Fermion zero modes for the mixed-flux AdS_3 giant magnon

Adam Varga

*Department of Mathematics, School of Mathematics, Computer Science and Engineering,
City, University of London,
London, EC1V 0HB, U.K.*

E-mail: Adam.Varga@city.ac.uk

ABSTRACT: We explicitly construct the four and two fermion zero modes for the mixed-flux generalization of the Hofman-Maldacena giant magnon on two of the AdS_3 backgrounds with maximal amount of supersymmetry, $\text{AdS}_3 \times \text{S}^3 \times \text{T}^4$ and $\text{AdS}_3 \times \text{S}^3 \times \text{S}^3 \times \text{S}^1$. We also show how to get the $\mathfrak{psu}(1|1)^4$ and $\mathfrak{su}(1|1)^2$ superalgebras from the semiclassically quantized fermion zero modes.

KEYWORDS: AdS-CFT Correspondence, Superstrings and Heterotic Strings

ARXIV EPRINT: [1901.00530](https://arxiv.org/abs/1901.00530)

Contents

1	Introduction	1
2	Bosonic solution	4
2.1	Strings on $\mathbb{R} \times S^3$ with mixed flux	4
2.2	The $SU(2)$ principal chiral model	5
2.3	Two-charge giant magnon on $\mathbb{R} \times S^3$	6
2.4	Stationary magnon on $\mathbb{R} \times S^3$	7
2.4.1	Parameter ranges	8
2.5	Mixed-flux giant magnon on $AdS_3 \times S^3 \times S^3 \times S^1$	8
3	Fermion zero mode equations	10
3.1	Fermionic equations of motion	10
3.2	Zero mode condition	12
3.3	Fixing kappa symmetry	13
3.4	Zero mode equations	15
3.4.1	Equations for $\Delta = 0$	16
3.4.2	The case of $\Delta \neq 0$	16
4	Mixed-flux fermion zero modes	17
4.1	Fixing kappa-gauge	17
4.2	Zero mode solutions	18
4.2.1	Solutions on the P_+ subspace	18
4.2.2	Solutions on the P_- subspace	19
4.3	Zero mode action	21
4.4	Zero-mode algebra for $AdS_3 \times S^3 \times S^3 \times S^1$	22
4.5	Zero-mode algebra for $AdS_3 \times S^3 \times T^4$	23
4.6	Zero modes in the $\alpha \rightarrow 0, 1$ limits	23
5	Fermion zero modes for $q = 1$	24
5.1	Bosonic solution	24
5.2	Zero mode equations for $AdS_3 \times S^3 \times T^4$	25
5.3	Zero mode solutions for $AdS_3 \times S^3 \times T^4$	26
5.4	Zero modes for $AdS_3 \times S^3 \times S^3 \times S^1$	27
5.5	The $q \rightarrow 1$ limit	28
6	Conclusions	28
A	Geometry of AdS_3 and S^3	29
A.1	AdS_3	30
A.2	S^3	31

B	String solutions on $\mathbb{R} \times S^3 \times S^3$	32
C	Pullback of the vielbein and spin connection to the worldsheet	33
D	Gamma matrices	33
E	Fermion derivatives	34
F	No normalizable solutions for $\Delta \neq 0$	36
G	Phase identities	41
H	The $\mathfrak{su}(1 1)_{\text{c.e.}}^2$ algebra and its representaions	41
	H.1 The $\mathfrak{su}(1 1)$ algebra	41
	H.2 The $\mathfrak{su}(1 1)^2$ algebra	42
	H.3 The centrally extended $\mathfrak{su}(1 1)^2$ algebra	43
	H.3.1 The left-moving representation	43
	H.3.2 The right-moving representation	44
I	The $\mathfrak{psu}(1 1)_{\text{c.e.}}^4$ algebra and its representations	44
	I.1 From $\mathfrak{su}(1 1)_{\text{c.e.}}^2$ to $\mathfrak{psu}(1 1)_{\text{c.e.}}^4$	44
	I.2 Bi-fundamental representations	45
J	$SU(2)$ currents for the $q = 1$ giant magnon	46
K	Terms appearing in the $q = 1$ fermion equations	47

1 Introduction

In the study of the holographic correspondence [1] integrability plays a key role, as it allows us to reduce the dynamics of interactions to diffractionless two-body scattering of elementary excitations, or magnons. In the context of the best understood example, the AdS_5/CFT_4 duality, integrable structures were found on both the gauge theory ($\mathcal{N} = 4$ super Yang-Mills) [2–4] and string theory (type IIB superstring theory on $AdS_5 \times S^5$) sides [5–9]. This shifted focus to the worldsheet S-matrix [10], which was determined (up to an overall phase) by Beisert, using only the $SU(2|2) \times SU(2|2)$ symmetry of the theory [11, 12]. The remaining phase factor was then calculated using crossing symmetry [13–17].

An important result of Beisert’s analysis was the dispersion relation for the magnon

$$\epsilon = \sqrt{1 + 4h^2 \sin^2 \frac{p}{2}}, \quad (1.1)$$

where

$$h = \frac{\sqrt{\lambda}}{2\pi}, \quad (1.2)$$

and p is the momentum of the magnon on the worldsheet. Hofman and Maldacena explicitly constructed a classical string configuration on $\mathbb{R} \times S^2$, naming it the giant magnon [18], with dispersion relation $\epsilon = 2h \sin \frac{p}{2}$, in agreement with the large coupling limit (this is where the string theory approximation is valid) of (1.1). Subsequently this was generalized to the dyonic giant magnon [19], living on $\mathbb{R} \times S^3$, with the exact dispersion relation already at the (semi-)classical level.

The giant magnon is a BPS state, and as such, it should be part of a 16 dimensional short multiplet of $SU(2|2) \times SU(2|2)$ [12]. Hofman and Maldacena argued that in order to reproduce this representation, the giant magnon should have eight fermion zero modes [18]. This was later explicitly shown by Minahan [20], who constructed these zero modes from the quadratic fermion fluctuation piece of the Green-Schwarz action, taking the giant magnon as the background. Quantizing these modes, he was also able to match them to the fermionic generators of the $SU(2|2) \times SU(2|2)$ residual algebra.

Remarkably, integrability persists to other, less symmetric classes of AdS/CFT duals. There has been significant progress in AdS_4/CFT_3 , for references see [21], but this paper is concerned with AdS_3/CFT_2 . In particular, we will focus on two¹ AdS_3 backgrounds with maximal supersymmetry (16 supercharges), $AdS_3 \times S^3 \times T^4$ and $AdS_3 \times S^3 \times S^3 \times S^1$. Supergravity equations relate the radii of AdS_3 and S^3 components, for $AdS_3 \times S^3 \times T^4$ they give

$$R_{AdS_3} = R_{S^3}, \quad (1.3)$$

while for $AdS_3 \times S^3 \times S^3 \times S^1$ the AdS radius R and the radii of the two 3-spheres $R_{\pm}$ must satisfy [22]

$$\frac{1}{R_+^2} + \frac{1}{R_-^2} = \frac{1}{R^2}. \quad (1.4)$$

Historically, these backgrounds were considered in two different settings: either supported by Ramond-Ramond (R-R) or Neveu-Schwarz-Neveu-Schwarz (NS-NS) fluxes. They were shown to be classically integrable with pure R-R flux [23–25], and in fact remain classically integrable even when supported by mixed R-R and NS-NS fluxes [26]

$$\begin{aligned} F &= 2\tilde{q} \left(\text{Vol}(AdS_3) + \cos \varphi \text{Vol}(S_+^3) + \sin \varphi \text{Vol}(S_-^3) \right), \\ H &= 2q \left(\text{Vol}(AdS_3) + \cos \varphi \text{Vol}(S_+^3) + \sin \varphi \text{Vol}(S_-^3) \right), \end{aligned} \quad (1.5)$$

where the overall factors satisfy $\tilde{q} = \sqrt{1 - q^2}$, and the range is given by $q \in [0, 1]$. Recently, it was found that the mixed-flux action is equivalent to the pure NS-NS theory with an R-R modulus turned on [27] upon identifying q and $\tilde{q}$ as

$$q = k \frac{\alpha'}{R^2}, \quad \tilde{q} = -g_s c_0 k \frac{\alpha'}{R^2}, \quad (1.6)$$

¹There is a third background with 16 supercharges: $AdS_3 \times S^3 \times K3$, which should also be possible to understand using integrable methods, at least in the orbifold limit of $K3$, and it would then be interesting to see what happens when blow-up modes are turned on.

where k is integral and c_0 is continuous. This is different to the conventional interpretation of the mixed-flux background as the near-horizon limit of bound states of D1/D5- and F1/NS5-branes carrying R-R and NS-NS charges, respectively. Signs of integrability on the CFT side have also been identified [28] in the CFT₂ dual to strings on $\text{AdS}_3 \times \text{S}^3 \times \text{T}^4$, however, finding the CFT₂ dual of $\text{AdS}_3 \times \text{S}^3 \times \text{S}^3 \times \text{S}^1$ string theory proved to be a difficult problem [29–32].

Assuming that integrability holds for the mixed-flux theory at the quantum level, Hoare and Tseytlin first calculated the tree-level S-matrix for the massive spectrum of $\text{AdS}_3 \times \text{S}^3 \times \text{T}^4$ in uniform light-cone gauge [33], then, by analysing the constraints of symmetry, they proposed an exact massive worldsheet S-matrix [34], generalizing the result of [35] to $q \neq 0$. They also found the magnon dispersion relation to be

$$\epsilon_{\pm} = \sqrt{M_{\pm}^2 + 4\tilde{q}^2 h^2 \sin^2 \frac{p}{2}}. \quad (1.7)$$

The dyonic giant magnon solutions [19] on $\mathbb{R} \times \text{S}^3$, with two angular momenta (J_1, J_2) , were lifted to $q \neq 0$ by Hoare, Stepanchuk and Tseytlin [36], fixing²

$$M_{\pm} = J_2 \pm qhp. \quad (1.8)$$

Quantization leads to $M_{\pm} = m \pm qhp$ with $m = 1$, and in fact, using symmetry arguments, the dispersion relation was shown to hold to all loops for both massive and massless magnons $m = 1, 0$ in [39], where the massless and mixed-mass S-matrices were also determined. In recent developments, integrable methods were used to derive the protected spectrum of these AdS_3 backgrounds, proving that the dispersion relations above receive no corrections to all orders in the string tension [40, 41]. The same protected spectrum in the $\text{AdS}_3 \times \text{S}^3 \times \text{S}^3 \times \text{S}^1$ background was derived independently using supergravity and WZW methods in [42]. In the case of the $\text{AdS}_3 \times \text{S}^3 \times \text{T}^4$ background the protected spectrum agrees with the older results of [43].

This paper is concerned with the fermion fluctuations around the AdS_3 giant magnon. The residual (off-shell) symmetry algebra of the ground state of $\text{AdS}_3 \times \text{S}^3 \times \text{T}^4$ superstring theory is the centrally extended $\mathfrak{psu}(1|1)^4$ superalgebra [35, 39, 44, 45], while on $\text{AdS}_3 \times \text{S}^3 \times \text{S}^3 \times \text{S}^1$ the elementary excitations transform under the centrally extended $\mathfrak{su}(1|1)^2$ algebra [46, 47]. Analogously to the case of $\text{AdS}_5 \times \text{S}^5$, the giant magnon is a BPS state, and should be part of the 4 and 2 dimensional short multiplets of $\mathfrak{psu}(1|1)^4$ and $\mathfrak{su}(1|1)^2$, respectively. To reproduce these representations, the mixed-flux giant magnon on $\text{AdS}_3 \times \text{S}^3 \times \text{T}^4$ and $\text{AdS}_3 \times \text{S}^3 \times \text{S}^3 \times \text{S}^1$ should have 4 and 2 fermion zero modes, respectively, and our main objective is to find these zero modes, following the calculation of Minahan [20]. The rest of this paper is structured as follows.

In section 2 we present the two-charge giant magnon on $\text{AdS}_3 \times \text{S}^3 \times \text{T}^4$ with mixed flux, found by Hoare, Stepanchuk and Tseytlin [36], and describe a one-parameter family generalizing the Hofman-Maldacena magnon of $q = 0$, that we call *stationary*. The reason

²The same mixed-flux classical giant magnon solution was also found as a limit of rigidly rotating strings [37], and using the dressing method [38].

Minahan [20] managed to find zero modes relatively easily is that he took the HM magnon, rather than the more general dyonic magnon, as starting point. Similarly, the stationary magnon is the bosonic background that will make subsequent calculations most simple. Finally, we outline how the magnon can be put on $\text{AdS}_3 \times \text{S}^3 \times \text{S}^3 \times \text{S}^1$.

In section 3 we discuss the quadratic fermionic action, which is obtained from the GS action by considering perturbations around the giant magnon as background. We will look at the zero mode condition and kappa-gauge fixing, before arriving at the zero mode equations of motion. These equations are then solved in section 4, to get the expected number of normalizable zero modes. After semiclassical quantization, we construct the fermionic generators of the corresponding superalgebras.

In section 5 we consider the special case of $q = 1$. In agreement with the chiral nature of the background, we find that all of the zero modes are non-normalizable. Since the notion of stationary magnon breaks down, we cannot simply take the $q \rightarrow 1$ limit of the zero modes found for $q < 1$, and the issue of semiclassical quantization also needs further attention. This is a question we hope to return to in the future. We conclude in section 6 and present some of the more technical details in appendices.

2 Bosonic solution

In the case of the $\text{AdS}_3 \times \text{S}^3 \times \text{S}^3 \times \text{S}^1$ background, choosing unit radius for AdS, the supergravity equations allow a one-parameter family of radii for the two spheres $\text{S}^3_{\pm}$:

$$\frac{1}{R_+^2} = \alpha \equiv \cos^2 \varphi, \quad \frac{1}{R_-^2} = 1 - \alpha \equiv \sin^2 \varphi, \quad (2.1)$$

and in fact this covers $\text{AdS}_3 \times \text{S}^3 \times \text{T}^4$ as well, for $\varphi = 0$. Another parameter of the theory is $q \in [0, 1]$, describing the amount of NS-NS background flux, or equivalently, the R-R modulus in the pure NS-NS theory via $\tilde{q} = \sqrt{1 - q^2}$, as described below (1.5).

Most of this section is a summary of the work done by Hoare, Stepanchuk and Tseytlin on the mixed-flux two-charge giant magnon on $\mathbb{R} \times \text{S}^3$ [36]. We make the contribution of pointing out that a certain restriction of their solution can be regarded as the mixed-flux generalization of the Hofman-Maldacena magnon, which will enable us to identify the fermion zero modes on the the AdS_3 backgrounds. Furthermore, we describe the corresponding solution on $\text{AdS}_3 \times \text{S}^3 \times \text{S}^3 \times \text{S}^1$.

2.1 Strings on $\mathbb{R} \times \text{S}^3$ with mixed flux

Using Hopf coordinates for the 3-sphere (see appendix A)

$$Z_1 = \sin \theta e^{i\phi_1}, \quad Z_2 = \cos \theta e^{i\phi_2}, \quad (2.2)$$

the $\mathbb{R} \times \text{S}^3$ bosonic string action in static conformal gauge, which sets the target-space time (coordinate on $\mathbb{R}$) proportional to the worldsheet time

$$t = \kappa \tau, \quad (2.3)$$

is given by

$$S_1[\Omega] = -\frac{\hbar}{2} \int_{\mathcal{M}} d^2\sigma \left[\partial_a \theta \partial^a \theta + \sin^2 \theta \partial_a \phi_1 \partial^a \phi_1 + \cos^2 \theta \partial_a \phi_2 \partial^a \phi_2 + q(\cos 2\theta + c)(\dot{\phi}_1 \phi'_2 - \dot{\phi}_2 \phi'_1) \right], \quad (2.4)$$

where $a = 0, 1$ correspond to τ, σ , $\dot{} = \partial_\tau$, $' = \partial_\sigma$, the worldsheet metric is $\eta = \text{diag}(-1, 1)$, $\hbar$ is the string tension, and we represent maps from the worldsheet to S^3 by $\Omega = (\theta, \phi_1, \phi_2)$. The expression proportional to q represents the Wess-Zumino term, and the parameter c was introduced by Hoare et al. [36]. This c -term is a total derivative and drops out of the equations of motion, but will affect the conserved charges for string solutions with non-periodic boundary conditions, e.g. the dyonic giant magnon.

From the action it is an easy exercise to derive the equations of motion

$$\begin{aligned} \ddot{\theta} - \theta'' - \sin \theta \cos \theta \left(\dot{\phi}_1^2 - \phi_1'^2 - \dot{\phi}_2^2 + \phi_2'^2 + 2q(\dot{\phi}_1 \phi'_2 - \dot{\phi}_2 \phi'_1) \right) &= 0, \\ \sin^2 \theta \left(\ddot{\phi}_1 - \phi_1'' \right) + 2 \sin \theta \cos \theta \left(\dot{\theta} \dot{\phi}_1 - \theta' \phi_1' + q(\theta' \dot{\phi}_2 - \dot{\theta} \phi_2') \right) &= 0, \\ \cos^2 \theta \left(\ddot{\phi}_2 - \phi_2'' \right) - 2 \sin \theta \cos \theta \left(\dot{\theta} \dot{\phi}_2 - \theta' \phi_2' + q(\theta' \dot{\phi}_1 - \dot{\theta} \phi_1') \right) &= 0, \end{aligned} \quad (2.5)$$

which need to be supplemented with the conformal gauge Virasoro constraints

$$\begin{aligned} V_1[\Omega] &\equiv \dot{\theta}^2 + \theta'^2 + \sin^2 \theta (\dot{\phi}_1^2 + \phi_1'^2) + \cos^2 \theta (\dot{\phi}_2^2 + \phi_2'^2) = \kappa^2, \\ V_2[\Omega] &\equiv \dot{\theta} \theta' + \sin^2 \theta \dot{\phi}_1 \phi_1' + \cos^2 \theta \dot{\phi}_2 \phi_2' = 0. \end{aligned} \quad (2.6)$$

Conserved charges. Classical string solutions on $\mathbb{R} \times S^3$ will have a number of conserved Noether charges, and the ones of particular interest to us are the spacetime energy E (due to translational invariance in AdS time t) and the angular momenta J_1 and J_2 (due to invariance under shifts in ϕ_1 and ϕ_2)

$$\begin{aligned} E &= 2\pi \hbar \kappa, \\ J_1[\Omega] &= \hbar \int_{-\pi}^{\pi} d\sigma \left[\sin^2 \theta \dot{\phi}_1 - \frac{q}{2}(\cos 2\theta + c)\phi_2' \right], \\ J_2[\Omega] &= \hbar \int_{-\pi}^{\pi} d\sigma \left[\cos^2 \theta \dot{\phi}_2 + \frac{q}{2}(\cos 2\theta + c)\phi_1' \right]. \end{aligned} \quad (2.7)$$

2.2 The SU(2) principal chiral model

The conformal gauge string action on $\mathbb{R} \times S^3$ is equivalent to that of the *principal chiral model* (PCM) with a Wess-Zumino term (proportional to $q \in [0, 1]$) and underlying group SU(2), which has a group manifold diffeomorphic to S^3 . In terms of the left currents $\mathfrak{J} = g^{-1}dg$, where $g \in \text{SU}(2)$, the action is given by

$$S = -\frac{\hbar}{2} \left[\int_{\mathcal{M}} d^2\sigma \frac{1}{2} \text{tr}(\mathfrak{J}_+ \mathfrak{J}_-) - q \int_{\mathcal{B}} d^3\sigma \frac{1}{3} \varepsilon^{abc} \text{tr}(\mathfrak{J}_a \mathfrak{J}_b \mathfrak{J}_c) \right], \quad \mathfrak{J}_a = g^{-1} \partial_a g, \quad (2.8)$$

where $\mathcal{M}$ is the decompactified string worldsheet, $\mathcal{B}$ is a 3d manifold with boundary $\mathcal{M}$, and $\sigma^\pm = \frac{1}{2}(\tau \pm \sigma)$. Using the parametrization

$$g = \begin{pmatrix} Z_1 & Z_2 \\ -Z_2^* & Z_1^* \end{pmatrix} \in \text{SU}(2), \quad (2.9)$$

and substituting (2.2) we get the action (2.4). Also introducing the right current $\mathfrak{K} = dg g^{-1}$, the PCM equations of motion can be written in the two equivalent forms

$$\begin{aligned} (1+q)\partial_- \mathfrak{J}_+ + (1-q)\partial_+ \mathfrak{J}_- &= 0, \\ (1-q)\partial_- \mathfrak{K}_+ + (1+q)\partial_+ \mathfrak{K}_- &= 0. \end{aligned} \quad (2.10)$$

From the action we get the left-invariant and right-invariant conserved SU(2) currents

$$L_a = \mathfrak{J}_a - q\epsilon_{ab}\mathfrak{J}^b, \quad R_a = \mathfrak{K}_a + q\epsilon_{ab}\mathfrak{K}^b, \quad \partial_a L^a = \partial_a R^a = 0, \quad (2.11)$$

which give rise to the conserved charges

$$Q_L = \hbar \int d\sigma (\mathfrak{J}_0 + q\mathfrak{J}_1), \quad Q_R = \hbar \int d\sigma (\mathfrak{K}_0 - q\mathfrak{K}_1). \quad (2.12)$$

From these we can define the following pair of scalar charges, of particular interest for the case of the giant magnon

$$J = -\frac{i}{4} (\text{tr}[Q_L \cdot \sigma_3] + \text{tr}[Q_R \cdot \sigma_3]), \quad M = -\frac{i}{4} (-\text{tr}[Q_L \cdot \sigma_3] + \text{tr}[Q_R \cdot \sigma_3]). \quad (2.13)$$

Substituting in the Hopf parametrization (2.2) we get

$$\begin{aligned} J &= \hbar \int_{-\pi}^{\pi} d\sigma \left[\sin^2 \theta \dot{\phi}_1 - \frac{q}{2} (\cos 2\theta + 1) \phi_2' \right], \\ M &= \hbar \int_{-\pi}^{\pi} d\sigma \left[\cos^2 \theta \dot{\phi}_2 + \frac{q}{2} (\cos 2\theta - 1) \phi_1' \right]. \end{aligned} \quad (2.14)$$

Comparing these SU(2) charges to those in (2.7) we have

$$J_1 = J - \frac{c-1}{2} \hbar q \Delta \phi_2, \quad J_2 = M + \frac{c+1}{2} \hbar q \Delta \phi_1, \quad (2.15)$$

where $\Delta \phi_i = \phi_i(\pi) - \phi_i(-\pi)$. We see that non-zero boundary twists $\Delta \phi_i$ break the SU(2) symmetry, there is no choice of c for which J, M are obtained as Noether charges of the local action (2.4).

2.3 Two-charge giant magnon on $\mathbb{R} \times \mathbb{S}^3$

The giant magnon is a solution in the Hofman-Maldacena limit [18], where E and J_1 are taken to infinity — i.e. $\kappa \rightarrow \infty$ — with their difference held fixed (thus finite)

$$E, J_1 \rightarrow \infty, \quad E - J_1, J_2 = \text{fixed}. \quad (2.16)$$

We can decompactify the worldsheet by rescaling³

$$x = \kappa\sigma, \quad t = \kappa\tau, \quad \kappa \rightarrow \infty, \quad x \in (-\infty, +\infty), \quad (2.17)$$

which essentially describes an open string with non-trivial boundary conditions. With this, the giant magnon on $\text{AdS}_3 \times \text{S}^3 \times \text{T}^4$ with mixed flux [36] is given by

$$Z_1 = \frac{e^{it} [b + i \tanh U]}{\sqrt{1 + b^2}}, \quad Z_2 = \frac{e^{iV} \text{sech} U}{\sqrt{1 + b^2}}, \quad (2.18)$$

where, with $\gamma = (1 - u^2)^{-1/2}$,

$$\begin{aligned} U &= \cos \rho \tilde{q} \gamma (x - ut), \\ V &= \sin \rho \tilde{q} \gamma (t - ux) - qx, \end{aligned} \quad (2.19)$$

and the parameters⁴ are related via

$$b = \tilde{q} \gamma u \sec \rho + q \tan \rho. \quad (2.20)$$

Hoare et al. also found that the worldsheet momentum of the magnon is related to the opening angle between the two endpoints on the equator $p = \Delta\phi_1$. Looking at (2.18) this is

$$p = 2 \arccot b. \quad (2.21)$$

Furthermore, the boundary term in the local action (2.4) was fixed to be $c = 1$ to keep the difference $E - J_1$ finite, in particular

$$\begin{aligned} E - J_1 &= 2h\tilde{q}\gamma \sec \rho \sin^2 \frac{p}{2}, \\ J_2 &= M + hqp, \quad M = 2h \sin^2 \frac{p}{2} (\tan \rho - q \cot \frac{p}{2}). \end{aligned} \quad (2.22)$$

Using (2.20), it is easy to see that the magnon satisfies the dispersion relation

$$E - J_1 = \sqrt{(J_2 - hqp)^2 + 4h^2 \tilde{q}^2 \sin^2 \frac{p}{2}}. \quad (2.23)$$

2.4 Stationary magnon on $\mathbb{R} \times \text{S}^3$

The parameter u can be regarded as the velocity of the magnon, and the boosted worldsheet coordinates

$$\mathcal{X} = \gamma(x - ut), \quad \mathcal{T} = \gamma(t - ux) \quad (2.24)$$

naturally appear in the solution (2.18). In the next section we want to obtain the fermion zero modes of the giant magnon. These are, in some loose sense, independent of time, but also require the bosonic solution to be stationary, i.e. have a time-independent shape. In

³Slightly abusing notation, in static gauge the target-space time is functionally the same as our rescaled τ .

⁴Throughout [36] another parameter v is used, which is the magnon speed only in the $q = 0$ limit, and related to our parameter u by the relativistic boost $u = \frac{v-q}{1-vq}$.

other words, apart from the e^{it} term in Z_1 , we want the solution to only depend on $\mathcal{X}$. This fixes the value of ρ

$$\sin \rho = \frac{qu}{\sqrt{1-q^2}\sqrt{1-u^2}} = \frac{\gamma u q}{\tilde{q}}, \quad (2.25)$$

and the solution becomes

$$\begin{aligned} Z_1 &= \frac{e^{it} \left[b + i \tanh\left(\gamma\sqrt{\tilde{q}^2 - u^2}\mathcal{X}\right) \right]}{\sqrt{1+b^2}}, \\ Z_2 &= \frac{e^{-iq\gamma\mathcal{X}} \operatorname{sech}\left(\gamma\sqrt{\tilde{q}^2 - u^2}\mathcal{X}\right)}{\sqrt{1+b^2}}, \quad b = \frac{u}{\sqrt{\tilde{q}^2 - u^2}}. \end{aligned} \quad (2.26)$$

It is worth noting that $M = 0$ for this choice of the parameter ρ , so the stationary magnon is a natural restriction from the perspective of the PCM, and the dispersion relation (2.23) takes the simpler form

$$E - J_1 = 2h\tilde{q} \sin \frac{p}{2}. \quad (2.27)$$

This is much like the dispersion relation of the single-charge magnon of Hofman and Maldacena, for which Minahan found the fermion zero modes in the AdS^5 geometry [20]. This further justifies taking the stationary magnon as the starting point of the zero-mode analysis.

2.4.1 Parameter ranges

Since u is a worldsheet speed, one might expect it to take values in the range $(-1, 1)$. This is certainly true for the general solution (2.18), but the stationary condition (2.25) further restricts

$$\sin^2 \rho \leq 1 \quad \Rightarrow \quad |u| \leq \tilde{q}. \quad (2.28)$$

It might look like we are missing some solutions, but in fact there will be a stationary magnon for each value of the worldsheet momentum p . This becomes obvious once we rewrite the stationary condition (2.25) using (2.20) as

$$u = \tilde{q} \cos \frac{p}{2}, \quad \tan \rho = q \cot \frac{p}{2}. \quad (2.29)$$

2.5 Mixed-flux giant magnon on $\text{AdS}_3 \times \text{S}^3 \times \text{S}^3 \times \text{S}^1$

We can construct the $\text{AdS}_3 \times \text{S}^3 \times \text{S}^3 \times \text{S}^1$ dyonic magnon using the prescription of appendix B, by putting the magnon (2.18) on S^3_+ and the BMN string on S^3_- . In Hopf coordinates for the two spheres $\text{S}^3_\pm$

$$Z_1^\pm = \sin \theta^\pm e^{i\phi_1^\pm}, \quad Z_2^\pm = \cos \theta^\pm e^{i\phi_2^\pm}, \quad (2.30)$$

the solution is given by

$$\begin{aligned}
\theta^+ &= \arccos \left(\frac{\operatorname{sech} [A \cos \rho \tilde{q} \mathcal{X}]}{\sqrt{1+b^2}} \right), \\
\phi_1^+ &= At + \arctan (b^{-1} \tanh [A \cos \rho \tilde{q} \mathcal{X}]), & \phi_2^+ &= A \sin \rho \tilde{q} \mathcal{T} - Aqx, \\
\theta^- &= \frac{\pi}{2}, & \phi_1^- &= Bt, & \phi_2^- &= 0. \\
\gamma^2 &= \frac{1}{1-u^2}, & b &= \tilde{q} \gamma u \sec \rho + q \tan \rho, & u &\in (0, 1),
\end{aligned} \tag{2.31}$$

The parameters A and B determine the angle at which the ends of the string move in the (ϕ_1^+, ϕ_1^-) plane, and they satisfy the Virasoro constraint

$$\frac{A^2}{\cos^2 \varphi} + \frac{B^2}{\sin^2 \varphi} = 1. \tag{2.32}$$

Noether charges. The $\text{AdS}_3 \times \text{S}^3 \times \text{S}^3 \times \text{S}^1$ action is invariant under four different angular shifts leading to four conserved angular momenta on top of the conserved energy, in terms of the $\mathbb{R} \times \text{S}^3$ charges (2.7)

$$E = 2\pi\hbar\kappa, \quad J_{1,2}^+ = \frac{1}{\cos^2 \varphi} J_{1,2}[\Omega_+], \quad \text{and} \quad J_{1,2}^- = \frac{1}{\sin^2 \varphi} J_{1,2}[\Omega_-]. \tag{2.33}$$

The physical angular momenta relevant to the giant magnon are

$$J_1 \equiv AJ_1^+ + BJ_1^-, \quad J_2 = J_2^+, \tag{2.34}$$

and with these the giant magnon has

$$\begin{aligned}
E - J_1 &= \frac{A}{\cos^2 \varphi} 2\hbar\tilde{q}\gamma \sec \rho \sin^2 \frac{\rho}{2}, \\
J_2 &= \frac{1}{\cos^2 \varphi} (M + \hbar q p), \quad M = 2\hbar \sin^2 \frac{\rho}{2} (\tan \rho - q \cot \frac{\rho}{2}).
\end{aligned} \tag{2.35}$$

The dispersion relation is therefore

$$E - J_1 = \frac{A}{\cos^2 \varphi} \sqrt{(\cos^2 \varphi J_2 - \hbar q p)^2 + 4\hbar^2 \tilde{q}^2 \sin^2 \frac{\rho}{2}}. \tag{2.36}$$

There are two conclusions to be made. Firstly, to match the correct dispersion relation derived from symmetry [47], we need to take $A = \cos^2 \varphi$, a choice that is also physically motivated if we recall that the giant magnon is an excitation above the BMN vacuum. The true vacuum of the theory should preserve maximal amount of supersymmetry, and this condition leaves the (up to signs) unique choice [23] $A = \cos^2 \varphi$, $B = \sin^2 \varphi$, which we will refer to as *maximally SUSY* solution. Secondly, we see that we have found one of the light magnons with mass $m = \cos^2 \varphi$. We can get the other light magnon of mass $\sin^2 \varphi$ by switching the two spheres, but we have not been able to find the massless ($m = 0$) or heavy ($m = 1$) magnons with this construction.

Stationary magnon. As the starting point of our fermion zero mode analysis we will take the maximally SUSY $\text{AdS}_3 \times \text{S}^3 \times \text{S}^3 \times \text{S}^1$ generalization of the stationary magnon (2.26) given by

$$\begin{aligned}\theta^+ &= \arccos\left(\frac{\text{sech}\mathcal{Y}}{\sqrt{1+b^2}}\right), \\ \phi_1^+ &= \cos^2\varphi t + \arctan(b^{-1} \tanh\mathcal{Y}), & \phi_2^+ &= -\frac{q\mathcal{Y}}{\sqrt{\tilde{q}^2 - u^2}}, \\ \theta^- &= \frac{\pi}{2}, & \phi_1^- &= \sin^2\varphi t, & \phi_2^- &= 0. \\ \gamma^2 &= \frac{1}{1-u^2}, & b &= \frac{u}{\sqrt{\tilde{q}^2 - u^2}}, & u &\in (-\tilde{q}, \tilde{q}),\end{aligned}\tag{2.37}$$

where we further defined the scaled and boosted worldsheet coordinate

$$\mathcal{Y} = \cos^2\varphi \gamma \sqrt{\tilde{q}^2 - u^2} \mathcal{X}.\tag{2.38}$$

3 Fermion zero mode equations

In this section we look at the equations of motion describing fermion perturbations around the stationary giant magnon (2.37). Note that this treatment includes both the $\text{AdS}_3 \times \text{S}^3 \times \text{S}^3 \times \text{S}^1$ and $\text{AdS}_3 \times \text{S}^3 \times \text{T}^4$ (for $\varphi = 0$) cases. We explain what is meant by zero modes, and describe in some detail the fixing of fermionic kappa-gauge. Finally, we write down the zero mode equations for kappa-fixed spinors, that will be solved in the next section.

3.1 Fermionic equations of motion

The quadratic fermionic action in conformal gauge is given by [48]

$$S_F = \hbar \int d^2\sigma \mathcal{L}_F, \quad \mathcal{L}_F = -i \left(\eta^{ab} \delta^{IJ} + \epsilon^{ab} \sigma_3^{IJ} \right) \bar{\vartheta}^I \rho_a \mathcal{D}_b \vartheta^J, \tag{3.1}$$

where $I, J = 1, 2$, the ϑ^I are ten-dimensional Majorana-Weyl spinors, and ρ_a are projections of the ten-dimensional Dirac matrices⁵

$$\rho_a \equiv e_a^A \Gamma_A, \quad e_a^A \equiv \partial_a X^\mu E_\mu^A(X). \tag{3.2}$$

X^μ are the coordinates of the target spacetime $\text{AdS}_3 \times \text{S}^3 \times \text{S}^3 \times \text{S}^1$ and will be evaluated on the classical solution (2.37). The giant magnon solution has non-constant components for $\mu = t, \theta^+, \phi_1^+, \phi_2^+, \phi_1^-$ corresponding to the tangent space components $A = 0, 3, 4, 5, 7$ respectively. The covariant derivative is given by

$$\mathcal{D}_a \vartheta^I = \left(D_a \delta^{IJ} + \frac{1}{48} \not{F} \rho_a \sigma_1^{IJ} + \frac{1}{8} \not{H}_a \sigma_3^{IJ} \right) \vartheta^J, \tag{3.3}$$

⁵The matrices ρ_a are not to be confused with the parameter ρ of the dyonic giant magnon (2.18), which has been fixed for the stationary magnon, and will not appear in this section.

where $D_a = \partial_a + \frac{1}{4}\omega_a^{AB}\Gamma_{AB}$ with the pullback of the spin connection $\omega_a^{AB} \equiv \partial_a X^\mu \omega_\mu^{AB}$. For a detailed review of the vielbein and spin connection the reader is referred to appendix A, while explicit expressions for the pullbacks e_a^A , ω_a^{AB} can be found in appendix C.

The tangent space components of the fluxes (1.5) are given by

$$F_{012} = 2\tilde{q}, \quad F_{345} = 2\tilde{q} \cos \varphi, \quad F_{678} = 2\tilde{q} \sin \varphi, \quad (3.4)$$

$$H_{012} = 2q, \quad H_{345} = 2q \cos \varphi, \quad H_{678} = 2q \sin \varphi, \quad (3.5)$$

and they appear in the action as $\not{F} \equiv F_{ABC}\Gamma^{ABC}$, $\not{H}_a \equiv e_a^A H_{ABC}\Gamma^{BC}$. Introducing

$$\Gamma_* \equiv \Gamma^{012}, \quad (\Gamma_*)^2 = \mathbb{1}, \quad (3.6)$$

$$\Gamma_+ \equiv \Gamma^{345}, \quad (\Gamma_+)^2 = -\mathbb{1}, \quad (3.7)$$

$$\Gamma_- \equiv \Gamma^{678}, \quad (\Gamma_-)^2 = -\mathbb{1}, \quad (3.8)$$

the contractions of the fluxes with the Dirac matrices are

$$\not{F} = 12\tilde{q} (\Gamma_* + \cos \varphi \Gamma_+ + \sin \varphi \Gamma_-), \quad (3.9)$$

$$\not{H} = 12q (\Gamma_* + \cos \varphi \Gamma_+ + \sin \varphi \Gamma_-).$$

The equations of motion derived from (3.1) are

$$\begin{aligned} (\rho_0 + \rho_1)(\mathcal{D}_0 - \mathcal{D}_1)\vartheta^1 &= 0, \\ (\rho_0 - \rho_1)(\mathcal{D}_0 + \mathcal{D}_1)\vartheta^2 &= 0. \end{aligned} \quad (3.10)$$

After expanding the covariant derivatives $\mathcal{D}_a$ we get

$$\begin{aligned} (\rho_0 + \rho_1) \left[(\mathcal{D}_1 - \mathcal{D}_0)\vartheta^1 - \frac{1}{48}\not{F}(\rho_0 - \rho_1)\vartheta^2 - \frac{1}{8}(\not{H}_0 - \not{H}_1)\vartheta^1 \right] &= 0, \\ (\rho_0 - \rho_1) \left[(\mathcal{D}_1 + \mathcal{D}_0)\vartheta^2 + \frac{1}{48}\not{F}(\rho_0 + \rho_1)\vartheta^1 - \frac{1}{8}(\not{H}_0 + \not{H}_1)\vartheta^2 \right] &= 0. \end{aligned} \quad (3.11)$$

At this point it is natural to change variables to the scaled and boosted worldsheet coordinates (2.38)

$$\mathcal{Y} = \cos^2 \varphi \zeta \mathcal{X}, \quad \mathcal{S} = \cos^2 \varphi \zeta \mathcal{T}, \quad \zeta = \gamma \sqrt{\tilde{q}^2 - u^2}, \quad (3.12)$$

satisfying

$$\partial_1 \mp \partial_0 = \cos^2 \varphi \zeta (1 \pm u) \gamma (\partial_{\mathcal{Y}} \mp \partial_{\mathcal{S}}). \quad (3.13)$$

With this, the equations become

$$\begin{aligned} (\rho_0 + \rho_1) [\zeta(1+u)\gamma(D - \partial_{\mathcal{S}})\vartheta^1 + \mathcal{O}\vartheta^2] &= 0, \\ (\rho_0 - \rho_1) [\zeta(1-u)\gamma(\tilde{D} + \partial_{\mathcal{S}})\vartheta^2 + \tilde{\mathcal{O}}\vartheta^1] &= 0, \end{aligned} \quad (3.14)$$

where

$$\mathcal{O} = -\frac{1}{48 \cos^2 \varphi} \not{F}(\rho_0 - \rho_1), \quad \tilde{\mathcal{O}} = \frac{1}{48 \cos^2 \varphi} \not{F}(\rho_0 + \rho_1), \quad (3.15)$$

and the fermion derivatives are

$$\begin{aligned} D &= \partial_{\mathcal{Y}} + \frac{1}{2}G\Gamma_{34} + \frac{1}{2}Q\Gamma_{35} - \frac{(1-u)\gamma}{48\cos^2\varphi\zeta}(\not{H}(\rho_0 - \rho_1) + (\rho_0 - \rho_1)\not{H}) , \\ \tilde{D} &= \partial_{\mathcal{Y}} + \frac{1}{2}\tilde{G}\Gamma_{34} + \frac{1}{2}Q\Gamma_{35} - \frac{(1+u)\gamma}{48\cos^2\varphi\zeta}(\not{H}(\rho_0 + \rho_1) + (\rho_0 + \rho_1)\not{H}) . \end{aligned} \quad (3.16)$$

A detailed derivation can be found in appendix E, together with explicit expressions for the scalar functions $G, \tilde{G}, Q$ in (E.5).

The operators in front of the equations (3.14) are nilpotent

$$(\rho_0 + \rho_1)^2 = (\rho_0 - \rho_1)^2 = 0 , \quad (3.17)$$

since they are evaluated on the classical solution, which satisfies the Virasoro constraints. If we further define

$$\bar{\rho}_0 \equiv e_0^0 \Gamma_0 - e_0^3 \Gamma_3 - e_0^4 \Gamma_4 - e_0^5 \Gamma_5 + e_0^7 \Gamma_7 , \quad (3.18)$$

which turns out to be $\bar{\rho}_0 = -\rho_0^\dagger$ for the gamma matrices described in appendix D, we get another set of nilpotent operators $(\bar{\rho}_0 + \rho_1)^2 = (\bar{\rho}_0 - \rho_1)^2 = 0$. However, the two sets differ by the nonsingular operator $\bar{\rho}_0 - \rho_0$, which squares to

$$(\bar{\rho}_0 - \rho_0)^2 = 4\cos^2\varphi\tilde{q}^{-2}(\zeta^2\tanh^2\mathcal{Y} + q^2u^2\gamma^2)\mathbb{1} . \quad (3.19)$$

The kernel of a $2m$ -dimensional nilpotent operator is of at least m dimensions since all its eigenvalues are zero. If the sum of two nilpotent operators is full-rank, as above, the kernels must be disjoint, therefore the sum of their nullities is at most the full $2m$. From this we see that the $(\rho_0 \pm \rho_1)$ are half-rank, an important observation for subsection 3.3.

3.2 Zero mode condition

Note that the fermion Lagrangian (3.1) has a dependence on the worldsheet coordinates only through the vielbein and spin connection. These quantities, on the other hand, depend only on $\mathcal{Y}$, i.e. the Lagrangian is independent of the temporal coordinate $\mathcal{S}$

$$\mathcal{L}_F = \mathcal{L}_F(\mathcal{Y}, \vartheta^J, \partial_{\tilde{a}}\vartheta^J) , \quad (3.20)$$

where $\tilde{a} = \tilde{0}, \tilde{1}$ correspond to the variables $\mathcal{S}$ and $\mathcal{Y}$, respectively.

Translations in $\mathcal{S}$ can be equivalently described as a transformations of the fields

$$\delta\vartheta^J = \varepsilon\partial_{\mathcal{S}}\vartheta^J , \quad \delta(\partial_{\tilde{a}}\vartheta^J) = \varepsilon\partial_{\mathcal{S}}(\partial_{\tilde{a}}\vartheta^J) , \quad (3.21)$$

and accordingly

$$\delta\mathcal{L}_F = \varepsilon\left(\frac{\partial\mathcal{L}_F}{\partial\vartheta^J}\partial_{\mathcal{S}}\vartheta^J + \frac{\partial\mathcal{L}_F}{\partial(\partial_{\tilde{a}}\vartheta^J)}\partial_{\mathcal{S}}(\partial_{\tilde{a}}\vartheta^J)\right) \quad (3.22)$$

$$= \varepsilon\partial_{\mathcal{S}}\mathcal{L}_F = \varepsilon\partial_{\tilde{a}}(\delta_{\tilde{0}}^{\tilde{a}}\mathcal{L}_F) . \quad (3.23)$$

The change in the Lagrangian is a total derivative, and applying Noether's theorem we get a conserved current

$$j^{\tilde{a}} = \frac{\partial \mathcal{L}_F}{\partial (\partial_{\tilde{a}} \vartheta^J)} \partial_S \vartheta^J - \delta_0^{\tilde{a}} \mathcal{L}_F, \quad (3.24)$$

where summation over $J = 1, 2$ is understood. However, for the fermionic action we have $\mathcal{L}_F = 0$ on-shell, and the current simply reduces to

$$j^{\tilde{a}} = \frac{\partial \mathcal{L}_F}{\partial (\partial_{\tilde{a}} \vartheta^J)} \partial_S \vartheta^J, \quad (3.25)$$

The explicit form of this current is unimportant for the present argument.

Since $\mathcal{S}$ is a time-like worldsheet coordinate, we might interpret the corresponding conserved quantity as the energy of the fermionic perturbation above the giant magnon background

$$E_F = \int d\mathcal{X} j^{\tilde{0}} = \int d\mathcal{X} \frac{\partial \mathcal{L}_F}{\partial (\partial_S \vartheta^J)} \partial_S \vartheta^J. \quad (3.26)$$

Zero modes, by definition, are zero energy fluctuations above the giant magnon, i.e. $E_F = 0$. Henceforth, we will take the zero mode condition to be

$$\partial_S \vartheta^J = 0, \quad (3.27)$$

and with this, the equations for the fermion zero modes are

$$\begin{aligned} (\rho_0 + \rho_1) [\zeta(1+u)\gamma D \vartheta^1 + \mathcal{O} \vartheta^2] &= 0, \\ (\rho_0 - \rho_1) [\zeta(1-u)\gamma \tilde{D} \vartheta^2 + \tilde{\mathcal{O}} \vartheta^1] &= 0. \end{aligned} \quad (3.28)$$

3.3 Fixing kappa symmetry

The Green-Schwarz superstring has a local fermionic symmetry, the so-called kappa-symmetry, that ensures spacetime supersymmetry of the physical spectrum. Let us take another look at the quadratic fermionic Lagrangian (3.1)

$$\mathcal{L}_F = -i \left(\eta^{ab} \delta^{IJ} + \epsilon^{ab} \sigma_3^{IJ} \right) \bar{\vartheta}^I \rho_a \mathcal{D}_b \vartheta^J, \quad (3.29)$$

$$= i \bar{\vartheta}^1 (\rho_0 + \rho_1) (\mathcal{D}_0 - \mathcal{D}_1) \vartheta^1 + i \bar{\vartheta}^2 (\rho_0 - \rho_1) (\mathcal{D}_0 + \mathcal{D}_1) \vartheta^2, \quad (3.30)$$

$$\begin{aligned} &= -i \cos^2 \varphi \bar{\vartheta}^1 (\rho_0 + \rho_1) \left(\zeta(1+u) \gamma (D - \partial_S) \vartheta^1 + \mathcal{O} \vartheta^2 \right) \\ &\quad + i \cos^2 \varphi \bar{\vartheta}^2 (\rho_0 - \rho_1) \left(\zeta(1-u) \gamma (\tilde{D} + \partial_S) \vartheta^2 + \tilde{\mathcal{O}} \vartheta^1 \right), \end{aligned} \quad (3.31)$$

where $D, \tilde{D}, \mathcal{O}$ and $\tilde{\mathcal{O}}$ are defined in (3.15)–(3.16). We see the nilpotent operators $(\rho_0 \pm \rho_1)$ acting on the conjugate spinors: components of ϑ^1 and ϑ^2 that are projected out by $(\rho_0 + \rho_1)$ and $(\rho_0 - \rho_1)$, respectively, do not contribute to the action, we can consider them non-dynamical.

To fully fix kappa-gauge, however, not only do we need to project out non-dynamical degrees of freedom, but also specify what happens to the rest, i.e. we need actual projectors:

$$K_1 = \frac{1}{2} \Pi (\rho_0 + \rho_1), \quad K_2 = \frac{1}{2} \Pi (\rho_0 - \rho_1), \quad (3.32)$$

for some invertible Π , that has to satisfy a number of conditions. A straightforward, albeit somewhat cumbersome,⁶ calculation gives $[\rho_0 + \rho_1, D] = [\rho_0 - \rho_1, \tilde{D}] = 0$, so we have

$$[K_1, D] = 0, \quad [K_2, \tilde{D}] = 0, \quad (3.33)$$

provided $[\Pi, D] = [\Pi, \tilde{D}] = 0$. Another condition of course, is that the K_J have to be genuine projectors — i.e. $K_J^2 = K_J$ —, which, with (3.15), would imply that

$$\mathcal{O} = \mathcal{O}K_2, \quad \tilde{\mathcal{O}} = \tilde{\mathcal{O}}K_1. \quad (3.34)$$

The most obvious choice would be $\Pi = \Gamma^0$, but taking this route one encounters technical difficulties when considering the $\text{AdS}_3 \times \text{S}^3 \times \text{S}^3 \times \text{S}^1$ geometry, arising from the appearance of Γ_7 in $(\rho_0 \pm \rho_1)$. Noting that in both of these operators Γ_7 only appears in the combination $\Gamma_0 + \sin \varphi \Gamma_7$, it is tempting to “boost” our gamma matrices in the 0-7 directions

$$\hat{\Gamma}^0 = \sec \varphi (\Gamma^0 - \sin \varphi \Gamma^7), \quad \hat{\Gamma}^7 = \sec \varphi (\Gamma^7 - \sin \varphi \Gamma^0), \quad (3.35)$$

leaving unchanged all the others $\hat{\Gamma}^A = \Gamma^A$, $A \neq 0, 7$. One can easily check that these satisfy the Clifford algebra. We lower the index on $\hat{\Gamma}^A$ with the Minkowski metric, in particular $\hat{\Gamma}_0 = -\hat{\Gamma}^0 = \sec \varphi (\Gamma_0 + \sin \varphi \Gamma_7)$ soaks up all the Γ_7 dependence in $(\rho_0 \pm \rho_1)$

$$\rho_0 \pm \rho_1 = \cos \varphi \left(\hat{\Gamma}_0 + \hat{e}_\pm^3 \hat{\Gamma}_3 + \hat{e}_\pm^4 \hat{\Gamma}_4 + \hat{e}_\pm^5 \hat{\Gamma}_5 \right) \quad (3.36)$$

where $\hat{e}_\pm^A = \sec \varphi (e_0^A \pm e_1^A)$. All of this is good motivation for the choice of $\Pi = \sec \varphi \hat{\Gamma}^0$, which can be easily shown to satisfy our conditions. Henceforth, we will take

$$K_1 = \frac{1}{2} \sec \varphi \hat{\Gamma}^0 (\rho_0 + \rho_1), \quad K_2 = \frac{1}{2} \sec \varphi \hat{\Gamma}^0 (\rho_0 - \rho_1). \quad (3.37)$$

The advantages of this choice will become obvious in the next subsection.

If we take a basis of gamma matrices such that $\hat{\Gamma}^A$ have definite hermiticity, e.g. the one described in appendix D, the projectors are Hermitian $K_J^\dagger = K_J$. Furthermore, in such a basis the Hermitian conjugate intertwiner (see appendix D) is given by $\hat{\Gamma}^0$, hence the Dirac conjugate is $\bar{\vartheta} = \vartheta^\dagger \hat{\Gamma}^0$. With this, and the properties listed above, we can write the Lagrangian as

$$\begin{aligned} \mathcal{L}_F = & -2i \cos^3 \varphi (\Psi^1)^\dagger \left(\zeta(1+u) \gamma(D - \partial_S) \Psi^1 + \mathcal{O} \Psi^2 \right) \\ & + 2i \cos^3 \varphi (\Psi^2)^\dagger \left(\zeta(1-u) \gamma(\tilde{D} + \partial_S) \Psi^2 + \tilde{\mathcal{O}} \Psi^1 \right), \end{aligned} \quad (3.38)$$

where we introduced the notation $\Psi^J = K_J \vartheta^J$ for the projected spinors, and we indeed see that only these components are dynamical.

Using the kappa-projectors, the zero mode equations (3.28) can be written as

$$\begin{aligned} K_1 [\zeta(1+u) \gamma D \vartheta^1 + \mathcal{O} \vartheta^2] &= 0, \\ K_2 [\zeta(1-u) \gamma \tilde{D} \vartheta^2 + \tilde{\mathcal{O}} \vartheta^1] &= 0. \end{aligned} \quad (3.39)$$

⁶One can easily convince themselves that it is sufficient to check the Γ_3 , Γ_4 and Γ_5 components of the operator equations, simplifying matters a great deal.

For the kappa-fixed spinors $\Psi^J = K_J \vartheta^J$, using (3.33)–(3.34), these equations become

$$\begin{aligned}\zeta(1+u)\gamma D\Psi^1 + K_1\mathcal{O}\Psi^2 &= 0, \\ \zeta(1-u)\gamma\tilde{D}\Psi^2 + K_2\tilde{\mathcal{O}}\Psi^1 &= 0.\end{aligned}\tag{3.40}$$

3.4 Zero mode equations

With the choice of kappa projectors (3.37) we get a commuting 6d chirality projector for free⁷

$$P_{\pm} = \frac{1}{2} \left(\mathbb{1} \pm \hat{\Gamma}_* \hat{\Gamma}_+ \right), \quad [P_{\pm}, K_J] = 0.\tag{3.41}$$

Using this, the contraction of the background fluxes $\not{F}$, $\not{H}$ (3.9) can be written as

$$\begin{aligned}\Gamma_* + \cos\varphi\Gamma_+ + \sin\varphi\Gamma_- &= \cos\varphi \left((\sec\varphi\Gamma^0 + \tan\varphi\Gamma^{1268}\Gamma^7)\Gamma^{12} + \Gamma_+ \right) \\ &= \cos\varphi \left(\hat{\Gamma}^0\Gamma^{12} + \Gamma_+ \right) - 2\cos\varphi\Delta\Gamma^{12} \\ &= 2\cos\varphi \left(\hat{\Gamma}_* P_+ - \Delta\hat{\Gamma}^{12} \right),\end{aligned}\tag{3.42}$$

where

$$\Delta = -\frac{1}{2}\tan\varphi \left(\hat{\Gamma}^{1268} + \mathbb{1} \right) \Gamma^7 \equiv \Delta_0\hat{\Gamma}^0 + \Delta_7\hat{\Gamma}^7,\tag{3.43}$$

with

$$\Delta_0 = -\frac{1}{2}\tan^2\varphi \left(\hat{\Gamma}^{1268} + \mathbb{1} \right), \quad \Delta_7 = \csc\varphi\Delta_0.\tag{3.44}$$

Even though Δ_0 and Δ_7 are matrices, we can essentially treat them as scalars, since they commute with the equations of motion.

Recalling $\bar{\rho}_0$ from (3.18), which also satisfies $\rho_0\hat{\Gamma}^0 = \hat{\Gamma}^0\bar{\rho}_0$, we can define an invertible operator from (3.19)

$$R = \frac{1}{2}\sec\varphi\hat{\Gamma}_*(\bar{\rho}_0 - \rho_0) \quad : \quad R^2 = -\tilde{q}^{-2}(\zeta^2\tanh^2\mathcal{Y} + q^2u^2\gamma^2)\mathbb{1}.\tag{3.45}$$

With all of this, the fermion derivatives (3.16) can be rewritten as (see appendix E)

$$\begin{aligned}D &= \partial_{\mathcal{Y}} + \frac{1}{2}G\hat{\Gamma}_{34} + \frac{1}{2}Q\hat{\Gamma}_{35} + \frac{q(1-u)\gamma}{\zeta} \left(RP_- - (R + \hat{\Gamma}_{12})P_+ + \Delta_0\hat{\Gamma}_{12} \right), \\ \tilde{D} &= \partial_{\mathcal{Y}} + \frac{1}{2}\tilde{G}\hat{\Gamma}_{34} + \frac{1}{2}Q\hat{\Gamma}_{35} + \frac{q(1+u)\gamma}{\zeta} \left(RP_- - (R + \hat{\Gamma}_{12})P_+ + \Delta_0\hat{\Gamma}_{12} \right),\end{aligned}\tag{3.46}$$

however, these expressions are only valid when acting on kappa-fixed spinors, i.e. in the form DK_1 and $\tilde{D}K_2$. As for the terms (3.15) mixing the two spinors in the equations of motion, we have

$$\begin{aligned}\mathcal{O} &= -\tilde{q} \left(\hat{\Gamma}^{12}P_- + \Delta\hat{\Gamma}_* \right) K_2, \\ \tilde{\mathcal{O}} &= \tilde{q} \left(\hat{\Gamma}^{12}P_- + \Delta\hat{\Gamma}_* \right) K_1.\end{aligned}\tag{3.47}$$

⁷In any spinor operator M , replace Γ^A by $\hat{\Gamma}^A$ to get $\hat{M}$.

Using the nilpotency relations $(\rho_0 \pm \rho_1)^2 = 0$, it is easy to see that

$$\Gamma^{12} K_1 K_2 = -R K_2, \quad \Gamma^{12} K_2 K_1 = -R K_1, \quad (3.48)$$

and the equations of motion (3.40) become

$$\begin{aligned} \zeta(1+u)\gamma D \Psi^1 + \tilde{q} \left(R P_- - K_1 \Delta \hat{\Gamma}_* \right) \Psi^2 &= 0, \\ \zeta(1-u)\gamma \tilde{D} \Psi^2 - \tilde{q} \left(R P_- - K_2 \Delta \hat{\Gamma}_* \right) \Psi^1 &= 0. \end{aligned} \quad (3.49)$$

3.4.1 Equations for $\Delta = 0$

Equation (3.42) might seem arbitrary at first, so let us elaborate on the advantages of this rearrangement. Our goal was to have $(\hat{\Gamma}_* + \hat{\Gamma}_+)$ — instead of $\hat{F}$ — in the equations, since $P_\pm$ commutes with K_J . After this rewriting we are left with an extra term $K \Delta K$, which does not in general commute with $P_\pm$. However, in the following two cases we have $\Delta = 0$

- $\varphi = 0$: corresponding to the $\text{AdS}_3 \times \text{S}^3 \times \text{T}^4$ geometry.
- “ $\hat{\Gamma}^{1268} = -1$ ”: i.e. the $\text{AdS}_3 \times \text{S}^3 \times \text{S}^3 \times \text{S}^1$ geometry, with the fermions restricted to the -1 eigenspace of $\hat{\Gamma}^{1268}$. Note that this is compatible with the equations, since $\hat{\Gamma}^{1268}$ commutes with all the terms.

Assuming $\Delta = 0$, the fermion derivatives take the simpler form

$$\begin{aligned} D &= \partial_{\mathcal{Y}} + \frac{1}{2} G \hat{\Gamma}_{34} + \frac{1}{2} Q \hat{\Gamma}_{35} + \frac{q(1-u)\gamma}{\zeta} \left(R P_- - (R + \hat{\Gamma}_{12}) P_+ \right), \\ \tilde{D} &= \partial_{\mathcal{Y}} + \frac{1}{2} \tilde{G} \hat{\Gamma}_{34} + \frac{1}{2} \tilde{Q} \hat{\Gamma}_{35} + \frac{q(1+u)\gamma}{\zeta} \left(R P_- - (R + \hat{\Gamma}_{12}) P_+ \right). \end{aligned} \quad (3.50)$$

Also note that the equations of motion have no explicit dependence on φ , only an implicit one via the rescaled variable $\mathcal{Y}$ (3.12). In other words, the following equations apply in both geometries

$$\begin{aligned} \zeta(1+u)\gamma D \Psi^1 + \tilde{q} R P_- \Psi^2 &= 0, \\ \zeta(1-u)\gamma \tilde{D} \Psi^2 - \tilde{q} R P_- \Psi^1 &= 0. \end{aligned} \quad (3.51)$$

as long as we impose the extra condition $\hat{\Gamma}^{1268} \vartheta^J = -\vartheta^J$ in the S^1 geometry.

3.4.2 The case of $\Delta \neq 0$

As we have seen above, we can treat the $\text{AdS}_3 \times \text{S}^3 \times \text{S}^3 \times \text{S}^1$ fermion zero modes in much the same way as those of the $\text{AdS}_3 \times \text{S}^3 \times \text{T}^4$ giant magnon, provided $\Delta = 0$. In section 4 this will allow us to find solutions for both geometries and general values of q in a single calculation. However, we need to make sure there are no zero modes that we are missing by restricting to $\Delta = 0$.

We can get an intuition for why this must be the case by looking at the near BMN spectrum of the $\text{AdS}_3 \times \text{S}^3 \times \text{S}^3 \times \text{S}^1$ superstring. Since in the BMN limit the zero modes

become the fermion superpartners of the magnon background, they must all have the same mass, hence definite chirality under $\hat{\Gamma}_{1268}$, according to equation (2.33) in [47].⁸ Given that in the next section we find normalizable solutions for $\Delta = 0$ ($\hat{\Gamma}_{1268} = -1$), we expect no zero modes for $\Delta \neq 0$ ($\hat{\Gamma}_{1268} = +1$). In appendix F we show that there are in fact no normalizable solutions to (3.49) for $\Delta \neq 0$.

4 Mixed-flux fermion zero modes

In this section we find exact solutions for the ($\Delta = 0$) zero mode equations (3.51). Our main aim is to write down the normalizable solutions, representing the perturbative zero modes over the giant magnon background. Using these normalizable zero modes, we then perform semiclassical quantization, and reproduce the algebra that the fermion excitations must satisfy.

4.1 Fixing kappa-gauge

We start by noting that the kappa-projectors (3.37) can be written as

$$\begin{aligned} K_1 &= \frac{1}{2} \left(\mathbb{1} - \sin(2\chi) \cos v_+ \hat{\Gamma}_{03} - \cos(2\chi) \cos v_+ \hat{\Gamma}_{04} + \sin v_+ \hat{\Gamma}_{05} \right), \\ K_2 &= \frac{1}{2} \left(\mathbb{1} + \sin(2\tilde{\chi}) \cos v_- \hat{\Gamma}_{03} + \cos(2\tilde{\chi}) \cos v_- \hat{\Gamma}_{04} - \sin v_- \hat{\Gamma}_{05} \right), \end{aligned} \quad (4.1)$$

where

$$\begin{aligned} \chi(\mathcal{Y}) &= \frac{1}{2} \left(\operatorname{arccot} \left(\frac{u \operatorname{csch} \mathcal{Y}}{\tilde{q}} \right) - \arcsin \left(\frac{\tanh \mathcal{Y}}{\sqrt{1 - Q_+^2 \operatorname{sech}^2 \mathcal{Y}}} \right) \right), \\ \tilde{\chi}(\mathcal{Y}) &= \frac{1}{2} \left(\operatorname{arccot} \left(\frac{u \operatorname{csch} \mathcal{Y}}{\tilde{q}} \right) + \arcsin \left(\frac{\tanh \mathcal{Y}}{\sqrt{1 - Q_-^2 \operatorname{sech}^2 \mathcal{Y}}} \right) \right), \end{aligned} \quad (4.2)$$

and we also introduced

$$Q_{\pm} = \frac{q \sqrt{\tilde{q}^2 - u^2}}{\tilde{q}(1 \pm u)}, \quad v_{\pm} = \arcsin(Q_{\pm} \operatorname{sech} \mathcal{Y}). \quad (4.3)$$

Ansatz. Since K_J , $\hat{\Gamma}_{12}$ and $\hat{\Gamma}_* \hat{\Gamma}_+$ all mutually commute, as our starting point we can take shared eigenvectors U

$$\hat{\Gamma}_{12} U^J = \lambda_{12} U^J, \quad \hat{\Gamma}_* \hat{\Gamma}_+ U^J = \lambda_P U^J, \quad (4.4)$$

where $\lambda_{12} = \pm i$, and $\lambda_P = \pm 1$ correspond to the $P_{\pm}$ projections. Accordingly, there are no restrictions on these eigenvalues for the kappa-fixed spinor. The operator $\hat{\Gamma}_{34}$ does not commute with K_J , hence a suitable combination of its opposite eigenvectors makes a good candidate for the general gauge-fixed spinor. This motivates the further restriction of $\hat{\Gamma}_{34} U^J = i U^J$ and the ansatz

$$\Psi^J = \left(\alpha_+^J(\mathcal{Y}) + \alpha_-^J(\mathcal{Y}) \hat{\Gamma}_{45} \right) U^J \quad (4.5)$$

⁸We are grateful to Bogdan Stefanski for pointing out this relation.

Solution. Substituting this into the equations $K_1 \Psi^1 = \Psi^1$, and using the various eigenvector relations of U^1 , we get one equation for each eigenspace of $\hat{\Gamma}_{34}$

$$\begin{aligned} \lambda e^{2i\chi} \cos v_+ \alpha_-^1 - \lambda \sin v_+ \alpha_+^1 &= \alpha_+^1, \\ \lambda e^{-2i\chi} \cos v_+ \alpha_+^1 + \lambda \sin v_+ \alpha_-^1 &= \alpha_-^1, \end{aligned} \quad (4.6)$$

where $\lambda = i\lambda_{12}\lambda_P = \pm 1$. What we have here are two equations for the single variable α_-/α_+ , corresponding to the fact that the norm of the eigenvector is not fixed. The equations are consistent, and a symmetric solution is given by

$$\alpha_+^1 = e^{i\chi} \sqrt{1 - \lambda Q_+ \operatorname{sech} \mathcal{Y}}, \quad \alpha_-^1 = e^{-i\chi} \lambda \sqrt{1 + \lambda Q_+ \operatorname{sech} \mathcal{Y}}. \quad (4.7)$$

A similar calculation gives

$$\alpha_+^2 = e^{i\tilde{\chi}} \sqrt{1 + \lambda Q_- \operatorname{sech} \mathcal{Y}}, \quad \alpha_-^2 = -e^{-i\tilde{\chi}} \lambda \sqrt{1 - \lambda Q_- \operatorname{sech} \mathcal{Y}}. \quad (4.8)$$

Written in a single expression, the most general gauge-fixed spinors are

$$\begin{aligned} \Psi^1 &= \sum_{\lambda=\pm} \left(e^{i\chi} \sqrt{1 - \lambda Q_+ \operatorname{sech} \mathcal{Y}} + e^{-i\chi} \lambda \sqrt{1 + \lambda Q_+ \operatorname{sech} \mathcal{Y}} \hat{\Gamma}_{45} \right) U_\lambda^1, \\ \Psi^2 &= \sum_{\lambda=\pm} \left(e^{i\tilde{\chi}} \sqrt{1 + \lambda Q_- \operatorname{sech} \mathcal{Y}} - e^{-i\tilde{\chi}} \lambda \sqrt{1 - \lambda Q_- \operatorname{sech} \mathcal{Y}} \hat{\Gamma}_{45} \right) U_\lambda^2, \end{aligned} \quad (4.9)$$

where $\hat{\Gamma}_{34} U_\pm^J = +i U_\pm^J$ and $i\hat{\Gamma}_{12} \hat{\Gamma}_* \hat{\Gamma}_+ U_\pm^J = i\hat{\Gamma}_{0345} U_\pm^J = \pm U_\pm^J$. The above analysis shows that these are kappa-fixed eigenvectors, and by counting the degrees of freedom (components of U^J) we see that there are no others.

4.2 Zero mode solutions

The projectors $P_\pm$ commute with the equations of motion (3.51), therefore we can consider solutions of definite $P_\pm$ “chirality”. In the following we obtain solutions on the two subspaces in turn, by letting $U_\pm^J$ depend on $\mathcal{Y}$, and substituting (4.9) into the equations. The identities listed in appendix G were useful in simplifying some of the more complicated expressions.

4.2.1 Solutions on the P_+ subspace

For this projection the spinors decouple

$$D \Psi^1 = 0, \quad \tilde{D} \Psi^2 = 0. \quad (4.10)$$

Substitution gives

$$\begin{aligned} \sum_{\lambda=\pm} \left(\alpha_+^1 + \alpha_-^1 \hat{\Gamma}_{45} \right) (\partial_{\mathcal{Y}} + C_+) U_\lambda^1 &= 0, \\ \sum_{\lambda=\pm} \left(\alpha_+^2 + \alpha_-^2 \hat{\Gamma}_{45} \right) (\partial_{\mathcal{Y}} + C_-) U_\lambda^2 &= 0, \end{aligned} \quad (4.11)$$

with c-numbers

$$C_{\pm} = \frac{i\lambda q}{2\sqrt{\tilde{q}^2 - u^2}} + \frac{i\lambda Q_{\pm}\sqrt{1 - Q_{\pm}^2}}{2(\cosh^2 \mathcal{Y} - Q_{\pm}^2)}, \quad (4.12)$$

and this simple form of the equations is a consequence (or proof in itself) of the fact that kappa-fixing commutes with the fermion derivative operators. The solution is

$$\begin{aligned} U_{\lambda}^1 &= e^{-\frac{i\lambda q}{2\sqrt{\tilde{q}^2 - u^2}}\mathcal{Y} - \frac{i}{2}\lambda \arctan\left(\frac{Q_+ \tanh \mathcal{Y}}{\sqrt{1 - Q_+^2}}\right)} V_{\lambda}^1, \\ U_{\lambda}^2 &= e^{-\frac{i\lambda q}{2\sqrt{\tilde{q}^2 - u^2}}\mathcal{Y} - \frac{i}{2}\lambda \arctan\left(\frac{Q_- \tanh \mathcal{Y}}{\sqrt{1 - Q_-^2}}\right)} V_{\lambda}^2, \end{aligned} \quad (4.13)$$

where V^J are (independent) constant MW spinors with $\hat{\Gamma}_{34}V_{\pm}^J = +iV_{\pm}^J$, $\hat{\Gamma}_{12}V_{\pm}^J = \mp iV_{\pm}^J$ and $P_+V_{\pm}^J = V_{\pm}^J$. However, with these, the spinors (4.9) are not normalizable and we discard them as perturbative zero modes.

4.2.2 Solutions on the P_- subspace

The equations on this subspace become

$$\begin{aligned} \zeta(1+u)\gamma D\Psi^1 + \tilde{q}R\Psi^2 &= 0, \\ \zeta(1-u)\gamma \tilde{D}\Psi^2 - \tilde{q}R\Psi^1 &= 0, \end{aligned} \quad (4.14)$$

with fermion derivatives

$$\begin{aligned} D &= \partial_{\mathcal{Y}} + \frac{1}{2}G\hat{\Gamma}_{34} + \frac{1}{2}Q\hat{\Gamma}_{35} + \frac{q(1-u)\gamma}{\zeta}R, \\ \tilde{D} &= \partial_{\mathcal{Y}} + \frac{1}{2}\tilde{G}\hat{\Gamma}_{34} + \frac{1}{2}Q\hat{\Gamma}_{35} + \frac{q(1+u)\gamma}{\zeta}R. \end{aligned} \quad (4.15)$$

After substitution, and a considerable amount of simplification, we get

$$\begin{aligned} \sum_{\lambda=\pm} \left(\alpha_+^1 + \alpha_-^1 \hat{\Gamma}_{45} \right) [(\partial_{\mathcal{Y}} + C_{11})U_{\lambda}^1 + C_{12}U_{\lambda}^2] &= 0, \\ \sum_{\lambda=\pm} \left(\alpha_+^2 + \alpha_-^2 \hat{\Gamma}_{45} \right) [(\partial_{\mathcal{Y}} + C_{21})U_{\lambda}^2 + C_{22}U_{\lambda}^1] &= 0, \end{aligned} \quad (4.16)$$

with

$$\begin{aligned} C_{11} &= -\frac{i\lambda q(1-2u)}{2\sqrt{\tilde{q}^2 - u^2}} + \frac{i\lambda Q_+\sqrt{1 - Q_+^2}}{2(\cosh^2 \mathcal{Y} - Q_+^2)}, \\ C_{21} &= -\frac{i\lambda q(1+2u)}{2\sqrt{\tilde{q}^2 - u^2}} + \frac{i\lambda Q_-\sqrt{1 - Q_-^2}}{2(\cosh^2 \mathcal{Y} - Q_-^2)}, \\ C_{12} &= (1-u)\gamma e^{\int (C_{21}-C_{11})d\mathcal{Y}} e^{+i2\lambda\xi\mathcal{Y}} (\lambda \tanh \mathcal{Y} - i\xi), \\ C_{22} &= (1+u)\gamma e^{\int (C_{11}-C_{12})d\mathcal{Y}} e^{-i2\lambda\xi\mathcal{Y}} (\lambda \tanh \mathcal{Y} + i\xi), \end{aligned} \quad (4.17)$$

where we also defined

$$\xi = \frac{qu}{\sqrt{\tilde{q}^2 - u^2}}. \quad (4.18)$$

The motivation for writing C_{12} and C_{22} in the above form becomes clear once we make the ansatz

$$\begin{aligned} U_\lambda^1 &= \frac{e^{-\int C_{11} d\mathcal{Y}}}{\sqrt{1+u}} \tilde{U}_\lambda^1 = \frac{1}{\sqrt{1+u}} e^{\frac{i\lambda q(1-2u)}{2\sqrt{\tilde{q}^2 - u^2}} \mathcal{Y} - \frac{i}{2} \lambda \arctan\left(\frac{Q_+ \tanh \mathcal{Y}}{\sqrt{1-Q_+^2}}\right)} \tilde{U}_\lambda^1, \\ U_\lambda^2 &= \frac{e^{-\int C_{12} d\mathcal{Y}}}{\sqrt{1-u}} \tilde{U}_\lambda^2 = \frac{1}{\sqrt{1-u}} e^{\frac{i\lambda q(1+2u)}{2\sqrt{\tilde{q}^2 - u^2}} \mathcal{Y} - \frac{i}{2} \lambda \arctan\left(\frac{Q_- \tanh \mathcal{Y}}{\sqrt{1-Q_-^2}}\right)} \tilde{U}_\lambda^2, \end{aligned} \quad (4.19)$$

and the equations in brackets (4.16) reduce to

$$\begin{aligned} \partial_{\mathcal{Y}} \tilde{U}_\lambda^1 + e^{+i2\lambda\xi\mathcal{Y}} (\lambda \tanh \mathcal{Y} - i\xi) \tilde{U}_\lambda^2 &= 0, \\ \partial_{\mathcal{Y}} \tilde{U}_\lambda^2 + e^{-i2\lambda\xi\mathcal{Y}} (\lambda \tanh \mathcal{Y} + i\xi) \tilde{U}_\lambda^1 &= 0. \end{aligned} \quad (4.20)$$

Inverting the first equation and substituting into the second we get a second-order ODE for $\tilde{U}_\lambda^1$

$$\partial_{\mathcal{Y}}^2 \tilde{U}_\lambda^1 - \left(2i\lambda\xi + \frac{\text{sech}^2 \mathcal{Y}}{\tanh \mathcal{Y} - i\lambda\xi} \right) \partial_{\mathcal{Y}} \tilde{U}_\lambda^1 - (\tanh^2 \mathcal{Y} + \xi^2) \tilde{U}_\lambda^1 = 0 \quad (4.21)$$

with solutions

$$\tilde{U}_\lambda^1 = \left(\text{sech} \mathcal{Y} V_\lambda + (\cosh \mathcal{Y} - i\lambda\xi \sinh \mathcal{Y} - i\lambda\xi \mathcal{Y} \text{sech} \mathcal{Y}) \tilde{V}_\lambda \right) e^{i\lambda\xi\mathcal{Y}}. \quad (4.22)$$

Taking $\tilde{V}_\lambda = 0$, we obtain the normalizable solutions

$$\tilde{U}_\lambda^1 = \text{sech} \mathcal{Y} e^{i\lambda\xi\mathcal{Y}} V_\lambda, \quad \tilde{U}_\lambda^2 = \lambda \text{sech} \mathcal{Y} e^{-i\lambda\xi\mathcal{Y}} V_\lambda, \quad (4.23)$$

and the (kappa-fixed) fermion zero modes are given by

$$\begin{aligned} \Psi^1 &= \sum_{\lambda=\pm} \frac{\text{sech} \mathcal{Y}}{4\sqrt{1+u}} e^{i\lambda\omega_+} \left(e^{i\chi} \sqrt{1 - \lambda Q_+ \text{sech} \mathcal{Y}} + e^{-i\chi} \lambda \sqrt{1 + \lambda Q_+ \text{sech} \mathcal{Y}} \hat{\Gamma}_{45} \right) V_\lambda, \\ \Psi^2 &= \sum_{\lambda=\pm} \frac{\lambda \text{sech} \mathcal{Y}}{4\sqrt{1-u}} e^{i\lambda\omega_-} \left(e^{i\tilde{\chi}} \sqrt{1 + \lambda Q_- \text{sech} \mathcal{Y}} - e^{-i\tilde{\chi}} \lambda \sqrt{1 - \lambda Q_- \text{sech} \mathcal{Y}} \hat{\Gamma}_{45} \right) V_\lambda, \end{aligned} \quad (4.24)$$

where

$$\omega_\pm(\mathcal{Y}) = \frac{q\mathcal{Y}}{2\sqrt{\tilde{q}^2 - u^2}} - \frac{1}{2} \arctan \left(\frac{Q_\pm \tanh \mathcal{Y}}{\sqrt{1 - Q_\pm^2}} \right), \quad (4.25)$$

and the constant MW spinors $V_\pm$ satisfy $P_- V_\pm = V_\pm$, $\hat{\Gamma}_{34} V_\pm = +i V_\pm$, and $\hat{\Gamma}_{12} V_\pm = \pm i V_\pm$.

Counting the zero modes. The normalizable zero modes above are parametrized by the constant spinor $V = V_+ + V_-$. An unconstrained 10-d MW spinor has 16 real degrees of freedom, but kappa-fixing (which in our parametrisation translates to $\hat{\Gamma}_{34}V = +iV$) and 6d-chirality ($P_-V = V$) both reduce the number of components by half. Recalling the further restriction $\hat{\Gamma}^{1268}V = -V$ for the S^1 case, we conclude that there are 4 and 2 normalizable solutions for the $\text{AdS}_3 \times S^3 \times T^4$ and $\text{AdS}_3 \times S^3 \times S^1$ backgrounds, respectively, i.e. we get the expected number of fermion zero modes.

4.3 Zero mode action

Now letting $V = V_+ + V_-$ depend on $\mathcal{T}$ and substituting these zero modes into (3.38) we get

$$\mathcal{L}_{F,0} = 2i \cos \varphi (1+u) \gamma \Psi^{1\dagger} \partial_{\mathcal{T}} \Psi^1 + 2i \tilde{A} (1-u) \gamma \Psi^{2\dagger} \partial_{\mathcal{T}} \Psi^2, \quad (4.26)$$

$$= \frac{i \cos \varphi \gamma}{2} \text{sech}^2 \mathcal{Y} V^\dagger \partial_{\mathcal{T}} V, \quad (4.27)$$

where, going to the second line, we implicitly used the fact that $V = \frac{1}{2}(\mathbb{1} - i\Gamma_{34})V$, and $(\mathbb{1} - i\Gamma_{34})\Gamma_{45}(\mathbb{1} - i\Gamma_{34}) = 0$. Integrating over $\mathcal{X}$ we get the zero mode action

$$S_{F,0} = h\tilde{\gamma} \sec \varphi \int d\mathcal{T} \left(i V^\dagger \partial_{\mathcal{T}} V \right), \quad (4.28)$$

with

$$\tilde{\gamma} = \frac{\gamma}{\zeta} = \frac{1}{\sqrt{\tilde{q}^2 - u^2}}. \quad (4.29)$$

We can further simplify this by considering a Majorana basis, where all (boosted) gamma-matrices are purely imaginary $\hat{\Gamma}_A^* = -\hat{\Gamma}_A^*$, and the Majorana condition reduces to reality of the spinors $\Psi^{I*} = \Psi^I$. Applying this to the solutions (4.24), we get

$$V_- = V_+^* \quad \Rightarrow \quad V^* = V, \quad (4.30)$$

and the zero mode action becomes

$$S_{F,0} = h\tilde{\gamma} \sec \varphi \int d\mathcal{T} \left(i V^T \partial_{\mathcal{T}} V \right). \quad (4.31)$$

As we have noted above, there are 2 and 4 real fermion zero modes for the giant magnons on $\text{AdS}_3 \times S^3 \times S^3 \times S^1$ and $\text{AdS}_3 \times S^3 \times T^4$ respectively. Quantization of these real fermions leads to the anticommutators

$$\{V_{\alpha a}, V_{\beta b}\} = \delta_{\alpha\beta} \delta_{ab} \frac{\cos \varphi}{h\tilde{\gamma}}, \quad (4.32)$$

where $a, \alpha = 1, 2$, and for $\varphi \neq 0$ only the $a = 1$ modes are present. After complexifying

$$V_{La} = \frac{1}{\sqrt{2}} (V_{1a} + i V_{2a}), \quad V_{Ra} = \frac{1}{\sqrt{2}} (V_{1a} - i V_{2a}), \quad (4.33)$$

the only non-trivial zero-mode anticommutator is

$$\{V_{La}, V_{Rb}\} = \delta_{ab} \frac{\cos \varphi}{h\tilde{\gamma}}. \quad (4.34)$$

In the remaining part of this section we will see, for both geometries, how the symmetry superalgebra of the ground state (BMN vacuum) arises from these zero modes.

4.4 Zero-mode algebra for $\text{AdS}_3 \times \text{S}^3 \times \text{S}^3 \times \text{S}^1$

By considering the corresponding spin-chain, it was argued that the fundamental excitations transform in the 2 dimensional short representations of the centrally extended $\mathfrak{su}(1|1)^2$ algebra [46]. This superalgebra has 4 fermionic generators and 4 central charges satisfying⁹

$$\begin{aligned} \{\mathbf{Q}_L, \mathbf{S}_L\} &= \mathbf{H}_L, & \{\mathbf{Q}_L, \mathbf{Q}_R\} &= \mathbf{C}, \\ \{\mathbf{Q}_R, \mathbf{S}_R\} &= \mathbf{H}_R, & \{\mathbf{S}_L, \mathbf{S}_R\} &= \overline{\mathbf{C}}. \end{aligned} \quad (4.35)$$

Consequently, the symmetry algebra of light-cone gauge superstring theory on $\text{AdS}_3 \times \text{S}^3 \times \text{S}^3 \times \text{S}^1$ was shown to take the same form, after lifting the level-matching condition [47]. Thus, it is important to see how the supercharges of the algebra can be constructed from the zero modes (4.34).

For an off-shell one-particle representation the values of the central charges are given by

$$\begin{aligned} \mathbf{H}_L &= \frac{1}{2}(\epsilon + M), & \mathbf{C} &= \frac{h\varsigma}{\tilde{\gamma}}, \\ \mathbf{H}_R &= \frac{1}{2}(\epsilon - M), & \overline{\mathbf{C}} &= \frac{h}{\tilde{\gamma}\varsigma}, \end{aligned} \quad (4.36)$$

where $M = m \pm qhp$, with mass m , ϵ is the energy of the magnon

$$\epsilon = \sqrt{M^2 + \frac{4h^2}{\tilde{\gamma}^2}}, \quad (4.37)$$

and ς can be removed by rescaling for a one-particle state, but plays an important role in constructing multi-particle representations [49]. Note that the momentum of the excitation enters into these expressions through (2.29)

$$(\tilde{q}\tilde{\gamma})^{-1} = \sin \frac{p}{2}. \quad (4.38)$$

These values satisfy the shortening condition $\mathbf{H}_L \mathbf{H}_R - \mathbf{C} \overline{\mathbf{C}} = 0$, therefore on this representation the supercharges must be related to each other. Assuming only $\{\mathbf{Q}_L, \mathbf{Q}_R\} = \frac{h\varsigma}{\gamma}$, it is not too hard to justify¹⁰ that the rest of (4.35) will follow from

$$\mathbf{S}_{L,R} = \varsigma^{-1} \left(\sqrt{\frac{\tilde{\gamma}^2 M^2}{4h^2} + 1} + \frac{\tilde{\gamma} M}{2h} (-1)^F \right) \mathbf{Q}_{R,L}, \quad (4.39)$$

where F is the fermion number operator, i.e. $(-1)^F$ anticommutes with the supercharges. This leaves us with the task of expressing $\mathbf{Q}_{L,R}$ in terms of the zero modes. We can make the general ansatz

$$\mathbf{Q}_{L,R} = \varsigma^{1/2} (\mathcal{A} - \mathcal{B}(-1)^F) V_{L,R}, \quad (4.40)$$

⁹For a detailed description of $\mathfrak{su}(1|1)_{\text{c.e.}}^2$ and its representations see appendix H.

¹⁰In doing so, one might find useful the fact that acting on the short representation (H.9), the supercharges satisfy: $[\mathbf{Q}_L, \mathbf{Q}_R] = -(-1)^F \mathbf{C}$, $[\mathbf{S}_L, \mathbf{S}_R] = (-1)^F \overline{\mathbf{C}}$.

where $\mathcal{A}$ and $\mathcal{B}$ are some c-numbers, and (4.34) guarantees that the condition $\{\mathbf{Q}_L, \mathbf{Q}_R\} = \frac{h\varsigma}{\tilde{\gamma}}$ will be satisfied as long as $\mathcal{A}^2 - \mathcal{B}^2 = \sec \varphi h^2$. Our freedom in choosing $\mathcal{A}$ is just basis dependence, and a symmetric identification is given by

$$\mathcal{A} = \sqrt{\frac{\sec \varphi h^2}{2}} \left(\sqrt{\frac{\tilde{\gamma}^2 M^2}{4h^2} + 1} + 1 \right)^{1/2}, \quad \mathcal{B} = \sqrt{\frac{\sec \varphi h^2}{2}} \left(\sqrt{\frac{\tilde{\gamma}^2 M^2}{4h^2} + 1} - 1 \right)^{1/2}, \quad (4.41)$$

with the supercharges taking the form

$$\begin{aligned} \mathbf{Q}_{L,R} &= \varsigma^{1/2} (\mathcal{A} - \mathcal{B}(-1)^F) V_{L,R}, \\ \mathbf{S}_{L,R} &= \varsigma^{-1/2} (\mathcal{A} + \mathcal{B}(-1)^F) V_{R,L}. \end{aligned} \quad (4.42)$$

4.5 Zero-mode algebra for $\text{AdS}_3 \times \text{S}^3 \times \text{T}^4$

The off-shell symmetry algebra of superstring theory on this background is the centrally extended $\mathfrak{psu}(1|1)^4$ [39], which is essentially a tensor product of two $\mathfrak{su}(1|1)_{\text{c.e.}}^2$ algebras with matching central charges.¹¹ The giant magnon is part of a 4 dimensional short representation, and we should be able to match the supercharges to the zero modes.

Having noted the tensor product structure of the algebra, the construction is trivial, since (4.34) gives us two non-interacting copies of $U_{L,R}$. The central charges take the same values as in (4.36), hence everything from the previous subsection holds for each copy of $\mathfrak{su}(1|1)_{\text{c.e.}}^2$, and the supercharges of $\mathfrak{psu}(1|1)_{\text{c.e.}}^4$ are simply

$$\begin{aligned} \mathbf{Q}_{L,R a} &= \varsigma^{1/2} (\mathcal{A} - \mathcal{B}(-1)^F) V_{L,R a}, \\ \mathbf{S}_{L,R a} &= \varsigma^{-1/2} (\mathcal{A} + \mathcal{B}(-1)^F) V_{R,L a}. \end{aligned} \quad (4.43)$$

where $\mathcal{A}$ and $\mathcal{B}$ are still given by (4.41).

4.6 Zero modes in the $\alpha \rightarrow 0, 1$ limits

The parameter $\alpha \in [0, 1]$ determines the radii of the 3-spheres in the $\text{AdS}_3 \times \text{S}^3 \times \text{S}^3 \times \text{S}^1$ geometry (2.1), and in the limits $\alpha \rightarrow 0, 1$, blowing up either of the spheres, we are left with — up to compactification of the flat directions — $\text{AdS}_3 \times \text{S}^3 \times \text{T}^4$. It is interesting to see what happens to the fermion zero modes (4.24) in the process.

Taking $\alpha \rightarrow 1$ (or $\varphi \rightarrow 0$) blows up S_-^3 , the sphere on which we have the BMN-like leg of the magnon (2.37). In this limit $\mathcal{Y} \rightarrow \gamma \sqrt{\tilde{q}^2 - u^2} \mathcal{X}$, i.e. the magnon becomes the T^4 magnon, and the zero modes reduce to two of the four real T^4 zero modes, the ones on the $\Gamma_{1268} = -1$ subspace. The remaining two we will find on the $\Gamma_{1268} = +1$ eigenspace, where Δ also becomes zero (3.43).

On the other hand, $\alpha \rightarrow 0$ (or $\varphi \rightarrow \frac{\pi}{2}$) blows up S_+^3 with the stationary magnon on it, and the bosonic solution becomes a BMN string on S_-^3 . Since the rescaled coordinate

$$\mathcal{Y} = \cos^2 \varphi \gamma \sqrt{\tilde{q}^2 - u^2} \mathcal{X} \rightarrow 0 \quad (4.44)$$

for all points on the string, the zero mode solution (4.24) reduces to constant spinors. The highest weight state of the massless magnon is fermionic [39] and should correspond to the

¹¹See appendix I for the construction and short representations of $\mathfrak{psu}(1|1)_{\text{c.e.}}^4$.

limit of our fermion fluctuations, but it appears we are unable to learn more about these modes from a semiclassical analysis. This shows that some aspects of the massless modes can only be captured by exact in α' results, in agreement with similar findings in the spin chain limit [50].

5 Fermion zero modes for $q = 1$

In this section we take a look at the special case of $q = 1$, as there are some subtleties not captured by our general discussion. The $q = 1$ fermion zero modes on the two AdS_3 backgrounds are more closely related than for $q < 1$, hence we will first focus on the $\text{AdS}_3 \times \text{S}^3 \times \text{T}^4$ case, then briefly describe the differences for $\text{AdS}_3 \times \text{S}^3 \times \text{S}^3 \times \text{S}^1$.

5.1 Bosonic solution

For $q = 1$, the giant magnon

$$\begin{aligned} Z_1 &= e^{it} \left[\cos \frac{p}{2} + i \sin \frac{p}{2} \tanh U \right], \\ Z_2 &= e^{iV} \sin \frac{p}{2} \text{sech} U, \end{aligned} \tag{5.1}$$

found in [36] is still a valid solution, but the magnon speed on the worldsheet is actually fixed to be the speed of light, and we will use a different parametrization¹²

$$\begin{aligned} U &= \cos \rho \beta (x + t), \\ V &= \sin \rho \beta (x + t) - x, \end{aligned} \tag{5.2}$$

where $\beta > 0$, $\rho \in [0, 2\pi)$, and the parameters are related via

$$b \equiv \cot \frac{p}{2} = \frac{\sin \rho - \beta}{\cos \rho}. \tag{5.3}$$

Already from this representation of the solution it seems like the main dependence is on the light-cone coordinate $x^+ = \frac{1}{2}(t + x)$. This is hinting at the magnon having a definite chirality, not completely unexpectedly considering that bosonic theory reduces to the conformal WZW model at $q = 1$. This statement will be made more precise shortly.

Conserved charges. For the above solution the conserved charges are

$$\begin{aligned} E - J_1 &= M = 2h \sin^2 \frac{p}{2} (\tan \rho - \cot \frac{p}{2}), \\ J_2 &= M + hp, \end{aligned} \tag{5.4}$$

with dispersion relation

$$E - J_1 = J_2 - hp. \tag{5.5}$$

¹²In the $q \rightarrow 1$ limit the parameter v of [36] is meaningless, instead we will use $\beta = \sqrt{\frac{1-v}{1+v}}$.

The WZW model. The $SU(2)$ PCM with WZ term (see section 2.2) simplifies significantly for the case of $q = 1$, with the equations of motion (2.10) now reading

$$\partial_- \mathfrak{J}_+ = 0, \quad \partial_+ \mathfrak{K}_- = 0. \quad (5.6)$$

The degrees of freedom separate based on chirality: the left-movers are described by $\mathfrak{J}_+(x^+)$, while $\mathfrak{K}_-(x^-)$ describes right-movers. Looking at the magnon's $SU(2)$ currents, listed in appendix J, we note that $\mathfrak{K}_-$ is in fact constant with no dynamical information (i.e. it can be gauged away). It is in this sense that the classical bosonic solution has a definite chirality.

5.2 Zero mode equations for $AdS_3 \times S^3 \times T^4$

The derivation of the fermion equations of motion is analogous to the $q \in [0, 1)$ case presented in section 3, and we omit the details here. In terms of the light-cone coordinates $x^\pm = \frac{1}{2}(t \pm x)$, we have

$$\begin{aligned} (\partial_- - 2\beta \cos \rho M(x^+)) \Psi^1 &= 0, \\ (\partial_+ + 2\beta \cos \rho \tilde{M}(x^+)) \Psi^2 &= 0, \end{aligned} \quad (5.7)$$

with

$$\begin{aligned} M &= \frac{1}{2\beta \cos \rho} \left(\frac{1}{2} G \Gamma_{34} + \frac{1}{2} Q \Gamma_{35} + R P_- - (R + \Gamma_{12}) P_+ \right) \\ \tilde{M} &= \frac{1}{2\beta \cos \rho} \left(\frac{1}{2} \tilde{G} \Gamma_{34} + \frac{1}{2} \tilde{Q} \Gamma_{35} + R P_- - (R + \Gamma_{12}) P_+ \right) \end{aligned} \quad (5.8)$$

where all the dependence is on x^+ via

$$\mathcal{Y} = 2\beta \cos \rho x^+. \quad (5.9)$$

The expressions for $G, \tilde{G}, Q$ and $\tilde{Q}$, along with the pullbacks of the vielbein and spin connection can be found in appendix K. These equations are the $q = 1$ versions of (3.14), but also after commuting the kappa projectors through. Note however, that they cannot be obtained as limits of the $q < 1$ analogues. In this general setting for $q = 1$ surely not (there are two parameters β, ρ here versus the one parameter u in section 3), but not even for any special case, since there is no $q = 1$ stationary magnon (see towards the end of this section).

Zero mode condition. As we have seen above, the bosonic background is itself chiral ($\partial_- \mathfrak{J}_+ = 0$), and it is reasonable to expect this to carry through to the fermionic zero modes, i.e. $\partial_- \vartheta^J = 0$. This can be viewed as the extension of the zero mode condition for $q \in [0, 1)$, and forces the first spinor to be trivial

$$\Psi^1 = 0. \quad (5.10)$$

Changing to the variable $\mathcal{Y}$, the remaining equation for Ψ^2 reads

$$(\partial_{\mathcal{Y}} + \tilde{M}) \Psi^2 = 0. \quad (5.11)$$

5.3 Zero mode solutions for $\text{AdS}_3 \times \mathbf{S}^3 \times \mathbf{T}^4$

We can find the solutions for Ψ^2 in much the same way we did in section 4. First we solve for the general kappa-fixed spinor, then substituting it into (5.11) we get a set of simpler equations on the $P_{\pm}$ subspaces, that we can easily solve.

Fixing kappa-gauge. The projector can be written as

$$K_2 = \frac{1}{2} (\mathbb{1} - \cos v \Gamma_{04} - \sin v \Gamma_{05}) , \quad (5.12)$$

with

$$v = \arcsin \left(\frac{\text{sech} \mathcal{Y}}{\sqrt{1+b^2}} \right) . \quad (5.13)$$

Making the ansatz

$$\Psi^2 = (\alpha_+(\mathcal{Y}) + \alpha_-(\mathcal{Y}) \Gamma_{45}) U , \quad (5.14)$$

with $\Gamma_{34}U = iU$ and $i\Gamma_{12}\Gamma_*\Gamma_+U = \lambda U$, the equation $K_2\Psi^2 = \Psi^2$ reduces to

$$\begin{aligned} \lambda \sin v \alpha_+ + \lambda \cos v \alpha_- &= \alpha_+ , \\ \lambda \cos v \alpha_+ - \lambda \sin v \alpha_- &= \alpha_- . \end{aligned} \quad (5.15)$$

A symmetric solution is given by

$$\alpha_+ = \sqrt{1 + \lambda \sin v} , \quad \alpha_- = \lambda \sqrt{1 - \lambda \sin v} , \quad (5.16)$$

and the most general gauge-fixed spinor is

$$\Psi^2 = \sum_{\lambda=\pm} \left(\sqrt{1 + \lambda \sin v} + \lambda \sqrt{1 - \lambda \sin v} \Gamma_{45} \right) U_{\lambda} , \quad (5.17)$$

where still $\Gamma_{34}U_{\pm} = \pm iU_{\pm}$ and $i\Gamma_{12}\Gamma_*\Gamma_+U_{\pm} = i\Gamma_{0345}U_{\pm} = \pm U_{\pm}$.

Solutions on the $P_{\pm}$ subspaces. Now letting U_{λ} depend on $\mathcal{Y}$ and substituting (5.17) into the $P_{\pm}$ projections of (5.11), after a considerable amount of simplification, we get

$$\begin{aligned} \sum_{\lambda=\pm} (\alpha_+ + \alpha_- \Gamma_{45}) (\partial_{\mathcal{Y}} + C_+) U_{\lambda} &= 0 \quad \text{on } P_+ , \\ \sum_{\lambda=\pm} (\alpha_+ + \alpha_- \Gamma_{45}) (\partial_{\mathcal{Y}} + C_-) U_{\lambda} &= 0 \quad \text{on } P_- , \end{aligned} \quad (5.18)$$

with the scalars¹³

$$C_{\pm} = -\frac{i\lambda}{4} \left(\frac{2b \text{sech}^2 \mathcal{Y}}{b^2 + \tanh^2 \mathcal{Y}} \pm \frac{\sec^2 \rho}{b - \tan \rho} + 2 \tan \rho \right) . \quad (5.19)$$

It is now a simple exercise to arrive at the solutions Ψ^{2+}, Ψ^{2-} on the P_+ and P_- subspaces, respectively,

$$\Psi^{2\pm} = \sum_{\lambda=\pm} e^{i\lambda\omega_{\pm}(\mathcal{Y})} \left(\sqrt{1 + \lambda \sin v} + \lambda \sqrt{1 - \lambda \sin v} \Gamma_{45} \right) V_{\lambda}^{\pm} , \quad (5.20)$$

¹³Note that these are different from (4.12).

where

$$\omega_{\pm}(\mathcal{Y}) = \frac{1}{2} \arctan\left(\frac{\tanh \mathcal{Y}}{b}\right) + \frac{1}{4} \left(2 \tan \rho \pm \frac{\sec^2 \rho}{b - \tan \rho}\right) \mathcal{Y}, \quad (5.21)$$

and the constant spinors V_{λ}^a satisfy $\Gamma_{34}V_{\lambda}^a = +iV_{\lambda}^a$, $P_{\pm}V_{\lambda}^{\pm} = V_{\lambda}^{\pm}$ and $i\Gamma_{0345}V_{\pm}^a = \pm V_{\pm}^a$. Starting with 16 (unconstrained) real MW spinors, these conditions leave us with 4+4 real zero modes on the P_+ and P_- subspaces. We see that none of the solutions are normalizable, which is to be expected given the chiral nature of the background. However, only looking at the solutions, and not extrapolating from the $q < 1$ case, it is unclear which 4 of these should be included in semiclassical quantization and the construction of the algebra.

5.4 Zero modes for $\text{AdS}_3 \times \mathbf{S}^3 \times \mathbf{S}^3 \times \mathbf{S}^1$

We can put the magnon (5.1) on $\text{AdS}_3 \times \mathbf{S}^3 \times \mathbf{S}^3 \times \mathbf{S}^1$ the same way we did in (2.31). Just like above, the zero modes satisfy $\Psi^1 = 0$ and

$$(\partial_{\mathcal{Y}} + \tilde{M}) \Psi^2 = 0, \quad (5.22)$$

where, similarly to (3.46)

$$\tilde{M} = \frac{1}{2\beta \cos \rho} \left(\frac{1}{2} \tilde{G} \hat{\Gamma}_{34} + \frac{1}{2} \tilde{Q} \hat{\Gamma}_{35} + R P_- - (R + \hat{\Gamma}_{12}) P_+ + \Delta_0 \hat{\Gamma}_{12} \right). \quad (5.23)$$

The boosted worldsheet coordinate is $\mathcal{Y} = 2 \cos^2 \varphi \beta \cos \rho x^+$, the scalar functions $G, \tilde{G}, Q, \tilde{Q}$ are still as given in appendix K, and from (3.43)

$$\Delta_0 = -\frac{\kappa^2}{2} \left(\hat{\Gamma}^{1268} + \mathbb{1} \right), \quad \kappa \equiv \tan \varphi. \quad (5.24)$$

$\hat{\Gamma}^{1268} = -1$. On the -1 eigenspace of $\hat{\Gamma}^{1268}$ the solutions are the same as for $\text{AdS}_3 \times \mathbf{S}^3 \times \mathbf{T}^4$ (5.20), with all Γ_A replaced by $\hat{\Gamma}_A$ (including $P_{\pm} = \frac{1}{2}(\mathbb{1} \pm \hat{\Gamma}_* \hat{\Gamma}_+)$) and imposing the extra condition $\hat{\Gamma}^{1268} V_{\lambda}^a = -V_{\lambda}^a$.

$\hat{\Gamma}^{1268} = +1$. On this subspace $\Delta_0 = -\kappa^2$, and after making the ansatz (5.17) we get

$$\begin{aligned} \left(\partial_{\mathcal{Y}} + C_+ + \frac{i\lambda\kappa^2}{2\beta \cos \rho} \right) U_{\lambda} &= 0 \quad \text{on } P_+, \\ \left(\partial_{\mathcal{Y}} + C_- - \frac{i\lambda\kappa^2}{2\beta \cos \rho} \right) U_{\lambda} &= 0 \quad \text{on } P_-. \end{aligned} \quad (5.25)$$

The zero mode solutions are

$$\Psi^{2\pm} = \sum_{\lambda=\pm} e^{i\lambda\tilde{\omega}_{\pm}(\mathcal{Y})} \left(\sqrt{1 + \lambda \sin v} + \lambda \sqrt{1 - \lambda \sin v} \Gamma_{45} \right) V_{\lambda}^{\pm}, \quad (5.26)$$

where

$$\tilde{\omega}_{\pm}(\mathcal{Y}) = \frac{1}{2} \arctan\left(\frac{\tanh \mathcal{Y}}{b}\right) + \frac{1}{4} \left(2 \tan \rho \pm \frac{(1 + \kappa^2) \sec^2 \rho}{b - \tan \rho} \right) \mathcal{Y}, \quad (5.27)$$

and $\Gamma_{34}V_{\lambda}^a = +iV_{\lambda}^a$, $P_{\pm}V_{\lambda}^{\pm} = V_{\lambda}^{\pm}$, $i\Gamma_{0345}V_{\pm}^a = \pm V_{\pm}^a$, $\hat{\Gamma}^{1268}V_{\lambda}^a = +V_{\lambda}^a$.

We have 8 real solutions in total, 2+2 for $P_{\pm}$ on each eigenspace of $\hat{\Gamma}^{1268}$. Once again, all of these zero modes are non-normalizable, and without extrapolating from the $q < 1$ analysis, we have not been able to find any distinguishing features of the 2 that would enter into canonical quantization.

5.5 The $q \rightarrow 1$ limit

We can go from the $q < 1$ dyonic magnon (2.18) to the $q = 1$ solution (5.1) by taking

$$\tilde{q} \rightarrow 0, \quad u \rightarrow -1, \quad \text{with } \tilde{q}\gamma = \beta \text{ fixed.} \quad (5.28)$$

However, to compare the zero modes above to those found in section 4, we need the $q = 1$ version of the stationary magnon (2.26) we used as a background for the $q < 1$ fermions. There are two natural ways of taking the $q \rightarrow 1$ limit, let us look at them in turn.

Our first instinct would be to take the same limit (5.28) for the stationary magnon (2.26), but this is not compatible with the condition (2.29), restricting $|u| \leq \tilde{q}$. Equivalently, we cannot make V in (5.2) only depend on x^+ (technically one could take $\beta \rightarrow \infty$, but this results in a discontinuous bosonic solution).

Alternatively, we can impose the second form of the stationary condition, and require the $SU(2)$ charge M to be zero. This would mean $\beta = 0$, and then $U \equiv 0$, with the endpoints not on the equator any more. Furthermore, the parameter p in (5.1) would not be the worldsheet momentum, as $\Delta\phi_1 = 0$.

Lacking a suitable generalization of the Hofman-Maldacena magnon for $q = 1$, it is not immediately clear how we can apply the analysis of previous sections. It would be interesting to further investigate the relation between the $q \rightarrow 1$ limit of zero modes found in section 4, to the $q = 1$ fermion fluctuations found here.

6 Conclusions

In this paper we wrote down the stationary giant magnon solution on $AdS_3 \times S^3 \times S^3 \times S^1$, which is the $q \in [0, 1)$ mixed-flux generalization of the Hofman-Maldacena magnon. We then explicitly constructed the fermion zero modes of this bosonic string solution from the quadratic action, and found that there are 4 and 2 zero modes for the $AdS_3 \times S^3 \times T^4$ and $AdS_3 \times S^3 \times S^3 \times S^1$ magnons, respectively, in agreement with the algebraic structure. We also showed how to get the generators of the centrally extended $\mathfrak{psu}(1|1)^4$ and $\mathfrak{su}(1|1)^2$ algebras from the semiclassically quantized fermion zero modes.

We treated the $q = 1$ limit separately, and found that there is no stationary magnon in this case. As expected from the chiral nature of the magnons at the $q = 1$ point, all of the zero modes we found are non-normalizable. We have the same number of fermionic generators in the off-shell algebra as for $q < 1$, and with the excess number of solutions, the issue of canonical quantization needs to be further addressed.

Our understanding of the AdS_3/CFT_2 duality is far from complete, and it is an area of active research. The CFT dual of the WZW model at $k = 1$, i.e. $AdS_3 \times S^3 \times T^4$ string theory with the smallest amount of quantized NS-NS flux, has been recently argued to be the limit of a symmetric product orbifold [32, 51, 52]. The complete Yangian algebra of the mixed-flux $AdS_3 \times S^3 \times S^3 \times S^1$ superstring has also been found [53], and it would be interesting to see how it relates to our semi-classical quantization. Semiclassical methods continue to be useful in probing the string theory side, present paper being one example, or the one-loop corrections to rigid spinning string dispersion relations [54]. It seems,

however, that massless modes cannot be captured in the semiclassical limit. Considering that in the T^4 theory the massless modes' highest weight state is a fermion [44], taking the $\alpha \rightarrow 0$ limit of the fermion zero modes might have been a good way to arrive at the solutions. The fact that this did not work indicates that the fermionic massless mode is inherently non-perturbative in nature. This is also in agreement with [50], where it was found that the $\alpha \rightarrow 0$ limit fails to capture the non-perturbative nature of the massless mode at the spin chain point (i.e. at the opposite limit of the duality). Furthermore, when the worldsheet is compactified, the presence of massless particles does not allow for perturbative computations of wrapping corrections [55]. Instead, such wrapping corrections can be computed using a non-perturbative TBA which allows for an alternative low-momentum expansion [56], based on the earlier observation of non-trivial massless scattering in the BMN limit [57].

We propose two directions for future research. Firstly, we want to follow up on the $q = 1$ limit, to get a deeper understanding of the magnon and its fermion zero modes. Secondly, we would like to generalize the dressing method to the case of $\mathbb{R} \times S^3 \times S^3$ and carry out an analysis of dressing phases for the mixed-flux $AdS_3 \times S^3 \times S^3 \times S^1$ background, similar to what was done for $AdS_3 \times S^3 \times T^4$ in [38].

Acknowledgments

AV would like to thank B. Stefański and O. Ohlsson-Sax for useful discussions. AV acknowledges the support of the George Daniels postgraduate scholarship.

A Geometry of AdS_3 and S^3

To understand string motion on $AdS_3 \times S^3 \times S^3$ one must first study its geometry. Let us take the time here to describe the components of this space in so-called Hopf coordinates, which are well-suited for describing the main object of our discussion, the giant magnon.

We can describe both AdS_3 and S^3 as hypersurfaces in four flat dimensions (with different signatures) and by parametrizing these embeddings we derive the metrics $g_{\mu\nu}$ in Hopf coordinates. From the metric we can easily read off (the natural choice for) the vielbein E_μ^A satisfying

$$E_\mu^A E_\nu^B \eta_{AB} = g_{\mu\nu}, \quad (A.1)$$

where A, B are tangent-space indices and η_{AB} is the flat (Minkowski or Euclidean) metric. The vielbein provides the most tractable construction of curved-space Dirac matrices (Γ_μ) from those of flat space (Γ_A):

$$\Gamma_\mu \equiv E_\mu^A \Gamma_A \quad \Rightarrow \quad \{\Gamma_\mu, \Gamma_\nu\} = E_\mu^A E_\nu^B \underbrace{\{\Gamma_A, \Gamma_B\}}_{2\eta_{AB}} = 2g_{\mu\nu}, \quad (A.2)$$

hence its appearance in the fermionic Lagrangian. Another object of similar importance is the *spin connection* ω_μ^{AB} , as it appears in the construction of the covariant derivative for spinors. It is given by the formula

$$\omega_\mu^{AB} = E_\nu^A \partial_\mu E^{\nu B} + E_\nu^A \Gamma_{\sigma\mu}^\nu E^{\sigma B}, \quad (A.3)$$

Figure 1. Images of a sphere and of a universal cover of AdS space.

where the Greek indices are raised by the inverse metric $g^{\mu\nu}$, and the Christoffel symbols are given by the usual $\Gamma_{\sigma\mu}^{\nu} = \frac{1}{2}g^{\nu\rho}(\partial_{\sigma}g_{\mu\rho} + \partial_{\mu}g_{\sigma\rho} - \partial_{\rho}g_{\sigma\mu})$.

Embedding into four flat dimensions (X_1, X_2, X_3, X_4) is equivalent to embedding into $\mathbb{C}^2$ via

$$Z_1 = X_1 + iX_2, \quad Z_2 = X_3 + iX_4, \quad (\text{A.4})$$

and we will use this correspondence below.

A.1 AdS_3

The three dimensional anti-de Sitter space AdS_3 can be represented as a hyperboloid (a constant negative curvature quadric)

$$\eta_{PQ}X^P X^Q = -X_1^2 - X_2^2 + X_3^2 + X_4^2 = -1 \quad (\text{A.5})$$

in $\mathbb{R}^{2,2}$ with the metric

$$ds^2 = \eta_{PQ} dX^P dX^Q, \quad \eta_{PQ} = \text{diag}(-1, -1, +1, +1). \quad (\text{A.6})$$

Equivalently in $\mathbb{C}^{1,1}$

$$|Z_1|^2 - |Z_2|^2 = 1, \quad (\text{A.7})$$

which has the global solution (the analogue of the Hopf coordinates for the sphere)

$$Z_1 = \cosh \rho e^{it}, \quad Z_2 = \sinh \rho e^{i\psi}, \quad (\text{A.8})$$

where the radius takes values $\rho \in [0, \infty)$ and $\psi \in [0, 2\pi)$. Note that $t \in [0, 2\pi)$ already covers the hyperboloid once. In the context of AdS/CFT however, it is standard to decompactify the t direction (to avoid closed time-like curves), i.e. to assume $t \in (-\infty, \infty)$. In the case of AdS_2 this “cutting open” along the circular time direction is depicted in figure 1.

Substituting into the $\mathbb{R}^{2,2}$ metric we get

$$ds^2 = -\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\psi^2, \quad (\text{A.9})$$

from which we can immediately read off the AdS_3 metric and the vielbein in coordinates (t, ρ, ψ)

$$g_{\mu\nu} = \text{diag}(-\cosh^2 \rho, 1, \sinh^2 \rho), \quad (\text{A.10})$$

$$E_\mu^A = \text{diag}(\cosh \rho, 1, \sinh \rho). \quad (\text{A.11})$$

Straightforward calculation yields the only non-zero Christoffel symbols:

$$\Gamma_{\tau\rho}^\tau = \Gamma_{\rho\tau}^\tau = \tanh \rho, \quad \Gamma_{\psi\rho}^\psi = \Gamma_{\rho\psi}^\psi = \coth \rho, \quad \Gamma_{\tau\tau}^\rho = -\Gamma_{\psi\psi}^\rho = \cosh \rho \sinh \rho, \quad (\text{A.12})$$

while some further crunching at the formulae gives the only non-zero components of the spin connection:

$$\omega_\tau^{01} = -\omega_\tau^{10} = \sinh \rho, \quad (\text{A.13})$$

$$\omega_\psi^{21} = -\omega_\psi^{12} = \cosh \rho. \quad (\text{A.14})$$

Note that the tangent space indices corresponding to (t, ρ, ψ) run from 0 to 2.

A.2 S^3

The 3-sphere is the hypersurface

$$\eta_{PQ} X^P X^Q = X_1^2 + X_2^2 + X_3^2 + X_4^2 = 1 \quad (\text{A.15})$$

in $\mathbb{R}^4$ with the metric

$$ds^2 = \eta_{PQ} dX^P dX^Q, \quad \eta_{PQ} = \text{diag}(+1, +1, +1, +1). \quad (\text{A.16})$$

Equivalently in $\mathbb{C}^2$

$$|Z_1|^2 + |Z_2|^2 = 1, \quad (\text{A.17})$$

which motivates the choice of the so-called *Hopf coordinates*

$$Z_1 = \sin \theta e^{i\phi_1}, \quad Z_2 = \cos \theta e^{i\phi_2}, \quad (\text{A.18})$$

where the range of θ is $[0, \frac{\pi}{2}]$, while ϕ_1 and ϕ_2 take values in $[0, 2\pi)$ with periodic identification at the endpoints. Substituting into the $\mathbb{R}^4$ metric we get

$$ds^2 = d\theta^2 + \sin^2 \theta d\phi_1^2 + \cos^2 \theta d\phi_2^2, \quad (\text{A.19})$$

hence the S^3 metric and the vielbein in coordinates (θ, ϕ_1, ϕ_2)

$$g_{\mu\nu} = \text{diag}(1, \sin^2 \theta, \cos^2 \theta), \quad (\text{A.20})$$

$$E_\mu^A = \text{diag}(1, \sin \theta, \cos \theta). \quad (\text{A.21})$$

From this we obtain the following non-zero Christoffel symbols

$$\Gamma_{\phi_1\theta}^{\phi_1} = \Gamma_{\theta\phi_1}^{\phi_1} = \cot \theta, \quad \Gamma_{\phi_2\theta}^{\phi_2} = \Gamma_{\theta\phi_2}^{\phi_2} = -\tan \theta, \quad \Gamma_{\phi_1\phi_1}^\theta = -\Gamma_{\phi_2\phi_2}^\theta = -\cos \theta \sin \theta, \quad (\text{A.22})$$

and putting all this together, the only non-zero spin connection components are

$$\omega_{\phi_1}^{12} = -\omega_{\phi_1}^{21} = -\cos \theta, \quad (\text{A.23})$$

$$\omega_{\phi_2}^{31} = -\omega_{\phi_2}^{13} = -\sin \theta. \quad (\text{A.24})$$

Note that the tangent space indices corresponding to (θ, ϕ_1, ϕ_2) run from 1 to 3.

B String solutions on $\mathbb{R} \times \mathbb{S}^3 \times \mathbb{S}^3$

In this appendix we describe a simple procedure that yields string solutions on $\mathbb{R} \times \mathbb{S}^3 \times \mathbb{S}^3$ from two $\mathbb{R} \times \mathbb{S}^3$ strings. Let us denote the maps from the decompactified worldsheet to the two spheres $\mathbb{S}^3_{\pm}$ by

$$\Omega_{\pm} : \mathcal{M} \rightarrow \mathbb{S}^3_{\pm}. \quad (\text{B.1})$$

We can think of such maps more explicitly in terms of Hopf coordinates (2.30) as

$$\Omega_{\pm}(\mathbf{x}) = (\theta^{\pm}(t, x), \phi_1^{\pm}(t, x), \phi_2^{\pm}(t, x)). \quad (\text{B.2})$$

The static conformal gauge string action on $\mathbb{R} \times \mathbb{S}^3 \times \mathbb{S}^3$ is a sum of two single-sphere actions (2.4), weighted by the squared radii of the spheres

$$S[\Omega_+, \Omega_-] = \frac{1}{\cos^2 \varphi} S_1[\Omega_+] + \frac{1}{\sin^2 \varphi} S_1[\Omega_-]. \quad (\text{B.3})$$

The equations of motion decouple for Ω_+ and Ω_- , taking the same form (2.5) on both. The Virasoro constraints (2.6), however, now read

$$\begin{aligned} \frac{1}{\cos^2 \varphi} V_1[\Omega_+] + \frac{1}{\sin^2 \varphi} V_1[\Omega_-] &= \kappa^2, \\ \frac{1}{\cos^2 \varphi} V_2[\Omega_+] + \frac{1}{\sin^2 \varphi} V_2[\Omega_-] &= 0, \end{aligned} \quad (\text{B.4})$$

and connect the two spheres.

If $\Omega_1(\mathbf{x})$, $\Omega_2(\mathbf{x})$ are two string solutions on $\mathbb{R} \times \mathbb{S}^3$, i.e. they satisfy (2.5) and (2.6), it is a simple exercise to see that

$$\Omega_+(\mathbf{x}) = \Omega_1(A\mathbf{x}), \quad \Omega_-(\mathbf{x}) = \Omega_2(B\mathbf{x}), \quad (\text{B.5})$$

constitute a solution on $\mathbb{R} \times \mathbb{S}^3 \times \mathbb{S}^3$, as long as the constants A, B satisfy

$$\frac{A^2}{\cos^2 \varphi} + \frac{B^2}{\sin^2 \varphi} = 1, \quad (\text{B.6})$$

which is nothing but the combined Virasoro constraint.

C Pullback of the vielbein and spin connection to the worldsheet

Putting the giant magnon (2.37) as background, one finds the following components for the pulled-back vielbein $e_a^A = E_\mu^A(X) \partial_a X^\mu$

$$e_0^0 = 1, \quad e_1^0 = 0, \quad (C.1)$$

$$e_0^3 = -\cos \varphi \frac{u\gamma^2 (\tilde{q}^2 - u^2) \tanh \mathcal{Y}}{\sqrt{\tilde{q}^2 \sinh^2 \mathcal{Y} + u^2}}, \quad e_1^3 = \cos \varphi \frac{\gamma^2 (\tilde{q}^2 - u^2) \tanh \mathcal{Y}}{\sqrt{\tilde{q}^2 \sinh^2 \mathcal{Y} + u^2}}, \quad (C.2)$$

$$e_0^4 = \cos \varphi \frac{(q^2 u^2 \gamma^2 + \tilde{q}^2 \sinh^2 \mathcal{Y}) \operatorname{sech} \mathcal{Y}}{\tilde{q} \sqrt{\tilde{q}^2 \sinh^2 \mathcal{Y} + u^2}}, \quad e_1^4 = \cos \varphi \frac{u\gamma^2 (\tilde{q}^2 - u^2) \operatorname{sech} \mathcal{Y}}{\tilde{q} \sqrt{\tilde{q}^2 \sinh^2 \mathcal{Y} + u^2}}, \quad (C.3)$$

$$e_0^5 = \cos \varphi \frac{qu\gamma^2 \sqrt{\tilde{q}^2 - u^2} \operatorname{sech} \mathcal{Y}}{\tilde{q}}, \quad e_1^5 = -\cos \varphi \frac{q\gamma^2 \sqrt{\tilde{q}^2 - u^2} \operatorname{sech} \mathcal{Y}}{\tilde{q}}, \quad (C.4)$$

$$e_0^7 = \sin \varphi, \quad e_1^7 = 0, \quad (C.5)$$

while the only non-zero components of the spin connection (pulled back to the worldsheet) are

$$\omega_0^{34} = -\omega_0^{43} = -\frac{\cos^2 \varphi \sqrt{\tilde{q}^2 - u^2} (q^2 u^2 \gamma^2 + \tilde{q}^2 \sinh^2 \mathcal{Y}) \operatorname{sech} \mathcal{Y}}{\tilde{q} \sqrt{\tilde{q}^2 \sinh^2 \mathcal{Y} + u^2}}, \quad (C.6)$$

$$\omega_1^{34} = -\omega_1^{43} = -\frac{\cos^2 \varphi u\gamma^2 (\tilde{q}^2 - u^2)^{3/2} \operatorname{sech} \mathcal{Y}}{\tilde{q} \sqrt{\tilde{q}^2 \sinh^2 \mathcal{Y} + u^2}}, \quad (C.7)$$

$$\omega_0^{35} = -\omega_0^{53} = \frac{\cos^2 \varphi qu\gamma^2}{\tilde{q}} \sqrt{\tilde{q}^2 \sinh^2 \mathcal{Y} + u^2} \operatorname{sech} \mathcal{Y}, \quad (C.8)$$

$$\omega_1^{35} = -\omega_1^{53} = -\frac{\cos^2 \varphi q\gamma^2}{\tilde{q}} \sqrt{\tilde{q}^2 \sinh^2 \mathcal{Y} + u^2} \operatorname{sech} \mathcal{Y}. \quad (C.9)$$

D Gamma matrices

In sections 3.3–3.4 we demonstrate that it is beneficial to work with a set of boosted gamma matrices, related to our original 10d Dirac matrices Γ^A , $A = 0, 1, \dots, 9$, by

$$\hat{\Gamma}^0 = \sec \varphi (\Gamma^0 - \sin \varphi \Gamma^7), \quad \hat{\Gamma}^7 = \sec \varphi (\Gamma^7 - \sin \varphi \Gamma^0), \quad \hat{\Gamma}^A = \Gamma^A \quad \text{for } A \neq 0, 7. \quad (D.1)$$

We pick the representation of Γ^A that yields the following forms for $\hat{\Gamma}^A$:

$$\hat{\Gamma}^\mu = \sigma^1 \otimes \gamma^\mu \otimes \mathbb{1} \otimes \sigma^2 \otimes \mathbb{1}, \quad \mu = 0, 1, 2 \quad (D.2)$$

$$\hat{\Gamma}^n = \sigma^1 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \sigma^1 \otimes \gamma^n, \quad n = 3, 4, 5 \quad (D.3)$$

$$\hat{\Gamma}^{\dot{n}} = \sigma^1 \otimes \mathbb{1} \otimes \gamma^{\dot{n}} \otimes \sigma^3 \otimes \mathbb{1}, \quad \dot{n} = 6, 7, 8 \quad (D.4)$$

$$\hat{\Gamma}^9 = -\sigma^2 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1}, \quad (D.5)$$

where, in terms of the Pauli matrices σ^i , the 3d gamma matrices γ^i are given by

$$\gamma^\mu = (-i\sigma^3, \sigma^1, \sigma^2), \quad \gamma^n = (\sigma^1, \sigma^2, \sigma^3), \quad \gamma^{\dot{n}} = (\sigma^2, -\sigma^3, -\sigma^1). \quad (\text{D.6})$$

In this basis,

$$\hat{\Gamma} = \sigma^3 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1}, \quad (\text{D.7})$$

$$\hat{\Gamma}^{12} = \mathbb{1} \otimes (i\sigma^3) \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1}, \quad (\text{D.8})$$

$$\hat{\Gamma}^{68} = \mathbb{1} \otimes \mathbb{1} \otimes (i\sigma^3) \otimes \mathbb{1} \otimes \mathbb{1}, \quad (\text{D.9})$$

$$\hat{\Gamma}^{012345} = \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \sigma^3 \otimes \mathbb{1}, \quad (\text{D.10})$$

$$\hat{\Gamma}^{34} = \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes (i\sigma^3), \quad (\text{D.11})$$

$$\hat{\Gamma}^{35} = \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes (-i\sigma^2), \quad (\text{D.12})$$

and in particular we see that $\hat{\Gamma}$ (Weyl matrix), $\hat{\Gamma}^{12}$, $\hat{\Gamma}^{68}$ and the projectors $\hat{P}_\pm = \frac{1}{2}(\mathbb{1} \pm \hat{\Gamma}^{012345})$ are simultaneously diagonalized.

Note that in this representation, instead of Γ^A , it is $\hat{\Gamma}^A$ that have definite hermiticity: $\hat{\Gamma}^0$ is anti-hermitian, while $\hat{\Gamma}^i$ is hermitian for $i = 1, 2, \dots, 9$. Accordingly, for the intertwiners B , T and C , defined by the relations¹⁴

$$(\Gamma^A)^* = B \Gamma^A B^{-1}, \quad (\text{D.13})$$

$$(\Gamma^A)^\dagger = -T \Gamma^A T^{-1}, \quad (\text{D.14})$$

$$(\Gamma^A)^T = -C \Gamma^A C^{-1}, \quad (\text{D.15})$$

we have

$$B = \Gamma^{1469}, \quad T = \hat{\Gamma}^0, \quad C = T B. \quad (\text{D.16})$$

E Fermion derivatives

Looking at equation (3.11) we can define the following fermion derivatives

$$\begin{aligned} D &= \frac{(1-u)\gamma}{\cos^2\varphi\zeta} \left(D_1 - D_0 - \frac{1}{8}(\#_0 - \#_1) \right)_{\partial_S \rightarrow 0}, \\ \tilde{D} &= \frac{(1+u)\gamma}{\cos^2\varphi\zeta} \left(D_1 + D_0 - \frac{1}{8}(\#_0 + \#_1) \right)_{\partial_S \rightarrow 0}, \end{aligned} \quad (\text{E.1})$$

¹⁴These relations must hold for Γ^A , not the boosted $\hat{\Gamma}^A$.

where $D_a = \partial_a + \frac{1}{4}\omega_a^{AB}\Gamma_{AB}$, and $\cos^2\varphi\zeta(1\pm u)\gamma$ were introduced to normalize the ∂_y term. The NS-NS flux appears as $\#_a \equiv e_a^A H_{ABC}\Gamma^{BC}$, which we can rewrite

$$\begin{aligned}
 \#_a &= \frac{1}{3}H_{ABC} (e_a^A\Gamma^{BC} + e_a^B\Gamma^{CA} + e_a^C\Gamma^{AB}) \\
 &= \sum_{ABC} \frac{1}{6}H_{ABC} \left(e_a^A(\Gamma_A\Gamma^{ABC} + \Gamma^{BCA}\Gamma_A) + e_a^B(\Gamma_B\Gamma^{BCA} + \Gamma^{CAB}\Gamma_B) \right. \\
 &\quad \left. + e_a^C(\Gamma_C\Gamma^{CAB} + \Gamma^{ABC}\Gamma_C) \right) \\
 &= \sum_{ABC} \frac{1}{6}H_{ABC} \sum_{D \in A,B,C} e_a^D(\Gamma_D\Gamma^{ABC} + \Gamma^{ABC}\Gamma_D) \\
 &= \frac{1}{6}H_{ABC} \sum_D e_a^D(\Gamma_D\Gamma^{ABC} + \Gamma^{ABC}\Gamma_D) \\
 &= \frac{1}{6}(\rho_a\# + \#\rho_a).
 \end{aligned} \tag{E.2}$$

On the first line we used the antisymmetry of H , going to the second that $\Gamma_A\Gamma^A = \mathbb{1}$ (no summation), on the third the antisymmetry of Γ^{ABC} , and lastly on the fourth line the fact that for $D \notin \{A, B, C\}$

$$\Gamma_D\Gamma^{ABC} + \Gamma^{ABC}\Gamma_D = 0. \tag{E.3}$$

Hence we have

$$\begin{aligned}
 D &= \partial_y + \frac{1}{2}G\Gamma_{34} + \frac{1}{2}Q\Gamma_{35} - \frac{(1-u)\gamma}{48\cos^2\varphi\zeta} (\#(\rho_0 - \rho_1) + (\rho_0 - \rho_1)\#), \\
 \tilde{D} &= \partial_y + \frac{1}{2}\tilde{G}\Gamma_{34} + \frac{1}{2}Q\Gamma_{35} - \frac{(1+u)\gamma}{48\cos^2\varphi\zeta} (\#(\rho_0 + \rho_1) + (\rho_0 + \rho_1)\#).
 \end{aligned} \tag{E.4}$$

with

$$\begin{aligned}
 G &= \frac{\omega_1^{34} - \omega_0^{34}}{\cos^2\varphi\zeta(1+u)\gamma} = \frac{\tilde{q}^2(1-u)\cosh^2\mathcal{Y} - \tilde{q}^2 + u^2}{\tilde{q}(\tilde{q}^2\sinh^2\mathcal{Y} + u^2)}\text{sech}\mathcal{Y}, \\
 \tilde{G} &= \frac{\omega_1^{34} + \omega_0^{34}}{\cos^2\varphi\zeta(1-u)\gamma} = -\frac{\tilde{q}^2(1+u)\cosh^2\mathcal{Y} - \tilde{q}^2 + u^2}{\tilde{q}(\tilde{q}^2\sinh^2\mathcal{Y} + u^2)}\text{sech}\mathcal{Y}, \\
 Q &= \frac{\omega_1^{35} \mp \omega_0^{35}}{\cos^2\varphi\zeta(1\pm u)\gamma} = -\frac{q}{\tilde{q}\sqrt{\tilde{q}^2 - u^2}} \sqrt{\tilde{q}^2\sinh^2\mathcal{Y} + u^2}\text{sech}\mathcal{Y}.
 \end{aligned} \tag{E.5}$$

The next step is to take (3.42)

$$\# = 24q\cos\varphi\left(\hat{\Gamma}_*P_+ - \Delta\hat{\Gamma}^{12}\right) \tag{E.6}$$

and substitute into (E.4), with the further restriction that the derivatives act on kappa

fixed spinors, as in (3.40). For DK_1 the relevant term is

$$\begin{aligned}
 & \frac{1}{48 \cos^2 \varphi} \left(\not{H}(\rho_0 - \rho_1) + (\rho_0 - \rho_1) \not{H} \right) K_1 \\
 &= -\frac{1}{24 \cos \varphi} \left(\not{H} \hat{\Gamma}^0 K_2 + \hat{\Gamma}^0 K_2 \not{H} \right) K_1 \\
 &= -q \left(\hat{\Gamma}_* P_+ \hat{\Gamma}^0 K_2 K_1 + \hat{\Gamma}^0 K_2 \hat{\Gamma}_* P_+ K_1 \right. \\
 &\quad \left. - \Delta \hat{\Gamma}^{12} \hat{\Gamma}^0 K_2 K_1 - \hat{\Gamma}^0 K_2 \Delta \hat{\Gamma}^{12} K_1 \right) \\
 &= q \left(P_- \hat{\Gamma}^{12} K_2 K_1 + (-K_2 + \mathbb{1}) P_+ \hat{\Gamma}^{12} K_1 \right. \\
 &\quad \left. - \left(\Delta_0 + \Delta_7 \hat{\Gamma}^{07} \right) \hat{\Gamma}^{12} K_2 K_1 + \left(\Delta_0 + \Delta_7 \hat{\Gamma}^{07} \right) \hat{\Gamma}^{12} K_2 K_1 - \Delta_0 \hat{\Gamma}^{12} K_1 \right) \\
 &= -q \left(R P_- - \left(R + \hat{\Gamma}_{12} \right) P_+ + \Delta_0 \hat{\Gamma}_{12} \right) K_1,
 \end{aligned} \tag{E.7}$$

where we have also used the definition of the kappa projectors (3.37), the form of Δ in (3.43), the relation $K_J \hat{\Gamma}^0 = -\hat{\Gamma}^0 K_J + \hat{\Gamma}^0$, and (3.48). Similarly, for $\tilde{D}K_2$ we have

$$\frac{1}{48 \cos^2 \varphi} \left(\not{H}(\rho_0 + \rho_1) + (\rho_0 + \rho_1) \not{H} \right) K_2 \tag{E.8}$$

$$= -q \left(R P_- - \left(R + \hat{\Gamma}_{12} \right) P_+ + \Delta_0 \hat{\Gamma}_{12} \right) K_2, \tag{E.9}$$

and with this, the fermion derivatives take the final form

$$\begin{aligned}
 D &= \partial_{\mathcal{Y}} + \frac{1}{2} G \hat{\Gamma}_{34} + \frac{1}{2} Q \hat{\Gamma}_{35} + \frac{q(1-u)\gamma}{\zeta} \left(R P_- - \left(R + \hat{\Gamma}_{12} \right) P_+ + \Delta_0 \hat{\Gamma}_{12} \right), \\
 \tilde{D} &= \partial_{\mathcal{Y}} + \frac{1}{2} \tilde{G} \hat{\Gamma}_{34} + \frac{1}{2} \tilde{Q} \hat{\Gamma}_{35} + \frac{q(1+u)\gamma}{\zeta} \left(R P_- - \left(R + \hat{\Gamma}_{12} \right) P_+ + \Delta_0 \hat{\Gamma}_{12} \right).
 \end{aligned} \tag{E.10}$$

Let us stress one last time, that these forms are only valid when acting on kappa-fixed spinors.

F No normalizable solutions for $\Delta \neq 0$

In section 4 we found the expected number of normalizable solutions in an analytic form for $\Delta = 0$. However, to complete the counting argument for fermion zero modes, it is necessary to demonstrate that there are no normalizable solutions at all for $\Delta \neq 0$. This happens for the maximally SUSY $\text{AdS}_3 \times S^3 \times S^3 \times S^1$ giant magnon, on the $\Gamma_{1268} = +1$ spinor subspace:

$$\begin{aligned}
 \Delta &= -\tan^2 \varphi \hat{\Gamma}^0 - \tan \varphi \sec \varphi \hat{\Gamma}^7 = \left(\kappa^2 - \kappa \tilde{\kappa} \hat{\Gamma}_{07} \right) \hat{\Gamma}_0, \\
 \kappa &= \tan \varphi, \quad \tilde{\kappa} = \sqrt{1 + \kappa^2} = \sec \varphi.
 \end{aligned} \tag{F.1}$$

The equations of motion are

$$\begin{aligned}
 \zeta(1+u)\gamma D \Psi^1 + \tilde{q} \left(R P_- - K_1 \Delta \hat{\Gamma}_* \right) \Psi^2 &= 0, \\
 \zeta(1-u)\gamma \tilde{D} \Psi^2 - \tilde{q} \left(R P_- - K_2 \Delta \hat{\Gamma}_* \right) \Psi^1 &= 0.
 \end{aligned} \tag{F.2}$$

with fermion derivatives

$$\begin{aligned} D &= \partial_{\mathcal{Y}} + \frac{1}{2}G\hat{\Gamma}_{34} + \frac{1}{2}Q\hat{\Gamma}_{35} + \frac{q(1-u)\gamma}{\zeta} \left(RP_- - (R + \hat{\Gamma}_{12})P_+ - \kappa^2\hat{\Gamma}_{12} \right), \\ \tilde{D} &= \partial_{\mathcal{Y}} + \frac{1}{2}\tilde{G}\hat{\Gamma}_{34} + \frac{1}{2}Q\hat{\Gamma}_{35} + \frac{q(1+u)\gamma}{\zeta} \left(RP_- - (R + \hat{\Gamma}_{12})P_+ - \kappa^2\hat{\Gamma}_{12} \right). \end{aligned} \quad (\text{F.3})$$

Our approach will be similar to section 4. First we write down general kappa-fixed spinors, which we then substitute into the equations of motion to get a system of simpler ODEs.

Kappa fixing. The main difference from $\Delta = 0$ is that the solutions will not have definite $P_{\pm}$ chirality, since Δ mixes the P_+ and P_- subspaces. Accordingly, the kappa-fixed ansatz generalizing (4.5) will have to relate the two projections. This is achieved by

$$\Psi^J = \sum_{\lambda=\pm} \left[\left(\alpha_+^J + \alpha_-^J \hat{\Gamma}_{45} \right) f_J(\mathcal{Y}) + \left(\bar{\alpha}_+^J + \bar{\alpha}_-^J \hat{\Gamma}_{45} \right) g_J(\mathcal{Y}) \hat{\Gamma}_{07} \right] U_{\lambda}, \quad (\text{F.4})$$

where the constant spinor U_{λ} is shared between $I = 1$ and 2, and has eigenvalues $\Gamma_{34}U_{\lambda} = +iU_{\lambda}$, $P_-U_{\lambda} = U_{\lambda}$ and $\hat{\Gamma}_{12}U_{\lambda} = i\lambda U_{\lambda}$. The functions f_J , g_J represent the parts of the solution on the P_- and P_+ subspaces respectively, and $\hat{\Gamma}_{07}$ transforms U_{λ} between the two. We take $\alpha_{\pm}^J$ to be defined by (4.7)–(4.8), and

$$\bar{\alpha}_{\pm}^J \equiv \alpha_{\pm}^J|_{\lambda \rightarrow -\lambda}. \quad (\text{F.5})$$

This is because the definition of λ here differs from that in section 4, the two agree on the P_- subspace, while on P_+ they are related by a minus sign.

The $K\Delta\hat{\Gamma}_*$ terms. For the most part, substitution yields equations that are familiar from section 4, the only new terms being $K_1\Delta\hat{\Gamma}_*\Psi^2$ and $K_2\Delta\hat{\Gamma}_*\Psi^1$. It is easy to see that

$$\Delta\hat{\Gamma}_*\Psi^J = \sum_{\lambda=\pm} \left[\left(\alpha_+^J + \alpha_-^J \hat{\Gamma}_{45} \right) i\lambda \left(\kappa^2 - \kappa\tilde{\kappa}\hat{\Gamma}_{07} \right) f_J \right. \quad (\text{F.6})$$

$$\left. + \left(\bar{\alpha}_+^J + \bar{\alpha}_-^J \hat{\Gamma}_{45} \right) i\lambda \left(\kappa^2 \hat{\Gamma}_{07} - \kappa\tilde{\kappa} \right) g_J \right] U_{\lambda}. \quad (\text{F.7})$$

On the other hand, from (4.1) and the definitions (4.7)–(4.8) one can derive the action of the kappa-projectors on a general spinor $V = V_+ + V_-$ on the P_- subspace, with components $\hat{\Gamma}_{34}V_{\pm} = \pm iV_{\pm}$

$$\begin{aligned} K_1V &= \left(\alpha_+^1 + \alpha_-^1 \hat{\Gamma}_{45} \right) \left[\frac{1}{2}e^{-i\chi}\sqrt{1 - \lambda Q_+ \text{sech}\mathcal{Y}} V_+ \right. \\ &\quad \left. - \frac{1}{2}\lambda e^{i\chi}\sqrt{1 + \lambda Q_+ \text{sech}\mathcal{Y}} \hat{\Gamma}_{45} V_- \right], \\ K_2V &= \left(\alpha_+^2 + \alpha_-^2 \hat{\Gamma}_{45} \right) \left[\frac{1}{2}e^{-i\tilde{\chi}}\sqrt{1 + \lambda Q_- \text{sech}\mathcal{Y}} V_+ \right. \\ &\quad \left. + \frac{1}{2}\lambda e^{i\tilde{\chi}}\sqrt{1 - \lambda Q_- \text{sech}\mathcal{Y}} \hat{\Gamma}_{45} V_- \right]. \end{aligned} \quad (\text{F.8})$$

The corresponding expressions for the P_+ subspace are obtained by sending $\lambda \rightarrow -\lambda$. Putting these together we get

$$\begin{aligned}
 K_1 \Delta \hat{\Gamma}_* \Psi^2 &= \sum_{\lambda=\pm} \left[\left(\alpha_+^1 + \alpha_-^1 \hat{\Gamma}_{45} \right) \left[i\lambda \tilde{q} \kappa^2 \delta_1 f_2 - i\lambda \tilde{q} \kappa \tilde{\kappa} \delta_2 g_2 \right] \right. \\
 &\quad \left. + \left(\bar{\alpha}_+^1 + \bar{\alpha}_-^1 \hat{\Gamma}_{45} \right) \left[i\lambda \tilde{q} \kappa^2 \bar{\delta}_1 g_2 - i\lambda \tilde{q} \kappa \tilde{\kappa} \bar{\delta}_2 f_2 \right] \hat{\Gamma}_{07} \right] U_\lambda, \\
 K_2 \Delta \hat{\Gamma}_* \Psi^1 &= \sum_{\lambda=\pm} \left[\left(\alpha_+^2 + \alpha_-^2 \hat{\Gamma}_{45} \right) \left[-i\lambda \tilde{q} \kappa^2 \bar{\delta}_1 f_1 - i\lambda \tilde{q} \kappa \tilde{\kappa} \delta_2 g_1 \right] \right. \\
 &\quad \left. + \left(\bar{\alpha}_+^2 + \bar{\alpha}_-^2 \hat{\Gamma}_{45} \right) \left[-i\lambda \tilde{q} \kappa^2 \delta_1 g_1 - i\lambda \tilde{q} \kappa \tilde{\kappa} \bar{\delta}_2 f_1 \right] \hat{\Gamma}_{07} \right] U_\lambda,
 \end{aligned} \tag{F.9}$$

with

$$\begin{aligned}
 \delta_1 &= \frac{1}{2} \left(e^{i(\tilde{x}-x)} \sqrt{(1-\lambda Q_+ \operatorname{sech} \mathcal{Y})(1+\lambda Q_- \operatorname{sech} \mathcal{Y})} \right. \\
 &\quad \left. - e^{-i(\tilde{x}-x)} \sqrt{(1+\lambda Q_+ \operatorname{sech} \mathcal{Y})(1-\lambda Q_- \operatorname{sech} \mathcal{Y})} \right), \\
 \delta_2 &= \frac{1}{2} \left(e^{i(\tilde{x}-x)} \sqrt{(1-\lambda Q_+ \operatorname{sech} \mathcal{Y})(1-\lambda Q_- \operatorname{sech} \mathcal{Y})} \right. \\
 &\quad \left. + e^{-i(\tilde{x}-x)} \sqrt{(1+\lambda Q_+ \operatorname{sech} \mathcal{Y})(1+\lambda Q_- \operatorname{sech} \mathcal{Y})} \right),
 \end{aligned} \tag{F.10}$$

and $\bar{\delta}_J = \delta_J|_{\lambda \rightarrow -\lambda}$.

Reduced equations. Substituting (F.4) into (F.2) we get

$$\begin{aligned}
 \sum_{\lambda=\pm} &\left[\left(\alpha_+^1 + \alpha_-^1 \hat{\Gamma}_{45} \right) \left[\partial_{\mathcal{Y}} f_1 + \left(C_{11} - i\lambda q \kappa^2 \frac{(1-u)\gamma}{\zeta} \right) f_1 \right. \right. \\
 &\quad \left. \left. + \left(C_{12} - i\lambda \tilde{q} \kappa^2 \delta_1 \frac{(1-u)\gamma}{\zeta} \right) f_2 + i\lambda \tilde{q} \kappa \tilde{\kappa} \delta_2 \frac{(1-u)\gamma}{\zeta} g_2 \right] \right. \\
 &\quad \left. + \left(\bar{\alpha}_+^1 + \bar{\alpha}_-^1 \hat{\Gamma}_{45} \right) \left[\partial_{\mathcal{Y}} g_1 - \left(C_+ + i\lambda q \kappa^2 \frac{(1-u)\gamma}{\zeta} \right) g_1 \right. \right. \\
 &\quad \left. \left. + i\lambda \tilde{q} \kappa \tilde{\kappa} \bar{\delta}_2 \frac{(1-u)\gamma}{\zeta} f_2 - i\lambda \tilde{q} \kappa^2 \bar{\delta}_1 \frac{(1-u)\gamma}{\zeta} g_2 \right] \hat{\Gamma}_{07} \right] U_\lambda = 0,
 \end{aligned} \tag{F.11}$$

$$\begin{aligned}
 \sum_{\lambda=\pm} &\left[\left(\alpha_+^2 + \alpha_-^2 \hat{\Gamma}_{45} \right) \left[\partial_{\mathcal{Y}} f_2 + \left(C_{21} - i\lambda q \kappa^2 \frac{(1+u)\gamma}{\zeta} \right) f_2 \right. \right. \\
 &\quad \left. \left. + \left(C_{22} - i\lambda \tilde{q} \kappa^2 \bar{\delta}_1 \frac{(1+u)\gamma}{\zeta} \right) f_1 - i\lambda \tilde{q} \kappa \tilde{\kappa} \bar{\delta}_2 \frac{(1+u)\gamma}{\zeta} g_1 \right] \right. \\
 &\quad \left. + \left(\bar{\alpha}_+^2 + \bar{\alpha}_-^2 \hat{\Gamma}_{45} \right) \left[\partial_{\mathcal{Y}} g_2 - \left(C_- + i\lambda q \kappa^2 \frac{(1+u)\gamma}{\zeta} \right) g_2 \right. \right. \\
 &\quad \left. \left. - i\lambda \tilde{q} \kappa \tilde{\kappa} \bar{\delta}_2 \frac{(1+u)\gamma}{\zeta} f_1 - i\lambda \tilde{q} \kappa^2 \delta_1 \frac{(1+u)\gamma}{\zeta} g_1 \right] \hat{\Gamma}_{07} \right] U_\lambda = 0,
 \end{aligned} \tag{F.12}$$

where $C_{\pm}$ and C_{ij} are as defined in (4.12), (4.17). If we make the ansatz

$$\begin{aligned} f_1 &= \frac{e^{-\int C_{11} d\mathcal{Y}}}{\sqrt{1+u}} \tilde{f}_1, & g_1 &= \frac{e^{\int C_+ d\mathcal{Y}}}{\sqrt{1+u}} \tilde{g}_1, \\ f_2 &= \frac{e^{-\int C_{21} d\mathcal{Y}}}{\sqrt{1-u}} \tilde{f}_2, & g_2 &= \frac{e^{\int C_- d\mathcal{Y}}}{\sqrt{1-u}} \tilde{g}_2, \end{aligned} \quad (\text{F.13})$$

we get the following four equations

$$\begin{aligned} \left(\partial_{\mathcal{Y}} - i\lambda q \kappa^2 \frac{(1-u)\gamma}{\zeta} \right) \tilde{f}_1 + \left(e^{+i2\lambda\xi\mathcal{Y}} (\lambda \tanh \mathcal{Y} - i\xi) - i\lambda \tilde{q} \kappa^2 \frac{\delta_1}{\zeta} e^{\int (C_{11}-C_{21}) d\mathcal{Y}} \right) \tilde{f}_2 \\ + i\lambda \tilde{q} \kappa \tilde{\kappa} \frac{\delta_2}{\zeta} e^{\int (C_{11}+C_-) d\mathcal{Y}} \tilde{g}_2 = 0, \end{aligned} \quad (\text{F.14})$$

$$\begin{aligned} \left(\partial_{\mathcal{Y}} - i\lambda q \kappa^2 \frac{(1+u)\gamma}{\zeta} \right) \tilde{f}_2 + \left(e^{-i2\lambda\xi\mathcal{Y}} (\lambda \tanh \mathcal{Y} + i\xi) - i\lambda \tilde{q} \kappa^2 \frac{\bar{\delta}_1}{\zeta} e^{\int (C_{21}-C_{11}) d\mathcal{Y}} \right) \tilde{f}_1 \\ - i\lambda \tilde{q} \kappa \tilde{\kappa} \frac{\delta_2}{\zeta} e^{\int (C_{21}+C_+) d\mathcal{Y}} \tilde{g}_1 = 0, \end{aligned} \quad (\text{F.15})$$

$$\begin{aligned} \left(\partial_{\mathcal{Y}} - i\lambda q \kappa^2 \frac{(1-u)\gamma}{\zeta} \right) \tilde{g}_1 + i\lambda \tilde{q} \kappa \tilde{\kappa} \frac{\bar{\delta}_2}{\zeta} e^{-\int (C_{21}+C_+) d\mathcal{Y}} \tilde{f}_2 \\ - i\lambda \tilde{q} \kappa^2 \frac{\bar{\delta}_1}{\zeta} e^{\int (C_- - C_+) d\mathcal{Y}} \tilde{g}_2 = 0, \end{aligned} \quad (\text{F.16})$$

$$\begin{aligned} \left(\partial_{\mathcal{Y}} - i\lambda q \kappa^2 \frac{(1+u)\gamma}{\zeta} \right) \tilde{g}_2 - i\lambda \tilde{q} \kappa \tilde{\kappa} \frac{\bar{\delta}_2}{\zeta} e^{-\int (C_{11}+C_-) d\mathcal{Y}} \tilde{f}_1 \\ - i\lambda \tilde{q} \kappa^2 \frac{\delta_1}{\zeta} e^{\int (C_+ - C_-) d\mathcal{Y}} \tilde{g}_1 = 0. \end{aligned} \quad (\text{F.17})$$

The first thing to observe is that setting $\kappa = 0$ the functions $\tilde{f}_1, \tilde{f}_2$ decouple from $\tilde{g}_1, \tilde{g}_2$, and indeed we recover the $\Delta = 0$ solutions found in section 4.

Pure R-R background. We have not been able to find exact solutions at general values of q and κ , nonetheless, we can give an argument for their non-normalizability if we consider an expansion in powers of q and κ . It turns out we can already see non-normalizability at leading order in q , i.e. at $q = 0$, with the equations simplifying to

$$\begin{aligned} \partial_{\mathcal{Y}} \tilde{f}_1 + \lambda \tilde{\kappa}^2 \tanh \mathcal{Y} \tilde{f}_2 + i\lambda \kappa \tilde{\kappa} \operatorname{sech} \mathcal{Y} \tilde{g}_2 &= 0, \\ \partial_{\mathcal{Y}} \tilde{f}_2 + \lambda \tilde{\kappa}^2 \tanh \mathcal{Y} \tilde{f}_1 - i\lambda \kappa \tilde{\kappa} \operatorname{sech} \mathcal{Y} \tilde{g}_1 &= 0, \\ \partial_{\mathcal{Y}} \tilde{g}_1 + \lambda \kappa^2 \tanh \mathcal{Y} \tilde{g}_2 + i\lambda \kappa \tilde{\kappa} \operatorname{sech} \mathcal{Y} \tilde{f}_2 &= 0, \\ \partial_{\mathcal{Y}} \tilde{g}_2 + \lambda \kappa^2 \tanh \mathcal{Y} \tilde{g}_1 - i\lambda \kappa \tilde{\kappa} \operatorname{sech} \mathcal{Y} \tilde{f}_1 &= 0. \end{aligned} \quad (\text{F.18})$$

Zeroth order in κ . The first thing to observe is that setting $\kappa = 0$ leads to a significant simplification of the equations. $\tilde{f}_1, \tilde{f}_2$ decouple from $\tilde{g}_1, \tilde{g}_2$, and the solutions take the form

$$\begin{aligned} \tilde{f}_1 &= c_1 \operatorname{sech} \mathcal{Y} + c_2 \cosh \mathcal{Y}, & \tilde{g}_1 &= c_3, \\ \tilde{f}_2 &= \lambda c_1 \operatorname{sech} \mathcal{Y} - \lambda c_2 \cosh \mathcal{Y}, & \tilde{g}_2 &= c_4. \end{aligned} \quad (\text{F.19})$$

This limit corresponds to the case of $\Delta = 0$, and the solutions match those found in section 4, after we set $q = 0$. Let us denote the only normalizable solution in the $\kappa \rightarrow 0$ limit by

$$\begin{aligned}\tilde{f}_1^{(0)} &= C_0 \operatorname{sech} \mathcal{Y}, & \tilde{g}_1^{(0)} &= 0, \\ \tilde{f}_2^{(0)} &= \lambda C_0 \operatorname{sech} \mathcal{Y}, & \tilde{g}_2^{(0)} &= 0.\end{aligned}\tag{F.20}$$

Expansion in κ . Introducing the vector notation $\mathbf{f} = (\tilde{f}_1, \tilde{f}_2, \tilde{g}_1, \tilde{g}_2)^\top$, the equations above, for general values of κ , can be written as

$$\partial_{\mathcal{Y}} \mathbf{f} + M_{\kappa}(\mathcal{Y}) \mathbf{f} = \mathbf{0}.\tag{F.21}$$

Since $M_{\kappa}(\mathcal{Y})$ is regular at $\kappa = 0$, we can make the ansatz

$$\mathbf{f} = \sum_{n=0}^{\infty} \kappa^n \mathbf{f}^{(n)},\tag{F.22}$$

where $\mathbf{f}^{(n)} = (\tilde{f}_1^{(n)}, \tilde{f}_2^{(n)}, \tilde{g}_1^{(n)}, \tilde{g}_2^{(n)})^\top$ are independent of κ . Substituting this into the equations, then expanding in κ , we get a system of ODEs for each power of κ : for all n the $\mathbf{f}^{(n)}$ equations will have the same homogeneous part as the $\kappa = 0$ system, and the forcing terms will be given by some linear combination of lower order solutions

$$\partial_{\mathcal{Y}} \mathbf{f}^{(n)} + M_0(\mathcal{Y}) \mathbf{f}^{(n)} = \sum_{k=0}^{n-1} F_{n,k}^1 \mathbf{f}^{(k)}.\tag{F.23}$$

We need to solve these order-by-order, and for normalizability at generic values of κ , we would need all $\mathbf{f}^{(n)}$ to be normalizable.

First order in κ . At zeroth order we simply have the homogeneous $\kappa = 0$ equations, and the normalizable $\mathbf{f}^{(0)}$ solution is (F.20). The first subleading solution $\mathbf{f}^{(1)}$ is obtained from (F.23) with

$$F_{1,0}^1 = i\lambda \operatorname{sech} \mathcal{Y} \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},\tag{F.24}$$

and is given by

$$\begin{aligned}\tilde{f}_1^{(1)} &= c_3 \operatorname{sech} \mathcal{Y} + c_4 \cosh \mathcal{Y}, & \tilde{g}_1^{(1)} &= c_1 - i C_0 \tanh \mathcal{Y}, \\ \tilde{f}_2^{(1)} &= c_3 \operatorname{sech} \mathcal{Y} - c_4 \cosh \mathcal{Y}, & \tilde{g}_2^{(1)} &= c_2 - i\lambda C_0 \tanh \mathcal{Y}.\end{aligned}\tag{F.25}$$

The terms with C_0 are fixed, they are the response to the zeroth order ($\kappa = 0$) solution (F.20), while the integration constants c_j for $j = 1, \dots, 4$ parametrize the homogeneous solution. We see that there is no combination of c_j that would make all components normalizable, in particular, $\tilde{g}_j^{(1)}$ can be chosen to decay at either $\mathcal{Y} \rightarrow \infty$ or $\mathcal{Y} \rightarrow -\infty$, but not both.

It is already impossible to find a decaying solution at first order in κ , and we conclude that there are no normalizable solutions for $\Delta \neq 0$.

G Phase identities

The following formulae are useful when deriving the reduced equations of motion (4.11) and (4.16). Using simple trigonometric and hyperbolic identities and Euler's formula it is easy to see that

$$e^{i \operatorname{arccot}(\alpha \operatorname{csch} \mathcal{Y})} = i \left(\frac{\sinh \mathcal{Y} - i\alpha}{\sinh \mathcal{Y} + i\alpha} \right)^{1/2}, \quad (\text{G.1})$$

$$e^{i \arcsin\left(\frac{\tanh \mathcal{Y}}{\sqrt{1-\alpha^2 \operatorname{sech}^2 \mathcal{Y}}}\right)} = i \frac{\tanh \mathcal{Y} - i\sqrt{1-\alpha^2} \operatorname{sech} \mathcal{Y}}{\sqrt{1-\alpha^2 \operatorname{sech}^2 \mathcal{Y}}}, \quad (\text{G.2})$$

and with these we have

$$\begin{aligned} e^{i\chi} &= \left(\frac{\tilde{q} \sinh \mathcal{Y} - iu}{\tilde{q} \sinh \mathcal{Y} + iu} \right)^{1/4} \left(\frac{\tanh \mathcal{Y} + i\sqrt{1-Q_+^2} \operatorname{sech} \mathcal{Y}}{\sqrt{1-Q_+^2 \operatorname{sech}^2 \mathcal{Y}}} \right)^{1/2}, \\ e^{i\tilde{\chi}} &= i \left(\frac{\tilde{q} \sinh \mathcal{Y} - iu}{\tilde{q} \sinh \mathcal{Y} + iu} \right)^{1/4} \left(\frac{\tanh \mathcal{Y} - i\sqrt{1-Q_-^2} \operatorname{sech} \mathcal{Y}}{\sqrt{1-Q_-^2 \operatorname{sech}^2 \mathcal{Y}}} \right)^{1/2}, \end{aligned} \quad (\text{G.3})$$

where, as defined in (4.2),

$$\begin{aligned} \chi &= \frac{1}{2} \left(\operatorname{arccot} \left(\frac{u \operatorname{csch} \mathcal{Y}}{\tilde{q}} \right) - \arcsin \left(\frac{\tanh \mathcal{Y}}{\sqrt{1-Q_+^2 \operatorname{sech}^2 \mathcal{Y}}} \right) \right), \\ \tilde{\chi} &= \frac{1}{2} \left(\operatorname{arccot} \left(\frac{u \operatorname{csch} \mathcal{Y}}{\tilde{q}} \right) + \arcsin \left(\frac{\tanh \mathcal{Y}}{\sqrt{1-Q_-^2 \operatorname{sech}^2 \mathcal{Y}}} \right) \right). \end{aligned} \quad (\text{G.4})$$

H The $\mathfrak{su}(1|1)_{\text{c.e.}}^2$ algebra and its representaions

In this section we will build up the centrally extended $\mathfrak{su}(1|1)^2$ algebra, and look at its short representations. This is the off-shell symmetry algebra of the light-cone gauge fixed superstring theory on $\text{AdS}_3 \times \text{S}^3 \times \text{S}^3 \times \text{S}^1$ [47], and as such, is of great importance in understanding the $\text{AdS}_3/\text{CFT}_2$ duality.

H.1 The $\mathfrak{su}(1|1)$ algebra

The algebra $\mathfrak{psu}(1|1)$ consist of the anticommuting supercharges $\mathbf{Q}$ and $\mathbf{S}$, and if we add the central charge $\mathbf{H}$, i.e. introduce the non-trivial anticommutation relation

$$\{\mathbf{Q}, \mathbf{S}\} = \mathbf{H}, \quad (\text{H.1})$$

we get the $\mathfrak{su}(1|1)$ algebra. In its simplest non-trivial representation a bosonic state $|\phi\rangle$ and a fermionic state $|\psi\rangle$ transform under the charges according to

$$\begin{aligned} \mathbf{Q} |\phi\rangle &= a |\psi\rangle, & \mathbf{S} |\phi\rangle &= 0, & \mathbf{H} |\phi\rangle &= H |\phi\rangle, \\ \mathbf{Q} |\psi\rangle &= 0, & \mathbf{S} |\psi\rangle &= b |\phi\rangle, & \mathbf{H} |\psi\rangle &= H |\psi\rangle. \end{aligned} \quad (\text{H.2})$$

For closure of the algebra the eigenvalue of the central charge must be $H = ab$. In fact the representation is labelled by H alone, the ratio of a and b is physically irrelevant, it only parametrizes the difference in normalization of the states $|\phi\rangle$ and $|\psi\rangle$. Let us denote this representation by $(\mathbf{1}|\mathbf{1})_H$.

H.2 The $\mathfrak{su}(1|1)^2$ algebra

In the direct product of two $\mathfrak{su}(1|1)$ algebras we have two copies (left and right) of each charge, satisfying

$$\begin{aligned} \{\mathbf{Q}_L, \mathbf{S}_L\} &= \mathbf{H}_L, & \{\mathbf{Q}_L, \mathbf{Q}_R\} &= 0, & \{\mathbf{Q}_L, \mathbf{S}_R\} &= 0, \\ \{\mathbf{Q}_R, \mathbf{S}_R\} &= \mathbf{H}_R, & \{\mathbf{S}_L, \mathbf{S}_R\} &= 0, & \{\mathbf{Q}_R, \mathbf{S}_L\} &= 0. \end{aligned} \quad (\text{H.3})$$

When coupling these two systems, we can introduce the total Hamiltonian $\mathbf{H}$ and the angular momentum $\mathbf{M}$

$$\mathbf{H} = \mathbf{H}_L + \mathbf{H}_R, \quad \mathbf{M} = \mathbf{H}_L - \mathbf{H}_R. \quad (\text{H.4})$$

In terms of these generators we have

$$\{\mathbf{Q}_L, \mathbf{S}_L\} = \frac{1}{2}(\mathbf{H} + \mathbf{M}), \quad \{\mathbf{Q}_R, \mathbf{S}_R\} = \frac{1}{2}(\mathbf{H} - \mathbf{M}). \quad (\text{H.5})$$

Representations. Irreducible representations will be tensor products of a left-moving and a right-moving part, since the algebra is a direct product. For later convenience we take $\mathbf{S}_L$ and $\mathbf{Q}_R$ to be raising operators, while $\mathbf{Q}_L$ and $\mathbf{S}_R$ will be lowering operators. A highest weight state then satisfies

$$\mathbf{S}_L |\text{h.w.}\rangle = 0, \quad \mathbf{Q}_R |\text{h.w.}\rangle = 0. \quad (\text{H.6})$$

In a *short* representation a highest weight state will be annihilated by additional supercharges. For the $\mathfrak{su}(1|1)^2$ algebra the two shortening conditions are $H_L = 0$ and $H_R = 0$. A representation where the h.w. state has vanishing H_R , and is therefore annihilated by $\mathbf{S}_R$, is called a *left-moving* representation. The simplest non-trivial example is given by $(\mathbf{1}|\mathbf{1})_H \otimes \mathbf{1}$, with a bosonic state $|\phi\rangle$ and a fermionic state $|\psi\rangle$ transforming as

$$\begin{aligned} \mathbf{Q}_L |\phi\rangle &= a |\psi\rangle, & \mathbf{S}_L |\phi\rangle &= 0, & \mathbf{H}_L |\phi\rangle &= H |\phi\rangle, \\ \mathbf{Q}_L |\psi\rangle &= 0, & \mathbf{S}_L |\psi\rangle &= b |\phi\rangle, & \mathbf{H}_L |\psi\rangle &= H |\psi\rangle, \\ \mathbf{Q}_R |\phi\rangle &= 0, & \mathbf{S}_R |\phi\rangle &= 0, & \mathbf{H}_R |\phi\rangle &= 0, \\ \mathbf{Q}_R |\psi\rangle &= 0, & \mathbf{S}_R |\psi\rangle &= 0, & \mathbf{H}_R |\psi\rangle &= 0. \end{aligned} \quad (\text{H.7})$$

with $H = ab$. We also have *right-moving* representations with $H_L = 0$, whose highest weight states are annihilated by $\mathbf{Q}_L$. An example is $\mathbf{1} \otimes (\mathbf{1}|\mathbf{1})_H$, in which the right generators act on the two states $|\bar{\phi}\rangle$ and $|\bar{\psi}\rangle$ as in (H.2), and all the left generators annihilate them.

H.3 The centrally extended $\mathfrak{su}(1|1)^2$ algebra

We can extend the $\mathfrak{su}(1|1)^2$ algebra by introducing two additional central charges $\mathbf{C}$ and $\overline{\mathbf{C}}$. These appear in anticommutators between the two sectors, and we take the choice¹⁵

$$\begin{aligned} \{\mathbf{Q}_L, \mathbf{S}_L\} &= \mathbf{H}_L, & \{\mathbf{Q}_L, \mathbf{Q}_R\} &= \mathbf{C}, & \{\mathbf{Q}_L, \mathbf{S}_R\} &= 0, \\ \{\mathbf{Q}_R, \mathbf{S}_R\} &= \mathbf{H}_R, & \{\mathbf{S}_L, \mathbf{S}_R\} &= \overline{\mathbf{C}}, & \{\mathbf{Q}_R, \mathbf{S}_L\} &= 0. \end{aligned} \quad (\text{H.8})$$

Note that $\mathfrak{su}(1|1)_{\text{c.e.}}^2$ is not of direct product form, i.e. we cannot construct its irreducible representations from irreps of the two sectors. To make connection to the physics, from now on we use the subscript p on the states and representation parameters, indicating that these depend on the momentum of the excitation. Let us now consider the short representations of this algebra.

H.3.1 The left-moving representation

The generalization of (H.7) compatible with the above deformation is given by

$$\boxed{\varrho_L :} \quad \begin{aligned} \mathbf{Q}_L |\phi_p^L\rangle &= a_p |\psi_p^L\rangle, & \mathbf{Q}_L |\psi_p^L\rangle &= 0, \\ \mathbf{S}_L |\phi_p^L\rangle &= 0, & \mathbf{S}_L |\psi_p^L\rangle &= b_p |\phi_p^L\rangle, \\ \mathbf{Q}_R |\phi_p^L\rangle &= 0, & \mathbf{Q}_R |\psi_p^L\rangle &= c_p |\phi_p^L\rangle, \\ \mathbf{S}_R |\phi_p^L\rangle &= d_p |\psi_p^L\rangle, & \mathbf{S}_R |\psi_p^L\rangle &= 0, \end{aligned} \quad (\text{H.9})$$

with central charges

$$\begin{aligned} \mathbf{H}_L |\phi_p^L\rangle &= a_p b_p |\phi_p^L\rangle, & \mathbf{C} |\phi_p^L\rangle &= a_p c_p |\phi_p^L\rangle, \\ \mathbf{H}_R |\phi_p^L\rangle &= c_p d_p |\phi_p^L\rangle, & \overline{\mathbf{C}} |\phi_p^L\rangle &= b_p d_p |\phi_p^L\rangle. \end{aligned} \quad (\text{H.10})$$

Shortening condition. The highest weight state $|\phi_p^L\rangle$ is annihilated by the raising operators $\mathbf{S}_L$ and $\mathbf{Q}_R$, but also satisfies the condition

$$(\mathbf{H}_R \mathbf{Q}_L - \mathbf{C} \mathbf{S}_R) |\phi_p^L\rangle = (a_p c_p d_p - a_p c_p d_p) |\psi_p^L\rangle = 0. \quad (\text{H.11})$$

Since this particular combination of the lowering operators $\mathbf{Q}_L$ and $\mathbf{S}_R$ annihilates the h.w. state, the representation is short. The state $|\phi_p^L\rangle$ must also be annihilated by the anti-commutator $\{\mathbf{S}_L, \mathbf{H}_R \mathbf{Q}_L - \mathbf{C} \mathbf{S}_R\} = \mathbf{H}_L \mathbf{H}_R - \mathbf{C} \overline{\mathbf{C}}$, but this is a central charge, implying that

$$(\mathbf{H}_L \mathbf{H}_R - \mathbf{C} \overline{\mathbf{C}}) |\chi_p^L\rangle = 0 \quad (\text{H.12})$$

for all states $\chi_p^L = \phi_p^L, \psi_p^L$ in the representation. This shortening condition, when applied to physical states, will play the role of the dispersion relation.

¹⁵Alternatively we could have taken $\{\mathbf{Q}_L, \mathbf{S}_R\} = \mathbf{C}$, but this deformation was ruled out for the case of $\text{AdS}_3/\text{CFT}_2$, by considering the length-changing effects on the spin-chain [46].

H.3.2 The right-moving representation

For this representation the role of $\mathbf{Q}_L$, $\mathbf{Q}_R$ and $\mathbf{S}_L$, $\mathbf{S}_R$ is exchanged, and the right-movers $|\phi_p^R\rangle$ and $|\psi_p^R\rangle$ transform according to

$$\boxed{\varrho_R :} \quad \begin{aligned} \mathbf{Q}_R |\phi_p^R\rangle &= a_p |\psi_p^R\rangle, & \mathbf{Q}_R |\psi_p^R\rangle &= 0, \\ \mathbf{S}_R |\phi_p^R\rangle &= 0, & \mathbf{S}_R |\psi_p^R\rangle &= b_p |\phi_p^R\rangle, \\ \mathbf{Q}_L |\phi_p^R\rangle &= 0, & \mathbf{Q}_L |\psi_p^R\rangle &= c_p |\phi_p^R\rangle, \\ \mathbf{S}_L |\phi_p^R\rangle &= d_p |\psi_p^R\rangle, & \mathbf{S}_L |\psi_p^R\rangle &= 0, \end{aligned} \quad (\text{H.13})$$

with the central charges acting as

$$\begin{aligned} \mathbf{H}_L |\phi_p^R\rangle &= c_p d_p |\phi_p^R\rangle, & \mathbf{C} |\phi_p^R\rangle &= a_p c_p |\phi_p^R\rangle, \\ \mathbf{H}_R |\phi_p^R\rangle &= a_p b_p |\phi_p^R\rangle, & \overline{\mathbf{C}} |\phi_p^R\rangle &= b_p d_p |\phi_p^R\rangle. \end{aligned} \quad (\text{H.14})$$

Shortening condition. The highest weight state, which is $|\psi_p^R\rangle$ in this case, again satisfies the condition

$$(\mathbf{H}_R \mathbf{Q}_L - \mathbf{C} \mathbf{S}_R) |\psi_p^R\rangle = 0, \quad (\text{H.15})$$

and the representation is short. The state $|\psi_p^R\rangle$ must also be annihilated by the anticommutator $\{\mathbf{S}_L, \mathbf{H}_R \mathbf{Q}_L - \mathbf{C} \mathbf{S}_R\} = \mathbf{H}_L \mathbf{H}_R - \mathbf{C} \overline{\mathbf{C}}$, and we have the same shortening condition in terms of the central charges as for the left-movers

$$(\mathbf{H}_L \mathbf{H}_R - \mathbf{C} \overline{\mathbf{C}}) |\chi_p^R\rangle = 0 \quad (\text{H.16})$$

for all states $\chi_p^R = \phi_p^R, \psi_p^R$.

I The $\mathfrak{psu}(1|1)_{\text{c.e.}}^4$ algebra and its representations

The centrally extended $\mathfrak{psu}(1|1)^4$ superalgebra is of particular interest to us, as it is the symmetry algebra of the ground state in the instance of $\text{AdS}_3/\text{CFT}_2$ duality where the string background is $\text{AdS}_3 \times \text{S}^3 \times \text{T}^4$ [44]. As it already appeared in the study of the massive sector of the theory in [35], the algebra can be obtained from two copies of the centrally extended $\mathfrak{su}(1|1)^2$. In this section we briefly review this construction.

I.1 From $\mathfrak{su}(1|1)_{\text{c.e.}}^2$ to $\mathfrak{psu}(1|1)^4_{\text{c.e.}}$

If we take two copies of the $\mathfrak{su}(1|1)_{\text{c.e.}}^2$ algebra (H.8) that share the four central charges, we get the centrally extended $\mathfrak{psu}(1|1)^4$ with generators

$$\{\mathbf{Q}_L^{\dot{a}}, \mathbf{S}_{L\dot{a}}, \mathbf{Q}_{R\dot{a}}, \mathbf{S}_{R\dot{a}}, \mathbf{H}_L, \mathbf{H}_R, \mathbf{C}, \overline{\mathbf{C}}\}_{\dot{a}=1,2} \quad (\text{I.1})$$

satisfying the anticommutation relations

$$\begin{aligned} \{\mathbf{Q}_L^{\dot{a}}, \mathbf{S}_{L\dot{b}}\} &= \delta_{\dot{b}}^{\dot{a}} \mathbf{H}_L, & \{\mathbf{Q}_L^{\dot{a}}, \mathbf{Q}_{R\dot{b}}\} &= \delta_{\dot{b}}^{\dot{a}} \mathbf{C}, \\ \{\mathbf{Q}_{R\dot{a}}, \mathbf{S}_{R\dot{b}}\} &= \delta_{\dot{a}}^{\dot{b}} \mathbf{H}_R, & \{\mathbf{S}_{L\dot{a}}, \mathbf{S}_{R\dot{b}}\} &= \delta_{\dot{a}}^{\dot{b}} \overline{\mathbf{C}}. \end{aligned} \quad (\text{I.2})$$

In other words, we are dealing with

$$\mathfrak{psu}(1|1)^4 \ltimes \mathfrak{u}(1)^4. \quad (\text{I.3})$$

Equivalently, we can consider a tensor product of two copies of (H.8)

$$\begin{aligned} \mathbf{Q}_L^1 &= \mathbf{Q}_L \otimes \mathbf{1}, & \mathbf{S}_{L1} &= \mathbf{S}_L \otimes \mathbf{1}, & \mathbf{Q}_L^2 &= \mathbf{1} \otimes \mathbf{Q}_L, & \mathbf{S}_{L2} &= \mathbf{1} \otimes \mathbf{S}_L, \\ \mathbf{Q}_{R1} &= \mathbf{Q}_R \otimes \mathbf{1}, & \mathbf{S}_R^1 &= \mathbf{S}_R \otimes \mathbf{1}, & \mathbf{Q}_{R2} &= \mathbf{1} \otimes \mathbf{Q}_R, & \mathbf{S}_R^2 &= \mathbf{1} \otimes \mathbf{S}_R, \end{aligned} \quad (\text{I.4})$$

also for the central elements

$$\begin{aligned} \mathbf{H}_L^1 &= \mathbf{H}_L \otimes \mathbf{1}, & \mathbf{H}_L^2 &= \mathbf{1} \otimes \mathbf{H}_L, & \mathbf{C}^1 &= \mathbf{C} \otimes \mathbf{1}, & \mathbf{C}^2 &= \mathbf{1} \otimes \mathbf{C}, \\ \mathbf{H}_R^1 &= \mathbf{H}_R \otimes \mathbf{1}, & \mathbf{H}_R^2 &= \mathbf{1} \otimes \mathbf{H}_R, & \overline{\mathbf{C}}^1 &= \overline{\mathbf{C}} \otimes \mathbf{1}, & \overline{\mathbf{C}}^2 &= \mathbf{1} \otimes \overline{\mathbf{C}}. \end{aligned} \quad (\text{I.5})$$

After identifying the central charges as

$$\mathbf{H}_L^1 = \mathbf{H}_L^2, \quad \mathbf{H}_R^1 = \mathbf{H}_R^2, \quad \mathbf{C}^1 = \mathbf{C}^2, \quad \overline{\mathbf{C}}^1 = \overline{\mathbf{C}}^2, \quad (\text{I.6})$$

and consequently dropping the indices 1, 2, we are left with $\mathfrak{psu}(1|1)_{\text{c.e.}}^4$. Looking at the algebra this way will be helpful in constructing its short representations.

I.2 Bi-fundamental representations

It was shown, first for the spin-chain and later for the $\text{AdS}_3 \times \text{S}^3 \times \text{S}^3 \times \text{S}^1$ superstring, that the massive off-shell excitations in both of the left- and right-moving sectors transform in *short* (four-dimensional) bi-fundamental representations of the centrally extended $\mathfrak{psu}(1|1)^4$. That is, we can obtain the relevant representations by tensoring the fundamental representations ϱ_L (H.9) and ϱ_R (H.13) of $\mathfrak{su}(1|1)_{\text{c.e.}}^2$.

Left module. Borrowing notation from [44], the four left-movers can be written as

$$Y^L = \phi^L \otimes \phi^L, \quad \eta^{L1} = \psi^L \otimes \phi^L, \quad \eta^{L2} = \phi^L \otimes \psi^L, \quad Z^L = \psi^L \otimes \psi^L, \quad (\text{I.7})$$

and they transform under the tensor product of two left-moving representations ϱ_L :

$$\begin{aligned} \mathbf{Q}_L^{\dot{a}} |Y_p^L\rangle &= a_p |\eta_p^{L\dot{a}}\rangle, & \mathbf{Q}_L^{\dot{a}} |\eta_p^{L\dot{b}}\rangle &= \epsilon^{\dot{a}\dot{b}} a_p |Z_p^L\rangle, \\ \mathbf{S}_{L\dot{a}} |Z_p^L\rangle &= -\epsilon_{\dot{a}\dot{b}} b_p |\eta_p^{L\dot{b}}\rangle, & \mathbf{S}_{L\dot{a}} |\eta_p^{L\dot{b}}\rangle &= \delta_{\dot{a}}^{\dot{b}} b_p |Y_p^L\rangle, \\ \mathbf{Q}_{R\dot{a}} |Z_p^L\rangle &= -\epsilon_{\dot{a}\dot{b}} c_p |\eta_p^{L\dot{b}}\rangle, & \mathbf{Q}_{R\dot{a}} |\eta_p^{L\dot{b}}\rangle &= \delta_{\dot{a}}^{\dot{b}} c_p |Y_p^L\rangle, \\ \mathbf{S}_R^{\dot{a}} |Y_p^L\rangle &= d_p |\eta_p^{L\dot{a}}\rangle, & \mathbf{S}_R^{\dot{a}} |\eta_p^{L\dot{b}}\rangle &= \epsilon^{\dot{a}\dot{b}} d_p |Z_p^L\rangle. \end{aligned} \quad (\text{I.8})$$

The representation coefficients of the two ϱ_L must match, since the central charges are shared, and we get a minus sign when charges of the second type act on a state with a fermion in the first part of the tensor product. Each central charge acts uniformly across all states

$$\begin{aligned} \mathbf{H}_L |\chi^L\rangle &= a_p b_p |\chi^L\rangle, & \mathbf{C} |\chi^L\rangle &= a_p c_p |\chi^L\rangle, \\ \mathbf{H}_R |\chi^L\rangle &= c_p d_p |\chi^L\rangle, & \overline{\mathbf{C}} |\chi^L\rangle &= b_p d_p |\chi^L\rangle. \end{aligned} \quad (\text{I.9})$$

Right module. Similarly we can introduce the right-moving excitations

$$Y^R = \phi^R \otimes \phi^R, \quad \eta_1^R = \psi^R \otimes \phi^R, \quad \eta_2^R = \phi^R \otimes \psi^R, \quad Z^R = \psi^R \otimes \psi^R, \quad (\text{I.10})$$

and these will transform in the representation

$$\boxed{\varrho_R \otimes \varrho_R :} \quad \begin{aligned} \mathbf{Q}_{R\dot{a}} |Y_p^R\rangle &= \epsilon_{\dot{a}b} a_p |\eta_p^{R\dot{b}}\rangle, & \mathbf{Q}_{R\dot{a}} |\eta_p^{R\dot{b}}\rangle &= \delta_{\dot{a}}^{\dot{b}} a_p |Z_p^R\rangle, \\ \mathbf{S}_R^{\dot{a}} |Z_p^R\rangle &= b_p |\eta_p^{R\dot{a}}\rangle, & \mathbf{S}_R^{\dot{a}} |\eta_p^{R\dot{b}}\rangle &= -\epsilon^{\dot{a}\dot{b}} b_p |Y_p^R\rangle, \\ \mathbf{Q}_L^{\dot{a}} |Z_p^R\rangle &= c_p |\eta_p^{R\dot{a}}\rangle, & \mathbf{Q}_L^{\dot{a}} |\eta_p^{R\dot{b}}\rangle &= -\epsilon^{\dot{a}\dot{b}} c_p |Y_p^R\rangle, \\ \mathbf{S}_{L\dot{a}} |Y_p^R\rangle &= \epsilon_{\dot{a}b} d_p |\eta_p^{R\dot{b}}\rangle, & \mathbf{S}_{L\dot{a}} |\eta_p^{R\dot{b}}\rangle &= \delta_{\dot{a}}^{\dot{b}} d_p |Z_p^R\rangle, \end{aligned} \quad (\text{I.11})$$

and for all right-movers

$$\begin{aligned} \mathbf{H}_L |\chi^R\rangle &= c_p d_p |\chi^R\rangle, & \mathbf{C} |\chi^R\rangle &= a_p c_p |\chi^R\rangle, \\ \mathbf{H}_R |\chi^R\rangle &= a_p b_p |\chi^R\rangle, & \overline{\mathbf{C}} |\chi^R\rangle &= b_p d_p |\chi^R\rangle. \end{aligned} \quad (\text{I.12})$$

Shortening condition. Naturally extending the choice made for $\mathfrak{su}(1|1)^2$, we take $\mathbf{S}_{L\dot{a}}$ and $\mathbf{Q}_{R\dot{a}}$ as our raising operators, while $\mathbf{Q}_L^{\dot{a}}$ and $\mathbf{S}_R^{\dot{a}}$ will be lowering operators. The highest weight states for $\varrho_L \otimes \varrho_L$ and $\varrho_R \otimes \varrho_R$ are $|Y_p^L\rangle$ and $|Z_p^R\rangle$ respectively, but they are also annihilated by two combinations of lowering operators, as should be the case for short representations

$$\begin{aligned} (\mathbf{H}_R \mathbf{Q}_L^{\dot{a}} - \mathbf{C} \mathbf{S}_R^{\dot{a}}) |Y_p^L\rangle &= (a_p c_p d_p - a_p c_p d_p) |\eta_p^{R\dot{a}}\rangle = 0, \\ (\mathbf{H}_R \mathbf{Q}_L^{\dot{a}} - \mathbf{C} \mathbf{S}_R^{\dot{a}}) |Z_p^R\rangle &= (a_p b_p c_p - a_p b_p c_p) |\eta_p^{R\dot{a}}\rangle = 0. \end{aligned} \quad (\text{I.13})$$

Similarly to the case of $\mathfrak{su}(1|1)_{\text{c.e.}}^2$, the anticommutator of this with $\mathbf{S}_{L\dot{b}}$ still annihilates the highest weight states, and in fact any state across both sectors, since it is a central element of the algebra

$$(\mathbf{H}_L \mathbf{H}_R - \mathbf{C} \overline{\mathbf{C}}) |\chi_p^{L,R}\rangle = 0. \quad (\text{I.14})$$

Note that this is the same as (H.12) and (H.16).

J SU(2) currents for the $q = 1$ giant magnon

Using the usual SU(2) embedding (2.9), it is a relatively simple exercise to derive the left- and right-currents for the $q = 1$ giant magnon (5.1):

$$\begin{aligned} \mathfrak{J}_+ &= \begin{pmatrix} ia & b \\ -b^* & -ia \end{pmatrix}, & \mathfrak{J}_- &= \begin{pmatrix} ic & d \\ -d^* & -ic \end{pmatrix}, \\ \mathfrak{K}_+ &= \begin{pmatrix} ie & f \\ -f^* & -ie \end{pmatrix}, & \mathfrak{K}_- &= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \end{aligned} \quad (\text{J.1})$$

where

$$\begin{aligned}
 a &= 1 - 2\beta^2 \sin^2 \frac{\rho}{2} \operatorname{sech}^2 \mathcal{Y}, \\
 b &= 2i\beta \sin^2 \frac{\rho}{2} \operatorname{sech} \mathcal{Y} (\sec \rho - \beta \tan \rho - i\beta \tanh \mathcal{Y}) e^{-2i(1-\beta \sin \rho)x^+}, \\
 c &= 1 - 2 \sin^2 \frac{\rho}{2} \operatorname{sech}^2 \mathcal{Y}, \\
 d &= 2i \sin \frac{\rho}{2} \operatorname{sech} \mathcal{Y} \sqrt{1 - \sin^2 \frac{\rho}{2} \operatorname{sech}^2 \mathcal{Y}} e^{-2i(1-\beta \sin \rho)x^+ - i \arctan\left(\tan \frac{\rho}{2} \tanh \mathcal{Y}\right)}, \\
 e &= 1 - 2 \cos^2 \rho \operatorname{sech}^2 \mathcal{Y}, \\
 f &= 2 \cos^2 \rho \operatorname{sech} \mathcal{Y} (\tanh \mathcal{Y} - i \tan \rho) e^{2i(\beta \sin \rho x^+ + x^-)}.
 \end{aligned} \tag{J.2}$$

K Terms appearing in the $q = 1$ fermion equations

With the bosonic solution from section 5 as background, the following are the components of the pulled-back vielbein $e_a^A = E_\mu^A(X) \partial_a X^\mu$

$$e_0^0 = 1, \quad e_1^0 = 0, \tag{K.1}$$

$$e_0^3 = \frac{\beta \cos \rho \tanh \mathcal{Y}}{\sqrt{b^2 + (1 + b^2) \sinh^2 \mathcal{Y}}}, \quad e_1^3 = \frac{\beta \cos \rho \tanh \mathcal{Y}}{\sqrt{b^2 + (1 + b^2) \sinh^2 \mathcal{Y}}}, \tag{K.2}$$

$$e_0^4 = \frac{b\beta \cos \rho + b^2 + (1 + b^2) \sinh^2 \mathcal{Y}}{\sqrt{b^2 + (1 + b^2) \sinh^2 \mathcal{Y}}} \frac{\operatorname{sech} \mathcal{Y}}{\sqrt{1 + b^2}}, \quad e_1^4 = \frac{b\beta \cos \rho}{\sqrt{b^2 + (1 + b^2) \sinh^2 \mathcal{Y}}} \frac{\operatorname{sech} \mathcal{Y}}{\sqrt{1 + b^2}}, \tag{K.3}$$

$$e_0^5 = \beta \sin \rho \frac{\operatorname{sech} \mathcal{Y}}{\sqrt{1 + b^2}}, \quad e_1^5 = (\beta \sin \rho - 1) \frac{\operatorname{sech} \mathcal{Y}}{\sqrt{1 + b^2}}, \tag{K.4}$$

while the only non-zero components of the spin connection (pulled back to the world-sheet) are

$$\omega_0^{34} = -\omega_0^{43} = -\frac{b\beta \cos \rho + b^2 + (1 + b^2) \sinh^2 \mathcal{Y}}{b^2 + (1 + b^2) \sinh^2 \mathcal{Y}} \frac{\operatorname{sech} \mathcal{Y}}{\sqrt{1 + b^2}}, \tag{K.5}$$

$$\omega_1^{34} = -\omega_1^{43} = -\frac{b\beta \cos \rho}{b^2 + (1 + b^2) \sinh^2 \mathcal{Y}} \frac{\operatorname{sech} \mathcal{Y}}{\sqrt{1 + b^2}}, \tag{K.6}$$

$$\omega_0^{35} = -\omega_0^{53} = \beta \sin \rho \sqrt{b^2 + (1 + b^2) \sinh^2 \mathcal{Y}} \frac{\operatorname{sech} \mathcal{Y}}{\sqrt{1 + b^2}}, \tag{K.7}$$

$$\omega_1^{35} = -\omega_1^{53} = (\beta \sin \rho - 1) \sqrt{b^2 + (1 + b^2) \sinh^2 \mathcal{Y}} \frac{\operatorname{sech} \mathcal{Y}}{\sqrt{1 + b^2}}. \tag{K.8}$$

Note that $\mathcal{Y} = 2\beta \cos \rho x^+$ and the three parameters are related by $\beta = -(b \cos \rho - \sin \rho)$. The combinations appearing in the fermion derivatives are

$$G = \omega_1^{34} - \omega_0^{34} = \frac{\text{sech} \mathcal{Y}}{\sqrt{1+b^2}}, \quad (\text{K.9})$$

$$Q = \omega_1^{35} - \omega_0^{35} = -\sqrt{b^2 + (1+b^2) \sinh^2 \mathcal{Y}} \frac{\text{sech} \mathcal{Y}}{\sqrt{1+b^2}}, \quad (\text{K.10})$$

$$\tilde{G} = \omega_1^{34} + \omega_0^{34} = -\frac{2b\beta \cos \rho + b^2 + (1+b^2) \sinh^2 \mathcal{Y}}{b^2 + (1+b^2) \sinh^2 \mathcal{Y}} \frac{\text{sech} \mathcal{Y}}{\sqrt{1+b^2}}, \quad (\text{K.11})$$

$$\tilde{Q} = \omega_1^{35} + \omega_0^{35} = (2\beta \sin \rho - 1) \sqrt{b^2 + (1+b^2) \sinh^2 \mathcal{Y}} \frac{\text{sech} \mathcal{Y}}{\sqrt{1+b^2}}. \quad (\text{K.12})$$

Open Access. This article is distributed under the terms of the Creative Commons Attribution License ([CC-BY 4.0](https://creativecommons.org/licenses/by/4.0/)), which permits any use, distribution and reproduction in any medium, provided the original author(s) and source are credited.

References

- [1] J.M. Maldacena, *The Large N limit of superconformal field theories and supergravity*, *Int. J. Theor. Phys.* **38** (1999) 1113 [[hep-th/9711200](#)] [[INSPIRE](#)].
- [2] J.A. Minahan and K. Zarembo, *The Bethe ansatz for $N = 4$ superYang-Mills*, *JHEP* **03** (2003) 013 [[hep-th/0212208](#)] [[INSPIRE](#)].
- [3] N. Beisert, C. Kristjansen and M. Staudacher, *The Dilatation operator of conformal $N = 4$ superYang-Mills theory*, *Nucl. Phys. B* **664** (2003) 131 [[hep-th/0303060](#)] [[INSPIRE](#)].
- [4] N. Beisert and M. Staudacher, *The $N = 4$ SYM integrable super spin chain*, *Nucl. Phys. B* **670** (2003) 439 [[hep-th/0307042](#)] [[INSPIRE](#)].
- [5] I. Bena, J. Polchinski and R. Roiban, *Hidden symmetries of the $AdS_5 \times S^5$ superstring*, *Phys. Rev. D* **69** (2004) 046002 [[hep-th/0305116](#)] [[INSPIRE](#)].
- [6] V.A. Kazakov, A. Marshakov, J.A. Minahan and K. Zarembo, *Classical/quantum integrability in AdS/CFT* , *JHEP* **05** (2004) 024 [[hep-th/0402207](#)] [[INSPIRE](#)].
- [7] G. Arutyunov, S. Frolov and M. Staudacher, *Bethe ansatz for quantum strings*, *JHEP* **10** (2004) 016 [[hep-th/0406256](#)] [[INSPIRE](#)].
- [8] N. Beisert, V.A. Kazakov, K. Sakai and K. Zarembo, *The Algebraic curve of classical superstrings on $AdS_5 \times S^5$* , *Commun. Math. Phys.* **263** (2006) 659 [[hep-th/0502226](#)] [[INSPIRE](#)].
- [9] G. Arutyunov and S. Frolov, *Integrable Hamiltonian for classical strings on $AdS_5 \times S^5$* , *JHEP* **02** (2005) 059 [[hep-th/0411089](#)] [[INSPIRE](#)].
- [10] M. Staudacher, *The factorized S -matrix of CFT/AdS* , *JHEP* **05** (2005) 054 [[hep-th/0412188](#)] [[INSPIRE](#)].
- [11] N. Beisert, *The $SU(2|2)$ dynamic S -matrix*, *Adv. Theor. Math. Phys.* **12** (2008) 945 [[hep-th/0511082](#)] [[INSPIRE](#)].

- [12] N. Beisert, *The Analytic Bethe Ansatz for a Chain with Centrally Extended $su(2-2)$ Symmetry*, *J. Stat. Mech.* **01** (2007) P01017 [[nlin/0610017](#)].
- [13] R.A. Janik, *The $AdS_5 \times S^5$ superstring worldsheet S -matrix and crossing symmetry*, *Phys. Rev. D* **73** (2006) 086006 [[hep-th/0603038](#)] [[INSPIRE](#)].
- [14] N. Beisert, R. Hernandez and E. Lopez, *A Crossing-symmetric phase for $AdS_5 \times S^5$ strings*, *JHEP* **11** (2006) 070 [[hep-th/0609044](#)] [[INSPIRE](#)].
- [15] N. Beisert, B. Eden and M. Staudacher, *Transcendentality and Crossing*, *J. Stat. Mech.* **0701** (2007) P01021 [[hep-th/0610251](#)] [[INSPIRE](#)].
- [16] N. Dorey, D.M. Hofman and J.M. Maldacena, *On the Singularities of the Magnon S -matrix*, *Phys. Rev. D* **76** (2007) 025011 [[hep-th/0703104](#)] [[INSPIRE](#)].
- [17] D. Volin, *Minimal solution of the AdS/CFT crossing equation*, *J. Phys. A* **42** (2009) 372001 [[arXiv:0904.4929](#)] [[INSPIRE](#)].
- [18] D.M. Hofman and J.M. Maldacena, *Giant Magnons*, *J. Phys. A* **39** (2006) 13095 [[hep-th/0604135](#)] [[INSPIRE](#)].
- [19] H.-Y. Chen, N. Dorey and K. Okamura, *Dyonic giant magnons*, *JHEP* **09** (2006) 024 [[hep-th/0605155](#)] [[INSPIRE](#)].
- [20] J.A. Minahan, *Zero modes for the giant magnon*, *JHEP* **02** (2007) 048 [[hep-th/0701005](#)] [[INSPIRE](#)].
- [21] T. Klose, *Review of AdS/CFT Integrability, Chapter IV.3: $N = 6$ Chern-Simons and Strings on $AdS_4 \times CP^3$* , *Lett. Math. Phys.* **99** (2012) 401 [[arXiv:1012.3999](#)] [[INSPIRE](#)].
- [22] J.P. Gauntlett, R.C. Myers and P.K. Townsend, *Supersymmetry of rotating branes*, *Phys. Rev. D* **59** (1998) 025001 [[hep-th/9809065](#)] [[INSPIRE](#)].
- [23] A. Babichenko, B. Stefański, Jr. and K. Zarembo, *Integrability and the AdS_3/CFT_2 correspondence*, *JHEP* **03** (2010) 058 [[arXiv:0912.1723](#)] [[INSPIRE](#)].
- [24] O. Ohlsson Sax and B. Stefański, Jr., *Integrability, spin-chains and the AdS_3/CFT_2 correspondence*, *JHEP* **08** (2011) 029 [[arXiv:1106.2558](#)] [[INSPIRE](#)].
- [25] P. Sundin and L. Wulff, *Classical integrability and quantum aspects of the $AdS_3 \times S^3 \times S_3 \times S_1$ superstring*, *JHEP* **10** (2012) 109 [[arXiv:1207.5531](#)] [[INSPIRE](#)].
- [26] A. Cagnazzo and K. Zarembo, *B -field in AdS_3/CFT_2 Correspondence and Integrability*, *JHEP* **11** (2012) 133 [*Erratum ibid.* **1304** (2013) 003] [[arXiv:1209.4049](#)] [[INSPIRE](#)].
- [27] O. Ohlsson Sax and B. Stefański, Jr., *Closed strings and moduli in AdS_3/CFT_2* , *JHEP* **05** (2018) 101 [[arXiv:1804.02023](#)] [[INSPIRE](#)].
- [28] O. Ohlsson Sax, A. Sfondrini and B. Stefański, Jr., *Integrability and the Conformal Field Theory of the Higgs branch*, *JHEP* **06** (2015) 103 [[arXiv:1411.3676](#)] [[INSPIRE](#)].
- [29] H.J. Boonstra, B. Peeters and K. Skenderis, *Brane intersections, anti-de Sitter space-times and dual superconformal theories*, *Nucl. Phys. B* **533** (1998) 127 [[hep-th/9803231](#)] [[INSPIRE](#)].
- [30] S. Gukov, E. Martinec, G.W. Moore and A. Strominger, *The Search for a holographic dual to $AdS_3 \times S^3 \times S^3 \times S^1$* , *Adv. Theor. Math. Phys.* **9** (2005) 435 [[hep-th/0403090](#)] [[INSPIRE](#)].
- [31] D. Tong, *The holographic dual of $AdS_3 \times S^3 \times S^3 \times S^1$* , *JHEP* **04** (2014) 193 [[arXiv:1402.5135](#)] [[INSPIRE](#)].

- [32] L. Eberhardt, M.R. Gaberdiel and R. Gopakumar, *The Worldsheet Dual of the Symmetric Product CFT*, [arXiv:1812.01007](#) [[INSPIRE](#)].
- [33] B. Hoare and A.A. Tseytlin, *On string theory on $AdS_3 \times S^3 \times T^4$ with mixed 3-form flux: tree-level S -matrix*, *Nucl. Phys. B* **873** (2013) 682 [[arXiv:1303.1037](#)] [[INSPIRE](#)].
- [34] B. Hoare and A.A. Tseytlin, *Massive S -matrix of $AdS_3 \times S^3 \times T^4$ superstring theory with mixed 3-form flux*, *Nucl. Phys. B* **873** (2013) 395 [[arXiv:1304.4099](#)] [[INSPIRE](#)].
- [35] R. Borsato, O. Ohlsson Sax, A. Sfondrini, B. Stefański, Jr. and A. Torrielli, *The all-loop integrable spin-chain for strings on $AdS_3 \times S^3 \times T^4$: the massive sector*, *JHEP* **08** (2013) 043 [[arXiv:1303.5995](#)] [[INSPIRE](#)].
- [36] B. Hoare, A. Stepanchuk and A.A. Tseytlin, *Giant magnon solution and dispersion relation in string theory in $AdS_3 \times S^3 \times T^4$ with mixed flux*, *Nucl. Phys. B* **879** (2014) 318 [[arXiv:1311.1794](#)] [[INSPIRE](#)].
- [37] A. Banerjee, K.L. Panigrahi and P.M. Pradhan, *Spiky strings on $AdS_3 \times S^3$ with NS-NS flux*, *Phys. Rev. D* **90** (2014) 106006 [[arXiv:1405.5497](#)] [[INSPIRE](#)].
- [38] A. Stepanchuk, *String theory in $AdS_3 \times S^3 \times T^4$ with mixed flux: semiclassical and 1-loop phase in the S -matrix*, *J. Phys. A* **48** (2015) 195401 [[arXiv:1412.4764](#)] [[INSPIRE](#)].
- [39] T. Lloyd, O. Ohlsson Sax, A. Sfondrini and B. Stefański, Jr., *The complete worldsheet S matrix of superstrings on $AdS_3 \times S^3 \times T^4$ with mixed three-form flux*, *Nucl. Phys. B* **891** (2015) 570 [[arXiv:1410.0866](#)] [[INSPIRE](#)].
- [40] R. Borsato, O. Ohlsson Sax, A. Sfondrini and B. Stefański, Jr., *On the spectrum of $AdS_3 \times S^3 \times T^4$ strings with Ramond-Ramond flux*, *J. Phys. A* **49** (2016) 41LT03 [[arXiv:1605.00518](#)] [[INSPIRE](#)].
- [41] M. Baggio, O. Ohlsson Sax, A. Sfondrini, B. Stefański, Jr. and A. Torrielli, *Protected string spectrum in AdS_3/CFT_2 from worldsheet integrability*, *JHEP* **04** (2017) 091 [[arXiv:1701.03501](#)] [[INSPIRE](#)].
- [42] L. Eberhardt, M.R. Gaberdiel, R. Gopakumar and W. Li, *BPS spectrum on $AdS_3 \times S^3 \times S^3 \times S^1$* , *JHEP* **03** (2017) 124 [[arXiv:1701.03552](#)] [[INSPIRE](#)].
- [43] J. de Boer, *Six-dimensional supergravity on $S^3 \times AdS_3$ and 2 – D conformal field theory*, *Nucl. Phys. B* **548** (1999) 139 [[hep-th/9806104](#)] [[INSPIRE](#)].
- [44] R. Borsato, O. Ohlsson Sax, A. Sfondrini and B. Stefański, Jr., *The complete $AdS_3 \times S^3 \times T^4$ worldsheet S matrix*, *JHEP* **10** (2014) 66 [[arXiv:1406.0453](#)] [[INSPIRE](#)].
- [45] R. Borsato, O. Ohlsson Sax, A. Sfondrini and B. Stefański, Jr., *Towards the All-Loop Worldsheet S Matrix for $AdS_3 \times S^3 \times T^4$* , *Phys. Rev. Lett.* **113** (2014) 131601 [[arXiv:1403.4543](#)] [[INSPIRE](#)].
- [46] R. Borsato, O. Ohlsson Sax and A. Sfondrini, *A dynamic $SU(1|1)^2$ S -matrix for AdS_3/CFT_2* , *JHEP* **04** (2013) 113 [[arXiv:1211.5119](#)] [[INSPIRE](#)].
- [47] R. Borsato, O. Ohlsson Sax, A. Sfondrini and B. Stefański, Jr., *The $AdS_3 \times S^3 \times S^3 \times S^1$ worldsheet S matrix*, *J. Phys. A* **48** (2015) 415401 [[arXiv:1506.00218](#)] [[INSPIRE](#)].
- [48] M. Cvetič, H. Lü, C.N. Pope and K.S. Stelle, *T duality in the Green-Schwarz formalism and the massless/massive IIA duality map*, *Nucl. Phys. B* **573** (2000) 149 [[hep-th/9907202](#)] [[INSPIRE](#)].

- [49] G. Arutyunov, S. Frolov and M. Zamaklar, *The Zamolodchikov-Faddeev algebra for $AdS_5 \times S^5$ superstring*, *JHEP* **04** (2007) 002 [[hep-th/0612229](#)] [[INSPIRE](#)].
- [50] O. Ohlsson Sax, B. Stefański Jr. and A. Torrielli, *On the massless modes of the AdS_3/CFT_2 integrable systems*, *JHEP* **03** (2013) 109 [[arXiv:1211.1952](#)] [[INSPIRE](#)].
- [51] G. Giribet, C. Hull, M. Kleban, M. Porrati and E. Rabinovici, *Superstrings on AdS_3 at $\parallel = 1$* , *JHEP* **08** (2018) 204 [[arXiv:1803.04420](#)] [[INSPIRE](#)].
- [52] M.R. Gaberdiel and R. Gopakumar, *Tensionless string spectra on AdS_3* , *JHEP* **05** (2018) 085 [[arXiv:1803.04423](#)] [[INSPIRE](#)].
- [53] A. Pittelli, *Yangian Symmetry of String Theory on $AdS_3 \times S^3 \times S^3 \times S^1$ with Mixed 3-form Flux*, *Nucl. Phys. B* **935** (2018) 271 [[arXiv:1711.02468](#)] [[INSPIRE](#)].
- [54] J.M. Nieto and R. Ruiz, *One-loop quantization of rigid spinning strings in $AdS_3 \times S^3 \times T^4$ with mixed flux*, *JHEP* **07** (2018) 141 [[arXiv:1804.10477](#)] [[INSPIRE](#)].
- [55] M.C. Abbott and I. Aniceto, *Massless Lüscher terms and the limitations of the AdS_3 asymptotic Bethe ansatz*, *Phys. Rev. D* **93** (2016) 106006 [[arXiv:1512.08761](#)] [[INSPIRE](#)].
- [56] D. Bombardelli, B. Stefański, Jr. and A. Torrielli, *The low-energy limit of AdS_3/CFT_2 and its TBA*, *JHEP* **10** (2018) 177 [[arXiv:1807.07775](#)] [[INSPIRE](#)].
- [57] R. Borsato, O. Ohlsson Sax, A. Sfondrini, B. Stefański, Jr., A. Torrielli and O. Ohlsson Sax, *On the dressing factors, Bethe equations and Yangian symmetry of strings on $AdS_3 \times S^3 \times T^4$* , *J. Phys. A* **50** (2017) 024004 [[arXiv:1607.00914](#)] [[INSPIRE](#)].