pi^2}{NN_1N_2} \left(\frac{\Delta_A^2}{2l_0^2 l_3^2} + \frac{\Delta_B^2}{2l_1^2 l_2^2} \right) \tag{3.19}$$

giving $\bar{f}_A = \Delta_A/(l_0 l_3)$ and $\bar{f}_B = \Delta_B/(l_1 l_2)$. The self-duality condition then becomes

$$\frac{\Delta_A}{l_0 l_3} = \frac{\Delta_B}{l_1 l_2} \tag{3.20}$$

Thus, the ratios of areas of the two twisted planes must be a particular rational number. Notice that, even within the set of diagonal matrices, there are many solutions since, for example, multiplying l_0 by any number and dividing l_3 by the same number does not alter the self-duality.

It is possible to obtain the most general constant symmetric matrix \hat{g}_0 for which the constant field strength connection is self-dual. For that purpose we realize that given an antisymmetric matrix X , one has

$$\frac{1}{2} \epsilon_{\mu\nu\rho\sigma} X_{\rho\sigma} = -\text{Pf}(X)(X^{-1})_{\mu\nu} \tag{3.21}$$

Hence, if we apply this expression to the antisymmetric tensor $f_{\mu\nu} = \text{Tr}(T\hat{F}_{\mu\nu})$ defined in eq. (3.12) we obtain

$$\tilde{f}_{\mu\nu} = -\frac{\text{Pf}(f)}{\sqrt{\det \hat{g}_0}} (\hat{g}_0 f^{-1} \hat{g}_0)_{\mu\nu} \tag{3.22}$$

From here we see that there is no constraint on the determinant of \hat{g}_0 (conformal invariance), and on the value of the Pfaffian of f . If we define $J = \hat{g}_0^{-1} f \cdot (\det(\hat{g}_0))^{1/4} / \sqrt{|\text{Pf}(f)|}$, then the self-duality condition becomes $J^2 = -1$. This defines an almost complex structure. Hence, f , J and \hat{g}_0 are a compatible triplet. Self-duality is achieved for all metrics of the form

$$\hat{g}_0 = -fJ \tag{3.23}$$

up an arbitrary multiplicative constant. The compatibility condition ensuring that the matrix \hat{g}_0 is symmetric reads

$$f = J^t f J \tag{3.24}$$

which expresses the fact that J is an element of the symplectic group $\text{Sp}(4, \mathbb{R})$. Thus, given any element of the group and inserting it in eq. 3.23, we get all the constant metric tensors for which the constant field strength is self-dual.

Concerning the choice of integers $M_{A1}, M_{A2}, M_{B1}, M_{B2}$, it is convenient to restrict ourselves to N_1 and N_2 being coprime. Otherwise, $\gcd(N_1, N_2)$ divides also N , Δ_A and Δ_B , and by dividing by this greater common divisor we can reduce the problem to this case. With the coprime condition we can easily solve for all quantities. We conclude that $k^{(a)} = M_{Aa} \hat{k}^{(a)}$ with $\hat{k}^{(a)}$ coprime with M_{Ba} , satisfying

$$\hat{k}^{(2)} M_{B1} - \hat{k}^{(1)} M_{B2} = 1 \tag{3.25}$$

In an analogous fashion $m^{(a)} = M_{Ba} \hat{m}^{(a)}$, with $\hat{m}^{(a)}$ coprime with M_{Aa} . These two conditions imply that the matrices $\Gamma_0^{(a)}$ and $\Gamma_3^{(a)}$ generate an M_{Ba}^2 dimensional irreducible algebra, while $\Gamma_1^{(a)}$ and $\Gamma_2^{(a)}$ generate an M_{Aa}^2 dimensional algebra. In other words we can write our twist-eating matrices as tensor products

$$\Gamma_0^{(a)} = \hat{\Gamma}_0^{(a)} \otimes \mathbb{I}_{M_{Aa}} ; \quad \Gamma_3^{(a)} = \hat{\Gamma}_3^{(a)} \otimes \mathbb{I}_{M_{Aa}} \tag{3.26}$$

$$\Gamma_1^{(a)} = \mathbb{I}_{M_{Ba}} \otimes \hat{\Gamma}_1^{(a)} ; \quad \Gamma_2^{(a)} = \mathbb{I}_{M_{Ba}} \otimes \hat{\Gamma}_2^{(a)} \tag{3.27}$$

where $\hat{\Gamma}_{0,3}^{(a)}$ are $M_{Ba} \times M_{Ba}$ matrices satisfying

$$\hat{\Gamma}_0^{(a)} \hat{\Gamma}_3^{(a)} = e^{2\pi i \hat{k}^{(a)} / M_{Ba}} \hat{\Gamma}_3^{(a)} \hat{\Gamma}_0^{(a)} \tag{3.28}$$

and $\hat{k}^{(a)} = k^{(a)} / M_{Aa}$. A similar relation follows for $\hat{\Gamma}_{1,2}^{(a)}$, replacing \hat{k} by \hat{m} and M_{Ba} by M_{Aa} .

More specifically, we can choose a basis in which $\hat{\Gamma}_0^{(a)}$ and $\hat{\Gamma}_1^{(a)}$ are diagonal. Then we can express all matrices $\hat{\Gamma}$ in terms of t Hooft clock matrices ($s \in \{0, 1, \dots, N-1\}$):

$$(Q_N)_{ss'} = e^{i\theta_N} e^{2\pi i s / N} \delta(s - s') \tag{3.29}$$

$$(P_N)_{ss'} = e^{i\theta_N} \delta(s + 1 - s') \tag{3.30}$$

where $\theta_N = \pi$ for odd N and equals π/N for even N to ensure that the matrices belong to $SU(N)$. Now we can write

$$\begin{aligned} \hat{\Gamma}_0^{(a)} &= Q_{M_{Ba}} ; & \hat{\Gamma}_1^{(a)} &= Q_{M_{Aa}} \\ \hat{\Gamma}_3^{(a)} &= (P_{M_{Ba}})^{-\hat{k}^{(a)}} ; & \hat{\Gamma}_2^{(a)} &= (P_{M_{Aa}})^{-\hat{m}^{(a)}} \end{aligned} \tag{3.31}$$

This means that all the basis vectors of the N_a dimensional space are labelled by a pair of integers (s_{Aa}, s_{Ba}) , with $0 \leq s_{X,a} \leq M_{Xa} - 1$. We will be using this basis in what follows.

We emphasize that our construction generates all constant field strength fractional instantons. This includes the $SU(2)$ case dealt in ref. [45], as well as the apparently different looking solutions appearing in ref. [51]. It is convenient to rewrite the main equations in matrix form as follows:

$$\hat{\mathbf{N}} \mathbf{M} \equiv \begin{pmatrix} \hat{k}^{(2)} & \hat{k}^{(1)} \\ -\hat{m}^{(1)} & \hat{m}^{(2)} \end{pmatrix} \begin{pmatrix} M_{B1} & M_{A2} \\ -M_{B2} & M_{A1} \end{pmatrix} = \begin{pmatrix} 1 & k \\ -m & 1 \end{pmatrix} \tag{3.32}$$

where the matrix \mathbf{M} has determinant equal to N , and the matrix $\hat{\mathbf{N}}$ has determinant $(km + 1)/N$. As a more complex example one might take $N = 43$ and split it into $N_1 = 15$,

$N_2 = 28$. This gives integers $M_{B1} = 3$ $M_{B2} = 4$ $M_{A1} = 5$ $M_{A2} = 7$, and hence $\Delta_A = 35$, $\Delta_B = 12$. This gives $\hat{k}^{(1)} = 2$, $\hat{k}^{(2)} = 3$, $\hat{m}^{(1)} = 2$, $\hat{m}^{(2)} = 3$ and hence $k = 31$ and $m = 18$.

A very interesting example is provided by the case in which $N_1 = M_{A1}$ and $N_2 = M_{B2}$ are two successive Fibonacci numbers. Then N becomes the next number in the sequence. As in other related problems [61], running over the index of the Fibonacci sequence defines a nice way to take the large N limit, in which the field strength tends to a finite value. Furthermore, the corresponding ratio of areas for the self-duality condition is given by the golden ratio. This and other possible choices involving generalized Fibonacci sequences are worth of being explored in greater detail.

4 Deforming constant field strength connections

Here we will address the case in which the constant field strength connection is not self-dual. Our strategy will be to construct the non-constant self-dual connection by deforming the previous constant connections obtained in the previous section.

Any vector potential defined on the bundles considered can be written as

$$A_\mu(x) = \hat{A}_\mu + \delta_\mu(x) \tag{4.1}$$

where \hat{A}_μ is the constant field strength associated to the transition matrices. The main advantage is that δ_μ transform homogeneously under translations by the generators of the lattice Λ . To express the twisted boundary conditions it is convenient to split δ_μ into the N_1 and N_2 rows and columns:

$$\delta_\mu(x) = \begin{pmatrix} S_\mu^{(1)}(x) & \mathcal{W}_\mu(x) \\ \mathcal{W}_\mu^\dagger(x) & S_\mu^{(2)}(x) \end{pmatrix} \tag{4.2}$$

Thus, $S_\mu^{(a)}(x)$ is an $N_a \times N_a$ hermitian matrix satisfying

$$S_\mu^{(a)}(x + e_\alpha) = \Gamma_\alpha^{(a)} S_\mu^{(a)}(x) \Gamma_\alpha^{\dagger(a)} \tag{4.3}$$

On the other hand the $N_1 \times N_2$ matrix \mathcal{W}_μ satisfies

$$\mathcal{W}_\mu(x + e_\alpha) = \exp\{i\pi N \hat{w}(e_\alpha, x) / (N_1 N_2)\} \Gamma_\alpha^{(1)} \mathcal{W}_\mu(x) \Gamma_\alpha^{\dagger(2)} \tag{4.4}$$

Following the choices done in the previous section, we will take the e_α that brings $\hat{w}(e_\alpha, e_\beta)$ to canonical form.

Now we can compute the field strength

$$F_{\mu\nu}(x) = \hat{F}_{\mu\nu} + \hat{D}_\mu \delta_\nu - \hat{D}_\nu \delta_\mu - i[\delta_\mu, \delta_\nu] \tag{4.5}$$

The operators \hat{D}_μ are the covariant derivatives (in the adjoint representation) with respect to the constant field strength connection \hat{A} . The field tensor can also be decomposed into blocks

$$F_{\mu\nu}(x) = \begin{pmatrix} F_{\mu\nu}^{(1)}(x) & \mathcal{F}_{\mu\nu}(x) \\ \mathcal{F}_{\mu\nu}^\dagger(x) & F_{\mu\nu}^{(2)}(x) \end{pmatrix} \tag{4.6}$$

with coefficients given by

$$\begin{aligned}
 F_{\mu\nu}^{(1)}(x) &= \hat{F}_{\mu\nu}^{(1)} + \partial_\mu S_\nu^{(1)}(x) - \partial_\nu S_\mu^{(1)}(x) - i[S_\mu^{(1)}, S_\nu^{(1)}] - i\mathcal{W}_\mu \mathcal{W}_\nu^\dagger + i\mathcal{W}_\nu \mathcal{W}_\mu^\dagger \\
 F_{\mu\nu}^{(2)}(x) &= \hat{F}_{\mu\nu}^{(2)} + \partial_\mu S_\nu^{(2)}(x) - \partial_\nu S_\mu^{(2)}(x) - i[S_\mu^{(2)}, S_\nu^{(2)}] - i\mathcal{W}_\mu^\dagger \mathcal{W}_\nu + i\mathcal{W}_\nu^\dagger \mathcal{W}_\mu \\
 \mathcal{F}_{\mu\nu}(x) &= \bar{D}_\mu \mathcal{W}_\nu - \bar{D}_\nu \mathcal{W}_\mu - iS_\mu^{(1)} \mathcal{W}_\nu + iS_\nu^{(1)} \mathcal{W}_\mu - i\mathcal{W}_\mu S_\nu^{(2)} + i\mathcal{W}_\nu S_\mu^{(2)}
 \end{aligned}
 \tag{4.7}$$

where \bar{D}_μ is the covariant derivative with respect to a U(1) gauge field whose constant field strength $f_{\mu\nu} = \text{Tr}(T\hat{F}_{\mu\nu})$ was defined in eq. (3.12).

Up to now everything is independent on the metric and hence on the choice of coordinates. The self-duality condition can be expressed by setting to zero the projection onto the antiself-dual part. This can be written as follows

$$\frac{1}{2} \bar{\eta}_i^{\mu\nu} F_{\mu\nu} = 0
 \tag{4.8}$$

where $\bar{\eta}_i^{\mu\nu}$ for $i = 1, 2, 3$ are a basis of the antiself-dual tensors. For unit metric tensor they coincide with the symbols $\bar{\eta}_i^{ab}$ introduced by 't Hooft. For the unit period metric they can be written as

$$\bar{\eta}_i^{\mu\nu} = W_a^\mu W_b^\nu \bar{\eta}_i^{ab}
 \tag{4.9}$$

The contribution of the constant field strength is then

$$\frac{1}{2} \bar{\eta}_i^{\mu\nu} \hat{F}_{\mu\nu} = \delta_{i3} (\hat{F}_{03} - \hat{F}_{12}) = \frac{2\pi}{N} (\bar{f}_A - \bar{f}_B) \delta_{i3} T
 \tag{4.10}$$

which vanishes in the self-dual case. The strategy put forward in our paper [45] is to treat the difference $\epsilon \equiv (\bar{f}_A - \bar{f}_B)$ as an expansion parameter and compute the self-dual connection as a power series expansion in this parameter. When only a few orders are computed the approximation becomes closer to the exact result the smaller the value of ϵ . Indeed, this was verified in ref. [45], for the SU(2) case with the diagonal metric, by computing the analytic expressions and comparing them with the numerical solution obtained by a minimization method. The solution now has a lumpy structure with a peak in the action density at a particular point. Obviously the 4 moduli parameters are associated with the space-time coordinates of the peak.

In what follows we will extend the previous construction to SU(N). For that purpose it is important to revise the details of the method as it appears for SU(N) case.

The first observation is that the off-block part of the deformation \mathcal{W}_μ becomes a power series in odd powers of $\sqrt{\epsilon}$:

$$\mathcal{W}_\mu = \sqrt{\epsilon} \sum_{n=0}^{\infty} \epsilon^n \mathcal{W}_{\mu,n}(x)
 \tag{4.11}$$

On the other hand the block terms $S_\mu^{(a)}$ become power series in ϵ starting at order 1:

$$S_\mu^{(a)} = \epsilon \sum_{n=0}^{\infty} \epsilon^n S_{\mu,n}^{(a)}
 \tag{4.12}$$

The even or odd powers of $\sqrt{\epsilon}$ apply to $F^{(a)}$ and \mathcal{F} as well, as can be seen from the expression (4.7).

Now, before going into the actual calculation of the coefficients $S_{\mu,n}^{(a)}$ and $W_{\mu,n}(x)$, let us explain how the first few terms in the expansion proceed, because that clarifies the general procedure with certain subtleties involved. The first term in the expansion of the self-dual part of the action is actually of order $\sqrt{\epsilon}$:

$$0 = \bar{\eta}_i^{\mu\nu} \bar{D}_\mu \mathcal{W}_{\nu,0} \quad (4.13)$$

This is an homogeneous equation so that the solution is only fixed up to a multiplicative constant. Thus, it is unclear to what extent is the contribution of order $\sqrt{\epsilon}$. This becomes clear when looking at the equation at order ϵ :

$$0 = \frac{2\pi}{N} T \delta_{i3} + \frac{1}{2} \bar{\eta}_i^{\mu\nu} \begin{pmatrix} \partial_\mu S_{\nu,0}^{(1)}(x) - \partial_\nu S_{\mu,0}^{(1)}(x) & 0 \\ 0 & \partial_\mu S_{\nu,0}^{(2)}(x) - \partial_\nu S_{\mu,0}^{(2)}(x) \end{pmatrix} + \frac{1}{2} \bar{\eta}_i^{\mu\nu} \begin{pmatrix} -i\mathcal{W}_{\mu,0}\mathcal{W}_{\nu,0}^\dagger + i\mathcal{W}_{\nu,0}\mathcal{W}_{\mu,0}^\dagger & 0 \\ 0 & -i\mathcal{W}_{\mu,0}^\dagger\mathcal{W}_{\nu,0} + i\mathcal{W}_{\nu,0}^\dagger\mathcal{W}_{\mu,0} \end{pmatrix} \quad (4.14)$$

The first term comes from the constant field strength part, which as we saw before is of order ϵ . Now if we integrate this equation over the torus, the term containing derivatives vanishes and we get

$$0 = \frac{2\pi\mathcal{V}}{N} T \delta_{i3} - \frac{i}{2} \bar{\eta}_i^{\mu\nu} \int_{\mathcal{T}} dx \begin{pmatrix} \mathcal{W}_{\mu,0}\mathcal{W}_{\nu,0}^\dagger - \mathcal{W}_{\nu,0}\mathcal{W}_{\mu,0}^\dagger & 0 \\ 0 & \mathcal{W}_{\mu,0}^\dagger\mathcal{W}_{\nu,0} - \mathcal{W}_{\nu,0}^\dagger\mathcal{W}_{\mu,0} \end{pmatrix} \quad (4.15)$$

where \mathcal{V} is the volume of the torus. Indeed, one can multiply the equation by the generator T and take the trace to obtain

$$0 = \frac{2\pi\mathcal{V}}{N} \delta_{i3} - i \bar{\eta}_i^{\mu\nu} \int_{\mathcal{T}} dx \text{Tr}(\mathcal{W}_{\mu,0}\mathcal{W}_{\nu,0}^\dagger) \quad (4.16)$$

It is now obvious that this equation fixes the normalization of $\mathcal{W}_{\mu,0}$ up to a phase.

The arbitrariness of the phase can be put into a wider context by investigating the multiplicity of solutions. Obviously symmetries imply that the solution is non-unique. First of all, one has gauge transformations. As in our previous paper we fix them by imposing the background field gauge $\hat{D}_\mu \delta_\mu = 0$, leading to

$$\partial_\mu S_\mu^{(a)} = 0 ; \quad \bar{D}_\mu \mathcal{W}_\mu = 0 \quad (4.17)$$

There is a remaining invariance under those global gauge transformations which are consistent with the boundary conditions. Indeed, this freedom is connected to the phase arbitrariness of \mathcal{W} .

Apart from the phase arbitrariness notice that in eq. (4.14) $S_\mu^{(a)}$ only enters through its derivative. Thus, one can always add a constant, which because of the boundary conditions must be proportional to the identity in each block. The traceless condition then fixes this to be a constant times the generator T . Finally, one realizes that the arbitrariness can be associated to space-time translation, being equivalent to a shift $x \rightarrow x - x_0$ in the original spatial constant solution. One can add a condition to fix this arbitrariness and obtain a

unique solution. This is very similar to the discussion and procedure employed in ref. [45] when dealing with the SU(2) case.

After this explanation we proceed to the actual calculation to first order in ϵ which is done in the next section.

5 Non-constant fractional instanton to order ϵ

In this section we present the calculation up to order ϵ , as was done in our previous paper for SU(2). The calculation will be split into two subsections dealing with $\mathcal{W}_{\mu,0}$ and $S_{\mu,0}^{(a)}$ respectively.

5.1 The first equation

The first part of the calculation involves the determination of $\mathcal{W}_{\mu,0}$. This function satisfies the boundary conditions eq. (4.4) and the equation

$$\frac{1}{2}\bar{\eta}_i^{\mu\nu}\bar{D}_\mu\mathcal{W}_{\nu,0}=0 \tag{5.1}$$

where \bar{D}_μ is the covariant derivative with respect to the abelian gauge field with constant field strength $f = \text{Tr}(T\hat{F})$. In unit-period coordinates y^μ and orthonormal coordinates z^a we can write

$$f = \frac{2\pi}{\Delta_B}dy^0 \wedge dy^3 + \frac{2\pi}{\Delta_A}dy^1 \wedge dy^2 = \frac{2\pi\bar{f}_A}{N_1N_2}dz^0 \wedge dz^3 + \frac{2\pi\bar{f}_B}{N_1N_2}dz^1 \wedge dz^2 \tag{5.2}$$

We can restate the boundary conditions by introducing operators \mathbf{O}_α as follows:

$$\mathbf{O}_\alpha = e^{-if_{\mu\nu}e_\alpha^\mu x^\nu/2} \delta_\alpha \tag{5.3}$$

where δ_α is the operator that shifts x by e_α :

$$\delta_\alpha\Psi(x) = \Psi(x + e_\alpha) \tag{5.4}$$

The operators satisfy the relations

$$\mathbf{O}_\alpha\mathbf{O}_\beta = e^{2\pi i\Delta_{\alpha\beta}/(N_1N_2)}\mathbf{O}_\beta\mathbf{O}_\alpha \tag{5.5}$$

Now the boundary conditions can be rewritten as

$$\mathbf{O}_\alpha\mathcal{W} = \Gamma_\alpha^{(1)}\mathcal{W}\Gamma_\alpha^{(2)\dagger} \tag{5.6}$$

Notice that the operators $\mathbf{O}_{0,3}$ commute with $\mathbf{O}_{1,2}$. We have

$$\mathbf{O}_0\mathbf{O}_3 = e^{2\pi i/\Delta_B}\mathbf{O}_3\mathbf{O}_0 \tag{5.7}$$

$$\mathbf{O}_1\mathbf{O}_2 = e^{2\pi i/\Delta_A}\mathbf{O}_2\mathbf{O}_1 \tag{5.8}$$

Now we can simultaneously diagonalize \mathbf{O}_1 and \mathbf{O}_0 which are unitary operators. Through the boundary conditions this is equivalent to diagonalizing $\Gamma_0^{(a)}$ and $\Gamma_1^{(a)}$. This is the

same as going to the basis that was presented in section 3. It is convenient to label the matrix elements of $\mathcal{W}(x)$ in terms of two indices $l_B = (s_{B1}M_{B2} - s_{B2}M_{B1})$ and $l_A = (s_{A1}M_{A2} - s_{A2}M_{A1})$. In this notation the function $\mathcal{W}_{l_A l_B}(x)$ satisfies the following boundary conditions

$$\mathbf{O}_0(\mathcal{W})_{l_A l_B} = e^{i\theta_B} e^{2\pi i l_B / \Delta_B} (\mathcal{W})_{l_A l_B} ; \quad \mathbf{O}_3(\mathcal{W})_{l_A l_B} = e^{i\theta_B} (\mathcal{W})_{l_A l_{B+1}} \quad (5.9)$$

$$\mathbf{O}_1(\mathcal{W})_{l_A l_B} = e^{i\theta_A} e^{2\pi i l_A / \Delta_A} (\mathcal{W})_{l_A l_B} ; \quad \mathbf{O}_2(\mathcal{W})_{l_A l_B} = e^{i\theta_A} (\mathcal{W})_{l_{A+1} l_B} \quad (5.10)$$

where $\theta_{A,B}$ is zero if $\Delta_{A,B}$ is odd. In general, we have $\theta_X = \pi \epsilon_X / \Delta_X$ with $\epsilon_X \equiv M_{X1} - M_{X2} \pmod{2}$. These boundary conditions imply that once $\bar{\mathcal{W}}(x) \equiv (\mathcal{W})_{00}(x)$ is known, we can immediately solve for $(\mathcal{W})_{l_A l_B}$ as follows

$$(\mathcal{W})_{l_A l_B}(x) = (\mathbf{O}_3)^{l_B} (\mathbf{O}_2)^{l_A} \bar{\mathcal{W}}(x) = e^{i\pi(y^0 l_B / \Delta_B + y^1 l_A / \Delta_A)} \bar{\mathcal{W}}(x + l_B e_3 + l_A e_2) \quad (5.11)$$

We recall that the function $\bar{\mathcal{W}}$ satisfies the following boundary conditions

$$\mathbf{O}_0 \bar{\mathcal{W}} = e^{i\theta_B} \bar{\mathcal{W}} ; \quad (\mathbf{O}_3)^{\Delta_B} \bar{\mathcal{W}} = e^{i\theta_B \Delta_B} \bar{\mathcal{W}} \quad (5.12)$$

$$\mathbf{O}_1 \bar{\mathcal{W}} = e^{i\theta_A} \bar{\mathcal{W}} ; \quad (\mathbf{O}_2)^{\Delta_A} \bar{\mathcal{W}} = e^{i\theta_A \Delta_A} \bar{\mathcal{W}} \quad (5.13)$$

Now notice that all operators \bar{D}_μ commute with \mathbf{O}_α and furthermore they are scalar and do not mix different components of \mathcal{W} . All the problem reduces to that of an abelian connection. In particular, our first equation reduces to a scalar equation involving only $\bar{\mathcal{W}}_\mu(x)$:

$$\frac{1}{2} \bar{\eta}_i^{\mu\nu} \bar{D}_\mu \bar{\mathcal{W}}_{\nu,0} = 0 \quad (5.14)$$

In the z coordinate system the projection operator becomes just 't Hooft symbol, so that using the same strategy as in our SU(2) paper, we can reformulate the problem by introducing 2×2 matrices $\sigma_a = (\mathbb{I}_2, -i\vec{\tau})$ and $\bar{\sigma}_a = \sigma_a^\dagger = (\mathbb{I}_2, i\vec{\tau})$, where τ_i are the Pauli matrices and \mathbb{I}_2 the 2×2 identity matrix. These matrices verify

$$\bar{\sigma}_a \sigma_b = \bar{\eta}_c^{ab} \sigma_c \quad (5.15)$$

where $\bar{\eta}_0^{ab} = \delta_{ab}$. Now we can rewrite the equation as

$$(\bar{D}_a \bar{\sigma}_a)(\bar{\mathcal{W}}_b \sigma_b) = 0 \quad (5.16)$$

where each of the parenthesis involves a 2×2 matrix. We have actually added one equation which expresses the condition of background field gauge $\bar{D}_a \bar{\mathcal{W}}_a = 0$. Being a matrix equation, the previous condition imposes 4 real equations. Just as for the SU(2) case the equation has a solution when $(\bar{\mathcal{W}}_b \sigma_b)$ consist only of the 11 element. This occurs for $\bar{\mathcal{W}}_1 = \bar{\mathcal{W}}_2 = 0$ and $\bar{\mathcal{W}}_3 = i\bar{\mathcal{W}}_0$. In that case, the matrix equation reduces just to 2 complex equations:

$$D_B \bar{\mathcal{W}}_0 \equiv (\bar{D}_0 + i\bar{D}_3) \bar{\mathcal{W}}_0 = 0 \Leftrightarrow \left(\frac{\partial}{\partial z^0} + i \frac{\partial}{\partial z^3} \right) \bar{\mathcal{W}}_0 = -\frac{\pi \bar{f}_A}{N_1 N_2} (z^0 + iz^3) \bar{\mathcal{W}}_0 \quad (5.17)$$

$$D_A \bar{\mathcal{W}}_0 \equiv (\bar{D}_1 + i\bar{D}_2) \bar{\mathcal{W}}_0 = 0 \Leftrightarrow \left(\frac{\partial}{\partial z^1} + i \frac{\partial}{\partial z^2} \right) \bar{\mathcal{W}}_0 = -\frac{\pi \bar{f}_B}{N_1 N_2} (z^2 + iz^2) \bar{\mathcal{W}}_0 \quad (5.18)$$

The choice of \bar{W}_a is justified precisely to keep only these two conditions. Why precisely these two is clear from our previous work on the subject [45, 46] and will be explained below. The equations are essentially two copies of the equations involved in the Bradlow expansion for vortices on the 2-torus. The treatment performed in ref. [46] is to write the equations in terms of complex coordinates, which fixes the solution up to a holomorphic function. The latter is fixed by the boundary conditions. In the two dimensional case these boundary conditions led to the Jacobi theta functions and those with rational characteristics. In our case, something very similar follows for the case in which \hat{g} (the metric tensor in unit-period coordinates) is diagonal.

In treating the general case, we consider more instructive to follow an alternative method which is more constructive. For that purpose we express eqs. (5.17)–(5.18) and the boundary conditions in terms of the unit-period coordinates y . The two equations can be written as

$$U_X^\alpha \left(\frac{\partial}{\partial y^\alpha} + i f_{\alpha\beta} y^\beta / 2 \right) \bar{W}_0 = 0 \tag{5.19}$$

where $X \in \{A, B\}$, $U_A = W_0 + iW_3$ and $U_B = W_1 + iW_2$. We remind the reader that the vectors W_a^α are the inverse of the vierbein, which in these coordinates coincide with the lattice generators e_α^a . Given the form of the equation we will try a solution which is the exponential of a quadratic form

$$Y = \exp \left\{ -\frac{1}{2} y^\alpha y^\beta R_{\alpha\beta} \right\} \tag{5.20}$$

Obviously, the matrix R is symmetric. Applying the previous equation to our ansatz we get

$$U_A^\alpha \left(-R_{\alpha\beta} + \frac{i}{2} f_{\alpha\beta} \right) y^\beta = 0 \tag{5.21}$$

This equation alone does not fix the matrix R uniquely. Now we should impose the boundary conditions.

We first impose the boundary conditions with respect to translations by e_0 and e_1 . This demands that

$$-R_{\alpha\beta} - \frac{i}{2} f_{\alpha\beta} = 0 \tag{5.22}$$

valid for $\alpha = 0, 1$ and β arbitrary. Given the symmetry of R this equation fixes the matrix R up to the 2×2 submatrix \bar{R} with $\alpha, \beta \in \{3, 2\}$. The next step is to return to eq. (5.21) with the information that we have obtained on the structure of R . The best way to obtain the solution is by expressing the equation in terms of 2×2 matrices. We write U_{11} for the 2×2 matrix with components U_X^α with $\alpha = 0, 1$. We call U_{12} the corresponding one matrix for $\alpha = 3, 2$. Now setting $F_2 = 2\pi \text{diag}(1/\Delta_B, 1/\Delta_A)$ we can write

$$iU_{11}F_2 - U_{12}\bar{R} = 0 \tag{5.23}$$

This allows us to solve for \bar{R}

$$\bar{R} = iU_{12}^{-1}U_{11}F_2 \tag{5.24}$$

To see the consistency of the solution we should still verify that the so obtained matrix is symmetric. This can be deduced from the form of f in both coordinate systems (eq. (5.2)). We leave the verification to the reader.

We now summarise the form of R :

$$R_{\alpha\beta}y^\alpha y^\beta = -\frac{2i\pi}{\Delta_B}y^0y^3 - \frac{2i\pi}{\Delta_A}y^1y^2 + \bar{R}_{AA}(y^2)^2 + \bar{R}_{BB}(y^3)^2 + \bar{R}_{AB}y^2y^3 \quad (5.25)$$

For the diagonal metric case $\bar{R}_{AB} = 0$, $\bar{R}_{AA} = \frac{2\pi l_2}{l_1 \Delta_A}$ and $\bar{R}_{BB} = \frac{2\pi l_3}{l_0 \Delta_B}$

We have succeeded in constructing a solution of the first equation that satisfies the right boundary conditions under translations in y^0 and y^1 , but we still have not enforced the rest of boundary conditions. This will be done constructively. Suppose that we have a function $\Psi(x)$ and we want to impose that it satisfies the eigenvalue equation:

$$(\mathbf{O}_3)^k \Psi = e^{i\lambda} \Psi \quad (5.26)$$

This can be done by projection as follows:

$$\Psi \longrightarrow \sum_{q \in \mathbb{Z}} (\mathbf{O}_3)^{kq} e^{-i\lambda q} \Psi \quad (5.27)$$

In our particular case we have just to perform the following projection

$$\sum_{q_B, q_A \in \mathbb{Z}} e^{-i\pi \epsilon_A q_A - i\pi \epsilon_B q_B} (\mathbf{O}_3)^{\Delta_B q_B} (\mathbf{O}_2)^{\Delta_A q_A} Y \quad (5.28)$$

We point out that \mathbf{O}_α commute with the covariant derivative, so that the new function is still a solution of the same equation. All we need to do is to apply the definitions to obtain the requested solution explicitly

$$\bar{\mathcal{W}}_0 = \mathcal{N} Y \sum_{q_A, q_B \in \mathbb{Z}} e^{2\pi i(u_B q_B + u_A q_A)} e^{-\frac{1}{2} \bar{R}_{BB} (q_B \Delta_B)^2 - \frac{1}{2} \bar{R}_{AA} (q_A \Delta_A)^2 - \bar{R}_{AB} \Delta_A \Delta_B q_A q_B} \quad (5.29)$$

where we have introduced two complex variables as follows

$$u_B = y_0 + i \frac{\Delta_B}{2\pi} (\bar{R}_{BB} y_3 + \bar{R}_{BA} y_2) - \frac{\epsilon_B}{2}; \quad u_A = y_1 + i \frac{\Delta_A}{2\pi} (\bar{R}_{AB} y_3 + \bar{R}_{AA} y_2) - \frac{\epsilon_A}{2}. \quad (5.30)$$

Now one easily recognizes that the sum in the expression of $\bar{\mathcal{W}}_0$ is just the Riemann theta function $\Theta(\vec{u}, \tau)$ where the 2×2 symmetric matrix τ is given by

$$\tau_{XY} = \frac{i}{2\pi} \Delta_X \bar{R}_{XY} \Delta_Y \quad (5.31)$$

for $X, Y \in \{A, B\}$. It can be easily proven that τ satisfies Siegel positivity condition and the function is well defined. In the special case in which $\bar{R}_{AB} = 0$, the function factorizes into a product of Jacobi theta functions. This is the case for the diagonal metric.

Now plugging our solution eq. 5.29 into eq. (5.11) we obtain the requested solution $\mathcal{W}_{0,0}$ up to a normalization \mathcal{N} which is not fixed. As explained in the previous section, this normalization can be fixed by the space-time integral of the second equation. This is

eq. (4.15) of the previous section. We will now compute it. First notice that only $\mathcal{W}_{0,0}$ and $\mathcal{W}_{3,0} = i\mathcal{W}_{0,0}$ are non-zero. Hence, the only non-zero trace is

$$\text{Tr}(\mathcal{W}_{0,0}\mathcal{W}_{3,0}^\dagger) = -i \sum_{l_A, l_B} |\bar{\mathcal{W}}_0(x - l_A e_2 - l_B e_3)|^2 \quad (5.32)$$

Now we can perform the integral over the torus giving

$$\int_{\mathcal{T}} dx \text{Tr}(\mathcal{W}_{0,0}\mathcal{W}_{3,0}^\dagger) = -i\mathcal{V} \int_0^1 dy^0 \int_0^1 dy^1 \int_{-\Delta_B}^0 dy^2 \int_{-\Delta_A}^0 dy^3 |\bar{\mathcal{W}}_0(y)|^2 \quad (5.33)$$

The next step is to substitute the expression (5.29) and perform the integration. The interesting thing is that the integral over x_0 and x_1 are very simple implying that the integers q_A and q_B for both factors should be the same. For the purpose of computing the final result it is much better to go back to Eq. (5.28) and realize that

$$(\mathbf{O}_a \bar{\mathcal{W}}_0)(\mathbf{O}_a \bar{\mathcal{W}}_0)^* = \delta_a (|\bar{\mathcal{W}}_0|^2) \quad (5.34)$$

The sum over q_A and q_B has then the effect of extending the integration over x^2 and x^3 to the full real axis. We then have

$$\int_{\mathcal{T}} dx \text{Tr}(\mathcal{W}_{0,0}\mathcal{W}_{3,0}^\dagger) = -i\mathcal{V} |\mathcal{N}|^2 \int_{-\infty}^{\infty} dy^2 \int_{-\infty}^{\infty} dy^3 |Y(y^2, y^3)|^2 \quad (5.35)$$

The final integral is gaussian and gives $2\pi/\sqrt{\det((\bar{R} + \bar{R}^*)/2)}$. Now introducing the result into the normalization equation we get

$$|\mathcal{N}|^2 = \frac{\sqrt{\det((\bar{R} + \bar{R}^*)/2)}}{2N} \quad (5.36)$$

For the case of the diagonal metric the determinant of \bar{R} can be easily determined and the result becomes

$$|\mathcal{N}|^2 = \frac{\pi}{N} \sqrt{\frac{l_3 l_2}{l_0 l_1 N_1 N_2}} \quad (5.37)$$

5.2 The second equation

Now we have to look at the equation to order ϵ . This is an equation where the unknowns are the matrices $S_{\mu,0}^{(a)}$. The boundary conditions on the $S^{(a)}$ are given in (4.3). This can be easily solved using a modified Fourier decomposition. There exist a basis of matrices $\hat{\mathbf{\Gamma}}^{(a)}(\vec{q}_c)$ satisfying [53]

$$\Gamma_\mu^{(a)} \hat{\mathbf{\Gamma}}^{(a)}(\vec{q}_c) \Gamma_\mu^{(a)\dagger} = e^{iq_c \mu} \hat{\mathbf{\Gamma}}^{(a)}(\vec{q}_c) \quad (5.38)$$

where $\vec{q}_c = 2\pi(n_0/M_{Ba}, n_1/M_{Aa}, n_2/M_{Aa}, n_3/M_{Ba})$ with $0 \leq n_0, n_3 \leq M_{Ba} - 1$ and $0 \leq n_1, n_2 \leq M_{Aa} - 1$ are integers. The total number of matrices is N_a^2 , so that the set represents a basis of the space of $N_a \times N_a$ matrices. It is more convenient to consider that the integers n_μ are actually defined modulo $M_{X,a}$ ($M_{A,a}$ or $M_{B,a}$ depending on the index). In addition, one needs a normalization condition on the basis matrices. One can take

$$\text{Tr}(\hat{\mathbf{\Gamma}}^{(a)}(\vec{q}_c)(\hat{\mathbf{\Gamma}}^{(a)}(\vec{p}_c)^\dagger) = \delta(\vec{q}_c - \vec{p}_c) \quad (5.39)$$

where the delta function is taken modulo the corresponding congruences. This defines the matrices up to a phase. Explicitly one can write

$$\begin{aligned} (\hat{\Gamma}^{(1)}(\vec{p}))_{ss'} &= \frac{1}{N_1} e^{is_{B1}M_{B2}p_3 + is_{A,1}M_{A,2}p_2} \delta\left(s'_{B1} - s_{B1} + \frac{p_0 M_{B1}}{2\pi}\right) \delta\left(s'_{A1} - s_{A1} + \frac{p_1 M_{A1}}{2\pi}\right) \\ (\hat{\Gamma}^{(2)}(\vec{p}))_{ss'} &= \frac{1}{N_2} e^{-is_{B2}M_{B1}p_3 - is_{A,2}M_{A,1}p_2} \delta\left(s'_{B2} - s_{B2} + \frac{p_0 M_{B2}}{2\pi}\right) \delta\left(s'_{A2} - s_{A2} + \frac{p_1 M_{A2}}{2\pi}\right) \end{aligned}$$

Finally, we can decompose any matrix satisfying the boundary conditions (4.3) as

$$S(x) = \sum_{p \in \Lambda_*} e^{ip_\alpha y^\alpha} \hat{\Gamma}^{(a)}(p_c) \hat{S}(p) \quad (5.40)$$

where $M_{Ba}p_0/(2\pi)$, $M_{Ba}p_3/(2\pi)$, $M_{Aa}p_1/(2\pi)$ and $M_{Aa}p_2/(2\pi)$ run over all integers, and p_c is the corresponding vector with congruent integers. In other words, $(p - p_c)/(2\pi)$ is an arbitrary vector of integers. Now given a matrix satisfying the boundary conditions $S(x)$, it is possible to obtain the Fourier coefficients by the following procedure

$$\hat{S}(p) = \prod_{\alpha} \left(\int_0^1 dy^\alpha e^{-ip_\alpha y^\alpha} \right) \text{Tr} \left((\hat{\Gamma}^{(a)}(p_c))^\dagger S(x) \right) \quad (5.41)$$

Now let us go to the second equation and proceed as before, by introducing the matrices $\bar{\sigma}_a$ and σ_b . Given an arbitrary vector v_a we can construct matrices $\bar{\mathbf{v}} = v_a \bar{\sigma}_a$ and $\tilde{\mathbf{v}} = v_a \sigma_a$. Then we can write

$$\bar{\partial} \tilde{\mathbf{S}} = \bar{\eta}_d^{bc} \frac{\partial}{\partial z^b} S_c \sigma_d \quad (5.42)$$

To parameterize $\tilde{\mathbf{S}}$ we write it as $\bar{\partial} \tilde{\mathbf{G}}$. This is always possible if $\hat{S}(p=0) = 0$, i.e. when S_b has no constant term. We then see that the equation for S transforms into an equation for G :

$$\bar{\eta}_d^{bc} \frac{\partial}{\partial z^b} S_c \sigma_d = \left(\frac{\partial}{\partial z^b} \right)^2 G_d \sigma_d \quad (5.43)$$

Now notice that, given that \mathcal{W}_0 and \mathcal{W}_3 are the only non-zero components of the other term in the equation, this implies that only G_3 could be non zero. Combining this information we write explicitly the form of $S_b^{(a)}$

$$S_{0,0}^{(a)} = -\frac{\partial G_3^{(a)}}{\partial z^3}; \quad S_{3,0}^{(a)} = \frac{\partial G_3^{(a)}}{\partial z^0} \quad (5.44)$$

$$S_{1,0}^{(a)} = \frac{\partial G_3^{(a)}}{\partial z^2}; \quad S_{2,0}^{(a)} = -\frac{\partial G_3^{(a)}}{\partial z^1} \quad (5.45)$$

Now we can write down the equation

$$\begin{pmatrix} \Delta G_3^{(1)} & 0 \\ 0 & \Delta G_3^{(2)} \end{pmatrix} = 2 \begin{pmatrix} \mathcal{W}_{0,0} \mathcal{W}_{0,0}^\dagger & 0 \\ 0 & \mathcal{W}_{0,0}^\dagger \mathcal{W}_{0,0} \end{pmatrix} \quad (5.46)$$

where Δ is the Laplacian operator. To solve this equation we use the Fourier decomposition. Both sides of the equation can be written as a Fourier sum and the equality

corresponds to the equality of the Fourier coefficients. The advantage of this procedure is that the Laplacian operator has a simple action on the Fourier coefficients.

To obtain an explicit solution we need to determine the Fourier coefficients of the left hand side of eq. (5.46). This can be done with our formulas:

$$\hat{C}^{(1)}(p) = 2 \prod_{\alpha} \left(\int_0^1 dy^{\alpha} e^{-ip_{\alpha} y^{\alpha}} \right) \text{Tr} \left((\hat{\Gamma}^{(1)}(p_c))^{\dagger} \mathcal{W}_{0,0} \mathcal{W}_{0,0}^{\dagger} \right) \quad (5.47)$$

$$\hat{C}^{(2)}(p) = 2 \prod_{\alpha} \left(\int_0^1 dy^{\alpha} e^{-ip_{\alpha} y^{\alpha}} \right) \text{Tr} \left((\hat{\Gamma}^{(2)}(p_c))^{\dagger} \mathcal{W}_{0,0}^{\dagger} \mathcal{W}_{0,0} \right) \quad (5.48)$$

Finally, the coefficients of $G_3^{(a)}$ are easily obtained as follows

$$\hat{G}_3^{(a)}(q) = \frac{\hat{C}^{(a)}(q)}{\|q\|^2} \quad (5.49)$$

where $\|q\|^2 = q_{\alpha} q_{\beta} \hat{g}^{\alpha\beta}$, with \hat{g} the metric (upper indices for the inverse metric). One has to exclude the coefficient for $q = 0$, for which the denominator is singular. The value of $\hat{C}^{(a)}(0)$ was determined earlier and used to fix the normalization of $\mathcal{W}_{0,0}$. Applying the derivatives (which is easily done in the Fourier decomposition) in eq. (5.44) we obtain the Fourier coefficients of $S_{\mu,0}^{(a)}$ which completes the solution of the second equation.

The only missing piece for an explicit solution is to determine the coefficients $\hat{C}^{(a)}(q)$ by means of the integrals (5.47). This poses no fundamental problem since all are simply Gaussian integrals. The most important thing is to determine how to do the calculation efficiently. Let me sketch very briefly how the calculation can be done and write down the final result. The first part is to write \mathcal{W} as follows

$$(\mathcal{W})_{l_A l_B} = \sum_{q_A, q_B \in \mathbb{Z}} \mathbf{O}_3^{\Delta_B q_B + l_B} \mathbf{O}_2^{\Delta_A q_A + l_A} Y \quad (5.50)$$

Then we can write down this expression factorizing the part which depends on y^0 and y^1 and a part that depends only on y^3 and y^2 . In this second part it is better to keep explicitly the δ_3 and δ_2 operators. Now one can combine the result with that of \mathcal{W}^{\dagger} and $(\hat{\Gamma}(p))^{\dagger}$. The main observation is that the dependence on y_0 and y_1 of the whole integral appears as an imaginary exponential. One can integrate on these two variables to give a delta function equating the q_A and q_B coming from \mathcal{W} and \mathcal{W}^{\dagger} . The final expression takes the following form

$$\int_0^1 dy^3 \int_0^1 dy^2 \sum_{n_3, n_2 \in \mathbb{Z}} \delta_3^{n_3} \delta_2^{n_2} H(y_3, y_2) \quad (5.51)$$

for a function H to be specified below. We then realize that the sum of the δ operators simply extends the integration region to infinity, giving

$$\int_{-\infty}^{\infty} dy^3 \int_{-\infty}^{\infty} dy^2 H(y_3, y_2) \quad (5.52)$$

The function H is just the exponential of a quadratic form. Introducing the 2 component column vectors \vec{Q} and $\vec{y} \equiv (y^2, y^3)$ we get

$$H(y_3, y_2) = \mathcal{N} \exp \left\{ -\frac{1}{2} \vec{y}^t (\bar{R} + \bar{R}^*) \vec{y} + \vec{Q} \vec{y} \right\} \quad (5.53)$$

Finally the result of the calculation is

$$\hat{C}^{(1)}(p) = 2 \exp \left\{ -\frac{i}{4\pi} (\Delta_B p_0 p_3 + \Delta_A p_1 p_2) - \frac{1}{4} \left(\frac{\bar{p}_0^2 + \bar{p}_3^2}{\bar{f}_B} + \frac{\bar{p}_1^2 + \bar{p}_2^2}{\bar{f}_A} \right) \right\} \quad (5.54)$$

$$\hat{C}^{(2)}(p) = -2 \exp \left\{ \frac{i}{4\pi} (\Delta_B p_0 p_3 + \Delta_A p_1 p_2) - \frac{1}{4} \left(\frac{\bar{p}_0^2 + \bar{p}_3^2}{\bar{f}_B} + \frac{\bar{p}_1^2 + \bar{p}_2^2}{\bar{f}_A} \right) \right\} \quad (5.55)$$

The \bar{p} are defined in the following way. We first introduce the 1-form $\hat{p} = p_\mu dy^\mu$, and then we express it in terms of the z^a coordinates

$$\hat{p} = \bar{p}_a dz^a = p_\alpha W_a^\alpha dz^a \quad (5.56)$$

The prefactor of $\hat{C}^{(A)}(p)$ is fixed by our previous normalization which coincides with the result for $p = 0$. Notice that the difference between \bar{f}_A and \bar{f}_B is order ϵ . Thus, if we neglect this term as higher order the result simplifies and the real quadratic form in the exponent becomes

$$\frac{\sqrt{N_1 N_2} \mathcal{V}}{2\pi} \|p\|^2 = \frac{\sqrt{N_1 N_2} \mathcal{V}}{2\pi} p_\alpha p_\beta \hat{g}^{\alpha\beta} \quad (5.57)$$

6 Computation to higher orders in ϵ

The computation of the vector potential for the fractional instanton can be continued to higher orders using essentially the same strategy that was used for the calculation to order ϵ . One has to sequentially solve for $\mathcal{W}_{\mu,n}$ by the equivalent of the first equation and then solve for $S_{\mu,n}^{(a)}$ by the equivalent of the second equation. The procedure to solve for this second equation would be based on the Fourier decomposition as before. This is an inhomogeneous equation, where the known part involves the coefficients determined already at lower orders. The treatment of the unknown term $\bar{\eta}_d^{bc} \partial_b S_{c,n}^{(a)}$ is done once more in a quaternionic fashion, introducing the matrices σ and $\bar{\sigma}$ and parameterizing $S_{c,n}^{(a)}$ as follows

$$\tilde{S}^{(a)}_{,n} = \tilde{\delta} \tilde{G}_{,n} \quad (6.1)$$

where now $\tilde{G}_{,n} = G_{b,n} \sigma_b$. The equation then involves the Laplacian of $G_{b,n}$, with Fourier coefficients easily expressible in terms of those of G . Again adding a constant term to $\tilde{S}_{c,n}^{(a)}$ gives also a solution of the equation, however this is related space-time translations (see our discussion in section 4). We fix the solution uniquely by setting this constant to 0.

Now we consider the odd equations, which fix the $\mathcal{W}_{\mu,n}$. For $n > 0$ this now becomes inhomogeneous too. The part containing the unknown to be determined looks like

$$\bar{\eta}_d^{bc} \bar{D}_b \mathcal{W}_{c,n} \quad (6.2)$$

This must be expressed in quaternionic form

$$\bar{D} \tilde{\mathcal{W}}_{,n} \quad (6.3)$$

We now proceed to analyze the structure of the quaternionic operator \bar{D} . Written in matrix form we have

$$\begin{pmatrix} \bar{D}_0 + i\bar{D}_3 & i\bar{D}_1 + \bar{D}_2 \\ i\bar{D}_1 - \bar{D}_2 & \bar{D}_0 - i\bar{D}_3 \end{pmatrix} \quad (6.4)$$

Notice that the commutator of the covariant derivatives is given in terms of the abelian field. With our choices the 0 and 3 components commute with the 1 and 2 components. Concerning the two combinations one has

$$[\bar{D}_0 + i\bar{D}_3, \bar{D}_0 - i\bar{D}_3] = -\frac{4\pi\bar{f}_A}{N} \tag{6.5}$$

This looks similar to the commutation relations of creation and annihilation operators. Indeed, if we write

$$\mathbf{a} = i\sqrt{\frac{4\pi\bar{f}_A}{N}}(\bar{D}_0 + i\bar{D}_3) \tag{6.6}$$

the previous relation becomes exactly

$$[\mathbf{a}, \mathbf{a}^\dagger] = 1 \tag{6.7}$$

This formulation is particularly inspiring for physicists since we are well acquainted with the properties of creation and annihilation operators. In particular, the operator $\mathbf{a}^\dagger\mathbf{a}$ has a spectrum given by the positive integers. The lowest eigenvalue is zero and its eigenvector is the state annihilated by \mathbf{a} . A similar thing can be done for \bar{D}_1 and \bar{D}_2 generating other creation-annihilation operators \mathbf{b}^\dagger and \mathbf{b} , commuting with the previous ones. If we write back the quaternionic operator in this notation we have

$$\bar{D} = -i\sqrt{\frac{N}{4\pi}} \begin{pmatrix} \mathbf{a}/\sqrt{\bar{f}_A} & i\mathbf{b}^\dagger/\sqrt{\bar{f}_B} \\ i\mathbf{b}/\sqrt{\bar{f}_B} & \mathbf{a}^\dagger/\sqrt{\bar{f}_A} \end{pmatrix} \tag{6.8}$$

It is now clear why, at the level of the first equation, the form of $\mathcal{W}_{\mu,0}$ was taken in that particular fashion. The only non-zero component was annihilated by \mathbf{b} and \mathbf{a} . For $n > 0$ however the creation operators also contribute. An appropriate basis of the space is given by the simultaneous eigenstates of the two number operators $\mathbf{a}^\dagger\mathbf{a}$ and $\mathbf{b}^\dagger\mathbf{b}$. Since these operators commute with \mathbf{O}_α , it is enough to consider the 0–0 element of the corresponding matrix and construct the complete $N_1 \times N_2$ matrix in the same way as we did for the calculation to order $\sqrt{\epsilon}$. Thus, we define (up to a normalization)

$$\mathbf{a}^\dagger\mathbf{a}\Psi(n, n', y) = n\Psi(n, n', y) ; \quad \mathbf{b}^\dagger\mathbf{b}\Psi(n, n', y) = n'\Psi(n, n', y) \tag{6.9}$$

Then we can expand the solution $\bar{\mathcal{W}}_0$ to any order in ϵ as a linear combination

$$\bar{\mathcal{W}}_0(y) = \sum_{n, n'=0}^{\infty} c_{nn'}\Psi(n, n', y) \tag{6.10}$$

It is not difficult to construct the basis functions $\Psi(n, n', y)$ explicitly starting from the solution for $n = n' = 0$ that we used before. This can be done with the well-known formulas for the harmonic oscillator

$$\Psi(n, n', y) = \frac{1}{\sqrt{n!n'}}(\mathbf{a}^\dagger)^n(\mathbf{b}^\dagger)^{n'}\Psi(0, 0, y) \tag{6.11}$$

The procedure to solve the corresponding equation to higher orders is essentially the same detailed in the appendix of the paper [46], but generalized from 2 dimensions to 4 dimensions. For diagonal metric tensor the 4 dimensions split naturally into two 2-dimensional planes and the formulas can be obtained readily from that reference. In the general case, the 4 dimensions are intermingled and the formulas become more involved. The explicit formulas necessary to implement the iterative procedure will be given elsewhere [62].

7 Conclusions

In this paper we have set up the formalism for writing analytic formulas for the gauge potentials and field strength of (minimum action) fractional instantons for $SU(N)$ gauge theories on a 4 dimensional torus. We have given the general solution of the constant field strength type studied by 't Hooft [16]. We have clarified how self-duality implies conditions on the flat metric tensor which determines the length and scalar product of the generators of the lattice defining the torus, and expressed the general solution for this metric. Other metric tensors can be written as deformations of these solutions. The self-duality condition then gives rise to a hierarchy of equations which allows to determine the gauge potentials as a power series in the parameter controlling the deformation. The study constitutes in itself a proof that there are indeed non-constant fractional instantons for metrics not too far from those giving directly self-duality. The whole procedure generalizes the structure already devised for the $SU(2)$ case [45], and deals with the multiple complications associated to the higher rank.

The method allows multiple extensions of this work which have been left out of this paper. First of all, one can set up a methodology to extend the computation to higher orders. Something very similar was already done in the case of two-dimensional abelian Higgs vortices in ref. [46]. There we were able to go to up to order 51 in the expansion. Here of course, everything becomes more complex, so maybe one cannot go that high in the expansion. An alternative possibility based on our construction is to use a variational method. The construction privileges a certain basis in the matrix functional space. Then one could use a truncated basis space and determine the optimal values of the parameters to minimize the anti-self-dual part of the field strength. This method is worth being explored.

The present work has concentrated on determining the self-dual configuration with minimal action in the sector with $Q = 1/N$. The reason is that this configuration is essentially unique up to space-time translations and gauge transformations. Higher values of the topological charge imply a much richer moduli space. These are essentially multi-fractional instanton solutions. Their general structure could be quite rich. Studying them with our method is however feasible as was done in the simpler two-dimensional abelian multi-vortex solutions [46].

Having an analytic control of the vector potential and field strength associated to the fractional instanton opens the way to many collateral analytic calculations. For example one can compute the zero-modes of the Dirac equation both in the fundamental and in the adjoint representations. In the former case, one can use these solutions to construct the Nahm dual of the fractional instantons. This might be particularly useful for the Nahm-

self-dual cases. The adjoint zero-modes might be useful in the context of Adjoint QCD with its many attractive properties. Last but not least the formulas developed here can be analysed to see simplifications occurring in certain limits which might give rise to compact analytic expressions and connections with other instanton solutions. Both the non-abelian self-dual vortices [36–38] and the calorons [32–35] can be obtained as limiting cases of these fractional instantons on the torus. The case of calorons is particularly interesting as there are analytic solutions to which to compare.

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