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Expanding 3d $\mathcal{N} = 2$ theories around the round sphere

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Abstract: We study a perturbative expansion of the squashed 3-sphere (S_b^3) partition function of 3d $\mathcal{N} = 2$ gauge theories around the squashing parameter $b = 1$. Our proposal gives the coefficients of the perturbative expansion as a finite sum over the saddle points of the supersymmetric-localization integral in the limit $b \to 0$ (the so-called Bethe vacua), and the contribution from each Bethe vacua can be systematically computed using saddle-point methods. Our expansion provides an efficient and practical method for computing basic CFT data (F, C_T, C_{JJ} and higher-point correlation functions of the stress-energy tensor) of the IR superconformal field theory without performing the localization integrals.

Keywords: Chern-Simons Theories, Conformal Field Theory, Supersymmetric Gauge Theory

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Contents

1 Introduction and summary

1.1 The problem

The three-sphere partition function [1–4] is a powerful quantity to characterize three-dimensional $\mathcal{N} \geq 2$ supersymmetric quantum field theories. This partition function is defined as the supersymmetric partition function on the ellipsoid [4]

$$S_b^3 := \left\{ b^2|z|^2 + \frac{1}{b^2}|w|^2 = 1 \ : \ z, w \in \mathbb{C} \right\} , \tag{1.1}$$

where the "squashing parameter" b parametrizes a family of transversely holomorphic foliations on the three-sphere [5]. This geometry becomes the round 3-sphere when $b = 1$:

$$S_{b=1}^3 = (\text{round 3-sphere}) . \tag{1.2}$$

The supersymmetric partition function on the geometry (1.1) depends on a set of real mass parameters $\vec{m}$ and the R-symmetry mixing parameters $\vec{\nu}$ of the theory (see eq. (2.2)). We will denote this partition function as $\mathcal{Z}_b(\vec{m}, \vec{\nu})$, and the associated free energy by

$$F_b(\vec{m}, \vec{\nu}) = -\mathrm{Re}\left[\log \mathcal{Z}_b(\vec{m}, \vec{\nu})\right] . \tag{1.3}$$

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The problem discussed in this paper is to compute a perturbative expansion of this partition function around the special point $b = 1$, where the three-sphere (1.1) has the round metric:[1]

$$F_b(\vec{m}, \vec{\nu}) = F_{b=1}(\vec{m}, \vec{\nu}) + \frac{1}{2}(1-b)^2 F^{(2)}(\vec{m}, \vec{\nu}) + \frac{1}{3!}(1-b)^3 F^{(3)}(\vec{m}, \vec{\nu}) + \dots \,. \tag{1.4}$$

Since integral expressions for the three-sphere partition function is already known in the literature [1–4], in principle this is a matter of expanding a known expression. The integral expression, however, is given as a complicated oscillatory integral, and this makes the expansion highly non-trivial and inefficient either analytically or numerically, especially if one wishes to go to higher orders in $b-1$. The goal of this paper is to propose a different method for expanding around the value $b = 1$, which in particular does not involve any integral.

1.2 Motivations

There are two motivations for this problem.

First, the quantities appearing in the expansion (1.4) contains useful quantities characterizing the system, and our method gives an efficient and practical method to compute these quantities.

At leading order the resulting free energy $F = F_{b=1}$ on the round three-sphere, when we choose $\vec{m} = 0$ and $\vec{\nu}$ to be the IR superconformal R-charge, is known to decrease along the renormalization group (RG) flow [6, 7] (see also [8]). At the next non-trivial order, the coefficient $F^{(2)}$ is identified [9, 10] with the "central charge" C_T,[2] defined from the two-point function of the stress-energy tensor $T_{\mu\nu}$ [12]:

$$\langle T_{\mu\nu}(x) T_{\rho\sigma}(0) \rangle = C_T \frac{I_{\mu\nu,\rho\sigma}(x)}{|x|^6} \,, \tag{1.5}$$

where

$$\begin{aligned} I_{\mu\nu,\rho\sigma}(x) &:= \frac{1}{2}\left(I_{\mu\nu}(x) I_{\rho\sigma}(x) + I_{\mu\rho}(x) I_{\nu\sigma}(x) \right) - \frac{\delta_{\mu\nu}\delta_{\rho\sigma}}{3} \,, \\ I_{\mu\nu}(x) &:= \delta_{\mu\nu} - 2\frac{x_\mu x_\nu}{x^2} \,. \end{aligned} \tag{1.6}$$

The quantity C_T is a useful input for the conformal bootstrap program, see e.g. [13]. More generally, we can compute higher order terms in the expansion around $b = 1$, and extract more detailed information of the system, which is related to higher-point correlation functions of the stress-energy tensor (see e.g. [14] for related discussion in the context of conformal bootstrap, albeit in non-supersymmetric settings).

When the system has global symmetries, the three-sphere partition function depends on the corresponding real mass parameters $\vec{m}$. We can then consider the expansion of $F_{b=1}$ with respect to the parameters $\vec{m}$ around $\vec{m} = 0$. This gives $C_{J_{\mathrm{U}(1)_A} J_{\mathrm{U}(1)_B}}$, which are defined by the two-point function of conserved currents [12],

$$\langle J_I^\mu(x) J_J^\nu(0) \rangle = C_{IJ} \frac{I_{\mu\nu}(x)}{|x|^4} \,, \tag{1.7}$$

[1] There is no term linear in $b-1$, as expected from the symmetry $b \to b^{-1}$ of the geometry (1.1).
[2] There are counterexamples to the conjecture that C_T decreases along the RG flow [11].

where J_I^μ is the conserved current for the I-th Cartan generator of the flavor symmetry.

The second motivation comes from the 3d-3d correspondence [15–18], which claims that the S_b^3-partition functions of a class of 3d $\mathcal{N}=2$ theories are identified with the partition functions of the complexified Chern-Simons theory on 3-manifolds (such as knot complements). The perturbative expansion at the value $b=1$ gives infinitely many topological invariants of the 3-manifold. Note that our expansion is different from the expansion around $b=0$, which has been studied in the context of the generalized volume conjectures [19–21].

1.3 Main result

The goal of the present paper is to propose a new, efficient method to systematically compute the higher-order expansion of the three-sphere partition function in $1-b$, and hence in particular to compute F, C_T and C_{IJ}. Our expression appears in eq. (3.20). The resulting expression is written as a finite sum, as opposed to an integral (as one might expect from supersymmetric localization).

For the quantity F, our expression nicely matches the result in [22]. We reproduce their result in a different approach. Moreover our result extends the result to all-order expansion in $1-b$, and hence to infinitely many quantities.

While we defer the detailed discussion to the following sections, the basis idea is to relate the expansion around $b=1$ to a different expansion around $b=0$. Contrary to the case of the former expansion, the integrand diverges in the latter expansion, and hence we can systematically compute the higher-order expansion by the saddle point method.

It is instructive to compare our discussion with the case of four-dimensional $\mathcal{N}=1$ supersymmetry. In this case, C_T-maximization (or τ_{RR}-maximization [23] for the U(1) R-symmetry current, as related by $\mathcal{N}=1$ supersymmetry) can also be implemented by a-maximization [24], which is a rather simple algebraic (in this case cubic) function of the trial R-charge. Our result is a similar in spirit, but now in three dimensions.

Organization of the paper. The rest of this paper is organized as follows. We first quickly review the supersymmetric partition functions in section 2. The next section, section 3, contains our main result concerning the expansion around $b=1$. We motivate this result from the factorization property of the supersymmetric partition function. This property relates the expansion around $b=1$ to a different expansion around $b=0$, which can then be computed by saddle-point methods. In section 4 we apply our method to several examples. In appendix A we summarize some properties of the quantum dilogarithm function needed for the main text.

2 Localization on S_b^3 and $S^2 \times_q S^1$: review

In this section, we review supersymmetric-localization of the squashed 3-sphere partition function and the superconformal index.

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2.1 Localization on S_b^3

In this section, we give a brief review of supersymmetric localization [1–4] of 3d $\mathcal{N}=2$ gauge theories on the squashed 3-sphere S_b^3. We choose notations that will be convenient for the discussion of perturbative expansions in later sections.

We consider a 3d $\mathcal{N}=2$ gauge theory $\mathcal{T}$ with a compact connected gauge group G. For simplicity, we further assume that G is a product of a torus and a simply-connected Lie group. The theory can be coupled to background vector multiplets for the flavor symmetry group G_F of rank r_F. Let Cartan generators of G_F be $\{F_I\}_{I=1}^{r_F}$. The vacuum expectation values (VEVs) of the scalar fields in these multiplets are the real masses m_I, $I=1,\ldots,r_F$. For later use, it is convenient to rescale the real mass parameters as follows

$$U_I := b m_I \,. \tag{2.1}$$

For the geometry S_b^3 to preserve some supercharges, we need to turn on the background gauge field coupled to a $\mathrm{U}(1)_R$ symmetry. The choice of R-symmetry is not unique and can be mixed with flavor symmetries

$$R^{\vec{\nu}} = R^{\vec{\nu}=\vec{0}} + \sum_{I=1}^{r_F} \nu_I F_I \,, \tag{2.2}$$

and the partition function depends on the mixing parameters $\vec{\nu}$. Here $R^{\vec{\nu}}$ is the Cartan charge of the mixed R-symmetry $\mathrm{U}(1)_R^{\vec{\nu}}$. The mixing shifts the values of $\vec{m}$ inside the partition function:

$$\mathcal{Z}_b(\vec{m},\vec{\nu}) = \mathcal{Z}_b\left(\vec{m} + i\pi\left(b+\frac{1}{b}\right)\vec{\nu}, \vec{\nu}=\vec{0}\right) \,. \tag{2.3}$$

The mixing exists (i.e. $\nu_I \neq 0$) only for F_I which corresponds to U(1) Cartan generators of the non-simple part of the gauge group. Note that G_F includes symmetries that act on the matter as well as the topological Abelian symmetries. The topological symmetry is usually denoted as $\mathrm{U}(1)_J$, and its conserved charges are the monopole fluxes for the Abelian gauge symmetries. This implies that real masses for the topological symmetries are the Fayet-Iliopoulos (FI) parameters.

The partition function is given by an integral of the form

$$\mathcal{Z}_b(\vec{m},\vec{\nu}) = \frac{1}{|\mathrm{Weyl}(G)|}\int_{\Gamma_{\mathbb{R}}} \frac{d^{r_G}Z}{(2\pi\hbar)^{\frac{r_G}{2}}}\,\Upsilon_\hbar\left(\vec{U} + \left(i\pi+\frac{\hbar}{2}\right)\vec{\nu}, \vec{Z}\right)\Bigg|_{\vec{U}=b\vec{m}} \tag{2.4}$$

over the Coulomb-branch parameters Z_i, $i=1,\ldots,r_G$, where $r_G := \mathrm{rank}(G)$. In this formula and further in this section we use the notation

$$\hbar := 2\pi i b^2 \,. \tag{2.5}$$

The integrand $\Upsilon_\hbar$ at $\vec{\nu}=0$ in eq. (2.4) contains the following factors:

- An $\mathcal{N}=2$ chiral multiplet, that transforms with weights $(\beta,\gamma)\in(\rho_G,\rho_F)$ under the maximal tori of the gauge and the flavor groups, contributes[3]

$$\prod_{(\beta,\gamma)\in(\rho_G,\rho_F)} \psi_\hbar(\beta\cdot Z+\gamma\cdot U)\,. \tag{2.6}$$

The special function $\psi_\hbar(X)$ is the quantum dilogarithm (Q.D.L). Its definition and properties are reviewed in appendix A.

- An Abelian Chern-Simons term with a matrix of integer levels k_{ij} contributes a factor $\exp\left(\frac{1}{2\hbar}\sum_{i,j}k_{ij}Z_iZ_j\right)$. In particular, a U(1) Chern-Simons term at level k_{ii} gives $\exp\left(\frac{1}{2\hbar}k_{ii}Z_i^2\right)$, while a mixed U(1) $\times$ U(1) Chern-Simons term at level k_{ij} gives $\exp\left(\frac{1}{\hbar}k_{ij}Z_iZ_j\right)$.

- FI parameter ζ, which is the real mass parameter for the topological U$(1)_J$ symmetry, contributes a factor $\exp\left(-\frac{1}{\hbar}ZU\right)|_{U=b\zeta}$.

- A level-k Chern-Simons term for a simple factor of the gauge group G contributes $k_{ij}=k\langle\alpha_i^\vee,\alpha_j^\vee\rangle$ to the matrix of Chern-Simons levels for the maximal torus. Here $\alpha_i^\vee$ are the coroots of the simple factor and $\langle,\rangle$ is the canonically-normalized Killing form.

- An $\mathcal{N}=2$ vector multiplet contributes

$$\prod_{\lambda\in\Lambda^+_{\text{adj}}} 4\sinh\left(\frac{1}{2}\lambda\cdot Z\right)\sinh\left(\frac{\pi i}{\hbar}\lambda\cdot Z\right)\,,$$

where the product goes over the set Λ^+_{adj} of positive roots of the gauge group.

There are also contributions from the Chern-Simons terms that involve background gauge fields coupled to flavor symmetries, which we did not write out. In general, there are also Chern-Simons terms for the Levi-Civita and the R-symmetry connections. A careful treatment of these terms can be found e.g. in [22, 25]. We will not keep track of them but instead will define the partition function $\mathcal{Z}_b$ only up to an overall factor

$$\exp\left(i\pi\mathbb{Q}\left(b^2+\frac{1}{b^2}\right)+i\pi\mathbb{Q}\right)\,. \tag{2.7}$$

The integration cycle $\Gamma_\mathbb{R}$ in eq. (2.4) is $\mathbb{R}^{r_G}\subset\mathbb{C}^{r_G}$. More precisely, it is infinitesimally deformed in a suitable manner to make the integral convergent.

[3] We use the so-called "$k=-1/2$ regularization" of the chiral multiplet path-integral. See e.g. [22] for a discussion.

2.2 Localization on $S^2 \times_q S^1$

The superconformal index [26] is defined as

$$\mathcal{I}_q(\mathbf{m}_I, u_I; \vec{\nu}) := \mathrm{Tr}_{\mathcal{H}(S^2;\mathbf{m}_I)}(-1)^{R^{\vec{\nu}}} q^{\frac{R^{\vec{\nu}}}{2}+j_3} \prod_{I=1}^{r_F} u_I^{F_I} \,. \tag{2.8}$$

Here $R^{\vec{\nu}}$ is the Cartan generator of the R-symmetry $\mathrm{U}(1)_R^{\vec{\nu}}$ in eq. (2.2). $\mathcal{H}(S^2;\mathbf{m}_I)$ is the Hilbert space of the radially-quantized 3d theory with background magnetic fluxes $\{\mathbf{m}_I\}$ coupled to the Cartan generators of G_F turned on the S^2. From supersymmetric localization, the index is given as a sum/integral [27, 28]

$$\mathcal{I}_q(\vec{\mathbf{m}}, \vec{u}; \vec{\nu}) = \frac{1}{|\mathrm{Weyl}(G)|} \sum_{\vec{\mathbf{n}}} \oint \prod_{i=1}^{r_G} \frac{dw_i}{(2\pi i w_i)} \Omega_q \left(\mathbf{n}_i, w_i, \mathbf{m}_I, u_I(-q^{1/2})^{\nu_I}\right) . \tag{2.9}$$

The integrand $\Omega_q(\vec{\mathbf{n}}, \vec{w}, \vec{\mathbf{m}}, \vec{u})$ at $\vec{\nu} = 0$ contains the following factors:

- An $\mathcal{N} = 2$ chiral multiplet that transforms with weights $(\beta, \gamma) \in (\rho_G, \rho_F)$ under the maximal tori of the gauge and the flavor groups contributes

  $$\prod_{(\beta,\gamma)\in(\rho_G,\rho_F)} \mathcal{I}_\Delta(\beta \cdot \vec{\mathbf{n}} + \gamma \cdot \vec{\mathbf{m}}, e^{\beta\cdot \log w + \gamma \cdot \log u}; q) \,, \tag{2.10}$$

  where the "tetrahedron index" $\mathcal{I}_\Delta$ is defined as [29]

  $$\mathcal{I}_\Delta(\mathbf{m}, u; q) := \prod_{r=0}^{\infty} \frac{1 - q^{r-\frac{\mathbf{m}}{2}+1} u^{-1}}{1 - q^{r-\frac{\mathbf{m}}{2}} u} \,. \tag{2.11}$$

- An Abelian Chern-Simons term with a matrix of integer levels k_{ij} contributes a factor $\prod u_i^{k_{ij}\mathbf{m}_j}$.

- The $\mathrm{U}(1)_J$ topological symmetry associated to an Abelian gauge symmetry contributes $u^{-\mathbf{n}} w^{-\mathbf{m}}$. Here, $(\mathbf{m}, u)$ is a pair (magnetic flux, fugacity) for the $\mathrm{U}(1)_J$ flavor symmetry, while $(\mathbf{n}, w)$ is for the Abelian gauge symmetry.

- A level-k Chern-Simons term for a simple factor of G contributes $k_{ij} = k\langle \alpha_i^\vee, \alpha_j^\vee \rangle$ to the matrix of Chern-Simons levels for the maximal torus. Here $\alpha_i^\vee$ are the coroots of the simple factor and $\langle , \rangle$ is the canonically-normalized Killing form.

- An $\mathcal{N} = 2$ vector multiplet contributes

  $$\prod_{\lambda \in \Lambda^+_{\mathrm{adj}}} (q^{\lambda\cdot(\frac{\mathbf{n}}{2}+\log w)} - q^{-\lambda\cdot(\frac{\mathbf{n}}{2}+\log w)})(q^{\lambda\cdot(\frac{\mathbf{n}}{2}-\log w)} - q^{-\lambda\cdot(\frac{\mathbf{n}}{2}-\log w)}) \,,$$

  where the product goes over the set Λ^+_{adj} of positive roots of the gauge group.

We need to sum over all the magnetic fluxes $\vec{\mathbf{n}}$ satisfying the following Dirac quantization conditions

$$\left\{ \vec{\mathbf{n}} \ : \ \beta \cdot \vec{\mathbf{n}} + \gamma \cdot \vec{\mathbf{m}} \in \mathbb{Z} \,, \lambda \cdot \vec{\mathbf{n}} \in \mathbb{Z} \,, \ \forall (\beta, \gamma) \in (\rho_G, \rho_F) \text{ and } \lambda \in \Lambda^+_{\mathrm{adj}} \right\} / \, \mathrm{Weyl}(G) \,. \tag{2.12}$$

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3 Expansion at $b \to 1$ from expansion at $b \to 0$

In this section, as the main result of this paper, we propose the $b \to 1$ expansion (3.20) of the squashed 3-sphere partition (2.4). Using the proposed expansion, we can express correlation functions of the stress-energy tensor (including the stress-energy tensor central charge) of a 3d $\mathcal{N} = 2$ theory as a finite sum over Bethe-vacua. Our proposal is motivated by the factorization property of supersymmetric partition functions [30, 31]. Interestingly, we find that the convergent expansion at $b \to 1$ can be reconstructed from the asymptotic expansion at $b \to 0$.

3.1 Expansion at $b \to 0$ (asymptotic expansion)

In this section, we provide general strategy for computing the perturbative expansion $S_n^{(\alpha,\vec{\ell})}$ ($n \geq 0$) of the localization integral (2.4) in the $b \to 0$ limit around saddle point $\vec{Z}^{(\alpha,\vec{\ell})}$ in eq. (3.11). Here the index α labels Bethe-vacua (3.8) of the 3d theory while integer-valued vector $\vec{\ell}$ labels the unphysical shift ambiguity (3.6). For a 3d theory associated to a hyperbolic 3-manifold [16, 32, 33], the perturbative invariant $S_n^{(\alpha,\vec{\ell})}$ computes the n-loop invariant of the three-dimensional $\mathrm{SL}(N,\mathbb{C})$ Chern-Simons theory around an irreducible flat connection $\mathcal{A}^\alpha$ (satisfying $d\mathcal{A}^\alpha + \mathcal{A}^\alpha \wedge \mathcal{A}^\alpha = 0$) on the 3-manifold. We refer to [19–21, 32, 34–38] for systematic study on the perturbative invariants in the context of the volume conjecture [39, 40].

Quantum effective twisted superpotential. We define the quantum effective twisted superpotential [31, 41–43] in perturbative expansion in $\hbar$,

$$\mathcal{W}_\hbar^{\vec{\ell}}(\vec{Z},\vec{U},\vec{\nu}) \xrightarrow{\hbar\to 0 \text{ with fixed } \vec{U}=b\vec{m}} \sum_{n=0}^{\infty} \hbar^n \mathcal{W}_n^{\vec{\ell}}(\vec{Z},\vec{U},\vec{\nu})\,. \tag{3.1}$$

It can be obtained from the limit of the integrand $\Upsilon_\hbar$ in eq. (2.4) at $\hbar \to 0$,

$$\Upsilon_\hbar\left(\vec{U} + \left(i\pi + \frac{\hbar}{2}\right)\vec{\nu}, \vec{Z}\right) \xrightarrow{\hbar\to 0 \text{ with fixed } \vec{U}=b\vec{m}} \exp\left(\hbar^{-1}\mathcal{W}_\hbar^{\vec{\ell}}(\vec{Z},\vec{U},\vec{\nu})\right)\,. \tag{3.2}$$

The contribution of Chern-Simons terms to $\mathcal{W}_n$ is clear. Most of them contribute only to $\mathcal{W}_0$, with the exception of the ones that involve the R-symmetry. Those can also contribute to $\mathcal{W}_1$. For example, a level k term for a U(1) symmetry with parameter Z gives

$$\mathcal{W}_\hbar^{\rm CS} = \frac{1}{2}kZ^2\,. \tag{3.3}$$

The contribution of a charged $\mathcal{N} = 2$ chiral multiplet to $\mathcal{W}_\hbar$ can be read off from the expansion of the quantum dilogarithm function $\psi_\hbar(Z)$, which is reviewed in appendix A. A chiral multiplet of charge one under a U(1) symmetry with parameter Z gives

$$\mathcal{W}_\hbar^{\rm chiral} = \sum_{n=0}^{\infty} \hbar^n \frac{B_n}{n!}\mathrm{Li}_{2-n}(e^{-Z})\,, \quad n = 0, 1, \dots\,, \tag{3.4}$$

where $B_n = (1, 1/2, 1/6, \dots)$ are the Bernoulli numbers. Finally, the W-bosons contribute

$$\mathcal{W}_\hbar^{\mathrm{W}} = 2\pi i \rho \cdot Z + \hbar \sum_{\lambda \in \Lambda_{\mathrm{adj}}^+} \log \left(e^{\lambda \cdot Z/2} - e^{-\lambda \cdot Z/2} \right) , \tag{3.5}$$

where $\rho = \frac{1}{2} \sum_{\lambda \in \Lambda_{\mathrm{adj}}^+} \lambda$ is the Weyl vector.

The quantum effective twisted superpotential $\mathcal{W}_\hbar^{\vec{\ell}}(\vec{U}, \vec{Z})$ is defined up to an unphysical shift,

$$\mathcal{W}_\hbar^{\vec{\ell}} = \mathcal{W}_\hbar^{\vec{\ell}=\vec{0}} + 2\pi i (Z_i \ell_z^i + U_I \ell_u^I + 2\pi i \ell_0) + i\pi\hbar\ell_1 \, , \quad \ell_z^i,\ \ell_u^I,\ \ell_0,\ \ell_1 \in \mathbb{Z} \, . \tag{3.6}$$

In particular, $\mathcal{W}_\hbar$ is not single-valued and has branch points at $2\pi i \mathbb{Z}$ both in Z_i and U_I, and in going around these, $\mathcal{W}_\hbar$ gets shifted by terms of the form (3.6). Also in the equation (3.2), the result actually depends on the direction (phase factor of $\hbar$) in which one takes the limit, but only by terms of the form (3.6).

There can also be contributions from Chern-Simons terms in background supergravity fields. These are physically meaningful, but we will not keep track of them. Instead, we will allow an ambiguity

$$\mathcal{W}_\hbar \sim \mathcal{W}_\hbar + \pi^2 \mathbb{Q} + i\pi\hbar\mathbb{Q} + \hbar^2 \mathbb{Q} \, . \tag{3.7}$$

Bethe-vacua $\mathcal{S}_{\mathrm{BE}}(\vec{U}, \vec{\nu})$. Bethe-vacua are defined by

$$\begin{aligned} \mathcal{S}_{\mathrm{BE}}(\vec{U}, \vec{\nu}) &:= \left\{ \vec{z}_0 \ : \ \exp(\partial_{\vec{Z}} \mathcal{W}_0(\vec{Z}, \vec{U}, \vec{\nu}))\big|_{\vec{Z} = \log \vec{z}_0} = \vec{1}, \ \text{triv. isotr.} \right\} / \, \mathrm{Weyl}(G) \\ &= \left\{ \vec{z}^{(\alpha)} \right\}_{\alpha=1}^{|\mathcal{S}_{\mathrm{BE}}(\vec{U},\vec{\nu})|} . \end{aligned} \tag{3.8}$$

Here "triv. isotr." stands to indicate that Bethe solutions that are invariant under a non-trivial subgroup of the Weyl group should be discarded, since they do not correspond to physical vacua. (See e.g. [22] for a discussion and references.) We further assume that the background parameters $\vec{U}$ can be chosen in such a way that all Bethe vacua are massive. Note that the above equations are independent of the unphysical shift ambiguities in eq. (3.6). The number of Bethe-vacua at generic $\vec{U}$ is equal to the Witten index [44, 45]

$$(\text{Witten index}) = |\mathcal{S}_{\mathrm{BE}}| \, . \tag{3.9}$$

Perturbative expansion $\{S_n^{(\alpha,\vec{\ell})}(\vec{U}, \vec{\nu})\}$. We consider a formal perturbative expansion of the localization integral

$$\begin{aligned} &\frac{1}{|\mathrm{Weyl}(G)|} \int \frac{d^{r_G} \delta Z}{(2\pi\hbar)^{r_G/2}} \Upsilon_\hbar \left(\vec{U} + \left(i\pi + \frac{\hbar}{2} \right) \vec{\nu}, \vec{Z} = \vec{Z}^{(\alpha,\vec{\ell})} + \delta \vec{Z} \right) \\ &\xrightarrow{\hbar \to 0 \text{ with fixed } \vec{U} = b\vec{m}} \mathcal{Z}_{\mathrm{pert}}^{(\alpha,\vec{\ell})}(\vec{U}, \vec{\nu}; \hbar) = \exp \left(\sum_{n=0}^{\infty} \hbar^{n-1} S_n^{(\alpha,\vec{\ell})}(\vec{U}, \vec{\nu}) \right) \end{aligned} \tag{3.10}$$

in the limit $\hbar \to 0$ around a saddle point $\vec{Z} = Z^{(\alpha,\vec{\ell})}(\vec{U}, \vec{\nu})$ which satisfies

$$\partial_{\vec{Z}} \mathcal{W}_0^{\vec{\ell}}(\vec{Z}, \vec{U}, \vec{\nu})\big|_{\vec{Z} = \vec{Z}^{(\alpha,\vec{\ell})}}(\vec{U}, \vec{\nu}) = 0 \, . \tag{3.11}$$

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For a given Bethe vacuum $\vec{z}^{\alpha} \in \mathcal{S}_{\rm BE}(\vec{U},\vec{\nu})$, there is an associated logarithmic saddle point $Z^{(\alpha,\vec{\ell})}$ satisfying $e^{\vec{Z}^{(\alpha,\vec{\ell})}} = \vec{z}^{\alpha}$ upon a proper choice of shift ambiguities $\vec{\ell}$. The first two perturbative coefficients $S_0^{(\alpha,\vec{\ell})}$ and $S_1^{(\alpha,\vec{\ell})}$ depend on the choices $\vec{\ell}$ while higher loop coefficients do not. The dependence is of the following form:

$$\sum_{n=0}^{\infty} S_n^{(\alpha,\vec{\ell})}(\vec{U},\vec{\nu})\hbar^n = \sum_{n=0}^{\infty} S_n^{(\alpha,\vec{\ell}=\vec{0})}(\vec{U},\vec{\nu})\hbar^n + 2\pi i(U_I \ell_u^I + 2\pi i \ell_0) + i\pi\hbar\ell_1\,, \quad \ell_u^I,\, \ell_0,\, \ell_1 \in \mathbb{Z}\,. \tag{3.12}$$

The actual localization integral (2.4) along the physical cycle $\Gamma_{\mathbb{R}}$ can be asymptotically expanded in terms of the formal perturbative expansion

$$\mathcal{Z}_b(\vec{m},\vec{\nu}) \xrightarrow{\;b^2\to 0 \text{ with fixed } \vec{U}=b\vec{m}\;} \sum_{\alpha} n_\alpha \mathcal{Z}_{\rm pert}^{(\alpha,\vec{\ell})}(\vec{U},\vec{\nu};\hbar)\big|_{\hbar=2\pi i b^2}\,, \tag{3.13}$$

where the integer coefficients n_α and the shift ambiguity $\vec{\ell}$ depends on the direction (phase factor of b^2) of the limit. For our purpose (obtaining the $b\to 1$ expansion), the actual values of them are not relevant and we only need to know the formal perturbative expansion modulo the ambiguities in eq. (3.12).

Now let us give a more explicit formula for the formal perturbative expansion. For the perturbative expansions, we first expand the integrand around a saddle point $\vec{Z}^{(\alpha,\vec{\ell})}$,

$$\begin{aligned}
&\exp\left(\sum_{n=0}^{\infty} \hbar^{n-1}\mathcal{W}_n(\vec{Z}^{(\alpha,\vec{\ell})} + \hbar^{\frac{1}{2}}\delta\vec{\zeta})\right) \\
&= \exp\Bigg(\sum_{n=0}^{\infty} \mathcal{W}_n(\vec{Z}^{(\alpha,\vec{\ell})})\hbar^{n-1} - \frac{1}{2}\delta\zeta_i \Pi^{ij}(\vec{Z}^{(\alpha,\vec{\ell})})\delta\zeta_j \\
&\qquad + \sum_{n=1}^{\infty} \sum_{\substack{1\le m\le n+2;\\ m-n\in 2\mathbb{Z}}} \hbar^{n/2} C_{n,m}^{i_1\ldots i_m}(\vec{Z}^{(\alpha,\vec{\ell})})\delta\zeta_{i_1}\ldots\delta\zeta_{i_m}\Bigg) \\
&= \exp\left(\sum_{n=0}^{\infty} \mathcal{W}_n(\vec{Z}^{(\alpha,\vec{\ell})})\hbar^{n-1} - \frac{1}{2}\delta\zeta_i \Pi^{ij}(\vec{Z}^{(\alpha,\vec{\ell})})\delta\zeta_j\right) \\
&\quad \times \left(1 + \sum_{n=1}^{\infty} \sum_{\substack{1\le m\le 3n\\ m-n\in 2\mathbb{Z}}} \hbar^{\frac{n}{2}} D_{n,m}^{i_1\ldots i_m}(\vec{Z}^{(\alpha,\vec{\ell})})\delta\zeta_{i_1}\ldots\delta\zeta_{i_m}\right).
\end{aligned} \tag{3.14}$$

In the expansion, the quadratic terms $O(\delta Z^2)$ of the classical part $\mathcal{W}_0$ play the role of the inverse propagator

$$\Pi^{ij}(\vec{Z}^{(\alpha,\vec{\ell})}) := -\frac{\partial^2 \mathcal{W}_0^{\vec{\ell}}}{\partial Z_i \partial Z_j}\bigg|_{\vec{Z}=\vec{Z}^{(\alpha,\vec{\ell})}} \qquad \text{(inverse propagator)}\,, \tag{3.15}$$

while other terms play the role of the interaction vertices

$$C_{n,m}^{i_1\ldots i_m}(\vec{Z}^{(\alpha,\vec{\ell})}) := \frac{1}{m!}\frac{\partial^m \mathcal{W}^{\vec{\ell}}_{\frac{n-m}{2}+1}}{\partial Z_{i_1}\ldots\partial Z_{i_m}}\bigg|_{\vec{Z}=\vec{Z}^{(\alpha,\vec{\ell})}} \qquad \text{(vertices)}\,. \tag{3.16}$$

The coefficients $D^{i_1,\ldots,i_n}_{n,m}$ in eq. (3.14) can be written as a finite sum of finite products of the C's. Then the formal expansion $\mathcal{Z}^{(\alpha,\vec{\ell})}_{\text{pert}}(\hbar)$ around the saddle point $\vec{Z}^{(\alpha,\vec{\ell})}$ is given by[4]

$$
\begin{aligned}
\mathcal{Z}^{(\alpha,\vec{\ell})}_{\text{pert}} &= \exp\left(\sum_{n=0}^{\infty}\hbar^{n-1}S_n^{(\alpha,\vec{\ell})}\right)\\
&= \exp\left(\sum_{n=0}^{\infty}\hbar^{n-1}\mathcal{W}_n^{\vec{\ell}}(\vec{Z}^{(\alpha,\vec{\ell})})\right)\int\prod_{i=1}^{r_G}\frac{d(\delta\zeta_i)}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}\delta\zeta_i\Pi^{ij}(\vec{Z}^{(\alpha,\vec{\ell})})\delta\zeta_j\right)\\
&\quad\times\left(1+\sum_{n=1}^{\infty}\sum_{m=1}^{3n}\hbar^n D^{i_1\ldots i_{2m}}_{2n,2m}(\vec{Z}^{(\alpha,\vec{\ell})})\delta\zeta_{i_1}\ldots\delta\zeta_{i_{2m}}\right), \qquad (3.17)\\
&= \exp\left(-\frac{1}{2}\log\det\Pi(\vec{Z}^{(\alpha,\vec{\ell})})+\sum_{n=0}^{\infty}\hbar^{n-1}\mathcal{W}_n^{\vec{\ell}}(\vec{Z}^{(\alpha,\vec{\ell})})\right)\\
&\quad\times\left(1+\sum_{n=1}^{\infty}\sum_{m=1}^{3n}\hbar^n D^{i_1\ldots i_{2m}}_{2n,2m}(\vec{Z}^{(\alpha,\vec{\ell})})G_{i_1,\ldots,i_{2m}}(\vec{Z}^{(\alpha,\vec{\ell})})\right).
\end{aligned}
$$

To arrive at the last line, we performed the formal Gaussian integrals

$$
\begin{aligned}
G_{i_1,\ldots,i_{2m}} &:= (\det\Pi)^{\frac{1}{2}}\int\prod_{i=1}^{r_G}\frac{d\delta\zeta_i}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}\delta\zeta_i\Pi^{ij}\delta\zeta_j\right)\delta\zeta_{i_1}\ldots\delta\zeta_{i_{2m}}\\
&= \frac{\partial^{2m}\left(\exp\left(\frac{1}{2}\mu^i(\Pi^{-1})_{ij}\mu^j\right)\right)}{\partial\mu^{i_1}\ldots\partial\mu^{i_{2m}}}\bigg|_{\mu_i=0}.
\end{aligned} \qquad (3.18)
$$

The classical and the one-loop contributions are

$$
S_0^{(\alpha,\vec{\ell})}(\vec{U},\vec{\nu}) = \mathcal{W}_0^{\vec{\ell}}(\vec{Z}^{(\alpha,\vec{\ell})}), \quad S_1^{(\alpha,\vec{\ell})}(\vec{U},\vec{\nu}) = \mathcal{W}_1^{\vec{\ell}}(\vec{Z}^{(\alpha,\vec{\ell})})-\frac{1}{2}\log\det\left(\Pi(\vec{Z}^{(\alpha,\vec{\ell})})\right). \qquad (3.19)
$$

The terms of higher order in $\hbar$ are defined using $\partial^{\geq 2}\mathcal{W}_0$, $\partial^{\geq 1}\mathcal{W}_1$ and $\partial^{\geq 0}\mathcal{W}_{\geq 2}$ only, and therefore are not affected by the unphysical shifts (3.6).

3.2 Expansion at $b \to 1$ (convergent expansion)

Unlike the $b \to 0$ limit, we can not use the saddle-point approximation around $b = 1$ since the integrand is smooth around the point. The analytic evaluation of the integral around $b = 1$ therefore seems to be intractable as we need to evaluate a generic finite-dimensional integral involving special functions. Most of the interesting physical quantities which can be extracted from the partition function come from expansion around $b = 1$, where the space-time geometry becomes conformally flat. So far, people have heavily relied on numerical approach in computing the physical quantities.

However, as already noticed in some literature, the localization integrals are not generic finite-dimensional integral but have several non-trivial hidden structures. One non-trivial property relevant to us is the factorization property [30, 31]. As we will see below, the

[4]The overall factor 1/—Weyl(G)— does not appear here since there are —Weyl(G)— many saddle points (related by the Weyl group action) of localization integral for each Bethe-vacua, all of which give the same contributions to the asymptotic expansion.

property is so restrictive that we can determine the localization integral around $b=1$ from the asymptotic expansion in the $b\to 0$ limit without performing actual integration.

As a main result of this paper, we propose that

$$\begin{aligned}\mathcal{Z}_b(\vec{m},\vec{\nu}_0)&\simeq \sum_{\alpha\in\mathcal{S}_{\rm BE}}\exp\left[\sum_{n=0}^{\infty}\epsilon^n s_n^{(\alpha)}(\vec{m},\vec{\nu}_0)\right]\quad(\text{for } |\epsilon=1-b|<1)\\ &:=\sum_{\alpha\in\mathcal{S}_{\rm BE}}\exp\left[\sum_{n=0}^{\infty}\left(\hbar_1^{n-1}S_n^{(\alpha,\vec{\ell})}(\vec{U}=b\vec{m},\vec{\nu}_0)+\hbar_2^{n-1}S_n^{(\alpha,\vec{\ell})}(\vec{U}=b^{-1}\vec{m},\vec{\nu}_0)\right)\right],\end{aligned}$$

where

$$\begin{aligned}&\hbar_1:=2\pi i(b^2-1)=2\pi i(-2\epsilon+\epsilon^2)\,,\\ &\hbar_2:=2\pi i(b^{-2}-1)=2\pi i\frac{2\epsilon-\epsilon^2}{(1-\epsilon)^2}=2\pi i(2\epsilon+3\epsilon^2+4\epsilon^3+\ldots)\,.\end{aligned}\tag{3.20}$$

Here $\simeq$ means equality up to an unphysical phase factor

$$\mathcal{Z}_1\simeq\mathcal{Z}_2 \text{ if } \mathcal{Z}_1=e^{i\pi\delta}\mathcal{Z}_2 \text{ with } \delta\in\mathbb{Q}\,.\tag{3.21}$$

The above formula holds only when the R-symmetry mixing parameters $\{\nu_I\}_{I=1}^{r_F}$ in eq. (2.2) are cleverly chosen $\vec{\nu}=\vec{\nu}_0$ such that

$$R^{\vec{\nu}_0}+2j_3\in 2\mathbb{Z}\,,\quad \text{for all 1/4-BPS local operators}\,.\tag{3.22}$$

This guarantees the following condition[5]

$$\text{only } q^{\rm integer} \text{ appears in } \mathcal{I}_q(\mathbf{m}_I=0,u_I;\vec{\nu}_0)\,.\tag{3.23}$$

Unlike $S_n^{(\alpha,\vec{\ell})}$, the perturbative coefficients $s_n^{(\alpha)}$ are independent on the choices of $\vec{\ell}$ if the above conditions (3.23) are met. Let us give explicit expressions for the first few coefficients $s_n^{(\alpha)}$ in terms of $S_n^{(\alpha,\vec{\ell})}$,

$$\begin{aligned}&s_0^{(\alpha)}(\vec{m},\vec{\nu}_0)=\frac{i}{2\pi}\left(S_0^{(\alpha,\vec{\ell})}-U_I\partial_I S_0^{(\alpha,\vec{\ell})}\right)+2S_1^{(\alpha,\vec{\ell})}\Big|_{U_I=m_I}\,,\\ &s_1^{(\alpha)}(\vec{m},\vec{\nu}_0)=0\,,\\ &s_2^{(\alpha)}(\vec{m},\vec{\nu}_0)=s_3^{(\alpha)}(\vec{m},\vec{\nu}_0)\\ &\quad=-32\pi^2S_3^{(\alpha)}+U_I\partial_I S_1^{(\alpha,\vec{\ell})}+8\pi i\left(S_2^{(\alpha)}+U_I\partial_I S_2^{(\alpha)}\right)+U_IU_J\partial_I\partial_J S_1^{(\alpha,\vec{\ell})}\\ &\qquad-\frac{i}{12\pi}U_IU_JU_K\partial_I\partial_J\partial_K S_0^{(\alpha,\vec{\ell})}\Big|_{U_I=m_I}\,.\end{aligned}\tag{3.24}$$

The main result above is valid even for complex $\vec{m}$. This means that if one wants to compute the $\mathcal{Z}_b(\vec{m},\vec{\nu})$ with general choice of R-charge mixing $\vec{\nu}$ which does not satisfy the above condition (3.23), we can use eq. (2.3)

$$\mathcal{Z}_b\left(\vec{m},\vec{\nu}\right)=\mathcal{Z}_b\left(\vec{m}+i\pi\left(b+\frac{1}{b}\right)(\vec{\nu}-\vec{\nu}_0)\,,\vec{\nu}_0\right)\,,\tag{3.25}$$

with $\vec{\nu}_0$ satisfying the condition in eq. (3.23).

[5]Upon a generic choice of $\vec{\nu}$, the superconformal index contains a term q^α with generic real number α.

Derivation from factorization. Our proposal can be derived from the conjectured factorization property of the S_b^3 partition function [30, 31]. This states the following factorization:

$$\begin{aligned}&\mathcal{Z}_b(\vec{m},\vec{\nu}) = \sum_{\alpha\in\mathcal{S}_{\text{BE}}} B^{(\alpha)}\left(b^2,\vec{u};\vec{\nu}\right) B^{(\alpha)}\left(b^{-2},\vec{\tilde{u}};\vec{\nu}\right) \text{ where}\\ &u_I := e^{U_I} := e^{bm_I},\ \tilde{u}_I := e^{\tilde{U}_I} := e^{b^{-1}m_I}\,.\end{aligned} \tag{3.26}$$

The building block $B^{(\alpha)}$, which is often called the holomorphic block in the literature, can be considered as the partition function of the 3d $\mathcal{N}=2$ theory on $\mathbb{R}^2_\hbar \times S^1$, with an omega-deformation parameter $\hbar := 2\pi i b^2$ and with an asymptotic boundary condition determined by the Bethe vacuum labelled by α:

$$B^{(\alpha)}(b^2,\vec{u};\vec{\nu}) = \text{Tr}_{\mathcal{H}(\mathbb{R}^2;\alpha)}(-1)^{R^{\vec{\nu}}} q^{\frac{R^{\vec{\nu}}}{2}+j_3}\prod_I u_I^{F_I}|_{u_I=e^{bm_I}}\,, \quad q := e^{2\pi i b^2}\,. \tag{3.27}$$

The perturbative expansion of the block $B^\alpha(b^2,\vec{u};\vec{\nu})$ in the limit $b^2\to 0$ with fixed $\vec{u}$ is expected to be identical to the asymptotic expansion $\mathcal{Z}^{(\alpha,\vec{\ell})}_{\text{pert}}(\vec{U},\vec{\nu};\hbar)$ in eq. (3.17) of the $\mathcal{Z}_b$ in the limit $b^2\to 0$ with fixed $U_I = bm_I$ around the Bethe vacuum:

$$\begin{aligned}&\mathcal{Z}^{(\alpha,\vec{\ell})}_{\text{pert}}(\vec{U},\vec{\nu};\hbar) \sim B^{(\alpha)}(b^2,\vec{u}=e^{\vec{U}};\vec{\nu})\\ &\xrightarrow{b^2\to 0 \text{ with fixed } U_I=bm_I} \exp\left(\sum_{n=1}^{\infty}\hbar^{n-1}S_n^{(\alpha,\vec{\ell})}(\vec{U},\vec{\nu})\right)\Bigg|_{\hbar=2\pi i b^2}\,.\end{aligned} \tag{3.28}$$

Here $\sim$ means equality as an asymptotic expansion in the above limit modulo unphysical factor of the form in eq. (3.12). This is because of that the anti-holomorphic part in (3.26) become trivial in the asymptotic limit:

$$B^{(\alpha)}\left(b^{-2},\vec{\tilde{u}}=e^{\vec{\tilde{U}}};\vec{\nu}\right) \xrightarrow{b^2=-i\delta\to 0 \text{ with fixed } U_I=b^2\tilde{U}_I\in\mathbb{R}} 1+o(e^{-\sharp\frac{2\pi}{\delta}})\,. \tag{3.29}$$

More precisely, the above triviality is expected when the limit $b^2\to 0$ is taken along the direction $b^2\in -i\mathbb{R}_+$ with purely imaginary $\tilde{U}_I$ and the R-charge mixing $\vec{\nu}$ is chosen such that

$$B^{(\alpha)}\left(b^2=\frac{1}{2\pi i}\log q,\vec{u};\vec{\nu}\right) = 1+(\ldots)q^{\sharp>0}+(\text{higher-order in } q)\,. \tag{3.30}$$

The form of the expansion above is expected when the R-charge mixing satisfies the unitarity constraints from superconformal algebra, such as $R^{\vec{\nu}}(\mathcal{O})\geq\frac{1}{2}$ for all chiral primary operator $\mathcal{O}$. When the condition (3.23) is satisfied, $\vec{\nu}=\vec{\nu}_0$, the block depends on only $q=e^{2\pi i b^2}$ (instead of b^2) and thus

$$\text{For } \vec{\nu}_0 \text{ in eq. (3.23)}\,,\qquad \begin{aligned}B^{(\alpha)}(b^2,\vec{u};\vec{\nu}_0) &= B^{(\alpha)}(b^2-1,\vec{u};\vec{\nu}_0)\,,\\ B^{(\alpha)}(b^{-2},\vec{\tilde{u}};\vec{\nu}_0) &= B^{(\alpha)}(b^{-2}-1,\vec{\tilde{u}};\vec{\nu}_0)\,.\end{aligned} \tag{3.31}$$

Combining eq. (3.26), (3.28), (3.31) and the fact that both of $\hbar_1 = 2\pi i(b^2-1)$ and $\hbar_2 = 2\pi i(b^{-2}-1)$ approach to 0 at $b\to 1$, we derive the expansion in eq. (3.20). Since the $b\to 1$ limit of $\mathcal{Z}_b$ should be smooth, the final result is valid for any direction in the limit $b\to 1$.

Comparison with Closset-Kim-Willett [22]. According to [22], the squashed 3-sphere partition function $\mathcal{Z}_b$ at $b=1$ can be written as follows

$$\mathcal{Z}_{b=1}(\vec{m},\vec{\nu}) = \mathcal{Z}^{\nu_R=0}_{\mathcal{M}_{g=0,p=1}}(\vec{m},\vec{\nu}) = \sum_{\alpha\in\mathcal{S}_{\text{BE}}} \left(\mathcal{H}^\alpha_{\nu_R=0}(\vec{m},\vec{\nu})\right)^{-1} \mathcal{F}^\alpha_{\nu_R=0}(\vec{m},\vec{\nu})\,. \tag{3.32}$$

Here $\mathcal{H}^\alpha_{\nu_R}$ and $\mathcal{F}^\alpha_{\nu_R}$ are so-called 'handle-gluing' and 'fibering' operators respectively. These operators serve as basic building blocks of more general supersymmetric partition functions $\mathcal{Z}^{\nu_R}_{\mathcal{M}_{g,p}}$ on $\mathcal{M}_{g,p}$:

$$\begin{aligned} \mathcal{M}_{g,p} &:= (S^1\text{-bundle of degree } p \text{ over a Riemann surface } \Sigma_g)\,, \\ \mathcal{Z}^{\nu_R}_{\mathcal{M}_{g,p}} &= \sum_{\alpha\in\mathcal{S}_{\text{BE}}} (\mathcal{H}^\alpha_{\nu_R})^{g-1}(\mathcal{F}^\alpha_{\nu_R})^p\,. \end{aligned} \tag{3.33}$$

For the case when the cycle $[S^1]$ along the fiber is nontrivial in $H_1(\mathcal{M}_{g,p},\mathbb{Z}_2)$ (i.e. p is even), there are two types of supersymmetric backgrounds depending on the choice of spin-structure along the S^1 [25]. Depending on the discrete choice, ν_R can be either 0 (periodic boundary condition for fermions) or 1/2 (anti-periodic boundary condition). When $\nu_R = 1/2$, we need to turn on the discrete $\mathbb{Z}_2$ holonomy along the $[S^1]$ coupled to a $\mathrm{U}(1)^{\vec{\nu}}_R$ symmetry to preserve some supercharges. When the fiber $[S^1]$ is trivial in $H_1(\mathcal{M}_{g,p},\mathbb{Z}_2)$ (i.e. p is odd), only the choice $\nu_R=0$ is allowed. For $\mathcal{M}_{g,p=0}=\Sigma_g\times S^1$ case, for example, the two different choices correspond to the following two types of twisted indices

$$\begin{aligned} \mathcal{Z}^{\nu_R=0}_{\mathcal{M}_{g,p=0}}(\vec{m},\vec{\nu}) &= \text{Tr}_{\mathcal{H}^{\text{top}}(\Sigma_g;\vec{\nu})}(-1)^{2j_3}\prod_{I=1}^{r_F} e^{m_I F_I}\,, \\ \mathcal{Z}^{\nu_R=1/2}_{\mathcal{M}_{g,p=0}}(\vec{m},\vec{\nu}) &= \text{Tr}_{\mathcal{H}^{\text{top}}(\Sigma_g;\vec{\nu})}(-1)^{R^{\vec{\nu}}}\prod_{I=1}^{r_F} e^{m_I F_I}\,. \end{aligned} \tag{3.34}$$

Here $\mathcal{H}^{\text{top}}(\Sigma_g,\vec{\nu})$ is the Hilbert space of the topologically-twisted 3d $\mathcal{N}=2$ theory on a Riemann surface Σ_g of genus g. To preserve some supercharges, we turn on the background magnetic flux coupled to the $\mathrm{U}(1)^{\vec{\nu}}_R$ R-symmetry in eq. (2.3).

$$\frac{1}{2\pi}\int_{\Sigma_g} dA_{\mathrm{U}(1)^{\vec{\nu}}_R} = (g-1)\,. \tag{3.35}$$

Due to the background magnetic flux, the twisted index is well-defined only when the mixing parameters $\vec{\nu}$ are chosen to satisfy following Dirac quantization condition

$$R^{\vec{\nu}}(\mathcal{O})\times(g-1)\in\mathbb{Z} \tag{3.36}$$

for all local operators $\mathcal{O}$ in the 3d theory.

From the comparison between eq. (3.19) and the explicit expression for $\mathcal{H}^\alpha$ and $\mathcal{F}^\alpha$ given in [25], it is straightforward to check that

$$\mathcal{F}^\alpha_{\nu_R=1/2} = \exp\left(i\frac{S_0^{(\alpha,\vec{\ell})} - U_I\partial_I S_0^{(\alpha,\vec{\ell})}}{2\pi}\right)\Bigg|_{U_I=m_I}\,, \quad \mathcal{H}^\alpha_{\nu_R=1/2} = \exp\left(-2S_1^{(\alpha,\vec{\ell})}\right)\Big|_{U_I=m_I}\,. \tag{3.37}$$

One simpler way to understand the second equality is using a factorization of refined twisted index on S^2 as explained in [25]. The refined index is defined as

$$\mathcal{I}_q^{\text{top}}(\vec{m},\vec{\nu}) := \text{Tr}_{\mathcal{H}^{\text{top}}(\Sigma_g,\vec{\nu})}(-1)^{R^{\vec{\nu}}} q^{\frac{1}{2}R^{\vec{\nu}}+j_3} \prod_{I=1}^{r_F} e^{m_I F_I}\,, \tag{3.38}$$

and is known to have the following factorization property [46, 47]

$$\mathcal{I}_q^{\text{top}}(\vec{m},\vec{\nu}) = \sum_{\alpha\in\mathcal{S}_{\text{BE}}} B^\alpha(b^2,\vec{u},\vec{\nu}) B^\alpha(-b^2,\vec{u},\vec{\nu})\big|_{u_I=e^{m_I},\, 2\pi i b^2=\log q}\,. \tag{3.39}$$

Taking the $b^2\to 0$ ($q\to 1$) limit, we have

$$\mathcal{I}_q^{\text{top}}(\vec{m},\vec{\nu})\big|_{q=e^{2\pi i b^2},\, b^2\to 0} = \sum_{\alpha\in\mathcal{S}_{\text{BE}}} e^{2S_1^\alpha(\vec{m},\vec{\nu})}(1+o(b^4))\,. \tag{3.40}$$

Here we use the asymptotic limit of the holomorphic blocks given in eq. (3.28). In the $q\to 1$ limit, the refined index becomes $\mathcal{Z}^{\nu_R=\frac{1}{2}}_{\mathcal{M}_{p=0,g=0}}$. Comparing the above expression with the general formula in eq. (3.33) with $p=g=0$ and $\nu_R=1/2$, we have the second equality in eq. (3.37).

When the condition in (3.22) is met, there is no difference between two discrete choices, $\nu_R=0$ or $1/2$, as is obvious in (3.34). So, in the case, we have

$$\begin{aligned}
\mathcal{Z}_{b=1} &= \sum_{\alpha\in\mathcal{S}_{\text{BE}}} (\mathcal{H}^\alpha_{\nu_R=0})^{-1}\mathcal{F}^\alpha_{\nu_R=0} \\
&= \sum_{\alpha\in\mathcal{S}_{\text{BE}}} (\mathcal{H}^\alpha_{\nu_R=1/2})^{-1}\mathcal{F}^\alpha_{\nu_R=1/2} \\
&= \sum_{\alpha\in\mathcal{S}_{\text{BE}}} \exp\left(i\frac{S_0^{(\alpha,\vec{\ell})} - U_I\partial_I S_0^{(\alpha,\vec{\ell})}}{2\pi} + 2S_1^{(\alpha,\vec{\ell})}\right)\,.
\end{aligned} \tag{3.41}$$

It matches the zero-th order approximation, s_0 in eq. (3.24), of the general perturbative expansion (3.20) in $\epsilon = 1-b$.

3.3 Current and stress tensor correlation functions

We define the free energy and its real part,

$$\begin{aligned}
\mathcal{F}_b(\vec{m},\vec{\nu}) &= -\log\mathcal{Z}_b(\vec{m},\vec{\nu}) = -\log\mathcal{Z}_b\left(\vec{m}+i\pi\left(b+\frac{1}{b}\right)(\vec{\nu}-\vec{\nu}_0),\vec{\nu}_0\right) \\
&= -\log\sum_\alpha \exp\left(\sum_{n=0}^\infty s_n\left(\vec{m}+i\pi\left(b+\frac{1}{b}\right)(\vec{\nu}-\vec{\nu}_0),\vec{\nu}_0\right)(1-b)^n\right)\,, \\
F_b(\vec{m},\vec{\nu}) &:= \text{Re}[\mathcal{F}_b(\vec{m},\vec{\nu})]\,.
\end{aligned} \tag{3.42}$$

Here $\vec{\nu}_0$ is a choice of R-charge mixing satisfying (3.22). The infra-red (IR) superconformal R-charge $R^{\vec{\nu}_{\text{IR}}}$ can be determined by the F-maximization principle [2] which states that[6]

$$F_{b=1}(\vec{m}=\vec{0},\vec{\nu}) \text{ is maximized at } \vec{\nu}=\vec{\nu}_{\text{IR}}\,. \tag{3.43}$$

[6]This only works when all the Cartan subalgebra of the IR flavor symmetry comes from the UV flavor symmetry G_F.

Then, basic CFT data at the IR fixed point can be obtained as follows [9, 10]

$$
\begin{aligned}
F &= F_{b=1}(\vec{m}=\vec{0}, \vec{\nu}_{\rm IR}) , \\
&= -\text{Re}\left[\log \sum_{\alpha} \exp(s_0^{(\alpha)}(\vec{m},\vec{\nu}_0))\right]\Bigg|_{\vec{m}=2\pi i(\vec{\nu}_{\rm IR}-\vec{\nu}_0)} , \\
C_{IJ} &= 8\partial_{m_I}\partial_{m_J} F_{b=1}\Big|_{\vec{m}=\vec{0},\vec{\nu}=\vec{\nu}_{\rm IR}} \\
&= -8\text{Re}\left[\frac{\sum_{\alpha\in\mathcal{S}_{\rm BE}} \exp\left(s_0^{(\alpha)}(\vec{m},\vec{\nu}_0)\right)\left(\partial_{m_I}\partial_{m_J} s_0^{(\alpha)}(\vec{m},\vec{\nu}_0)\right)}{\sum_{\alpha\in\mathcal{S}_{\rm BE}} \exp\left(s_0^{(\alpha)}(\vec{m},\vec{\nu}_0)\right)}\right]_{\vec{m}=2\pi i(\vec{\nu}_{\rm IR}-\vec{\nu}_0)} , \\
C_T &= \frac{8}{\pi^2}\partial_b\partial_b F_b\Big|_{\vec{m}=\vec{0},\vec{\nu}=\vec{\nu}_{\rm IR},b=1} \\
&= -\frac{8}{\pi^2}\text{Re}\left[\frac{\sum_{\alpha\in\mathcal{S}_{\rm BE}} \exp\left(s_0^{(\alpha)}(\vec{m},\vec{\nu}_0)\right)\left(2s_2^{(\alpha)}(\vec{m},\vec{\nu}_0)+m_I\partial_{m_I} s_0^{(\alpha)}(\vec{m},\vec{\nu}_0)\right)}{\sum_{\alpha\in\mathcal{S}_{\rm BE}} \exp\left(s_0^{(\alpha)}(\vec{m},\vec{\nu}_0)\right)}\right]_{\vec{m}=2\pi i(\vec{\nu}_{\rm IR}-\vec{\nu}_0)} .
\end{aligned}
\tag{3.44}
$$

The central charges C_{IJ} and C_T are defined in eq. (1.5) and (1.7). We fix the normalization of $T_{\mu\nu}$ and J_μ such that

$$
\begin{aligned}
&(C_T \text{ of a free theory of single chiral } \Phi) = 1 , \\
&(C_{JJ} \text{ for the U(1) symmetry of a free chiral theory under which } \Phi \text{ has charge } +1) = 1 .
\end{aligned}
\tag{3.45}
$$

4 Examples

In this section we present some concrete computations of the $b \to 1$ expansion (3.20). In addition to analytical computation as presented above, we independently confirm the computations by numerically evaluating the coefficients directly from the localization integral. Using the expansion, we present analytic expression of the stress-energy tensor central charge, C_T, for candidates for minimal (with lowest C_T) 3d $\mathcal{N}=2$ SCFTs. It include the candidates for minimal 3d $\mathcal{N}=2$ theory [48], minimal 3d $\mathcal{N}=4$ theory [49] and minimal 3d $\mathcal{N}=2$ theory [33, 50–52] with SU(3) flavor symmetry.

4.1 Free chiral multiplet and critical Wess-Zumino model

The S_b^3-partition function and the superconformal index for the theory of an $\mathcal{N}=2$ free chiral multiplet are

$$
\begin{aligned}
\mathcal{Z}_b(m,\nu) &:= \psi_\hbar\left(U + \left(i\pi + \frac{\hbar}{2}\right)\nu\right)\Big|_{U=bm} , \\
\mathcal{I}_q(\mathbf{m}, u; q) &= \mathcal{I}_\Delta(\mathbf{m}, u(-q^{1/2})^\nu; q) .
\end{aligned}
\tag{4.1}
$$

The theory has a $\mathrm{U}(1)_\Phi$ flavor symmetry under which Φ has charge $+1$. The parameters m and ν are the real mass parameter for the flavor symmetry and the R-symmetry mixing

	ϕ	ψ
$\mathrm{U}(1)_R^{\nu=0}$	0	-1
$\mathrm{U}(1)_\Phi$	1	1

Table 1. $\mathrm{U}(1)_R \times \mathrm{U}(1)_\Phi$ symmetry of the free chiral theory. ϕ and ψ are the complex scalar and the fermion field inside the $\mathcal{N}=2$ chiral multiplet respectively.

parameter $R^\nu = R^{\nu=0} + \nu \mathrm{U}(1)_\Phi$, respectively, and $(\mathbf{m}, u)$ is a pair (background magnetic flux, fugacity) for the flavor symmetry. The condition (3.23) is satisfied when

$$\nu_0 \in 2\mathbb{Z}\,. \tag{4.2}$$

In this example, we do not need to solve the saddle point equations to obtain an asymptotic expansion in $b \to 0$ limit since there is no integration.[7] The asymptotic expansion is given in (A.7),

$$\begin{aligned}
&\sum_{n=0}^{\infty} \hbar^{n-1} S_n^{\vec{\ell}}(U,\nu) = \sum_{n=0}^{\infty} \frac{B_n}{n!} \hbar^{n-1} \mathrm{Li}_{2-n}(e^{-U+(i\pi+\frac{\hbar}{2})\nu}) + \frac{1}{\hbar}(4\pi^2\ell_0 + 2\pi i \ell_u U + i\pi\hbar\ell_1)\,,\\
&\quad\Rightarrow S_0^{\vec{\ell}}(U,\nu) = \mathrm{Li}_2(e^{-U-i\pi\nu}) + 4\pi^2\ell_0 + 2\pi i\ell_u U\,,\\
&\qquad S_1^{\vec{\ell}}(U,\nu) = -\frac{1}{2}\log(1-e^{-U-i\pi\nu}) + \frac{1}{2}\nu\log(1-e^{-U-i\pi\nu}) + \pi i \ell_1\,,\\
&\qquad S_2(U,\nu) = \frac{2-6\nu+3\nu^2}{24(e^{U+i\pi\nu}-1)}\,, \quad S_3(U,\nu) = \frac{e^{U+i\pi\nu}\nu(2-3\nu+\nu^2)}{48(1-e^{U+i\pi\nu})^2}\,.
\end{aligned} \tag{4.3}$$

Then using eq. (3.24), we have

$$\begin{aligned}
&\exp\big(s_0(m,\nu)\big) = \exp\left[\frac{i}{2\pi}\left((2\pi i(1-\nu)-m)\log(1-e^{-m-i\pi\nu}) + \mathrm{Li}_2(e^{-m-i\pi\nu})\right)\right]\,,\\
&s_2(m,\nu) = s_3(m,\nu)\\
&= \frac{e^{m+i\pi\nu}\left[im^3 - 6\pi(\nu-1)m^2 + 4\pi^2(2\pi\nu^3 + 3i\nu^2 - 6\pi\nu^2 - 6i\nu + 4\pi\nu + 2i)\right]}{12\pi\left(-1+e^{m+i\pi\nu}\right)^2}\\
&\quad + \frac{me^{m+i\pi\nu}(-6i\pi\nu^2 + 12i\pi\nu + 3\nu - 4i\pi - 3) + (-6i\pi\nu^2 - 3m\nu + 12i\pi\nu + 3m - 4i\pi)}{6\left(-1+e^{m+i\pi\nu}\right)^2}\,.
\end{aligned} \tag{4.4}$$

Note that the $\exp(s_0)$ is independent on the choice of $\ell_u \in \mathbb{Z}$. We finally have ($\nu_0 \in 2\mathbb{Z}$)

$$\begin{aligned}
\log \mathcal{Z}_b(m,\nu) &= \log \mathcal{Z}_b\left(m + i\pi(b+b^{-1})(\nu-\nu_0), \nu_0\right)\\
&= s_0\big(m + 2\pi i(\nu-\nu_0), \nu_0\big) + \big[s_2\big(m+2\pi i(\nu-\nu_0),\nu_0\big)\big)\\
&\quad + i\pi(\nu-\nu_0)s_0^{(1,0)}\big(m+2\pi i(\nu-\nu_0),\nu_0\big)\big]\big((1-b)^2 + (1-b)^3\big) + o\big((b-1)^4\big)\,.
\end{aligned} \tag{4.5}$$

[7] According to a 3d $\mathcal{N}=2$ mirror duality [53], as we will see below, the theory is dual to a theory with a U(1) vector multiplet coupled to a single chiral multiplet. In the dual description, there is a single Bethe vacuum and the asymptotic expansion here can be thought as the expansion around the unique vacuum in the dual description.

Here

$$s_0^{(1,0)}(m,\nu) := \partial_m s_0(m,\nu) = \frac{im - 2\pi\nu + 2\pi}{2\pi - 2\pi e^{m+i\pi\nu}}\,. \tag{4.6}$$

As a consistency check, the expression is actually independent on the choice of $\nu_0 \in 2\mathbb{Z}$. For a free chiral theory, the superconformal IR R-symmetry corresponds to $\mathrm{U}(1)_R^\nu$ with $\nu = \frac{1}{2}$ and the stress-energy tensor central charge is

$$\begin{aligned}(C_T \text{ of a free } \Phi) &= -\frac{8}{\pi^2}\mathrm{Re}\partial_b^2\left[\log \mathcal{Z}_b\left(m=0,\nu=\frac{1}{2}\right)\right]_{b=1} \\ &= -\frac{8}{\pi^2}\mathrm{Re}\partial_b^2\left[\log \mathcal{Z}_b\left(m=\frac{i\pi}{2}(b+b^{-1}),\nu_0=0\right)\right]_{b=1} \\ &= -\frac{16}{\pi^2}\mathrm{Re}\left[s_2(i\pi,\nu_0=0)+\frac{i\pi}{2}s_0^{(1,0)}(i\pi,\nu_0=0)\right] = 1\,.\end{aligned} \tag{4.7}$$

In the second line, we use eq. (3.25). The C_{JJ} for the flavor $\mathrm{U}(1)_\Phi$ symmetry is

$$\begin{aligned}C_{JJ} &= -8\mathrm{Re}\partial_m^2\left[\log \mathcal{Z}_b\left(m,\nu=\frac{1}{2}\right)\right]_{m=0,b=1} \\ &= -8\mathrm{Re}\partial_m^2\left[\log \mathcal{Z}_b\left(m+\frac{i\pi}{2}(b+b^{-1}),\nu_0=0\right)\right]_{m=0,b=1} \\ &= -8\mathrm{Re}\partial_m^2 s_0(m+\pi i,\nu_0=0)|_{m=0} = -8\mathrm{Re}\frac{e^m(im-i+2\pi)+i}{2\pi\,(e^m-1)^2}|_{m=i\pi} \\ &= 1\,.\end{aligned} \tag{4.8}$$

From the above computations for the free chiral theory, we confirmed the normalization in eq. (3.45).

On the other hand, the IR R-symmetry of the critical Wess-Zumino model (cWZ, a chiral multiplet Φ with superpotential $W_{\text{sup}} = \Phi^3$) corresponds to $\mathrm{U}(1)_R^\nu$ with $\nu = 2/3$ and

$$\begin{aligned}(C_T \text{ of cWZ}) &= -\frac{8}{\pi^2}\mathrm{Re}\partial_b^2\left[\log \mathcal{Z}_b\left(m=0,\nu=\frac{2}{3}\right)\right] \\ &= -\frac{8}{\pi^2}\mathrm{Re}\partial_b^2\left[\log \mathcal{Z}_b\left(m=\frac{2\pi i}{3}(b+b^{-1}),\nu_0=0\right)\right] \\ &= -\frac{16}{\pi^2}\mathrm{Re}\left[s_2\left(\frac{4i\pi}{3},\nu_0=0\right)+\frac{2i\pi}{3}s_0^{(1,0)}\left(\frac{4i\pi}{3},\nu_0=0\right)\right] \\ &= \frac{16}{243}\left(16-\frac{9\sqrt{3}}{\pi}\right) \simeq 0.726785\,.\end{aligned} \tag{4.9}$$

The result nicely matches the analytic result in [54] obtained from direct integration using eq. (A.6). We reproduce the result from a drastically simpler computation.

4.2 $\mathrm{U}(1)_k$ coupled to a chiral multiplet of charge +1

The squashed 3-sphere partition function and the superconformal index of theory are

$$\begin{aligned}\mathcal{Z}_b(m,\nu) &= \int \frac{dZ}{\sqrt{2\pi\hbar}}\, e^{\frac{(k+\frac{1}{2})Z^2-2Z(U+(i\pi+\frac{\hbar}{2})\nu)}{2\hbar}}\psi_\hbar(Z)\Big|_{U=bm}\,, \\ \mathcal{I}_q(\mathbf{m},u;\nu) &= \sum_{\mathbf{n}\in\mathbb{Z}}\oint_{|v|=1}\frac{dv}{2\pi i v}\, v^{-\mathbf{m}}((-q^{\frac{1}{2}})^\nu u)^{-\mathbf{n}} v^{(k+\frac{1}{2})\mathbf{n}}\mathcal{I}_\Delta(\mathbf{n},v;q)\,.\end{aligned} \tag{4.10}$$

The theory has a topological $\mathrm{U}(1)_J$ flavor symmetry associated to the Abelian gauge symmetry. As in the previous example, m and ν are the real mass parameter and the R-symmetry mixing parameter for the flavor symmetry respectively, and $(\mathbf{m}, u)$ are (background magnetic flux, fugacity) for the flavor symmetry. The proper Chern-Simons level quantization of the theory is [55, 56]

$$k \in \mathbb{Z} + \frac{1}{2}\,. \tag{4.11}$$

The condition (3.23) is satisfied when

$$\nu_0 + \left(k + \frac{1}{2}\right) \in 2\mathbb{Z}\,. \tag{4.12}$$

The Bethe-vacua of the theory are determined by the following algebraic equations

$$\mathcal{S}_{\mathrm{BE}}(U,\nu) = \left\{ z \ : \ z^{k+\frac{1}{2}} \left(1 - \frac{1}{z}\right) = e^{U}(-1)^{\nu} \right\}\,. \tag{4.13}$$

Note that, for generic U

$$|\mathcal{S}_{\mathrm{BE}}| = |k| + \frac{1}{2}\,, \tag{4.14}$$

which matches the Witten index computation in [57]. The $b \to 0$ perturbative expansion coefficients $\{S_n^{(\alpha,\vec{\ell})}(U,\nu)\}_{n=0}^{\infty}$ can be computed using the method summarized in the section 3.1. Up to 3-loop, the perturbative invariants are as follows

$$\begin{aligned}
&S_0^{(\alpha,\vec{\ell})}(U,\nu) = \mathrm{Li}_2(e^{-Z}) + 4\pi^2\ell_0 + 2\pi i \ell_z Z - (U + i\pi\nu)Z + \frac{k+1/2}{2}Z^2\Big|_{Z=Z^{(\alpha,\vec{\ell})}(U,\nu)}\,,\\
&e^{2S_1^{(\alpha,\vec{\ell})}(U,\nu)} = \frac{2z^{1-\nu}}{A}\Big|_{z=\exp(Z^{(\alpha,\vec{\ell})}(U,\nu))}\,,\\
&S_2^{\alpha}(U,\nu=0) = \frac{A^2(2k-5)+A(-14k+6z+17)+10(2k-2z-1)}{12A^3}\Big|_{z=\exp\left(Z^{(\alpha,\vec{\ell})}(U,\nu)\right)}\,,\\
&S_2^{\alpha}(U,\nu=-1) = \frac{A^2(2k-3z+4)+A(-14k-6z+17)+10(2k-2z-1)}{12A^3}\Big|_{z=\exp\left(Z^{(\alpha,\vec{\ell})}(U,\nu)\right)}\,,\\
&S_3^{\alpha}(U,\nu=0) = \frac{(4k^2-1)(1-z)z\left(A^3+A^2(6k-z-15)+10A(2z-4k+5)+30(2k-2z-1)\right)}{12A^6}\Big|_{z=\exp\left(Z^{(\alpha,\vec{\ell})}(U,\nu)\right)}\,,\\
&S_3^{\alpha}(U,\nu=-1) = \frac{(2k-1)(1-z)z\left(A^2k(6k+1)+Ak(20-40k)+15(1-2k)^2\right)}{6A^6}\Big|_{z=\exp\left(Z^{(\alpha,\vec{\ell})}(U,\nu)\right)}\,.
\end{aligned} \tag{4.15}$$

Here we define

$$A := 2k(z-1) + z + 1\,. \tag{4.16}$$

The $Z^{(\alpha,\vec{\ell})}$ and $\ell_z^{\alpha} \in \mathbb{Z}$ is chosen such that

$$\partial_Z \mathcal{W}_0^{\vec{\ell}}\big|_{Z=Z^{(\alpha,\vec{\ell})}} = \left(k + \frac{1}{2}\right) Z + \log(1 - e^{-Z}) - U - i\pi\nu + 2\pi i \ell_z^{\alpha}\big|_{Z=Z^{(\alpha,\vec{\ell})}} = 0\,. \tag{4.17}$$

For given $z^{\alpha} \in \mathcal{S}_{\mathrm{BE}}(U,\nu)$ in eq. (4.13), the choice of $(Z^{\alpha} = \log z^{\alpha}, \ell_z^{\alpha})$ is not unique but has following shift ambiguity

$$Z^{\alpha} \to Z^{\alpha} + 2\pi i t\,, \quad \ell_z^{\alpha} \to \ell_z^{\alpha} - \left(k + \frac{1}{2}\right) t\,, \quad t \in \mathbb{Z}\,. \tag{4.18}$$

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Then, the $b\to 1$ expansion coefficients $s_n^\alpha(m,\nu)$ in eq. (3.24) up to $n=3$ are

$$
\begin{aligned}
\exp\left(s_0^\alpha(m,\nu)\right) &= \frac{2\exp\left(\frac{i\mathrm{Li}_2\left(e^{-Z}\right)}{2\pi}+\frac{i(k+\frac{1}{2})Z^2}{4\pi}-\frac{\nu}{2}Z+(1-\ell_z^\alpha)Z\right)}{A}\Bigg|_{Z=Z^{(\alpha,\vec{\ell})}(U=m,\nu)},\\
s_2^\alpha(m,\nu=0) &= \frac{1}{3\pi A^6}\Bigg(A^5\left(4i\pi^2k-3\pi m-10i\pi^2\right)\\
&\quad +A^4\left(-28i\pi^2k+6\pi m^2z+6\pi mz+12i\pi^2z+34i\pi^2\right)\\
&\quad +A^3\big(-16i\pi^2k^2mz^2+16i\pi^2k^2mz+32\pi^3k^2z^2-32\pi^3k^2z+32i\pi^2kmz^2-32i\pi^2kmz\\
&\qquad -16\pi^3kz^2+16\pi^3kz+40i\pi^2k+2im^3z^2-2im^3z-12\pi m^2z^2-24\pi m^2z\\
&\qquad -4i\pi^2mz^2+4i\pi^2mz-40i\pi^2z-20i\pi^2\big)\\
&\quad +A^2\big(192\pi^3k^3z^2-192\pi^3k^3z+224i\pi^2k^2mz^2-224i\pi^2k^2mz-512\pi^3k^2z^2+512\pi^3k^2z\\
&\qquad -256i\pi^2kmz^2+256i\pi^2kmz+304\pi^3kz^2-304\pi^3kz+48\pi m^2z^2\\
&\qquad +72i\pi^2mz^2-72i\pi^2mz-48\pi^3z^2+48\pi^3z\big)\\
&\quad +A\big(-1280\pi^3k^3z^2+1280\pi^3k^3z-480i\pi^2k^2mz^2+480i\pi^2k^2mz+2240\pi^3k^2z^2\\
&\qquad -2240\pi^3k^2z+480i\pi^2kmz^2-480i\pi^2kmz-1280\pi^3kz^2\\
&\qquad +1280\pi^3kz-120i\pi^2mz^2+120i\pi^2mz+240\pi^3z^2-240\pi^3z\big)\\
&\quad +1920\pi^3k^3z^2-1920\pi^3k^3z-2880\pi^3k^2z^2+2880\pi^3k^2z+1440\pi^3kz^2\\
&\quad -1440\pi^3kz-240\pi^3z^2+240\pi^3z\Bigg)\Bigg|_{z=\exp\left(Z^{(\alpha,\vec{\ell})}(U=m,\nu)\right)},\\
s_2^\alpha(m,\nu=-1) &= \frac{1}{3\pi A^6}\Bigg(A^5\left(4i\pi^2k+3\pi mz-6\pi m-6i\pi^2z+8i\pi^2\right)\\
&\quad +A^4\left(-28i\pi^2k+6\pi m^2z+6\pi mz-12i\pi^2z+34i\pi^2\right)\\
&\quad +A^3\big(-16i\pi^2k^2mz^2+16i\pi^2k^2mz-16i\pi^2kmz^2+16i\pi^2kmz+40i\pi^2k+2im^3z^2-2im^3z\\
&\qquad -36\pi m^2z-4i\pi^2mz^2+4i\pi^2mz-40i\pi^2z-20i\pi^2\big)\\
&\quad +A^2\big(192\pi^3k^3z^2-192\pi^3k^3z+224i\pi^2k^2mz^2-224i\pi^2k^2mz-64\pi^3k^2z^2+64\pi^3k^2z\\
&\qquad -64i\pi^2kmz^2+64i\pi^2kmz-16\pi^3kz^2+16\pi^3kz+48\pi m^2z^2-24i\pi^2mz^2+24i\pi^2mz\big)\\
&\quad +A\big(-1280\pi^3k^3z^2+1280\pi^3k^3z-480i\pi^2k^2mz^2+480i\pi^2k^2mz+1280\pi^3k^2z^2-1280\pi^3k^2z\\
&\qquad +480i\pi^2kmz^2-480i\pi^2kmz-320\pi^3kz^2+320\pi^3kz-120i\pi^2mz^2+120i\pi^2mz\big)\\
&\quad +1920\pi^3k^3z^2-1920\pi^3k^3z-2880\pi^3k^2z^2+2880\pi^3k^2z+1440\pi^3kz^2\\
&\quad -1440\pi^3kz-240\pi^3z^2+240\pi^3z\Bigg)\Bigg|_{z=\exp\left(Z^{(\alpha,\vec{\ell})}(U=m,\nu)\right)}.
\end{aligned}
\tag{4.19}
$$

Here A is defined in eq. (4.16) and we use the followings

$$
\begin{aligned}
\partial_U Z^{(\alpha,\vec{\ell})}(U,\nu) &= \frac{2(z-1)}{A}\Bigg|_{z=\exp\left(Z^{(\alpha,\vec{\ell})}\right)},\\
\partial_U\partial_U Z^{(\alpha,\vec{\ell})}(U,\nu) &= \frac{8(z-1)z}{A^3}\Bigg|_{z=\exp\left(Z^{(\alpha,\vec{\ell})}\right)},\\
\partial_U\partial_U\partial_U Z^{(\alpha,\vec{\ell})}(U,\nu) &= -\frac{16(z-1)z(Az+A-6z)}{A^5}\Bigg|_{z=\exp\left(Z^{(\alpha,\vec{\ell})}\right)}.
\end{aligned}
\tag{4.20}
$$

Under the shift (4.18), the expression $\exp\left(s_0^\alpha(m,\nu)\right)$ transforms as

$$
\exp\left(s_0^\alpha(m,\nu)\right)\to e^{i\pi t(\nu-(k+\frac{1}{2})t)}\exp\left(s_0^\alpha(m,\nu)\right). \tag{4.21}
$$

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Thanks to the condition in eq. (4.12), the phase factor is just 1 and the $\exp\left(s_0^\alpha(m,\nu)\right)$ is invariant under the shift. So, we finally have

$$
\begin{aligned}
&\log \mathcal{Z}_b(m,\nu)\\
&=\log \mathcal{Z}_b\left(m+i\pi(b+b^{-1})(\nu-\nu_0),\nu_0\right)\\
&=\log\sum_\alpha \exp\Big[s_0^\alpha\left(m+2\pi i(\nu-\nu_0),\nu_0\right)+\left((1-b)^2+(1-b)^3\right)\times\Big(s_2^\alpha\left(m+2\pi i(\nu-\nu_0),\nu_0\right)\\
&\qquad\qquad +\pi i(\nu-\nu_0)s_0^{\alpha,(1,0)}\left(m+2\pi i(\nu-\nu_0),\nu_0\right)\Big)+o\left((1-b)^4\right)\Big], \qquad (4.22)
\end{aligned}
$$

where ν_0 is chosen as in eq. (4.13). Here we define

$$
\begin{aligned}
&s_0^{\alpha,(1,0)}(m,\nu):=\partial_m s^\alpha(m,\nu)\\
&=\left.\frac{iAm(z-1)-2\pi A(\nu(z-1)+1)+4\pi z}{\pi A^2}\right|_{z=\exp\left(Z^{(\alpha,\vec{\ell})}\right)},\\
&s_0^{\alpha,(2,0)}(m,\nu):=\partial_m\partial_m s^\alpha(m,\nu) \qquad (4.23)\\
&=\left.\frac{iA^3(z-1)+4\pi A^2z+4iAm(z-1)z-8\pi Az(-\nu+\nu z+z+2)+32\pi z^2}{\pi A^4}\right|_{z=\exp\left(Z^{(\alpha,\vec{\ell})}\right)}.
\end{aligned}
$$

$k=-1/2$ case: dual to free chiral multiplet. In the case, the squashed three-sphere partition function is

$$\mathcal{Z}_b(m,\nu=0)=\int\frac{dZ}{\sqrt{2\pi\hbar}}e^{-\frac{ZU}{\hbar}}\psi_\hbar(Z)\Big|_{U=bm}. \qquad (4.24)$$

Recall that $\hbar:=2\pi ib^2$. The logarithmic Bethe-vacua equation is

$$
\begin{aligned}
&\log(1-e^{-Z})-m+2\pi i\ell_z=0,\\
&\Rightarrow Z=-\log(1-e^m),\quad \ell_z=0.
\end{aligned} \qquad (4.25)
$$

From the computations in eq. (4.19) (we choose $\nu_0=0$),

$$
\begin{aligned}
&\exp\left(s_0(m,\nu_0=0)\right)=-\exp\left[\frac{i}{2\pi}\mathrm{Li}_2(e^{-Z})+Z\right]=-\exp\left[\frac{i}{2\pi}\mathrm{Li}_2(1-e^m)-\log(1-e^m)\right]\\
&=\exp\left[\frac{i}{2\pi}\left(\mathrm{Li}_2(e^{-m})+(2\pi i-m)\log(1-e^{-m})-\frac{1}{2}m^2+2\pi im-\frac{\pi^2}{6}\right)\right],\\
&s_2(m,\nu_0=0)=\frac{i\left(m^3(z-1)z-6i\pi m^2(z-1)z-2\pi m(z-1)(4\pi z+3i)+4\pi^2(1-2z)\right)}{12\pi}\\
&=\frac{e^m m\left(im^2+6\pi m+2\pi(3-4i\pi)\right)+2i\pi e^{2m}(2\pi+3im)-4i\pi^2}{12\pi\left(e^m-1\right)^2}. \qquad (4.26)
\end{aligned}
$$

Here we use following identity

$$\mathrm{Li}_2(1-u)=\mathrm{Li}_2\left(\frac{1}{u}\right)+\frac{\pi^2}{3}+\frac{1}{2}\log^2(-u)-\log u\log(1-u). \qquad (4.27)$$

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Comparing with s_0 and s_1 for a free chiral theory in eq. (4.4),

$$\begin{aligned} s_0^{\mathrm{U}(1)_{-\frac{1}{2}}+\Phi}(m,\nu_0=0) &= s_0^{\Phi}(m,\nu_0=0)+\frac{m^2}{4\pi i}-m-\frac{i\pi}{12}\,,\\ s_2^{\mathrm{U}(1)_{-\frac{1}{2}}+\Phi}(m,\nu_0=0) &= s_2^{\Phi}(m,\nu_0=0)-\frac{m}{2}+\frac{i\pi}{3}\,. \end{aligned} \tag{4.28}$$

This is compatible with following identity ($\mathcal{Z}_1\simeq\mathcal{Z}_2$ means $\mathcal{Z}_1=e^{i\pi\delta}\mathcal{Z}_2$ with $\delta\in\mathbb{Q}$)

$$\begin{aligned} &\int\frac{dZ}{\sqrt{2\pi\hbar}}\,e^{-\frac{ZU}{\hbar}}\psi_\hbar(Z)\simeq e^{\frac{U^2-(2\pi i+\hbar)U}{2\hbar}+\frac{i\pi(b^2+b^{-2})}{12}}\psi_\hbar(U)\,,\\ \Rightarrow\ &\mathcal{Z}_b^{\mathrm{U}(1)_{-\frac{1}{2}}+\Phi}(m,\nu_0=0)\simeq e^{\frac{m^2-2\pi i(b+b^{-1})m}{4\pi i}+\frac{i\pi(b^2+b^{-2})}{12}}\mathcal{Z}_b^{\Phi}(m,\nu_0=0)\,. \end{aligned} \tag{4.29}$$

$k=-3/2$ case: SUSY enhancement. The partition function is

$$\mathcal{Z}_b(m,\nu)=\int\frac{dZ}{\sqrt{2\pi\hbar}}\,e^{-\frac{Z^2+2Z\left(U+\nu\left(i\pi+\frac{\hbar}{2}\right)\right)}{2\hbar}}\psi_\hbar(Z)\Big|_{U=bm}\,. \tag{4.30}$$

The saddle point equation and Bethe-vacua equation are

$$\begin{aligned} &(\text{saddle point equation})\ :\ -Z^{(\alpha,\ell_z)}+\log(1-e^{-Z^{(\alpha,\ell_z)}})-U+2\pi i\ell_z^\alpha-i\pi\nu=0\,,\\ &(\text{Bethe-vacua equation})\ :\ \frac{1-\frac{1}{z}}{z}=(-1)^\nu e^U\,. \end{aligned} \tag{4.31}$$

There are two Bethe-vacua $\{z^\alpha\}_{\alpha=1,2}$ and there are corresponding two saddle points $\{Z^{(\alpha,\ell_z)}(U,\nu)\}$ with a proper choice of $\ell_z\in\mathbb{Z}$. The round 3-sphere free energy $F_{b=1}(m,\nu):=-\mathrm{Re}\log\mathcal{Z}_{b=1}(m,\nu)$ with general choice of the R-charge mixing ν can be computed using eq. (3.44) and (4.19):

$$\begin{aligned} F_{b=1}(m,\nu) &= -\mathrm{Re}\log\sum_{\alpha=1,2}\exp\left(s_0^\alpha\left(m+2\pi i(\nu+1),\nu_0=-1\right)\right)\\ &= -\mathrm{Re}\sum_{\alpha=1,2}\frac{2\exp\left(\frac{i\mathrm{Li}_2(e^{-Z})}{2\pi}-\frac{iZ^2}{4\pi}+\frac{1}{2}Z+(1-\ell_z^\alpha)Z\right)}{-3(e^Z-1)+e^Z+1}\Bigg|_{Z=Z^{(\alpha,\ell_z)}(U=m+2\pi i(\nu+1),\nu_0=-1)}\,. \end{aligned} \tag{4.32}$$

The free energy has $\mathbb{Z}_2$ symmetry

$$F_{b=1}(m=0,\nu)=F_{b=1}(m=0,-\nu)\,, \tag{4.33}$$

and has maximum at

$$\nu_{\mathrm{IR}}=0\,. \tag{4.34}$$

At the IR fixed point $\nu_{\mathrm{IR}}=0$, there are two Bethe-vacua

$$z^{(\alpha=1)}=\frac{1}{2}(-1+\sqrt{5})\,,\quad z^{(\alpha=2)}=\frac{1}{2}(-1-\sqrt{5})\,. \tag{4.35}$$

Here we choose $\nu_0 = -1$ and $U = 2\pi i(\nu_{\rm IR} - \nu_0) = 2\pi i$. Basic CFT data at the IR fixed point can be computed using eq. (3.44), (4.19) and (4.23) with $k = -3/2, \nu_0 = -1$ and $\nu_{\rm IR} = 0$

$$
\begin{aligned}
F &= F_{b=1}(m=0,\nu=0) = -\mathrm{Re}\log\left(\frac{1+\exp(\frac{3\pi i}{5})}{\sqrt{5}}\right) = 0.642965\,, \\
C_{JJ} &= 8\,\partial_m^2 F_{b=1}(m=0,\nu=0) = \frac{2}{25}\left(8-\frac{5\sqrt{2\sqrt{5}+5}}{\pi}\right) = 0.248137\,, \\
C_T &= \frac{8}{\pi^2}\partial_b^2 F_{b=1}(m=0,\nu=0) = \frac{8}{25}\left(8-\frac{5\sqrt{2\sqrt{5}+5}}{\pi}\right) = 0.992549\,.
\end{aligned}
\tag{4.36}
$$

Note the equality $4C_{JJ} = C_T$ which is a strong evidence for the IR $\mathcal{N}=4$ supersymmetry of the theory. Refer to [49] for more evidences for the IR enhancement.

4.3 $\mathrm{U}(1)_k$ coupled to two chiral multiplets of charge +1

The squashed 3-sphere partition function and the superconformal index are

$$
\begin{aligned}
\mathcal{Z}_b(m,\nu) &= \int \frac{dZ}{\sqrt{2\pi\hbar}}\, e^{\frac{(k+1)Z^2-2Z(U+(i\pi+\frac{\hbar}{2})\nu)}{2\hbar}}\psi_\hbar(Z)\psi_\hbar(Z)\Big|_{U=bm}\,, \\
\mathcal{I}_q(\mathbf{m},u;\nu) &= \sum_{\mathbf{n}\in\mathbb{Z}}\oint_{|v|=1}\frac{dv}{2\pi i v}\, v^{-\mathbf{m}}((-q^{\frac{1}{2}})^\nu u)^{-\mathbf{n}} v^{(k+1)\mathbf{n}}\mathcal{I}_\Delta(\mathbf{n},v;q)\mathcal{I}_\Delta(\mathbf{n},v;q)\,.
\end{aligned}
\tag{4.37}
$$

The theory has $\mathrm{U}(1)_J\times \mathrm{SU}(2)_\Phi$ flavor symmetry. $\mathrm{U}(1)_J$ is the topological symmetry while $\mathrm{SU}(2)_\Phi$ is the symmetry rotating the two chiral multiplets. The R-symmetry can be mixed only with the $\mathrm{U}(1)_J$. m and ν are the real mass and the R-symmetry mixing parameter for the $\mathrm{U}(1)_J$ respectively, while $(\mathbf{m},u)$ are (background magnetic flux, fugacity) for the $\mathrm{U}(1)_J$. In the index formula, we turned off the (background magnetic flux, fugacity) for the $\mathrm{SU}(2)_\Phi$ symmetry. The proper CS level quantization is

$$k\in\mathbb{Z}\,. \tag{4.38}$$

The condition (3.23) is satisfied when

$$\nu_0 + (k+1)\in 2\mathbb{Z}\,. \tag{4.39}$$

The twisted superpotential at leading order is

$$\mathcal{W}_0^{\vec{\ell}}(Z,U,\nu) = \frac{(k+1)}{2}Z^2 + 2\mathrm{Li}_2(e^{-Z}) + 2\pi i\ell_z Z - ZU - i\pi\nu Z\,. \tag{4.40}$$

The saddle points and Bethe-vacua of the theory are determined by following equations

$$
\begin{aligned}
&\text{Saddle point equation}: \ (k+1)Z + 2\log(1-e^{-Z}) - U - i\pi\nu + 2\pi i\ell_z = 0\,, \\
&\text{Bethe-vacua equation}: \ z^{k+1}\left(1-\frac{1}{z}\right)^2 = e^U(-1)^\nu\,.
\end{aligned}
\tag{4.41}
$$

For each Bethe-vacuum $z^{(\alpha)}$, there is an associated saddle point $Z^{(\alpha,\ell_z)}$ upon a proper choice of ℓ_z. Up to 3-loop, the perturbative invariants are

$$
\begin{aligned}
S_0(U,\nu) &= \mathcal{W}_0^{\vec{\ell}}(Z,U,\nu)\,,\\
e^{2S_1} &= \frac{z^{1-\nu}}{B(1-z)}\,,\\
S_2(U,\nu=0) &= \frac{4B^3-12B^2(z+1)+6B(z^2+5z)-20z^2}{24B^3(z-1)}\,,\\
S_2(U,\nu=-1) &= \frac{4B^3-3B^2\left(z^2-2z+9\right)-6B(z^2-7z)-20z^2}{24B^3(z-1)}\,,\\
S_3(U,\nu=0) &= \frac{-2B^5z(z+3)+B^4z\left(3z^2+34z+27\right)-B^3z\left(z^3+49z^2+127z+27\right)}{24B^6(z-1)^2}\\
&\quad+\frac{2B^2z^2\left(11z^2+98z+67\right)-20B\left(5z^4+11z^3\right)+120z^4}{24B^6(z-1)^2}\,,\\
S_3(U,\nu=-1) &= \frac{-8B^5z+16B^4z(z+3)-2B^3\left(5z^3+65z^2+32z\right)}{24B^6(z-1)^2}\\
&\quad+\frac{2B^2z^2\left(z^2+62z+113\right)-40B\left(z^4+7z^3\right)+120z^4}{24B^6(z-1)^2}\,.
\end{aligned}
\tag{4.42}
$$

Here we define

$$
B := (k+1)z+1-k\,. \tag{4.43}
$$

The $b\to 1$ expansion coefficients $s_n^{\alpha}(m,\nu)$ in eq. (3.24) are

$$
\begin{aligned}
\exp\left(s_0(m,\nu)\right) &= \frac{\exp\left(\frac{Z\left(im-2i\log\left(1-e^{-Z}\right)+\pi(-3\nu-2\ell_z+8)\right)+4i\mathrm{Li}_2\left(e^{-Z}\right)}{4\pi}\right)}{B(1-z)}\,,\\
\partial_m s_0(m,\nu) &= \frac{iBm(z-1)+2\pi B(\nu+\nu(-z)-2)+4\pi z}{2\pi B^2}\,,\\
\partial_m^2 s_0(m,\nu) &= \frac{iB^3(z-1)+4\pi B^2z+2iBm(z-1)z-4\pi Bz(\nu(z-1)+z+3)+16\pi z^2}{2\pi B^4}\,,
\end{aligned}
\tag{4.44}
$$

and

$$
\begin{aligned}
s_2(m,\nu=0) = \frac{1}{6\pi B^6(z-1)^2}\Bigg(&8\pi^2 iB^6(z-1)-960\pi^3z^4\\
&+B^5\left(-8i\pi^2mz^2-6\pi mz^2+8i\pi^2mz+12\pi mz-6\pi m+16\pi^3z^2-24i\pi^2z^2+48\pi^3z+24i\pi^2\right)\\
&+B^4\big(6\pi m^2z^3-12\pi m^2z^2+6\pi m^2z+24i\pi^2mz^3+6\pi mz^3+48i\pi^2mz^2-12\pi mz^2-72i\pi^2mz\\
&\qquad+6\pi mz-24\pi^3z^3+12i\pi^2z^3-272\pi^3z^2+48i\pi^2z^2-216\pi^3z-60i\pi^2z\big)\\
&+B^3\big(im^3z^4-3im^3z^3+3im^3z^2-im^3z-6\pi m^2z^4-6\pi m^2z^3+30\pi m^2z^2-18\pi m^2z\\
&\qquad-12i\pi^2mz^4-180i\pi^2mz^3+84i\pi^2mz^2+108i\pi^2mz+8\pi^3z^4+392\pi^3z^3\\
&\qquad-40i\pi^2z^3+1016\pi^3z^2+40i\pi^2z^2+216\pi^3z\big)\\
&+B^2\big(24\pi m^2z^4-48\pi m^2z^3+24\pi m^2z^2+128i\pi^2mz^4+192i\pi^2mz^3-320i\pi^2mz^2\\
&\qquad-176\pi^3z^4-1568\pi^3z^3-1072\pi^3z^2\big)\\
&+B\left(-240i\pi^2mz^4+240i\pi^2mz^3+800\pi^3z^4+1760\pi^3z^3\right)\Bigg)\,,
\end{aligned}
$$

$$
\begin{aligned}
s_2(m,\nu=-1)=\frac{1}{6\pi B^6(z-1)^2}\Big(&8\pi^2 iB^6(z-1)-960\pi^3z^4\\
&+B^5\big(3\pi mz^3-8i\pi^2mz^2-15\pi mz^2+8i\pi^2mz+21\pi mz-9\pi m-6i\pi^2z^3\\
&\qquad+18i\pi^2z^2+64\pi^3z-66i\pi^2z+54i\pi^2\big)\\
&+B^4\big(6\pi m^2z^3-12\pi m^2z^2+6\pi m^2z+6\pi mz^3+96i\pi^2mz^2-12\pi mz^2-96i\pi^2mz+6\pi mz\\
&\qquad-12i\pi^2z^3-128\pi^3z^2+96i\pi^2z^2-384\pi^3z-84i\pi^2z\big)\\
&+B^3\big(im^3z^4-3im^3z^3+3im^3z^2-im^3z-24\pi m^2z^3+48\pi m^2z^2-24\pi m^2z-120i\pi^2mz^3\\
&\qquad-72i\pi^2mz^2+192i\pi^2mz+80\pi^3z^3-40i\pi^2z^3+1040\pi^3z^2+40i\pi^2z^2+512\pi^3z\big)\\
&+B^2\big(24\pi m^2z^4-48\pi m^2z^3+24\pi m^2z^2+32i\pi^2mz^4+384i\pi^2mz^3\\
&\qquad-416i\pi^2mz^2-16\pi^3z^4-992\pi^3z^3-1808\pi^3z^2\big)\\
&+B\big(-240i\pi^2mz^4+240i\pi^2mz^3+320\pi^3z^4+2240\pi^3z^3\big)\Big).
\end{aligned}
\tag{4.45}
$$

***k* = 0 case: SU(3) symmetry enhancement.** In the case, the $\mathrm{U}(1)_J \times \mathrm{SU}(2)_\Phi$ symmetry is enhanced to SU(3) at IR [33, 50–52]. Under the symmetry enhancement, the Cartan $\mathfrak{u}(1)_J$ for the topological $\mathrm{U}(1)_J$ symmetry is embedded into $\mathfrak{su}(3)$ as follows

$$
\mathfrak{u}(1)_J = \frac{1}{3}\mathrm{diag}\{1,1,-2\} \in \mathfrak{su}(3)\,. \tag{4.46}
$$

Using eq. (3.44), (4.44) and (4.45), we can compute $F_{b=1}(m,\nu)$ and check that

$$
F_{b=1}(m=0,\nu) = F_{b=1}(m=0,2-\nu) \text{ is maximized at } \nu=\nu_{\mathrm{IR}}=1. \tag{4.47}
$$

At the IR fixed point $\nu_{\mathrm{IR}}=1$, there are two Bethe-vacua

$$
z^{(\alpha=1)} = e^{i\pi/3}\,, \quad z^{(\alpha=2)} = e^{-i\pi/3}\,. \tag{4.48}
$$

Here, we choose $\nu_0=1$ and $U=2\pi i(\nu_{\mathrm{IR}}-\nu_0)=0$. Using eq. (3.44), (4.44) and (4.45), we can compute the basic CFT data of the SCFT at the fixed point

$$
\begin{aligned}
F &= F_{b=1}(m=0,\nu=1) = -\log\frac{1}{\sqrt{3}}\left(e^{\frac{V}{2\pi}}-e^{-\frac{V}{2\pi}}\right) \simeq 0.968723\,,\\
C_{JJ} &= 8\partial_m^2 F_{b=1}(m=0,\nu=1) = \frac{4\coth\left(\frac{V}{2\pi}\right)}{\sqrt{3}\pi}-\frac{16}{9} \simeq 0.576242\,,\\
C_T &= \frac{8}{\pi^2}\partial_b^2 F_{b=1}(m=0,\nu=1) = \frac{16}{27}\left(\frac{11\sqrt{3}\coth\left(\frac{V}{2\pi}\right)}{\pi}-6\right) \simeq 2.02706\,.
\end{aligned}
\tag{4.49}
$$

Here

$$
V := 2\,\mathrm{Im}[\mathrm{Li}_2(e^{i\pi/3})] = 2.02988\,. \tag{4.50}
$$

The F matches the computation in [58]. Note that the C_T is less than that of the free theory of 3 chiral multiplets, which also has SU(3) symmetry.

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A Quantum dilogarithm

The quantum dilogarithm function (Q.D.L) $\psi_\hbar(Z)$ is defined by [59] ($\hbar = 2\pi i b^2$)

$$\psi_\hbar(Z) := \begin{cases} \prod_{r=1}^{\infty} \dfrac{1 - q^r e^{-Z}}{1 - \tilde{q}^{-r+1} e^{-\tilde{Z}}} & \text{if } |q| < 1\,, \\ \prod_{r=1}^{\infty} \dfrac{1 - \tilde{q}^r e^{-\tilde{Z}}}{1 - q^{-r+1} e^{-Z}} & \text{if } |q| > 1\,, \end{cases} \tag{A.1}$$

with

$$q := e^{2\pi i b^2}\,, \quad \tilde{q} := e^{2\pi i b^{-2}}\,, \quad \tilde{Z} := \frac{1}{b^2} Z\,. \tag{A.2}$$

The function satisfies the following difference equations

$$\psi_\hbar(Z + 2\pi i b^2) = (1 - e^{-Z})\psi_\hbar(Z)\,, \quad \psi_\hbar(Z + 2\pi i) = \left(1 - e^{-\frac{Z}{b^2}}\right)\psi_\hbar(Z)\,. \tag{A.3}$$

At the special value $b = 1$, the Q.D.L simplifies as

$$\log \psi_{\hbar = 2\pi i}(Z) = \frac{-(2\pi + iZ)\log(1 - e^{-Z}) + i\mathrm{Li}_2(e^{-Z})}{2\pi}\,. \tag{A.4}$$

Poles and zeros of Q.D.L are

$$\begin{aligned} &\text{simple poles}: \ 2\pi i \mathbb{Z}_{\leq 0} + 2\pi i b^2 \mathbb{Z}_{\leq 0}\,, \\ &\text{simple zeros}: \ 2\pi i \mathbb{Z}_{\geq 1} + 2\pi i b^2 \mathbb{Z}_{\geq 1}\,. \end{aligned} \tag{A.5}$$

We have an integral representation:

$$\log \psi_\hbar(Z) = \int_{\mathbb{R}+i0^+} \frac{e^{\frac{itZ}{\pi b} + t(b+b^{-1})}}{\sinh(bt)\sinh(b^{-1}t)} \frac{dt}{4t}\,, \quad \text{for } 0 < \mathrm{Im}[Z] < 2\pi(1 + b^2)\,. \tag{A.6}$$

The asymptotic expansion when $\hbar = 2\pi i b^2 \to 0$ is given by

$$\log \psi_\hbar(Z) \xrightarrow{b^2 \to 0} \sum_{n=0}^{\infty} \frac{B_n \hbar^{n-1}}{n!} \mathrm{Li}_{2-n}(e^{-Z}, \ell_0, \ell_z)\,. \tag{A.7}$$

Here B_n is the n-th Bernoulli number with $B_1 = 1/2$.

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