Rectangular W-algebras, extended higher spin gravity and dual coset CFTs

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ABSTRACT: We analyze the asymptotic symmetry of higher spin gravity with $M \times M$ matrix valued fields, which is given by rectangular W-algebras with $\mathfrak{su}(M)$ symmetry. The matrix valued extension is expected to be useful for the relation between higher spin gravity and string theory. With the truncation of spin as $s = 2, 3, \dots, n$, we evaluate the central charge c of the algebra and the level k of the affine currents with finite c, k . For the simplest case with $n = 2$, we obtain the operator product expansions among generators by requiring their associativity. We conjecture that the symmetry is the same as that of Grassmannian-like coset based on our proposal of higher spin holography. Comparing c, k from the both theories, we obtain the map of parameters. We explicitly construct low spin generators from the coset theory, and, in particular, we reproduce the operator product expansions of the rectangular W-algebra for $n = 2$. We interpret the map of parameters by decomposing the algebra in the coset description.

KEYWORDS: AdS-CFT Correspondence, Conformal and W Symmetry, Higher Spin Gravity

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1 Introduction

Higher spin symmetry is believed to be useful to investigate the tensionless limit of string theory [1]. A higher spin gauge theory on AdS space is given by a Vasiliev theory [2] with a gauge field for each spin $s = 2, 3, \dots$, and the theory is expected to describe the first Regge trajectory of strings. In order to explain the higher Regge trajectories as well, we may consider the Vasiliev theories with $M \times M$ matrix valued fields, see [3] for higher tensor generalizations. It was proposed in [4] that the matrix extension of 4d Vasiliev theory is dual to the Aharony-Bergman-Jafferis(-Maldacena) theory [5, 6], and the duality implies a connection between the higher spin gravity and superstrings on $\text{AdS}_4 \times \mathbb{CP}^3$. In our previous work [7], we examined the 3d Prokushkin-Vasiliev theory with matrix valued fields [8], and we claimed that the theory is dual to a 2d coset model.¹ In this paper,

¹Several generalizations of the holographic duality were discussed in [9].

we extend the analysis on the asymptotic symmetry of the 3d higher spin gravity beyond the classical limit. The symmetry is infinite dimensional and should be useful to analyze quantum corrections. In [7], we have considered a bosonic and a $\mathcal{N} = 2$ supersymmetric holography. Without the matrix extension, i.e., with $M = 1$, the bosonic holography essentially reduces to the one by [10]. Moreover, the $\mathcal{N} = 2$ holography with $M = 1$ is identical to that of [11]. In this paper, we mainly focus on the bosonic case.

For $M = 1$, the gauge algebra of the 3d higher spin theory is called as $\text{hs}[\lambda]$, which can be truncated to $\text{sl}(n)$ for $\lambda = n$ ($n = 2, 3, \dots$). After the matrix extension, the gauge algebra becomes

$$\text{sl}(M) \otimes \mathbb{1}_n \oplus \mathbb{1}_M \otimes \text{sl}(n) \oplus \text{sl}(M) \otimes \text{sl}(n) \simeq \text{sl}(Mn), \quad (1.1)$$

see, e.g., [7, 12]. The gravitational sector corresponds to $\mathbb{1}_M \otimes \text{sl}(2) \subset \mathbb{1}_M \otimes \text{sl}(n)$ with the principal embedding of $\text{sl}(2)$ into $\text{sl}(n)$.² The $\text{sl}(2)$ embedding corresponds to the partition of Mn ;

$$Mn = n + n + \dots + n. \quad (1.2)$$

The asymptotic symmetry of higher spin gravity has been analyzed in [15–18]. Utilizing the arguments, the asymptotic symmetry can be obtained as the rectangular W-algebra³ from the Hamiltonian reduction of $\text{sl}(Mn)$ with the $\text{sl}(2)$ embedding (1.2). We can see that the affine $\text{su}(M)$ Lie algebra is included as a sub-algebra. We first compute the central charge c of the W-algebra and the level k of the affine $\text{su}(M)$ at the classical limit, and then obtain the exact results of c, k by applying the quantum Hamiltonian reduction. We further determine the operator product expansions (OPEs) of generators for $n = 2$ by assigning their associativity and show the uniqueness of the algebra, see [12] for the analysis at the classical limit.

In our previous work [7], we have proposed that the 3d Vasiliev-Prokushkin theory with $M \times M$ matrix valued fields is dual to the Grassmannian-like coset

$$\frac{\text{su}(N+M)_k}{\text{su}(N)_k \oplus \text{u}(1)_{kNM(N+M)}}. \quad (1.3)$$

The classical higher spin theory corresponds to the large N limit with the 't Hooft parameter⁴

$$\lambda = \frac{k}{k+N} \quad (1.4)$$

and M kept finite. The parameter is identified with λ for the higher spin algebra $\text{hs}[\lambda]$. As a strong support for our conjecture, we have shown the match of one-loop partition

²Different embeddings of $\text{sl}(2)$ were analyzed in [13, 14].

³The partition (1.2) corresponds to the rectangular Young tableau consisting of M columns of n boxes of height each, hence the name rectangular W-algebra.

⁴The definition of 't Hooft parameter in [7] is slightly different from the one here at finite k, N , though they are the same at the large N limit. The difference is important since we would like to analyze with finite k, N .

functions at the 't Hooft limit among others. Based on this duality, we propose here that the W-algebra obtained from the Hamiltonian reduction of $\mathfrak{sl}(Mn)$ is the same as the one for the coset (1.3) with $\lambda = n$. This leads to $k = -nN/(n-1)$, which implies that the coset theory is in a non-unitary regime. We provide the map of parameters by comparing the central charge of the algebra and the level of affine $\mathfrak{su}(M)$. We explicitly construct low spin currents transforming in the adjoint representation of $\mathfrak{su}(M)$, and reproduce the OPEs for the rectangular W-algebra with $n = 2$. We can use another 't Hooft parameter

$$\lambda = -\frac{k}{k+N+M} \quad (1.5)$$

in stead of (1.4), and this implies a duality relation of the coset model (1.3) as in the cases with $M = 1$ [19–21]. We give an explanation of the duality by decomposing the W-algebra in terms of the coset description as in [22–26].

1.1 The main conjecture

The main conjecture of this work is that the coset (1.3) at level k is isomorphic to the simple rectangular W-algebra of $\mathfrak{sl}(Mn)$ at level $-t$ where the levels are related via $k = -tn + Mn(n-1)$, if

$$k = -\frac{nN}{n-1} \quad \text{or} \quad k = -\frac{n(N+M)}{n+1}. \quad (1.6)$$

In the first case the first 't Hooft parameter (1.4) is

$$\lambda = \frac{k}{k+N} = n \quad (1.7)$$

while in the second case the second 't Hooft parameter (1.5) is

$$\lambda = -\frac{k}{k+N+M} = n \quad (1.8)$$

as well. We expect it to be a very difficult problem to give a general proof of this conjecture. We verify that central charges of theories and levels of current algebras agree. Furthermore in the second case $k = -n(N+M)/(n+1)$ we observe from the characters that the coset indeed has the required null vectors at conformal weight $n+1$. We then prove a uniqueness result of this type of W-algebra for the case $n = 2$, and various N, M . This means that a simple chiral algebra of this type with strong generators only in weight one and two is completely determined by the level of the current algebra and the central charge. We then also check that indeed there are no fields of higher spin for various cosets at $\lambda = 2$ so that we have proven our conjecture in these cases.

1.2 Organization

The organization of this paper is as follows; in the next section, we introduce the higher spin gravity with $M \times M$ matrix valued fields and the rectangular W-algebra as its asymptotic symmetry near the AdS boundary. For this, we simply apply the general procedure in [15–18]. In particular, we compute the central charge c of the W-algebra and the level k of

$\mathfrak{su}(M)$ currents at the classical limit as was done in [7] for the $\mathcal{N} = 2$ supersymmetry setup. In section 3, we start by providing some overview of W-algebras. We then propose the actions for the symmetry algebras. We obtain the exact expressions of c, k both from the general prescription and the proposed actions, and we find agreement. For the simplest case with $n = 2$, we compute the OPEs of generators using their associativity. In section 4, we realize the W-algebra in terms of coset description based on our proposed holography in [7]. Comparing c, k , we obtain the map of parameters between the algebras from the reduction of $\mathfrak{sl}(Mn)$ and the coset model (1.3). We then explicitly construct spin 2 currents transforming in the adjoint representation of $\mathfrak{su}(M)$ and reproduce the OPEs for $n = 2$. Decomposing the W-algebra in terms of coset model, we give an interpretation of the map of parameters. Section 5 is devoted to the summary of this paper and discussions on open problems. In appendix A, we construct spin 3 currents of the W-algebra from the coset model (1.3). In appendix B, we reproduce the decomposition of the W-algebra in terms of the algebra obtained from the reduction of $\mathfrak{sl}(Mn)$ for the simplest setup with $n = 2$ and $M = 2$.

2 Asymptotic symmetry of higher spin gravity

In this section, we examine the classical asymptotic symmetry of higher spin gravity near the AdS boundary. Higher spin gauge theory can be constructed from Chern-Simons gauge theory based on $\mathfrak{g} \oplus \mathfrak{g}$ Lie algebra. The action is given by

$$S = S_{\text{CS}}[A] - S_{\text{CS}}[\tilde{A}], \quad S_{\text{CS}}[A] = \frac{k_{\text{CS}}}{4\pi} \int \text{Tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right), \quad (2.1)$$

which is invariant under the gauge transformations

$$\delta A = d\Lambda + [A, \Lambda], \quad \delta \tilde{A} = d\tilde{\Lambda} + [\tilde{A}, \tilde{\Lambda}]. \quad (2.2)$$

The 1-form gauge fields $A, \tilde{A}$ and the gauge parameters $\Lambda, \tilde{\Lambda}$ take values in $\mathfrak{g}$. The theory describes pure AdS_3 gravity for $\mathfrak{g} = \mathfrak{sl}(2)$ [27, 28] and higher spin gravity for a higher rank $\mathfrak{g}$ [29]. In the next subsection, we introduce a higher spin algebra $\mathfrak{g}$ relevant for the higher spin gravity with the matrix extension. In subsection 2.2, we examine the asymptotic symmetry of the higher spin gravity at the classical limit applying the method of [15–18].

2.1 Higher spin algebra

We start from the 3d Prokushkin-Vasiliev theory without the matrix extension in [8]. The gauge algebra is given by a higher spin algebra $\text{hs}[\lambda]$. One definition is

$$\text{hs}[\lambda] = B[\lambda] \ominus \mathbb{1}, \quad (2.3)$$

where $B[\lambda]$ is the universal enveloping algebra of $\mathfrak{sl}(2)$ divided by the ideal $C_2 - \frac{1}{4}(\lambda^2 - 1)\mathbb{1}$ (C_2 is the $\mathfrak{sl}(2)$ Casimir);

$$B[\lambda] = \frac{U(\mathfrak{sl}(2))}{\langle C_2 - \frac{1}{4}(\lambda^2 - 1)\mathbb{1} \rangle}. \quad (2.4)$$

At $\lambda = n$ with $n = 2, 3, \dots$, an ideal appears and $\text{hs}[\lambda]$ can be truncated to $\text{sl}(n)$ by dividing the ideal. We may express the generators of $B[\lambda]$ as $V_0^1 \equiv \mathbb{1}$ and V_m^s with $s = 2, \dots$ and $m = -s + 1, \dots, s - 1$, where the commutation relations are of the form

$$[V_m^s, V_l^t] = \sum_{u=2,4,\dots}^{s+t-|s-t|-1} g_u^{st}(m, l; \lambda) V_{m+l}^{s+t-u}. \quad (2.5)$$

We use the definition of trace;

$$\begin{aligned} \text{tr}_V(V_m^s V_l^t) &= N_s \frac{(-1)^{s-m-1}}{(2s-2)!} \Gamma(s+m) \Gamma(s-m) \delta^{st} \delta_{m,-l}, \\ N_s &= \frac{3 \cdot 4^{1-s} \sqrt{\pi} \Gamma(s) \Gamma(s-\lambda) \Gamma(s+\lambda)}{(\lambda^2-1) \Gamma(s+\frac{1}{2}) \Gamma(1-\lambda) \Gamma(1+\lambda)}. \end{aligned} \quad (2.6)$$

In particular, we need

$$\text{tr}_V(V_0^1 V_0^1) = -\frac{6}{1-\lambda^2}, \quad \text{tr}_V(V_0^2 V_0^2) = \frac{1}{2}. \quad (2.7)$$

See [17, 30] for more details of the algebra.

We extend the algebra by multiplying the $M \times M$ matrix algebra as (see, e.g., [7, 31])

$$\text{hs}_M[\lambda] = \text{gl}(M) \otimes B[\lambda] \oplus \mathbb{1}_M \otimes \mathbb{1}. \quad (2.8)$$

This algebra can be decomposed as

$$\text{hs}_M[\lambda] = \text{sl}(M) \otimes \mathbb{1} \oplus \mathbb{1}_M \otimes \text{hs}[\lambda] \oplus \text{sl}(M) \otimes \text{hs}[\lambda]. \quad (2.9)$$

Notice that only the identity operator $\mathbb{1}_M \otimes \mathbb{1}$ can be decoupled from the algebra. There are two sub-algebras $\text{sl}(M) \otimes \mathbb{1}$ and $\mathbb{1}_M \otimes \text{hs}[\lambda]$, and the sub-sectors with these sub-algebras do not mix with the others by the gauge transformations in (2.2). We denote the generators of $\text{gl}(M)$ as $t^A = \mathbb{1}_M, t^a$ ($a = 1, \dots, M^2 - 1$) with the normalization

$$\text{tr}_M(t^a t^b) = \delta^{ab}, \quad [t^a, t^b] = i f^{ab}_c t^c. \quad (2.10)$$

Here we express t^A by a $M \times M$ matrix and define the trace tr_M as the sum over the diagonal elements. The generators of $\text{hs}_M[\lambda]$ are given by $t^A \otimes V_m^s$, and the trace is defined as the product

$$\text{Tr}[(t^A \otimes V_m^s)(t^B \otimes V_l^t)] = \text{tr}_M(t^A t^B) \text{tr}_V(V_m^s V_l^t). \quad (2.11)$$

In the next subsection, we examine the asymptotic symmetry at the classical limit, and in the following sections, we extend the analysis including quantum effects. In order to deal with quantum corrections, it is convenient to work with $\lambda = n$. The algebra after the matrix extension is then given by $\text{sl}(Mn)$ as in (1.1). We may express the generators of the algebra as $t^A \otimes V_m^s$ as before but with a restriction $s \leq n$. For the generators of $\text{gl}(n)$, we can use $n \times n$ matrices, see, e.g., [32] for the explicit realizations. The trace $\text{tr}_V = \text{tr}_n$ is given as in the $\text{gl}(M)$ sector, and the current convention leads to

$$\text{tr}_V(V_0^1 V_0^1) = \text{tr}_n(\mathbb{1}_n) = n, \quad \text{tr}_V(V_0^2 V_0^2) = \frac{1}{12} n(n^2 - 1) \quad (2.12)$$

for example.

2.2 Classical asymptotic symmetry

In order to define a higher spin gravity, we need to assign proper boundary conditions to higher spin fields. For the purpose, we introduce the coordinate system $(\rho, z, \bar{z})$, where ρ is the radial coordinate and the boundary is located at $\rho \rightarrow \infty$. Moreover, $(z, \bar{z})$ are the coordinates for the plane parallel to the boundary. The gravitational sector is identified as $\mathbb{1}_M \otimes \mathfrak{sl}(2) \subset \mathbb{1}_M \otimes \mathfrak{hs}[\lambda]$ with the principally embedded $\mathfrak{sl}(2) \subset \mathfrak{hs}[\lambda]$. The solutions to the equations of motion from the Chern-Simons theory are given by flat connections, which may be expressed by

$$A = e^{-\rho(\mathbb{1}_M \otimes V_0^2)} a(z) e^{\rho(\mathbb{1}_M \otimes V_0^2)} dz + (\mathbb{1}_M \otimes V_0^2) d\rho \quad (2.13)$$

in a gauge fixing and similarly for $\tilde{A}$. The AdS background corresponds to the configuration of gauge field A_{AdS} with $a(z) = \mathbb{1}_M \otimes V_1^2$. We assign the boundary condition such that [15, 16]

$$(A - A_{\text{AdS}})|_{\rho \rightarrow \infty} = \mathcal{O}(1). \quad (2.14)$$

With this condition and the residual gauge symmetry, we can put

$$\begin{aligned} a(z) = & \mathcal{J}_a(z)(t^a \otimes V_0^1) + \mathbb{1}_M \otimes V_1^2 + \sum_{s=2}^{\infty} \mathcal{W}^{(s)}(z)(\mathbb{1}_M \otimes V_{-s+1}^s) \\ & + \sum_{s=2}^{\infty} \mathcal{Q}_a^{(s)}(z)(t^a \otimes V_{-s+1}^s), \end{aligned} \quad (2.15)$$

see [12, 33, 34].

The asymptotic symmetry can be obtained as a classical Hamiltonian reduction of $\mathfrak{hs}_M[\lambda]$ for the Chern-Simons theory by following the standard procedure in [15–18]. We can read off the algebra generated by $\mathcal{J}_a, \mathcal{W}^{(s)}, \mathcal{Q}_a^{(s)}$ from the gauge transformations in (2.2) preserving the form of gauge fields as in (2.15). We can see that $\mathcal{J}_a$ generates affine $\mathfrak{sl}(M)$ (or $\mathfrak{su}(M)$) Lie algebra. Moreover, $\mathcal{W}^{(s)}$ and $\mathcal{Q}_a^{(s)}$ are spin s currents transforming in the trivial and adjoint representation of $\mathfrak{sl}(M)$, respectively. It is quite complicated to obtain the full commutation relations among the generators, see [12] for the simplest case with $\lambda = 2$. Thus, in the following, we focus on the two sub-sectors including $\mathcal{J}_a$ and $\mathcal{W}^{(s)}$, and compute the level of affine $\mathfrak{sl}(M)$ and the central charge of the algebra. Similar analysis has already been done in [7] for the $\mathcal{N} = 2$ supersymmetric case.

First, we restrict ourselves to the sub-sector with $\mathcal{J}_a$. We set the other elements $\mathcal{W}^{(s)}, \mathcal{Q}_a^{(s)}$ to be zero and consider the gauge transformations (2.2) with the generators $t^a \otimes V_0^1$. With this setup, the system becomes the same as the $\mathfrak{sl}(M)$ Chern-Simons gauge theory, so the boundary symmetry is the affine $\mathfrak{sl}(M)$ Lie algebra. From the definition of trace in (2.11), we have

$$\text{Tr}[(t^a \otimes V_0^1)(t^b \otimes V_0^1)] = \text{tr}_M(t^a t^b) \text{tr}_V(V_0^1 V_0^1). \quad (2.16)$$

This means that the overall factor in front of the action is multiplied by $\text{tr}_V(V_0^1 V_0^1)$. Therefore, the level of affine $\mathfrak{sl}(M)$ can be read off as

$$\ell = k_{\text{CS}} \text{tr}_V(V_0^1 V_0^1) = -\frac{6k_{\text{CS}}}{1 - \lambda^2}. \quad (2.17)$$

Here we have used (2.7). For the comparison with the symmetry of the coset (1.3), it is convenient to use the term of $\mathfrak{su}(M)$ instead of $\mathfrak{sl}(M)$. The level k of affine $\mathfrak{su}(M)$ is related as $k = -\ell$, thus we have

$$k = \frac{6k_{\text{CS}}}{1 - \lambda^2} \quad (2.18)$$

instead of (2.17).

Next, we restrict ourselves to the sub-sector with $\mathcal{W}^{(s)}$ in a similar way. We set the other elements $\mathcal{J}_a, \mathcal{Q}_a^{(s)}$ to be zero and consider the gauge transformations (2.2) with the generators $\mathbb{1}_M \otimes V_m^s$. Then the setup is the same as the $\mathfrak{hs}[\lambda]$ Chern-Simons theory with the principal embedding of $\mathfrak{sl}(2)$ in $\mathfrak{hs}[\lambda]$. The W-algebra obtained this way is called as $W_\infty[\lambda]$ but now at the classical limit. The definition of the trace in (2.11) leads to

$$\text{Tr}[(\mathbb{1}_M \otimes V_m^s)(\mathbb{1}_M \otimes V_l^t)] = \text{tr}_M(\mathbb{1}_M) \text{tr}_V(V_m^s V_l^t) = M \text{tr}_V(V_m^s V_l^t). \quad (2.19)$$

Therefore, the central charge of the algebra $W_\infty[\lambda]$ is [15, 16]

$$c = 12k_{\text{CS}} M \text{tr}_V(V_0^2 V_0^2) = 6k_{\text{CS}} M \quad (2.20)$$

using (2.7). Since $\mathcal{W}^{(2)}$ is the only spin 2 generator in the singlet of $\mathfrak{sl}(M)$, the central charge is the one for the whole algebra as well. Removing k_{CS} , we find the relation

$$c = M(1 - \lambda^2)k \quad (2.21)$$

from (2.18) and (2.20). In particular, the level k is positive when $c > 0$ and $0 < \lambda < 1$ as in the setups of the holography [7, 10].

Since the two sub-sectors decouples, the affine $\mathfrak{su}(M)$ generators and the energy-momentum tensor commute with each other. Therefore, we should redefine the energy-momentum tensor for the affine $\mathfrak{su}(M)$ generators to be Virasoro primary as in [35]. Let us denote the affine $\mathfrak{su}(M)$ generators as $\mathcal{J}_m^a$ and the Virasoro generators as $\mathcal{L}_m$. We may choose the normalizations such that their commutation relations become

$$\begin{aligned} i\{\mathcal{L}_m, \mathcal{L}_l\} &= (m-l)\mathcal{L}_{m+l} + \frac{c}{12}m(m^2-1)\delta_{m+l,0}, \\ i\{\mathcal{J}_m^a, \mathcal{J}_l^b\} &= k\delta^{ab}m\delta_{m+l,0} + if_c^{ab}\mathcal{J}_{m+l}^c, \quad i\{\mathcal{L}_m, \mathcal{J}_l^a\} = 0. \end{aligned} \quad (2.22)$$

Here the structure constant f_c^{ab} is given as in (2.10). Redefining the Virasoro generators as⁵

$$\mathcal{L}_m + \frac{1}{2k} \sum_l J_l^a J_{m-l}^a \rightarrow \mathcal{L}_m, \quad (2.23)$$

we obtain the commutation relations in (2.22) but the third one is replaced by

$$i\{\mathcal{L}_m, \mathcal{J}_l^a\} = -l\mathcal{J}_{m+l}^a \quad (2.24)$$

for large c, k .

⁵With the notation in (2.10), we use δ^{ab} to raise or lower the $\mathfrak{su}(M)$ indices.

From the next sections, we mainly focus on the case with $\lambda = n$. The relation between c and k is given by (2.21) with setting $\lambda = n$ as

$$c = -M(n^2 - 1)k. \quad (2.25)$$

For $c > 0$ and $n = 2, 3, \dots$, the level k become negative. It might be worth to derive the same result with the conventional notation in terms of $\mathfrak{sl}(n)$ instead of $\mathfrak{hs}[\lambda]$. The sub-sector with $\mathcal{J}_a$ leads to the affine $\mathfrak{su}(M)$ symmetry with the level

$$k = -k_{\text{CS}}n, \quad (2.26)$$

where we have used (2.17) with $k = -\ell$ and (2.12). The sub-sector with $\mathcal{W}^{(s)}$ provides the W_n -algebra, which can be regarded as a truncation of $W_\infty[\lambda]$ with $\lambda = n$. The central charge of the algebra is

$$c = k_{\text{CS}}Mn(n^2 - 1) \quad (2.27)$$

using (2.20) and (2.12). Using $k = -k_{\text{CS}}n$, we reproduce (2.25).

3 Quantum Hamiltonian reduction

In the previous section, we have examined the asymptotic symmetry of the higher spin gravity at the classical limit. In the case with $\lambda = n$, the asymptotic symmetry is given by the classical Hamiltonian reduction of $\mathfrak{sl}(Mn)$ with the $\mathfrak{sl}(2)$ embedding corresponding to the partition (1.2). In this section, we study the W-algebra obtained as the quantum Hamiltonian reduction instead of the classical one, see [36] for an introduction. Before we start, let us give some short overview about W-algebras.

By a W-algebra we mean the chiral algebra or vertex algebra of a CFT that has generating fields of higher conformal weight, i.e. not only a current algebra. There are three standard constructions of such algebras, namely as BRST-cohomology, as joint intersection of kernels of screening operators and as coset. Each approach has its advantages and difficulties. In the cohomological approach it is easy to determine the generating fields of the algebra and their spin, the kernel of screening picture gives a concrete action of the theory and is most suitable for concrete computations while cosets are often quite useful for understanding the representation theory. The spin content of coset theories can be determined using the theory of [37, 38]. The most common BRST-cohomology is the quantum Hamiltonian reduction, see [39], and recently Genra [40] has shown that there is also a kernel of screening realization of these W-algebras. The most common W-algebras are the principal W-algebras corresponding to the principal or regular quantum Hamiltonian reduction. The ADE-series of these W-algebras is now finally also known to be realized as a coset theory [41]. This is important as the coset realization of the A -series is precisely the dual theory for the ordinary bosonic higher spin gravity correspondence of Gaberdiel and Gopakumar [10]. Moreover the proof of [41] consists of finding a kernel of screenings realization of the coset theory and this step generalizes and some generalizations are currently work in progress.

Now, the bosonic higher spin algebra of Gaberdiel and Gopakumar is of type $2, 3, 4, \dots$ and Linshaw has proven that there exists a two parameter family of such W-algebras [42] and there are certain curves (in the parameter space) of ideals where the simple quotient truncates to an algebra of type $2, 3, \dots, n$ for some given n . Intersections of such curves correspond to isomorphisms of algebras and this is quite important to us. A similar Theorem in the even spin case also exists [43]. Let us now turn to W-algebras with $\mathfrak{su}(M)$ symmetry. By this we mean a vertex algebra that has a $\mathfrak{su}(M)$ current algebra together with M^2 generating fields of spin $2, 3, \dots, n$ for some n . These algebras will be parameterized by their field content, the central charge, the level of the current algebra and probably further parameters will be necessary for larger n . There are two natural ways to realize such W-algebras, namely rectangular W-algebras via quantum Hamiltonian reduction and the coset algebras of Grassmannian type cosets. We believe that there exists a $W_\infty(m)$ -algebra, that is a multi parameter family of vertex algebras with an $\mathfrak{su}(M)$ current algebra and M^2 generators of spin $2, 3, \dots$ up to infinity. Further in analogy to W_∞ there should be certain curves of ideals for which the simple quotient of the algebra truncates to an algebra with M^2 fields of spin $2, 3, \dots, n$ together with the $\mathfrak{su}(M)$ current algebra. Moreover the expectation is that both the Grassmannian type coset as well as the rectangular W-algebras are described by such quotients and moreover we expect coincidences, i.e. intersections of these curves and interesting values of the parameters. In the following we will explore this idea.

We start by introducing rectangular W-algebras and proposing an action for these theories. Especially we compute the two obvious parameters, the central charge c and the level k of the current algebra. In subsection 3.3, we restrict our attention to the case $n = 2$; we obtain the OPEs among the currents by requiring the associativity. Furthermore, we show the uniqueness of the algebra with one parameter, e.g., the level k of the affine symmetry fixed.

3.1 Rectangular W-algebras

In this subsection we give a quick overview of the mathematics of rectangular W-algebras. They haven't been studied much, but one reference is [44]. We use [39] as reference on quantum Hamiltonian reduction. Consider a simple Lie algebra $\mathfrak{g}$, then the chiral algebra of the Wess-Zumino-Novikov-Witten (WZNW) theory of $\mathfrak{g}$ at level t , $\mathfrak{g}_t$, is the affine vertex algebra of $\mathfrak{g}$ at level $-t$ (note that normalization of bilinear form between the physics notation that we use and standard notation in vertex algebras differ by a sign and hence we have this sign difference in meaning of level). The notation for the affine vertex algebra is $V_{-t}(\mathfrak{g})$. We are interested in the special case $\mathfrak{g} = \mathfrak{sl}(L)$. Consider an embedding ρ of $\mathfrak{sl}(2)$. This then determines a representation of $\mathfrak{sl}(2)$ on $\mathfrak{sl}(L)$ as well as on the standard representation of $\mathfrak{sl}(L)$. The decomposition of the standard representation of $\mathfrak{sl}(L)$ into $\mathfrak{sl}(2)$ -modules can be labelled by a partition of L according to the dimensions of the irreducible summands appearing or equivalently by a Young tableau with L boxes. The Young tableau consists of as many columns as irreducible summands appear in the decomposition of $\mathfrak{sl}(L)$ and the height of the columns is given by the dimensions of the corresponding irreducible summands. Quantum Hamiltonian reduction then associates a new vertex algebra,

a W-algebra, to this data as a certain semi-infinite cohomology. The main point for us is that every lowest-weight vector for the action of $\mathfrak{sl}(2)$ on $\mathfrak{sl}(L)$ via ρ gives rise to exactly one generator of the W-algebra of conformal weight $(d+1)/2$ with d the dimension of the corresponding irreducible representation.

We now further specialize to $L = nM$ and first decompose $\mathfrak{sl}(nM)$ into a module for $\mathfrak{sl}(n) \oplus \mathfrak{sl}(M)$ in the obvious way,

$$\mathfrak{sl}(M) \otimes \mathbb{1}_n \oplus \mathbb{1}_M \otimes \mathfrak{sl}(n) \oplus \mathfrak{sl}(M) \otimes \mathfrak{sl}(n) \simeq \mathfrak{sl}(Mn), \quad (3.1)$$

and then consider the embedding of $\mathfrak{sl}(2)$ in $\mathfrak{sl}(nM)$ given by the composition of the regular embedding of $\mathfrak{sl}(2)$ in $\mathfrak{sl}(n)$ and above embedding of $\mathfrak{sl}(n)$ in $\mathfrak{sl}(nM)$. The corresponding Young tableau is then of rectangular type, i.e. it has M columns each of height n . Especially $\mathfrak{sl}(nM)$ decomposes as $\mathfrak{sl}(2)$ representations as

$$\mathfrak{sl}(nM) \cong (M^2 - 1)\mathbb{1} \oplus M^2\mathfrak{3} \oplus \dots \oplus M^2\mathfrak{2n} - \mathbb{1}$$

with $\mathfrak{s}$ the s -dimensional irreducible representation of $\mathfrak{sl}(2)$. This rectangular W-algebra is thus of desired type, i.e. it has a $\mathfrak{sl}(M)$ current algebra and additional M^2 fields of spin $2, 3, \dots, n$. The M^2 fields of a given spin carry the adjoint plus trivial representation of $\mathfrak{sl}(M)$. Let us denote the W-algebra constructed in this way from the affine vertex algebra of $\mathfrak{sl}(nM)$ at level $-t$ by $W(-t, n, M)$. The central charge of this W-algebra is extracted from equation (2.3) of [39]. It consists of three summands, the central charge of $V_{-t}(\mathfrak{sl}(Mn))$, a dilaton shift in the Cartan direction of the image of ρ of the $\mathfrak{sl}(2)$ Cartan sub-algebra element h in $\mathfrak{sl}(nM)$ and various ghost contributions. Note that $\rho(h)$ has norm $Mn(n^2 - 1)/12$ and so the central charge contribution of the dilaton shift is $Mtn(n^2 - 1)$. A pair of ghosts of conformal weight $\lambda, 1 - \lambda$ has central charge $1 - 12(\lambda - 1/2)^2$ and there are $(n - 1)M^2$ ghosts of weight $(1, 0)$, $(n - 2)M^2$ ghosts of weight $(2, -1)$ and so on up to M^2 ghosts of weight $(n - 1, -n + 2)$ giving a total contribution to the central charge of $M^2(n - 1)(1 - (n - 1)^2(n + 1))$. This is then evaluated to

$$\begin{aligned} c_W(-t, n, M) &= \frac{t(n^2 M^2 - 1)}{t - Mn} + Mtn(n^2 - 1) + M^2(n - 1)(1 - (n - 1)^2(n + 1)) \\ &= \frac{Mn(n^2 M^2 - 1)}{t - Mn} + (t - Mn)Mn(n^2 - 1) + M^2n(2n^2 - 1) - 1. \end{aligned} \quad (3.2)$$

Moreover the level $-\ell$ of the affine vertex algebra of $\mathfrak{sl}(M)$ is shifted due to ghost contributions and it is

$$-\ell (= k) = -tn + Mn(n - 1). \quad (3.3)$$

This follows since the $M^2 n(n - 1)$ ghosts carry $n(n - 1)$ copies of the adjoint plus trivial representation of $\mathfrak{sl}(M)$ and each contributing to a level shift by the dual Coxeter number M of $\mathfrak{sl}(M)$. Setting $c = c_W(-t, n, M)$, we find the relation among c, k as

$$c = -\frac{(k^2 - 1)n^2 M}{k + nM} + kM - 1, \quad (3.4)$$

whose leading behaviour recovers the central charge of the gravity computation (2.25).

3.2 Actions for rectangular W-algebras

We begin with the simplest case with $n = 2$. We consider the $\mathfrak{sl}(2M)$ WZNW model with the level t . The reduction procedure in [45] (see also [46–48]) would lead to the action

$$S_t[\varphi, g_1, g_2] = S_{t-M}^{\text{WZNW}}[g_1] + S_{t-M}^{\text{WZNW}}[g_2] + \frac{1}{2\pi} \int d^2z \left[\partial\varphi \bar{\partial}\varphi + \frac{Q_\varphi}{4} \sqrt{g} \mathcal{R}\varphi + \frac{1}{t} \text{tr} \left(e^{-2b\varphi} g_1^{-1} g_2 \right) \right]. \quad (3.5)$$

Here g_1, g_2 are the elements of $\mathfrak{sl}(M)$, and $S_t^{\text{WZNW}}[g]$ denotes the action of the $\mathfrak{sl}(M)$ WZNW model with level t . The background charge for φ is

$$Q_\varphi = -M^2 b - \frac{1}{b}, \quad b = \frac{1}{\sqrt{M(t-2M)}}. \quad (3.6)$$

The central charge of the theory can be computed as

$$c = 2 \cdot \frac{(M^2 - 1)(t - M)}{t - M - M} + 1 + 6Q_\varphi^2 = \frac{6t^2 M - 10tM^2 - t + 4M^3}{t - 2M}, \quad (3.7)$$

which is the same as $c_W(-t, 2, M)$ in (3.2). Moreover, the diagonal part of the affine symmetries with g_1 and g_2 survives in the interaction of (3.5). Thus the theory admits the symmetry of affine $\mathfrak{sl}(M)$ with the level

$$\ell = 2(t - M), \quad (3.8)$$

see (3.3).

Next, we study the case with generic n . The realization of rectangular W-algebras as intersection of kernels of screenings of Genra [40, Prop. 3.7 and Thm. 3.8] suggests that the action is given by

$$\begin{aligned} S_t &= S_t[\varphi_1, \dots, \varphi_{n-1}, g_1, \dots, g_n] \\ &= \sum_{i=1}^n S_{t-M(n-1)}^{\text{WZNW}}[g_i] \\ &\quad + \sum_{j=1}^{n-1} \frac{1}{2\pi} \int d^2z \left[\partial\varphi_j \bar{\partial}\varphi_j + \frac{Q_\varphi}{4} \sqrt{g} \mathcal{R}\varphi_j + \frac{1}{t} \text{tr} \left(e^{-2b\varphi_j} g_j^{-1} g_{j+1} \right) \right]. \end{aligned}$$

The background charge for φ is

$$Q_\varphi = -\sqrt{\frac{n(n+1)}{6}} \left(M^2 b + \frac{1}{b} \right), \quad b = \frac{1}{\sqrt{M(t-nM)}}. \quad (3.9)$$

The central charge of the theory can be computed as

$$c = n \cdot \frac{(M^2 - 1)(t - M(n-1))}{t - nM} + (n-1)(1 + 6Q_\varphi^2), \quad (3.10)$$

which simplifies to (3.2). The level of the affine $\mathfrak{sl}(M)$ sub-algebra is

$$\ell = n(t - M(n-1)) \quad (3.11)$$

as in (3.3).

3.3 The simplest example with $n = 2$

As discussed above, the asymptotic symmetry algebra of the higher spin theory includes spin s currents $W^{(s)}$ ($s = 2, 3, \dots, n$) and the affine $\mathfrak{su}(M)$ currents J^a . There are spin s currents $Q_a^{(s)}$ ($s = 2, 3, \dots, n$) in the adjoint representation of $\mathfrak{su}(M)$ in addition to $W^{(s)}$ in the trivial one. We would like to know the OPEs among these currents, which may be determined uniquely by requiring their associativity. In this subsection, we would like to show that this is the case for $n = 2$.⁶

The algebra with $n = 2$ includes the energy-momentum tensor $T \equiv W^{(2)}$ and the affine $\mathfrak{su}(M)$ currents J^a with the OPEs

$$\begin{aligned} T(z)T(0) &\sim \frac{c/2}{z^4} + \frac{2T(0)}{z^2} + \frac{\partial T(0)}{z}, \\ T(z)J^a(0) &\sim \frac{J^a(0)}{z^2} + \frac{\partial J^a(0)}{z}, \quad J^a(z)J^b(0) \sim \frac{k\delta^{ab}}{z^2} + \frac{if_c^{ab}J^c(0)}{z}. \end{aligned} \quad (3.12)$$

Here the structure constant f_c^{ab} is defined as in (2.10). For a while, we do not assign any relation among the central charge c and the level of $\mathfrak{su}(M)$ currents k . Along with them, there are also spin 2 currents $Q^a \equiv Q_a^{(2)}$. We choose their basis such that the OPEs among T and J^a become

$$T(z)Q^a(0) \sim \frac{2Q^a(0)}{z^2} + \frac{\partial Q^a(0)}{z}, \quad J^a(z)Q^b(0) \sim \frac{if_c^{ab}Q^c(z)}{z}. \quad (3.13)$$

In the following, we determine the OPEs among Q^a and Q^b by requiring their associativity and show that they are unique.⁷ We first classify all possible terms generated by the operator products of Q^a , and then show that the associativity uniquely fixes the coefficients of the terms up to an overall normalization. In particular, we obtain a relation between the central charge c and the level of $\mathfrak{su}(M)$ currents as

$$c = -\frac{4(k^2 - 1)M}{k + 2M} + kM - 1, \quad (3.14)$$

which is the same as the one in (3.4) with $n = 2$.

First we complete the list of operators consisting of T, J^a, Q^a with conformal dimension $h = 0, 1, 2, 3$, since the operator product $Q^a(z)Q^b(0)$ is of dimension 4. The operator with $h = 0$ is the identity, and the operators with $h = 1$ are the $\mathfrak{su}(M)$ currents J^a . For $h = 2$, there are quasi-primary operators such as the energy-momentum tensor T , the spin 2 charged currents Q^a . Along with them, there are composite quasi-primary operators

$$(J^a J^b)(z) \equiv \frac{1}{2} \left[(J^a J^b)(z) + (J^b J^a)(z) \right]. \quad (3.15)$$

The normal ordering is defined as

$$(AB)(z) = \frac{1}{2\pi i} \oint \frac{dw}{w - z} A(w)B(z), \quad (3.16)$$

⁶At the classical limit, it was shown in [12] that the Poisson brackets among generators are rigid by requiring the Jacobi identities with some ansatz. At $\lambda = 0, 1$, free field realizations are possible, and the linear versions of W-algebras with affine $\mathfrak{su}(M)$ symmetry were obtained in [49, 50].

⁷We utilize the Mathematica package of [51].

and the brackets $(a_1, \dots, a_p)$ and $[a_1, \dots, a_p]$ represent the symmetric and anti-symmetric indices, respectively, with the pre-factor $1/(p!)$. The composite quasi-primary operators with $h = 3$ are

$$\begin{aligned} (J^a J^b J^c), \quad (T J^a) - \frac{1}{2} \partial^2 J^a, \quad (J^a Q^b) - \frac{i}{4} f^{ab}_c \partial Q^c, \\ \Xi^{ab} = (\partial J^a J^b) - (\partial J^b J^a) - \frac{i}{3} f^{ab}_c \partial^2 J^c. \end{aligned} \quad (3.17)$$

There are several ways to make the product $(J^a J^b J^c)$ to be quasi-primary by adding the products of J^a . Here we have chosen one of them with symmetric indices. The differences can be expressed by linear combinations of other quasi-primaries Ξ^{ab} .

The operator product $Q^a(z)Q^b(0)$ has two labels $\{a, b\}$, which are symmetric or anti-symmetric depending on the even or odd powers in the $1/z$ -expansions. Moreover, we have chosen quasi-primary composite operators such as to include only symmetric or anti-symmetric indices. Therefore, the coefficients in front of the operators have indices with specific properties, and we would like to classify all of them. We can construct invariant tensors from the $\text{su}(M)$ generators t^a such as

$$\text{tr}_M([t^a, t^b]t^c) = i f^{abc}, \quad \text{tr}_M(\{t^a, t^b\}t^c) = d^{abc}, \quad (3.18)$$

where the indices are totally anti-symmetric for f^{abc} and symmetric for d^{abc} . In terms of these tensors, the products of t^a are written as

$$t^a t^b = \frac{\delta^{ab}}{M} \mathbb{1}_M + \frac{1}{2} \left(i f^{ab}_c + d^{ab}_c \right) t^c, \quad (3.19)$$

where we have used (2.10).

The invariant tensors are given by products of traces of multiple t^a , which can be expressed in terms of (2.10) and (3.18). The coefficients in front of operators generated should be proportional to them. Because the $\text{su}(M)$ generators are traceless, there is no invariant vector. For tensors with two indices, we have δ^{ab} as in (2.10). For tensors with three indices, we have f^{abc} and d^{abc} defined in (3.18). For tensors with label $\{a, b, c, d\}$, we need those with $(ab)(cd)$, $(ab)[cd]$, and $[ab][cd]$. With the single trace, we have the independent bases

$$\begin{aligned} d_{4SS1}^{abcd} &\equiv 4 \text{tr}_M(t^{(a} t^b) t^{(c} t^{d)}), & d_{4SS2}^{abcd} &\equiv 4 \text{tr}_M(t^{(a} t_{(c} t^{b)} t_{d)}), \\ d_{4SA}^{abcd} &\equiv -4i \text{tr}_M(t^{(a} t^b) t^{[c} t^{d]}) = d^{abe} f_e^{cd}, & & \\ d_{4AA1}^{abcd} &\equiv -4 \text{tr}_M(t^{[a} t^b] t^{[c} t^{d]}) = f^{abe} f_e^{cd}, & d_{4AA2}^{abcd} &\equiv 4 \text{tr}_M(t^{[a} t_{[c} t^{b]} t_{d]}), \end{aligned} \quad (3.20)$$

where we should notice that

$$\text{tr}_M(t^{(a} t_{[c} t^{b]} t_{d]}) = 0. \quad (3.21)$$

There are also tensors consisting of double traces as

$$\delta^{ab} \delta^{cd}, \quad \delta^{ac} \delta^{bd} \pm \delta^{ad} \delta^{bc}. \quad (3.22)$$

For tensors with five indices, we need those with $[ab](cde)$. The two independent bases with single trace are

$$d_{51}^{abcde} \equiv 12\text{tr}_M(t^{[a}t^b]t^{(c}t^d t^{e)}), \quad d_{52}^{abcde} \equiv 12\text{tr}_M(t^{[a}t_{(c}t^{b]}t_d t_{e)}). \quad (3.23)$$

Those with double traces are

$$f^{ab}{}_{(c}\delta_{de)}, \quad \delta^{[a}{}_{(c}d^{b]}{}_{de)}. \quad (3.24)$$

With these preparations, we can write down our ansatz for the OPEs;

$$\begin{aligned} Q^a(z)Q^b(0) \sim & \frac{c_1}{c}\delta^{ab}\left(\frac{c/2}{z^4} + \frac{2a_1T(0)}{z^2} + \frac{a_1\partial T(0)}{z}\right) \\ & + c_2f^{ab}{}_c\left(\frac{J^c(0)}{z^3} + \frac{1/2\partial J^c(0)}{z^2} + \frac{a_2\partial^2 J^c(0) + a_3(TJ^c)(0)}{z}\right) \\ & + (c_{31}\delta^{ab}\delta^{cd} + c_{32}d_{4SS1}^{abcd} + c_{33}d_{4SS2}^{abcd} + c_{34}\delta^{ac}\delta^{bd}) \\ & \quad \times \left(\frac{(J_{(c}J_{d)})(0)}{z^2} + \frac{1/2\partial(J_{(c}J_{d)})(0)}{z}\right) \\ & + (c_{41}d_{4AA1}^{abcd} + c_{42}d_{4AA2}^{abcd} + c_{43}\delta^{ac}\delta^{bd})\frac{\Xi_{cd}}{z} \\ & + (c_{51}f^{ab}{}_c\delta_{de} + c_{52}d_{51}^{abcde} + c_{53}d_{52}^{abcde} + c_{54}\delta^{[a}{}_cd^{b]}{}_{de})\frac{(J_{(c}J_dJ_{e)})(0)}{z} \\ & + c_6d^{ab}{}_c\left(\frac{2Q^c(0)}{z^2} + \frac{\partial Q^c(0)}{z}\right) \\ & + (c_{71}d_{4AA1}^{abcd} + c_{72}d_{4AA2}^{abcd} + c_{73}d_{4SA}^{cdab} + 2c_{74}\delta^{[a}{}_cd^{b]}{}_d)\frac{(J^cQ^d)(0)}{z}. \end{aligned} \quad (3.25)$$

Requiring the associativity of OPEs, we find that these coefficients are uniquely fixed up to an overall normalization $\pm\sqrt{c_1}$ as

$$\begin{aligned} a_1 &= \frac{(2k+M)(k(3kM-2M^2+1)-2M)}{2(k-1)(k+1)M(3k+2M)}, \\ a_2 &= \frac{k(k(3kM+2M^2+12)+27M)+10M^2}{6(k-1)(k+1)M(3k+2M)}, \quad a_3 = -2a_2 + \frac{1}{3}, \quad c_2 = \frac{ic_1}{2k}, \\ c_{31} &= \frac{c_1(k(4k+5M)+2M^2)}{2(k-1)(k+1)M(2k+M)(3k+2M)}, \quad c_{32} = \frac{c_1k}{4(k-1)(k+1)(2k+M)}, \\ c_{33} &= \frac{c_1}{8-8k^2}, \quad c_{34} = \frac{c_1}{2k-2k^3}, \quad c_{41} = \frac{c_1(k+M)(k(kM+8)+4M)}{8(k-1)k(k+1)M(2k+M)(3k+2M)}, \\ c_{42} &= 0, \quad c_{43} = -\frac{c_1M}{4(k-1)k(k+1)(2k+M)}, \quad c_{51} = \frac{ic_1(k+M)}{(k-1)k(k+1)M(3k+2M)}, \\ c_{52} &= \frac{c_1k}{12(k-1)(k+1)(2k+M)(3k+2M)}, \quad c_{53} = -\frac{c_1}{12(k-1)(k+1)(2k+M)}, \\ c_{54} &= 0, \quad c_6 = \pm \frac{i\sqrt{c_1k}(k+M)}{\sqrt{2(k-1)(k+1)(2k+M)(3k+2M)}}, \quad c_{71} = 0, \quad c_{72} = \frac{c_6}{k}, \\ c_{73} &= \frac{ic_6}{k+M}, \quad c_{74} = 0 \end{aligned} \quad (3.26)$$

along with (3.14). We have checked this for $M = 2, 3, 4, 5$ and claim that the above expressions also hold for $M > 5$.

All bases used above are independent for $M = 4, 5$, but this is not the case for $M = 2, 3$. Therefore, we should be careful to claim the uniqueness of the algebra. For $M = 3$, we have

$$d_{4SS2}^{abcd} = 2\delta^{ab}\delta^{cd} - 2d_{4SS1}^{abcd} + 2(\delta^{ac}\delta^{bd} + \delta^{ad}\delta^{bc}), \quad d_{51}^{abcde} = 3if_{(c}^{ab}\delta_{de)}. \quad (3.27)$$

Using these relations, we can set, say, c_{33} and c_{51} arbitrarily by changing other parameters. There is no other freedom to tune parameters and the associativity uniquely fixes them as in the above expressions. For $M = 2$, we have more relations. Using

$$d^{abc} = 0, \quad d_{4AA2}^{abcd} = 0, \quad d_{4SA}^{abcd} = 0, \quad (3.28)$$

we can use arbitrary $c_6, c_{53}, c_{72}, c_{42}, c_{73}$. There are more relations as

$$\begin{aligned} d_{51}^{abcde} &= -d_{52}^{abcde} = 3if_{(c}^{ab}\delta_{de)}, & d_{4AA1}^{abcd} &= 2(\delta^{ac}\delta^{bd} - \delta^{ad}\delta^{bc}), \\ d_{4SS1}^{abcd} &= 2\delta^{ab}\delta^{cd}, & d_{4SS2}^{abcd} &= -2\delta^{ab}\delta^{cd} + 2(\delta^{ac}\delta^{bd} + \delta^{ad}\delta^{bc}). \end{aligned} \quad (3.29)$$

With them, we can set, say, $c_{52}, c_{54}, c_{41}, c_{71}, c_{32}, c_{33}$ arbitrarily by changing other parameters as in the above expressions. Others are fixed by the associativity of OPEs as for other M .

4 Symmetry of Grassmannian-like coset

In the previous section, we have examined the asymptotic symmetry of the higher spin gravity with $\lambda = n$ beyond the classical limit. The holographic duality of [7] suggests that the symmetry can be identified with that of the coset (1.3) with $\lambda = n$. In the next subsection, we confirm this proposal by comparing the central charge c of the model and the level k of the $\mathfrak{su}(M)$ current. In subsection 4.3, we explicitly construct the spin 2 currents in terms of the coset model and reproduce the OPEs among them for $n = 2$. In subsection 4.4, we provide an interpretation of the map of parameters by decomposing the coset algebra.

4.1 Map of parameters

The symmetry algebra of the coset (1.3) includes the energy-momentum tensor, which can be obtained from the standard coset construction. The central charge is computed as

$$\begin{aligned} c_{\text{coset}}(k, N, M) &= \frac{((N+M)^2 - 1)k}{k + N + M} - \frac{(N^2 - 1)k}{k + N} - 1 \\ &= \frac{kM(k(M+2N) + N(M+N) + 1)}{(k+N)(k+M+N)} - 1. \end{aligned} \quad (4.1)$$

The coset also has the symmetry generated by $\mathfrak{su}(M)$ currents with level k . They come from $\mathfrak{su}(M)$ within $\mathfrak{su}(N+M)$ in the numerator of (1.3), which are regular with respect

to the action of the currents in the denominator. In terms of 't Hooft parameter in (1.4), the central charge (4.1) is written as

$$c = -\frac{(k^2 - 1)\lambda^2 M}{k + \lambda M} + kM - 1, \quad (4.2)$$

which reduces to (2.21) at the 't Hooft limit with k large but λ, M fixed. Therefore, we find the agreement of c, k at the 't Hooft limit as shown in [7] for the $\mathcal{N} = 2$ supersymmetric case.

We would like to extend the comparison beyond the 't Hooft limit with setting $\lambda = n$ as before. We require that the level k of the affine $\mathfrak{su}(M)$ and the central charge $c_{\text{coset}}(k, N, M)$ are the same as (3.3) and (3.2) obtained from the quantum Hamiltonian reduction of $\mathfrak{sl}(Mn)$. That is, for fixed n, M, N , we look for the solutions for k such that

$$k = -tn + Mn(n - 1), \quad c_{\text{coset}}(k, N, M) = c_W(-t, n, M). \quad (4.3)$$

Removing t , the above condition can be written as

$$\frac{(k^2 - 1) M(k(n - 1) + nN)(k(n + 1) + n(M + N))}{(k + N)(k + Mn)(k + M + N)} = 0. \quad (4.4)$$

For $M \neq 0$, the solutions are

$$k = 1, -1, -\frac{nN}{n - 1}, -\frac{n(N + M)}{n + 1}. \quad (4.5)$$

The first two solutions correspond to the free fermion and $\beta\gamma$ -ghost realization of the coset (1.3) as explained in, e.g., [52], where the fields transform as the fundamental and anti-fundamental representations of $\mathfrak{su}(M)$ and $\mathfrak{su}(N)$, respectively. In these cases one expects that the simple rectangular W -algebra is just the current algebra or a conformal extension of the current algebra, see [53, Remark 5.3], [54, section 5.1.1] and [55] for the example $k = -1$ especially at $n = M = 2$.

The non-trivial solutions are thus the latter two. The third solution can be written as $n = k/(N + k)$, which implies that $\lambda = n$ from (1.4). We may compare the expressions of central charge in (3.4) and (4.2). This case may be explained from the AdS/CFT correspondence at the 't Hooft limit as a kind of analytic continuation. The fourth solution may be expressed as $n = -k/(k + N + M)$, which motivates us to define another 't Hooft parameter (1.5). In fact, the central charge of the coset (1.3) in terms of λ' is given by (4.2) with λ replaced by λ' . This implies that there is a duality relation as in [19]. We will make some comments on the duality relation soon but let us first comment on this matching of central charges and levels.

4.2 Singular vectors for boundary admissible level theories

This very nice matching of central charges and levels of course leads us to conjecture that the coset at these special levels is precisely isomorphic to the corresponding rectangular W -algebra. Proving such a conjecture is rather difficult. On the other hand in order for the conjecture to be true the coset has to have singular vectors at conformal weight $n + 1$, which of course implies that the affine vertex algebra of $\mathfrak{su}(N + M)$ with this k has

singular vectors at conformal weight $n+1$. Unfortunately the level $-nN/(n-1)$ is only an admissible level if $n \geq 2(M+N)/M$ and it is rather difficult to study. On the other hand the level $-n(N+M)/(n+1)$ is always a boundary admissible level of $\mathfrak{su}(N+M)$ [56].

Since admissible level WZW theories are not as well-known in the CFT community as the positive integer theories we decided to start with a small overview of features and known results. The point is that admissible level theories have so-called ordinary modules that form a tensor subcategory of modules that is finite and semi-simple, i.e. very similar to the positive integer case. But there are many more modules that are not always completely reducible and thus lead to logarithmic theories, see [57] for an introduction and [58–60] for the example of $\mathfrak{sl}(2)$ at admissible level. We are here only interested in ordinary modules. Firstly, a level k is called principal admissible if it is a rational number satisfying $k + h^\vee \geq h^\vee/u$ and u is both coprime to the dual Coxeter number $h^\vee$ of the Lie algebra $\mathfrak{g}$ as well as to its lacity $r^\vee$ [61]. The category of ordinary modules consists of simple highest-weight modules at level k of those weights that are also weights of the WZW theory of $\mathfrak{g}$ at level $\ell = u - h^\vee$ [62, Main Theorem]. Characters of modules can be meromorphically continued to components of vector-valued meromorphic Jacobi forms, i.e. they have some nice modularity properties [63]. However these characters do not form a closed representation of the modular group and one has to include characters of more modules. Nonetheless, the tensor category of these ordinary modules is nice in the sense that they form a braided fusion category and a weak version of Verlinde’s formula holds [64, 65].

Using the character formulae of Kac and Wakimoto one can thus possibly detect singular vectors of WZW theories at admissible level. This works particularly well if we are at the boundary of admissibility, i.e. $k + h^\vee = h^\vee/u$. Fortunately, we are in that situation and by Remark 1 of [56] the character of $\mathfrak{su}(N+M)$ at level $k = -n(N+M)/(n+1)$ is just

$$\text{ch}\left[\mathfrak{su}(N+M)_{-\frac{n(N+M)}{n+1}}\right] = q^{nd} \frac{\Pi(u, \tau)}{\Pi(u, (n+1)\tau)}, \quad d = (N+M)^2 - 1 \quad (4.6)$$

with $\Pi(u, \tau)$ the vacuum character of the universal affine vertex algebra of $\mathfrak{su}(N+M)$, that is

$$\Pi(u, \tau) = \frac{1}{\prod_{n=1}^{\infty} (1 - q^n)^{N+M-1} \prod_{\alpha \in \Delta_+} (1 - z^\alpha q^n)(1 - z^{-\alpha} q^{n-1})} \quad (4.7)$$

where Δ_+ denotes the set of positive roots and z^α is short-hand for $e^{2\pi i \alpha(u)}$ and u in the Cartan sub-algebra of $\mathfrak{su}(N+M)$. We thus see that the character $\text{ch}[\mathfrak{su}(N+M)_{-\frac{n(N+M)}{n+1}}]$ agrees with the character of the universal affine vertex algebra up to conformal weight n but at conformal weight $n+1$ they differ by the character of the adjoint representation of the finite dimensional Lie algebra $\mathfrak{su}(N+M)$. There must thus be a singular vector at conformal weight $n+1$ in this representation and so especially there are M^2 null fields of conformal weight $n+1$ that decouple. In principle it would be nice to compute the precise coset character and this would amount to computing the Fourier coefficient of a negative index meromorphic Jacobi form in N -variables. Technology is only available for the one

variable case [66, 67], but the higher rank case is work in progress so that we hopefully can return to a more detailed character analysis in the future.

4.3 Charged spin 2 currents

In [7] (see also [68]), the match of partition function was shown for the holographic duality at the 't Hooft limit. This, in particular, implies that the spin contents of the symmetry generators agree with each other. In this subsection, we obtain the exact expression of spin 2 currents K^a with finite c in terms of the coset model (1.3). We require the OPEs as in (3.13), which shall uniquely determine the form of the spin 2 currents up to an overall factor.

In order to obtain the explicit form of generators, we set up our notation of $\text{su}(N+M)$. We decompose $\text{su}(N+M)$ as

$$\text{su}(N+M) = \text{su}(N) \oplus \text{su}(M) \oplus \text{u}(1) \oplus (N, \bar{M}) \oplus (\bar{N}, M), \quad (4.8)$$

where L and $\bar{L}$ denote the fundamental and anti-fundamental representations of $\text{su}(L)$. We use the generators $t^A = (t^\alpha, t^a, t^{\text{u}(1)}, t^{(\rho\bar{i})}, t^{(\bar{\rho}i)})$, respectively. We express the generators in terms of $(N+M) \times (N+M)$ matrices and use the trace for the matrices as $\text{tr} = \text{tr}_{N+M}$. We use the normalization of generators such that the metric $g^{AB} = \text{tr}(t^A t^B)$ becomes

$$\text{tr}(t^\alpha t^\beta) = \delta^{\alpha\beta}, \quad \text{tr}(t^a t^b) = \delta^{ab}, \quad \text{tr}(t^{\text{u}(1)} t^{\text{u}(1)}) = 1, \quad \text{tr}(t^{(\rho\bar{i})} t^{(\bar{\rho}i)}) = \delta^{\rho\bar{\rho}} \delta^{i\bar{i}}. \quad (4.9)$$

We also need the invariant tensors

$$\text{tr}([t^A, t^B] t^C) = i f^{ABC}, \quad \text{tr}(\{t^A, t^B\} t^C) = d^{ABC} \quad (4.10)$$

as in (3.18). Several explicit expressions are (see, e.g., appendix B of [69])

$$\begin{aligned} i f^{(\rho\bar{i})(\bar{\sigma}j)\text{u}(1)} &= \sqrt{\frac{M+N}{MN}} \delta^{j\bar{i}} \delta^{\rho\bar{\sigma}}, & d^{(\rho\bar{i})(\bar{\sigma}j)\text{u}(1)} &= \frac{M-N}{\sqrt{MN(N+M)}} \delta^{j\bar{i}} \delta^{\rho\bar{\sigma}}, \\ i f^{(\rho\bar{i})(\bar{\sigma}j)\alpha} &= d^{(\rho\bar{i})(\bar{\sigma}j)\alpha} = \delta^{\rho\bar{\rho}} \delta^{\sigma\bar{\sigma}} (t^\alpha)_{\sigma\bar{\rho}} \delta^{j\bar{i}}, & i f^{(\rho\bar{i})(\bar{\sigma}j)a} &= -d^{(\rho\bar{i})(\bar{\sigma}j)a} = -\delta^{\rho\bar{\rho}} (t^a)_{i\bar{j}} \delta^{i\bar{i}} \delta^{j\bar{j}}. \end{aligned} \quad (4.11)$$

Important properties of the invariant tensors may be found in appendix B of [70].

With the preparation, we can write down low spin generators explicitly. The spin one generators are the $\text{su}(M)_k$ currents given by J^a . The energy-momentum tensor can be constructed as

$$T = \frac{g_{AB}}{2(k+N+M)} (J^A J^B) - \frac{1}{2(k+N)} (J^\alpha J^\alpha) - \frac{1}{2kNM(N+M)} (\tilde{J}^{\text{u}(1)} \tilde{J}^{\text{u}(1)}), \quad (4.12)$$

where $\tilde{J}^{\text{u}(1)} = \sqrt{NM(N+M)} J^{\text{u}(1)}$. Here the normal ordering is defined as in (3.16). We look for symmetry generators, which are regular with respect to the action of the currents in the denominator of (1.3). The spin 2 generators K^a in the adjoint representation of $\text{su}(M)$ should be given by linear combinations of

$$(J^b J^c), \quad (J^{(\rho\bar{i})} J^{(\bar{\rho}j)}) \delta_{\rho\bar{\rho}}, \quad (J^{(\bar{\rho}j)} J^{(\rho\bar{i})}) \delta_{\rho\bar{\rho}}, \quad (J^a J^{\text{u}(1)}), \quad \partial J^a. \quad (4.13)$$

We would like to have linear combinations which are consistent with the OPEs in (3.13). It is useful to use the formula

$$J^C(z)K^{AB}(0) \sim \frac{ikf_D^{CA}g^{DB}}{z^3} + \frac{kg^{CA}J^B(0) + kg^{CB}J^A(0) - f_D^{CA}f_F^{DB}J^F(0)}{z^2} + \frac{if_D^{CA}K^{DB}(0) + if_D^{CB}K^{AD}(0)}{z}, \quad (4.14)$$

which can be obtained from

$$J^A(z)J^B(0) \sim \frac{kg^{AB}}{z^2} + \frac{if_C^{AB}J^C(0)}{z}, \quad K^{AB}(z) = (J^A J^B)(z). \quad (4.15)$$

We then obtain the expression of spin 2 currents as

$$K^a = [(J^{(\rho\bar{i})}J^{(\bar{\rho}j)}) + (J^{(\bar{\rho}j)}J^{(\rho\bar{i})})]\delta_{\rho\bar{\rho}}t_{j\bar{i}}^a - \frac{N}{M+2k}d^{abc}(J^b J^c) + \frac{2}{k}\sqrt{\frac{N(N+M)}{M}}(J^a J^{u(1)}) \quad (4.16)$$

up to an overall normalization. In a similar manner, we can construct spin 3 currents in the coset language, see appendix A.

With the explicit expression of K^a in terms of $\text{su}(N+M)$ currents, we can compute the OPEs among them. In general, the operator products produce spin 3 currents, and the new operators would generate higher spin currents as well. However, at $\lambda = 2$, the operator products of spin 2 currents Q^a in the rectangular W-algebra do not generate new spin 3 currents, and in fact we computed that the algebra is uniquely fixed by requiring the associativity of OPEs. Therefore, once we accept the decoupling of the spin 3 currents, then the operator products of K^a and those of Q^a should be identical. We have computed the operator products of K^a with explicit values of N, M ,⁸ and deduced the relation as

$$K^a(z)K^b(0) \sim Q^a(z)Q^b(0) + \frac{if_c^{ab}P^c(0)}{z} \quad (4.17)$$

at $\lambda = 2$. Here P^a are given by

$$P^a = 4(J^\alpha(J^{\rho\bar{i}}J^{\bar{\rho}j}))t_{\rho\bar{\rho}}^\alpha t_{j\bar{i}}^a - \frac{2}{N}(J^\alpha(J^\alpha J^a)). \quad (4.18)$$

We can check that they satisfy

$$T(z)P^a(0) \sim \frac{3P^a(0)}{z^2} + \frac{\partial P^a(0)}{z}, \quad J^a(z)P^b(0) \sim \frac{if_c^{ab}P^c(z)}{z}, \quad (4.19)$$

which means that they can be interpreted as the charged spin 3 currents. See appendix A for the expressions with generic k . A crucial point here is that the operator products of P^a are of the form

$$P^a(z)P^b(0) \sim \frac{0 \cdot \delta^{ab}}{z^6} + \mathcal{O}(z^{-5}). \quad (4.20)$$

That is, the states corresponding to the spin 3 currents have zero norm, and hence they can be regarded as null vectors. Therefore, we can decouple P^a by setting $P^a = 0$. In this way, we can confirm the equality of the symmetries from the affine $\text{sl}(2M)$ and the coset (1.3) with $\lambda = 2$.

⁸We have computed with $N = 2, \dots, 6$ and $M = 2, \dots, 5$ except for $M = N$ where the central charge diverges.

4.4 Decomposition of rectangular W-algebra

In subsection 4.1, we argued that the W_n -algebra with $\mathfrak{su}(M)$ symmetry can be realized as the symmetry of the coset (1.3) if $k/(k+N) = n$ or $-k/(k+N+M) = n$. In this subsection, we would like to examine the meaning of the map of parameters more closely. In order to do so, it is convenient to decompose the coset algebra as

$$\frac{\mathfrak{su}(N+M)_k}{\mathfrak{su}(N)_k \oplus \mathfrak{u}(1)} \supset \frac{\mathfrak{su}(N+M)_k}{\mathfrak{su}(N+M-1)_k \oplus \mathfrak{u}(1)} \oplus \cdots \oplus \frac{\mathfrak{su}(N+1)_k}{\mathfrak{su}(N)_k \oplus \mathfrak{u}(1)} \oplus (M-1)\mathfrak{u}(1). \quad (4.21)$$

The decomposition can be nicely explained in terms of brane junctions [22–26]. We would like to examine the properties of the coset (1.3) in terms of the sum of the cosets.

Each coset in (4.21) can be interpreted in terms of $W_\infty[\lambda]$ with a specific value of λ . Here we define $W_\infty[\lambda]$ as the quantum Hamiltonian reduction of $\mathfrak{hs}[\lambda]$ with the principal embedding of $\mathfrak{sl}(2)$. Because of the truncation of $\mathfrak{hs}[\lambda]$, the algebra $W_\infty[\lambda]$ can be reduced to W_L at $\lambda = L$. The W_L -algebra can be realized as the symmetry of the coset (finally proven in [41, Main Theorem 1 and 2])

$$\frac{\mathfrak{su}(L)_K \oplus \mathfrak{su}(L)_1}{\mathfrak{su}(L)_{K+1}} \quad (4.22)$$

with the central charge

$$c_{L,K} = (L-1) \left(1 - \frac{L(L+1)}{(K+L)(K+L+1)} \right). \quad (4.23)$$

In [19] (see also [20]), a triality relation of $W_\infty[\lambda]$ was found, and the algebra was suggested to be the same with the three choices of λ as

$$\lambda_1 = \frac{L}{L+K}, \quad \lambda_2 = -\frac{L}{L+K+1}, \quad \lambda_3 = L \quad (4.24)$$

and with fixed $c = c_{L,K}$. There is an isomorphism between the universal enveloping algebra of $W_\infty[\lambda]$ and the affine Yangian of $\mathfrak{gl}(1)$ in [71] as explained in [72, 73]. In the Yangian description, there are types of representation expressed by plane partitions. We need three axes, e.g., x_1, x_2, x_3 to express the plane partitions, and the invariance under the rotation of axes corresponds to the triality relation.

It has been known for a long time [74, 75], see [41, Theorem 13.1] for a proof, that the coset

$$\frac{\mathfrak{su}(K+1)_L}{\mathfrak{su}(K)_L \oplus \mathfrak{u}(1)} \quad (4.25)$$

is level-rank dual to the coset (4.22). Therefore, the decomposition of (4.21) implies that

$$\frac{\mathfrak{su}(N+M)_k}{\mathfrak{su}(N)_k \oplus \mathfrak{u}(1)} \supset W_\infty[\mu^{(M-1)}] \oplus \cdots \oplus W_\infty[\mu^{(0)}] \oplus (M-1)\mathfrak{u}(1), \quad (4.26)$$

where the parameters $\mu^{(i)}$ are one of three possibilities

$$\mu_1^{(i)} = \frac{k}{k+N+i}, \quad \mu_2^{(i)} = -\frac{k}{k+N+i+1}, \quad \mu_3^{(i)} = k. \quad (4.27)$$

Here the each W-algebra is labeled with the central charge $C_i = c_{k,N+i}$. We can check that the central charge of the coset (1.3) can be reproduced from the right hand side of (4.26) as

$$c_{\text{coset}}(k, N, M) = \sum_{i=0}^{M-1} C_i + M - 1. \quad (4.28)$$

In appendix B, we explicitly decompose the symmetry algebra with $\lambda = 2$ and $M = 2$ as in (4.26) using the results obtained in subsection 3.3.

In order to express the whole coset algebra in terms of the sum of component W-algebras, we need to consider operators connecting two of the components. The symmetry algebra should be generated by currents with integer conformal dimensions. In the present case, the conformal dimensions of operators connecting two W-algebras can be integer with the help of the relation

$$\mu_2^{(i)} + \mu_1^{(i+1)} = 0, \quad (4.29)$$

see discussions in subsection 4.4.1 of [24]. In the Yangian description, we need to connect plane partitions with coordinates $(x_1^{(i)}, x_2^{(i)}, x_3^{(i)})$ [24, 76, 77]. The relation (4.24) implies that the $x_2^{(i)}$ -direction should be connected with the $x_1^{(i+1)}$ -direction for all $i = 0, 1, \dots, M-1$. Therefore, among the product of the rotational symmetries of axes, only $\mathbb{Z}_2$ -symmetry reversing the connected direction is remained. This exchanges, in particular, $\mu_1^{(0)}$ and $\mu_2^{(M-1)}$, which explain the duality of the coset models (1.3) with $\mu_1^{(0)} = \mu$ and $\mu_2^{(M-1)} = \mu$ (μ is a number) but with a fixed k .

At the 't Hooft limit, all the component W-algebras become the same as $W_\infty[\lambda]$ with the central charge $c = (1 - \lambda^2)k$. The same result can be obtained from the higher spin theory by following the analysis in section 2. In this case, we restrict ourselves to the subsectors generated by $t^H \otimes V_m^s$ ($H = 1, 2, \dots, M$), where t^H are generators of the Cartan sub-algebra of $\mathfrak{gl}(M)$. At $\lambda = n$ with $n = 2, 3, \dots$, we have $\mu_1^{(0)} = n$, thus the component $W_\infty[\mu^{(0)}]$ can be truncated to be a W_n -algebra. A similar statement holds also for $\lambda' = n$. This fact should be related to the truncation of the higher spin currents as argued in subsection 4.1.

A unitary model can be realized when $k = n$ with an integer n , where all $W_\infty[\mu^{(i)}]$ can be truncated to W_k -algebra. However, this is not a good choice in order to realize the W-algebra with $\mathfrak{su}(M)$ symmetry, since the level k of $\mathfrak{su}(M)$ becomes the same as the upper bound for the spin of currents. For $M = 1$, the restriction from the level of $\mathfrak{su}(M)$ disappears, and we can set $k = n$ as

$$\frac{\mathfrak{su}(N+1)_n}{\mathfrak{su}(N)_n \oplus \mathfrak{u}(1)} \simeq \frac{\mathfrak{su}(n)_N \oplus \mathfrak{su}(n)_1}{\mathfrak{su}(n)_{N+1}}. \quad (4.30)$$

However, we may consider an analytic continuation of N in order to relate with the symmetry of $\mathfrak{sl}(n)$ Chern-Simons theory [19, 32, 78].

5 Conclusion

In this paper, we examined the asymptotic symmetry of the 3d Prokushkin-Vasiliev theory with $M \times M$ matrix valued fields. The gauge algebra is given by $\text{hs}_M[\lambda]$ defined in (2.8),

and it can be truncated to be $\mathfrak{sl}(Mn)$ at $\lambda = n$ as in (1.1). At $\lambda = n$, the symmetry algebra is obtained by the Hamiltonian reduction of $\mathfrak{sl}(Mn)$ with the $\mathfrak{sl}(2)$ embedding corresponding to the partition (1.2). The W-algebra includes the affine $\mathfrak{su}(M)$ Lie algebra as a sub-algebra. We computed the central charge c of the W-algebra and the level k of the affine $\mathfrak{su}(M)$ as in (3.2) and (3.3). We first obtained the expressions at the classical limit from the Chern-Simons description of higher spin theory and then found the exact results from the quantum Hamiltonian reduction of $\mathfrak{sl}(Mn)$. For $n = 2$, we obtained the OPEs of the symmetry generators by requiring their associativity.

Based on the holographic duality in [7], we claim that the symmetry algebra is the same as the one of the Grassmannian-like coset in (1.3) if the 't Hooft parameter in (1.4) or in (1.5) is set as $\lambda = n$ or $\lambda' = n$. We obtained the map of parameters from the comparison of the central charge c of the algebra and the level k of the affine $\mathfrak{su}(M)$. We constructed low spin currents explicitly in terms of the coset model and reproduce OPEs for $n = 2$. In order to interpret the meaning of the 't Hooft parameters, we decomposed the W-algebra as in (4.26). We can explain the duality of the coset model (1.3) by combining the triality relation of [19] and the Yangian description of [72, 73] for the component W-algebras.

We need further study in order to confirm our claim on the match of the symmetry algebra with generic $n > 2$. We may be able to obtain the OPEs of higher spin by requiring their associativity, and this would lead to the relation of c, k in (3.4) with generic n . If we work with generic λ , the duality relation would be derived from the invariance of the structure constants as in [19–21]. The symmetry algebra could be reconstructed from the coset model as in subsection 4.3. For the purpose, we need the explicit form of higher spin currents and the OPEs among these currents. Since the explicit computations are straightforward but quite tedious, it would be better if there is an abstract way to prove our proposal. In subsection 4.4, we briefly comment on the Yangian description of the W-algebra with $\mathfrak{su}(M)$ symmetry, but we have not worked out any details there. It is important to develop the description in order to understand the properties of the W-algebra with deformable parameters. In particular, we would like to understand more the truncation of spectrum at $\lambda = n$. We also want to construct conical defect geometry in the $\mathfrak{sl}(Mn)$ Chern-Simons theory and compare them to some states in the W-algebra as in [19, 32, 78, 79]. We hope to report on some developments in near future.

We have examined the matrix extension of higher spin gravity in order to see the relation to string theory. In order to proceed furthermore, it would be useful to introduce an extended supersymmetry. In our previous works [69, 80], we introduced the $\mathcal{N} = 3$ supersymmetry by dealing with a critical level of Grassmannian model. For the extended supersymmetry, we embedded the Clifford algebra into the matrix algebra of higher spin theory as in [8, 81]. Thus, it is natural to expect that the analysis in this paper can be applied to the case with extended supersymmetry in a rather straightforward manner. The $\mathcal{N} = 4$ supersymmetry was introduced in [31, 82, 83], and the relation to the symmetric orbifold was discussed. The symmetry of the symmetric orbifold was named as “higher spin square,” and it is interesting to see how the algebra is related to the one examined in this paper.

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A Spin 3 currents from the coset

In this appendix, we obtain spin 3 currents from the coset model (1.3). The currents should be constructed from composite operators of the currents in the numerator, which are regular with the currents $J^\alpha, J^{u(1)}$ in the denominator. We first study the charged ones and then examine the singlet one with respect to the $\mathfrak{su}(M)$ action.

For the charged currents, we choose the composite operators transforming in the trivial and adjoint representations of $\mathfrak{su}(N)$ and $\mathfrak{su}(M)$, respectively, and use the ansatz

$$\begin{aligned}
 P^a = & a_1(J^\alpha(J^{\bar{\rho}i}J^{\bar{\sigma}j}))t_{\bar{\rho}\bar{\sigma}}^\alpha t_{j\bar{i}}^a + a_2(J^\alpha(J^\alpha J^a)) + a_3(J^b(J^b J^a)) + a_4(J^a(J^{u(1)}J^{u(1)})) \\
 & + (a_5d^{abc} + a_6if^{abc})[(J^c(J^{\bar{\rho}i}J^{\bar{\rho}j})) + (J^c(J^{\bar{\rho}j}J^{\bar{\rho}i}))]\delta_{\bar{\rho}\bar{\rho}}t_{j\bar{i}}^b \\
 & + a_7[(J^{u(1)}(J^{\bar{\rho}i}J^{\bar{\rho}j})) + (J^{u(1)}(J^{\bar{\rho}j}J^{\bar{\rho}i}))]\delta_{\bar{\rho}\bar{\rho}}t_{j\bar{i}}^a \\
 & + a_8[(J^a(J^{\bar{\rho}i}J^{\bar{\rho}i})) + (J^a(J^{\bar{\rho}i}J^{\bar{\rho}i}))]\delta_{\bar{\rho}\bar{\rho}}\delta_{i\bar{i}} \\
 & + a_9d^{abc}(J^b(J^cJ^{u(1)})) + (a_{10}d^{abc} + a_{11}if^{abc})((\partial J^b)J^c) \\
 & + [a_{12}((\partial J^{\bar{\rho}i})J^{\bar{\rho}j}) + a_{13}((\partial J^{\bar{\rho}j})J^{\bar{\rho}i})]\delta_{\bar{\rho}\bar{\rho}}t_{j\bar{i}}^a + a_{14}(J^a(\partial J^{u(1)})) + a_{15}((\partial J^a)J^{u(1)}) \\
 & + a_{16}\partial^2 J^a + a_{17}6\text{tr}_M(t^a t^b t^c t^d)(J^b(J^c J^d)). \tag{A.1}
 \end{aligned}$$

Along with the condition with $J^\alpha, J^{u(1)}$, we require for these currents to satisfy the OPEs (4.19). Then we can fix the coefficients as

$$\begin{aligned}
 a_2 = \frac{a_1}{k}, \quad a_3 = \frac{a_1 N(k+2N)}{k(k+M)(3k+2M)}, \quad a_4 = \frac{a_1(k+2N)(M+N)}{k^2 M}, \\
 a_5 = -\frac{a_1(k+2N)}{4(k+M)}, \quad a_6 = 0, \quad a_7 = \frac{a_1(k+2N)}{2k}\sqrt{\frac{M+N}{MN}}, \quad a_8 = -\frac{a_1(k+2N)}{2kM}, \\
 a_9 = -\frac{a_1(k+2N)}{2k(k+M)}\sqrt{\frac{N(M+N)}{M}}, \quad a_{10} = 0, \quad a_{11} = \frac{a_1(k^2-8)N(k+2N)}{4k(k+M)(3k+2M)}, \tag{A.2} \\
 a_{12} = -\frac{a_1(k+2N)}{2}, \quad a_{13} = \frac{a_1(k+2N)}{2}, \quad a_{14} = 0, \quad a_{15} = 0, \\
 a_{16} = -\frac{a_1 N(6k^3 + 9k^2 M + 4kM^2 + 12M)(k+2N)}{12k(k+M)(3k+2M)}, \quad a_{17} = \frac{a_1 N(k+2N)}{6(k+M)(3k+2M)}
 \end{aligned}$$

up to an overall factor a_1 . At $k = -2N$, all coefficients vanish except for a_1, a_2 , and the charged spin 3 currents reduces to (4.18). We have checked the above expressions for $N = 2, \dots, 6$ and $M = 2, \dots, 5$, and claim that they are also true for generic N, M . For $M > 3$, the condition uniquely fixes all the parameters except for a_1 . For $M = 2, 3$ we

can change

$$a_{17} \rightarrow a_{17} - \delta, \quad a_3 \rightarrow a_3 - 3\delta, \quad a_{11} \rightarrow a_{11} + 3\delta, \quad a_{16} \rightarrow a_{16} + 2M\delta. \quad (\text{A.3})$$

Moreover, for $M = 2$, we can use arbitrary a_5, a_9, a_{10} since $d^{abc} = 0$. Other parameters can be fixed by the condition.

For the singlet currents, we need the composite operators in the trivial representation both of $\text{su}(N)$ and $\text{su}(M)$, and use the ansatz

$$\begin{aligned} W = & a_1 d^{\alpha\beta\gamma} (J^\alpha (J^\beta J^\gamma)) + a_2 d^{abc} (J^a (J^b J^c)) + a_3 (J^{u(1)} (J^{u(1)} J^{u(1)})) \\ & + a_4 (J^\alpha (J^\alpha J^{u(1)})) + a_5 (J^a (J^a J^{u(1)})) + a_6 [(J^\alpha (J^{\rho\bar{i}} J^{\bar{\sigma}i}) + (J^\alpha (J^{\bar{\sigma}i} J^{\rho\bar{i}}))] t_{\rho\bar{\sigma}}^\alpha \delta_{i\bar{i}} \\ & + a_7 [(J^a (J^{\rho\bar{i}} J^{\bar{\rho}j}) + (J^a (J^{\bar{\rho}j} J^{\rho\bar{i}}))] \delta_{\rho\bar{\rho}} t_{j\bar{i}}^a \\ & + a_8 [(J^{u(1)} (J^{\bar{\rho}i} J^{\rho\bar{i}}) + (J^{u(1)} (J^{\rho\bar{i}} J^{\bar{\rho}i}))] \delta_{\rho\bar{\rho}} \delta_{i\bar{i}} \\ & + a_9 ((\partial J^\alpha) J^\alpha) + a_{10} ((\partial J^a) J^a) + a_{11} ((\partial J^{u(1)}) J^{u(1)}) \\ & + [a_{12} ((\partial J^{\rho\bar{i}}) J^{\bar{\rho}i}) + a_{13} ((J^{\bar{\rho}i}) J^{\rho\bar{i}})] \delta_{\rho\bar{\rho}} \delta_{i\bar{i}} + a_{14} \partial^2 J^{u(1)}. \end{aligned} \quad (\text{A.4})$$

We require that W is spin 3 primary with respect to the Virasoro algebra and has the regular OPE with J^a as

$$T(z)W(0) \sim \frac{3W(0)}{z^2} + \frac{\partial W(0)}{z}, \quad J^a(z)W(0) \sim 0 \quad (\text{A.5})$$

in addition to the condition with $J^\alpha, J^{u(1)}$. This determines the coefficients as

$$\begin{aligned} a_2 = & -\frac{a_1 N(k+N)(k+2N)}{M(k+M)(k+2M)}, \quad a_3 = \frac{2a_1(k+N)(k+2N)(M+N)}{k^2 M} \sqrt{\frac{M+N}{MN}}, \\ a_4 = & \frac{6a_1(k+N)}{k} \sqrt{\frac{M+N}{MN}}, \quad a_5 = \frac{6a_1(k+N)(k+2N)}{Mk(k+2M)} \sqrt{\frac{N(M+N)}{M}}, \\ a_6 = & -\frac{3a_1(k+N)}{M}, \quad a_7 = \frac{3a_1(k+N)(k+2N)}{M(k+2M)}, \\ a_8 = & -\frac{3a_1(k+N)(k+2N)}{kM} \sqrt{\frac{M+N}{MN}}, \quad a_9 = 0, \quad a_{10} = 0, \quad a_{11} = 0, \\ a_{12} = & \frac{3a_1(k+N)(k+2N)}{M}, \quad a_{13} = -\frac{3a_1(k+N)(k+2N)}{M}, \\ a_{14} = & -a_1(k+N)(k+2N) \sqrt{\frac{N(M+N)}{M}}. \end{aligned} \quad (\text{A.6})$$

We have obtained the above expressions with $N = 2, \dots, 6$ and $M = 2, \dots, 5$, and propose that they are true also for generic N, M . For $N, M > 2$, these coefficients are uniquely fixed up to an overall normalization a_1 . For $N = 2$, there is no relation between a_1 and other coefficients because of $d^{\alpha\beta\gamma} = 0$. Similarly, for $M = 2$, we can use arbitrary a_2 due to $d^{abc} = 0$. Utilizing these fact, the coefficients can be chosen as in (A.6), and others are fixed uniquely. At $k = -2N$, the expressions become simplified and only a_1, a_4, a_6 remain non-zero. In this case, we have checked for several examples⁹ that $W(z)W(0) \sim 0 \cdot z^{-6} + \mathcal{O}(z^{-5})$. Therefore, we can say that the spin 3 current W is decoupled at $k = -2N$ just as in the case of P^a .

⁹We have examined the cases with $N, M = 2, 3, 4$.

B Decomposition of W-algebra with $(n, M) = (2, 2)$

In this appendix, we reproduce the decomposition in (4.26) from the W_2 -algebra with $\text{su}(2)$ symmetry obtained in subsection 3.3. Similar analysis for other type of W-algebras has been done in [24], see also [76].

First, we look for two commuting energy-momentum tensors T_η ($\eta = 0, 1$), whose OPEs are

$$T_\eta(z)T_\eta(0) \sim \frac{C_i/2}{z^4} + \frac{2T_\eta}{z^2} + \frac{\partial T_\eta}{z}, \quad T_0(z)T_1(0) \sim 0, \quad T_\eta(z)J^3(0) \sim 0. \quad (\text{B.1})$$

Here we have chosen t^3 as the generator of Cartan sub-algebra and required that the spin 1 current J^3 decouples from the algebras. We use the ansatz

$$\begin{aligned} T_0 &\equiv a_1 T + a_2 Q^3 + a_3 (J^a J^a) + a_4 (J^3 J^3), \\ T_1 &\equiv b_1 T + b_2 Q^3 + b_3 (J^a J^a) + b_4 (J^3 J^3). \end{aligned} \quad (\text{B.2})$$

Then the OPEs in (B.1) are reproduced with

$$\begin{aligned} a_1 &= \frac{1}{k+2} + \frac{1}{2}, & a_2 &= \frac{\sqrt{-3k^3 - k^2 + 4k}}{\sqrt{2}(k+2)}, & a_3 &= -\frac{1}{2k+4}, & a_4 &= \frac{1}{4(k+2)}, \\ b_1 &= \frac{k}{2k+4}, & b_2 &= -\frac{\sqrt{-3k^3 - k^2 + 4k}}{\sqrt{2}(k+2)}, & b_3 &= \frac{1}{2k+4}, & b_4 &= -\frac{3k+4}{4k^2+8k}. \end{aligned} \quad (\text{B.3})$$

The central charges are $C_0 = c_{k,N}$ and $C_1 = c_{k,N+1}$ as they should be.

Next, we search for spin 3 and 4 currents, which are primary with respect to T_0 or T_1 and commute with J^3 . There are no such currents primary to T_0 . The three parameters $\mu_0^{(i)}$ ($i = 1, 2, 3$) for the W-algebra including T_0 are

$$\mu_1^{(0)} = 2, \quad \mu_2^{(0)} = -\frac{2k}{k+2}, \quad \mu_3^{(0)} = k. \quad (\text{B.4})$$

This implies that the allowed spin of current is only 2, and this is consistent with our result. We can find out spin 3 and 4 currents primary to T_1 . We normalize the currents such as

$$W^{(3)} = (J^a Q^a) + \dots, \quad W^{(4)} = (Q^a Q^a) + \dots. \quad (\text{B.5})$$

Some of the coefficients of OPEs are

$$\begin{aligned} C_{33}^0 &= \frac{2(k-2)^2(k+4)}{3k^2}, \quad C_{33}^4 = k+4, \\ C_{44}^0 &= -\frac{4(k-4)(k-3)(k-2)^2(k+1)(k+4)(k+6)(3k+2)(5k+12)}{(k-1)k^4(3k+4)(k(k(15k-47)-162)-136)}, \end{aligned} \quad (\text{B.6})$$

where C_{ij}^k are the coefficient in front of $W^{(k)}$ in the OPE of $W^{(i)}$ and $W^{(j)}$. According to [20] (see also [19]), their normalization invariant combination is related to the parameters λ_i ($i = 1, 2, 3$) and the central charge c of the W-algebra as

$$\frac{(C_{33}^4)^2 C_{44}^0}{(C_{33}^0)^2} = \frac{144(c+2)(\lambda_1-3)(\lambda_2-3)(\lambda_3-3)}{c(5c+22)(\lambda_1-2)(\lambda_2-2)(\lambda_3-2)}. \quad (\text{B.7})$$

This equation is consistent with the values $\lambda_i = \mu_i^{(1)}$ and $c = c_{k,N+1}$ if we set $k/(k+N) = 2$.

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References

- [1] D.J. Gross, *High-energy symmetries of string theory*, *Phys. Rev. Lett.* **60** (1988) 1229 [[INSPIRE](#)].
- [2] M.A. Vasiliev, *Nonlinear equations for symmetric massless higher spin fields in $(A)dS(d)$* , *Phys. Lett. B* **567** (2003) 139 [[hep-th/0304049](#)] [[INSPIRE](#)].
- [3] M.A. Vasiliev, *From Coxeter higher-spin theories to strings and tensor models*, *JHEP* **08** (2018) 051 [[arXiv:1804.06520](#)] [[INSPIRE](#)].
- [4] C.-M. Chang, S. Minwalla, T. Sharma and X. Yin, *ABJ triality: from higher spin fields to strings*, *J. Phys. A* **46** (2013) 214009 [[arXiv:1207.4485](#)] [[INSPIRE](#)].
- [5] O. Aharony, O. Bergman, D.L. Jafferis and J. Maldacena, *$N = 6$ superconformal Chern-Simons-matter theories, M2-branes and their gravity duals*, *JHEP* **10** (2008) 091 [[arXiv:0806.1218](#)] [[INSPIRE](#)].
- [6] O. Aharony, O. Bergman and D.L. Jafferis, *Fractional M2-branes*, *JHEP* **11** (2008) 043 [[arXiv:0807.4924](#)] [[INSPIRE](#)].
- [7] T. Creutzig, Y. Hikida and P.B. Ronne, *Extended higher spin holography and Grassmannian models*, *JHEP* **11** (2013) 038 [[arXiv:1306.0466](#)] [[INSPIRE](#)].
- [8] S.F. Prokushkin and M.A. Vasiliev, *Higher spin gauge interactions for massive matter fields in $3 - D$ AdS space-time*, *Nucl. Phys. B* **545** (1999) 385 [[hep-th/9806236](#)] [[INSPIRE](#)].
- [9] L. Eberhardt, M.R. Gaberdiel and I. Rienacker, *Higher spin algebras and large $N = 4$ holography*, *JHEP* **03** (2018) 097 [[arXiv:1801.00806](#)] [[INSPIRE](#)].
- [10] M.R. Gaberdiel and R. Gopakumar, *An AdS_3 dual for minimal model CFTs*, *Phys. Rev. D* **83** (2011) 066007 [[arXiv:1011.2986](#)] [[INSPIRE](#)].
- [11] T. Creutzig, Y. Hikida and P.B. Ronne, *Higher spin AdS_3 supergravity and its dual CFT*, *JHEP* **02** (2012) 109 [[arXiv:1111.2139](#)] [[INSPIRE](#)].
- [12] E. Joung, J. Kim, J. Kim and S.-J. Rey, *Asymptotic symmetries of colored gravity in three dimensions*, *JHEP* **03** (2018) 104 [[arXiv:1712.07744](#)] [[INSPIRE](#)].
- [13] S. Gwak, E. Joung, K. Mkrtchyan and S.-J. Rey, *Rainbow valley of colored (Anti-)de Sitter gravity in three dimensions*, *JHEP* **04** (2016) 055 [[arXiv:1511.05220](#)] [[INSPIRE](#)].
- [14] S. Gwak, E. Joung, K. Mkrtchyan and S.-J. Rey, *Rainbow vacua of colored higher-spin $(A)dS_3$ gravity*, *JHEP* **05** (2016) 150 [[arXiv:1511.05975](#)] [[INSPIRE](#)].
- [15] M. Henneaux and S.-J. Rey, *Nonlinear W_∞ as asymptotic symmetry of three-dimensional higher spin Anti-de Sitter gravity*, *JHEP* **12** (2010) 007 [[arXiv:1008.4579](#)] [[INSPIRE](#)].
- [16] A. Campoleoni, S. Fredenhagen, S. Pfenninger and S. Theisen, *Asymptotic symmetries of three-dimensional gravity coupled to higher-spin fields*, *JHEP* **11** (2010) 007 [[arXiv:1008.4744](#)] [[INSPIRE](#)].
- [17] M.R. Gaberdiel and T. Hartman, *Symmetries of holographic minimal models*, *JHEP* **05** (2011) 031 [[arXiv:1101.2910](#)] [[INSPIRE](#)].

- [18] A. Campoleoni, S. Fredenhagen and S. Pfenninger, *Asymptotic W -symmetries in three-dimensional higher-spin gauge theories*, *JHEP* **09** (2011) 113 [[arXiv:1107.0290](#)] [[INSPIRE](#)].
- [19] M.R. Gaberdiel and R. Gopakumar, *Triality in minimal model holography*, *JHEP* **07** (2012) 127 [[arXiv:1205.2472](#)] [[INSPIRE](#)].
- [20] T. Procházka, *Exploring W_∞ in the quadratic basis*, *JHEP* **09** (2015) 116 [[arXiv:1411.7697](#)] [[INSPIRE](#)].
- [21] C. Candu and M.R. Gaberdiel, *Duality in $N = 2$ minimal model holography*, *JHEP* **02** (2013) 070 [[arXiv:1207.6646](#)] [[INSPIRE](#)].
- [22] D. Gaiotto and M. Rapčák, *Vertex algebras at the corner*, *JHEP* **01** (2019) 160 [[arXiv:1703.00982](#)] [[INSPIRE](#)].
- [23] T. Creutzig and D. Gaiotto, *Vertex algebras for S -duality*, [arXiv:1708.00875](#) [[INSPIRE](#)].
- [24] T. Procházka and M. Rapčák, *Webs of W -algebras*, *JHEP* **11** (2018) 109 [[arXiv:1711.06888](#)] [[INSPIRE](#)].
- [25] T. Procházka and M. Rapčák, *W -algebra modules, free fields and Gukov-Witten defects*, [arXiv:1808.08837](#) [[INSPIRE](#)].
- [26] K. Harada and Y. Matsuo, *Plane partition realization of (web of) W -algebra minimal models*, *JHEP* **02** (2019) 050 [[arXiv:1810.08512](#)] [[INSPIRE](#)].
- [27] A. Achúcarro and P.K. Townsend, *A Chern-Simons action for three-dimensional Anti-de Sitter supergravity theories*, *Phys. Lett. B* **180** (1986) 89 [[INSPIRE](#)].
- [28] E. Witten, *$(2+1)$ -dimensional gravity as an exactly soluble system*, *Nucl. Phys. B* **311** (1988) 46 [[INSPIRE](#)].
- [29] M.P. Blencowe, *A consistent interacting massless higher spin field theory in $D = (2+1)$* , *Class. Quant. Grav.* **6** (1989) 443 [[INSPIRE](#)].
- [30] C.N. Pope, L.J. Romans and X. Shen, *W_∞ and the Racah-Wigner algebra*, *Nucl. Phys. B* **339** (1990) 191 [[INSPIRE](#)].
- [31] M.R. Gaberdiel and R. Gopakumar, *Large $N = 4$ holography*, *JHEP* **09** (2013) 036 [[arXiv:1305.4181](#)] [[INSPIRE](#)].
- [32] A. Castro, R. Gopakumar, M. Gutperle and J. Raeymaekers, *Conical defects in higher spin theories*, *JHEP* **02** (2012) 096 [[arXiv:1111.3381](#)] [[INSPIRE](#)].
- [33] J. Balog et al., *Toda theory and W algebra from a gauged WZNW point of view*, *Annals Phys.* **203** (1990) 76 [[INSPIRE](#)].
- [34] A. Campoleoni, S. Fredenhagen and J. Raeymaekers, *Quantizing higher-spin gravity in free-field variables*, *JHEP* **02** (2018) 126 [[arXiv:1712.08078](#)] [[INSPIRE](#)].
- [35] M. Henneaux, L. Maoz and A. Schwimmer, *Asymptotic dynamics and asymptotic symmetries of three-dimensional extended AdS supergravity*, *Annals Phys.* **282** (2000) 31 [[hep-th/9910013](#)] [[INSPIRE](#)].
- [36] T. Arakawa, *Introduction to W -algebras and their representation theory*, [arXiv:1605.00138](#) [[INSPIRE](#)].
- [37] T. Creutzig and A.R. Linshaw, *The super $W_{1+\infty}$ algebra with integral central charge*, *Trans. Amer. Math. Soc.* **367** (2015) 5521 [[arXiv:1209.6032](#)].

- [38] T. Creutzig and A.R. Linshaw, *Cosets of affine vertex algebras inside larger structures*, *J. Algebra* **517** (2019) 396 [[arXiv:1407.8512](#)] [[INSPIRE](#)].
- [39] V.G. Kac and M. Wakimoto, *Quantum reduction and representation theory of superconformal algebras*, [math-ph/0304011](#) [[INSPIRE](#)].
- [40] N. Genra, *Screening operators for W -algebras*, [arXiv:1606.00966](#) [[INSPIRE](#)].
- [41] T. Arakawa, T. Creutzig and A.R. Linshaw, *W -algebras as coset vertex algebras*, [arXiv:1801.03822](#) [[INSPIRE](#)].
- [42] A.R. Linshaw, *Universal two-parameter W_∞ -algebra and vertex algebras of type $W(2, 3, \dots, N)$* , [arXiv:1710.02275](#) [[INSPIRE](#)].
- [43] S. Kanade and A.R. Linshaw, *Universal two-parameter even spin W_∞ -algebra*, [arXiv:1805.11031](#) [[INSPIRE](#)].
- [44] T. Arakawa and A. Molev, *Explicit generators in rectangular affine W -algebras of type A* , *Lett. Math. Phys.* **107** (2017) 47 [[arXiv:1403.1017](#)] [[INSPIRE](#)].
- [45] T. Creutzig, Y. Hikida and P.B. Rønne, *Correspondences between WZNW models and CFTs with W -algebra symmetry*, *JHEP* **02** (2016) 048 [[arXiv:1509.07516](#)] [[INSPIRE](#)].
- [46] Y. Hikida and V. Schomerus, *H_3^+ WZNW model from Liouville field theory*, *JHEP* **10** (2007) 064 [[arXiv:0706.1030](#)] [[INSPIRE](#)].
- [47] Y. Hikida and V. Schomerus, *Structure constants of the $OSp(1|2)$ WZNW model*, *JHEP* **12** (2007) 100 [[arXiv:0711.0338](#)] [[INSPIRE](#)].
- [48] T. Creutzig, Y. Hikida and P.B. Ronne, *Supergroup-extended super Liouville correspondence*, *JHEP* **06** (2011) 063 [[arXiv:1103.5753](#)] [[INSPIRE](#)].
- [49] I. Bakas and E. Kiritsis, *Grassmannian coset models and unitary representations of W_∞* , *Mod. Phys. Lett. A* **5** (1990) 2039 [[INSPIRE](#)].
- [50] S. Odake and T. Sano, *$W_{1+\infty}$ and super W_∞ algebras with $SU(N)$ symmetry*, *Phys. Lett. B* **258** (1991) 369.
- [51] K. Thielemans, *A Mathematica package for computing operator product expansions*, *Int. J. Mod. Phys. C* **2** (1991) 787 [[INSPIRE](#)].
- [52] P. Di Francesco, P. Mathieu and D. Senechal, *Conformal field theory*, *Graduate Texts in Contemporary Physics*, Springer, Germany (1997).
- [53] T. Creutzig, S. Kanade, A.R. Linshaw and D. Ridout, *Schur-Weyl duality for Heisenberg cosets*, [arXiv:1611.00305](#) [[INSPIRE](#)].
- [54] T. Creutzig, *W -algebras for Argyres-Douglas theories*, [arXiv:1701.05926](#) [[INSPIRE](#)].
- [55] D. Adamovic and A. Milas, *On some vertex algebras related to $V_{-1}(\mathfrak{sl}(n))$ and their characters*, [arXiv:1805.09771](#).
- [56] V.G. Kac and M. Wakimoto, *A remark on boundary level admissible representations*, [arXiv:1612.07423](#).
- [57] T. Creutzig and D. Ridout, *Logarithmic conformal field theory: beyond an introduction*, *J. Phys. A* **46** (2013) 4006 [[arXiv:1303.0847](#)] [[INSPIRE](#)].
- [58] D. Adamović and A. Milas, *Vertex operator algebras associated to modular invariant representations for $A_1^{(1)}$* , *Math. Res. Lett.* **2** (1995) 563.

- [59] T. Creutzig and D. Ridout, *Modular data and Verlinde formulae for fractional level WZW models II*, *Nucl. Phys. B* **875** (2013) 423.
- [60] D. Ridout and S. Wood, *Relaxed singular vectors, Jack symmetric functions and fractional level $\widehat{sl}(2)$ models*, *Nucl. Phys. B* **894** (2015) 621.
- [61] V.G. Kac and M. Wakimoto, *Classification of modular invariant representations of affine algebras*, in *Infinite-dimensional Lie algebras and groups (Luminy-Marseille, 1988)*, V.G. Kac ed., Advanced Series Mathematical Physics volume 7, World Scientific, Singapore (1989).
- [62] T. Arakawa, *Rationality of admissible affine vertex algebras in the category $\mathcal{O}$* , *Duke Math. J.* **165** (2016) 67 [[arXiv:1207.4857](#)] [[INSPIRE](#)].
- [63] V.G. Kac and M. Wakimoto, *Modular invariant representations of infinite-dimensional Lie algebras and superalgebras*, *Proc. Nat. Acad. Sci. U.S.A.* **85** (1988) 4956.
- [64] T. Creutzig, Y.-Z. Huang and J. Yang, *Braided tensor categories of admissible modules for affine Lie algebras*, *Commun. Math. Phys.* **362** (2018) 827 [[arXiv:1709.01865](#)] [[INSPIRE](#)].
- [65] T. Creutzig, *Fusion categories for affine vertex algebras at admissible levels*, [arXiv:1807.00415](#).
- [66] K. Bringmann, T. Creutzig and L. Rolin, *Negative index Jacobi forms and quantum modular forms*, *Res. Math. Sci.* **1** (2014) 11.
- [67] K. Bringmann, L. Rolin and S. Zwegers, *On the fourier coefficients of negative index meromorphic jacobi forms*, *Res. Math. Sci.* **3** (2016) 5.
- [68] C. Candu and C. Vollenweider, *On the coset duals of extended higher spin theories*, *JHEP* **04** (2014) 145 [[arXiv:1312.5240](#)] [[INSPIRE](#)].
- [69] T. Creutzig, Y. Hikida and P.B. Ronne, *Higher spin AdS_3 holography with extended supersymmetry*, *JHEP* **10** (2014) 163 [[arXiv:1406.1521](#)] [[INSPIRE](#)].
- [70] F.A. Bais, P. Bouwknegt, M. Surridge and K. Schoutens, *Extensions of the Virasoro algebra constructed from Kac-Moody algebras using higher order Casimir invariants*, *Nucl. Phys. B* **304** (1988) 348 [[INSPIRE](#)].
- [71] O. Tsymbaliuk, *The affine Yangian of gl_1 , and the infinitesimal Cherednik algebras*, Ph.D. thesis, MIT, Department of Mathematics, Cambridge, U.S.A. (2014).
- [72] T. Procházka, *$\mathcal{W}$ -symmetry, topological vertex and affine Yangian*, *JHEP* **10** (2016) 077 [[arXiv:1512.07178](#)] [[INSPIRE](#)].
- [73] M.R. Gaberdiel, R. Gopakumar, W. Li and C. Peng, *Higher spins and Yangian symmetries*, *JHEP* **04** (2017) 152 [[arXiv:1702.05100](#)] [[INSPIRE](#)].
- [74] D. Altschuler, M. Bauer and C. Itzykson, *The branching rules of conformal embeddings*, *Commun. Math. Phys.* **132** (1990) 349 [[INSPIRE](#)].
- [75] M.A. Walton, *Conformal branching rules and modular invariants*, *Nucl. Phys. B* **322** (1989) 775 [[INSPIRE](#)].
- [76] M.R. Gaberdiel, W. Li, C. Peng and H. Zhang, *The supersymmetric affine Yangian*, *JHEP* **05** (2018) 200 [[arXiv:1711.07449](#)] [[INSPIRE](#)].
- [77] M.R. Gaberdiel, W. Li and C. Peng, *Twin-plane-partitions and $N = 2$ affine Yangian*, *JHEP* **11** (2018) 192 [[arXiv:1807.11304](#)] [[INSPIRE](#)].

- [78] E. Perlmutter, T. Prochazka and J. Raeymaekers, *The semiclassical limit of W_N CFTs and Vasiliev theory*, *JHEP* **05** (2013) 007 [[arXiv:1210.8452](#)] [[INSPIRE](#)].
- [79] Y. Hikida, *Conical defects and $N = 2$ higher spin holography*, *JHEP* **08** (2013) 127 [[arXiv:1212.4124](#)] [[INSPIRE](#)].
- [80] Y. Hikida and P.B. Rønne, *Marginal deformations and the Higgs phenomenon in higher spin AdS_3 holography*, *JHEP* **07** (2015) 125 [[arXiv:1503.03870](#)] [[INSPIRE](#)].
- [81] M. Henneaux, G. Lucena Gómez, J. Park and S.-J. Rey, *Super- W_∞ asymptotic symmetry of higher-spin AdS_3 supergravity*, *JHEP* **06** (2012) 037 [[arXiv:1203.5152](#)] [[INSPIRE](#)].
- [82] M.R. Gaberdiel and R. Gopakumar, *Higher spins $\mathcal{E}$ strings*, *JHEP* **11** (2014) 044 [[arXiv:1406.6103](#)] [[INSPIRE](#)].
- [83] M.R. Gaberdiel and R. Gopakumar, *Stringy symmetries and the higher spin square*, *J. Phys. A* **48** (2015) 185402 [[arXiv:1501.07236](#)] [[INSPIRE](#)].