Gauge-independent approach to resonant dark matter annihilation

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ABSTRACT: In spontaneously broken gauge theories, transition amplitudes describing dark-matter (DM) annihilation processes through a resonance may become highly inaccurate close to a production threshold, if a Breit-Wigner (BW) ansatz with a constant width is used. To partially overcome this problem, the BW propagator needs to be modified by including a momentum dependent decay width. However, such an approach to resonant transition amplitudes generically suffers from gauge artefacts that may also give rise to a bad or ambiguous high-energy behaviour for such amplitudes. We address the two problems of gauge dependence and high-energy unitarity within a gauge-independent framework of resummation implemented by the so-called Pinch Technique. We study DM annihilation via scalar resonances in a gauged $U(1)_X$ complex-scalar extension of the Standard Model that features a massive stable gauge field which can play the role of the DM. We find that the predictions for the DM abundance may vary significantly from previous studies based on the naive BW ansatz and propose an alternative simple approximation which leads to the correct DM phenomenology. In particular, our results do not depend on the gauge-fixing parameter and are consistent with considerations from high-energy unitarity.

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1 Introduction

Unveiling the nature of Dark Matter (DM) constitutes one of the biggest challenges in Cosmology and possibly in Particle Physics. Even though several pieces of evidence coming from both cosmological and astrophysical scales confirm its presence, the actual composition of the DM itself remains elusive to us thus far. In numerous models, the DM is assumed to be a new kind of massive particles that were in thermal equilibrium with the Standard Model (SM) particles in the early Universe. In such thermal DM models, the DM relic abundance crucially depends on the rate at which DM particles annihilate into thermal bath states. This rate is proportional to the thermally averaged DM-pair-annihilation cross-section summed over all possible final states. Most remarkably, if this cross-section has a value which is typical to the one governing SM weak interactions, then the predicted DM density turns out to be in the right ballpark in agreement with observations. Note that the most accurate determination of the DM relic density comes from measurements of the Cosmic Microwave Background (CMB) by the Planck satellite in conjunction with a number of other experiments and astrophysical data (for a recent analysis, see [1]).

A popular DM scenario [2–16] is based on the working hypothesis that there exists a mediator with couplings to both DM and SM particles, and whose mass happens to be

approximately twice that of the DM. In such scenarios, one has to consider DM annihilation processes at energies where the tree-level propagator of the mediator becomes singular. A standard way to avoid such singularities is to employ a Breit-Wigner (BW) ansatz for the propagator with a constant decay width [17]. Then, the amplitude gets regularized by the non-zero width of the mediator and can thus get drastically enhanced by many orders of magnitude, when compared to other non-resonant contributions. In Quantum Field Theory (QFT), the BW form of the propagator usually arises from a Dyson series summation of self-energy graphs of the mediator. In the so-called on-mass-shell (OS) scheme of renormalization [18], the dispersive parts of the self-energies renormalize the masses, whilst their absorptive part is related to the decay width of the mediator.

From the phenomenological point of view, the channel of resonant DM annihilation turns out to be an attractive option, as it leads to suppressed DM-nucleon cross-sections thereby avoiding the tight constraints emanating from the null results of direct DM searches. Therefore, it should not be too surprising that this resonant region in question may become the only viable region in the parameter space of a given model with DM mass in the GeV range that survives after all direct detection limits on the DM-nucleon cross-section were imposed [19, 20].

As was first noted in [14], the BW approximation with a constant decay width in the propagator of the mediator can become very inaccurate close to the DM production threshold, especially when the respective DM channel contributes significantly to the decay width of the mediator. To partially remedy this problem, one is compelled to use an effective BW propagator with momentum-dependent or running width for the exchanged particle in the s -channel. In this article, we will go beyond the previously used non-relativistic approach [14]. In general, an s -dependent width results from the imaginary (absorptive) part of the self-energy of the mediator. This quantity is only gauge-independent when evaluated at the pole of the propagator, but becomes gauge-variant in the off-shell region. This was a well-known problem in QFT and pertains to the question whether a consistent gauge-independent definition of off-shell Green's function for unstable particles exists in spontaneously broken gauge theories [21]. To deal with this issue, a number of recipes and methods have been put forward by several authors, such as the Laurent series expansion [22, 23], the complex mass scheme [24, 25], the fermion loop scheme [26], and the effective theory approach [27].

An elegant and equally consistent gauge-independent framework to address the aforementioned problem is the so-called Pinch Technique (PT) [28–32]. The PT preserves basic properties of QFT, such as analyticity, unitarity and the gauge invariance of the classical action. The PT resummation approach to unstable particles [30, 31, 33] was extensively studied in the literature originally within the context of the SM [34–38] and more recently in two Higgs-doublet models [39–42].

In this paper we discuss the problems that arise in the relativistic treatment of resonant DM annihilation processes in spontaneously broken gauge theories and show how these can be avoided in a resummation approach implemented by the PT. As an archetypal model, we consider a gauged $U(1)_X$ complex-scalar extension of the SM that includes a massive stable gauge boson X as a candidate particle for a Vector DM (VDM). We explicitly demon-

strate how the transition amplitude for DM-pair annihilation has the proper high-energy unitarity limit in compliance with the Equivalence Theorem [43, 44] and its generalized version (GET) [45, 46]. By virtue of the PT resummation method adopted in this paper, we obtain predictions for the yield of DM abundance that may vary significantly from previous considerations based on the naive BW approximation. In particular, we illustrate the gauge independence of our results and their consistency with high-energy unitarity. In the same context, we present an alternative simple approximation which leads to the correct DM phenomenology.

The paper is organized as follows. In section 2 we discuss the process of resonant DM-pair annihilation and the problems that arise in the relativistic treatment of a VDM in the vicinity of a resonance. In section 3 we describe the PT resummation approach and apply it to transition amplitudes that are typical to DM annihilation processes, such as $XX \rightarrow \bar{f}f$, where f is a SM fermion. By means of this approach, we obtain a Born-improved amplitude which is gauge independent and possesses the proper asymptotics in the high-energy limit. In section 4 we use the Born-improved amplitude derived in the previous section to compute the annihilation cross-sections and DM relic density. This enables us to assess the significance of our results by comparing them with those derived with other methods. In section 5 the key points of our analysis were summarised. Technical details of this study were relegated to a number of appendices. Specifically, appendix A provides further details of the VDM model under study, appendix B presents the Feynman rules for this model in the R_ξ and background field gauges (BFG), and appendix C contains analytical expressions of one-loop vertex corrections in the PT. Finally, appendix D complements our proof of the GET for the DM annihilation process $XX \rightarrow \bar{f}f$.

2 Resonant dark matter annihilation

Annihilation of DM in the vicinity of a resonance has certain features that renders it distinct from a generic scenario of a thermal DM. First, the thermally averaged cross-section displays strong dependence on the temperature T that can lead to a prolonged period of effective DM annihilation which substantially changes the comoving DM density, even though the DM itself may have already decoupled from the thermal bath. Second, the cross-section may be significantly enhanced at small velocities leading to a strengthening of the signal probed by DM indirect searches [5, 8]. Finally, the coupling of the DM to SM particles can become rather suppressed. This last property can be quite challenging for collider or direct detection experiments as it usually gives rise to weaker constraints for this area of parameter space. It also raises the temperature of kinetic decoupling which, in connection with the high T -variability of the annihilation cross-section, may affect the predictions of relic density [14, 47, 48].

In the Born approximation, the transition amplitude for a resonant DM annihilation process does not describe consistently the dynamics of a possible unstable mediator with mass M , because its tree-level propagator $\Delta^0(s) \equiv (s - M^2)^{-1}$ becomes singular at $s = M^2$. The standard way to treat this singularity is to perform a Dyson series summation of the mediator's self-energy $\Pi(s)$ which results in a replacement of the tree-level propaga-

tor $\Delta^0(s)$ with a resummed one, $\Delta(s) \equiv (s - M^2 + \Pi(s))^{-1}$. In the OS scheme of renormalization [18], this modification amounts to the well-known BW approximation [17], with $\Pi(s) \approx \Im m \Pi(s = M^2) = iZ^{-1} M\Gamma$, where Z is the wave-function renormalization (set to 1 at the one-loop level), and M and Γ are the renormalized mass and width of the resonant particle in this scheme. In this way, one obtains a finite analytic expression for the amplitude in the resonance region. For instance, the cross-section for a $2 \rightarrow 2$ annihilation process, which proceeds via the s -channel and has two identical DM particles of mass m_i in the initial state and two particles of equal mass m_f in the final state, is approximately given by

$$\sigma \simeq \frac{1}{s} \sum_{f \neq i} \frac{M^2 \Gamma^2 B_i B_f}{(s - M^2)^2 + M^2 \Gamma^2} . \quad (2.1)$$

In the above, we followed the notation and conventions of [5], after taking into account an extra factor of 2 because of identical particles in the initial state. Hence, B_i and B_f in (2.1) are the branching ratios for the mediator state to decay into initial and final states, denoted as i and f , respectively.

Nevertheless, when the dominant contribution to the mediator's self-energy $\Pi(s)$ comes from particles which are also the initial states of the transition amplitude, then the BW approximation is in general not applicable [14]. In this case, one finds high inaccuracies in the transition amplitudes that result from the phase space factor $\sqrt{1 - s/(4m^2)}$ present in $\Pi(s)$, which varies substantially for $s \gtrsim 4m^2$. To deal with this problem, one has to consider the energy-dependence of the imaginary part of the self-energy, which is sometimes described by the running width of the mediator $\Gamma(s) \equiv \Im m \Pi(s)/M$. However, this approach encounters other serious theoretical problems. As the resummation process relies on taking a subset of graphs from each order of the perturbative expansion, the subtle cancellations which guarantee the gauge-independence of the amplitudes at each order of the expansion are spoiled and the resulting expression has explicit dependence on the gauge fixing parameter(s) chosen to calculate $\Pi(s)$. The gauge-dependent amplitude results in an ambiguous annihilation cross-section, and as such its physical relevance becomes rather obscured.

In order to address the above issue, we adopt the PT framework which enables us to calculate such resonant DM annihilation amplitudes following a gauge-independent and self-consistent approach. As mentioned in the introduction, the so-derived Born-improved amplitudes will satisfy several desirable field-theoretic properties as a consequence of the analyticity, unitarity and gauge-invariance of the S -matrix. Our approach implemented by the PT will be illustrated within a VDM model [49–54]. This is an extension of the SM augmented by local $U(1)_X$ symmetry and by a complex scalar field S . The field S has a portal interaction with the SM Higgs doublet, and also develops a non-zero vacuum expectation value (VEV) that breaks $U(1)_X$ spontaneously. The resulting gauge boson X acquires a mass due to the Higgs mechanism and can be made stable, provided its kinetic mixing with the hypercharge gauge boson is forbidden by a Z_2 symmetry. In this $U(1)_X$ extension of the SM, the vector boson X can play the role of a viable DM. This VDM model contains two scalar mass eigenstates, denoted as h_1 and h_2 , where the first scalar represents the SM-like Higgs boson, while the second one will serve as a resonant mediator.

Figure 1. The tree-level (left) and the Born-improved (right) Feynman diagram for the process $XX \rightarrow f\bar{f}$.

The couplings of the h_1 and h_2 scalars to SM matter and VDM are governed by the interaction Lagrangian,

$$\begin{aligned} \mathcal{L}_{\text{int}} = & \frac{h_1 \cos \alpha + h_2 \sin \alpha}{v} \left(2M_W^2 W_\mu^+ W^{\mu-} + M_Z^2 Z_\mu Z^\mu - \sum_f m_f \bar{f} f \right) \\ & + 2g_x M_X \left(-h_1 \sin \alpha + h_2 \cos \alpha \right) X_\mu X^\mu, \end{aligned} \quad (2.2)$$

where α is the scalar mixing angle and g_x is the $U(1)_X$ gauge coupling. Our primary focus in this analysis will be on the resonance region, where $m_2 \approx 2M_X$ with a scalar mixing angle $\alpha \ll 1$. Further details of the VDM model under consideration may be found in appendix A.

3 Gauge independence and high energy unitarity

In this section, we will describe a gauge-invariant resummation approach to resonant transition amplitudes, within the Pinch Technique (PT) framework. As an illustrative example, we will be studying the annihilation process $XX \rightarrow \bar{f}f$, where X is a $U(1)_X$ gauge field which assumes the role of the DM and f collectively stands for a SM fermion. In the $U(1)_X$ scenario under study, the DM annihilation process, $XX \rightarrow \bar{f}f$, proceeds via the exchange of a coupled system of two mixed scalar resonances h_1 and h_2 . As illustrated in the left diagram of figure 1, the tree-level amplitude of such a reaction reads

$$\epsilon^\mu(p_1) \epsilon^\nu(p_2) i\mathcal{A}_{\mu\nu}^{XX \rightarrow \bar{f}f} = \epsilon^\mu(p_1) \epsilon^\nu(p_2) V_{\mu\nu}^{XX h_i} \frac{i\delta_{ij}}{s - m_i^2} V^{h_j \bar{f}f} \bar{u}(r_2) v(r_1). \quad (3.1)$$

Here, summation over the repeated indices $i, j = 1, 2$ is implied, $m_{1,2}$ are the masses of the Higgs scalars $h_{1,2}$, and $V_{\mu\nu}^{XX h_i}$ and $V^{h_j \bar{f}f}$ denote the tree-level expressions for the $XX h_i$ and $h_j \bar{f}f$ vertices, respectively. Moreover, $\epsilon^\mu(p_1)$ and $\epsilon^\nu(p_2)$ denote the polarizations of the X -vector bosons with four-momenta p_1 and p_2 , and $u(r_2)$ and $v(r_1)$ are the usual Dirac spinors for the fermion-antifermion pair f and $\bar{f}$ with momenta r_2 and r_1 , respectively.

As stated in section 2, the tree-level amplitude (3.1) becomes singular near the resonance regions $s \approx m_{1,2}^2$, requiring resummation of an infinite series of self-energies $\Pi_{ij}(s)$. Our aim is to calculate a Born-improved amplitude for the DM annihilation process $XX \rightarrow \bar{f}f$, which does not depend on the gauge-fixing parameter ξ_X used to parameterize the unphysical degrees of freedom of the gauge field X . In this context, we will adopt a gauge-independent resummation approach (illustrated in the right diagram of figure 1)

which is implemented by the PT in order to properly treat the coupled system of the two resonances, h_1 and h_2 , in a fashion analogous to [33]. In this framework, in section 3.1, we obtain a resummed propagator which is a 2×2 matrix and is denoted as $\hat{\Delta}_{ij}$. Subsequently, in section 3.2 we discuss the necessity of including one-loop corrections to the XXh_i vertices and present the tree-level Ward Identities (WIs) satisfied by the PT self-energies and PT vertices which ensure the validity of the ET [43, 44], as well as of the GET [45, 46]. Finally, we show in section 3.3 that the so-derived Born-improved amplitude $XX \rightarrow \bar{f}f$ exhibits proper high-energy asymptotic behaviour as expected from unitarity considerations.

3.1 Pinch technique resummation

To start with, we first calculate the X -field contribution to the $h_i h_j$ self-energies $\Pi_{ij}(s)$ in the VDM model of interest to us. In the renormalizable class of R_ξ gauges, upon inclusion of the would-be Goldstone bosons G_X and the pertinent ghost fields, these are given by

$$\Pi_{ij}^{(XX)}(s) = \frac{g_x^2 R_{2i} R_{2j}}{32\pi^2 M_X^2} \left[(s^2 - 4M_X^2 s + 12M_X^4) B_0(s, M_X^2, M_X^2) - (s^2 - m_i^2 m_j^2) B_0(s, \xi_X M_X^2, \xi_X M_X^2) \right], \quad (3.2)$$

where ξ_X is the gauge-fixing parameter associated with the $U(1)_X$ gauge field. Resummation of this self-energy inevitably leads to a gauge-dependent resonant amplitude and the appearance of unphysical thresholds at $s = 4\xi_X M_X^2$, when $\xi_X \neq 1$.

We may now calculate X , Z , W , t , h_k , h_l contributions to the $h_i h_j$ self-energies $\hat{\Pi}_{ij}(s)$ employing the PT, as done in [37, 38]. For the case of the VDM, these are given by

$$\hat{\Pi}_{ij}^{(XX)}(s) = \frac{g_x^2 R_{2i} R_{2j}}{8\pi^2} \left[\frac{(m_i m_j)^2}{4M_X^2} + \frac{m_i^2 + m_j^2}{2} - (2s - 3M_X^2) \right] B_0(s, M_X^2, M_X^2), \quad (3.3)$$

$$\hat{\Pi}_{ij}^{(WW)}(s) = \frac{g^2 R_{1i} R_{1j}}{16\pi^2} \left[\frac{(m_i m_j)^2}{4M_X^2} + \frac{m_i^2 + m_j^2}{2} - (2s - 3M_W^2) \right] B_0(s, M_W^2, M_W^2), \quad (3.4)$$

$$\hat{\Pi}_{ij}^{(ZZ)}(s) = \frac{g^2 R_{1i} R_{1j} M_Z^2}{32\pi^2 M_W^2} \left[\frac{(m_i m_j)^2}{4M_X^2} + \frac{m_i^2 + m_j^2}{2} - (2s - 3M_Z^2) \right] B_0(s, M_Z^2, M_Z^2), \quad (3.5)$$

$$\hat{\Pi}_{ij}^{(tt)}(s) = \frac{3g^2 R_{1i} R_{1j} m_t^2}{32\pi^2 M_W^2} (s - 4m_t^2) B_0(s, m_t^2, m_t^2), \quad (3.6)$$

$$\hat{\Pi}_{ij}^{(h_k h_l)}(s) = -\frac{V_{ikl}^h V_{jkl}^h}{32\pi^2} B_0(s, m_{h_k}^2, m_{h_l}^2), \quad (3.7)$$

where

$$B_0(p^2, m_a^2, m_b^2) \equiv (2\pi\mu)^{4-n} \int \frac{d^n k}{i\pi^2} \frac{1}{(k^2 - m_a^2) [(k+p)^2 - m_b^2]} \quad (3.8)$$

is the 't Hooft-Veltman function [55] defined in $n = 4 - 2\epsilon$ dimensions and the coupling V_{ikl}^h is specified in table 1. Note that the above results for the PT self-energies $\hat{\Pi}_{ij}(s)$ are also in agreement with that obtained in [39], upon appropriately replacing the couplings of that model. Moreover, the same results can be derived by making use of the established equivalence between the PT and the covariant Background Field Gauge (BFG) for $\xi_Q = 1$ [56–59]. For reader's convenience, the Feynman rules in the BFG are presented in appendix B.

Evidently, the $h_i h_j$ self-energies $\widehat{\Pi}_{ij}(s)$ do not depend on the gauge-fixing parameters ξ_X , ξ_W , ξ_Z and all display an absorptive part at the *physical* thresholds $s = 4M_X^2$, $4M_W^2$, $4M_Z^2$, as expected.

The full PT self-energies $\widehat{\Pi}_{ij}(s)$ can then be systematically resummed, yielding the PT resummed propagator

$$i\widehat{\Delta} = i\Delta^0 + i\Delta^0 i\widehat{\Pi} i\Delta^0 + i\Delta^0 (i\widehat{\Pi} i\Delta^0)^2 + \dots \quad (3.9)$$

In the above, we have suppressed the indices $i, j = 1, 2$ labelling the Higgs scalars $h_{1,2}$ and the s -dependence of the propagators and self-energies. Moreover, $\Delta^0(s) = \text{diag}[(s - m_1^2)^{-1}, (s - m_2^2)^{-1}]$ is the diagonal tree-level propagator matrix. From (3.9), one then gets

$$\widehat{\Delta}(s) = \frac{1}{D(s)} \begin{pmatrix} s - m_2^2 + \widehat{\Pi}_{22}(s) & -\widehat{\Pi}_{12}(s) \\ -\widehat{\Pi}_{21}(s) & s - m_1^2 + \widehat{\Pi}_{11}(s) \end{pmatrix}, \quad (3.10)$$

with $D(s) = [s - m_1^2 + \widehat{\Pi}_{11}(s)] [(s - m_2^2 + \widehat{\Pi}_{22}(s)) - \widehat{\Pi}_{12}(s)\widehat{\Pi}_{21}(s)]$. As we will see in the next section, in addition to its gauge independence, a Born-improved amplitude for $XX \rightarrow \bar{f}f$ must have the proper high-energy asymptotic behaviour, as dictated by the ET.

3.2 The generalized equivalence theorem

Replacing naively the tree-level propagator matrix $\Delta^0(s)$ with the resummed one $\widehat{\Delta}(s)$ in (3.1) modifies the transition amplitude $\mathcal{A}^{XX \rightarrow \bar{f}f}(s)$ not only in the vicinity of the resonance region, but also changes drastically its high-energy limit as $s/(4M_X^2) \rightarrow \infty$. This is a generic problem for most forms of BW propagators [60, 61] and is due to the presence of non-zero self-energies $\widehat{\Pi}_{ij}(s)$ or decay widths in $\widehat{\Delta}(s)$ that may possess non-trivial s -dependence. The latter distort subtle cancellations which are triggered by the WIs that result from the gauge invariance of the classical action, thus leading to a different high-energy limit from the one expected in the Born approximation.

To ensure the proper high-energy asymptotic behaviour of a scattering process involving massive gauge fields in the initial or final state, the amplitude of such process must obey the GET [45, 46], which is a consequence of the classical WIs mentioned above. Applying the GET to the amplitude of the process $XX \rightarrow \bar{f}f$ gives

$$\begin{aligned} \mathcal{A}_{X_L(p_1)X_L(p_2) \rightarrow \bar{f}f} &= -\mathcal{A}_{G_X(p_1)G_X(p_2) \rightarrow \bar{f}f} - i\mathcal{A}_{x^\mu(p_1)G_X(p_2) \rightarrow \bar{f}f} \\ &\quad - i\mathcal{A}_{G_X(p_1)x^\nu(p_2) \rightarrow \bar{f}f} + \mathcal{A}_{x^\mu(p_1)x^\nu(p_2) \rightarrow \bar{f}f}. \end{aligned} \quad (3.11)$$

Equation (3.11) establishes a relation between the amplitude with initial longitudinally polarized X -bosons, denoted here as $X_L^\mu(p_1) = \epsilon_L^\mu(p_1)$ and $X_L^\nu(p_2) = \epsilon_L^\nu(p_2)$, and the amplitudes with corresponding would-be Goldstone bosons $G_X(p_{1,2})$ or the energetically suppressed remainders, $x^\mu(p_1)$ and $x^\nu(p_2)$. Specifically, $x^\mu(p_1)$ is defined as

$$x^\mu(p_1) \equiv \epsilon_L^\mu(p_1) - \frac{p_1^\mu}{M_X}, \quad (3.12)$$

which satisfies the identities

$$x_\mu(p_1)p_1^\mu = -M_X, \quad x_\mu(p_1)x^\mu(p_1) = 0. \quad (3.13)$$

Note that an analogous definition and set of identities hold for $x^\nu(p_2)$ as well.

Proceeding as in [37, 38], we may reinforce the validity of the GET stated in (3.11) by considering one-loop h_iXX , h_iXG_X and $h_iG_XG_X$ vertices within the PT framework. More explicitly, the following substitutions for the tree-level h_i -couplings need to be considered:

$$V_{\mu\nu}^{h_iXX} \rightarrow \hat{\Gamma}_{\mu\nu}^{h_iXX}(q, p_1, p_2) = V_{\mu\nu}^{h_iXX} + \hat{V}_{\mu\nu}^{h_iXX}(q, p_1, p_2), \quad (3.14)$$

$$V_\mu^{h_iXG_X} \rightarrow \hat{\Gamma}_\mu^{h_iXG_X}(q, p_1, p_2) = V_\mu^{h_iXG_X} + \hat{V}_\mu^{h_iXG_X}(q, p_1, p_2), \quad (3.15)$$

$$V^{h_iG_XG_X} \rightarrow \hat{\Gamma}^{h_iG_XG_X}(q, p_1, p_2) = V^{h_iG_XG_X} + \hat{V}^{h_iG_XG_X}(q, p_1, p_2), \quad (3.16)$$

where the relevant PT one-loop vertices, $\hat{V}_{\mu\nu}^{h_iXX}$, $\hat{V}_\mu^{h_iXG_X}$ and $\hat{V}^{h_iG_XG_X}$, were calculated by means of the BFG method and presented in appendix C. One can show that when PT resummed vertices as given in (3.14)–(3.16) are employed, the relation (3.11) is satisfied in terms of the PT resummed (Born-improved) amplitudes.

The proof of (3.11) for the Born-improved amplitudes relies on the fact that the PT one-loop vertices $\hat{V}_{\mu\nu}^{h_iXX}$, $\hat{V}_\mu^{h_iXG_X}$ and $\hat{V}^{h_iG_XG_X}$, as well as the PT self-energies $\hat{\Pi}_{ij}$, $\hat{\Pi}_\mu^{XG_X}$ and $\hat{\Pi}^{G_XG_X}$, satisfy tree-like WIs that are identical to those derived from the classical action. In detail, these tree-like PT WIs read

$$p_2^\nu \hat{V}_{\mu\nu}^{h_iXX}(q, p_1, p_2) + iM_X \hat{V}_\mu^{h_iXG_X}(q, p_1, p_2) = -g_x R_{2i} \hat{\Pi}_\mu^{XG_X}(p_1), \quad (3.17)$$

$$p_1^\mu \hat{V}_\mu^{h_iXG_X}(q, p_1, p_2) + iM_X \hat{V}^{h_iG_XG_X}(q, p_1, p_2) = -g_x \left[R_{2j} \hat{\Pi}_{ji}(q^2) + R_{2i} \hat{\Pi}^{G_XG_X}(p_2) \right], \quad (3.18)$$

$$p_1^\mu p_2^\nu \hat{V}_{\mu\nu}^{h_iXX}(q, p_1, p_2) + M_X^2 \hat{V}^{h_iG_XG_X}(q, p_1, p_2) = ig_x M_X \left[R_{2j} \hat{\Pi}_{ji}(q^2) + R_{2i} \left(\hat{\Pi}^{G_XG_X}(p_1) + \hat{\Pi}^{G_XG_X}(p_2) \right) \right], \quad (3.19)$$

$$\hat{\Pi}_\mu^{XG_X}(p) = -\frac{iM_X p_\mu}{p^2} \hat{\Pi}^{G_XG_X}(p^2). \quad (3.20)$$

It is not difficult to verify that the PT WIs (3.17)–(3.20) are indeed satisfied by the tree-level couplings of the theory, $V_{\mu\nu}^{h_iXX}$, $V_\mu^{h_iXG_X}$ and $V^{h_iG_XG_X}$, after making the replacements:

$$\hat{\Pi}_{ij}(s) \rightarrow \delta_{ij}(s - m_i^2), \quad \hat{\Pi}_\mu^{XG_X}(p) \rightarrow iM_X p_\mu, \quad \hat{\Pi}^{G_XG_X}(p) \rightarrow -p^2.$$

Further details pertinent to the proof of (3.11) are given in appendix D.

3.3 High-energy asymptotics of the Born-improved amplitude

We will now study the high-energy behaviour of the PT resummed amplitude for the process $XX \rightarrow \bar{f}f$. The high-energy asymptotics of such an amplitude is similar to the one of the SM process $\bar{f}f \rightarrow ZZ$ mediated by the Higgs boson, which was considered in [37, 38] and studied in more detail in [62].

As we are interested in deriving a minimal Born-improved amplitude, we will ignore the real (dispersive) parts of the self-energies and vertex corrections. In the OS scheme, these are either subdominant or suppressed by higher orders in the resonance region [33]. Hence, we will only consider the imaginary (absorptive) parts of the PT resummed propagators

$\hat{\Delta}_{ij}(s)$ and the PT vertices $\hat{\Gamma}_{\mu\nu}^{h_i XX}$. In this minimal self-consistent PT framework, the Born-improved amplitude takes on the form

$$\begin{aligned} i\hat{\mathcal{A}}_{XX \rightarrow \bar{f}f} &\equiv \epsilon^\mu(p_1)\epsilon^\nu(p_2) i\hat{\mathcal{A}}_{\mu\nu}^{XX \rightarrow \bar{f}f} \\ &= \epsilon^\mu(p_1)\epsilon^\nu(p_2) \hat{\Gamma}_{\mu\nu}^{h_i XX}(q, p_1, p_2) i\hat{\Delta}_{ij}(s) V^{h_j \bar{f}f} \bar{u}(r_2)v(r_1), \end{aligned} \quad (3.21)$$

where $\hat{\Gamma}_{\mu\nu}^{h_i XX}(q, p_1, p_2) = V_{\mu\nu}^{h_i XX} + \hat{V}_{\mu\nu}^{h_i XX}(q, p_1, p_2)$ are the PT one-loop vertices, whose absorptive parts are given in appendix C.

To analyse the high-energy asymptotics of $\hat{\mathcal{A}}_{XX \rightarrow \bar{f}f}$, we consider the longitudinal X boson polarizations which in the centre-of-mass (CoM) frame can be expanded as follows

$$\epsilon_L^\mu(p_1) = \frac{p_1^\mu}{M_X} - 2M_X \frac{p_2^\mu}{s} + \mathcal{O}\left(\frac{16M_X^4}{s^2}\right), \quad \epsilon_L^\mu(p_2) = \frac{p_2^\mu}{M_X} - 2M_X \frac{p_1^\mu}{s} + \mathcal{O}\left(\frac{16M_X^4}{s^2}\right). \quad (3.22)$$

For instance, taking into account the absorptive effects due to the opening of the XX threshold only, the amplitude exhibits the following asymptotic behaviour:

$$\begin{aligned} \hat{\mathcal{A}}_{X_L(p_1)X_L(p_2) \rightarrow \bar{f}f}^{(XX)} &= gg_x m_f \bar{u}(r_2)v(r_1) \sin(2\alpha)(m_1^2 - m_2^2) \\ &\times \frac{64\pi M_X^2 - i6g_x^2(m_1^2 \sin^2 \alpha + m_2^2 \cos^2 \alpha)}{128\pi M_W M_X^3 s} + \mathcal{O}\left(\frac{16M_X^4}{s^2}\right), \end{aligned} \quad (3.23)$$

where g is $SU(2)_L$ gauge coupling and M_W is the $W^\pm$ -boson mass. This should be contrasted with the corresponding tree-level result,

$$\mathcal{A}_{X_L(p_1)X_L(p_2) \rightarrow \bar{f}f}^{\text{tree}} = \frac{igg_x m_f \bar{u}(r_2)v(r_1) \sin(2\alpha)(m_1^2 - m_2^2)}{2sM_W M_X} + \mathcal{O}\left(\frac{16M_X^4}{s^2}\right). \quad (3.24)$$

It is not difficult to see that up to higher order terms $\mathcal{O}(gg_x^3)$, the PT resummed amplitude given in (3.23) approaches the tree-level result (3.24) in the high-energy limit. Evidently, the Born-improved amplitude (3.21) displays the expected high-energy asymptotics.

It is instructive to see how the energetically constant terms $\propto s^0$ cancel in the high-energy limit of both the tree-level and Born-improved amplitude by virtue of the PT WIs. In the Born approximation, such cancellation is a consequence of the orthogonality of the mixing matrix R , since

$$s R_{2i} \Delta_{ij}^0(s) R_{1j} \stackrel{s \rightarrow \infty}{=} R_{2i} \delta_{ij} R_{1j} = 0. \quad (3.25)$$

Beyond the Born approximation, we may employ the PT WI (3.19) to derive an equivalent WI for the PT resummed vertex $\hat{\Gamma}_{\mu\nu}^{h_i XX}$ in the high-energy limit,

$$p_1^\mu p_2^\nu \hat{\Gamma}_{\mu\nu}^{h_i XX}(q, p_1, p_2) = ig_x M_X R_{2j} \hat{\Delta}_{ji}^{-1}(q^2) + \mathcal{O}[\ln(s/M_X^2)], \quad (3.26)$$

where we have ignored contributions from $\hat{V}^{h_i G_X G_X}(q, p_1, p_2)$ and $\hat{\Pi}^{G_X G_X}(p_{1,2})$ that grow as $\ln(s/M_X^2)$ or go to s^0 , for $p_1^2 = p_2^2 = M_X^2$ [62]. With the help of (3.26), the high-energy limit of the Born-improved amplitude for the longitudinal X bosons may then easily be evaluated as follows:

$$\hat{\mathcal{A}}_{X_L(p_1)X_L(p_2) \rightarrow \bar{f}f} \propto R_{2i} \hat{\Delta}_{ik}^{-1}(s) \hat{\Delta}_{kj}(s) R_{1j} \stackrel{s \rightarrow \infty}{=} R_{2i} \delta_{ij} R_{1j} = 0. \quad (3.27)$$

Figure 2. The Feynman diagram for the Born-improved amplitudes $XX \rightarrow VV$ (with $V = Z, W$) and $XX \rightarrow h_k h_l$.

Consequently, the leading high-energy asymptotics of $\hat{\mathcal{A}}_{X_L(p_1)X_L(p_2) \rightarrow \bar{f}f}$ is expected to be proportional to $s^{-1} \ln(s/M_X^2)$ or s^{-1} , as given in (3.23). The energetically subleading terms are not arbitrary either, but obey the GET stated in (3.11) [cf. appendix D].

In addition to the resonant process $XX \rightarrow f\bar{f}$, there are also the vector and scalar annihilation channels, $XX \rightarrow (ZZ, W^+W^-)$ and $XX \rightarrow h_k h_l$, in the VDM model, as shown in figure 2. In order to obtain the Born-improved amplitudes for these processes, we proceed as before. We replace the tree-level propagator $\Delta^0(s)$ in the s -channel graphs with the PT resummed propagator $\hat{\Delta}(s)$ given in (3.10). Likewise, we replace the tree-level couplings with their PT resummed counterparts [38, 62]:

$$V_{\mu\nu}^{h_i ZZ} \rightarrow \hat{\Gamma}_{\mu\nu}^{h_i ZZ}(q, p_1, p_2) = V_{\mu\nu}^{h_i ZZ} + \hat{V}_{\mu\nu}^{h_i ZZ}(q, p_1, p_2), \quad (3.28)$$

$$V_{\mu\nu}^{h_i W^+ W^-} \rightarrow \hat{\Gamma}_{\mu\nu}^{h_i W^+ W^-}(q, p_1, p_2) = V_{\mu\nu}^{h_i W^+ W^-} + \hat{V}_{\mu\nu}^{h_i W^+ W^-}(q, p_1, p_2). \quad (3.29)$$

Note that the PT vertex corrections to $h_i ZZ$ and $h_i W^+ W^-$ satisfy tree-like Ward identities analogous to those given in (3.17)–(3.20).

As can be seen from figure 2, besides the resonant s -channel graphs, there exist also non-resonant t - and u -channel diagrams and four-point vertices contributing to the processes $XX \rightarrow h_k h_l$ which do not require any improvement. The high-energy behaviour of these amplitudes can be studied using the GET. Because the structure of the vertex V_{jkl}^h contains a piece proportional to R_{2j} , there is no cancellation of the leading terms in energy for both tree-level and Born-improved s -channel diagrams as opposed to what happens for the process with $f\bar{f}$ in the final state. Therefore, we expect that such graphs and the total amplitude will go asymptotically to a constant ($\propto s^0$) at high energies. On the other hand, if we replace the initial states X with their respective Goldstone bosons G_X , the presence of the propagator suppresses the s -channel amplitude and makes the latter fall as s^{-1} . Consequently, thanks to the ET, the high energy behaviour of the total amplitude is dictated by the non-resonant part of the amplitude with Goldstone bosons in the initial state, i.e. $G_X G_X \rightarrow h_k h_l$. This part is not affected by resummation and coincides with the tree-level result, as can easily be inferred from the Feynman diagrams depicted in the second line of figure 2. This implies that for $s/(4M_X^2) \rightarrow \infty$, the s -channel contribution to the total Born-improved amplitude, which we denote here as $\hat{\mathcal{A}}_{X_L X_L \rightarrow h_k h_l}^s$, has to behave

exactly as its tree-level counterpart, i.e. $\mathcal{A}_{X_L X_L \rightarrow h_k h_l}^s \propto s^0$. Indeed, this is the case, as the PT ensures that the classical WI (3.26) be satisfied at both tree and Born-improved levels. Therefore, we obtain the same high-energy limit for the Born-improved amplitude $\hat{\mathcal{A}}_{X_L X_L \rightarrow h_k h_l}^s$ as the one given by the tree level amplitude, i.e.

$$\hat{\mathcal{A}}_{X_L X_L \rightarrow h_k h_l}^s \stackrel{s \rightarrow \infty}{=} i g_x M_X R_{2i} \hat{\Delta}_{im}^{-1}(s) \hat{\Delta}_{mj}(s) V_{jkl}^h = i g_x M_X R_{2i} V_{ikl}^h = \text{const} . \quad (3.30)$$

Observe that according to the ET, the high-energy limit of the total Born-improved amplitude $\hat{\mathcal{A}}_{X_L X_L \rightarrow h_k h_l}$ will be given by the contact diagram $G_X G_X h_k h_l$ which tends to a constant as $s/(4M_X^2) \rightarrow \infty$.

4 Annihilation cross-sections and evolution of dark matter density

In this section, we apply the PT resummation approach presented in the previous section in order to compute the DM annihilation cross-sections, as well as the evolution of the DM density within the VDM model. To this end, we adopt the Born-improved amplitude (3.21) with the PT resummed propagator and one-loop dressed vertices for all the annihilation channels, *viz.*

$$i \hat{\mathcal{A}}_{\mu\nu}^{XX \rightarrow \bar{w}w} = \sum_{ij} (V_{\mu\nu}^{XX h_i} + \hat{V}_{\mu\nu}^{XX h_i}) i \hat{\Delta}_{ij} \Gamma^{h_j \bar{w}w} , \quad (4.1)$$

where w specifies the SM final state, i.e. $w = Z, W, h_i, f$, and $\Gamma^{h_j \bar{w}w}$ represents the proper $h_j \bar{w}w$ -vertex upon appropriate contraction with polarization vectors and spinors.¹ For the annihilation channel into scalars (when $w = h_i$), we also include the non-resonant contributions shown in figure 2. The total annihilation cross-section σ is obtained as usual, by averaging the squared amplitude over initial X -boson polarizations and summing over all annihilation channels w in the final state, their polarizations and other possible internal degrees of freedom. Furthermore, we have introduced in our analysis the parameter

$$\delta \equiv \frac{4M_X^2}{m_2^2} - 1 , \quad (4.2)$$

which provides a measure of proximity of the h_2 -boson mass to the DM XX threshold.

Figure 3 shows the product σv of the annihilation cross-section σ with the relative velocity v in the CoM frame for the processes $XX \rightarrow t\bar{t}$ and $XX \rightarrow h_1 h_1$, as functions of the normalized energy squared parameter

$$\bar{s} \equiv \frac{s}{4M_X^2} . \quad (4.3)$$

¹Strictly speaking, one should include one-loop corrections to $h_i ZZ$ - and $h_i W^+ W^-$ -vertices within the PT framework similarly to what was done in section 3. However, these effects are found to be numerically negligible. Specifically, the one-loop corrections to $h_i XX$ -vertex change the cross-sections by as much as 2 %, when the dark gauge coupling g_x approaches its perturbativity limit $\sqrt{2\pi}$. The respective one-loop absorptive corrections to h_i vertices with the SM Z and $W^\pm$ vector bosons scale as g^4 , so simple estimates indicate that these are at most at per mille level. Consequently, for most applications of phenomenological interest, the absorptive loop corrections to $h_i ZZ$ - and $h_i W^+ W^-$ -vertices may safely be neglected.

Figure 3. Resonant annihilation cross-section (times the relative velocity v) for $XX \rightarrow t\bar{t}$ (upper panels with negative and positive value of δ) and $XX \rightarrow h_1 h_1$ (bottom panel) as a function of the energy ratio $s/(4M_X^2)$ obtained using the PT (solid line), a resummed propagator with self-energies calculated in the R_ξ gauge, for $\xi = 0, 1, 5$, in the unitary gauge (UG), with the s -dependent Higgs width (SDW) as well as in the BW and tree-level approximation. For the VDM model parameters, we set $M_X = 1$ TeV, $|\delta| = 10^{-4}$, $\alpha = 10^{-4}$ and $g_x = \sqrt{2}\pi$. Note that the inset plots display the respective cross-sections at the close vicinity of the DM threshold region.

The results are obtained by utilising the PT resummed propagator, a resummed BW propagator with self-energies calculated in the R_ξ class of gauges with $\xi = 0, 1, 5$ and in the unitary gauge (UG). In addition, figure 3 displays the quantity σv computed at the tree-level and in the BW approximation assuming a constant or an s -dependent decay width (SDW). In the BW case, we set $\Gamma_i = \Im m \Pi_{ii}(m_i^2)/m_i$. In the SDW approximation, we include in the partial decay width $\Gamma_{h_2 \rightarrow XX}$ the exact form of the phase space factor corresponding to the XX channel. This gives the leading energy-dependent correction to Γ_2 near the XX threshold, i.e.

$$\Gamma_{h_2 \rightarrow XX} = \sqrt{\frac{1 - 4M_X^2/s}{1 - 4M_X^2/m_2^2}} \frac{\Im m \Pi_{22}^{(XX)}(m_2^2)}{m_2} \theta(s - 4M_X^2). \quad (4.4)$$

For $\delta > 0$, the above expression has to be analytically continued. This SDW approximation works exceptionally well near the Higgs pole, where $s \approx m_2^2$. Moreover, it does not modify the high-energy behaviour of the tree-level amplitudes, because $\Gamma_2(s)$ approaches a constant value when $s/(4M_X^2) \rightarrow \infty$.

From figure 3, we can see that in the region where $s/(4M_X^2) \simeq 1$, the PT result is close to the one that has been evaluated in the R_ξ gauges with $\xi = 0, 1, 5$. However, at large s , the Born-improved (PT) cross-section (times the relative velocity v) for the process $XX \rightarrow t\bar{t}$ varies between the one computed in the R_ξ gauge and that in the tree-level approximation, whereas the cross-section calculated in the unitary gauge is suppressed. The latter is a result of the Z - and $W^\pm$ -boson contributions to $\Pi_{ij}(s)$ which display a scaling behaviour $\propto s^2/(16M_X^4)$ when the energy is well above the corresponding W^+W^- and ZZ thresholds. For the annihilation channel $XX \rightarrow h_1 h_1$, the differences between the various results are smaller, because the leading (in s) parts of the propagator do not cancel asymptotically. Moreover, the PT result approaches exactly the tree-level one as discussed earlier. Observe that near the threshold the naive BW and tree-level results deviate dramatically from those found in the PT, in contrast to the approximation with s -dependent XX phase space factor (4.4) which leads to a very good agreement in this region. For $\delta < 0$, the large width Γ_2 used in the BW approximation results in an underestimated cross-section right at the threshold. On the other hand, for $\delta > 0$, the width Γ_2 contains only SM contributions which are suppressed by the small mixing angle α . For this reason, the BW approximation is close to the tree-level result. Finally, we note the presence of the fictitious threshold for $\xi = 5$. This is an example of a gauge artifact that originates from a naive resummation of self-energies in the R_ξ class of gauges.

The evolution of DM density is mainly governed by the size of the thermally averaged annihilation cross-section $\langle \sigma v \rangle$, which may be computed as a function of $x \equiv M_X/T$ by employing the standard formula [63]:

$$\langle \sigma v \rangle = \frac{2x}{K_2^2(x)} \int_1^\infty d\bar{s} \, \sigma v \sqrt{\bar{s} - 1} \bar{s} K_1(2x\sqrt{\bar{s}}), \quad (4.5)$$

where the dimensionless parameter $\bar{s}$ is defined in (4.3) and $K_n(x)$ denotes the modified Bessel function of order n . In figure 4, we present the values of $\langle \sigma v \rangle$ for $g_x = 0.3, 1$ and

Figure 4. Thermally averaged total annihilation cross-section $\langle\sigma v\rangle(x)$ as a function of $x \equiv M_X/T$, for $g_x = 0.3, 1$ and $\sqrt{2\pi}$ and the remaining model parameters same as in figure 3. The solid line corresponds to the cross-section (4.5) obtained using the PT with resummed propagator and corrected $h_i XX$ vertex while the dashed lines are for the self-energies calculated using R_ξ gauges for several values of the gauge parameter ξ , the unitary gauge (UG), standard BW approximation with constant (BW) or s-dependent widths (SDW) given by (4.4).

$\sqrt{2\pi}$.² The results were calculated in the PT, the Feynman gauge ($\xi = 1$), the unitary gauge (UG) and in the BW approximation with the usual constant decay widths and s -dependent widths (SDW) as stated in (4.4). We see that as the dark gauge coupling g_x decreases, the standard BW approximation starts to describe more accurately the resonant cross-section. This follows from the fact that XX contribution to the self-energy $\Pi_{ij}(s)$, which strongly depends on the energy close to the threshold, ceases to dominate over the nearly constant contributions from the SM states. Then, as presented in the lower panel of figure 4 for $g_x = 0.3$, the thermally averaged cross-section $\langle\sigma v\rangle$ forms a hilltop located at $x = 1/|\delta|$. This is the value of the temperature for which the thermal distribution (4.5) is centered at the resonance peak $s = m_2^2$.

In figure 5, we display numerical comparisons of thermally averaged cross-sections computed by means of the PT and the standard BW approximation on the plane defined by the parameters g_x and δ . We observe that the BW approximation is applicable only if the parameters g_x and $|\delta|$ are sufficiently small. Otherwise, its simplistic use for relative velocities $v = 100$ km/s (typical in astrophysical searches of DM annihilation signal) can lead to annihilation cross-sections that are much smaller (larger) than the PT result, for negative (positive) values of δ . On the other hand, for thermally averaged cross-sections calculated at temperatures close to that of chemical decoupling ($x = 20$), the BW approximation gives results that are comparable to or larger than those derived by the PT resummation method.

In the following, we will analyze the evolution of the relic DM density. Following the recent studies [14, 48], we take into consideration the effect of early kinetic decoupling, which turns out to be an important phenomenon for scenarios with resonant DM annihilation. This effect appears, because the resonantly enhanced annihilation rate corresponds to scattering processes of suppressed strength, whose role is to maintain the kinetic equilibrium between the DM and the thermal bath in the early Universe. Moreover, as readily observed from figure 4, the thermally averaged cross-section $\langle\sigma v\rangle$ grows by many orders of magnitude when x varies from the chemical decoupling temperature of $x \approx 20$ until $x \sim \mathcal{O}(10^4)$. Therefore, the effective DM annihilation can be prolonged to a period when the DM does no longer have the same temperature as that of the SM thermal bath.

In order to determine the evolution of DM density, we assume that the DM distribution does not deviate significantly from the thermal one. Therefore, we may describe it with the DM temperature T_X . To compute the DM relic abundance, we solve numerically a set of coupled Boltzmann equations for the DM yield Y and the temperature parameter y [48, 64]

$$\frac{dY}{dx} = -\frac{s}{xH} \left(1 + \frac{T}{3h} \frac{dh}{dT} \right) \left(Y^2 \langle\sigma v\rangle_{T=T_X} - Y_{\text{eq}}^2 \langle\sigma v\rangle \right), \quad (4.6)$$

$$\begin{aligned} \frac{dy}{dx} = & -\frac{\gamma(T)}{xH} \left(1 + \frac{T}{3h} \frac{dh}{dT} \right) (y - y_{\text{eq}}) + \left(1 + \frac{T}{3h} \frac{dh}{dT} \right) \frac{\langle p^4/E^3 \rangle_{T=T_X}}{3xT_X} \\ & - \frac{sy}{xHY} \left(1 + \frac{T}{3h} \frac{dh}{dT} \right) \left[Y^2 \left(\langle\sigma v\rangle_2 - \langle\sigma v\rangle \right)_{T=T_X} - Y_{\text{eq}}^2 \left(\frac{y_{\text{eq}}}{y} \langle\sigma v\rangle_2 - \langle\sigma v\rangle \right) \right], \end{aligned} \quad (4.7)$$

²The value $g_x = \sqrt{2\pi}$ corresponds to the saturation of the perturbativity bound for the quartic coupling $\lambda_{H,S} \leq 4\pi$ for the resonant region of the VDM model, for which $2M_X \approx m_2$.

Figure 5. Numerical comparisons in the plane (δ, g_x) of the cross-sections obtained using standard BW approximation and PT method. In the upper panels contours of $\sigma v_{BW}/\sigma v_{PT}$ for $v = 100$ km/s are plotted, while in the bottom panels contours of $\langle \sigma v_{BW} \rangle / \langle \sigma v_{PT} \rangle$ at $x = 20$ are shown. The panels on the l.h.s. and r.h.s. present results for $\delta < 0$ and $\delta > 0$ respectively.

where

$$\begin{aligned}
 Y &= \frac{n_X}{s}, & y &= \frac{M_X T_X}{s^{2/3}}, & T_{X,\text{eq}} &= T, & n_{X,\text{eq}} &= \frac{g_X M_X^3}{2\pi^2 x} K_2(x), \\
 H &= \sqrt{\frac{4\pi^3 g}{45 m_{\text{Pl}}^2}} T^2, & s &= \frac{2\pi^2}{45} h T^3.
 \end{aligned} \tag{4.8}$$

In the above, m_{Pl} is the Planck mass, n_X denotes the number density of DM, s is the entropy density, H is the Hubble parameter, $g_X = 3$, whereas g and h are respectively the effective numbers of relativistic degrees of freedom for energy density and entropy. Besides

the collision terms $\propto \langle \sigma v \rangle$, the coupled equations (4.6) and (4.7) contain also the following thermal averages:

$$\langle \sigma v \rangle_2 = \int_1^\infty d\bar{s} \sigma v \frac{4x^3 \bar{s}^2}{3K_2^2(\bar{s})} \int_1^\infty d\epsilon_+ e^{-2\epsilon_+ \sqrt{\bar{s}}} \left[\epsilon_+ z + \frac{1}{2\sqrt{\bar{s}}} \log \left(\frac{\sqrt{\bar{s}}\epsilon_+ - z}{\sqrt{\bar{s}}\epsilon_+ + z} \right) \right], \quad (4.9)$$

$$\langle p^4/E^3 \rangle = \frac{g_X}{2\pi^2 n_{X,\text{eq}}(T)} \int dp \frac{p^6}{E^3} e^{-E/T}, \quad (4.10)$$

where E and p are the energy and momentum of the DM and $z \equiv \sqrt{(\bar{s} - 1)(\epsilon_+^2 - 1)}$. Finally, the scattering momentum exchange rate $\gamma(T)$ may be expressed as

$$\gamma(T) = \frac{1}{3\pi^2 g_X M_X} \sum_{\text{SM}} \int_{m_{\text{SM}}}^\infty d\omega g^\pm(\omega) \partial_\omega [(\omega^2 - m_{\text{SM}}^2)^2 \sigma_T]_{s=M_X^2+2\omega M_X+m_{\text{SM}}^2}, \quad (4.11)$$

where m_{SM} is the mass of the SM state upon which the DM scatters, $g^\pm(\omega) = 1/[\exp(\omega/T) \pm 1]$ denotes its phase-space distribution and $\sigma_T = \int d\Omega (1 - \cos \theta) d\sigma/d\Omega$ is the standard transfer cross-section for elastic scattering. The sum runs over all relativistic degrees of freedom present in the SM thermal bath.

The different predictions for the thermally averaged cross-sections $\langle \sigma v \rangle$ calculated using various methods manifest themselves in different evolutions of the DM yield $Y(x)$. As can be seen from figure 6, the result obtained in the unitary gauge or in the Landau gauge $\xi = 0$ deviates significantly from the PT evaluation. Likewise, the standard BW approximation completely fails to describe the underlying dynamics if $g_x = \sqrt{2\pi}$. Instead, an evaluation in the usual Feynman gauge $\xi = 1$ (where no unphysical thresholds occur) or in an s -dependent width approximation for the BW propagator of the resonant mediator turns out to be both very good approximations of the PT resummation approach.

5 Conclusions

In processes of resonant DM annihilation, one central difficulty arises from the use of a BW approximation with a constant width for the propagator of the exchanged particle in the s -channel. In particular, if the respective DM channel contributes significantly to the decay width of this mediator particle, the evaluation of the resonant transition amplitude can become very inaccurate close to the DM production threshold. This problem may be partially circumvented by utilising a running s -dependent width for the mediator [14], which results from the imaginary (absorptive) part of the mediator's self-energy. However, in spontaneously broken gauge theories, such a running width becomes gauge-dependent in the off-shell region.

In this paper, we addressed this issue within the gauge-independent framework of the Pinch Technique (PT). The PT has many desirable field-theoretic properties, including analyticity, unitarity and the gauge invariance of the classical action. As an illustrative scenario, we considered the SM extended by a local $U(1)_X$ symmetry which may lead to a massive stable gauge field X that could successfully play the role of a Vector DM. By adopting a resummation approach implemented by the PT, we calculated the PT resummed

Figure 6. Dark matter yield $Y(x)$ for $g_x = \sqrt{2}\pi$ (upper panel) and $g_x = 1$ (lower panel) obtained using the PT, R_ξ gauge ($\xi = 0, 1$), unitary gauge (UG), the standard Breit-Wigner propagator or approximate s -dependent width (SDW).

propagator for the scalar $h_i h_j$ system (with $i, j = 1, 2$). In addition, we calculated the relevant one-loop corrections to the $h_i X X$ -coupling in the PT. As we have shown in this work, the latter are necessary in order to obtain a transition amplitude for DM-pair annihilation which has the proper high-energy limit in agreement with both the Equivalence Theorem and the Generalized Equivalence Theorem.

The PT resummation method enabled us to derive a Born-improved amplitude which was used to obtain self-consistent and accurate predictions for the DM abundance of the X vector boson. We have illustrated how these predictions differ from those that one finds using other methods of resummation. In our comparative analysis, we have taken

into account the effect of early kinetic decoupling of the dark-sector particles from the SM thermal bath. We have shown that the standard BW approximation is not applicable to large part of the parameter space in which the DM contribution to the mediator's self-energy dominates over the SM one. Its mere use results not only in distorted predictions for the DM relic abundance, but also in wrong annihilation cross-sections for indirect detection experiments. To deal with this problem, we devised an alternative simple approximation that utilises an energy-dependent width which yields the correct DM annihilation rates and relic abundance. In this study, we presented results for resonant DM annihilation processes only. Nevertheless, it is straightforward to extend our considerations to DM elastic scatterings. We postpone such an analysis to a future communication.

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A The Abelian vector dark matter model

The simplest model of vector DM (VDM) is an extension of the SM by an extra $U(1)_X$ gauge group together with a complex scalar S that is charged under $U(1)_X$ [49–54]. The extra scalar S is singlet under the SM gauge group and has non-zero VEV, and as such it provides, via the standard Higgs mechanism, mass for the $U(1)_X$ vector boson X_μ which plays here the role of the VDM. Its stability can be ensured by a ‘dark’ charge symmetry which acts in the hidden DM sector as follows:

$$X_\mu \rightarrow -X_\mu, \quad S \rightarrow S^*. \quad (\text{A.1})$$

In the non-linear representation of the complex scalar field $S = \phi e^{i\sigma}$, the ‘dark’ charge symmetry implies that $\phi \rightarrow \phi$ and $\sigma \rightarrow -\sigma$, whereas all other fields are neutral under this symmetry.

In this VDM model, the VEVs of the scalar fields are given by $(\langle H \rangle, \langle S \rangle) = \frac{1}{\sqrt{2}}(v, v_x)$ which generate the gauge-boson masses

$$M_W = \frac{1}{2}gv, \quad M_Z = \frac{1}{2}\sqrt{g^2 + g'^2}v, \quad M_X = g_x v_x, \quad (\text{A.2})$$

where g , g' and g_x are the $SU(2)_L$, $U(1)_Y$ and $U(1)_X$ gauge coupling constants, respectively.

The scalar potential of the VDM model reads

$$V = -\mu_H^2 |H|^2 + \lambda_H |H|^4 - \mu_S^2 |S|^2 + \lambda_S |S|^4 + \kappa |S|^2 |H|^2, \quad (\text{A.3})$$

with $H = (H^+, H^0)^T$. To determine the physical scalar spectrum, we expand the scalar fields linearly about their respective VEVs as follows:

$$S = \frac{1}{\sqrt{2}}(v_x + \phi_S + iG_X), \quad H^0 = \frac{1}{\sqrt{2}}(v + \phi_H + iG_Z). \quad (\text{A.4})$$

Our next step is to diagonalise the square mass matrix $\mathcal{M}^2$ for the fluctuations (ϕ_H, ϕ_S) . This can be done by carrying out an orthogonal O(2) rotation R which is usually parameterised by a mixing angle α , such that $\mathcal{M}_{\text{diag}}^2 = R^{-1}\mathcal{M}^2 R$. In this way, we obtain

$$\mathcal{M}_{\text{diag}}^2 = \begin{pmatrix} m_1^2 & 0 \\ 0 & m_2^2 \end{pmatrix}, \quad R = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix}, \quad \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = R^{-1} \begin{pmatrix} \phi_H \\ \phi_S \end{pmatrix}, \quad (\text{A.5})$$

where h_1 is the observed Higgs particle with a mass $m_1 = 125 \text{ GeV}$. All scalar vertices of the theory may be described in terms of the four extra parameters: M_X , m_2 , g_x and α , in addition to the SM ones. Specifically, the quartic couplings of the VDM potential may be expressed as follows:

$$\lambda_h = g^2 \frac{m_1^2 c_\alpha^2 + m_2^2 s_\alpha^2}{8M_W^2}, \quad \lambda_s = g_x^2 \frac{m_1^2 s_\alpha^2 + m_2^2 c_\alpha^2}{8M_W^2}, \quad \kappa = gg_x \frac{(m_1^2 - m_2^2)s_{2\alpha}}{4M_W M_X}, \quad (\text{A.6})$$

with $c_\alpha \equiv \cos \alpha$ and $s_\alpha \equiv \sin \alpha$.

B Feynman rules in the R_ξ and covariant background field gauges

$i2g_x M_X R_{2i}$	$ig \frac{M_Z^2}{M_W} R_{1i}$	$ig M_W R_{1i}$	$igm_f 2M_W R_{1i}$
$V_{ijk}^h = i[\kappa v(R_{1i}R_{2j}R_{2k} + R_{2i}R_{1j}R_{2k} + R_{2i}R_{2j}R_{1k}) + \kappa v_x(R_{2i}R_{1j}R_{1k} + R_{1i}R_{2j}R_{1k} + R_{1i}R_{1j}R_{2k}) + 6\lambda v(R_{1i}R_{1j}R_{1k}) + 6\lambda_s v_x(R_{2i}R_{2j}R_{2k})]$

$-ig_x \frac{m_i^2}{M_X} R_{2i}$	$-ig_x M_X \xi_X$	$i2g_x^2 R_{2i} R_{2j}$	$-g(k_\mu - p_\mu) R_{2i}$
$i2g_x^2 g_{\mu\nu}$	$-i3g_x^2 \frac{R_{21}^2 m_1^2 + R_{22}^2 m_2^2}{M_X^2}$	$-igg_x \frac{\sin(2\alpha)(m_1^2 - m_2^2)}{2MM_W}$	$-ig \frac{m_i^2}{2M_W^2} R_{1i}$

Table 1. The vertices relevant for the calculation of amplitudes in the R_ξ gauge within the vector dark matter model. The Goldstone field of the $U(1)_X$ symmetry is denoted by G_X .

 $2g_x R_{2i} p_\mu$	 $-2g_x R_{2i} k_\mu$	 $-ig_x R_{2i} (\frac{m_i^2}{M_X^2} + 2\xi_Q M_X)$	 $-ig_x R_{2i} (\frac{m_i^2}{M_X^2} - \xi_Q M_X)$
 $-i2g_x M_X \xi_Q R_{2i}$	 $-i2g_x^2 \xi_Q$	 $-ig \left(\frac{m_i^2}{2M_W^2} + \frac{M_Z^2}{M_W^2} \xi \right)$	 $-ig \left(\frac{m_i^2}{2M_W^2} + M_W^2 \xi \right)$

Table 2. Vertices in the covariant background field gauge that differ from their counterparts in the R_ξ gauge. We follow the conventions of [56]. A hat over a symbol denotes a background field.

Figure 7. Feynman diagrams related to $\hat{V}_{\mu\nu}^{h_i X X}$.

C Analytic expressions of the PT vertices

In order to compute the PT vertices $\hat{V}_{\mu\nu}^{h_i X X}$, $\hat{V}_\mu^{h_i X G_X}$ and $\hat{V}^{h_i G_X G_X}$, we will exploit the equivalence between the PT and the covariant background field gauge for $\xi_Q = 1$, as shown in figures 7, 8 and 9. For our purposes, we only need to take into account the absorptive parts. Note that these are absent for the self-energies $\hat{\Pi}^{G_X G_X}$ and $\hat{\Pi}^{X G_X}$. When calculating $\hat{V}_{\mu\nu}^{h_i X X}$ and $\hat{V}_\mu^{h_i X G_X}$, we omit all those terms that vanish upon contraction with the polarization vectors of the external X bosons. To this end, the following abbreviations will be used:

$$\beta_X = \sqrt{1 - 4M_X^2/s}, \quad \beta_{h_{W/Z}} = \sqrt{1 - 4m_{W/Z}^2/s}, \quad \lambda(x, y, z) = (x - y - z)^2 - 4yz. \quad (\text{C.1})$$

After a lengthy but straightforward computation of the Feynman diagrams shown in figures 7, 8 and 9, the following analytic results for the absorptive parts of the PT vertices

Figure 8. Feynman diagrams contributing to $\hat{V}_\mu^{h_i X G_X}$.

Figure 9. Feynman diagrams pertinent to $\hat{V}^{h_i G_X G_X}$.

$\hat{V}_{\mu\nu}^{h_i X X}$, $\hat{V}_\mu^{h_i X G_X}$ and $\hat{V}^{h_i G_X G_X}$ are obtained:

$$\begin{aligned}
 \hat{V}_{\mu\nu}^{h_i X X} = & \sum_j \frac{g_x^3 R_{2i} R_{2j}^2}{8\pi s^3 \beta_X^3} \theta(s - 4M_X^2) \left[-s\beta_X^2 m_j^2 (m_i^2 + 2M^2) g_{\mu\nu} - 2[8M_X^6 \right. \\
 & - 2M_X^4 (9s - 2m_i^2 + 2m_j^2) + M_X^2 (4s(s + m_j^2) - m_i^2(s + 2m_j^2)) + 2m_i^2 m_j^2 s] r_{2\mu} r_{1\nu} \\
 & + \log \left[1 + \frac{s\beta_X^2}{m_j^2} \right] \left([16M_X^6 - 4M_X^4 (2m_j^2 + s) + 2M_X^2 m_j^2 (s + m_j^2 - 2m_i^2) \right. \\
 & \left. \left. + m_i^2 m_j^2 (s + m_j^2)] g_{\mu\nu} - 2 \frac{m_j^2}{s\beta_X^2} [16M_X^6 + 4M_X^4 (2m_i^2 - m_j^2 - 7s) \right. \right.
 \end{aligned}$$

$$+ M_X^2 (2s(3s+2m_j^2) - 2m_i^2(3s+m_j^2)) + sm_i^2(s+2m_j^2)] r_{2\mu} r_{1\nu} \Big] \quad (\text{C.2})$$

$$\begin{aligned}
 & + \sum_{k,l} \frac{ig_x^2 R_{2k} R_{2l} V_{ikl}^h}{16\pi s^4 \beta_X^4} \theta[s - (m_k + m_l)^2] \left[\lambda^{1/2}(s, m_k^2, m_l^2) \right. \\
 & \times \left(-s^2 \beta_X^2 (4M_X^2 - m_k^2 - m_l^2) g_{\mu\nu} + 4[M_X^2(3s + m_k^2 + m_l^2) - s(m_k^2 + m_l^2)] r_{2\mu} r_{1\nu} \right) \\
 & + \frac{2}{s\beta_X} \log \left[\frac{s - (m_k^2 + m_l^2) - s\beta_X \lambda^{1/2}(s, m_k^2, m_l^2)}{s - (m_k^2 + m_l^2) + s\beta_X \lambda^{1/2}(s, m_k^2, m_l^2)} \right] \\
 & \times \left(s\beta_X^2 [-8M_X^4 s + M_X^2(2s(m_k^2 + m_l^2) - (m_k^2 - m_l^2)^2 + s^2) - m_k^2 m_l^2 s] g_{\mu\nu} \right. \\
 & + 2[2M_X^4(2s(m_k^2 + m_l^2) - 3(m_k^2 - m_l^2)^2 + s^2) \\
 & + sM_X^2(s^2 - 4s(m_k^2 + m_l^2) + 3(m_k^4 + m_l^4) - 8m_k^2 m_l^2) + 2m_k^2 m_l^2 s^2] r_{2\mu} r_{1\nu} \Big) \Big], \\
 \widehat{V}_\mu^{h_i X G_X} = & - \sum_j \frac{ig_x^3 R_{2i} R_{2j}^2 r_{2\mu}}{8\pi M_X^2 s^2 \beta_X} \theta[s - 4M_X^2] \left[10M_X^2 + M_X^2(m_i^2 - 2m_j^2 - 4s) - m_i^2 m_j^2 \right. \\
 & + \beta_X^{-2} [8M_X^6 - 2M_X^4(s - 5m_j^2) + M_X^2 m_j^2(m_i^2 - 2m_j^2 - 4s) - m_i^2 m_j^4] \log \left[1 + \frac{s\beta_X^2}{m_j^2} \right] \Big] \\
 & + \sum_{k,l} \frac{g_x^2 R_{2k} R_{2l} V_{ikl}^h r_{2\mu}}{16\pi \beta_X^2 s^3 M_X^2} \theta[s - (m_k + m_l)^2] \left[\lambda^{1/2}(s, m_k^2, m_l^2) s [2M_X^2 - (m_k^2 + m_l^2)] \right. \\
 & + \beta_X^{-2} \log \left[\frac{s - (m_k^2 + m_l^2) - s\beta_X \lambda^{1/2}(s, m_k^2, m_l^2)}{s - (m_k^2 + m_l^2) + s\beta_X \lambda^{1/2}(s, m_k^2, m_l^2)} \right] \\
 & \times [2m_k^2 m_l^2 s + M_X^2(4s^2 - 3s(m_k^2 + m_l^2) + 2(m_k^2 - m_l^2)^2) - 12M_X^4 s] \Big], \quad (\text{C.3})
 \end{aligned}$$

$$\begin{aligned}
 \widehat{V}_\mu^{h_i G_X G_X} = & \sum_j \frac{g_x^3 R_{2i} R_{2j}^2}{32\pi s \beta_X} \theta[s - 4M_X^2] \left[s\beta_X^2 (8M_X^4 + 3m_j^2(2M_X^2 + m_i^2)) \right. \\
 & - 2[2M_X^2 m_j^2(-m_i^2 + m_j^2 + 2s) + M_X^4(m_i^2 - 4m_j^2) + m_i^2 m_j^4 - 6M_X^6] \\
 & \times \log \left[1 - \frac{s\beta_X^2}{m_j^2} \right] \Big] + \sum_{k,l} \frac{g_x V_{ikl}^h}{128\pi s M_X^2} \theta[s - (m_k + m_l)^2] \left[\lambda^{1/2}(s, m_k^2, m_l^2) [2g_x R_{2k} R_{2l} \right. \\
 & \times (4M^2 + m_1^2 + m_2^2 + \cos(2\alpha)(m_2^2 - m_1^2)) + g \frac{M_X}{M_W} R_{1k} R_{1l} (m_1^2 - m_2^2) \sin(2\alpha)] \\
 & + 8\beta_X^{-1/2} \log \left[\frac{s - (m_k^2 + m_l^2) - s\beta_X \lambda^{1/2}(s, m_k^2, m_l^2)}{s - (m_k^2 + m_l^2) + s\beta_X \lambda^{1/2}(s, m_k^2, m_l^2)} \right] \\
 & \times [3M_X^4 - M_X^2(2s - m_k^2 - m_l^2) - m_k^2 m_l^2] \Big] \\
 & + \sum_j \frac{g^2 g_X R_{1i} R_{1j} R_{2j} m_j^2}{32\pi M_Z M_X} \theta[s - 4M_Z^2] \beta_Z \\
 & + \sum_j \frac{g^2 g_X R_{1i} (m_2^2 - m_1^2) \sin(2\alpha)}{128\pi M_W^2 M_X} \theta[s - 4M_W^2] \beta_W. \quad (\text{C.4})
 \end{aligned}$$

D Generalized equivalence theorem for resummed amplitudes

The proof of the Generalized Equivalence Theorem (GET) for the amplitude $\mathcal{A}_{XX \rightarrow \bar{f}f}$ goes along the lines of the one given in [38] for the SM process $\bar{f}f \rightarrow ZZ$. It is based on the classical WIs that are satisfied by the PT Higgs self-energies and the PT hZZ vertex.

In order to prove the GET for the process $XX \rightarrow \bar{f}f$,

$$\begin{aligned} \mathcal{A}_{X_L(p_1)X_L(p_2) \rightarrow \bar{f}f} &= -\mathcal{A}_{G_X(p_1)G_X(p_2) \rightarrow \bar{f}f} - i\mathcal{A}_{x^\mu(p_1)G_X(p_2) \rightarrow \bar{f}f} \\ &\quad - i\mathcal{A}_{G_X(p_1)x^\nu(p_2) \rightarrow \bar{f}f} + \mathcal{A}_{x^\mu(p_1)x^\nu(p_2) \rightarrow \bar{f}f}, \end{aligned} \quad (\text{D.1})$$

we have to check that the following identity holds true (summation over the repeated indices i, j, k implied):

$$\begin{aligned} \epsilon^\mu(p_1)\epsilon^\nu(p_2)(V_{\mu\nu}^{h_iXX} + \widehat{V}_{\mu\nu}^{h_iXX})\widehat{\Delta}_{ij}V^{h_j\bar{f}f} &= -(V^{h_iG_XG_X} + \widehat{V}^{h_iG_XG_X})\widehat{\Delta}_{ij}V^{h_j\bar{f}f} \\ &\quad - ix^\mu(p_1)(V_\mu^{h_iXG_X} + \widehat{V}_\mu^{h_iXG_X})\widehat{\Delta}_{ij}V^{h_j\bar{f}f} - ix^\nu(p_2)(V_\nu^{h_iG_XX} + \widehat{V}_\nu^{h_iG_XX})\widehat{\Delta}_{ij}V^{h_j\bar{f}f} \\ &\quad + x^\mu(p_1)x^\nu(p_2)(V_{\mu\nu}^{h_iXX} + \widehat{V}_{\mu\nu}^{h_iXX})\widehat{\Delta}_{ij}V^{h_j\bar{f}f}. \end{aligned} \quad (\text{D.2})$$

Writing $\epsilon^\mu(p_{1,2}) = p_{1,2}^\mu/M_X + x^\mu(p_{1,2})$ and using (3.18) and (3.19), one arrives from (D.2) at an equivalent expression

$$\begin{aligned} i\frac{g}{M_X} \left(R_{2k} \left[(s - m_k^2)\delta_{ki} + \widehat{\Pi}_{ki} \right] + R_{2i} \left[-p_1^2 + \widehat{\Pi}^{G_XG_X}(p_1^2) - p_2^2 + \widehat{\Pi}^{G_XG_X}(p_2^2) \right] \right) \widehat{\Delta}_{ij}V^{h_j\bar{f}f} \\ - \frac{g}{M_X} R_{2i} \left(x^\mu(p_1) \left[iM_X p_{1\mu} + \widehat{\Pi}_\mu^{XG_X}(p_1^2) \right] + x^\nu(p_2) \left[iM_X p_{2\nu} + \widehat{\Pi}_\nu^{G_XX}(p_1^2) \right] \right) \widehat{\Delta}_{ij}V^{h_j\bar{f}f} = 0. \end{aligned} \quad (\text{D.3})$$

This last expression can be shown to be valid for both the tree-level and its PT extension. To do so, we first observe that $(s - m_k^2)\delta_{ki} + \widehat{\Pi}_{ki} = \widehat{\Delta}_{ki}^{-1}$. Then, using (3.13) and (3.20), one finds that the main bulk of the terms in (D.3) cancel. Thus, one is only left to prove that

$$i\frac{g}{M_X} R_{2k}\delta_{kj}V^{h_j\bar{f}f} = 0. \quad (\text{D.4})$$

The latter is true due to the orthogonality of the mixing matrix R . This concludes our proof of the GET for the amplitude $\mathcal{A}_{XX \rightarrow \bar{f}f}$.

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