Twin Turtles

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ABSTRACT: We present an ultraviolet extension of the Twin Higgs in which the radial mode of twin symmetry breaking is itself a pseudo-goldstone boson. This “turtle” structure raises the scale of new colored particles in exchange for additional states in the Higgs sector, making multiple Higgs-like scalars the definitive signature of naturalness in this context. We explore the parametrics and phenomenology of a concrete Twin Turtle model and demonstrate its robustness in two different supersymmetric completions. Along the way, we also introduce a new mechanism for inducing hard twin symmetry-breaking quartics via soft supersymmetry breaking.

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1 Introduction

The electroweak hierarchy problem is one of the primary motivators of physics beyond the Standard Model, with most proposed solutions predicting an abundance of new states near the weak scale. But the discovery of an apparently-elementary Higgs at the LHC, coupled with the non-discovery of any additional particles, raises a pressing question: if the hierarchy problem is solved by new physics, *where is everyone?* Attempts to answer this question have given rise to a variety of new approaches to the hierarchy problem, including selection of the weak scale by cosmological dynamics [1] and stabilization of the weak scale by discrete symmetries, rather than continuous ones [2–4]. Typically these new approaches are at most successful in stabilizing the Higgs mass up to a cutoff intermediate between the weak scale and the Planck scale, and still require one of the two conventional solutions (supersymmetry or a low cutoff, possibly from compositeness [5, 6]) to cover the remaining ground between the intermediate cutoff and the Planck scale.

The problem is particularly acute in models of ‘neutral naturalness’ that exploit discrete symmetries. While the lowest-lying states stabilizing the weak scale in these models are neutral under the Standard Model, they only succeed in raising the cutoff (at fixed fine-tuning) by a modest amount of order g_*/g_{SM} [7], where g_* is the coupling strength of

new dynamics associated with the discrete symmetry and g_{SM} is a relevant Standard Model coupling (typically the top yukawa or Higgs quartic, depending on the model). Thus supersymmetry or compositeness must still enter at a scale only modestly higher than before, albeit high enough to put typical SM-charged partner particles (just) outside the reach of the LHC [8–15].

The success of global symmetries in raising the cutoff of a natural Higgs sector has long hinted at a more ambitious possibility: that the ultimate cutoff could be raised further by successive patterns of global symmetry breaking, in which the scalar spontaneously breaking a global symmetry is itself a pseudo Nambu-Goldstone boson (pNGB). Theories in which the scale associated with the breaking of a global symmetry is stabilized by the breaking of another global symmetry are referred to as ‘turtles’ [16], borrowed from Hindu mythology by way of Hawking [17]. Turtles involving two successive Little Higgs models were constructed in [16, 18], pushing the ultimate cutoff of the perturbative, nonsupersymmetric construction up to ~ 100 TeV. Further raising the cutoff by additional symmetry-breaking structures was ultimately limited by the geometric increase in the number of fields at each level of the turtle, which reduce the corresponding expansion parameter. The possibility of constructing novel multi-level turtles with perturbative ‘t Hooft coupling was sketched in [18] but has yet to be exploited further.

In this paper we revive the turtle mechanism in the context of the simplest model of neutral naturalness, the Twin Higgs [2]. There are some natural advantages to this approach, as the use of discrete symmetries (rather than continuous ones) reduces the multiplicity of fields transforming under a given gauge group at each level of the turtle. But the introduction of twin turtles is also strongly motivated by LHC null results, in that it promises to further raise the scale at which new particles carrying Standard Model QCD quantum numbers must appear. Whereas the pre-LHC goal of the original turtle models was to push off the scale of supersymmetry or compositeness as far as possible by introducing a host of Standard Model-charged states, our objective here is more modest: to raise the scale associated with new charged states as far as possible, whether or not they are associated with supersymmetry or compositeness. The irreducible scale at which new colored particles appear in a natural completion of the weak scale is of particular relevance to the physics case for future colliders (see, e.g. [19]).

However, there is an immediate obstruction to naively concatenating patterns of global symmetry breaking in simple models of neutral naturalness such as the Twin Higgs: raising the cutoff with turtles is only manifestly sensible in Little Higgs-like theories where there is no additional tuning penalty associated with vacuum alignment. In conventional composite Higgs models or the Twin Higgs, the separation between two scales of spontaneous symmetry breaking $f_2 \gg f_1$ is typically associated with a tuning of order f_2^2/f_1^2 . In this case the concatenation of n successive symmetry breaking scales would accumulate a tuning of order $(f_n^2/f_{n-1}^2) \times \cdots \times (f_2^2/f_1^2) = f_n^2/f_1^2$, no less tuned (and significantly more baroque) than a two-scale theory with large separation of scales.

Progress is possible if the tuning associated with vacuum alignment can be reduced. For Twin Higgs models, there are three possibilities: tadpoles [20, 21]; collective symmetry breaking *à la* Little Higgs [22]; or hard breaking of the discrete symmetry [9]. Here we

will focus on the latter option, which is relatively straightforward to implement and entails fairly minimal field content.¹ Indeed, hard symmetry breaking is particularly well-suited to turtle extensions of the Twin Higgs. In simple Twin Higgs models, the size of the hard-breaking quartic — which in turn dictates the reduction in vacuum alignment tuning — is bounded by the known small size of the Standard Model-like Higgs quartic. But in turtle extensions of the Twin Higgs, hard breaking can be used to separate intermediate symmetry breaking scales, for which the associated quartics are unfixed.

This motivates us to study Twin Turtle models in which the vacuum alignment separating the weak scale from the first scale of global symmetry breaking is achieved by the soft breaking of a discrete symmetry, while the separation of this first scale from higher scales of global symmetry breaking is achieved by the hard breaking of discrete symmetries. In what follows, we will refer to this particular construction as the Twin Maximally Natural Turtle (TMNT). Ultimately, we will find that this particular structure is capable of parametrically increasing the scale associated with the appearance of new colored particles relative to the Twin Higgs, albeit at the cost of lowering the scale associated with the appearance of new particles in the Higgs sector. In this respect, the TMNT represents a class of models in which additional Higgs-like scalars become the definitive signature of naturalness.²

Thanks to the particular choice of symmetry breaking patterns, the Twin Turtle models explored here exhibit improved tuning by typical metrics such as the Barbieri-Giudice fine-tuning measure [25], but are perhaps unlikely to satisfy the Potter Stewart measure [26]. It is not our goal to satisfy every possible measure of tuning, but rather to understand the extent to which the scale of new colored particles may be delayed in natural completions of the weak scale, and to additionally lay the groundwork for renewed exploration of turtle models motivated by LHC null results. While these constructions demonstrate an improvement in tuning at the inarguable cost of complexity, they illustrate new parametric possibilities that nature may find a more elegant means to saturate.

The paper is organized as follows: in section 2 we discuss the parametrics of fine tuning in various incarnations of the Twin Higgs and its turtled extensions, motivating the particular combination of hard- and soft-breaking terms that characterize the Twin Maximally Natural Turtle. We then explore the parametrics and phenomenology of the TMNT in section 3 from the perspective of an effective theory with unspecified physics at the cutoff. In 4 we construct two supersymmetric UV completions of the TMNT, which serve to both justify the symmetry-breaking pattern and validate the tuning expectations of the effective theory. We explore further directions for twin turtle model-building and conclude in section 5.

2 Tunings in twins and turtles

To motivate the structure of the Twin Turtle in general, and the Twin Maximally Natural Turtle in particular, we begin by reviewing aspects of fine-tuning in the original Twin Higgs

¹By using hard breaking to eliminate vacuum alignment tuning, we depart from the original spirit of [16], which aimed to reduce all cutoff dependence to the greatest possible extent; here our focus is primarily on raising the scale associated with the top sector.

²For other Twin Higgs-inspired models in which additional Higgs bosons improve electroweak naturalness, albeit in different ways, see [23, 24].

model before introducing the simplest turtle extension and illustrating its parametric advantages. This necessarily entails a careful accounting of the full sensitivity of electroweak symmetry breaking to UV physics, expanding upon previous treatments of fine-tuning in Twin Higgs models.

2.1 Twin Higgs tuning

To understand tuning in the original Twin Higgs model, we begin with the most general linear sigma potential for a pair of complex scalars h_a, h_b , each of which transforms as a doublet under respective $SU(2)_a$ and $SU(2)_b$ gauge groups, which in turn are related by a discrete $\mathbb{Z}_2$ exchange symmetry:

$$V = \lambda(|h_a|^2 + |h_b|^2)^2 + m_h^2(|h_a|^2 + |h_b|^2) + \kappa(|h_a|^4 + |h_b|^4) + \rho|h_a|^4 + \mu^2|h_a|^2. \quad (2.1)$$

When $\kappa/\lambda, \rho/\lambda, \mu^2/m_h^2 \ll 1$, this theory has an approximate $SU(4)$ global symmetry. For appropriate signs in the potential, both h_a and h_b acquire vacuum expectation values, $\langle |h_a|^2 \rangle + \langle |h_b|^2 \rangle = f^2$, leading to a pseudo-goldstone of the spontaneously broken $SU(4)$ that is identified with the SM-like Higgs. In a full Twin Higgs model, we identify $SU(2)_a$ and $SU(2)_b$ with the weak gauge groups of two copies of the Standard Model, and extend the discrete symmetry to exchange SM_a with SM_b . We moreover identify SM_a as “our” copy of the Standard Model, and $SU(2)_a$ as our corresponding weak gauge group.

It’s convenient to study the properties of the SM-like Higgs ϕ in the pNGB limit, in which case it can be related to h_a, h_b and the scale of $SU(4)$ breaking f via

$$h_a = f \sin \frac{\phi}{\sqrt{2}f}, \quad h_b = f \cos \frac{\phi}{\sqrt{2}f}. \quad (2.2)$$

Inserting this parameterization into eq. (2.1) leads to expressions for the electroweak symmetry breaking v and SM-like Higgs mass m_{SM}^2 of the form

$$\begin{aligned} 2v^2 &= f^2 \frac{2\kappa}{2\kappa + \rho} - \frac{\mu^2}{2\kappa + \rho}, \\ m_{\text{SM}}^2 &= 4v^2(2\kappa + \rho) \left(1 - \frac{v^2}{f^2} \right). \end{aligned} \quad (2.3)$$

At this point it is straightforward to determine the tuning of the electroweak scale with respect to underlying parameters. Treating the Twin Higgs model as an effective theory up to some cutoff scale Λ , we would ultimately like to determine the sensitivity of the Higgs v to the cutoff, allowing for the possibility that the scale f is intermediate between the two.

It is common to approximate the overall tuning of v by factorizing it in terms of the sensitivity of v to the scale f , $\Delta_{v/f}$, and the sensitivity of f scale to the cutoff scale $\Delta_{f/\Lambda}$, $\Delta_{v/\Lambda}^{\text{TH}} = \Delta_{v/f} \times \Delta_{f/\Lambda}^{\text{TH}}$ (e.g. [10]). Here the factorization of tuning is valid insofar as the weak scale only depends implicitly on the cutoff through an intermediate scale, as is the case for the Twin Higgs potential in eq. (2.1) at tree level. This factorization can be violated by a variety of effects, including both logarithmic dependence on the cutoff that arises

from radiative corrections to quartic couplings and quadratic dependence on the cutoff from radiative corrections via hard-breaking quartics. In what follows, the (in)validity of factorization will often be important in understanding the fine tuning of Twin Turtles.

Both factorized and non-factorized contributions to the tuning of the weak scale can be accounted for succinctly by treating the fine-tuning in terms of the total derivative of the Higgs vev with respect to the cutoff, which in the current case takes the schematic form

$$\Delta_{v/\Lambda}^{\text{TH}} \equiv \frac{d \log v^2}{d \log \Lambda^2} = \frac{\partial \log v^2}{\partial \log f^2} \frac{d \log f^2}{d \log \Lambda^2} + \frac{\partial \log v^2}{\partial \log \Lambda^2}. \quad (2.4)$$

The first term reproduces the usual factorized result, while the second term accommodates direct sensitivity of v to Λ due to violations of factorization. Let us consider them each in turn.

The first term involves direct sensitivity of v to the scale f , and sensitivity of f to the cutoff(s) Λ . For the Twin Higgs, the sensitivity of f to the cutoff scale is simply that of a complex scalar with various couplings to SM-like fields. The two largest couplings are those associated with the top yukawa y_t in each sector and the SU(4)-symmetric quartic λ , and so it is convenient to distinguish the cutoff Λ_t associated with the top sector from the cutoff Λ_ρ associated with the scalar sector. (Here we will not distinguish between the cutoffs associated with the SU(4)-symmetric quartic λ and the SU(4)- and $\mathbb{Z}_2$ -breaking quartic ρ , though of course the two may be quite distinct in a UV completion.) Then the tuning of the scale f with respect to these cutoffs is simply

$$\frac{\partial \log f^2}{\partial \log \Lambda_t^2} = \frac{3}{32\pi^2} \frac{y_t^2}{\lambda} \frac{\Lambda_t^2}{f^2}, \quad (2.5)$$

$$\frac{\partial \log f^2}{\partial \log \Lambda_\rho^2} = -\frac{5}{32\pi^2} \frac{\Lambda_\rho^2}{f^2}. \quad (2.6)$$

The tuning of v with respect to f depends on the nature of the parameters breaking the $\mathbb{Z}_2$ symmetry. If the $\mathbb{Z}_2$ symmetry is only broken softly via the μ^2 term in eq. (2.1), such that $\rho = 0$, we have

$$\Delta_{v/f}^{\text{soft}} = \left. \frac{\partial \log v^2}{\partial \log f^2} \right|_{\rho=0} = \frac{f^2}{2v^2}, \quad (2.7)$$

corresponding to the familiar tuning of typical Twin Higgs models. But as was illustrated in [14], if the $\mathbb{Z}_2$ symmetry is broken instead through the hard-breaking quartic ρ , such that $\mu^2 = 0$, we have

$$\Delta_{v/f}^{\text{hard}} = \left. \frac{\partial \log v^2}{\partial \log f^2} \right|_{\rho \neq 0} = \frac{2\kappa}{2\kappa + \rho} \times \left. \frac{\partial \log v^2}{\partial \log f^2} \right|_{\rho=0} = \frac{2\kappa}{2\kappa + \rho} \frac{f^2}{2v^2}. \quad (2.8)$$

When $\kappa \ll \rho$, $\Delta_{v/f}$ can be improved significantly, in principle erasing tuning associated with the separation between v and f . However, $\frac{2\kappa}{2\kappa + \rho}$ cannot be arbitrarily small thanks to two effects. The first is the Higgs mass constraint, i.e. the contribution of $2\kappa + \rho$ to the SM-like Higgs quartic, whose value is known. The second is the irreducible contribution

to the quartic κ from fermion loops. The main radiative correction from the top loop to κ is given by

$$\delta\kappa = \frac{3y_t^4}{16\pi^2} \log \frac{\Lambda_t^2}{y_t^2 f^2}, \quad (2.9)$$

which on its own is not far from the SM-like Higgs quartic. These two considerations bound the improvement in fine-tuning from the introduction of a hard-breaking quartic ρ to

$$1 \gtrsim \frac{\Delta_{v/f}^{\text{hard}}}{\Delta_{v/f}^{\text{soft}}} \gtrsim \frac{3}{8\pi^2} \frac{y_t^4}{\lambda_{\text{SM}}} \log \frac{\Lambda_t^2}{y_t^2 f^2} \gtrsim \frac{1}{\text{few}}. \quad (2.10)$$

Now we turn to the second contribution to the tuning of v coming from the direct sensitivity of v to the cutoff. When the $\mathbb{Z}_2$ symmetry is only broken softly via the μ^2 term, this contribution vanishes, and we obtain the familiar factorized result. However, additional hard breaking of the $\mathbb{Z}_2$ symmetry via the quartic ρ restores quadratic sensitivity of the SM-like Higgs mass to the cutoff Λ_ρ at one loop,

$$\delta\mu^2 = \epsilon_\pm \frac{3\rho}{16\pi^2} \Lambda_\rho^2, \quad (2.11)$$

where $\epsilon_\pm = \pm 1$ depends on the UV completion. This amounts to a modest violation of the factorization assumption, made tolerable only by the smallness of $\rho \lesssim \lambda_{\text{SM}}$, and

$$\frac{\partial \log v^2}{\partial \log \Lambda_\rho^2} \approx -\frac{3\epsilon_\pm}{32\pi^2} \frac{\rho}{2\kappa + \rho} \frac{\Lambda_\rho^2}{v^2} \quad (2.12)$$

where we neglect contributions coming from logarithmic dependence on the cutoffs.

Putting everything together, the total tuning in the case of soft $\mathbb{Z}_2$ breaking, up to logarithmic cutoff dependence, is thus

$$\Delta_{v/\Lambda}^{\text{TH,soft}} = \frac{1}{64\pi^2} \left(\frac{3y_t^2}{\lambda} \frac{\Lambda_t^2}{v^2} - 5 \frac{\Lambda_\rho^2}{v^2} \right) \quad (2.13)$$

while the tuning in the presence of hard $\mathbb{Z}_2$ breaking is

$$\Delta_{v/\Lambda}^{\text{TH,hard}} = \frac{1}{64\pi^2} \frac{2\kappa}{2\kappa + \rho} \left(\frac{3y_t^2}{\lambda} \frac{\Lambda_t^2}{v^2} - \left(5 + 3\epsilon_\pm \frac{\rho}{\kappa} \right) \frac{\Lambda_\rho^2}{v^2} \right). \quad (2.14)$$

2.2 A twin turtle

It is apparent in eq.s (2.8)–(2.10) that a significant improvement in tuning would be possible if hard breaking were responsible for separating scales unrelated to the SM-like Higgs. This naturally suggests constructing a Twin Turtle in which a Twin Higgs model with a softly-broken discrete symmetry is UV completed by a Twin Higgs structure involving hard breaking of a discrete symmetry. This setup is illustrated in figure 1. As there are no mass constraints on the radial mode of the new twin structure, the constraints on the size of the hard breaking in the new upstairs twin (owing to the SM-like Higgs mass) are removed and we can potentially reach a larger improvement in the total fine tuning.

Figure 1. The symmetry structure of the Twin Turtle model pursued in this work, in which a softly-broken Twin Higgs model is UV completed by yet another Twin Higgs model with certain hard-breaking quartics. Only the $SU(2)$ groups are gauged; the $SU(4)$ and $SU(8)$ are approximate global symmetries.

To this end, we envision a model with four identical copies of the Standard Model. Each copy contains a complex scalar, which we will respectively label h_a, h_b, H_a , and H_b , transforming as a doublet under its own $SU(2)$ gauge symmetry. As with the Twin Higgs, we will associate the Higgs doublet h_a , with corresponding weak group $SU(2)_a$, with “our” copy of the Standard Model.

The four identical copies of the Standard Model in isolation nominally enjoy an S_4 symmetry acting on the field labels, but this is generally broken by turning on couplings between Higgs scalars from different copies. Heuristically, with the symmetry breaking structure of turtles in mind, it is convenient to organize the theory around the following pattern of discrete symmetries relating the Higgs scalars (and their corresponding copies of the Standard Model): the doublets h_a and h_b are related by a discrete $\mathbb{Z}_2$ symmetry, as are H_a and H_b , leading to two approximate $SU(4)$ global symmetries acting on $h \equiv (h_a, h_b)$ and $H \equiv (H_a, H_b)$. An exchange symmetry relating these two sectors then gives rise to an approximate $SU(8)$ global symmetry of the theory. In this respect it is useful to think of the symmetry of the Higgs scalars as $[SU(2) \times SU(2) \times \mathbb{Z}_2] \times [SU(2) \times SU(2) \times \mathbb{Z}_2] \times \mathbb{Z}_2$. The potential for the Higgs scalars will break these discrete symmetries in a hierarchical fashion. Needless to say, while this is a natural extension of the Twin Higgs model to a turtle structure, it is not the unique way of constructing a Twin Turtle, and other interesting configurations of symmetry embeddings are possible.

In what follows, we will often refer to the approximately SU(8)-symmetric sector as the “upstairs theory”, and the approximate SU(4)-symmetric sector containing the Standard Model SU(2)_a as the “downstairs theory.” The approximate SU(8) global symmetry is broken at a scale F , and the SU(4) global symmetry containing the Standard Model SU(2)_a is broken at a scale $f < F$.

Starting from the approximate SU(8) global symmetry, the relevant terms in the potential for this model are given by

$$\begin{aligned}
 V \supset & \bar{\lambda}(|h|^2 + |H|^2)^2 + m^2(|h|^2 + |H|^2) + \bar{\kappa}(|h|^4 + |H|^4) + \bar{\rho}|h|^4 + m_h'^2|h|^2 \quad (2.15) \\
 & + \rho''(|h_a|^2 + |H|^2)^2 + \kappa''(|h_a|^4 + |h_b|^4 + |H_a|^4 + |H_b|^4) \\
 & + \kappa'(|h_a|^4 + |h_b|^4) + \rho'|h_a|^4 + \mu'^2|h_a|^2.
 \end{aligned}$$

This potential includes the full set of independent mass terms and quartic couplings consistent with the four gauged SU(2) symmetries and the $\mathbb{Z}_2$ symmetry relating H_a and H_b . While we could relax the latter requirement, terms breaking this $\mathbb{Z}_2$ have no real impact on the properties of the SM-like Higgs and the scale of SU(2)_a breaking, and so we omit them for clarity.

Let us briefly summarize the role of each coupling, emphasizing both the discrete subgroups of the underlying S_4 symmetry and the continuous symmetries that they preserve:

- m^2 and $\bar{\lambda}$ respect the full S_4 discrete symmetry as well as the accidental SU(8) global symmetry of the model. They determine the scale F of SU(8) symmetry breaking.
- $\bar{\kappa}$ respects a D_4 (i.e., dihedral group of order 8) subgroup of the S_4 and an accidental SU(4) $\times$ SU(4) global symmetry. This term is crucial for developing additional scales below F , and in particular determines the scale f .
- $\bar{\rho}$ and $m_h'^2$ preserve a D_2 (equivalently $\mathbb{Z}_2 \times \mathbb{Z}_2$) subgroup of the S_4 and an accidental SU(4) $\times$ SU(4) global symmetry.
- ρ'' preserves a D_3 (equivalently S_3) subgroup of the S_4 and an accidental SU(6) $\times$ SU(2) global symmetry.
- κ'' preserves the original S_4 discrete symmetry but no accidental continuous symmetries. This is crucial for developing *vevs* below the scales of SU(4) symmetry breaking on both sides.
- κ' respects a D_2 subgroup of the S_4 and a partly-accidental SU(2) $\times$ SU(2) $\times$ SU(4) global symmetry
- ρ' and μ'^2 preserve a D_3 subgroup of the S_4 and an accidental SU(2) $\times$ SU(6) global symmetry. Crucially, this D_3 subgroup (and hence also the accidental continuous symmetry) is distinct from that preserved by ρ'' . ρ' is analogous to the hard quartic responsible for $\mathbb{Z}_2$ breaking in the original Twin Higgs model, while μ'^2 is analogous to the soft-breaking mass term.

The couplings $\bar{\lambda}$, $\bar{\kappa}$, and $\bar{\rho}$ evidently play the same role as λ , κ , and ρ , respectively, in eq. (2.1). In what follows, we will assume the ρ'' coupling is negligible compared to other symmetry-breaking quartics. This will be justified by the UV completions presented in section 4, for which this quartic only arises as a two-loop effect.

Working in the pNGB limit at each step, the vacuum expectation values and mass spectrum can be approximated as

$$\begin{aligned} 2f^2 &\sim \frac{2\bar{\kappa}}{2\bar{\kappa} + \bar{\rho}} F^2 - \frac{m_h'^2}{2\bar{\kappa} + \bar{\rho}}, & 2v^2 &\sim f^2 - \frac{\mu'^2}{2\kappa_1}, \\ m_1^2 &\sim 4\bar{\lambda}F^2, & m_2^2 &\sim 4(2\bar{\kappa} + \bar{\rho})f^2, & m_3^2 &\sim 4\kappa'' \frac{F^2 - f^2}{2}, \\ m_{\text{SM}}^2 &\sim 4v^2(2\kappa_1) \left(1 - \frac{v^2}{f^2}\right), \end{aligned} \quad (2.16)$$

where we have defined $\kappa_1 = \kappa'' + \kappa'$.

The tuning of the weak scale with respect to the cutoffs Λ_t and Λ_ρ of the top and Higgs sectors takes the general form

$$\Delta_{v/\Lambda}^{\text{TMNT}} = \frac{\partial \log v^2}{\partial \log f^2} \frac{d \log f^2}{d \log \Lambda^2} + \frac{\partial \log v^2}{\partial \log F^2} \frac{d \log F^2}{d \log \Lambda^2} + \frac{\partial \log v^2}{\partial \log \Lambda^2} \quad (2.17)$$

where we are taking care to accommodate violations of factorization. For the pattern of hard and soft breaking taken here, we have $\partial v / \partial F \approx 0$ and $\partial v / \partial \Lambda \approx 0$ up to logarithmic dependence. Since the separation of v from f arises only through soft breaking of a $\mathbb{Z}_2$, we have the usual

$$\frac{\partial \log v^2}{\partial \log f^2} = \frac{f^2}{2v^2}. \quad (2.18)$$

The nontrivial contribution to the tuning lies in the dependence of the scale f on the cutoff, where (in analogy with the above discussion of fine-tuning in the Twin Higgs with hard $\mathbb{Z}_2$ breaking)

$$\frac{d \log f^2}{d \log \Lambda^2} = \frac{\partial \log f^2}{\partial \log F^2} \frac{d \log F^2}{d \log \Lambda^2} + \frac{\partial \log f^2}{\partial \log \Lambda^2} \quad (2.19)$$

$$\approx \left[\frac{2\bar{\kappa}}{2\bar{\kappa} + \bar{\rho}} \frac{F^2}{2f^2} \right] \left[\frac{1}{32\pi^2} \left(\frac{3y_t^2}{\bar{\lambda}} \frac{\Lambda_t^2}{F^2} - 9 \frac{\Lambda_\rho^2}{F^2} \right) \right] - \frac{1}{32\pi^2} \frac{5\epsilon_\pm \bar{\rho}}{2\bar{\kappa} + \bar{\rho}} \frac{\Lambda_\rho^2}{f^2}. \quad (2.20)$$

Thus the tuning of the weak scale with respect to the cutoff in this Twin Turtle is, up to logarithmic corrections,

$$\Delta_{v/\Lambda}^{\text{TMNT}} \approx \frac{1}{64\pi^2} \frac{\bar{\kappa}}{2\bar{\kappa} + \bar{\rho}} \left(\frac{3y_t^2}{\bar{\lambda}} \frac{\Lambda_t^2}{v^2} - \left(9 + 5\epsilon_\pm \frac{\bar{\rho}}{\bar{\kappa}} \right) \frac{\Lambda_\rho^2}{v^2} \right). \quad (2.21)$$

2.3 Comparison with SM and twin Higgs

What have we gained? As a reminder, the tuning in the SM and a Twin Higgs model (with either soft or hard breaking of the $\mathbb{Z}_2$ symmetry, keeping track of the factorization

violation in the latter case) with respect to cutoffs Λ_t and Λ_ρ takes the form

$$\begin{aligned}\Delta_{v/\Lambda}^{\text{TH,soft}} &\approx \frac{1}{64\pi^2} \left(\frac{3y_t^2}{\lambda} \frac{\Lambda_t^2}{v^2} - 5 \frac{\Lambda_\rho^2}{v^2} \right), \\ \Delta_{v/\Lambda}^{\text{TH,hard}} &\approx \frac{1}{64\pi^2} \frac{2\kappa}{2\kappa + \rho} \left(\frac{3y_t^2}{\lambda} \frac{\Lambda_t^2}{v^2} - \left(5 + 3\epsilon_\pm \frac{\rho}{\kappa} \right) \frac{\Lambda_\rho^2}{v^2} \right), \\ \Delta_{v/\Lambda}^{\text{SM}} &\approx \frac{1}{32\pi^2} \left(\frac{3y_t^2}{\lambda_{\text{SM}}} \frac{\Lambda_t^2}{v^2} - 3 \frac{\Lambda_\rho^2}{v^2} \right).\end{aligned}\tag{2.22}$$

where λ is the approximately SU(4)-symmetric quartic in the Twin Higgs and λ_{SM} is the SM Higgs quartic.

Let us first consider the sensitivity to Λ_t . In many respects, this is the most important cutoff from the perspective of searches for new colored states associated with naturalness of the weak scale, in that it corresponds to the mass scale of colored top partners in a UV completion.³ The fine tuning improvement in the TMNT over a Twin Higgs model or the SM will be

$$\frac{\Delta_{v/\Lambda_t}^{\text{TMNT}}}{\Delta_{v/\Lambda_t}^{\text{TH,soft}}} = \frac{\lambda}{\bar{\lambda}} \frac{\bar{\kappa}}{2\bar{\kappa} + \bar{\rho}},\tag{2.23}$$

$$\frac{\Delta_{v/\Lambda_t}^{\text{TMNT}}}{\Delta_{v/\Lambda_t}^{\text{TH,hard}}} = \frac{\lambda}{\bar{\lambda}} \frac{\bar{\kappa}}{2\bar{\kappa}} \frac{2\kappa + \rho}{2\bar{\kappa} + \bar{\rho}} \approx \frac{1}{2} \frac{\lambda}{\bar{\lambda}} \frac{\lambda_{\text{SM}}}{2\bar{\kappa} + \bar{\rho}},\tag{2.24}$$

$$\frac{\Delta_{v/\Lambda_t}^{\text{TMNT}}}{\Delta_{v/\Lambda_t}^{\text{SM}}} = \frac{\lambda_{\text{SM}}}{2\bar{\lambda}} \frac{\bar{\kappa}}{2\bar{\kappa} + \bar{\rho}}.\tag{2.25}$$

The tuning improvement relative to the Standard Model is straightforward. The tuning improvement over the Twin Higgs depends on the Twin Higgs quartic λ relative to the TMNT quartic $\bar{\lambda}$. Since the TMNT involves more states that contribute to the running of $\bar{\lambda}$, we expect $\bar{\lambda} < \lambda$ assuming a common Landau pole scale, which undermines the gain in tuning. For example, in a SUSY UV completion, to have the same Landau poles, $\bar{\lambda} \simeq \frac{1}{2}\lambda$. To compensate, we require $\frac{\bar{\kappa}}{2\bar{\kappa} + \bar{\rho}} < \frac{1}{2}$ in order for the tuning of the TMNT with respect to Λ_t to improve over soft-breaking Twin Higgs models, which is straightforward. To see an improvement relative to hard-breaking Twin Higgs models, we require $\frac{\bar{\kappa}}{2\kappa} \frac{2\kappa + \rho}{2\bar{\kappa} + \bar{\rho}} \approx \frac{1}{2} \frac{\lambda_{\text{SM}}}{2\bar{\kappa} + \bar{\rho}} < \frac{1}{2}$, which is also quite feasible.

Crucially, there are two structural features that allow both $2\bar{\kappa} + \bar{\rho} \ll 2\kappa + \rho$ and $\bar{\kappa} \ll \bar{\rho}$, apart from any constraints arising from the need to reproduce the Standard Model Higgs quartic. First, Yukawa interactions where all the colored top partners are at the scale of Λ_t , preserving all discrete symmetries but breaking the accidental $\text{SU}(4) \times \text{SU}(4)$, would not correct $\bar{\kappa}$ at one loop. This is in contrast to the κ parameter in Twin Higgs models, which is generated from the top Yukawa. It bears emphasizing that this statement is dependent on the matter content of the UV completion. For example, radiative corrections

³The top quarks in the additional copies of the Standard Model serve as the top partners in the low-energy effective theory, but are themselves not charged under our copy QCD.

to $\bar{\kappa}$ could arise at one loop in UV completions where the top quarks and their partners fill out complete multiplets of the accidental $SU(4)$ symmetries. It is not, however, strictly necessary for the top quarks and their partners to fill out complete $SU(4)$ multiplets, as we demonstrate in the supersymmetric UV completions presented in section 4; in such cases, the lack of one-loop contributions to $\bar{\kappa}$ proportional to the top yukawa is robust. Second, the dominant radiative correction to $\bar{\kappa}$ is proportional to $\bar{\rho}$,

$$\delta\bar{\kappa} = \frac{5\bar{\rho}\bar{\lambda}}{32\pi^2} \log \frac{\Lambda_\rho^2}{m_2^2} \quad (2.26)$$

which permits $\bar{\kappa}$ to be small, and $\bar{\kappa} \ll \bar{\rho}$, without additional fine-tuning.

Next we consider the sensitivity to Λ_ρ . The tuning improvement over Twin Higgs models and the SM with respect to Λ_ρ is

$$\frac{\Delta_{v/\Lambda_\rho}^{\text{TMNT}}}{\Delta_{v/\Lambda_\rho}^{\text{TH,soft}}} = \frac{1}{5} \frac{9\bar{\kappa} + 5\epsilon_\pm \bar{\rho}}{2\bar{\kappa} + \bar{\rho}}, \quad (2.27)$$

$$\frac{\Delta_{v/\Lambda_\rho}^{\text{TMNT}}}{\Delta_{v/\Lambda_\rho}^{\text{TH,hard}}} = \frac{9\bar{\kappa} + 5\epsilon_\pm \bar{\rho}}{2\bar{\kappa} + \bar{\rho}} \frac{2\kappa + \rho}{10\kappa + 6\epsilon_\pm \rho}, \quad (2.28)$$

$$\frac{\Delta_{v/\Lambda_\rho}^{\text{TMNT}}}{\Delta_{v/\Lambda_\rho}^{\text{SM}}} = \frac{1}{6} \frac{9\bar{\kappa} + 5\epsilon_\pm \bar{\rho}}{2\bar{\kappa} + \bar{\rho}}. \quad (2.29)$$

For $\bar{\rho} \gg \bar{\kappa}$ — a sensible limit, as noted above — the sensitivity of the electroweak scale to Λ_ρ is comparable to the SM and softly-broken Twin Higgs, and potentially somewhat better than the hard-breaking Twin Higgs. In order to avoid severe fine-tuning, new electroweak states associated with the Higgs sector should appear around 2 TeV.

One interesting feature is apparent from these comparisons: while improving the sensitivity to the cutoff of the top sector relative to the Standard Model, the Twin Higgs model does not improve the sensitivity with respect to new electroweak states associated with the cutoff of the Higgs sector. Moreover, the price paid by the TMNT model in order to push the cutoff of the top sector to even higher scales is the introduction of *two* additional radial modes around the TeV range. This trade-off is even more apparent when working in terms of the fine-tuning of the Higgs mass instead of the electroweak scale, which we discuss in the next subsection.

2.4 Fine tuning *Lausannois*

We conclude our discussion of fine-tuning by re-phrasing the results of the previous subsection in the language of [7], which frames fine-tuning in Twin Higgs models intuitively in terms of leading corrections to the Higgs doublet mass parameter. While the tuning of the doublet mass parameter is equivalent to the tuning of the *vev* discussed above (as the quartic is not tuned), this provides a clear setting for understanding the key features of tuning in turtled models, as well as straightforward comparison with the authoritative discussion of Twin Higgs tuning in [7]. Here we assume that vacuum alignment is such that the Higgs mass parameter is identifiable with the electroweak doublet h_a in both the Twin Higgs and TMNT models.

In the Standard Model, the dominant radiative corrections to the Higgs doublet mass parameter are

$$\delta m_{\text{SM}}^2 = -\frac{3y_t^2}{16\pi^2}\Lambda_t^2 + \frac{3\lambda_{\text{SM}}}{16\pi^2}\Lambda_\rho^2. \quad (2.30)$$

In the Twin Higgs model with soft $\mathbb{Z}_2$ breaking, the analogous expression is

$$\delta m_{\text{SM}}^2 = \frac{\kappa}{\lambda} \left(m_h^2 - \frac{3y_t^2}{16\pi^2}\Lambda_t^2 + \frac{5\lambda}{16\pi^2}\Lambda_\rho^2 \right) + \mu^2, \quad (2.31)$$

where $2\kappa \sim \lambda_{\text{SM}}$. As we have seen, compared to the Standard Model, the sensitivity to the colored top partner scale is reduced by the factor of $\frac{\kappa}{\lambda}$ and the fine tuning with respect to the scalar cutoff is roughly the same as that of the SM. Finally, the price of improved tuning in the top sector is the introduction of a radial mode of mass $\sim |m_h|$.

Extending the Twin Higgs to include the hard $\mathbb{Z}_2$ breaking term $\rho|h_a|^4$, we have

$$\delta m_{\text{SM}}^2 = \frac{\kappa}{\lambda} \left(m_h^2 - \frac{3y_t^2}{16\pi^2}\Lambda_t^2 + \frac{5\lambda}{16\pi^2}\Lambda_\rho^2 \right) + \frac{3\epsilon_{\pm}\rho}{16\pi^2}\Lambda_\rho^2 + \mu^2, \quad (2.32)$$

with $(2\kappa+\rho)$ fixed to be roughly λ_{SM} . One would like to have as small a value of κ as possible to reduce the fine tuning. However, as we have discussed above, the radiative correction to κ from top loops almost saturate the SM quartic λ_{SM} , limiting the improvement of fine tuning.

Turning now to the TMNT potential in eq. (2.15) (without the ρ' and ρ'' quartics for simplicity), we have

$$\begin{aligned} \delta m_{\text{SM}}^2 &= \frac{\kappa_1}{2\bar{\kappa} + \bar{\rho}} \left[\frac{\bar{\kappa}}{\bar{\lambda}} \left(m^2 - \frac{3y_t^2}{16\pi^2}\Lambda_t^2 + \frac{9\bar{\lambda}}{16\pi^2}\Lambda_\rho^2 \right) + \frac{5\epsilon_{\pm}\bar{\rho}}{16\pi^2}\Lambda_\rho^2 \right] + \mu'^2 \\ &= \frac{\kappa_1}{\bar{\lambda}} \frac{\bar{\kappa}}{2\bar{\kappa} + \bar{\rho}} \left(m^2 - \frac{3y_t^2}{16\pi^2}\Lambda_t^2 + \frac{9\bar{\lambda}}{16\pi^2}\Lambda_\rho^2 \right) + \frac{\epsilon_{\pm}\bar{\rho}}{2\bar{\kappa} + \bar{\rho}} \frac{5\kappa_1\Lambda_\rho^2}{16\pi^2} + \mu'^2, \end{aligned} \quad (2.33)$$

where $2\kappa_1 \sim \lambda_{\text{SM}}$. Insofar as $2\bar{\kappa} + \bar{\rho}$ is not constrained by known quartics, the sensitivity to Λ_t can be improved significantly by taking $\bar{\rho} \gg \bar{\kappa}$. As discussed earlier, this limit is readily attainable since radiative corrections to $\bar{\kappa}$ from Yukawa couplings are absent, and $\bar{\kappa}$ may be separated from $\bar{\rho}$ by as much as a loop factor. This limit has little impact on the sensitivity to the cutoff associated with Higgs quartics, which is again comparable to the Standard Model. Finally, there are now *two* radial modes appearing in the theory, both of which lie at or below the scale $|m|$.

To better illustrate the sensitivity of the electroweak scale to different UV scales in each of these models, in figure 2 we show the natural values of the top sector cutoff Λ_t (corresponding to an un-tuned Higgs mass, $\Delta_{v/\Lambda_t, v/\Lambda_\rho} = 1$) and the mass scale of new radial modes (m_h in the Twin Higgs, m in the TMNT) as a function of the parameter $\alpha \equiv \frac{\bar{\kappa}}{2\bar{\kappa} + \bar{\rho}}$. For smaller values of α , the cutoff Λ_t associated with new colored states in the top sector can be raised well above the corresponding scale in Twin Higgs model and the Standard Model. In contrast, the tree-level symmetric scales setting the masses of the radial modes are required to be lighter than the scale associated with colored top partners, emphasizing the tradeoff between colored top partners and new states in the Higgs sector.

Figure 2. Natural values for the top sector cutoff Λ_t and radial modes in the TMNT model, the softly-broken Twin Higgs model, and the Standard Model as a function of $\alpha \equiv \frac{\bar{\kappa}}{2\bar{\kappa} + \bar{\rho}}$. The solid red line, dotted red line, and dashed red line denote the respective top sector cutoff scale in each model corresponding to no tuning of the Higgs mass. The natural values for the Higgs quartic cutoff are comparable in each model — ~ 2 TeV — and are not shown here. The solid blue line and dotted blue line denote the tree-level mass parameters m and m_h of the TMNT and softly-broken Twin Higgs, respectively, which are proxies for the scale of new radial modes. Here we have taken $\kappa_1 = \kappa = 0.065$, $\bar{\lambda} = 0.8$, $\lambda = 1.0$, $y_t = 0.85$.

It is interesting to notice that the cutoff Λ_ρ associated with scalar loops is roughly of the same magnitude in all three models, which indicates that we should expect new electroweak states at the few-TeV level irrespective of other considerations. The experimental bounds on the heavy scalars in a Mirror or a Fraternal twin Higgs model [27] can estimate the current bounds on the scalar sector of our setup as well. However, a detailed study is necessary to obtain the current experimental bounds on these states in the TMNT model.

Given the success of improving fine-tuning with respect to Λ_t by introducing an additional radial mode, one might be tempted to further improve fine-tuning by stacking additional turtles with hard $\mathbb{Z}_2$ -breaking quartics on top of each other. However, as discussed earlier, the hard-breaking quartics reintroduce sensitivity to the Higgs quartic cutoff at one loop, so no improvement in the scale of new electroweak-charged states can be expected. Moreover, the one-loop logarithmic dependence of the Higgs quartic on the cutoff of the top sector ultimately limits the extent to which the scale of colored top partners can be pushed off. Taken together, these effects tend to saturate the improvement in fine-tuning with one turtle unless extensive model-building gymnastics (relative to those already in play) are employed. As such, we will limit ourselves to one additional level of symmetry breaking in the TMNT. Having demonstrated the parametric improvement in fine-tuning, we now turn to a more detailed exploration of the model.

3 A bottom-up twin turtle

In this section we explore the TMNT model illustrated in figure 1 in further detail as an effective theory with cutoffs Λ_t and Λ_ρ , with an eye towards radiative correlations between the parameters of the potential and the qualitative phenomenology. In its entirety, the model

consists of four copies of the Standard Model, each with a Higgs doublet (h_a, h_b, H_a, H_b , respectively). The matter content, gauge groups, gauge couplings, and yukawa couplings of each copy are identical. The only couplings between these copies of the Standard Model are via the Higgs sector, for which the potential relating the different Higgs fields (subject to simplifications and assumptions discussed in section 2) is

$$\begin{aligned}
V \supset & \bar{\lambda}(|h|^2 + |H|^2)^2 + m^2(|h|^2 + |H|^2) + \bar{\kappa}(|h|^4 + |H|^4) + \bar{\rho}|h|^4 + m_h'^2|h|^2 \\
& + \rho''(|h_a|^2 + |H|^2)^2 + \kappa''(|h_a|^4 + |h_b|^4 + |H_a|^4 + |H_b|^4) \\
& + \kappa'(|h_a|^4 + |h_b|^4) + \rho'|h_a|^4 + \mu'^2|h_a|^2.
\end{aligned} \tag{3.1}$$

Radiative corrections correlate the parameters in the potential and give rise to both quadratic and logarithmic dependence on the cutoffs Λ_t and Λ_ρ . In particular, the dominant one-loop corrections take the form

$$\begin{aligned}
\delta m^2 &= -\frac{3y_t^2}{16\pi^2}\Lambda_t^2 + \frac{9\bar{\lambda}}{16\pi^2}\Lambda_\rho^2, \\
\delta \bar{\kappa} &= \frac{5\bar{\rho}\bar{\lambda}}{32\pi^2}\log\frac{\Lambda_\rho^2}{m_2^2}, \\
\delta \kappa'' &= \frac{3y_t^4}{16\pi^2}\log\frac{\Lambda_t^2}{m_{T_a}^2}, \\
\delta \kappa' &= \frac{3y_t^4}{16\pi^2}\log\frac{m_{T_a}^2}{m_{t_b}^2}, \\
\delta \rho' &= \frac{3y_t^4}{16\pi^2}\log\frac{m_{t_b}^2}{m_{t_a}^2}, \\
\delta m_h'^2 &= \epsilon_\pm \frac{5\bar{\rho}}{16\pi^2}\Lambda_\rho^2,
\end{aligned} \tag{3.2}$$

where m_2 is given by eq. (2.16), m_{t_b} (m_{T_a}) is the mass of the top twin in the h_b (H_a) side, and we have used the fact that the mass m_{T_b} of the top twin in the H_b side is equal to m_{T_a} as the $\mathbb{Z}_2$ within H is respected. We have neglected the log term correction for $\delta m_h'^2$ assuming that the quadratically divergent contribution dominates. In addition, we have neglected radiative contributions coming from electroweak gauge couplings, which are generally subdominant.

3.1 Fine tuning

Having discussed the parametrics of fine-tuning in section 2, we now perform a more detailed numerical study that takes into account logarithmic sensitivity to the cutoff and accommodates the observed value of the Higgs mass. As before, we concentrate on tuning with respect to dimensionful parameters, as tuning with respect to dimensionless parameters is always subleading in the regime of interest.

For simplicity, we illustrate the numerical tuning with respect to a generic benchmark point, corresponding to the parameters in table 1. To explore the tuning we consider varying Λ_t and the tree-level value of $\bar{\rho}$, using μ' to then set the electroweak vev . As we will justify in the next section, it is possible to have only some of the couplings generated

m [TeV]	Λ_ρ [TeV]	$\bar{\lambda}$	$\bar{\kappa}$	κ''	κ'	ρ'	m_h^2	ρ''
1.8	3.0	0.8	0.0(0.15)	0.0	0.005	0.0	0.0	0.0

Table 1. Tree-level parameters used for our numerical study of fine-tuning with respect Λ_t and tree-level $\bar{\rho}$. Numbers in parentheses are for ϵ_+ .

at tree-level and others induced purely radiatively. Given the tree level value of each parameter in table 1, we include the radiative corrections in eq. (3.2).

With these corrections included, we investigate the fine-tuning of the scalar sector in the $(\Lambda_t - \bar{\rho})$ plane. At this stage we will adopt simplified notation and drop the “TMNT” superscripts on the fine-tuning measure Δ , as henceforth all fine-tunings are calculated for the TMNT model. In order to capture a parametrization-independent result, we add the contribution of all the parameters in quadrature and define the total tuning as

$$\Delta_v = \sqrt{\max \left\{ 1, \sum_x \left(\frac{x}{v^2} \frac{\partial v^2}{\partial x} \right)^2 \right\}}, \quad (3.3)$$

where $x = \{\mu'^2, m^2, m_h'^2, \Lambda_t^2, \Lambda_\rho^2\}$ runs over all of the dimensionful inputs. This will give us a conservative measure of the tuning of the EW scale and separate the irreducible tuning of the structure in figure 1 from the tuning due to any specific underlying model. We will turn to tuning calculations in UV-complete supersymmetric models in the next section.

As we have already discussed in the previous section, it is often convenient to discuss the overall tuning of the electroweak scale by factorizing it into the product of tunings capturing the sensitivity of v to the scale f , and then f to the cutoff scale Λ , although this fails to capture general logarithmic cutoff dependence as well as additional quadratic sensitivity of v to Λ in the presence of hard breaking. To illustrate the extent to which the factorization approximation captures the fine-tuning, we also define a factorized tuning measure of the form

$$\Delta_v \sim \Delta_{v/f} \times \Delta_{f/\Lambda} = \sqrt{\max \left\{ 1, \sum_{x_v} \left(\frac{x_v}{v^2} \frac{\partial v^2}{\partial x_v} \right)^2 \right\}} \times \sqrt{1 + \sum_{x_f} \left(\frac{x_f}{f^2} \frac{\partial f^2}{\partial x_f} \right)^2}, \quad (3.4)$$

where $x_v = \{\mu'^2\}$, while $x_f = \{m^2, m_h'^2, \Lambda_t^2, \Lambda_\rho^2\}$. This will allow us to illustrate the extent to which the factorization approximation captures the leading contributions to fine-tuning.

For the case of ϵ_+ , the radiative correction $\delta m_h'^2$ from the $\bar{\rho}$ term tends to stabilise the vacuum of h around zero. A relatively large tree-level $\bar{\kappa}$ is needed to compensate this and induce spontaneous breaking in the h sector. We thus choose relatively large $\bar{\kappa} \sim 0.15$ to study the total fine tuning in this case. The total tuning of the electroweak vev Δ_v and the factorization approximation $\Delta_v \times \Delta_{v/f}$ are shown in figure 3. Also shown are the contours of the scale f . For the chosen parameters, the factorization approximation is in good agreement with the total tuning. As can be seen, the total tuning Δ_v parallels contours of increasing f , reflecting the fact that the dominant contribution to fine-tuning is coming from the soft breaking separating v from f .

Figure 3. Solid contours denote the total tuning Δ_v of v with respect to all parameters added in quadrature (left) and the factorized approximation $\Delta_{v/f} \times \Delta_f$ (right) as a function of $\bar{\rho}$ and Λ_t for ϵ_+ . The parameters fixed numerically are reported in table 1. The red band denotes the Higgs mass window (122 GeV, 128 GeV). The dashed line denotes the scale f in unit of GeV. Compared to a vanilla Twin Higgs model, we find a factor of 1.5–3.5 improvement in the tuning across this parameter space; see the text for further details.

It is straightforward to compare the tuning of the TMNT to the vanilla TH model. To that end, we focus on the limit where the sensitivity to Λ_t dominates the tuning, i.e. the regime where both the TMNT and TH give rise to the best improvement over the tuning in the SM. In this regime the improvement in the tuning is given by eq. (2.23). Using this equation and the numbers from table 1, we find an improvement of 1.5–3.5 in tuning in the TMNT setup compared to a vanilla TH model across the parameter space of figure 3.

The case of ϵ_- turns out to be somewhat different. As there are no requirements on the tree-level $\bar{\kappa}$, we choose $\bar{\kappa} = 0$ for the sake of reducing fine tuning. As before, the total tuning of the electroweak vev Δ_v and the factorization approximation $\Delta_v \times \Delta_{v/f}$ are shown in figure 4, demonstrating good agreement between the two. For this parameter space, $F \sim 1.5$ TeV and slightly increases with Λ_t . Since $\bar{\kappa}$ and $m_h'^2$ are purely radiatively generated from $\bar{\rho}$, while $m_h'^2$ is quadratically sensitive to Λ_ρ^2 , the scale f is primarily determined by $\delta m_h'^2$ and doesn't change much as a function of $\bar{\rho}$.

Similar to the ϵ_+ case, we compare the tuning of a TMNT to a generic TH in the regime where the sensitivity to Λ_t dominates the tuning using eq. (2.23). Across the parameter space of figure 3, we find an improvement of 8–15 in tuning in the TMNT setup compared to a vanilla TH model. Notice that this improvement is even better than the case of ϵ_+ studied earlier, which can be attributed to the fact that we are using $\bar{\kappa} = 0$ in table 1 as the benchmark point.

We next turn to consider tuning of the electroweak vev with respect to the three high energy scales m^2 , Λ_t^2 and Λ_ρ^2 in the $\bar{\rho} - \Lambda_t$ plane, illustrated in figure 5. It is apparent from the tuning with respect to Λ_t that the colored top partner can be as high as 9 TeV without introducing much tuning, provided a sufficiently large $\bar{\rho}$. Also noteworthy is the fact that the tuning with respect to Λ_t is not quadratically decreasing with Λ_t at low values of the cutoff, which is quite distinct from the vanilla Twin Higgs model. This is because the reduced quadratic sensitivity to Λ_t implies that the primary sensitivity of the weak scale to Λ_t in this regime comes from the one loop logarithmic dependence in eq. (3.2). Although the

Figure 4. Solid contours denote the total tuning Δ_v of v with respect to all parameters added in quadrature (left) and the factorized approximation $\Delta_{v/f} \times \Delta_{f/\Lambda}$ (right) as a function of $\bar{\rho}$ and Λ_t for ϵ_- . The parameters fixed numerically are reported in table 1. The red band denotes the Higgs mass window (122 GeV, 128 GeV). The dashed line denotes the scale f in unit of GeV. Compared to a vanilla Twin Higgs model, we find a factor of 8–15 improvement in the tuning across this parameter space; see the text for further details.

Figure 5. The total tuning of v with respect to the mass parameters m^2 , Λ_t^2 and Λ_ρ^2 as a function of $\bar{\rho}$ and Λ_t . The red band denotes the Higgs mass window (122 GeV, 128 GeV), while the dashed line indicates the value of the scale f in unit of GeV.

factorization approximation is not shown, it remains a good proxy for the total fine-tuning in this range of parameters despite factorization violations from $\delta\kappa'$ and $\delta\kappa''$.

As the TMNT structure doesn't improve the quadratic sensitivity to Λ_ρ^2 , the tuning with respect to Λ_ρ^2 is still dominated by the quadratic dependence and requires Λ_ρ to be much smaller than Λ_t for $\mathcal{O}(1)$ tuning. The natural scale of the tree-level symmetric mass parameter m must also be much smaller than Λ_t since the separation of the electroweak scale from this scale is set by $\frac{\kappa_1}{\lambda} \frac{\delta\tilde{\kappa}}{2\delta\tilde{\kappa} + \bar{\rho}}$. While the tuning with respect to m may in principle be further reduced relative to the values in figure 5 by decreasing m , this is ultimately limited by the need to maintain sufficiently large m (and hence f) to satisfy precision Higgs coupling constraints.

Our discussion of tuning in this section neglects the origin of the terms in the general potential from eq. (2.15). In this respect, they represent the intrinsic and irreducible tuning of the electroweak scale in our setup, which may be further increased by detailed UV completions.

Figure 6. From left to right: the mass m_2 of the approximate radial mode of $SU(4) \supset SU(2)_a$; the mass m_3 of the scalar associated with the other $SU(4)$; the mixing angle $\sin^2 \theta_{a1}$ between the SM doublet h_a and the radial mode of $SU(8)$; and the mixing angle $\sin^2 \theta_{a2}$ between the SM doublet h_a and the approximate radial mode of $SU(4) \supset SU(2)_a$. In each frame, the red band denotes the Higgs mass window (122 GeV, 128 GeV), while the dashed line indicates the scale of f in unit of GeV. The numerical values used in this plot are those reported in table 1 for ϵ_- .

3.2 Phenomenology

We next turn to a brief exploration of the infrared phenomenology of the TMNT model. As noted earlier, the improvement in cutoff sensitivity comes at the price of additional Higgs scalars. In addition to the SM-like Higgs boson, there are three new CP-even neutral scalars h_1 , h_2 and h_3 , corresponding to the radial mode associated with breaking the approximate $SU(8)$ and the two other uneaten goldstones of $SU(8) \rightarrow SU(7)$ breaking. Of these two additional goldstones, h_2 can be roughly thought of as the radial mode of $SU(4) \supset SU(2)_a$, while h_3 is a scalar associated with the other $SU(4)$ sector. The masses of these scalars are roughly given by m_1 , m_2 and m_3 , respectively in eq. (2.16). For ϵ_- with the parameters shown in table 1, m_1 is around 2.5 TeV. The masses m_2 , m_3 of the two lighter modes and the two largest mixing angles θ_{a1} and θ_{a2} relating the SM doublet h_a to the $SU(4) \supset SU(2)_a$ and $SU(8)$ radial modes are illustrated in figure 6. As a consequence of mixing, the SM-like Higgs couplings are modified relative to the Standard Model by an amount

$$g_{hii} \sim g_{h_{SM}ii} \left(1 - \frac{1}{2} (\sin^2 \theta_{a1} + \sin^2 \theta_{a2}) \right) \sim g_{h_{SM}ii} \left(1 - \frac{v^2}{2f^2} \right) \quad (3.5)$$

where the second approximation, much as in the vanilla Twin Higgs, is numerically apparent in figure 6. This leads to the dominant bound on the scale f due to constraints on Higgs coupling deviations. Given Higgs coupling measurements consistent with Standard Model predictions at the $\mathcal{O}(10\%)$ level [10, 28], the scale f is constrained to lie above ~ 500 GeV. By increasing m^2 , one can increase the scale f without changing the sensitivity with respect to the cutoffs Λ_t and Λ_ρ . While smaller $\bar{\lambda}$ or larger $\bar{\kappa}$ would also increase the scale f , the fine tuning with respect to the cutoff Λ_t would correspondingly increase.

With the colored top partners associated with the scale Λ_t well out of reach of the LHC, the additional Higgs scalars are the most promising avenue for discovering the Twin Turtle at the LHC. All three additional neutral Higgs scalars inherit their couplings to the Standard Model via mixing with the SM doublet h_a . Since the $SU(8)$ radial mode h_1 is very heavy and contains only a small admixture of h_a , the prospects for probing h_1 are somewhat remote, but in some sense constitutes the smoking gun signal of the Twin Turtle.

The lighter mode h_2 plays a role similar to the radial model in the vanilla Twin Higgs model if we identify $2\bar{\kappa} + \bar{\rho}$ with the vanilla Twin Higgs SU(4) symmetric quartic coupling. In contrast to the vanilla Twin Higgs, however, for a given SM-like Higgs coupling deviation, the mixing between h_2 and h_a is reduced by roughly a factor of f^2/F^2 owing to the mixing of h_1 and h_a . This indicates that the coupling of h_2 to SM particles is reduced by the same factor, compared to the vanilla Twin Higgs radial mode. Thus, given a certain SM-like Higgs coupling deviation, the production cross section of h_2 at LHC is smaller. As the coupling reduction is universal, the branching ratios of h_2 to different Standard Model channels is not changed. This highlights the value of making precise rate and branching ratio measurements of any heavy Higgs scalars produced in the future, as it allows differentiation between a vanilla Twin Higgs scenario and a Twin Turtle.

In addition to the radial modes of SU(8) and $\text{SU}(4) \supset \text{SU}(2)_a$, there is the third scalar h_3 associated with the other SU(4) sector. If κ'' is loop-generated, this could be the lightest of the new scalar modes, as shown in figure 6. However, its mixing with h_a is highly suppressed because of the degeneracy between H_a and H_b , and its production rate at the LHC is correspondingly small.

4 A top-down twin turtle

Thus far, we have treated the Twin Turtle purely as an effective theory up to cutoffs Λ_t and Λ_ρ in the top sector and Higgs sector. We now turn to the construction of supersymmetric UV completions of this framework, which demonstrate the robustness of the mechanism and validate several parametric choices made in our exploration of the bottom-up theory.

The leading quartic couplings in the TMNT model may be generated via either F -term or D -term quartics in the supersymmetric UV completion [8–10, 15, 29], and we explore both options. Along the way, we present a mechanism for generating the hard-breaking quartics purely from soft breakings of $\mathbb{Z}_2$ symmetries. This has the advantage of reducing UV sensitivity and restoring all of the relevant $\mathbb{Z}_2$ symmetries as exact symmetries of the dimensionless couplings in the theory. While we make use of the mechanism for the TMNT, it is generally applicable to supersymmetric UV completions of Twin Higgs models employing hard-breaking quartics.

4.1 An F -term model

We begin with a supersymmetric UV completion of the TMNT model that uses F -terms to generate the leading quartics. As always, the introduction of supersymmetry also mandates the doubling of the Higgs spectrum, with each SU(2) doublet now promoted to a pair of doublets, which we denote with u and d subscripts and refer to as up-type and down-type doublets, respectively. The $\mathbb{Z}_2$ symmetries acting on the doublets of the low-energy TMNT model are naturally extended to act on both the up-type and down-type doublets in the UV completion.

For the F -term supersymmetric model, we need to introduce additional singlets to generate different quartics. To generate a sizable SU(8) quartic, one singlet S_0 is introduced to couple to the SU(8)-symmetric combination of Higgs doublets. In order to ensure that

this singlet generates an SU(8)-symmetric quartic, we need only require that its superpotential coupling respect the exchange symmetry relating $h_u \equiv (h_u^a, h_u^b)$ and $h_d \equiv (h_d^a, h_d^b)$ to $H_u \equiv (H_u^a, H_u^b)$ and $H_d \equiv (H_d^a, H_d^b)$. This is sufficient to generate an SU(8)-invariant quartic, in analogy with supersymmetric completions of the vanilla Twin Higgs [8–10].

The SU(8)-symmetric superpotential and the soft masses are

$$W_{\text{SU}(8)} = (\mu + \lambda_{s0} S_0)(H_u H_d + h_u h_d) + \frac{\mu_{s0}}{2} S_0^2, \quad (4.1)$$

$$V_{\text{SU}(8)} = m_u^2(|H_u|^2 + |h_u|^2) + m_d^2(|H_d|^2 + |h_d|^2) - b(H_u H_d + h_u h_d + \text{h.c.}) + m_{s0}^2 S_0^2. \quad (4.2)$$

Assuming that the singlet soft mass m_{s0} is much larger than the supersymmetric mass μ_{s0} , we can integrate out S_0 non-supersymmetrically to obtain the following potential for the Higgs degrees of freedom:

$$V_{\text{SU}(8)}^{\text{eff}} = (m_u^2 + \mu^2)(|H_u|^2 + |h_u|^2) + (m_d^2 + \mu^2)(|H_d|^2 + |h_d|^2) - b(H_u H_d + h_u h_d + \text{h.c.}) + \lambda_{s0}^2 |H_u H_d + h_u h_d|^2 \quad (4.3)$$

where we are suppressing corrections subleading in μ_{s0}^2/m_{s0}^2 .

In order to generate quartics respecting the two $\text{SU}(4) \subset \text{SU}(8)$ symmetries, we introduce an additional two singlet superfields S_h and S_H . In contrast to S_0 , these superfields are related by the same $\mathbb{Z}_2$ that exchanges $H \leftrightarrow h$. If the potential for S_h and S_H fully respected the same exchange symmetry, then this would suffice to generate the $\bar{\kappa}$ quartic but not the $\bar{\rho}$ quartic. To generate $\bar{\rho}$, we allow for a soft breaking of the exchange symmetry relating S_h and S_H . Upon integrating out S_h and S_H , this will feed down into an effective quartic that violates the $\mathbb{Z}_2$ symmetry relating the two SU(4)s. Since $\mathbb{Z}_2$ is broken in the UV lagrangian only by soft masses, the one loop radiative corrections to the dimensionless couplings still preserve the $\mathbb{Z}_2$.

The superpotential and the soft masses for this level of the theory are

$$W_{\text{SU}(8)} = \lambda_{sh} S_h h_u h_d + \lambda_{sH} S_H H_u H_d + \frac{\mu_{sh}}{2} S_h^2 + \frac{\mu_{sH}}{2} S_H^2, \quad (4.4)$$

$$V_{\text{SU}(8)} = m_{sh}^2 |S_h|^2 + m_{sH}^2 |S_H|^2, \quad (4.5)$$

where $\lambda_{sh} = \lambda_{sH}$ and $\mu_{sh} = \mu_{sH}$ at tree level because of $\mathbb{Z}_2$ symmetry. A further restriction is necessary in order to forbid additional terms in the superpotential or soft potential involving both S_h and S_H . The simplest possibility is to impose independent PQ symmetries on both the h and H sectors.

Integrating out S_h and S_H , we obtain the SU(8)-breaking potential for the Higgs sector,

$$V_{\text{SU}(8)}^{\text{eff}} = \frac{\lambda_{sh}^2 m_{sh}^2}{\mu_{sh}^2 + m_{sh}^2} |h_u h_d|^2 + \frac{\lambda_{sH}^2 m_{sH}^2}{\mu_{sH}^2 + m_{sH}^2} |H_u H_d|^2. \quad (4.6)$$

The spectrum of the theory now includes a variety of additional Higgs scalars implied by the additional Higgs doublets. However, these additional scalars can be lifted in the decoupling limit of the 2HDM in each copy of the MSSM, so that the spectrum of light

states is precisely that of the TMNT model. Assuming equal t_β for each sectors, we can match the full potential to the non-supersymmetric potential V of the low-energy TMNT by rotating to the Higgs basis

$$\begin{aligned} h_u &= h s_\beta, & h_d &= h c_\beta, \\ H_u &= H s_\beta, & H_d &= H c_\beta, \end{aligned} \quad (4.7)$$

with

$$t_\beta^2 = \frac{m_d^2 + \mu^2}{m_u^2 + \mu^2}, \quad (4.8)$$

The matching between UV parameters and the low-energy TMNT parameters is simply

$$\begin{aligned} m^2 &= (m_u^2 + \mu^2)s_\beta^2 + (m_d^2 + \mu^2)c_\beta^2 - b s_{2\beta}, \\ \bar{\lambda} &= \frac{\lambda_{s0}^2}{4} s_{2\beta}^2, \\ \bar{\kappa} &= \frac{\lambda_{sH}^2 m_{sH}^2}{\mu_{sH}^2 + m_{sH}^2} \frac{s_{2\beta}^2}{4}, \\ \bar{\rho} &= \left(\frac{\lambda_{sh}^2 m_{sh}^2}{\mu_{sh}^2 + m_{sh}^2} - \frac{\lambda_{sH}^2 m_{sH}^2}{\mu_{sH}^2 + m_{sH}^2} \right) \frac{s_{2\beta}^2}{4}. \end{aligned} \quad (4.9)$$

We see that in the region where $m_{sH}^2 \ll \mu_{sH}^2$ and $m_{sh}^2 \gg \mu_{sh}^2$, the parametric ratio $\bar{\kappa} \ll \bar{\rho}$ is realized. Notice that m_{sH}^2 cannot be arbitrarily small, as the soft mass terms m_u^2 and m_d^2 contribute to m_{sH}^2 at one loop level, setting the natural size for m_{sH}^2 .

Electroweak gauge couplings and Yukawa interactions, while preserving $\mathbb{Z}_2$ symmetries, will contribute to SU(8) breaking quartics at one loop. These contributions are

$$\begin{aligned} V_{\text{SU}(8)} &= \frac{g_1^2 + g_2^2}{8} \left[(|h_u^a|^2 - |h_d^a|^2)^2 + (|h_u^b|^2 - |h_d^b|^2)^2 \right. \\ &\quad \left. + (|H_u^a|^2 - |H_d^a|^2)^2 + (|H_u^b|^2 - |H_d^b|^2)^2 \right] \\ &\quad + \frac{3y^4}{16\pi^2} \left[\log \left(\frac{m_{\tilde{t}_a}^2}{m_{t_a}^2} \right) |h_u^a|^4 + \log \left(\frac{m_{\tilde{t}_b}^2}{m_{t_b}^2} \right) |h_u^b|^4 \right. \\ &\quad \left. + \log \left(\frac{m_{\tilde{T}_a}^2}{m_{T_a}^2} \right) |H_u^a|^4 + \log \left(\frac{m_{\tilde{T}_b}^2}{m_{T_b}^2} \right) |H_u^b|^4 \right], \end{aligned} \quad (4.10)$$

where $m_{\tilde{t}_a}, m_{\tilde{t}_b}, m_{\tilde{T}_a}$ and $m_{\tilde{T}_b}$ are the stop masses⁴ in the h_a, h_b, H_a and H_b sectors, respectively. At tree level, we have $m_{\tilde{t}_a} = m_{\tilde{t}_b} = m_{\tilde{T}_a} = m_{\tilde{T}_b} = m_{\tilde{t}}$. These radiative corrections contribute to the effective potential for the TMNT model via the matching

$$\begin{aligned} \kappa'' &= \frac{g_{ew}^2}{8} c_{2\beta}^2 + \frac{3y_t^4}{16\pi^2} \log \left(\frac{m_{\tilde{t}}^2}{m_{T_a}^2} \right), \\ \kappa' &= \frac{3y_t^4}{16\pi^2} \log \left(\frac{m_{\tilde{T}_a}^2}{m_{\tilde{t}_b}^2} \right), \end{aligned} \quad (4.11)$$

⁴Common LH and RH stop masses in each sector are assumed.

where $y_t = y \sin(\beta)$, $g_{ew}^2 = g_1^2 + g_2^2$ is the electroweak gauge coupling in the SM, and we have used the relation $m_{T_a}^2 = m_{T_b}^2$ as the $\mathbb{Z}_2$ within H is respected.

The most important radiative corrections to the $SU(8)$ symmetric mass are from the stop loop and singlet S_0 loop,

$$\begin{aligned}\delta m_u^2 &\sim -\frac{3y^2}{16\pi^2} m_t^2 \log\left(\frac{\Lambda_{\text{mess}}^2}{m_t^2}\right) - \frac{\lambda_{s0}^2}{16\pi^2} m_{s0}^2 \log\left(\frac{\Lambda_{\text{mess}}^2}{m_{s0}^2}\right), \\ \delta m_d^2 &\sim -\frac{\lambda_{s0}^2}{16\pi^2} m_{s0}^2 \log\left(\frac{\Lambda_{\text{mess}}^2}{m_{s0}^2}\right),\end{aligned}\quad (4.12)$$

where Λ_{mess} is the messenger scale. In what follows we will typically take $\Lambda_{\text{mess}} = 10m_{\tilde{t}}$.

The soft mass terms m_{sh}^2 and m_{sH}^2 will also feed into $m_h'^2$ at one loop via, for example,

$$m_h'^2 = -\frac{\lambda_{sh}^2 m_{sh}^2}{16\pi^2} \log\left(\frac{\Lambda_{\text{mess}}^2}{m_{sh}^2 + \mu_{sh}^2}\right). \quad (4.13)$$

Radiative corrections from m_u^2 and m_d^2 to $m_h'^2$ are finite and neglected in this section. The sign of $m_h'^2$ is negative as we have $m_{sh}^2 \gg m_{sH}^2$ and positive soft mass terms m_{sh}^2 , m_{sH}^2 in order not to induce non-zero vacuum expectation values in the singlet sectors. Though will not be studied here, we note that corrections from the singlet soft mass terms could be improved by introducing additional singlets as in [30]. The $m_h'^2$ term further contributes to t_β on the h side as

$$t_{\beta_h}^2 = \frac{m_d^2 + \mu^2 + m_h'^2}{m_u^2 + \mu^2 + m_h'^2}. \quad (4.14)$$

As the correction only makes $t_{\beta_h}^2$ slightly different from t_β^2 , we will neglect it in what follows.

As the $\mathbb{Z}_2$ within h has to be softly broken to generate the misalignment between f and electroweak vev , we introduce the following soft mass terms

$$V_{SU(4)} = \delta m_u^2 |h_u^a|^2 + \delta m_d^2 |h_d^a|^2 + \delta b (h_u^a h_d^a + \text{h.c.}) \quad (4.15)$$

which amount to soft breaking of the $\mathbb{Z}_2$ symmetry relating h_a and h_b . Matching to the TMNT potential V gives $\mu'^2 = (\delta m_u^2 s_\beta^2 + \delta m_d^2 c_\beta^2)$, with $\delta b = 0$. This misalignment also leads to a modest contribution to the downstairs quartic,

$$\Delta V' = \frac{3y_t^4}{16\pi^2} \log\left(\frac{m_{t_b}^2}{m_{t_a}^2}\right) |h_a|^4 = \rho' |h_a|^4. \quad (4.16)$$

4.1.1 Constraints & fine tuning

We are now in a position to consider the tuning of the electroweak scale in the context of a UV completion. Before studying the tuning in detail, it is necessary to consider the viable range of couplings in the UV completion. In particular, in order to obtain sizeable quartics in the infrared TMNT model, the UV couplings $(\lambda_{s0}, \lambda_{sh}, \lambda_{sH})$ should also be sizeable. However, these couplings cannot be too large without inducing a Landau pole close to the scale of matching, compromising the validity of the UV completion.

m_d^2 [TeV ²]	μ^2 [TeV ²]	b [TeV ²]	m_{s0} [TeV]	λ_{s0}
0.8^2	0.48^2	1.96	1.6	1.3
m_{sh} [TeV]	m_{sH} [TeV]	$\mu_{sh} = \mu_{sH}$ [TeV]	$\lambda_{sh} = \lambda_{sH}$	y_t
1.0	0.2	0.8	1.7	0.85

Table 2. Tree-level parameters used for our numerical study in the F -term supersymmetric UV completion of the TMNT model with μ'^2 solved to get $vev = 174$ GeV. As the IR scale in our analysis is above TeV, we use $y_t = 0.85$ to account for the running effects.

The one-loop beta functions for the couplings λ_{s0} , λ_{sh} and λ_{sH} are

$$\begin{aligned}
 16\pi^2 \frac{d}{dt} \lambda_{s0} &= \lambda_{s0} (10\lambda_{s0}^2 + 6\lambda_{sh}^2 + 3y_t^2 - g_Y^2 - 3g_2^2), \\
 16\pi^2 \frac{d}{dt} \lambda_{sh} &= \lambda_{sh} (6\lambda_{sh}^2 + 6\lambda_{s0}^2 + 3y_t^2 - g_Y^2 - 3g_2^2), \\
 16\pi^2 \frac{d}{dt} \lambda_{sH} &= \lambda_{sH} (6\lambda_{sH}^2 + 6\lambda_{s0}^2 + 3y_t^2 - g_Y^2 - 3g_2^2).
 \end{aligned} \tag{4.17}$$

If we require that neither coupling hits a Landau pole below 100 TeV, this implies that the couplings in the infrared (5 TeV, for concreteness) must satisfy

$$\lambda_{s0} < 1.3, \quad \lambda_{sh} = \lambda_{sH} < 1.9. \tag{4.18}$$

For comparison, in the F -term supersymmetric completion of the vanilla Twin Higgs model, the singlet coupling is constrained to satisfy the weaker condition $\lambda_{s0} < 2.1$ to avoid a Landau pole beneath 100 TeV. This exemplifies the point made in section 2 regarding the relative sizes of symmetric quartics in the TMNT and vanilla Twin Higgs.

To illustrate the fine-tuning in the F -term UV completion, we fix the UV parameters as in table 2 and study the fine tuning in the plane t_β - $m_{\tilde{t}}$, where we translate the free parameter m_u^2 into the parameter t_β . The electroweak scale is fixed by the appropriate choice of μ'^2 whose value is shown in figure 7. Also shown in figure 7 are couplings for the low-energy TMNT potential. The dependence of $\bar{\lambda}$ on t_β determines that F increases with t_β . κ'' would fix the mass of the lighter mode on H side in the decoupling limit and ρ' contributes to the SM-like Higgs quartic. The red band indicates the Higgs mass window (122 GeV, 128 GeV).

We also show the mass spectrum for the other 3 scalars in figure 8 assuming the decoupling limit. Notice that the m_1 mass decreases with t_β as the symmetric mass m is driven by the b term and thus decreases with t_β , while m_3 and m_2 increase with t_β because of the increasing of κ'' and the scale f , respectively.

The fine-tuning of the electroweak vev is calculated by taking logarithmic derivatives with respect to the full set of dimensionful parameters in the UV completion, namely $P = \{m_u^2, m_d^2, b, \mu^2, m_{s0}^2, m_{sh}^2, m_{sH}^2, \mu_{sh}^2, \mu_{sH}^2, \mu'^2, m_{\tilde{t}}^2\}$, and adding them in quadrature:

$$\Delta_v = \sqrt{\sum_P \left(\frac{\partial \log v^2}{\partial \log P} \right)^2}. \tag{4.19}$$

Figure 7. The effective quartics and soft $\mathbb{Z}_2$ breaking term in the low-energy TMNT potential arising from a F -term supersymmetric UV completion with the specific values of input parameters listed in table 2. The red band indicates the Higgs mass window (122 GeV, 128 GeV). The scale F increases slightly with $\bar{\lambda}$ and is around 1.3 TeV everywhere.

Figure 8. The mass spectrum of the scalars for the numerical benchmark illustrated for the F -term model in this section. m_1 and m_3 are dominantly from the scalars on the H side and m_2 is dominantly from the h_b side. The red band indicates the Higgs mass window (122 GeV, 128 GeV). The scale F increases slightly with $\bar{\lambda}$ and is around 1.3 TeV everywhere.

Figure 9. The total tuning Δ_v of the electroweak scale v in the F -term supersymmetric UV completion of the TMNT model with respect to all dimensional parameters added in quadrature. The red band indicates the Higgs mass window (122 GeV, 128 GeV), while the dashed line denotes the scale of f in unit of GeV. The scale F increases slightly with $\bar{\lambda}$ and is around 1.3 TeV everywhere.

The total fine-tuning Δ_v in the F -term supersymmetric completion is shown in figure 9. In the non-decoupling F -term model, as t_β increases, the matched quartic couplings in eq. (4.9) decrease and thus the scales F and f increase. The weak dependence of the scale f on $m_{\tilde{t}}$ is due to the fact that $m_u^2 + \delta m_u^2$ is roughly fixed for a certain t_β even as $m_{\tilde{t}}$ is varied, so that the scale f is approximately constant. As such, the scale f is tuned between m_u^2 and δm_u^2 and thus the total fine tuning Δ_v doesn't necessarily parallel the f contour. Compared to the parameter space in [10], we obtain the observed SM-like Higgs mass for lower values of t_β given the same scale f . The F -term supersymmetric UV completion of the TMNT model improves the fine tuning by a factor of 3 compared to the F -term completion of the vanilla Twin Higgs for $m_{\tilde{t}} \sim 5$ TeV.

4.2 A D -term model

In the previous section, we introduced a supersymmetric model relying on F -terms to generate the terms required for a UV completion of the TMNT model. We can similarly use new U(1) gauge groups and their non-decoupling D -terms to generate the symmetry breaking terms in our model, in analogy with the D -term supersymmetric completion of the vanilla Twin Higgs [15]. We introduce such a model, carry out the same fine-tuning analysis as above, and additionally discuss various phenomenological constraints arising from the additional gauge sectors. The overall structure is similar to figure 1 but with new U(1) gauge groups, some singlets, and, of course, supersymmetry.

The D -term supersymmetric completion entails one U(1) gauge group to generate the SU(8) symmetric quartic and two additional U(1) gauge groups to generate the SU(4) symmetric quartics. We also introduce a SM singlet charged under each U(1). Let's begin with the SU(4) symmetric quartics. Their singlets can have the following superpotential

$$W \supset \lambda_{ph(H)} S_{h(H)} \left(P_{h(H)} \bar{P}_{h(H)} - M_{ph(H)}^2 \right), \quad (4.20)$$

where $S_{h(H)}$ is a singlet of $U(1)_{h(H)}$ and P fields have charge $\pm q_{ph(H)}$, with the subscript in each case indicating the $SU(4)$ group of interest. These fields can have soft terms as well, which for simplicity we take to be equal for P and $\bar{P}$:

$$V_{\text{soft}} \supset m_{ph(H)}^2 \left(P_{h(H)}^2 + \bar{P}_{h(H)}^2 \right). \quad (4.21)$$

These soft terms ensure that the D -term quartics induced by each $U(1)$ do not decouple [29]. The superpotential leads to nonzero vacuum expectation values for the $P, \bar{P}$ fields that Higgs each of the $U(1)$ s. The resulting gauge boson mass on each side is given by [15, 29]

$$m_{Zh(H)} = 2g_{h(H)}q_{ph(H)}v_{ph(H)}, \quad (4.22)$$

where $v_{ph(H)}$ and $g_{h(H)}$ denote the *vevs* and the gauge couplings of each side, respectively. As in the F -term case, we can match on to the low-energy TMNT model by rotating to the Higgs basis, thereby isolating contributions to the light scalars that remain in the decoupling limit of the supersymmetric 2HDMs. Assigning a charge of $q = 1/2$ to h and H fields under their respective $U(1)$ s, the non-decoupling D -term quartics on each side are

$$\begin{aligned} V_{Dh} &\supset \frac{g_h^2}{8} h^4 \cos^2(2\beta_h) \left(\frac{2m_{ph}^2}{2m_{ph}^2 + m_{Zh}^2} \right), \\ V_{DH} &\supset \frac{g_H^2}{8} H^4 \cos^2(2\beta_H) \left(\frac{2m_{pH}^2}{2m_{pH}^2 + m_{ZH}^2} \right). \end{aligned} \quad (4.23)$$

For the rest of our study, we will concentrate on equal β ($= \beta_h = \beta_H$) angle on each side.

We repeat the exercise for the $SU(8)$ symmetric quartic, introducing an analogous set of multiplets; we denote these with the subscript “0”. The quartic resulting from this $U(1)$ is simply

$$V_D \supset \frac{g_0^2}{8} (h^2 + H^2)^2 \cos^2(2\beta) \left(\frac{2m_{p0}^2}{2m_{p0}^2 + m_{Z0}^2} \right). \quad (4.24)$$

The D -term in eq. (4.24) can be mapped to the coupling $\bar{\lambda}$ in the general potential in eq. (2.15), while those in eq. (4.23) generate $\bar{\kappa}$ and $\bar{\rho}$. In particular, $\bar{\rho}$ can be generated by taking either the soft scales m_{ph}^2, m_{pH}^2 to be unequal, or the supersymmetric scales m_{Zh}^2, m_{ZH}^2 to be unequal, thereby generating an effective hard-breaking quartic via soft terms.

To simplify our analysis, we assume the new singlet *vevs* and the gauge couplings at $SU(4)$ level are equal at different sides. Hence, the $\mathbb{Z}_2$ -symmetry breaking is sourced only through unequal soft terms. Assuming $g_H = g_h$, the generated quartics are then given by

$$\begin{aligned} \bar{\lambda} &= \frac{g_0^2}{8} \frac{2m_{p0}^2}{2m_{p0}^2 + m_{Z0}^2} \cos^2(2\beta), \\ \bar{\kappa} &= \frac{g_h^2}{8} \frac{2m_{pH}^2}{2m_{pH}^2 + m_{ZH}^2} \cos^2(2\beta), \\ \bar{\rho} &= \frac{g_h^2}{8} \left(\frac{m_{ZH}^2}{2m_{pH}^2 + m_{ZH}^2} - \frac{m_{Zh}^2}{2m_{ph}^2 + m_{Zh}^2} \right) \cos^2(2\beta). \end{aligned} \quad (4.25)$$

The symmetric mass of the general potential at this level can arise from the usual mass terms in a SUSY model, namely m_u, m_d, μ^2 , and b , as in the F -term model. We assume these terms are the same on each side so they only contribute to the totally symmetric mass term at tree-level.⁵ In what follows, we will work with the tree-level expressions above for the respective quartics, as the loop corrections will be proportional to these quantities as well and are suppressed compared to the leading pieces.

The SM gauge group (and its twin) make an additional D -term contribution to the quartics at tree-level, while top loops give an appreciable one-loop contribution as well. These corrections have the form

$$\begin{aligned}\kappa'' &= \frac{g_{ew,\text{twin}}^2}{8} \cos^2(2\beta) + \frac{3y_t^4}{16\pi^2} \log\left(\frac{m_t^2}{m_{T_a}^2}\right), \\ \kappa' &= \frac{g_{ew}^2}{8} \cos^2(2\beta) + \frac{3y_t^4}{16\pi^2} \log\left(\frac{m_t^2}{m_{t_b}^2}\right) - \kappa'', \\ \rho' &= \frac{3y_t^4}{16\pi^2} \log\left(\frac{f^2}{v^2}\right),\end{aligned}\tag{4.26}$$

where we will use $g_{ew,\text{twin}} = g_{ew}$.

We include the correction to the soft terms in the general potential from the loops of the new gauge bosons and the stops loops. In contrast to the F -term model, where corrections from the singlet sector soft mass terms have a logarithmic dependence on the messenger scale Λ_{mess} , the corrections from the new gauge sector soft mass terms do not depend on Λ_{mess} at one loop. Hence, the loop corrections from the scalar and fermionic partners of the h supermultiplets will not be sub-dominant anymore. We include the contribution of these loops to $m_h'^2$. Due to supersymmetry, the quadratic pieces of these fermionic and bosonic loops will cancel; the log piece, however, gives rise to an analog of eq. (2.11), where instead of a quadratic cutoff we will have the soft mass of the scalar component of the h supermultiplet. Collectively, these effects contribute to the TMNT mass parameters as

$$\begin{aligned}\delta m^2 &= \frac{g_0^2}{64\pi^2} m_{Z0}^2 \log\left(\frac{2m_{p0}^2 + m_{Z0}^2}{m_{Z0}^2}\right) + \frac{g_h^2}{64\pi^2} m_{ZH}^2 \log\left(\frac{2m_{pH}^2 + m_{ZH}^2}{m_{ZH}^2}\right) \\ &\quad - \frac{3y_t^2}{16\pi^2} m_t^2 \log\frac{\Lambda_{\text{mess}}^2}{m_t^2}, \\ \delta m_h'^2 &= \frac{5\bar{\rho}}{16\pi^2} \tilde{\Lambda}_\rho^2 + \frac{g_h^2}{64\pi^2} m_{Zh}^2 \log\left(\frac{2m_{ph}^2 + m_{Zh}^2}{2m_{pH}^2 + m_{ZH}^2}\right),\end{aligned}\tag{4.27}$$

where in the decoupling limit

$$\tilde{\Lambda}_\rho^2 \sim -m_u^2 \log\left(\frac{m_{ph}^2}{M_h^2}\right),\tag{4.28}$$

M_h denotes the heavy Higgs mass of the h field that is decoupled, m_{ph} is the mass of the P singlet mass, the factor of 5 in the second expression in eq. (4.27) is due to the

⁵Through loop corrections, the $\mathbb{Z}_2$ -breaking terms do generate some misalignment in these parameters between different sites.

scalar multiplicity, and similar to the previous section we assume the messenger scale is at $\Lambda_{\text{mess}} = 10m_{\tilde{t}}$. Note should be taken that even though $\tilde{\Lambda}_\rho$ emulates a cutoff as in eq. (2.11), it is in fact a log correction. To stabilize the *vevs* in our setup we have $m_u^2 < 0$, thus $\tilde{\Lambda}_\rho^2 > 0$.

We have not introduced any source for generating the soft $\mathbb{Z}_2$ -breaking μ'^2 term. We assume it is generated by another mechanism not included here and determine its value numerically such that v has the correct numerical value.

4.2.1 Constraints & fine tuning

As with the F -term model, we now turn to the tuning of the electroweak scale in the D -term scenario. The viable parameter space is subject to somewhat stronger experimental constraints due to the effects of the additional U(1) gauge groups, which we must accommodate before determining the numerical fine-tuning. In particular, we must not only consider the constraints imposed by avoiding low-scale Landau poles associated with the new gauge groups, but also bounds coming from electroweak precision observables (in particular oblique parameters S and T [31–33]) due to Higgs mixing and the new Z' gauge bosons.

Let us begin with the Landau poles. We are introducing three new U(1) gauge groups, each of whose gauge couplings can develop a Landau pole in the UV. As we will focus on large gauge couplings (to generate sizable quartics), this threat arises at very low scales, i.e. below 1000 TeV. Avoiding these low-scale Landau poles requires further UV-completion of these gauge groups, e.g. embedding them inside a larger group. We postpone further study of such UV completions for now and merely demand the Landau poles to be above a fiducial value, say 100 TeV.

The one-loop Landau pole for a particular U(1) gauge group is given by

$$\Lambda_L = \Lambda_{\text{IR}} \exp \left(\frac{2\pi}{A\alpha_{\text{IR}}} \right), \quad (4.29)$$

where A stands for the quadrature sum of field charges under that group. We assume only the Higgs supermultiplets (and not other new or SM matter content) are charged under each gauge group, all with charge $1/2$.⁶ The Landau poles as a function of the IR gauge couplings (taken at 5 TeV) are shown in figure 10. This illustrates the upper bound on the gauge couplings implied by avoiding a Landau pole beneath a certain scale. In our numerical studies we will fix $g_0 = 1.6$ to have a Landau pole above 100 TeV, and take $g_h = 1.4$.

The most pressing experimental constraints arise from precision electroweak limits; here we focus on S and T . The contributions to S and T from the mixing of the SM-like Higgs with other sectors can be made sufficiently small for large enough f . More pressing is the contribution from the new Z' gauge bosons, which contribute to the T parameter at tree level with the charge assignments articulated above. Using the current bounds on

⁶By not charging SM field content under these new gauges, the Standard Model yukawas are necessarily generated by some non-renormalizable operators suppressed by a UV scale. This could explain the small value and the hierarchy of the SM yukawa couplings; see, for example, [34] for a similar idea in the context of deconstruction models [35].

Figure 10. Location of the one-loop Landau pole for the U(1) gauge groups associated with the SU(4) symmetric quartics (solid line) and the gauge group associated with the SU(8) symmetric quartic (dashed line). Here we assume only the supermultiplets containing Higgs-like fields are charged under the new groups, and the IR scale is taken to be $\sim 5\text{TeV}$. Further UV completion is required at or below the scale of these poles. Demanding the Landau poles to be above a certain value, e.g. 100TeV , limits the size of the D -term quartics in our model.

this parameter [36] and the formalism of [33], we find that staying within 1σ of the current central value implies

$$\frac{m_{Z'}}{g_{Z'}} \sim v_p \gtrsim 3\text{TeV}. \quad (4.30)$$

In what follows, we assume the v evs v_p are a factor of $\sqrt{2}$ greater than the bound above, to accommodate the fact that there are multiple vectors contributing to T .

We are now in a position to consider the numerical fine-tuning of the weak scale, following the prescription detailed in section 2. We add the contributions from all of the dimensionful underlying parameters in quadrature. In this model, these parameters are

$$\{v_p^2 (= v_{ph}^2 = v_{pH}^2), v_{p0}^2, m_{ph}^2, m_{pH}^2, m_{p0}^2, m_u^2, m_d^2, \mu^2, b, \mu'^2, m_{\tilde{t}}^2\}. \quad (4.31)$$

As there are many underlying parameters, to demonstrate the qualitative features of fine-tuning we fix many of them to particular numerical values and illustrate the tuning as a function of two parameters. Other underlying parameters are fixed to numerical values indicated in table 3. The couplings in the low-energy TMNT potential, derived from these input parameters and as a function of the cutoff scale $m_{\tilde{t}}$ and $\tan\beta$ (t_β), are illustrated in figure 11. Notice the different t_β dependence of $\bar{\lambda}$ coupling compared to the F -term model; as a result, in this model the scales f and F have a different dependence on t_β , which consequently gives rise to a different behavior for the electroweak scale tuning in this model. These figures also indicate the dependence of κ'' and ρ' on β and a slight dependence on $m_{\tilde{t}}$ due to the log factor in eq. (4.26). The red lines in this figure indicate the range of parameters that generates a higgs mass between $(122, 128)\text{GeV}$.

Assuming the decoupling limit, there are 3 other scalar masses in the spectrum. These eigenvalues are shown in figure 12. We observe a level-crossing between two of the eigenvalues in these plots. For almost all the range of parameters we are studying the lightest

g_0	$g_h = g_H$	$v_p = v_{p0}$ [TeV]	m_{ph} [TeV]	m_{pH} [TeV]
1.6	1.4	4.5	14.3	2.25
m_{p0} [TeV]	m_u^2 [TeV ²]	m_d^2 [TeV ²]	μ^2 [TeV ²]	y_t
14.3	-1.4^2	-0.1^2	0.48^2	0.85

Table 3. Numerical values of the input parameters used in our study. As the IR scale in our analysis is above a TeV, we use a smaller value of y_t than its weak-scale value to account for running effects. The only source of $\mathbb{Z}_2$ breaking between the two sides of our tower are the soft masses of the P fields.

Figure 11. The effective quartics and soft $\mathbb{Z}_2$ breaking term in the low-energy TMNT potential arising from a D -term supersymmetric UV completion with the specific values of input parameters listed in table 3. The red lines in this figure indicate the range of parameters that generates a higgs mass between (122, 128) GeV.

eigenvalue (the SM-like Higgs) has small enough mixing with other sectors to evade the electroweak bounds due to its mixing with the twin sector [10, 14].

The total tuning of the electroweak scale with respect to all the parameters in our model is shown in figure 13. The values of the vev f are also shown in blue dashed lines. The scale F behaves similar to the f vev ; for the parameters in table 3 and the range of

Figure 12. The mass spectrum of the scalars for the numerical benchmark illustrated in this section. m_1 is dominantly from the scalars on the H side of the tower. Notice the level-crossing between m_2 and m_3 in the 900 GeV contour; one of these eigenvalues is mostly comprised of h_b , while the other one is mainly made of the H fields.

Figure 13. The total fine-tuning of the electroweak scale v with respect to all the underlying parameters in the D -term model. We also show the values of the vev f in unit of TeV with blue dashed lines. The numerical benchmark values used for this plot are included in table 3. The contours of constant tuning have a different behavior compared to the F -term model in figure 9. This difference can be mainly attributed to the dependence of the symmetric coupling $\bar{\lambda}$ on the β angle in the two models.

$m_{\tilde{t}}$ and $\tan \beta$ we are studying, F varies between 2 – 4 TeV. Figure 13 indicates similar fine-tuning improvement to the F -term model, which is improved by a factor of 3-4 compared to a generic supersymmetric Twin Higgs model. As argued earlier, the TMNT gain in fine-tuning and its ability to raise the scale of new colored physics (denoted by the stop mass $m_{\tilde{t}}$) comes at the cost of additional Higgs scalars and a relatively low scale of UV completion for the Higgs quartic.

In contrast to the F -term model in section 4.1, the symmetric quartic in this model increases with $\tan \beta$. As a result, as we go to larger $\tan \beta$ the scale f decreases as indicated in figure 13. This, in turn, suggests an improvement in the tuning as we go to larger $\tan \beta$.

Using the factorization arguments for the tuning in TMNT in section 3, the dominant tuning is between the scales f and v , thus the contours of the tuning, to a good extent, follow the constant f contours. As the $m_{\tilde{t}}$ scale increases, the fine-tuning should naturally worsen. These effects give rise to the behavior of the fine-tuning as a function of $\tan \beta - m_{\tilde{t}}$ in figure 13. Notice the different behavior of contours of constant tuning compared to the F -term supersymmetric models in the previous section; as explained, this difference should be attributed to the dependence of the symmetric quartic $\bar{\lambda}$ and, consequently, the *vevs* f and F on $\tan \beta$.

Numerical investigation shows that a dominant source of tuning in our model is the μ'^2 term in the downstairs theory. This is indeed in accordance with our factorization analysis in section 3 that suggests the softly-broken downstairs $\mathbb{Z}_2$ symmetry is the prevailing source of tuning in the TMNT.

While the overall tuning is worsened in these UV completions, this is a common feature of all supersymmetric UV completions of the Twin Higgs (e.g. [10, 14]). Crucially, however, the UV completions validate the parametric improvement in tuning suggested by the analysis of the effective theory, in the sense that they improve tuning compared to supersymmetric UV completions of the vanilla Twin Higgs. Moreover, these supersymmetric UV completions validate some of the parametric choices made in constructing the bottom-up TMNT model, demonstrating that they can be preserved by more complete theories.

5 Conclusions

In this work, we have extended the framework of neutral naturalness by demonstrating that the hierarchical breaking of discrete symmetries can stabilize the Higgs mass and further increase the natural scale of new colored states associated with compositeness or supersymmetry as an ultimate UV completion. The increased scale of colored states is obtained at the price of introducing additional Higgs-like scalars that couple weakly to the Standard Model, making additional Higgs-like scalars the most immediate signature of naturalness in this framework. Additionally, the scale associated with new electroweak-charged states responsible for the UV completion of quartic couplings is roughly the same as in the Standard Model or conventional Twin Higgs models. We have not only illustrated the mechanism in terms of an effective extension of the Standard Model up to some further cutoff, but also provided two explicit supersymmetric UV completions that validate the expectations of the effective theory. Along the way, we have uncovered a novel mechanism for UV completing hard $\mathbb{Z}_2$ -breaking quartics in Twin Higgs models via soft supersymmetry breaking masses, which may be fruitfully applied to UV completions of more conventional Twin Higgs models as well.

The Twin Turtle joins tadpole-induced electroweak symmetry breaking [20, 21] and collective symmetry breaking [22] in the class of models that increase the natural scale of new colored states relative to the vanilla Twin Higgs. Although the detailed mechanism is distinct in each case, the models in this class are nonetheless broadly similar: structure is added in order to improve vacuum alignment, and raising the scale of colored partner particles requires new Higgs-like scalars. In the case of tadpoles and collective symme-

try breaking, some of the new Higgs-like scalars come from multiplets carrying Standard Model electroweak quantum numbers, while in the Twin Turtle the new multiplets containing Higgs-like scalars are exclusively charged under other copies of the Standard Model. Although the absence of new electroweak multiplets may be considered a relative advantage of the Twin Turtle, it comes at the unquestionably added complexity of introducing more copies of the Standard Model.

There are many possible opportunities and advantages in such ‘turtle’ constructions, of which the framework presented here is only one example. While we have focused on Twin Turtles in which vacuum alignment is achieved via hard-breaking quartics, further improvements might be realized by turtle models employing the above-mentioned tadpoles or collective symmetry breaking. While both of these alternative approaches would involve additional ingredients relative to the hard-breaking quartics employed here, they may lead to an even greater improvement in the parametric fine-tuning of the weak scale. In addition, they may provide promising avenues for pushing off all divergences to as many loops as possible, in the spirit of the original turtle models. In this respect, turtle models based on tadpoles or collective symmetry breaking could allow for the separation of intermediate scales of discrete symmetry breaking without reintroducing quadratic sensitivity to the cutoff, and merit further study.

Another promising direction is the construction of other patterns of discrete symmetry breaking. In this work we have constructed what is in some sense the most naive turtle extension of the Twin Higgs by squaring the underlying $SU(2) \times SU(2) \times \mathbb{Z}_2$ structure. But other patterns of discrete symmetry may yield greater simplicity and further improvement in tuning. More broadly, given the pressing need for innovation in our approach to the electroweak hierarchy problem, the further exploration of turtle constructions in general is likely to bear significant fruit.

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