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Conformal properties of soft-operators. Part II. Use of null-states

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ABSTRACT: Representations of the (Lorentz) conformal group with the soft operators as highest weight vectors have two universal properties, which we clearly state in this paper. Given a soft operator with a certain dimension and spin, the first property is about the existence of “(large) gauge transformation” that acts on the soft operator. The second property is the decoupling of (large) gauge-invariant null-states of the soft operators from the S -matrix elements. In each case, the decoupling equation has the form of zero field-strength condition with the soft operator as the (gauge) potential. Null-state decoupling effectively reduces the number of polarisation states of the soft particle and is crucial in deriving soft-theorems from the Ward identities of asymptotic symmetries. To the best of our understanding, these properties are not directly related to the Lorentz invariance of the S -matrix or the existence of asymptotic symmetries. We also verify that the results obtained from the decoupling of null-states are consistent with the leading and subleading soft-theorems with finite energy massive and massless particles in the external legs.

KEYWORDS: Scattering Amplitudes, Space-Time Symmetries

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1 Soft-operators

Soft operators are central in the study of the relationship between soft theorems and asymptotic symmetries in flat space-time [1–20]. These operators live on the celestial sphere S^{D-2} and transform as primaries of the Lorentz group $SO(D-1, 1)$, which acts on the celestial sphere by conformal transformation. In this paper, following [20], we continue our study of the properties of the conformal representation with a soft operator as the highest weight state.

For our purpose we consider primary operators $\{O_R^\Delta(x)\}$ which transform in the representation R of the rotation group $SO(D-2)$ and has conformal dimension Δ . Here x denotes the Cartesian coordinates of R^{D-2} which is the stereographic image of the celestial

sphere S^{D-2} . To maximise our freedom we do not restrict the representation R or the dimension Δ in any way.

Now, the origin of a subset of the primary operators $\{O_R^\Delta(x)\}$ with integer Δ and transforming in the symmetric traceless tensor representation of $SO(D-2)$, can be explained in the following way. Suppose we formally expand the physical (creation) annihilation operator of a massless particle of integer spin- l in a Laurent series as,¹

$$A_{a_1 \dots a_l}(\omega, x) = \sum_{p \in \mathbb{Z}} \frac{S_{a_1 \dots a_l}^p(x)}{\omega^p} \quad (1.1)$$

Now, using the Lorentz transformation property of $A_{a_1 \dots a_l}(\omega, x)$ it is easy to check that the operator $S_{a_1 \dots a_l}^p(x)$ transforms like a conformal primary operator of scaling dimension $\Delta = p$ and spin- l . $S_{a_1 \dots a_l}^p(x)$ is an example of a spin- l soft operator and a member of the set $\{O_R^\Delta(x)\}$.

Let us now substitute this expansion in an S -matrix element and for concreteness let us take $l = 1$,

$$\langle \text{out} | A_a(\omega, x) | \text{in} \rangle = \sum_{p \in \mathbb{Z}} \frac{\langle \text{out} | S_a^p(x) | \text{in} \rangle}{\omega^p} \quad (1.2)$$

Let us emphasise that *the $\langle \text{out} |$ and the $| \text{in} \rangle$ states may contain both massive and massless particles*. Now from the calculation of S -matrix elements we know that:

- 1) $\langle \text{out} | S_a^p(x) | \text{in} \rangle = 0$ for $p > 1$. In other words, the primary operators $S_{a_1 \dots a_l}^{p>1}(x)$ decouple from the S -matrix elements.
- 2) The $(\Delta = 1, l = 1)$ primary operator $S_a^1(x)$ is the leading soft-photon operator.
- 3) Similarly the $(\Delta = 0, l = 1)$ primary operator $S_a^0(x)$ is the subleading soft-photon operator.
- 4) The *primary* operators $S_a^p(x)$ with $\Delta = p < 0$ may also be called soft-operators, although, this does *not* (necessarily) imply factorisation of the matrix element $\langle \text{out} | S_a^p(x) | \text{in} \rangle$.

The same discussion applies to particles of higher spin.

So we can see that some of the primary operators have interpretation as soft-operators arising from *physical* creation-annihilation operators of massless gauge particles, but, for our purpose, we have to assume the existence of more general primary operators which do not have any such interpretation. We will see that they play the role of “transformation parameters”.

Before we leave this section we would like to mention that in $D = 4$ the (formal) expansion (1.1) requires some interesting modification due to the existence of the logarithmic terms [24, 25] in the soft expansion. But we leave this for future work.

¹The null momentum of the massless particle is parametrized as, $p^\mu(\omega, \vec{x}) = \omega(1 + \vec{x}^2, 2\vec{x}, 1 - \vec{x}^2)$. For details please see the appendix.

2 Universal properties of representation of the conformal group with soft-operator as the highest weight vector

In [20] it was shown that Lorentz invariance of the S -matrix, together with the Ward-identities of the asymptotic symmetries, require the soft-operators to satisfy certain differential equations. For example, the leading soft-photon operator $S_a(x)(=S_a^1(x))$ was shown to satisfy the (Euclidean) Maxwell's equation, $\partial_a(\partial_a S_b - \partial_b S_a) = 0$. Similarly, the leading soft-graviton operator $h_{ab}(x)(=S_{ab}^1(x))$, in $D = 6$, satisfies the equation,

$$\partial^2 h_{ab} - \frac{2}{3} \left(\partial_a \partial_c h_{cb} + \partial_b \partial_c h_{ca} \right) + \frac{1}{3} \delta_{ab} \partial_c \partial_d h_{cd} = 0 \quad (2.1)$$

These equations are additional constraints which need to be satisfied if we want to derive the soft-theorems from Ward-Identities of the asymptotic symmetries.

Now, the curious fact is that both these equations are equations of (Euclidean) gauge theories. Maxwell's equations have obvious gauge invariance whereas the second equation (2.1) is invariant under the transformation [27–29],

$$h_{ab}(x) \rightarrow h_{ab}(x) + \left(\partial_a \partial_b - \frac{1}{4} \delta_{ab} \partial^2 \right) \phi(x) \quad (2.2)$$

where $\phi(x)$ is a $(\Delta = -1, l = 0)$ primary.

In this paper, *instead of Lorentz invariance and Ward-identity, we take this “gauge-structure” as the starting point.* It turns out that the conformal representation built on any soft-operator $S_{a_1 \dots a_l}^\Delta(x)$ obeys two simple *universal* rules stated below,

1) Symmetry transformation. Given a (Δ, l) primary field $S_{a_1 \dots a_l}^\Delta(x)$, we look for another primary field $O_R^\delta(x)$ which transforms in the representation R of the rotation group $SO(D-2)$ and has dimension $\delta < \Delta$. The *defining property of the primary $O_R^\delta(x)$ is that it should have a primary descendant of weight Δ and spin l .* The primary field $O_R^\delta(x)$ may or may not exist depending on the quantum numbers (Δ, l) . Let us consider the interesting case where it exists and denote the corresponding primary descendant by $N_{a_1 \dots a_l}^\Delta(x)$.

The existence of the (Δ, l) primary descendant $N_{a_1 \dots a_l}^\Delta(x)$ allows us to define a “global symmetry transformation” by,

$$S_{a_1 \dots a_l}^\Delta(x) \rightarrow S_{a_1 \dots a_l}^\Delta(x) + N_{a_1 \dots a_l}^\Delta(x) \quad (2.3)$$

Since $N_{a_1 \dots a_l}^\Delta(x)$ can be written as a differential operator acting on the primary $O_R^\delta(x)$, one can think of $O_R^\delta(x)$ as the transformation function or parameter.

2) Decoupling of Null-states. The next step is to construct the conformal representation with the soft operator $S_{a_1 \dots a_l}^\Delta(x)$ as the highest-weight vector. For our purpose, the interesting states in the representation space are the primary descendants. The primary descendants have two possible fates, which we now describe:

Case-A (Δ, l) is such that the primary $O_R^\delta(x)$ does *not* exist. In that case, *we cannot set any of the primary descendants of $S_{a_1 \dots a_l}^\Delta(x)$ to zero.*

Case-B (Δ, l) is such that the primary field $O_R^\delta(x)$ exists and the symmetry transformation (2.3) is defined. In this case, we can *set only those primary descendants of $S_{a_1 \dots a_l}^\Delta(x)$ to zero which are invariant under the symmetry transformation (2.3)*. In other words, the primary descendants of $S_{a_1 \dots a_l}^\Delta(x)$, which are invariant under the symmetry transformation (2.3) *decouple from the S -matrix element*.

The transformations (2.3) that we are talking about, if they exist, are *large* gauge transformations. From the dual field theory point of view these are emergent in some sense because these are remnants of space-time gauge transformations. This also suggests we can only set those null-states to zero which are invariant under (large) gauge transformations because otherwise we will break the asymptotic or the infinite dimensional global symmetry of the dual field theory.

Invariance of the S -matrix has no obvious connection to the decoupling. Without decoupling there will be a mismatch between the number of Ward-identities of the asymptotic symmetries and the number of soft S -matrix elements. So, in order to derive soft-theorems from the corresponding Ward-identities, decoupling is crucial. It seems that the correct explanation of the *universal* rules we have stated, goes beyond the “asymptotic symmetry — soft theorem” correspondence. In fact there are some indications, that an Euclidean CFT living on the celestial sphere holographically computes the S -matrix elements in the asymptotically flat space [10–12, 35–52, 54]. If this is true then the dual CFT may provide a *microscopic* explanation of the universality. Another avenue will be to understand the asymptotic symmetries in flat space-time from the string theory point of view along the line of [31–34, 50, 51]. We leave these questions for future study.

In the rest of the paper we derive consequences of this decoupling and check their consistency with soft-theorems. This demonstration also shows that the above mentioned properties of the gauge particles are intrinsic and do not depend on whether the rest of the particles in the scattering process are massive or massless.

3 Consequences of decoupling

In this section we study the consequences of decoupling for various soft operators. We will assume throughout the rest of the paper that the space-time dimension $D = n + 2$ is even.

3.1 Leading soft-photon or infinite-dimensional $U(1)$

The leading soft-photon $S_a(x)$ is a $(\Delta = 1, l = 1)$ conformal primary and it has a primary descendant at level-1, given by,

$$F_{ab} = \partial_a S_b - \partial_b S_a \quad (3.1)$$

The symmetry transformation acting on S_a is given by,²

$$S_a \rightarrow S_a + \partial_a \phi \quad (3.2)$$

²See the appendix for a discussion of this.

where the transformation parameter ϕ is a $(\Delta = 0, l = 1)$ conformal primary. The null state F_{ab} is invariant under this transformation and so we set it to zero, i.e.,

$$F_{ab} = \partial_a S_b - \partial_b S_a = 0 \quad (3.3)$$

If we think of this as a classical field equation then this has solution,

$$S_a(x) = \partial_a \chi(x) \quad (3.4)$$

Now, at the level of S -matrix this implies that,

$$\langle \{p^{\text{out}}\} | S_a(x) | \{p^{\text{in}}\} \rangle = \partial_a \chi(x, \{p^{\text{out}}\}, \{p^{\text{in}}\}) \quad (3.5)$$

So the S -matrix elements corresponding to different polarisations of the soft-photon are determined in terms of a *single scalar function* χ . So, the *decoupling of null-states* $F_{ab}(x)$ *effectively reduces the number of polarization states of the soft-photon from* $n = D - 2$ *to* 1. This is consistent with the fact that the transformation parameter $\phi(x)$ is a scalar.

Let us discuss this a bit more. Setting $F_{ab} = 0$ *effectively* reduces the number of degrees of freedom of *soft photons* to 1, but the number of degrees of freedom of hard or energetic photons remain the same. That, such *effective* reduction is necessary, can be seen also from Weinberg's soft photon theorem. For example, in any space-time dimension D , it has been observed [35] that the combination $\partial_a S_b - \partial_b S_a$ decouples from the S -matrix. From the point of view of asymptotic symmetry, this means, for example in $D = 3 + 1$, that *the positive and negative helicity soft photons generate essentially the same symmetry*. $S_a = \partial_a \phi$ encodes this fact in the sense that the soft photons of every helicity is essentially a derivative of a scalar function ϕ which is the (large) gauge transformation parameter.

Another point of view will be the solvability of the Ward-identity for asymptotic symmetries. For $U(1)$ there is only one Ward-identity or to be more precise, one Ward-identity for every point on the celestial sphere. This follows from the fact that in this case the transformation parameter ϕ is a scalar and so has one degree of freedom per point on the celestial sphere. On the other hand, the number of soft theorems is equal to the number of helicity states of the soft photon. Now if we want a one-one correspondence between Ward-identity and soft theorem, which turns out to be the case in practice, then it is clear that we need an *effective* reduction in the number of helicity states of the soft photons. $S_a = \partial_a \phi$ essentially achieves that.

3.2 Leading soft-graviton or supertranslation

3.2.1 $n = D - 2 > 2$

The leading soft-graviton is a $(\Delta = 1, l = 2)$ conformal primary which we denote by $S_{ab}(x)$. For $D > 4$, $S_{ab}(x)$ has a unique level-1 null-state given by,

$$O_{abc}(x) = \partial_a S_{bc} - \partial_b S_{ac} + \frac{1}{n-1} \left(\delta_{bc} \partial_d S_{da} - \delta_{ac} \partial_d S_{db} \right) \quad (3.6)$$

The operator O_{abc} satisfies,

$$O_{abc} = -O_{bac}, \quad O_{aba} = O_{abb} = 0 \quad (3.7)$$

The symmetry transformation acting on S_{ab} is given by,

$$S_{ab} \rightarrow S_{ab} + \left(\partial_a \partial_b - \frac{1}{n} \delta_{ab} \partial^2 \right) f(x) \quad (3.8)$$

where the transformation parameter $f(x)$ is a $(\Delta = -1, l = 0)$ conformal primary. The null-state $O_{abc}(x)$ is invariant under this transformation, i.e.,

$$O[S_{ab}] = O \left[S_{ab} + \left(\partial_a \partial_b - \frac{1}{n} \delta_{ab} \partial^2 \right) f(x) \right] \quad (3.9)$$

So we have to set the null-state O_{abc} to zero, i.e.,

$$O_{abc}(x) = \partial_a S_{bc} - \partial_b S_{ac} + \frac{1}{n-1} \left(\delta_{bc} \partial_d S_{da} - \delta_{ac} \partial_d S_{db} \right) = 0 \quad (3.10)$$

Thinking of this as a classical field equation, let us now show that,

$$O_{abc}(x) = 0 \Rightarrow S_{ab}(x) = \left(\partial_a \partial_b - \frac{1}{n} \delta_{ab} \partial^2 \right) F(x) \quad (3.11)$$

for some function $F(x)$. We now give a formal proof of this.

Proof. We have to solve the equation

$$O_{abc}(x) = \partial_a S_{bc} - \partial_b S_{ac} + \frac{1}{n-1} \left(\delta_{bc} \partial_d S_{da} - \delta_{ac} \partial_d S_{db} \right) = 0 \quad (3.12)$$

Now,

$$\partial_c O_{abc}(x) = 0 \Rightarrow \partial_a \partial_c S_{cb} - \partial_b \partial_c S_{ca} = 0 \quad (3.13)$$

Define

$$V_a = \partial_b S_{ab} \quad (3.14)$$

In terms of V_a , (3.13) becomes

$$\partial_a V_b - \partial_b V_a = 0 \Rightarrow V_a = \partial_b S_{ab} = \partial_a \phi(x) \quad (3.15)$$

Now we also have,

$$\partial_a O_{abc}(x) = \partial^2 S_{bc} - \frac{n}{n-1} \left(\partial_b \partial_c - \frac{1}{n} \delta_{bc} \partial^2 \right) \phi(x) = 0 \quad (3.16)$$

where we have used (3.15). The solution of (3.16) can be written as,

$$S_{ab} = \frac{n}{n-1} \left(\partial_a \partial_b - \frac{1}{n} \delta_{ab} \partial^2 \right) \psi(x) + \tilde{S}_{ab}(x) \quad (3.17)$$

where,

$$\partial^2 \psi = \phi, \quad \partial^2 \tilde{S}_{ab}(x) = 0, \quad \partial_a \tilde{S}_{ab}(x) = 0 \quad (3.18)$$

Now substituting (3.17) into (3.12) we get,

$$\partial_a \tilde{S}_{bc} - \partial_b \tilde{S}_{ac} = 0 \quad (3.19)$$

where we have used $\partial_a \tilde{S}_{ab} = 0$. The general solution of (3.19) can be written as,

$$\tilde{S}_{ab} = \partial_a \xi_b(x) \quad (3.20)$$

where ξ_a is some vector field. Now we impose the constraints that $\tilde{S}_{ab}$ is symmetric and traceless. This gives us,

$$\tilde{S}_{ab} = \tilde{S}_{ba} \Rightarrow \partial_a \xi_b - \partial_b \xi_a = 0 \Rightarrow \xi_a(x) = \partial_a \chi(x) \quad (3.21)$$

for some scalar field $\chi(x)$. Therefore,

$$\tilde{S}_{ab}(x) = \partial_a \partial_b \chi(x) \quad (3.22)$$

Tracelessness of $\tilde{S}_{ab}(x)$ further implies that

$$\partial^2 \chi(x) = 0 \quad (3.23)$$

Therefore we can write,

$$\tilde{S}_{ab}(x) = \left(\partial_a \partial_b - \frac{1}{n} \delta_{ab} \partial^2 \right) \chi(x) \quad (3.24)$$

Now if we define

$$F(x) = \frac{n}{n-1} \psi(x) + \chi(x) \quad (3.25)$$

then we can write,

$$S_{ab}(x) = \left(\partial_a \partial_b - \frac{1}{n} \delta_{ab} \partial^2 \right) F(x) \quad (3.26)$$

So $S_{ab}(x)$ has the “pure-gauge” form and $O_{abc}(x)$ can be thought of as the “field-strength” for the Supertranslation transformation.

Now at the level of S -matrix this implies that,

$$\langle \{p^{\text{out}}\} | S_{ab}(x) | \{p^{\text{in}}\} \rangle = \left(\partial_a \partial_b - \frac{1}{n} \delta_{ab} \partial^2 \right) \psi(x, \{p^{\text{out}}\}, \{p^{\text{in}}\}) \quad (3.27)$$

So again we can see that the decoupling of the null-state effectively reduces the number of polarization states of soft-graviton to 1. In the case of leading soft graviton, the asymptotic symmetry is Supertranslation which is generated by one function on the celestial sphere. Therefore this reduction is consistent with the existence of supertranslation symmetry.

3.2.2 $D = 4$

In this case there are two leading soft gravitons with scaling dimension and spin given by $(\Delta = 1, l = \pm 2)$. They are $\text{SL}(2, \mathbb{C})$ primaries with weights given by, $(h = 3/2, \bar{h} = -1/2)$ and $(h = -1/2, \bar{h} = 3/2)$, respectively. Let us denote the primaries by $S_{zz}(z, \bar{z})$ and $S_{\bar{z}\bar{z}}(z, \bar{z})$, respectively. Now one can check that there are two primary descendants or null-states at level-2 given by,

$$O_1(z, \bar{z}) = \partial_{\bar{z}}^2 S_{zz}, \quad O_2(z, \bar{z}) = \partial_z^2 S_{\bar{z}\bar{z}} \quad (3.28)$$

Now the symmetry transformation acting on the soft-gravitons is given by,

$$S_{zz} \rightarrow S_{zz} + \partial_z^2 f(z, \bar{z}), \quad S_{\bar{z}\bar{z}} \rightarrow S_{\bar{z}\bar{z}} + \partial_{\bar{z}}^2 f(z, \bar{z}) \quad (3.29)$$

where $f(z, \bar{z})$ is a $\text{SL}(2, \mathbb{C})$ primary of weight $(h = -1/2, \bar{h} = -1/2)$. Now the unique linear combinations of the null-states O_1 and O_2 , which is invariant under the symmetry transformation, is given by,

$$O(z, \bar{z}) = O_1(z, \bar{z}) - O_2(z, \bar{z}) = \partial_{\bar{z}}^2 S_{zz} - \partial_z^2 S_{\bar{z}\bar{z}} \quad (3.30)$$

This is the null-state which decouples, i.e.,

$$O(z, \bar{z}) = \partial_{\bar{z}}^2 S_{zz} - \partial_z^2 S_{\bar{z}\bar{z}} = 0 \quad (3.31)$$

3.3 Subleading soft graviton or superrotation

3.3.1 $D = 4, n = 2$

In $D = 4$ there are two subleading soft gravitons with scaling dimension $\Delta = 0$ and spin $l = \pm 2$. They are $\text{SL}(2, \mathbb{C})$ primaries with weights given by, $(h = 1, \bar{h} = -1)$ and $(h = -1, \bar{h} = 1)$, respectively. We denote them by S_{zz} and $S_{\bar{z}\bar{z}}$, respectively. Now there are two null-states at level-3, given by,

$$O_z^{(1)}(z, \bar{z}) = \partial_{\bar{z}}^3 S_{zz}, \quad O_z^{(2)}(z, \bar{z}) = \partial_z^3 S_{\bar{z}\bar{z}} \quad (3.32)$$

Now the symmetry transformation acting on the subleading soft-gravitons is given by,

$$S_{zz} \rightarrow S_{zz} + 2\partial_z V_z, \quad S_{\bar{z}\bar{z}} \rightarrow S_{\bar{z}\bar{z}} + 2\partial_{\bar{z}} V_{\bar{z}} \quad (3.33)$$

where V_z is a $\text{SL}(2, \mathbb{C})$ primary with weight $(h = 0, \bar{h} = -1)$ and $V_{\bar{z}}$ is the complex conjugate with weight $(h = -1, \bar{h} = 0)$.

Now we can see that both the primary descendants, $O_z^{(1)}$ and $O_z^{(2)}$, are not invariant under the symmetry transformation and also, we cannot take any linear combination of them because they carry different quantum numbers. Therefore, *in $D = 4$, no primary descendant of the subleading soft graviton decouples from the S -matrix.*

This is an example where there are null-states but still *they do not decouple from the S -matrix elements because none of them are invariant under the symmetry transformation (3.33).* In other words, we cannot set them equal to zero. This is consistent with the fact that the transformation parameter is a vector field with two components and graviton also has two polarization states.

3.3.2 $D = 6, n = 4$

The situation is different in $D = 6$ (and higher) dimensions. The subleading soft graviton, denoted by $S_{ab}(x)$ is a conformal primary with $\Delta = 0$ and $l = 2$ and the symmetry transformation acting on S_{ab} is given by,

$$S_{ab} \rightarrow S_{ab} + \partial_a V_b + \partial_b V_a - \frac{1}{2} \delta_{ab} \partial \cdot V \quad (3.34)$$

where V_a is a $(\Delta = -1, l = 1)$ conformal primary.

Now the transformation parameter $V_a(x)$ is a vector field with 4 components and the number of polarization states of the graviton is 9. So there is a mismatch between the number of Ward-identities and the number of soft S -matrix elements. This will be cured if some null-state of $S_{ab}(x)$ decouples from the S -matrix elements.

Now, $S_{ab}(x)$ has a null state at level-2 given by,

$$\begin{aligned}
 W_{abcd} = & \partial_a \partial_c S_{bd} - \partial_a \partial_d S_{bc} - \partial_b \partial_c S_{ad} + \partial_b \partial_d S_{ac} \\
 & - \frac{1}{2} \delta_{ac} \partial^2 S_{bd} + \frac{1}{2} \delta_{ad} \partial^2 S_{bc} + \frac{1}{2} \delta_{bc} \partial^2 S_{ad} - \frac{1}{2} \delta_{bd} \partial^2 S_{ac} \\
 & + \frac{1}{2} \delta_{ac} \partial_b \partial_e S_{de} + \frac{1}{2} \delta_{ac} \partial_d \partial_e S_{be} - \frac{1}{2} \delta_{ad} \partial_b \partial_e S_{ce} - \frac{1}{2} \delta_{ad} \partial_c \partial_e S_{be} \\
 & - \frac{1}{2} \delta_{bc} \partial_a \partial_e S_{de} - \frac{1}{2} \delta_{bc} \partial_d \partial_e S_{ae} + \frac{1}{2} \delta_{bd} \partial_a \partial_e S_{ce} + \frac{1}{2} \delta_{bd} \partial_c \partial_e S_{ae} \\
 & - \frac{1}{3} \delta_{ac} \delta_{bd} \partial_e \partial_f S_{ef} + \frac{1}{3} \delta_{ad} \delta_{bc} \partial_e \partial_f S_{ef}
 \end{aligned} \tag{3.35}$$

As the notation suggests, the null (field) state W_{abcd} is essentially the *linearization of the Weyl tensor* around flat space δ_{ab} and S_{ab} is the trace-free part of the “metric fluctuation”.

Now it is obvious that,

$$W[S_{ab}] = W\left[S_{ab} + \partial_a V_b + \partial_b V_a - \frac{1}{2} \delta_{ab} \partial \cdot V\right] \tag{3.36}$$

Therefore we have to set the null state W_{abcd} to zero, i.e.,

$$W_{abcd}[S] = 0 \tag{3.37}$$

is the decoupling equation. Thought of as a classical field equation this has the solution,

$$S_{ab} = \partial_a \xi_b + \partial_b \xi_a - \frac{1}{2} \delta_{ab} \partial \cdot \xi \tag{3.38}$$

Here we have taken into account that S_{ab} is traceless.

At the level of S -matrix this implies that,

$$\begin{aligned}
 \langle \{p^{\text{out}}\} | S_{ab}(x) | \{p^{\text{in}}\} \rangle = & \partial_a \psi_b(x, \{p^{\text{out}}\}, \{p^{\text{in}}\}) + \partial_b \psi_a(x, \{p^{\text{out}}\}, \{p^{\text{in}}\}) \\
 & - \frac{1}{2} \delta_{ab} \partial \cdot \psi(x, \{p^{\text{out}}\}, \{p^{\text{in}}\})
 \end{aligned} \tag{3.39}$$

So we can see that in $D = 6$, the decoupling of the null state W_{abcd} , effectively reduces the 9 polarisation states of the subleading soft-graviton to 4. From the symmetry and soft-theorem point of view this is expected. Superrotation generators are vector fields on the celestial sphere. In $D = 5 + 1$ the number of independent vector fields at every point on the celestial sphere S^4 is 4 and the number of degrees of freedom of soft graviton in $D = 5 + 1$ is 9. So if we want to derive the soft theorem from the superrotation Ward-identity then some null state must decouple to (effectively) reduces the number of degrees of freedom of the subleading soft graviton. Indeed, the vanishing of the linearised Weyl tensor guarantees this reduction.

We would like to mention that one of the interesting problems to study will be the extension of this picture to *odd* space-time dimensions. Recently, significant progress has been made in understanding asymptotic symmetries in odd space-time dimensions [53]. We leave this problem for future study.

4 Consistency with soft-theorems: massive case

In this section we will explicitly verify that the forms of the soft S -matrix elements obtained from the decoupling equations match with that obtained from the soft theorems. We work with massive external states. Massless case can be studied by taking appropriate limit [37].

In this section we have defined $\eta_k = +1$ for the k -th outgoing particle and $\eta_k = -1$ for the k -th incoming particle. We have also taken the soft particle to be outgoing and for simplicity we have assumed the external states to be scalars. The parametrization of the momentum of a massive particle is discussed in the appendix.

4.1 Leading soft photon or U(1)

In terms of the leading soft-photon operator $S_a(\vec{w})$, Weinberg's soft photon theorem [21, 22] can be written as,

$$\begin{aligned} & \langle \{y_i, \vec{x}_i, Q_i\}, \text{out} | S_a(\vec{w}) | \{y_j, \vec{x}_j, Q_j\}, \text{in} \rangle \\ &= \left[\sum_k \eta_k Q_k \frac{p(y_k, \vec{x}_k) \cdot \epsilon_a(\vec{w})}{p(y_k, \vec{x}_k) \cdot \hat{q}(\vec{w})} \right] \langle \{y_i, \vec{x}_i, Q_i\}, \text{out} | \{y_j, \vec{x}_j, Q_j\}, \text{in} \rangle \end{aligned}$$

Here Q_k is the charge of the k -th particle and $\epsilon_a(\vec{w})$ is the polarization vector of the soft photon.

The soft-factor can be calculated to be,

$$\left[\sum_k \eta_k Q_k \frac{p(y_k, \vec{x}_k) \cdot \epsilon_a(\vec{w})}{p(y_k, \vec{x}_k) \cdot \hat{q}(\vec{w})} \right] = \frac{\partial}{\partial w_a} \left[\sum_k \eta_k \frac{1}{2} Q_k \ln(y_k^2 + |\vec{w} - \vec{x}_k|^2) \right] \quad (4.1)$$

To arrive at this expression we have used the explicit formulas of the momenta and polarisation vector given in the appendix.

Using this we can write the matrix element of the soft-photon operator as,

$$\langle \{y_i, \vec{x}_i, Q_i\}, \text{out} | S_a(\vec{w}) | \{y_j, \vec{x}_j, Q_j\}, \text{in} \rangle = \frac{\partial}{\partial w_a} \Lambda(\vec{w}, \{y, \vec{x}, Q\}) \quad (4.2)$$

where,

$$\Lambda(\vec{w}, \{y, \vec{x}, Q\}) = \left[\sum_k \eta_k \frac{1}{2} Q_k \ln(y_k^2 + |\vec{w} - \vec{x}_k|^2) \right] \langle \{y_i, \vec{x}_i, Q_i\}, \text{out} | \{y_j, \vec{x}_j, Q_j\}, \text{in} \rangle \quad (4.3)$$

(4.2) matches with the form (3.5) of the leading soft photon theorem obtained from the decoupling of the null-state.

4.2 Leading soft graviton or supertranslation

In terms of the leading soft graviton operator $S_{ab}(\vec{w})$, Weinberg's soft graviton theorem [21, 22] can be written as,

$$\begin{aligned} & \langle \{y_i, \vec{x}_i\}, \text{out} | S_{ab}(\vec{w}) | \{y_j, \vec{x}_j\}, \text{in} \rangle \\ &= \left[\sum_k \eta_k \frac{p_\mu(y_k, \vec{x}_k) p_\nu(y_k, \vec{x}_k) \epsilon_{ab}^{\mu\nu}(\vec{w})}{p(y_k, \vec{x}_k) \cdot \hat{q}(\vec{w})} \right] \langle \{y_i, \vec{x}_i\}, \text{out} | \{y_j, \vec{x}_j\}, \text{in} \rangle \end{aligned}$$

where $\epsilon_{ab}^{\mu\nu}(\vec{w})$ is the graviton polarization vector.

Now the soft-factor can be computed to be,

$$\gamma \left[\sum_k \eta_k \frac{m_k}{y_k} \left(\delta_{ab} - n \frac{(w_a - x_{ka})(w_b - x_{kb})}{y_k^2 + |\vec{w} - \vec{x}_k|^2} \right) \right] = \left(\partial_a^w \partial_b^w - \frac{1}{n} \delta_{ab} \partial_w^2 \right) F(\vec{w}, \{y, \vec{x}, m\}) \quad (4.4)$$

where,

$$F(\vec{w}, \{y, \vec{x}, m\}) = \frac{-n\gamma}{2} \left[\sum_k \eta_k m_k \frac{y_k^2 + |\vec{w} - \vec{x}_k|^2}{2y_k} \ln(y_k^2 + |\vec{w} - \vec{x}_k|^2) \right] \quad (4.5)$$

Here m_k is the mass of the k -th particle and γ is a numerical constant whose precise value is not important for us. Now, using this we can write,

$$\langle \{y_i, \vec{x}_i\}, \text{out} | S_{ab}(\vec{w}) | \{y_j, \vec{x}_j\}, \text{in} \rangle = \left(\partial_a^w \partial_b^w - \frac{1}{n} \delta_{ab} \partial_w^2 \right) \Lambda(\vec{w}, \{y, \vec{x}, m\}) \quad (4.6)$$

where we have defined,

$$\Lambda(\vec{w}, \{y, \vec{x}, m\}) = F(\vec{w}, \{y, \vec{x}, m\}) \langle \{y_i, \vec{x}_i\}, \text{out} | \{y_j, \vec{x}_j\}, \text{in} \rangle \quad (4.7)$$

(4.6) matches with the form (3.27) of the leading soft graviton theorem obtained from the decoupling of the null-state

4.3 Subleading soft graviton theorem in $D = 6$ ($n = 4$) or superrotation

In terms of the subleading soft graviton operator — again denoted by $S_{ab}(\vec{w})$ — Cachazo-Strominger subleading soft graviton theorem [23] can be written as,

$$\begin{aligned} & \langle \{y_i, \vec{x}_i\}, \text{out} | S_{ab}(\vec{w}) | \{y_j, \vec{x}_j\}, \text{in} \rangle \\ &= -i \sum_k \eta_k \frac{\epsilon_{ab}^{\mu\nu}(\vec{w}) p_\mu^k q^\rho}{p^k \cdot q} J_{\rho\nu}^k \langle \{y_i, \vec{x}_i\}, \text{out} | \{y_j, \vec{x}_j\}, \text{in} \rangle \end{aligned} \quad (4.8)$$

Since all the external particles are scalars, the (orbital) angular momentum operator is given by,

$$J_{\rho\nu}^k = -i \left(p_\rho^k \frac{\partial}{\partial p_\nu^k} - p_\nu^k \frac{\partial}{\partial p_\rho^k} \right) \quad (4.9)$$

Now a straightforward calculation gives,

$$\langle \{y_i, \vec{x}_i\}, \text{out} | S_{ab}(\vec{w}) | \{y_j, \vec{x}_j\}, \text{in} \rangle = \partial_a^w \zeta_b + \partial_b^w \zeta_a - \frac{1}{2} \delta_{ab} \partial^w \cdot \zeta \quad (4.10)$$

where the vector ζ is given by,

$$\begin{aligned} \zeta_a(\vec{w}, \{y, \vec{x}\}) = & \frac{\gamma}{2} \sum_k \eta_k \left[\ln(y_k^2 + |\vec{w} - \vec{x}_k|^2) \left\{ -(w_a - x_{ka}) y_k \frac{\partial}{\partial y_k} \right. \right. \\ & \left. \left. + \left((w_a - x_{ka})(w_c - x_{kc}) - \frac{1}{2}(y_k^2 + |\vec{w} - \vec{x}_k|^2) \delta_{ac} \right) \frac{\partial}{\partial x_{kc}} \right\} \langle \{y_i, \vec{x}_i\}, \text{out} | \{y_j, \vec{x}_j\}, \text{in} \rangle \right] \end{aligned} \quad (4.11)$$

Here γ is a numerical constant whose exact value is not important for us.

(4.10) matches with the form (3.39) of the subleading soft-graviton theorem obtained from the decoupling of the null-state.

5 Comments on the construction of the soft-charge

In *even* space-time dimensions the *soft-charge* [1–19] corresponding to a particular symmetry transformation can always be written as,

$$Q_S[\xi] = \int d^{D-2}x \, \xi(x) \cdot O(x) \quad (5.1)$$

where $\xi(x)$ is the transformation parameter and $O(x)$ is a local operator, which is also a *primary* of the (Lorentz) conformal group [20]. The tensor indices of $O(x)$ are dual to those of $\xi(x)$, so that the *contraction* $\xi(x) \cdot O(x)$ is a *scalar of dimension* $(D-2)$. In even space-time dimension $O(x)$ can be constructed as a descendant of the soft operator. Let us now illustrate this with an example.

Consider the case of subleading soft-graviton in $D=6$. In this case we know that the transformation parameter is a $(\Delta = -1, l = 1)$ primary which we denote by $\xi_a(x)$. So the soft-charge can be written as,

$$Q_S[\xi] = \int d^4x \, \xi_a(x) O_a(x) \quad (5.2)$$

Since $\xi_a(x) O_a(x)$ should have dimension 4 and $\xi(x)$ has dimension $\Delta = -1$, $O_a(x)$ *should be a vector primary with dimension* $\Delta = 5$. Now $O_a(x)$ can be constructed as a descendant of the subleading soft-graviton $S_{ab}(x)$ and is given by,

$$O_a(x) \sim \partial^2 \partial_a \partial_b \partial_c S_{bc} - \frac{3}{4} \partial^4 \partial_b S_{ab}(x) \quad (5.3)$$

Here we have used $\sim$ because the overall normalization of the charge cannot be determined in this way. Now since the subleading soft graviton is a $(\Delta = 0, l = 2)$ primary, $O_a(x)$ constructed in this way clearly has dimension $\Delta = 5$ and spin $l = 1$. One can check by straightforward calculation that $O_a(x)$, *given in (5.3), is a primary operator as long as the decoupling equation (3.37) for the subleading soft graviton $S_{ab}(x)$ holds.*

6 Comparison with [20]

Before we conclude we would like to compare the results of [20] with the current paper. For the sake of concreteness let us compare the decoupling equations for the leading soft-graviton $S_{ab}(x)$ which is a $(\Delta = 1, l = 2)$ primary.

In our current approach the decoupling equation (3.10) is given by,

$$O_{abc}(x) = \partial_a S_{bc} - \partial_b S_{ac} + \frac{1}{3} \left(\delta_{bc} \partial_d S_{da} - \delta_{ac} \partial_d S_{db} \right) = 0 \quad (6.1)$$

where we have specialized to the case of $D = 6$ or $n = 4$.

Now in [20], starting from the Lorentz invariance of the S -matrix and the Ward identity for the supertranslation, we arrived at a somewhat different decoupling equation,

$$\tilde{O}_{ab} = \partial^2 S_{ab} - \frac{2}{3} \left(\partial_a \partial_c S_{cb} + \partial_b \partial_c S_{ca} \right) + \frac{1}{3} \delta_{ab} \partial_c \partial_d S_{cd} = 0 \quad (6.2)$$

Now one can easily check that,

$$\tilde{O}_{bc} = \frac{1}{2} (\partial_a O_{abc} + \partial_a O_{acb}) \quad (6.3)$$

Therefore $\tilde{O}_{ab}$ is a descendant of O_{abc} — in fact a primary descendant — and the decoupling equation

$$O_{abc} = 0 \Rightarrow \tilde{O}_{ab} = 0 \quad (6.4)$$

But,

$$\tilde{O}_{ab} = 0 \not\Rightarrow \tilde{O}_{abc} = 0 \quad (6.5)$$

Therefore (6.1) is a stronger condition than (6.2).

Now, in the case of leading soft graviton or supertranslation, the operator $O(x)$ appearing in the soft charge must be a $(\Delta = 5, l = 0)$ primary in $D = 6$. Therefore a potential candidate for $O(x)$ is the unique scalar operator $\partial^2 \partial_a \partial_b S_{ab}$ of dimension $\Delta = 5$. In [20], the decoupling equation (6.2) was obtained by demanding that the operator $\partial^2 \partial_a \partial_b S_{ab}$ be a primary. Now the same thing can also be achieved by using the decoupling equation (6.1). To be more precise, under an infinitesimal special conformal transformation, $O(x) = \partial^2 \partial_a \partial_b S_{ab}$ transforms as [20],

$$O'(x') = (1 + 10 \epsilon \cdot x) O(x) + 8 \epsilon_a \partial_c (\partial_a \partial_b S_{bc}(x) - \partial_b \partial_c S_{ab}(x)) \quad (6.6)$$

Now,

$$O_{abc} = 0 \Rightarrow \partial_c O_{abc}(x) = 0 \Rightarrow \partial_a \partial_c S_{cb} - \partial_b \partial_c S_{ca} = 0 \quad (6.7)$$

Therefore, owing to the decoupling equation (6.1), the inhomogeneous term in (6.6) vanish and $\partial^2 \partial_a \partial_b S_{ab}$ indeed transforms like a primary.

Although the two formulations seem to be mutually consistent, the current one is more general and cannot be simply derived from Lorentz invariance and the existence of infinite dimensional asymptotic symmetries. For example, the decoupling equation (6.1) of this paper is *sufficient* for the Lorentz invariance of the supertranslation Ward identity. Now, it is natural to conjecture that this is also a *necessary* condition, although the reason may not be Lorentz invariance and asymptotic symmetry. In fact, the current formulation is very natural from a dual CFT perspective, if there is one. We hope to return to these interesting questions in future.

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A Notation and conventions

For the convenience of the reader we collect some useful formulas here. For details we refer to [35, 37].

In D dimensional Minkowski space-time we parametrize the null momentum $p^\mu(\omega, \vec{x})$ of a massless particle as,

$$p^\mu = \omega(1 + \vec{x}^2, 2\vec{x}, 1 - \vec{x}^2), \quad \vec{x} \in R^{D-2} = R^n \quad (\text{A.1})$$

where ω is a real number. The Lorentz group $\text{SO}(D-1, 1) = \text{SO}(n+1, 1)$ acts on R^n as the group of conformal transformations. The corresponding transformation of ω is given by,

$$\omega' = \left| \frac{\partial \vec{x}'}{\partial \vec{x}} \right|^{-\frac{1}{n}} \omega \quad (\text{A.2})$$

The polarisation vector of the photon is given by [37],

$$\epsilon_a^\mu(\vec{x}) = 2(x_a, \delta_{ab}, -x_a) \quad (\text{A.3})$$

The graviton polarisation tensor $\epsilon_{ab}^{\mu\nu}(\vec{x})$ is defined in terms of the photon polarization as, $\epsilon_{ab}^{\mu\nu}(\vec{x}) = \epsilon_a^\mu(\vec{x})\epsilon_b^\nu(\vec{x})$.

We parametrize the energy-momentum vector of a massive particle of mass m as [36, 37],

$$p^\mu(y, \vec{x}) = m \left(\frac{1 + y^2 + \vec{x}^2}{2y}, \frac{\vec{x}}{y}, \frac{1 - y^2 - \vec{x}^2}{2y} \right) \quad (\text{A.4})$$

Here $(y, \vec{x})$ are the Poincare coordinates on the unit (mass) hyperboloid, $p^2 = -1$. So $\vec{x}$ are the Cartesian coordinates of the (conformal) boundary R^n at infinity, on which the soft-operators live.

In the paper we omit the vector sign on $\vec{x}$ and simply denote it by x .

B Symmetry transformations

In this section we will see which primary fields $O_R^\delta(x)$ can give a null field $N_{a_1 \dots a_l}^\Delta$. The (Δ, l) we are interested in are $(1, 1)$, $(1, 2)$ and $(0, 2)$ corresponding to leading photon, leading graviton and subleading graviton respectively. If $O_R^\delta(x)$ has a primary descendant in the irreducible representation I of $\text{SO}(n)$ at level N then the following necessary but not sufficient condition, valid for $n > 2$, is satisfied,

$$\delta = \frac{1}{2N} (C_R + NC_V - N(N-1) - C_I), \quad N \in \{1, 2, 3 \dots\} \quad (\text{B.1})$$

$$\begin{aligned}
\text{for } l = 1 : & \quad \boxed{1 \ 2 \ 3 \ \dots \ s}, \quad \boxed{\begin{array}{cccc} 1 & 2 & 3 & \dots & i \\ 1 & & & & \end{array}}, \quad i \geq 1 \\
\text{for } l = 2 : & \quad \boxed{1 \ 2 \ 3 \ \dots \ s}, \quad \boxed{\begin{array}{cccc} 1 & 2 & 3 & \dots & i \\ 1 & & & & \end{array}}, \quad \boxed{\begin{array}{cccc} 1 & 2 & 3 & \dots & i \\ 1 & 2 & & & \end{array}}, \quad i \geq 2
\end{aligned}$$

Figure 1. Representations R of $O_R^\delta(x)$ which can have a descendant of spin- l .

Here $C_R, C_V(\text{vector})$ and C_I are the quadratic Casimirs for respective representations of $SO(n)$. This formula can be obtained by a simple extension of the method by [55].

If representations R and I are symmetric-traceless then this condition is consistent with the necessary and also sufficient constraints in [26]. Since we are interested in primary descendants (Δ, l) so

$$\delta = \Delta - N \quad \text{and} \quad C_I = l(l + n - 2) \quad (\text{B.2})$$

The derivatives generating the descendant are symmetric so $O_R^\delta(x)$ can give a spin $l = 1$ or 2 descendant only for the representations in figure 1. Casimir for representation R_{ij} is

$$\begin{aligned}
R_{ij} : & \quad \boxed{\begin{array}{cccc} 1 & 2 & 3 & \dots & \dots & i \\ 1 & 2 & \dots & j & & \end{array}} \\
C_{ij} = & \quad i(i + n - 2) + j(j + n - 4) \quad (\text{B.3})
\end{aligned}$$

Now, from the condition (B.1) we find that only one representation of $O_R^\delta(x)$ remains for each case in interest of (Δ, l) and the only possible $N_{a_1 a_2 a_3 \dots a_l}^\Delta$ are

$$\begin{aligned}
N_a^1 &= \partial_a O^0(x) \\
N_{ab}^1 &= \left(\partial_a \partial_b - \frac{1}{n} \delta_{ab} \partial^2 \right) O^{-1}(x) \\
N_{ab}^0 &= \left(\partial_a O_b^{-1} + \partial_b O_a^{-1} - \frac{2}{n} \delta_{ab} \partial \cdot O^{-1} \right) \quad (\text{B.4})
\end{aligned}$$

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