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Wormholes and masses for Goldstone bosons

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ABSTRACT: There exist non-trivial stationary points of the Euclidean action for an axion particle minimally coupled to Einstein gravity, dubbed wormholes. They explicitly break the continuous global shift symmetry of the axion in a non-perturbative way, and generate an effective potential that may compete with QCD depending on the value of the axion decay constant. In this paper, we explore both theoretical and phenomenological aspects of this issue. On the theory side, we address the problem of stability of the wormhole solutions, and we show that the spectrum of the quadratic action features only positive eigenvalues. On the phenomenological side, we discuss, beside the obvious application to the QCD axion, relevant consequences for models with ultralight dark matter and black hole superradiance. We conclude discussing wormhole solutions for a generic coset and the potential they generate.

KEYWORDS: Global Symmetries, Beyond Standard Model, Nonperturbative Effects

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1 Introduction

The explicit breaking of global symmetries due to gravitational effects is a topic of great relevance in physics, both for theoretical and phenomenological reasons. Its study might shed light on the reason why gravity is so different from the other fundamental interactions, and what rôle did these special features play in the history of the Universe.

Let us focus for simplicity and definiteness on the case of a U(1) Peccei-Quinn (PQ) global symmetry that is spontaneously broken at some scale f_a by the vacuum expectation value (VEV) of a complex scalar field Φ . By introducing the polar parametrization $\Phi = \rho e^{i\phi/f_a}$, the VEV of the complex field is $\langle |\Phi| \rangle = f_a$, and the axion can be identified with the angular direction ϕ — the Nambu-Goldstone boson arising from the spontaneous

symmetry breaking. Below the scale f_a , the axion enjoys the non-linearly realized global U(1) symmetry

$$\phi \rightarrow \phi + \alpha f_a, \quad (1.1)$$

for arbitrary values of the parameter α describing the shift in field space. The conserved Noether current implied by the symmetry in eq. (1.1) is $J^\mu = f_a \partial^\mu \phi$, and the associated charge $\mathcal{Q}$ generates the symmetry transformation. For the purpose of the present discussion, it is important to notice that the subgroup of eq. (1.1) given by the discrete shift

$$\phi \rightarrow \phi + 2k\pi f_a, \quad k \in \mathbb{Z}, \quad (1.2)$$

represents a gauge symmetry of the system. This symmetry is inherent to the mere definition of the axion field in terms of an angular dynamical variable. Referring for definiteness to the U(1) case discussed above, the discrete shift in eq. (1.2) maps the complex field Φ into itself, and thus corresponds to a redundancy in the description of the physical degrees of freedom.

At the QCD scale Λ_{QCD} , Yang-Mills instanton effects break the continuous shift in eq. (1.1) into its discrete subgroup $\phi \rightarrow \phi + 2k\pi f_a$ generating the potential

$$V_{\text{QCD}}(\phi) = -\Lambda_{\text{QCD}}^4 \cos\left(\frac{\phi}{f_a}\right), \quad (1.3)$$

minimized at $\phi = 0$ in agreement with the axion solution of the strong CP problem.

What does gravity do with global symmetries, like the axion shift symmetry in eq. (1.1)?

A number of ‘folk theorems’ state the impossibility to have global symmetries in a consistent theory of quantum gravity. To support this hypothesis, there exists a general semi-classical argument based on black hole physics. The reasoning is that if some amount of global charge is thrown into a black hole, then the subsequent thermal decay of the black hole into photons and gravitons via Hawking radiation destroys the global charge without any possibility to reconstruct it in the final state, thus defining a process by means of which the global charge is violated. To this respect, one also understands the crucial difference between global and local symmetries. Local symmetries obey the Gauss’s law, and any observer outside the black hole can determine its charge. The electric charge of a Kerr-Newman black hole represents an explicit example of this fact. On the contrary, there is no Gauss’s law associated with global symmetries, and when charged particles are thrown into the black hole, there is no way to track them from the outside: the global charge appears to be deleted, in obvious contradiction to its conservation.

In addition to the aforementioned theorem related to black hole physics, various arguments in perturbative string theory [1] — where global symmetries on the world-sheet become gauge symmetries in the target space — and AdS/CFT [2] — where global symmetries on the boundary correspond to gauge symmetries in the bulk — seem to corroborate this common belief.

Motivated by these arguments, many authors studied the consequences of the explicit breaking induced by higher-dimensional operators suppressed by the Planck scale

$M_{\text{Pl}} = \sqrt{1/G_N}$ of the form [3, 4]

$$V_{\text{gravity}}(\Phi) = g_{2m+n} \frac{|\Phi|^{2m} \Phi^n}{M_{\text{Pl}}^{2m+n-4}} + h.c., \quad (1.4)$$

with mass-dimension $2m + n$, and generic coupling g_{2m+n} that in principle is of order unity. Needless to say, the impact of these operators is catastrophic. To give an idea, in the QCD axion case including dimension-5 symmetry breaking operator induced by Planck scale physics would require a coupling of order $|g_5| \lesssim 10^{-55}$ in order to avoid dangerous CP violation effects for $f_a \sim 10^{12}$ GeV. This is of course unacceptable, since it introduces a fine-tuning by far more severe than the one the axion claims to solve.

The higher-dimensional operators in eq. (1.4) therefore represent, if present, a threat for the axion solution of the strong CP problem. To solve the issue, it is possible to tailor suitable extensions of the simple U(1) model discussed at the beginning of this section. For instance, if the axion global symmetry is an accidental symmetry descending from an exact (discrete) gauge symmetry, then the problematic higher-dimensional operators can be forbidden to very high order [5–12] (a similar conclusion remains true considering the axion global symmetry as an accidental symmetry descending from exact discrete global symmetries [13, 14]).

However, before engineering possible solutions, it is important to pose the following question: does gravity really generate power-suppressed symmetry breaking operators like those in eq. (1.4)?

Let us tackle the problem from the simplest perspective, and consider the action describing an axion field minimally coupled to Einstein gravity. In this setup, it is possible to show that the global symmetry in eq. (1.1) remains intact at any finite order in a perturbative expansion in G_N , and no power-suppressed operators are generated. Albeit surprising, this is not in contrast with the arguments presented at the beginning of this section about the breaking of global symmetries due to gravity. What is the origin of this conundrum? First of all, the reader should keep in mind that the ‘folk theorems’ mentioned above have to do with black holes, that is with non-perturbative objects. This fact suggests that the breaking of global symmetries induced by gravity is to be found at the non-perturbative level, and the lack of perturbative effects is not, after all, surprising. At the same time, this may sound discouraging — in particular from a phenomenological perspective — since referring to non-perturbative gravitational effects seems to be nothing but a vague and unclear indication. Is there a computable non-perturbative effect showing indisputably that gravity breaks explicitly global symmetries?

A careful analysis confirms the expectation sketched above, and gives a positive answer to the aforementioned question. Gravity breaks the global symmetry in eq. (1.1) down to its discrete version in eq. (1.2) in a non-perturbative way [15]. In more detail, Euclidean wormhole solutions [16] swallow axionic charge, break the shift symmetry it generates, and create a potential for the axion that may compete, depending on the value of the axion decay constant, with the QCD effect in eq. (1.3). These effects are controllably calculable in the context of Einstein gravity. Before proceeding, an important caveat must be stressed and should be always kept in mind throughout the reading of this paper. Wormhole physics

is described by means of a semi-classical analysis based on the Euclidean path integral approach to quantum gravity. The rationale here is the following. We cannot compute the path integral since we do not have a theory of quantum gravity. Nevertheless, an approximate solution to the path integral can be developed by considering classical Euclidean solutions of the theory and then performing a perturbative expansion around them. The underlying hope is that — since the importance of a given path integral contribution is weighted by the exponential of minus its action — the classical Euclidean solutions with minimum action dominate the path integral. The validity of such approach is of course subject to important theoretical restrictions: approaching the Planck scale, the new degrees of freedom implied by the UV completion of general relativity will start playing an important rôle, thus undermining the validity of the semi-classical analysis. However, as we shall discuss, the virtue of wormhole physics is that the classical solutions are controlled by one single length scale. By comparing the latter with the Planck scale one can, therefore, extract the limit of validity of the classical expansion.

This paper is structured as follows. In section 2 we review the Euclidean wormhole solutions [16] in the context of Einstein gravity. Section 3 contains the computation for the spectrum of the quadratic action, and in section 4 the effective potential for a generic U(1) Goldstone boson generated by Euclidean wormholes can be found. In section 5 possible UV completions are discussed, and their impact on the existence of wormhole solutions assessed. This concludes the theoretical part of this work. In section 6 we illustrate several phenomenological implications, including the QCD axion and the case of ultralight scalar dark matter. In section 7, we derive wormhole solutions for a generic coset and look into $O(n+1)/O(n)$ as an specific example. Conclusions can be found in 8.

2 The Euclidean wormhole solution

In this section we review the wormhole instanton solution first discussed in [16]. The computation we are about to discuss is not novel and can be found in the literature (see, e.g., [15, 17–20]). However, we consider useful and important to re-elaborate its derivation both for completeness and to highlight the points that will matter most for the rest of this work.

2.1 The bulk action

The starting point is the Euclidean action of a three-form H coupled to Einstein gravity¹

$$\mathcal{S}_E = \int d^4x \sqrt{g} \left(-\frac{M_{\text{Pl}}^2}{16\pi} \mathcal{R} + \frac{\mathcal{F}}{2} H_{\mu\nu\rho} H^{\mu\nu\rho} \right), \quad (2.1)$$

where $M_{\text{Pl}} = 1/\sqrt{G_N} \simeq 1.22 \times 10^{19}$ GeV is the Planck mass, and $\mathcal{F} \equiv 1/(3! f_a^2)$, with f_a the PQ scale. The (dimensionless) pseudo-scalar axion field θ is related to the three-form H

¹Unlike the case of Yang-Mills theories, the Euclidean action in pure gravity is unbounded from below. Naïvely, this fact seems to prevent from a well-defined path integral formulation of gravity. We shall discuss further this issue in section 3.1.

via the relation $H_{\mu\nu\rho} = f_a^2 \epsilon_{\mu\nu\rho\sigma} (\partial^\sigma \theta)$.² The canonically normalized axion field is $\phi \equiv f_a \theta$. The variation of $\mathcal{S}_E$ w.r.t. the metric is

$$\delta \mathcal{S}_E|_{\delta g_{\mu\nu}} = \int d^4x \sqrt{g} \left[\frac{M_{\text{Pl}}^2}{16\pi} \left(-\frac{1}{2} g^{\mu\nu} \mathcal{R} + \mathcal{R}^{\mu\nu} \right) + \frac{\mathcal{F}}{4} g^{\mu\nu} H^2 - \frac{3\mathcal{F}}{2} H^\mu_{\alpha\beta} H^{\nu\alpha\beta} \right] \delta g_{\mu\nu}, \quad (2.4)$$

and the corresponding Einstein equation reads

$$\mathcal{G}^{\mu\nu} \equiv \mathcal{R}^{\mu\nu} - \frac{1}{2} g^{\mu\nu} \mathcal{R} = \frac{8\pi}{M_{\text{Pl}}^2} T_H^{\mu\nu}, \quad T_H^{\mu\nu} \equiv \mathcal{F} \left(3H^\mu_{\alpha\beta} H^{\nu\alpha\beta} - \frac{1}{2} g^{\mu\nu} H^2 \right). \quad (2.5)$$

By taking the trace of eq. (2.5) we find $-\mathcal{R} = 8\pi G_N \mathcal{F} H^2$. The equation of motion for the 3-form takes the form

$$\delta \mathcal{S}_E|_{\delta B_{\nu\rho}} = \int d^4x \delta B_{\nu\rho} \left[-3\mathcal{F}^2 \partial_\mu (\sqrt{g} H^{\mu\nu\rho}) \right] = 0,$$

where we write the three form as the field strength of an antisymmetric tensor $B_{\mu\nu}$, i.e. $H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\rho B_{\mu\nu} + \partial_\nu B_{\rho\mu}$.

We now look for a spherically symmetric solution with the following ansatz

$$ds^2 = \alpha^2(r) dr^2 + \beta^2(r) d\Omega_{D-1, \mathcal{K}}^2, \quad H_{\mu\nu\rho} = \mathcal{H}(r) \epsilon_{ijk}, \quad (2.6)$$

where r is the time coordinate in Euclidean space, and the $D-1$ space has curvature $\mathcal{K}$. Without loss of generality, we write $d\Omega_{D-1, \mathcal{K}}^2 = \hat{g}_{ij} d\varphi^i d\varphi^j$, with metric determinant $\sqrt{g} = \alpha \beta^3 \sqrt{\hat{g}}$. $\mathcal{K} = 1$ corresponds to a closed geometry, and for $D = 4$ $d\Omega_{D-1, \mathcal{K}=1}^2$ reduces to the ordinary three-sphere whose metric can be parametrized in terms of three angles as $d\Omega_{3,1}^2 = (d\psi^2, \sin^2 \psi d\phi^2, \sin^2 \psi \sin^2 \phi d\varphi^2)$. In eq. (2.6) the indices in ϵ_{ijk} run over the three-dimensional space $d\Omega_{3, \mathcal{K}}^2$, and they are raised and lowered by the induced metric $\hat{g}$. The Einstein equations are

$$G_{rr} = -\frac{3\mathcal{K}\alpha^2}{\beta^2} + \frac{3(\beta')^2}{\beta^2} = \frac{8\pi}{M_{\text{Pl}}^2} (-3\mathcal{F}\alpha^2 \mathcal{H}^2), \quad (2.7)$$

$$G_{ij} = \hat{g}_{ij} \left[-\mathcal{K} - \frac{2\beta\beta'\alpha'}{\alpha^3} + \frac{(\beta')^2}{\alpha^2} + \frac{2\beta\beta''}{\alpha^2} \right] = \hat{g}_{ij} \frac{8\pi}{M_{\text{Pl}}^2} (3\mathcal{F}\beta^2 \mathcal{H}^2). \quad (2.8)$$

²Notice that ϵ is a genuine tensor under general coordinate transformations, and we have the relation

$$\epsilon_{i_1, \dots, i_n} = \sqrt{|g|} \varepsilon_{i_1, \dots, i_n}, \quad \epsilon^{i_1, \dots, i_n} = \frac{1}{\sqrt{|g|}} \varepsilon^{i_1, \dots, i_n}, \quad (2.2)$$

where the tensor density ε of weight -1 is the usual total antisymmetric Levi-Civita symbol, defined to be ± 1 and 0 in *all* frames. In eq. (2.2), $\sqrt{|g|} = \sqrt{g}$ if $t = 0$ (or more generally if t is even), and $\sqrt{|g|} = \sqrt{-g}$ if $t = 1$ (or more generally if t is odd).

Finally, in full generality, we have

$$\epsilon_{i_1, \dots, i_k, i_{k+1}, \dots, i_n} \epsilon^{i_1, \dots, i_k, j_{k+1}, \dots, j_n} = (-1)^t k! \delta_{i_{k+1}, \dots, i_n}^{j_{k+1}, \dots, j_n}, \quad \text{with} \quad \delta_{\nu_1, \dots, \nu_p}^{\mu_1, \dots, \mu_p} \equiv \begin{vmatrix} \delta_{\nu_1}^{\mu_1} & \dots & \delta_{\nu_1}^{\mu_p} \\ \vdots & \ddots & \vdots \\ \delta_{\nu_p}^{\mu_1} & \dots & \delta_{\nu_p}^{\mu_p} \end{vmatrix}. \quad (2.3)$$

Notice that the field ansatz in eq. (2.6) automatically solves the equation of motion $\partial_\mu (\sqrt{g} H_{\alpha\nu\rho} g^{\alpha\mu}) = 0$. On the contrary, the Bianchi identity $\epsilon^{\mu\nu\rho\sigma} (\partial_\rho H_{\sigma\mu\nu}) = 0$ provides a non-trivial constraint. Using the field ansatz, and bearing in mind that $\epsilon_{ijk} = \beta^3 \sqrt{\tilde{g}} \varepsilon_{ijk}$ (see footnote 2), we find $\partial_r (H \epsilon_{ijk}) = \partial_r (\mathcal{H} \beta^3 \sqrt{\tilde{g}} \varepsilon_{ijk}) = 0$. As a direct consequence, we have the r dependence $\mathcal{H}(r) = \mathcal{H}_0 / \beta^3(r)$, with $\mathcal{H}_0$ dimensionless constant. To make contact with the original solution proposed in [16], we now restrict the analysis to the three-sphere S_3 , with $\mathcal{K} = 1$ and $\beta^2(r) = r^2$. The constant $\mathcal{H}_0$ can be fixed by normalizing the integral of the field strength over S_3 . We have

$$\int_{S_3} \mathcal{H}_0 d\Omega_{3,1} = 2\pi^2 \mathcal{H}_0 = n, \quad (2.9)$$

where — as we shall see later — n is an integer number. We can now use eq. (2.7). Solving for α , we find

$$\alpha^2(r) = \frac{1}{1 - L^4/r^4}, \quad L \equiv \left(\frac{n^2}{3\pi^3 M_{\text{Pl}}^2 f_a^2} \right)^{1/4}. \quad (2.10)$$

The metric in eqs. (2.6), (2.10) describes the Giddings-Strominger wormhole configuration. The wormhole connects two asymptotically flat Euclidean region. At any fixed r , the cross-section of the wormhole is a three-sphere. The length scale L in eq. (2.10) defines the minimal radius of the wormhole throat. Let us note that there is no singularity at $L = 0$; this is indeed nothing but a coordinate singularity, as we shall discuss later.

Using the relation $H_{\mu\nu\rho} = f_a^2 \epsilon_{\mu\nu\rho\sigma} (\partial^\sigma \theta) = \epsilon_{\mu\nu\rho\sigma} J^\sigma$ and the field ansatz in eq. (2.6) we can compute the axion charge stored in a cross-section of the wormhole throat. We find

$$\mathcal{Q} \equiv \int_{S^3} f_a^2 (\partial^r \theta) \beta^3(r) d\Omega_{3,1} = \int_{S_3} \mathcal{H}_0 d\Omega_{3,1} = 2\pi^2 \mathcal{H}_0 = n. \quad (2.11)$$

The wormhole throat is therefore characterized by the axion charge n . This characteristic offers the possibility to consider the wormhole as the combination of an instanton with axion charge $+n$ and an anti-instanton with axion charge $-n$. Furthermore, as we shall discuss at the end of this section, n is quantized, and it only takes integer values. As discussed in [16], separating the wormhole in two halves has an important topological interpretation. On the one hand, the wormhole interpolates between two $\mathbb{R}^3$ geometries widely separated in time. On the other one, the instanton (anti-instanton) represents a topology change from $\mathbb{R}^3$ to $\mathbb{R}^3 \oplus S_3$ (from $\mathbb{R}^3 \oplus S_3$ to $\mathbb{R}^3$).

Instead of considering the wormhole as connecting two separate asymptotic Euclidean spaces — a configuration that has an unclear physical interpretation — it is possible to consider wormholes that connect back to the same space, as shown in figure 1. By slicing the wormhole in half, the instanton/anti-instanton represents, as before, a topology change from $\mathbb{R}^3$ to $\mathbb{R}^3 \oplus S_3$ and viceversa. From eq. (2.1), we can compute the action of an instanton

Figure 1. Pictorial representation of an euclidean wormhole with endpoints connected to the same asymptotically flat space. The Euclidean time runs in the direction of the arrow. At each instant of time, two space dimensions have been suppressed. The cross-section of the wormhole throat is a three-sphere S_3 .

configuration with charge n . Using $\mathcal{R} = -8\pi G_N \mathcal{F} H^2$, we find³

$$\mathcal{S}_{\text{inst}} = \int d^4x \sqrt{g} \mathcal{F} H^2 = \frac{n^2}{4\pi^4 f_a^2} \int_L^\infty dr \int d\Omega_{3,1} \frac{1}{r^3 \sqrt{1 - \frac{L^4}{r^4}}} = \frac{\sqrt{3\pi} n M_{\text{Pl}}}{8 f_a} = \frac{3\pi^2 M_{\text{Pl}}^2 L^2}{8}. \quad (2.12)$$

This is the right moment to do some useful dimensional analysis, and it is convenient to restore the appropriate powers of $\hbar$ (while keeping $c = 1$). Equivalently, we can introduce a unit of energy E and length L , with $[\hbar] = EL$. Natural units correspond to $E = L^{-1}$. It is straightforward to realize — using the fact that a generic Lagrangian density has dimension $[\mathcal{L}] = EL^{-3}$ — that the canonically normalized axion field has dimension $[a] = E^{1/2} L^{-1/2}$. Similarly, the dimension of a coupling constant (like a gauge coupling) is $[g] = E^{-1/2} L^{-1/2}$. The axion decay constant f_a is a VEV, and it shares the same dimensionality of the electroweak VEV and the Planck scale: $[f_a] = [v] = [M_{\text{Pl}}] = E^{1/2} L^{-1/2}$. From eq. (2.11) the dimensionality of the axion charge is $[n] = EL$, and it corresponds to an inverse coupling squared. Finally, from eq. (2.10) we see that L is a genuine length scale. From this simple analysis we learn a number of interesting things. First, we can exploit the analogy with the electroweak theory. The Fermi constant G_F (or equivalently the electroweak VEV) is related to the ratio between the electroweak mass (defined here by the mass of the W boson, and corresponding to the physical threshold of energy $E \sim M_W$ at which the new degrees of freedom in the electroweak sector become dynamical) and the electroweak coupling constant g_W by means of $[v] = [G_F]^{-1/2} = [M_W]/[g_W]$. Similarly, in gravity we expect the relation

$$[M_{\text{Pl}}] = [G_N]^{-1/2} = \frac{[M_S]}{[g_S]}, \quad (2.13)$$

where we introduced a string mass scale M_S and a string coupling g_S . If the UV completion of general relativity is weakly coupled, we expect new degrees of freedom to occur

³Notice that our result agrees with the original computation of [16], and with [15, 18]. However, we report a factor of 2 discrepancy if compared with the result of [19, 20], where the action of the half-wormhole configuration is half of $\mathcal{S}_{\text{inst}}$ in eq. (2.12).

even below the Planck scale. If, on the contrary, the UV completion of general relativity is strongly coupled, $g_S \gtrsim 1$, we expect the new dynamics of quantum gravity to occur above M_{Pl} . We shall return on this point later, discussing the validity of the wormhole solutions (see section 5.2). Second, from eq. (2.12) we recover the correct dimensionality of an instanton action since we have that $[\mathcal{S}_{\text{inst}}] = 1/[g]^2 = [\hbar]$. On general ground, this also explains why instantons are non-perturbative. If $g \rightarrow 0$, $\exp(-\mathcal{S}_{\text{inst}}/\hbar)$ goes to zero faster than any power of g , and instanton effects cannot be captured at any finite order in perturbation theory.

Finally, notice that the wormholes with action given in eq. (2.12) realize the Weak Gravity Conjecture (WGC) proposed in [21]. The generalization of the WGC to the case of axions states that for an axion with decay constant f_a there exists an instanton with action $\mathcal{S}_{\text{inst}} \lesssim M_{\text{Pl}}/f_a$. In the context of Einstein gravity, this is precisely the wormhole with $n = 1$ and $\mathcal{S}_{\text{inst}} \simeq 0.38 M_{\text{Pl}}/f_a$.

Let us now explore in more detail our solution. The wormhole metric is conformally equivalent to a flat metric on $S_3 \times \mathbb{R}$. The coordinate transformation

$$\left(\frac{d\tau}{dr}\right)^2 = \frac{r^2}{r^4 - L^4} \quad \Longrightarrow \quad \tau(r) = \frac{1}{2} \ln \left[\frac{r^2 + \sqrt{r^4 - L^4}}{L^2} \right], \quad (2.14)$$

brings the metric into the form

$$ds^2 = a^2(\tau) (d\tau^2 + d\Omega_{3,1}^2), \quad a^2(\tau) = L^2 \cosh(2\tau). \quad (2.15)$$

Notice that $r = L$ is mapped to $\tau = 0$ and in this coordinate system is plain to see that there is no singularity and the wormhole connects the two asymptotically flat regions at $\tau = \pm\infty$. With respect to this metric, the Einstein equations and the equation of motion for the axion field, which can be obtained from taking $\alpha = \beta = a$ in eqs. (2.7), (2.8), are

$$-3 + 3 \left(\frac{a'}{a}\right)^2 = -\frac{f_a^2 \kappa}{2} (\theta')^2, \quad 2 \frac{a''}{a} - \left(\frac{a'}{a}\right)^2 - 1 = \frac{f_a^2 \kappa}{2} (\theta')^2, \quad \theta'' + (2a'/a)\theta' = 0, \quad (2.16)$$

where we defined $\kappa \equiv 8\pi/M_{\text{Pl}}^2$. From the first Einstein equation, we find

$$f_a^2 \kappa (\theta')^2 = \frac{6}{\cosh^2(2\tau)}. \quad (2.17)$$

Integrating over τ , we find $f_a \theta(\tau) = \sqrt{6/\kappa} \arctan[\tanh(\tau)]$. In the left panel of figure 2 we show the instanton solution $\theta(\tau)$. Asymptotically, the axion field configuration approaches the constant values $f_a \theta(\tau)/\sqrt{6/\kappa} \xrightarrow{\tau \rightarrow \pm\infty} \pm\pi/4$, and features a sharp transition in between. Notice that the asymptotic limit is exponentially fast, $\tanh(\tau) \xrightarrow{\tau \rightarrow \pm\infty} \pm(1 - e^{-2\tau})$. Since instantons are well-localized objects, there are also approximate solutions consisting of strings of widely separated instantons and anti-instantons centered at $\tau_1, \dots, \tau_m$. For illustrative purposes, we show a multi-instanton solution in the right panel of figure 2. Since the m instantons are widely separated, the classical action is just $m\mathcal{S}_{\text{inst}}$.

Finally, we close this section with the following remark. As already anticipated, the axion charge in eq. (2.11) turns out to be quantized. This property can be understood as

Figure 2. Axion field profile corresponding to the wormhole solution. The blue and red colors describe the instanton and anti-instanton part of the solution (see figure 1). A string of widely separated instantons and anti-instantons (right panel) provides an approximate stationary point of the Euclidean action (which becomes an exact solution of the equation of motion only asymptotically, in the limit of infinite separation).

follows. We consider the action for the axion field in the background metric of eq. (2.15), and, after introducing the new variable $d\tau/d\sigma \equiv a^2$ and considering radially symmetric field configurations $\theta(\tau(\sigma))$, we find

$$\mathcal{S}_\theta = 2\pi^2 \int d\sigma \frac{f_a^2}{2} \left(\frac{d\theta}{d\sigma} \right)^2. \quad (2.18)$$

With respect to the new variable, the equation of motion of the axion field is simply given by $d^2\theta/d\sigma^2 = 0$. Equivalently, $d\theta/d\sigma = \text{const} = n/2\pi^2 f_a^2$. We can now expand eq. (2.18) around the instanton solution, $\theta \rightarrow \theta + \delta\theta$. We find, using the equation of motion, the following change in the action⁴

$$\delta\mathcal{S}_\theta = 2\pi^2 f_a^2 \int d\sigma \frac{d}{d\sigma} \left(\delta\theta \frac{d\theta}{d\sigma} \right) = 2\pi^2 f_a^2 \delta\theta \left. \frac{d\theta}{d\sigma} \right|_{\text{boundary}} = n\delta\theta. \quad (2.19)$$

In the presence of wormholes, the action is no longer invariant under the generic shift $\theta \rightarrow \theta + \delta\theta$. We now restrict to the discrete transformations $\delta\theta = 2k\pi$, with $k \in \mathbb{Z}$. Since θ is an angular variable, we know that $\theta \rightarrow \theta + 2k\pi$ corresponds to the same physical point, as discussed in section 1. This is a gauge redundancy, and, as such, it cannot be broken. The simple observation is that quantum mechanical transition amplitudes involve $e^{i\mathcal{S}_\theta}$, and the phase exponential must be gauge invariant. This means that the only change $\delta\mathcal{S}_\theta$ that can be tolerated corresponds to an integer multiple of 2π . We therefore find the quantization condition $n \in \mathbb{Z}$. The quantization of the wormhole charge $\mathcal{Q}$ — and the subsequent invariance of the action under the transformation $\theta \rightarrow \theta + 2k\pi$, with $k \in \mathbb{Z}$ — is a crucial point, since it is related to the fact that in the presence of wormholes the continuous shift symmetry in eq. (1.1) is broken down to its discrete subgroup in eq. (1.2). We shall elaborate more on this point in section 4. The quantization of the wormhole charge $\mathcal{Q}$ is tightly related to the periodicity of the axion field.

⁴For simplicity, we consider the perturbation $\delta\theta$ in the boundary at ∞ , setting $\delta\theta$ at the throat to 0; including both changes nothing as far as the quantization condition is concerned.

2.2 The Gibbons-Hawking-York boundary term

In the presence of a manifold with boundary, the Einstein-Hilbert action in eq. (2.1) must be supplemented by a boundary term so that the variational principle is well-defined [24–26].

It is therefore necessary to add the Gibbons-Hawking-York boundary term, whose general form is given by

$$\mathcal{S}_{\text{GHY}} = -\frac{M_{\text{Pl}}^2}{8\pi} \int_{\partial M} d^3\vec{x} \sqrt{h} (K - K_0), \quad (2.20)$$

where h is the determinant of the induced metric on the boundary manifold ∂M , K is the trace of the extrinsic curvature tensor, and K_0 is the trace of the extrinsic curvature tensor of the same boundary embedded in flat spacetime. The explicit computation of $\mathcal{S}_{\text{GHY}}$ is straightforward. We find [15]

$$\mathcal{S}_{\text{GHY}} = -\frac{3M_{\text{Pl}}^2\pi}{4} |aa' - a^2|_{\tau=0}^{\tau=\infty} = -\frac{3M_{\text{Pl}}^2\pi L^2}{4} = -\left(\frac{2}{\pi}\right) \frac{\sqrt{3\pi}nM_{\text{Pl}}}{8f_a}, \quad (2.21)$$

where only the boundary at the wormhole throat gives a non-vanishing contribution. Taking into account the boundary term, the action of an instanton configuration with charge n is

$$\mathcal{S}_{\text{inst}} = \frac{\sqrt{3\pi}nM_{\text{Pl}}}{8f_a} \left(1 - \frac{2}{\pi}\right). \quad (2.22)$$

3 Quadratic action and the absence of negative modes

In order to understand the true nature of the wormhole solution, it is necessary to inspect the action at the quadratic order. Let us clarify this point with an explicit example. In Euclidean time, the quantum probability amplitude from the initial state $|i\rangle$ at time $-T/2$ to the final state $|f\rangle$ at time $T/2$ is given by

$$\langle f|e^{-TH}|i\rangle = \sum_n e^{-TE_n} \langle f|n\rangle \langle n|i\rangle \approx \frac{\mathcal{N}}{\sqrt{\det' \left(\frac{\delta^2 \mathcal{S}_{\text{E}}}{\delta x^2} \right)}} e^{-\mathcal{S}_{\text{E}}[x_{\text{cl}}(\tau)]}, \quad (3.1)$$

where $\mathcal{N}$ is a normalization constant, $\mathcal{S}_{\text{E}}[x_{\text{cl}}(\tau)]$ is the Euclidean action computed for the classical instanton solution $x_{\text{cl}}(\tau)$, and the prefactor encodes the quantum fluctuation around the classical trajectory. The latter, as a result of a Gaussian integration, takes the shape of a determinant with a prime denoting the subtraction of zero eigenvalues. If we take T to be very large, the lowest energy state dominates the sum, and we can extract the ground state energy from the computation of the path integral. The spectrum of the differential operator $\delta^2 \mathcal{S}_{\text{E}}/\delta x^2$ is therefore crucial. In the presence of one (or an odd number of) negative eigenvalue(s), there is an imaginary contribution to the vacuum energy, and the instanton plays the rôle of a bounce. If the spectrum of the quadratic action is positive (or if the number of negative eigenvalues is even) the vacuum energy is real, and the instanton describes the transition between degenerate vacua.

3.1 First-order action for homogeneous scalar perturbations

In this section we study, motivated by the previous discussion, the spectrum of the quadratic action describing perturbations around the Euclidean wormhole solution. Our approach is close in spirit to the usual analysis of cosmological curvature perturbations in the case of a scalar field coupled to gravity in a Friedmann-Robertson-Walker metric [27]. This study requires the canonical formalism in order to correctly get rid of the non-dynamical degrees of freedom, as in any other gauge theory. The canonical formulation, although non-explicitly, preserves gauge invariance.

However, let us also stress that we depart from the traditional analysis, usually carried out in Minkowski space, for the following reasons: (i) wormhole solutions are Euclidean objects, and as such a clear-cut computation of the spectrum of the quadratic action must be performed in Euclidean space; (ii) we use the three-form language to correctly deal with perturbations in the scalar field. As stressed in section 2.1, the three-form language captures the correct description of the wormhole solution, and it represents, therefore, the best-suited tool to compute perturbations around it.

We start by considering metric perturbations in Euclidean space around the background line element $ds^2 = a^2(\tau)(d\tau^2 + \hat{g}_{ij}dx^i dx^j)$. Including fluctuations, in full generality we write $ds^2 = a^2(\tau) [(1 + 2A)d\tau^2 + 2B_i dx^i d\tau + (\hat{g}_{ij} + h_{ij})dx^i dx^j]$. This perturbed line element can be further analyzed by means of the scalar-vector-tensor decomposition, which separates the fluctuations into components according to their transformations under spatial rotations. In particular, the three-vector B_i splits into the gradient of a scalar, and a divergence-free vector $B_i = \partial_i B + \hat{B}_i$ while the symmetric tensor h_{ij} can be decomposed into a scalar part, a divergence-free vector, and a trace-free transverse tensor $h_{ij} = -2\psi\hat{g}_{ij} + 2\partial_{(i}\partial_{j)}E + 2\partial_{(i}\hat{E}_{j)} + t_{ij}$.⁵ The advantage of this decomposition is that scalar, vector and tensor Einstein equations are decoupled at the linear order and can therefore be treated separately. Vector perturbations are pure gauge modes, and they carry no dynamics [27]. As far as tensor perturbations are concerned, they do not couple to scalar modes, and we can therefore rely on standard computations of gravitational waves in instanton background [28]. We found a positive spectrum.

We focus here on scalar perturbations. The perturbed line element is therefore

$$ds^2 = a^2(\tau) \left\{ (1 + 2A)d\tau^2 + 2\partial_i B dx^i d\tau + [\hat{g}_{ij}(1 - 2\psi) + 2\partial_{(i}\partial_{j)}E] dx^i dx^j \right\}. \quad (3.2)$$

Furthermore, in the following we focus on the spatially homogeneous $O(4)$ invariant modes. As discussed in [29, 30], the homogeneous modes are those that are potentially responsible for the presence of negative eigenvalues in the spectrum of the quadratic action, and they need to be carefully analyzed. Indeed, in [31] a negative mode was found in the spectrum of the homogeneous scalar perturbations. In the following, we shall restrict to this case. The perturbed metric reduces to

$$ds^2 = \alpha^2(\tau)d\tau^2 + \beta^2(\tau)\hat{g}_{ij}dx^i dx^j \quad (3.3)$$

⁵We use the following short-hand notation: $\partial_{(i}\partial_{j)}E \equiv (\partial_i\partial_j - \frac{1}{3}\hat{g}_{ij}\nabla^2)E$, and $\partial_{(i}\hat{E}_{j)} \equiv (\partial_i\hat{E}_j + \partial_j\hat{E}_i)/2$.

where we set the short-hand notation $\alpha^2(\tau) \equiv a^2(\tau)[1 + 2A(\tau)]$, $\beta^2(\tau) \equiv a^2(\tau)[1 - 2\psi(\tau)]$. We start considering the gravitational part of the action in eq. (2.1) proportional to the Ricci scalar, $\mathcal{S}_E^R \equiv - \int d^4x \sqrt{g} \mathcal{R} / 2\kappa$. At the quadratic order, we find

$$\delta_2 \mathcal{S}_E^R = \frac{3}{2\kappa} \int d^4x \sqrt{\hat{g}} \left\{ A^2 [\mathcal{K}a^2 - 3a'^2] + A [\psi(2\mathcal{K}a^2 - 6a'^2) - 4aa'\psi'] \right. \\ \left. - 2a^2\psi'^2 + \psi^2(\mathcal{K}a^2 - a'^2 + 2aa'') \right\}, \quad (3.4)$$

where $\mathcal{K}$ is the curvature of the space-like manifold as defined in eq. (2.6). The derivation of eq. (3.4) implies integration by parts which is needed to get rid of second derivatives of the metric perturbations. The purpose of the Gibbons-Hawking-York boundary term is precisely cancelling the total derivative terms produced in this way so that the above equation captures the full second order gravity action. We now move to consider the action describing the total antisymmetric three form $H_{\mu\nu\rho}$. The latter is the field strength of an antisymmetric tensor $B_{\mu\nu}$, and we have the action

$$\mathcal{S}_E^B = \int d^4x \sqrt{g} \frac{\mathcal{F}}{2} H_{\mu\nu\rho} H^{\mu\nu\rho} = \int d^4x \frac{\sqrt{g}}{4f_a^2} [(\partial_\mu B_{\nu\rho})(\partial^\mu B^{\nu\rho}) + 2(\partial_\rho B_{\mu\nu})(\partial^\mu B^{\nu\rho})], \quad (3.5)$$

with $\sqrt{g} = \alpha\beta^3\sqrt{\hat{g}}$. Note that the covariant derivatives reduce to ordinary ones due to the antisymmetry in the Lorentz indices of forms. We now separate τ - and spatial-dependent components to find

$$\mathcal{S}_E^B = \int d^4x \frac{\alpha\beta^3\sqrt{\hat{g}}}{4f_a^2} \left\{ [\partial_0 B_{ij} - (\partial_i B_{0j} - \partial_j B_{0i})] [\partial^0 B^{ij} - (\partial^i B^{0j} - \partial^j B^{0i})] + \frac{1}{3} H_{ijk} H^{ijk} \right\}, \quad (3.6)$$

where $H_{ijk} = \partial_i B_{jk} + \partial_j B_{ki} + \partial_k B_{ij}$. Eq. (3.6) is well suited for perturbations. Now we expand on the 2-form field, $B_{\mu\nu} = \bar{B}_{\mu\nu} + b_{\mu\nu}$, $H_{ijk} = \bar{H}_{ijk} + h_{ijk}$, around the background of the wormhole solution, i.e. $\partial_0 \bar{B}_{ij} = 0$, $\bar{B}_{0i} = 0$ and $\bar{H}_{ijk} = \epsilon_{ijk} \mathcal{H}_0 / a^3 = \epsilon_{ijk} \sqrt{\hat{g}} \mathcal{H}_0$ with $\mathcal{H}_0 = n/2\pi^2$ as given in eq. (2.11), and eq. (3.6) yields the quadratic action

$$\delta_2 \mathcal{S}_E^B = \int d^4x \frac{\sqrt{\hat{g}} a^4}{4f_a^2} \left\{ [\partial_0 b_{ij} - (\partial_i b_{0j} - \partial_j b_{0i})] [\partial^0 b^{ij} - (\partial^i b^{0j} - \partial^j b^{0i})] + \frac{1}{3} h_{ijk} h^{ijk} \right. \\ \left. + \frac{2}{3} (A + 3\psi) h_{ijk} \bar{H}^{ijk} + \frac{1}{6} \bar{H}_{ijk} \bar{H}^{ijk} (-A^2 + 6A\psi + 15\psi^2) \right\}. \quad (3.7)$$

Since the perturbation h_{ijk} is a antisymmetric 3-form in 3-space we can write $h_{ijk} = \epsilon_{ijk} h / a^3$, where h is a constant that does not depend on τ . This is consistent with the Bianchi identity $\epsilon^{\mu\nu\rho\sigma} (\partial_\rho H_{\sigma\mu\nu}) = 0$ (that must be satisfied at all order) and the fact the we restricted the analysis to homogenous perturbations. In more detail, the Bianchi identity restricted to homogenous perturbations implies $\epsilon^{ijk} (\partial_k h_{0ij}) = 0$, with $h_{0ij} = \partial_0 b_{ij} + \partial_i b_{j0} + \partial_j b_{0i}$. The only non-trivial constraint is $\partial_0 (a^3 \epsilon^{ijk} \partial_k b_{ij}) = 0$, from which it follows that $\partial_0 h = 0$. Eq. (3.7) simplifies to

$$\delta_2 \mathcal{S}_E^B = \int d^4x \frac{\sqrt{\hat{g}}}{4f_a^2} \left\{ a^4 [\partial_0 b_{ij} - \partial_{[i} b_{0j]}]^2 + \frac{2h^2}{a^2} + \frac{4\mathcal{H}_0 h}{a^2} (A + 3\psi) + \frac{\mathcal{H}_0^2}{a^2} (-A^2 + 6A\psi + 15\psi^2) \right\}, \quad (3.8)$$

where we introduced the short-hand notation $\partial_{[i}b_{0j]} \equiv \partial_i b_{0j} - \partial_j b_{0i}$, and the square in the first term in the curly brackets implies a contraction with the metric $\hat{g}$ for each one of the spatial indices. The full quadratic Euclidean action is $\mathcal{S}_E^{(2)} \equiv \delta_2 \mathcal{S}_E^{\mathcal{R}} + \delta_2 \mathcal{S}_E^{\mathcal{B}}$. We are now in the position to switch to the Hamiltonian description to deal with non-dynamical degrees of freedom and their associated constraints. This formulation in euclidean requires a quick revision, we define conjugate momenta so that the path integral reads:

$$\int d\Psi e^{-\int d^4x_E \mathcal{L}_E(\psi, \psi')} = \int d\Psi d\Pi e^{\int d^4x_E \Pi \psi' - \int d^4x_E H_E(\Pi, \psi)}, \quad (3.9)$$

which implies a modified momentum definition and EOM in phase space:

$$\Pi = \iota \frac{\partial \mathcal{L}}{\partial \psi'}, \quad \iota \Psi' = \frac{\partial H}{\partial \Pi}, \quad \iota \Pi' = -\frac{\partial H}{\partial \psi}. \quad (3.10)$$

For the case at hand we have

$$\Pi_\psi = -\frac{6\iota\sqrt{\hat{g}}a^2}{\kappa} (\mathcal{H}A + \psi'), \quad \Pi^{ij} = \frac{\iota\sqrt{\hat{g}}a^4}{2f_a^2} (\partial^0 b^{ij} - \partial^{[i}b^{0j]}), \quad (3.11)$$

whereas the variables A and b_{0i} are non-dynamical and have no associated momentum. One therefore has $-\mathcal{L}_E = \iota \Pi_\psi \psi' + \iota \Pi_{ij} b'_{ij} - H_E$ with

$$\begin{aligned} H_E^{(2)} = & \frac{f_a^2 \Pi_{ij} \Pi^{ij}}{\sqrt{\hat{g}}a^2} - \iota \Pi^{ij} \partial_{[i}b_{0j]} - \frac{\kappa \Pi_\psi^2}{12\sqrt{\hat{g}}a^2} + \frac{\sqrt{\hat{g}}h^2}{2f_a^2 a^2} + \frac{3\sqrt{\hat{g}}\mathcal{H}_0 h \psi}{f_a^2 a^2} + \frac{\sqrt{\hat{g}}\mathcal{H}_0 h A}{f_a^2 a^2} \\ & + \frac{3\sqrt{\hat{g}}}{\kappa} (10\mathcal{K}a^2 - 9a'^2) \psi^2 + A \left\{ \frac{6\sqrt{\hat{g}}}{\kappa} [2\mathcal{K}a^2 - 3a'^2] \psi - \iota \mathcal{H} \Pi_\psi \right\}, \end{aligned} \quad (3.12)$$

where we have used Einstein's equations for the background fields, $3\mathcal{K}a^2 - 3a'^2 = \kappa \mathcal{H}_0^2 / 2f_a^2 a^2$ and $2aa'' - a'^2 - \mathcal{K}a^2 = \kappa \mathcal{H}_0^2 / 2f_a^2 a^2$, which in particular lead to the cancelation of the A^2 term. In this Hamiltonian the subset of phase space (Π_{ij}, b_{0i}) is disconnected from the rest; its study is customary, b_{0j} sets a constraint on the momenta Π_{ij} which have a positive definite Hamiltonian and so will the constrained momentum. We, therefore, focus on the remaining Hamiltonian in our search for negative modes. Since h is constant — i.e. it is a non-dynamical variable — we can set it to zero. This choice corresponds to the case in which $\mathcal{H}_0$ is frozen to its background value (being h its perturbation). From eq. (2.11), we see that $h = 0$ is a sensible condition since it is ensured by charge conservation along the wormhole throat. The background value is fixed by $\mathcal{H}_0 = n/2\pi^2$, with $n \in \mathbb{Z}$. A non-zero perturbation around this quantity would be consistent with the wormhole quantization condition only if one takes integer values of h . A change of this kind, however, would not correspond to a perturbation around the background solution but it would trade the latter with a new one, characterized by a different axion charge. This possibility falls outside the very same definition of small perturbations, and for this reason we set $h = 0$ in the following analysis. We find

$$H_E^{(2)} = -\frac{\kappa \Pi_\psi^2}{12\sqrt{\hat{g}}a^2} + \frac{3\sqrt{\hat{g}}}{\kappa} (10\mathcal{K}a^2 - 9a'^2) \psi^2 + \frac{6\sqrt{\hat{g}}}{\kappa} [2\mathcal{K}a^2 - 3a'^2] \psi A - \iota \mathcal{H} \Pi_\psi A. \quad (3.13)$$

We can solve the constraint associated with A . We find (after using Einstein's equations for the background fields)

$$\Pi_\psi = -\frac{6\sqrt{\hat{g}}aa'}{\kappa}\psi\left(-3 + \frac{2\mathcal{K}a^2}{a'^2}\right). \quad (3.14)$$

Given $\Pi_\psi \propto \psi$, the dynamics of the system will be trivial, as it turns out they are not only trivial, they are non-existent. Using eq. (3.14), we have an action:

$$\begin{aligned} -\mathcal{L}_E &= i\Pi_\psi\psi' - H_E^{(2)} = \frac{6\sqrt{\hat{g}}a}{\kappa a'}(2\mathcal{K}a^2 - 3a'^2)\psi\psi' - \frac{3\sqrt{\hat{g}}}{\kappa}\left[\frac{(2\mathcal{K}a^2 - 3a'^2)^2}{a'^2} + 10\mathcal{K}a^2 - 9a'^2\right]\psi^2 \\ &= \frac{3\sqrt{\hat{g}}}{\kappa}\left[-\left(\frac{a}{a'}(2\mathcal{K}a^2 - 3a'^2)\right)' - \frac{(2\mathcal{K}a^2 - 3a'^2)^2}{a'^2} - 10\mathcal{K}a^2 + 9a'^2\right]\psi^2 \\ &= \frac{3\sqrt{\hat{g}}}{\kappa}\left[\frac{4\mathcal{K}^2a^4 - 2\mathcal{K}a^2a'^2}{a'^2} - \frac{4\mathcal{K}^2a^4 - 2\mathcal{K}a^2a'^2}{a'^2}\right]\psi^2. \end{aligned} \quad (3.15)$$

The action cancels up to a total derivative, that is there is no dynamics or modes to look for! This outcome can be understood from the fact that ‘frozen’ matter perturbations leave us with gravity alone and as it is well known there are no homogeneous modes and ψ, A can be gauged away. This result brings us to the conclusion that in the homogeneous sector there are no physical perturbations at all, and hence no negative mode. This result supports the interpretation of the wormhole as an instanton mediating tunneling transitions between degenerate vacua. Equipped with this result, we can now move to compute the effective potential generated by gravity.

Before proceeding, let us comment about the discrepancy with the result presented in [31] where a negative eigenvalue was found. The metric studied in [31] has the form $ds^2 = N^2(\rho)d\rho^2 + R^2(\rho)d\Omega^2$, with the wormhole solution corresponding to $N(\rho) = 1$ and $R(\rho) = R_{\text{wh}}(\rho)$, where $R_{\text{wh}}(\rho)$ satisfies the equation $[R'_{\text{wh}}(\rho)]^2 = 1 - 1/R_{\text{wh}}^4(\rho)$ with $R_{\text{wh}}(\rho = 0) = L$. The authors of [31] focused the analysis on homogeneous perturbations. They defined the perturbed metric element as $ds^2 = [1 + n(\rho)]^2 d\rho^2 + [R_{\text{wh}}(\rho) + r(\rho)]^2$, and they fixed the gauge with the choice $n(\rho) = 0$ and $\phi = 0$, where the last condition applies to the perturbation of the scalar field. Furthermore, in [31] the conformal factor problem was circumvented by using the Gibbons-Hawking-Perry rotation [32]. In synthesis, the ‘wrong’ negative kinetic term in the quadratic action for the perturbation $r(\rho)$ changes its sign as a consequence of the replacement $r \rightarrow ir$, with the new r real.

In this paper, we tackled the problem considering the full set of scalar perturbations, and we eliminated the unphysical gauge freedom by means of the constrained Hamiltonian approach. Furthermore, we decided to work with the antisymmetric tensor field $B_{\mu\nu}$ (together with its field strength) as field variables. This is the most natural choice since it avoids possible subtleties and complications associated with the direct use of the scalar field in the Euclidean action. Most importantly, using the three-form language reveals in a very simple way that there are no propagating homogeneous perturbations. As explained, this is a consequence of the Bianchi identity and the quantization of the axion charge along the wormhole throat. Our approach, therefore, shows that the negative mode found in [31] is an artifact of the specific gauge choice adopted.

Figure 3. Half-wormholes as instantons in Euclidean space. In the left (right) panel we show an instanton (anti-instanton) mediating the topology change $\mathbb{R}^3 \rightarrow \mathbb{R}^3 \oplus S_3$, and carrying away an axion charge $+n$ ($-n$). The axion charge is overall conserved but an observer on $\mathbb{R}^3$ experiences, as Euclidean time passes by, a change $\Delta Q = -n$ ($\Delta Q = +n$). The non-conservation of the axion charge Q on $\mathbb{R}^3$ implies an explicit breaking of the symmetry it generates.

For completeness, let us mention that we also performed the analysis of the spectrum of the quadratic action by means of the scalar field formulation. We confirmed the main result of this section, that is the absence of negative modes. However, while in the three-form language this conclusion was achieved in a relatively simple way, in the scalar field formulation more work is needed because of the presence of the scalar field perturbation $\delta\theta$ that is, in general, non-vanishing. For the sake of clarity, we decided to present here only the three-form derivation, in line with the approach pursued throughout the paper.

4 Effective potential from Euclidean wormholes

The non-linearly realized global $U(1)$ symmetry $\phi \rightarrow \phi + \alpha f_a$ is generated by the axion charge Q . The ground state — corresponding to the bottom of the mexican hat potential — is degenerate along the angular direction, and vacua are described by continuously connected field configurations with minimum energy.

Half-wormholes induce quantum tunneling between classical vacuum states with different axion charge Q . The situation is schematically represented in figure 3. As Euclidean time passes by, in the presence of an instanton (anti-instanton) an observer on $\mathbb{R}^3$ experiences a change $\Delta Q = -n$ ($\Delta Q = +n$) since there is a net flux of axion charge equal to $+n$ ($-n$) through the wormhole throat [16, 17, 34]. Eventually, we will consider the case with $n = 1$, since the instanton action in eq. (2.12) is minimized. Consequently, from the point of view of the observer on $\mathbb{R}^3$, the axion charge is not conserved, and the associated symmetry explicitly broken down to the discrete gauge symmetry $\phi \rightarrow \phi + 2k\pi f_a$, with $k \in \mathbb{Z}$, which remains intact in the presence of wormhole instantons, as discussed in section 2.1. As a result of this breaking, an effective potential for the axion is generated.

A rigorous treatment in the context of wormhole physics was proposed in [34–36] (see also [37–39]). The goal of these papers was to compute the influence of wormholes on physics at low energy, or at distances large compared to the thickness of the wormhole throat, L . The key idea is that at energy scales below L^{-1} wormholes can be integrated out.⁶ We do not repeat here this computation explicitly, and we refer the interested readers to the original literature and the recent review [41] for details. Nevertheless, we shall briefly comment on the final result. In the low-energy effective theory, the leading breaking effects due to wormholes are described by the Lagrangian density

$$\mathcal{L}_{\text{wh}} = K e^{-\mathcal{S}_{\text{inst}}} \mathcal{O}_S \cos\left(\frac{\phi}{f_a} + \delta\right), \quad (4.1)$$

$$\mathcal{O}_S = 1 + a L^4 (\partial_\mu \phi)(\partial^\mu \phi) + \dots, \quad (4.2)$$

where in the second line we explicitly wrote the most general combination of operators singlet under the U(1) global symmetry. In eq. (4.2), we used as mass suppression scale the cutoff $1/L$, and a is expected to be order one numbers. The dimensionful constant K is also expected to be of order $K \sim 1/L^4$. Its actual value, that in general cannot be computed exactly, is however not crucial. The important point in eq. (4.1) is that breaking effects are *always* proportional to the factor $e^{-\mathcal{S}_{\text{inst}}}$, a trademark representing their non-perturbative origin. Because of the exponential dependence, the factor $e^{-\mathcal{S}_{\text{inst}}}$ dominates the size of the breaking effect irrespectively from the exact value of K . We remark that this point seems to be quite often overlooked in the literature, especially as far as phenomenological applications are concerned. Very often breaking effects generated by gravity are treated in a naïve way, along the line of the discussion outlined in the introduction, see eq. (1.4). On the contrary, the presence of the suppression factor $e^{-\mathcal{S}_{\text{inst}}}$ plays a crucial rôle. In the second part of this paper we shall discuss in detail the most relevant phenomenological applications.

5 Comments on possible UV completions

In this section we discuss possible UV completions of the axion Goldstone mode, and their consequences on wormhole physics.

5.1 Dynamical radial mode

We start our investigation considering the simplest UV completion of the axion theory. This is the case of a U(1) global symmetry that is spontaneously broken by the VEV $\langle |\Phi| \rangle = f_a/\sqrt{2}$ of a complex scalar field $\Phi(x) = [f(x)/\sqrt{2}] e^{i\phi(x)/f_a}$. The mexican hat potential responsible for the spontaneous symmetry breaking is $V(f) = \lambda(f^2 - f_a^2)^2/4$. The radial field $f(x)$ gets a mass $M_f^2 = \lambda f_a^2$. The physics discussed in the previous sections corresponds to the case in which the radial mode does not participate in the dynamics, and its value remains frozen at $f = f_a$. We expect this situation to be strictly true if

⁶The validity of this approach relies on the fact that large wormholes, with size $L' > L$, can be safely neglected. We refer the interested reader to ref. [40] for a discussion of this issue in the axion case.

Figure 4. Left panel. On-shell action in the presence of a dynamical radial mode as a function of its mass M_f . Right panel. Instanton action as a function of the string coupling g_s . On the top x-axis we show the value of the string mass scale M_S . In solid red, we show the on-shell action $\mathcal{S}_{\text{inst}}$ for the instanton at the critical value $f_a = 10^{16}$ GeV. The blue dashed line represents the string contribution $8\pi^2/g_S^2$. The region shaded in gray corresponds to $M_S < L^{-1}$.

$M_f > 1/L$. For the critical value $f_a \simeq 10^{16}$ GeV that we identified in section 6.1, we have $1/L \simeq 10^{18}$ GeV. It is therefore hard to imagine that the radial field f plays no dynamical rôle, since we expect its mass — for reasonable small coupling λ — to lie below the scale set by the wormhole throat. In order to validate our conclusions, it is necessary to generalize the wormhole solutions, and compute their action, to the case in which f is a dynamical field [15, 42].

We find that including the dynamics of the radial mode results in a net decrease of the action, in agreement with the result of [15]. This is because in the presence of a light dynamical radial mode the size of the wormhole shrinks, and, consequently, the wormhole action (which is proportional to $M_{\text{Pl}}^2 L^2$) decreases. For illustrative purposes, in figure 4 we extend the plot to large values of λ in order to show that, as M_f moves towards the threshold $1/L$ above which it can be safely integrated out, the action increases. The inclusion of the radial mode in the U(1) model shows that the presence of dynamical degrees of freedom belonging to the UV completion of the axion Lagrangian does not weaken the strength of the non-perturbative global symmetry breaking induced by gravity. On the contrary, the total action in figure 4 is significantly smaller if compared to the one in which the radial mode does not fluctuate around its VEV. We conclude that the bound derived in section 6.1 presumably represents a conservative estimate, since extra dynamics related to the UV completion of the axion Lagrangian may even strengthen the global symmetry breaking. It should be noted that the reduction of the action in this case, given $S \sim (L/L_p)^2$ implies a stronger assumption of the validity of the EFT description, take $\lambda \sim 0.1$ in figure 4 which implies L is a few below M_{Pl} .

5.2 The UV completion of general relativity

In this section we try to estimate the impact of the UV completion of general relativity. The lack of a quantum theory of gravity, however, makes any argument highly speculative. We can gain some insight considering the case of heterotic string theory in which the string gauge coupling g_S , the string mass scale M_S are related by [49] $M_S = g_S M_{\text{Pl}} / \sqrt{8\pi}$ in close analogy with the relation discussed in eq. (2.13). On purely dimensional grounds, we expect — in analogy with the typical instanton action in the Yang-Mills case — non-perturbative string contributions to the wormhole action of the form $\gamma = 8\pi^2/g_S^2$. We compare γ and $\mathcal{S}_{\text{inst}}$ in the right panel of figure 4 as a function of the string coupling. We fix $f_a = 10^{16}$ GeV (a representative value that, as we shall explore in the next section, is particularly relevant for phenomenological applications). On the top y-axis we put the values of the string mass scale M_S , and the region shaded in gray corresponds to the condition $M_S < 1/L$. In the presence of a weak string coupling our computation can be completely invalidated. The string mass scale becomes lower than L^{-1} — thus breaking the effective description — and the topological action generated by string theory dominates over the gravitational instanton contribution. The punch line is therefore very simple and can be summarized as follows. If the UV completion of general relativity is weakly coupled, new degrees of freedom will appear at or below the wormhole scale $1/L$ thus invalidating the semi-classical result. Bearing this important theoretical caveat in mind, we now move to discuss some phenomenological implications.

6 Phenomenological implications

In this section we discuss several phenomenological situations characterized by the presence of light axions with a decay constant such that non-perturbative gravity effects become relevant.

6.1 The QCD axion

We start discussing the implication of eq. (4.1) for the QCD axion. We consider only wormholes with unit charge, since they give the dominant contribution. The effective axion potential, including both QCD and non-perturbative gravitational contributions, is schematically given by

$$V(a) = \Lambda_{\text{QCD}}^4 \cos\left(\frac{\phi}{f_a}\right) + (1/L)^4 e^{-\mathcal{S}_{\text{inst}}} \cos\left(\frac{\phi}{f_a} + \delta\right), \quad (6.1)$$

where we estimated the prefactor of the gravitational contribution to be of order $(1/L)^4$. This is nothing but an order-of-magnitude estimate based on dimensional arguments but it does not affect the final result since the exponential term $e^{-\mathcal{S}_{\text{inst}}}$ dominates. In this regard, it is worth emphasizing the size of this suppression. Considering the benchmark value $f_a = 10^{10}$ GeV for a QCD axion in the classical window, we have $\mathcal{S}_{\text{inst}} \simeq 1.7 \times 10^8$. The explicit breaking generated by gravity is therefore negligible for all purposes.

Let us explore the implications of eq. (6.1) in more detail. First, notice that we defined the axion field so that the low energy QCD contribution to the axion potential is minimized

Figure 5. Non-perturbative gravity corrections (red dot-dashed line) on θ_{eff} (left panel) and the axion mass m_a (right panel) as a function of the axion decay constant f_a . In the regions shaded in red non-perturbative gravity dominates. In the right panel, the dashed blue line represents the contribution to the axion mass generated by QCD, computed according to [51].

at $\phi = 0$, and we redefined accordingly the phase δ in eq. (4.1). The phase shift δ arises from a mismatch between the gravitational and the low energy QCD term. In the absence of CP violation in the gravitational sector, the phase δ is just proportional to $\text{Arg Det}[Y_u Y_d]$, with $Y_{u,d}$ the Yukawa matrices for up- and down-type quarks, and it arises from the chiral rotation needed to move the phase of the fermion mass matrix into the θ -term. In the following, we interpret δ as an arbitrary order one phase.

The effect of the gravitational correction is twofold: it shifts the axion mass, and produces a non-zero θ_{eff} . We find

$$m_a^2 \simeq \frac{\Lambda_{\text{QCD}}^4}{f_a^2} + \frac{(1/L)^4}{f_a^2} e^{-S_{\text{inst}}} , \quad (6.2)$$

$$\theta_{\text{eff}} \simeq \frac{(1/L)^4}{\Lambda_{\text{QCD}}^4} \sin \delta e^{-S_{\text{inst}}} , \quad (6.3)$$

where we remind the reader that the action scales with M_{Pl}/f_a . The gravitational correction to the axion mass features a very interesting property if compared with the contribution generated by QCD. As f_a increases, the QCD axion mass $\Lambda_{\text{QCD}}^4/f_a^2$ decreases but the gravitational term becomes more and more important, and eventually it overcomes the former. This means that we expect a lower bound on the axion mass. This is shown in the right panel of figure 5 where we shaded in red the region in which non-perturbative gravitational corrections dominate. We find the lower bound on the axion mass (or, equivalently, the upper bound on the axion decay constant)

$$m_a \gtrsim 4.8 \times 10^{-10} \text{ eV} , \quad f_a \lesssim 10^{16} \text{ GeV} . \quad (6.4)$$

In the left panel of figure 5 we show the impact of the non-zero θ_{eff} in eq. (6.3). For reference, we consider the experimental bound $\theta_{\text{eff}} < 10^{-10}$ (horizontal gray line). We find that if $f_a \gtrsim 10^{16}$ GeV then the contribution induced by gravity becomes too large. Notice that, by accident, the bounds on f_a extracted from m_a and θ_{eff} are of the same order, and assuming $\delta_1 = 0$ does not change quantitatively our conclusion.

Figure 6. QCD axion parameter space (see text for details). The region shaded in red with horizontal meshes is excluded by non-perturbative gravitational effects, eq. (6.4). Note that the ticks on the right y-axis follow the relation between m_a and f_a dictated by QCD that ceases to be valid when non-perturbative gravitational effects become relevant.

In figure 6 we show the impact of the bound in eq. (6.4) on the QCD axion parameter space. As customary, the condition $f_a \leq H_I/2\pi$, with H_I the Hubble expansion rate at the end of inflation distinguishes between the cases in which the PQ symmetry is broken before ($>$) or after ($<$) the end of inflation. The first possibility defines the so-called anthropic axion window, in which the possibility to reproduce the observed dark matter relic density $\Omega_a h^2 \simeq 0.1$ relies on a fine-tuned choice of the initial misalignment angle θ_{in} (black dot-dashed lines in figure 6 for different θ_{in}). Notice that a substantial part of the anthropic axion window is excluded by the presence of axion isocurvature perturbations in the early Universe (blue region with vertical meshes) that we compute following [52, 53] (see also [54, 55] for recent progress for non-standard cosmologies and generalization to axion-like particles). The region $f_a < H_I/2\pi$ defines the classical axion window. In the region (i) shaded in brown with diagonal meshes $\Omega_a h^2 > 0.1$, and the Universe is over-closed. The dashed brown lines reproduce 1% and 10% of the observed dark matter abundance. In this region a sizable contribution from decay of topological defects is expected [56]. For completeness, we also show the region excluded by white dwarf cooling time [57] and upper limit on the tensor-to-scalar ratio [58]. The bound in eq. (6.4) applies in the anthropic axion window, and disfavors a region of the parameter space (shaded in red with horizontal meshes) previously allowed. We remark that the bound in eq. (6.4) was derived without specific assumption but only considering minimal coupling of the axion field with Einstein gravity. As a consequence, we remark that the caveat put forward in section 5.2 applies. Here, however, we opt for an optimistic view: the observation of an axion with a large ($f_a \gtrsim 10^{16}$ GeV) decay constant would imply an indirect evidence for a modification of general relativity at or below the Planck scale. We now move to briefly discuss few scenarios of phenomenological relevance in which non-perturbative gravitational corrections to the mass of the QCD axion may play an important rôle.

6.1.1 QCD axions and black hole superradiance

Spin and mass measurements of stellar-size black holes exclude the QCD axion mass window [59]

$$6 \times 10^{-13} \lesssim m_a [\text{eV}] \lesssim 2 \times 10^{-11}, \quad (6.5)$$

corresponding to $3 \times 10^{17} \lesssim f_a [\text{GeV}] \lesssim 10^{19}$. This is because superradiance effects become efficient [60, 61] when the Compton wavelength of the axion is comparable with the horizon size of the black hole. The axionic field forms a quasi-stationary configuration around the black hole at the expense of its rotational energy, giving birth to a quasi-bound system that shares remarkable similarities — such as energy orbitals and level transitions — with the hydrogen atom. From $\lambda_{\text{Compton}} = h/m_a c \sim R$, we have

$$m_a \sim 6 \times 10^{-12} \left(\frac{30 \text{ km}}{R} \right) \text{eV}, \quad (6.6)$$

where R is the typical radius of a stellar-mass black hole, thus justifying the mass range excluded in eq. (6.5). At the two sides of this interval, and within the mass range $m_a = [10^{-14}, 10^{-10}] \text{eV}$, stellar black hole superradiance in the presence of the QCD axion may produce in the next few years spectacular signatures — both direct and indirect — in gravitational wave detectors such as Advanced LIGO [62, 63]. Indirect signatures refer to the observation of gaps in the spin-mass distribution of final state black holes produced by binary black hole mergers. Direct signatures refer to monochromatic gravitational wave signals produced during the dissipation of the scalar condensate after the superradiant condition is saturated.

As far as direct signatures are concerned, a careful assess of the detection prospects in Advanced LIGO and LISA was recently proposed in [64, 65]. The outcome of the analysis is that, considering optimistic astrophysical models for black hole populations, the gravitational wave signal produced by superradiant clouds of scalar bosons with mass in the range $m_a = [2 \times 10^{-13}, 10^{-12}] \text{eV}$ is observable — i.e. it is characterized by a signal-to-noise ratio larger than the experimental threshold — by Advanced LIGO. Notice that this region seems to be ruled out if one considers at face value the bound in eq. (6.5). However, it is worth emphasizing that the bound in eq. (6.5) is most likely only indicative since it is based on black hole spin measurements that are extracted indirectly from X-ray observations of accretion disks in X-ray binaries. We only have very few of such measurements at our disposal, and it is difficult to extract a bound with significant statistical confidence. The discussion of this matter however escapes the scope of the present work.

As clear from the right panel of figure 5 and from the parameter space in figure 6, the QCD axion in the mass range $m_a = [10^{-14}, 10^{-10}] \text{eV}$ violates the bound in eq. (6.4), and fits into a region where non-perturbative gravitational effects dominate over QCD. We therefore conclude that — working under the very same hypothesis, that is an axion minimally coupled to Einstein gravity — the phenomenologically interesting mass range $m_a = [10^{-14}, 10^{-10}] \text{eV}$ motivated by black hole superradiance is theoretically forbidden for the QCD axion. This is not, of course, a lapidary conclusion. To be more optimistic, observing the QCD axion in connection with black hole mergers at the Advanced LIGO

Figure 7. Gravitational mass generated by non-perturbative Euclidean wormholes as a function of the Goldstone boson decay constant f_a . In the inset plot, we show, in the same interval of f_a , the inverse of the wormhole size L . The horizontal blue lines represent the mass range favored by the analysis in [64, 65] considering both Advanced LIGO ($m_a = [2 \times 10^{-13}, 10^{-12}]$ eV) and LISA ($m_a = [5 \times 10^{-19}, 5 \times 10^{-16}]$ eV) gravitational wave detectors.

could imply an evidence for modifications of Einstein gravity. Alternatively, a scalar boson different from the QCD axion could still leave its imprint in the texture of gravitational wave signatures. In this last case, the mass term generated by gravity can be used to justify the lightness of the scalar boson. By turning off the QCD contribution in eq. (6.2), we show the gravitational mass $m_a^2 \simeq (1/L)^4 e^{-S_{\text{inst}}} / f_a^2$ in figure 7. Clearly, if $f_a \simeq 10^{16}$ GeV it is possible to span, thanks to the exponential factor $e^{-S_{\text{inst}}}$ a large range of allowed values, including the regions favored by the analysis in [64, 65].

6.1.2 Axion stars as black hole seeds

The idea is to consider axion stars as black hole seeds, which are supermassive in the case of ultralight axions [66]. By studying numerically the collapse of axion stars in the context of general relativity, the authors of [66] identified the critical value of the axion decay constant $f_{\text{TP}} \sim 0.06 M_{\text{Pl}}$ above which black hole formation occurs. The mass of the typical black hole formed from axion star collapse is $M_{\text{BH}} \sim 3.4 (f_a / 0.12 M_{\text{Pl}})^{1.2} M_{\odot}$.

As before, the condition $f_a \gtrsim f_{\text{TP}} \sim 0.06 M_{\text{Pl}}$ is not compatible with the bound in eq. (6.4). We stress that this bound was derived considering a QCD axion minimally coupled to Einstein gravity, therefore without any additional assumptions compared to the ones in [66]. Said differently, it would be interesting to include in the analysis of [66] non-perturbative gravitational corrections to the axion potential since they play a relevant rôle in the region of parameter space in which black hole formation may occur.

6.2 Ultralight scalars as cosmological dark matter

Dark matter must behave sufficiently classically to be confined on galaxy scales. If we suppose dark matter to be a boson with mass m_a and velocity v , we can require to behave

classically down to the typical size of Milky Way satellite galaxies, and obtain the condition

$$\lambda_{\text{De Broglie}} = \frac{1}{m_a v} \lesssim 1 \text{ kpc} \quad \implies \quad m_a \gtrsim 10^{-22} \left(\frac{120 \text{ km/s}}{v} \right) \text{ eV}. \quad (6.7)$$

For a typical halo with size $R \sim \text{kpc}$, and mass $M \sim 10^9 M_\odot$, we expect a virial velocity $v \sim \sqrt{G_N M/R} \sim 70 \text{ km/s}$. From eq. (6.7), we extract a lower limit for the mass of bosonic dark matter of about 10^{-22} eV . The dark matter saturating this value is known as *fuzzy dark matter* (FDM) [67]. This kind of ultralight dark matter could form a Bose-Einstein condensate on galactic scales, providing a possible solution to the tensions that arise when the standard cold dark matter paradigm is probed into the deep non-linear regime at redshift $z \sim 0$ [68–70]. Apart from this motivation, an ultralight dark matter condensate features peculiar observational astrophysical properties, and it catalyzed increasing attention in the dark matter community (see, e.g., [71–78]). From a particle physics perspective, the extreme lightness of the FDM is well suited by an axion-like particle. Furthermore, reproducing the observed value of relic abundance via the misalignment mechanism gives a clue about the value of its decay constant f_a . One finds the parametric order-of-magnitude scaling

$$\Omega_a h^2 \approx 0.1 \left(\frac{f_a}{10^{17} \text{ GeV}} \right)^2 \left(\frac{m_a}{10^{-22} \text{ eV}} \right)^{1/2}, \quad (6.8)$$

to be compared with the Planck measurement $\Omega_a h^2 \approx 0.12$ [79]. Eq. (6.8) points towards a very specific interplay between f_a and m_a , and it is interesting to further investigate its origin. In [77] this relation was justified referring to string theory models in which one expects an explicit breaking of the axionic shift symmetry due to the existence of world-sheet or membrane instantons [80]. Even without invoking string theory constructions, we point out that the mass term generated by gravity in eq. (4.1) provides a nice explanation for the relic abundance in eq. (6.8). Indeed, using $m_a^2 \simeq (1/L)^4 e^{-S_{\text{inst}}} / f_a^2$, the observed abundance can be reproduced with $f_a \simeq 8 \times 10^{15} \text{ GeV}$ and $m_a \simeq 2.5 \times 10^{-18} \text{ eV}$. Notice that this value falls within the mass range that will be explored by the LISA gravitational wave interferometer, see figure 7. This is an interesting observation, since it shows that a dark matter candidate with nothing more than gravitational interactions may explain in a natural way the lightness of its mass. We finally note that, as figure 7 shows, a value of the axion decay constant above $f_a \simeq 10^{16} \text{ GeV}$ would generate an unacceptably large contribution in eq. (6.8).

7 Wormhole solutions for a generic Goldstone coset

The goal of this section is to find a generalization of the Giddings-Strominger wormhole for a generic Goldstone coset. The case of the axion explored in the previous part of the paper corresponds to the Goldstone of a spontaneously broken $U(1)$ symmetry, with conserved Noether current $J^\mu = f_a^2 (\partial^\mu \theta)$. The first step, therefore, is to generalize the definition of the Goldstone fields and the corresponding currents for a generic coset.

7.1 CCWZ formalism review

Let us start with a lightning review of the treatment of Goldstone bosons from the coset $\mathfrak{G}/\mathfrak{H}$ in the CCWZ formalism [81, 82]. The broken part of the Lie algebra will be denoted by the generators X_A whereas the unbroken symmetry generators are T_a with the following structure

$$[T_a, T_b] = \imath f_{ab}^c T_c, \quad [T_a, X_B] = \imath f_{aB}^C X_C, \quad (7.1)$$

where the last equation follows for compact groups and we take the generators to be hermitian. The Goldstones parametrize the broken part of the group; one can write $\eta = \exp(iX\theta)$ with $\eta \in \mathfrak{G}/\mathfrak{H}$ in terms of $n = \dim(\mathfrak{G}) - \dim(\mathfrak{H})$ θ coordinates, although different parametrizations are useful and often used. The transformation properties of η are

$$\eta \rightarrow \mathbf{G} \eta \mathfrak{h}^{-1}(\eta), \quad (7.2)$$

with $\mathbf{G} \in \mathfrak{G}$ and $\mathfrak{h} \in \mathfrak{H}$. This expression follows from the result that any element of the group can be factorized into a broken times an unbroken element and the composition law $\mathbf{G}\eta = \eta' \mathfrak{h}(\eta)$. The basic building block is the Lie-dragged derivative which is a one-form ω living in the broken Lie algebra, or tangent space of the broken group manifold. Explicitly

$$\eta^{-1} d\eta = \imath \omega^A X_A + \imath V^a T_a, \quad d = dx^\mu \partial_\mu, \quad (7.3)$$

where $\omega^A = \omega_\mu^A dx^\mu$. In terms of the θ coordinates one has the vierbein and metric (we use the convention of summed repeated indices unless otherwise stated)

$$\omega_\mu^A = e_i^A(\theta) \partial_\mu \theta^i, \quad G_{ij}(\theta) = e_i^A e_j^B \delta_{AB}. \quad (7.4)$$

In order to extract the symmetry transformations in terms of the θ coordinates we make eq. (7.2) infinitesimal

$$\delta_\epsilon \eta = (i\epsilon^a T_a + i\epsilon^A X_A) \eta - i\eta (\epsilon \cdot F(\eta))^a T_a, \quad \eta^{-1} \delta_\epsilon \eta = \eta^{-1} (i\epsilon T + i\epsilon X) \eta - i(\epsilon \cdot F(\eta))^a T_a, \quad (7.5)$$

and we project out the non-linear piece $F(\eta)$ by restricting to broken generators only. We find

$$\eta^{-1} X_A \eta|_X = X_B e_i^B \xi_A^i, \quad \eta^{-1} T_a \eta|_X = X_B e_i^B \xi_a^i, \quad (7.6)$$

where by $|_X$ we denote projection onto the broken generators. The above result yields the Killing vectors in θ -space ($\delta_\epsilon \theta^i = \epsilon_A \xi_A^i$); there are n Killing vectors ξ_A associated to the coset and $\dim(\mathfrak{H})$ Killing vectors ξ_a for the unbroken symmetries. The transformation of the one-form ω^A under the unbroken part of the group is specially simple since the symmetry is linearly realized

$$\delta_\epsilon \omega^A \rightarrow \epsilon^a f_{aB}^A \omega^B, \quad (7.7)$$

where we have used the second relation in eq. (7.1). In coordinate space the Killing vectors by construction satisfy

$$\frac{\partial \xi^k}{\partial \theta^i} G_{kj} + G_{ik} \frac{\partial \xi^k}{\partial \theta^j} + \left(\xi^k \frac{\partial}{\partial \theta^k} \right) G_{ij} = 0, \quad (7.8)$$

whereas the Noether currents for the broken generators read

$$J_A^\mu = \sqrt{g} f^2 \xi_A^i e_i^B e_j^B \partial^\mu \theta^j, \quad (7.9)$$

where we introduced the Goldstones decay constant f and equivalent expressions apply for the unbroken currents. We note that, if a coordinate system can be found for which $\xi_A^i e_i^B = \delta_B^A$, the Noether currents read $\sqrt{g} \omega_A^\mu$ and their conservation can be written as the equation of motion for a one-form although the Bianchi identity is not satisfied in general.

7.2 Solution of the euclidean EOM

We now turn to the action. By using the variational principle, one should be able to find non-trivial solutions of the equation of motion, and subsequently obtain the corresponding non-perturbative action by substituting back the solution in the original action — in parallel with what was done in section 2.1. It is often remarked how the axion gets an extra sign in its kinetic term when turning the action from Minkowski to Euclidean; here we observe that regardless of the sign, starting from the variational principle in Euclidean space yields a vanishing wormhole action, as we shall show below. This is remedied either using a three-form description — as done in section 2.1 — or introducing Lagrange multipliers [17, 50]. Since a general coset does not seemingly accept a three-form description we opt for the second option and write an action

$$\mathcal{S}_E = \int d^4x \left[\sqrt{g} \left(-\frac{\mathcal{R}}{2\kappa} + \frac{f^2}{2} \partial_\mu \theta G^{\mu\theta} \right) + \lambda^A(x) \partial_\mu (\sqrt{g} f^2 \xi_A G^{\mu\theta}) \right], \quad (7.10)$$

where $\kappa = 8\pi G_N$ and $\lambda^A(x)$ are Lagrange multipliers. To avoid cumbersome notation, we have suppressed scalar indices (which are summed univocally to build invariants), and the coefficient of each Lagrange multiplier is the divergence of the corresponding Noether current ($J^\mu = \sqrt{g} f^2 \xi G^{\mu\theta}$) which vanishes on-shell.

The variational principle yields the equations of motion $\delta \mathcal{S}_E|_{\delta g_{\mu\nu}} = 0$, $\delta \mathcal{S}_E|_{\delta \theta} = 0$, $\delta \mathcal{S}_E|_{\delta \lambda^A} = 0$ with

$$\begin{aligned} \delta \mathcal{S}_E|_{\delta g_{\mu\nu}} = \int d^4x \frac{\sqrt{g}}{2\kappa} \left[\mathcal{R}^{\mu\nu} - \frac{g^{\mu\nu}}{2} \mathcal{R} + \kappa f^2 \left(\frac{g^{\mu\nu}}{2} \partial_\rho \theta G^{\rho\theta} - \partial^\mu \theta G^{\nu\theta} \right) \right. \\ \left. - \kappa f^2 (g^{\mu\nu} \partial_\rho \lambda \xi G^{\rho\theta} - 2 \partial^\mu \lambda \xi G^{\nu\theta}) \right] \delta g_{\mu\nu}, \end{aligned} \quad (7.11)$$

$$\begin{aligned} \delta \mathcal{S}_E|_{\delta \theta} = \int d^4x f^2 \left[-\delta \theta \partial_\mu (\sqrt{g} G^{\mu\theta}) + \frac{\sqrt{g}}{2} \partial_\mu \theta \left(\delta \theta \frac{\partial}{\partial \theta} \right) G^{\mu\theta} + \delta \theta \partial_\mu (\sqrt{g} \xi G^{\mu\theta}) \right. \\ \left. - \partial_\mu \lambda \left(\delta \theta \frac{\partial}{\partial \theta} \right) \xi G^{\mu\theta} \right], \end{aligned} \quad (7.12)$$

$$\delta \mathcal{S}_E|_{\delta \lambda^A} = \int d^4x f^2 \delta \lambda^A \partial_\mu (\sqrt{g} \xi_A G^{\mu\theta}). \quad (7.13)$$

Taking the trace in Einstein equations, if we omit λ terms, yields $-\mathcal{R} + \kappa f^2 \partial_\mu \theta G^{\mu\theta} = 0$ and a vanishing action (regardless of the sign that one can change with κf^2). The rôle of the extra λ term, as we shall see in the following, is exactly to turn this result into a properly defined variational principle for wormhole solutions with non-vanishing action. In

this regard, our result is the generalization of the construction put forward in [17] to the case of a generic coset.

Most of the computation now involves manipulations of the equations of motion, the reader uninterested in this derivation can skip to the solution in eq. (7.22). We proceed as follows. In the equation for the Goldstones we perform the substitution $\delta\theta^i = \partial_\nu \lambda^B \xi_B^i$, and, after repeated use of the definition of Killing vectors with respect to the metric in eq. (7.8), we find

$$\partial_\nu \lambda^B \partial_\mu [\sqrt{g} \xi_B G (\xi^\mu \lambda - \partial^\mu \theta)] + \partial_\mu \lambda^A \partial_\nu \lambda^B \left(\xi_A \frac{\partial \xi_B}{\partial \theta} - \xi_B \frac{\partial \xi_A}{\partial \theta} \right) G \partial^\mu \theta = 0, \quad (7.14)$$

where the last term is antisymmetric in the summed indices A, B and proportional to the structure constants. This means that, for spherically symmetric solutions which depend only on the radius r , such a term cancels since $\partial_r \lambda^A \partial_r \lambda^B$ is symmetric in A, B . From now on, we focus on this type of solutions, $\theta(r)^i$, $\lambda^A(r)$, and we adopt the spherically symmetric ansatz in eq. (2.6) for the metric. As remarked above, eq. (7.14) reduces in this case to

$$\partial_r \lambda_B \partial_r [\sqrt{g} \xi_B G (\xi_A \partial^r \lambda_A - \partial^r \theta)] = 0, \quad (7.15)$$

solved by $\xi_A \partial_r \lambda^A = \partial_r \theta$. On passing, we note that this substitution in Einstein equations yields $\mathcal{R} + \kappa f^2 \partial_r \theta G \partial^r \theta = 0$, and a non-zero on-shell action. This relation for λ back into the Goldstone equation of motion eq. (7.12) yields, substituting $\delta\theta = \partial_r \theta$

$$\frac{1}{2} \sqrt{g} \partial_r \theta (\partial_r G) \partial^r \theta - \sqrt{g} \partial_r \theta (\partial_r G \xi) \partial^r \lambda = 0. \quad (7.16)$$

We now use the eq. (7.13) for the variation of the Lagrange multiplier to substitute $\partial_r \xi$. Eq. (7.16) above becomes

$$-\frac{1}{2} \sqrt{g} \partial_r \theta (\partial_r G) \partial^r \theta + \partial_r \theta (\partial_r \sqrt{g} G \partial^r \theta) = 0, \quad (7.17)$$

or, rearranging radial derivatives,

$$\frac{1}{2} \partial_r (\partial_r \theta \sqrt{g} G \partial^r \theta) + \frac{1}{2} \partial_r \theta \sqrt{g} G \partial^r \theta \frac{\alpha^2}{\sqrt{g}} \partial_r \left(\frac{\sqrt{g}}{\alpha^2} \right) = 0, \quad (7.18)$$

which can be integrated (note it can be written as $\partial_r [\log(\partial_r \theta \sqrt{g} G \partial^r \theta) + \log(\sqrt{g}/\alpha^2)] = 0$) to find

$$\frac{\beta^3}{\alpha} \partial_r \theta G \partial_r \theta = \frac{C^2}{f^4} \frac{\alpha}{\beta^3}, \quad (7.19)$$

with a real dimensionless integration constant C . Finally, we substitute the Goldstone boson kinetic term in eq. (7.19) back in Einsteins equations and we find

$$-\frac{3\mathcal{K}\alpha^2}{\beta^2} + \frac{3(\partial_r \beta)^2}{\beta^2} + \frac{\kappa}{2f^2} C^2 \frac{\alpha^2}{\beta^6} = 0, \quad (7.20)$$

$$\hat{g}_{ij} \left[-\mathcal{K} - \frac{2\beta \partial_r \beta \partial_r \alpha}{\alpha^3} + \left(\frac{\partial_r \beta}{\alpha} \right)^2 + 2 \frac{\beta \partial_r^2 \beta}{\alpha^2} \right] - \frac{\kappa}{f^2} \hat{g}_{ij} \frac{C^2}{2\beta^4} = 0, \quad (7.21)$$

which have the same structure and solution for the metric as found in the axion case, see eqs. (2.7)–(2.8), with the connection $C \rightarrow \mathcal{H}_0$.

Let us pause for a moment to collect the pieces of our result. We have a solution that reads, if we take for concreteness $\beta = r$ as in section 2.1:

$$\alpha^2 = \frac{1}{1 - L_C^4/r^4}, \quad \xi_A^i(\theta) G_{ij}(\theta) \partial_r \theta^j = \frac{I_A}{r^3 f^2 \sqrt{1 - L_C^4/r^4}}, \quad L_C^4 \equiv \frac{4\pi C^2}{3M_{\text{PL}}^2 f^2}, \quad (7.22)$$

where the conservation of the n -Noether currents gives n constants I_A , $\sqrt{g} \xi_A G \partial^r \theta = I_A$. An explicit expression for the constant C in certain cases takes a simple form but a general expression for any coset is not known to us. Note also that the non-linear dependence of the Noether currents on the Goldstone variables θ and the fact that the n equations are coupled does not allow for explicit integration.

The action of this solution, recalling that the solution satisfies $\mathcal{R} + \kappa f^2 \partial_r \theta G \partial^r \theta = 0$, reads:

$$\mathcal{S}_{\text{inst}} = \int d^4x f^2 \sqrt{g} \partial_r \theta G \partial^r \theta = \frac{\pi^3 C^2}{2f^2 L_C^2} = \frac{3\pi^2 M_{\text{PL}}^2 L_C^2}{8}, \quad (7.23)$$

which is to say the action in the axion case times $C/\mathcal{H}_0$. In the axion case we found that the periodicity of the axion, i.e. the fact that it lives in a closed manifold, imposed $\mathcal{H}_0 = k/2\pi^2$, $k \in \mathbb{Z}$. We, therefore, expect a similar quantization in the case of a generic coset given that it is also a closed manifold. As in the axion case, we would have that a given instanton solution is characterized by a set of charges $Q_A = \int J_A$, and mediates a process in which these units of charge escape through the throat in violation of the global symmetry conservation law as seen by an observer in asymptotically flat space. To elaborate on the quantization condition we move now to a explicit type of coset: $O(n+1)/O(n)$.

7.3 The coset $O(n+1)/O(n)$ as an explicit case

In order to make an educated coset choice, as ever, symmetry is helpful, in particular the unbroken symmetry. In this respect the maximal unbroken symmetry allowed is $O(n)$ for n Goldstones, which is the largest linearly realised symmetry that the associated tangent space, ω_μ^A , admits. A breaking pattern that yields this unbroken symmetry is $O(n+1)/O(n)$, which we will adopt as an explicit case to exemplify our discussion. One has n broken generators X_A and $n(n-1)/2$ unbroken ones T_a that read (we will use antisymmetric generators and therefore there is no need for factors of i in the following)

$$(X_A)_{\alpha\beta} = \delta_\beta^{n+1} \delta_\alpha^A - \delta_\alpha^{n+1} \delta_\beta^A, \quad (T_a)_{\alpha n+1} = (T_a)_{n+1\beta} = 0, \quad \text{with } T_a^T = -T_a. \quad (7.24)$$

The Goldstone fields can be taken as θ^i with $i = 1, \dots, n$ but, motivated by the symmetries of the unbroken group, we shall use in the following ‘spherical’ coordinates $\theta = \rho u(\varphi)$, with $u \cdot u = 1$. We, therefore, have one ‘radius’ ρ and $n-1$ angles φ^i . The Goldstone matrix reads

$$\eta = e^{X \cdot \theta} = e^{\rho X \cdot u} = \begin{pmatrix} 1 - (1 - \cos \rho) u u^T & \sin \rho u \\ -\sin \rho u^T & \cos \rho \end{pmatrix}, \quad (7.25)$$

where we note that $\rho + 2\pi$ gives the same group element as ρ and unbroken transformations are of the form $e^{\alpha T} = \text{diag}(\mathcal{O}, 1)$ and hence affect $u(\varphi)$ but not ρ . This η yields the one-form

$$\eta^{-1}d\eta|_X = \omega^A X_A = u^A X_A d\rho + \sin \rho \frac{\partial u^A}{\partial \varphi^i} X_A d\varphi^i, \quad (7.26)$$

which, in turn, gives a kinetic term for the Goldstones

$$\frac{f^2}{2} \omega_\mu^A \omega_A^\mu = \frac{f^2}{2} \left(\partial_\mu \rho \partial^\mu \rho + \sin^2 \rho \partial_\mu \varphi^i \frac{\partial u^A}{\partial \varphi^i} \delta_{AB} \frac{\partial u^B}{\partial \varphi^j} \partial^\mu \varphi^j \right), \quad (7.27)$$

where $(\partial u / \partial \varphi)^2$ is the metric in a S_{n-1} sphere. Using eq. (7.9), the Noether currents can be found to be

$$J_A^\mu = f^2 \sqrt{g} (\xi_A \cdot e)^B \omega_B^\mu = f^2 \sqrt{g} \left(u_A \partial^\mu \rho + \sin \rho \cos \rho \frac{\partial u_A}{\partial \varphi^i} \partial^\mu \varphi^i \right), \quad (7.28)$$

$$J_a^\mu = f^2 \sqrt{g} (\xi_A \cdot e)^B \omega_B^\mu = f^2 \sqrt{g} \sin^2 \rho u^T T_a \frac{\partial u}{\partial \varphi} \partial^\mu \varphi, \quad (7.29)$$

and in this case we can find the explicit connection between kinetic term and Noether currents, which also yields a relation between the constants $C, I_{A,a}$:

$$\sqrt{g} f^2 \omega_\mu^A \omega_A^\mu = \frac{1}{\sqrt{g} f^2} (J_\mu^A J_A^\mu + J_\mu^a J_a^\mu), \quad C^2 = \sum_A I_A^2 + \sum_a I_a^2. \quad (7.30)$$

For simplicity, we first look at solutions where the broken currents only are non-vanishing. From eqs. (7.28), (7.29) we see that these are trajectories in which ρ only varies; in this limit the Noether currents J_A^r turn linear in ρ and so does the EOM in eq. (7.22) which can be integrated to find,

$$\rho_{\text{inst}}(r) = \frac{C}{2f^2 L_C^2} \text{arcsec} \left(\frac{r^2}{L_C^2} \right) = \frac{C}{f^2 L_C^2} \arctan[\tanh(\tau)], \quad (7.31)$$

where the second form of the solution is given in terms of the variable $\tau(r)$ (defined in eq. (2.14)) to connect with the axion solution of eq. (2.17) and $C = \sqrt{I_A^2}$. If we now expand around this solution with $\delta\rho, \delta\varphi$, we find at first order, after using the equations of motion for the background,

$$\delta\mathcal{S} = \int dr d\Omega_{3,1} \frac{d}{dr} [\sqrt{g} f^2 \delta\rho \partial^r \rho_{\text{inst}}] = \int d\Omega_3 \delta\rho \sqrt{g} f^2 \partial^r \rho_{\text{inst}}|_{\text{boundary}}. \quad (7.32)$$

If we take the variation $\delta\rho$ to be a constant at the asymptotically flat boundary we obtain

$$\delta\mathcal{S} = \delta\rho \int d\Omega_3 \sqrt{g} f^2 \partial^r \rho = \delta\rho 2\pi^2 C \quad (7.33)$$

and given that ρ and $\rho + 2\pi$ are the same point in the Goldstone manifold (cf. eq. (7.25)) the quantization $C = k/2\pi^2$ with $k \in n\mathbb{Z}$ follows. We have therefore found an action which is the same as that of the axion case. Then the analogy with the axion case yields a potential of the form

$$V(\rho) = -K e^{-\mathcal{S}_{\text{inst}}} \cos(\rho + \delta), \quad (7.34)$$

where δ is a constant. The case for solutions with nonzero broken currents J_a can be obtained by simply changing the coordinate system ρ, φ to some new system $\hat{\rho}, \hat{\varphi}$ where $\hat{\rho}$ only varies for the solutions. In other words the solutions for the fields are geodesics in the sphere S^n where the Goldstone live parametrized by τ and these can have gradients, that is Noether currents, in any direction, J_A^τ, J_a^τ . The general solution will be characterized by a set of constants, I_A, I_a , and by the analysis above and the relation in eq. (7.30), the quantization condition reads

$$C = \sqrt{\sum_A I_A^2 + \sum_a I_a^2} = \frac{k}{2\pi^2}, \quad k \in \mathbb{Z}. \quad (7.35)$$

Therefore the lowest non-trivial solution has $k = 1$ and action as given in eq. (7.23), the same as in the axion case. To the potential generated in eq. (7.34) now we should add the contribution from these other class of solutions, which will depend on $\cos \hat{\rho}$ in the new coordinates. As a consequence of this sum, the flat directions in eq. (7.34) will be lifted. The final potential that is generated is best expressed in terms of the θ coordinates and expanded around the vacuum as

$$V(\rho) \simeq -K e^{-\mathcal{S}_{\text{inst}}} \left[1 - \frac{(\theta^i)^2}{2} \right]. \quad (7.36)$$

We, therefore, conclude this section by summarizing the results of our study. Gravity breaks the full $\mathfrak{G}$ symmetry and produces a potential which generates masses for all Goldstone bosons.

8 Conclusions and outlook

The explicit breaking of global symmetries induced by gravitational effects is a mesmerising phenomenon. Furthermore, far from academic, this issue could have direct implication in phenomenology.

In this paper we considered the specific case of the global shift symmetry of a Goldstone boson, and we explored the non-perturbative breaking due to Euclidean gravitational instantons in the context of Einstein gravity (aka wormholes).

This field has been studied in a decades-long endeavour, and in this regard the main novelties of our analysis are the following.

- *Theoretical analysis.* On the theory side, we addressed the problem of stability of wormhole solutions by computing the spectrum of the quadratic action, and we found a positive spectrum. This result is of fundamental importance since only in the absence of negative eigenvalues the gravitational instantons mediate tunneling transitions between degenerate vacua, in parallel with other known situations both in Quantum Mechanics and Quantum Field Theory. In turn, this property allows to consistently compute the effective potential generated by gravitational instantons. The latter, as a consequence of its non-perturbative nature, is characterized by the suppression factor $e^{-\mathcal{S}_{\text{inst}}}$, where $\mathcal{S}_{\text{inst}}$ is the wormhole action which scales with the Goldstone decay constant and Planck mass as $\mathcal{S}_{\text{inst}} \propto M_{\text{Pl}}/f_a$.

- *Phenomenological analysis.* On the phenomenological side, we focused our analysis on the compelling case of the QCD axion. By computing non-perturbative gravitational corrections to the QCD axion potential, we found the following lower bound on the mass of the QCD axion

$$m_a \gtrsim 4.8 \times 10^{-10} \text{ eV}, \quad f_a \lesssim 10^{16} \text{ GeV}. \quad (8.1)$$

As an application, we discussed important consequences related to black hole superradiance, and we showed that the mass range $m_a = [10^{-14}, 10^{-10}] \text{ eV}$ motivated by experimental searches and phenomenological considerations is theoretically disfavored.

In addition to the QCD axion, we discussed the case of ultralight scalars as cosmological dark matter. We showed that non-perturbative gravitational effects due to Einstein gravity generate a mass term in agreement with the estimate of the relic abundance based on the misalignment mechanism. In numbers, we found a dark matter candidate with mass $m_a \simeq 2.5 \times 10^{-18} \text{ eV}$ corresponding to the decay constant $f_a \simeq 8 \times 10^{15} \text{ GeV}$.

We concluded with a derivation of the of wormhole solution for a generic Goldstone coset, and we inspected the breaking pattern $O(n+1)/O(n)$ to find that all Goldstone bosons in the coset acquire a mass via gravitational effects.

Note added. After our work, ref. [83] appeared. In this paper the authors study perturbations around the wormhole solution. As far as homogenous scalar perturbations are concerned, our results agree with those of ref. [83]: there are no homogeneous propagating degrees of freedom. However, ref. [83] finds multiple negative modes in the inhomogeneous sector.

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