Double scaling limit of $\mathcal{N} = 2$ chiral correlators with Maldacena-Wilson loop

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ABSTRACT: We consider $\mathcal{N} = 2$ conformal QCD in four dimensions and the one-point correlator of a class of chiral primaries with the circular $\frac{1}{2}$ -BPS Maldacena-Wilson loop. We analyze a recently introduced double scaling limit where the gauge coupling is weak while the R-charge of the chiral primary Φ is large. In particular, we consider the case $\Phi = (\text{Tr}\varphi^2)^n$, where φ is the complex scalar in the vector multiplet. The correlator defines a non-trivial scaling function at fixed $\kappa = n g_{\text{YM}}^2$ and large n that may be studied by localization. For any gauge group $\text{SU}(N)$ we provide the analytic expression of the first correction $\sim \zeta(3) \kappa^2$ and prove its universality. In the $\text{SU}(2)$ and $\text{SU}(3)$ theories we compute the scaling functions at order $\mathcal{O}(\kappa^6)$. Remarkably, in the $\text{SU}(2)$ case the scaling function is equal to an analogous quantity describing the chiral 2-point functions $\langle \Phi \bar{\Phi} \rangle$ in the same large R-charge limit. We conjecture that this $\text{SU}(2)$ scaling function is computed at all-orders by a $\mathcal{N} = 4$ SYM expectation value of a matrix model object characterizing the one-loop contribution to the 4-sphere partition function. The conjecture provides an explicit series expansion for the scaling function and is checked at order $\mathcal{O}(\kappa^{10})$ by showing agreement with the available data in the sector of chiral 2-point functions.

KEYWORDS: Conformal Field Theory, Extended Supersymmetry, Supersymmetric Gauge Theory

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1 Introduction

Recently, a certain interest has been devoted to the large R-charge limit of four dimensional $\mathcal{N} = 2$ superconformal theories [1, 2]. Besides, this limit may be conveniently combined with a weak coupling expansion and tuned in order to provide a non-trivial scaling behaviour. In this mixed regime we can neglect instanton contributions while keeping some interesting (scaled) coupling dependence. Typically, one has a vanishing Yang-Mills coupling $g_{\text{YM}} \rightarrow 0$ while the R-charge grows like $1/g_{\text{YM}}^2$. Extremal correlators of chiral primaries in conformal $\mathcal{N} = 2$ QCD have been computed in such a double scaling limit [3]. If Φ_n is a chiral primary whose R-charge increases linearly with n , one may consider the normalized ratio between the 2-point functions in the $\mathcal{N} = 2$ theory and in the $\mathcal{N} = 4$ SYM universal parent theory. This defines the scaling function for gauge group $SU(N)$

$$F^\Phi(\kappa; N) = \lim_{n \rightarrow \infty} \frac{\langle \Phi_n \bar{\Phi}_n \rangle^{\mathcal{N}=2}}{\langle \Phi_n \bar{\Phi}_n \rangle^{\mathcal{N}=4}} \bigg|_{\kappa=\text{fixed}}, \quad (1.1)$$

where $\kappa = n g_{\text{YM}}^2$ is the fixed coupling at large R-charge. For certain classes of chiral primaries Φ_n , it is possible to compute the perturbative expansion of the function F^Φ by exploiting the integrable structure of the $\mathcal{N} = 2$ partition function. In the simplest setup, Φ_n is the maximal multi-trace tower $\Phi_n = \Omega_n \equiv (\text{Tr} \varphi^2)^n$ where φ is the complex scalar field belonging to the $\mathcal{N} = 2$ vector multiplet. The 2-point functions $\langle \Omega_n \bar{\Omega}_n \rangle$ are then captured by a Toda equation following from the four dimensional tt^* equations [4], i.e. the counterpart of the topological anti-topological fusion equations of 2d SCFTs [5, 6]. By exploiting the Toda equation in order to control the R-charge dependence, it is possible to compute the scaling function in (1.1) at rather high perturbative order [7]. This approach, based on decoupled semi-infinite Toda equations, has been proved to admit generalizations to broader classes of primaries and is believed to be a general feature of Lagrangian $\mathcal{N} = 2$ superconformal theories [8].

In this paper, we reconsider the double scaling limit in (1.1), but for a different class of correlators, i.e. the 1-point function $\langle \Phi_n W \rangle$ of large R-charge chiral primaries Φ_n in presence of a circular $\frac{1}{2}$ -BPS Maldacena-Wilson loop W [9, 10]. In conformal SQCD, it is possible to compute such correlators by localization as thoroughly studied in [11]. Other applications of localization to Wilson loops in $\mathcal{N} = 2$ superconformal theories may be found in [12–15]. For our purposes, we shall again be interested in the maximal multi-trace case, i.e. $\Phi = \Omega_n$.

As is well known, the localization computation is based on the partition function of a suitable deformation of the $\mathcal{N} = 2$ theory on S^4 [16–18]. However, the map from S^4 to flat space requires to disentangle a peculiar operator mixing induced by the conformal anomaly [19]. This is an annoying feature as far as the analysis of the $n \rightarrow \infty$ limit is concerned. Indeed, the mixing structure becomes more and more involved with growing n . In particular, one should need results like those in [11], but with a fully parametric dependence on n . Besides, as in the study of the large R-charge limit of chiral 2-point functions, we want to work with a generic $\text{SU}(N)$ gauge group with (finite) fixed N .¹ The drawback is that finite N results may display a deceiving complexity for large R-charge. For a discussion of what simplifications occur in the mixing problem at large N see [20–23].

To overcome these difficulties, we exploit special features of the Ω_n operators. For generic R-charge n and gauge group $\text{SU}(N)$, we provide the solution to the mixing problem in the maximally supersymmetric $\mathcal{N} = 4$ SYM theory. Similarly, we give as well exact expressions for the first genuine $\mathcal{N} = 2$ correction $\sim \zeta(3)$. These findings are a useful guide to study higher transcendentality contributions. Our results allows to consider the $n \rightarrow \infty$ limit of the ratio of 1-point functions

$$G(\kappa; N) = \lim_{n \rightarrow \infty} \frac{\langle W \Omega_n \rangle^{\mathcal{N}=2}}{\langle W \Omega_n \rangle^{\mathcal{N}=4}}, \quad (1.2)$$

taken with fixed $\kappa = n g_{\text{YM}}^2$ in the $\text{SU}(N)$ theory. The quantity in (1.2) is the simplest natural object to be studied in presence of the Wilson loop and corresponding to (1.1). Our analysis shows that the limiting scaling function $G(\kappa; N)$ is well defined and non-trivial.

¹In particular, the $N \rightarrow \infty$ limit is not relevant to our study and will be taken just to check our computation against some known results.

In the SU(2) theory, we check the remarkable equality $G(\kappa; N = 2) = F(\kappa; N = 2)$ at least at order $\mathcal{O}(\kappa^6)$. This identity does not hold for $N > 2$ as follows from a study of the G function in the SU(3) theory again at order $\mathcal{O}(\kappa^6)$.

The case of SU(2) is definitely special. Based on certain universality arguments we formulate a simple conjecture for the all-order expansion of $F(\kappa; 2)$ in terms of a certain $\mathcal{N} = 4$ expectation value of the one-loop contribution to the $\mathcal{N} = 2$ matrix model partition function. We have checked the conjecture at order $\mathcal{O}(\kappa^{10})$ by reproducing the results of [7].

The plan of the paper is the following. In section 2 we summarize recent developments about the large R-charge double scaling limit in the sector of chiral 2-point correlators. In section 3 we briefly set up the calculation with the Maldacena-Wilson loop and recall the localization algorithm to map flat space correlators to definite matrix model integrals. In section 4 we exploit some special features of the operators Ω_n to solve the associated mixing problem in the $\mathcal{N} = 4$ theory, to compute the correlator with the Wilson loop for a generic R-charge, and to evaluate the first correction $\sim \zeta(3)$ for a general gauge algebra rank. Section 5 is devoted to a focused study of the large R-charge limit. We first discuss in a rigorous way the universality of the $\zeta(3)$ correction. Next, simple educated assumptions allow us to compute the scaling function $G(\kappa; N)$ at sixth order in κ for the SU(2) and SU(3) theories. In the final section 6 we present a conjecture for the all-order expression of the scaling function in the SU(2) theory based on the identification of a particular universal and natural object in the $\mathcal{N} = 2$ matrix model which turns out to encode the scaling function itself.

2 Large R-charge double scaling in $\mathcal{N} = 2$ conformal SQCD

We work in flat 4d space and consider $\mathcal{N} = 2$ conformal SQCD, i.e. SYM with gauge group SU(N) and $2N$ fundamental hypermultiplets. Chiral primary operators are primaries annihilated by half of the supercharges. They have scaling dimension Δ and quantum numbers (R, r) of the R-symmetry $SU(2)_R \times U(1)_r$ obeying $R = 0$ and $\Delta = \frac{r}{2}$. If φ is the complex scalar field in the vector multiplet, a generic scalar chiral primary is labeled by a vector of integers $\mathbf{n} = (n_1, \dots, n_\ell)$ and reads²

$$\mathcal{O}_{\mathbf{n}}(x) = \prod_{k=1}^{\ell} \text{Tr}[\varphi(x)^{n_k}], \quad \Delta = |\mathbf{n}| = n_1 + \dots + n_\ell. \quad (2.1)$$

Superconformal symmetry predicts the diagonal 2-point function

$$\langle \mathcal{O}_{\mathbf{n}}(\infty) \overline{\mathcal{O}}_{\mathbf{m}}(0) \rangle = G_{\mathbf{n}, \mathbf{m}} \delta_{|\mathbf{n}|, |\mathbf{m}|}, \quad (2.2)$$

where $G_{\mathbf{n}, \mathbf{m}}$ is a function of the gauge coupling g . In the recent papers [3, 7] a special role has been played by the operators

$$\Omega_n \equiv \underbrace{\mathcal{O}_{2, \dots, 2}}_n = (\text{Tr} \varphi^2)^n, \quad (2.3)$$

²See [24] for a recent discussion about general constraints on chiral primaries in $\mathcal{N} = 2$ superconformal theories prohibiting non-scalar chiral operators to a large extent.

and the associated 2-point function coefficients $G_{2n} = \langle (\text{Tr} \varphi^2)^n (\text{Tr} \bar{\varphi}^2)^n \rangle$ have been computed. The functions G_{2n} are captured by a semi-infinite Toda equation [3, 8, 19] that allows to compute them, after normalization to their $\mathcal{N} = 4$ SYM value, in the large R-charge limit³

$$n \rightarrow \infty, \quad g \rightarrow 0, \quad \kappa = n g^2 = \text{fixed}. \quad (2.4)$$

One finds the expansion

$$\log F(\kappa; N) = \lim_{n \rightarrow \infty} \log \frac{G_{2n}^{\mathcal{N}=2}}{G_{2n}^{\mathcal{N}=4}} = \sum_{\ell=2}^{\infty} \left(\frac{\kappa}{8\pi^2} \right)^\ell \sum_{\mathbf{s}} c_{\mathbf{s}}^{(\ell)}(N) \zeta(\mathbf{s}), \quad \zeta(\mathbf{s}) = \zeta(s_1) \zeta(s_2) \dots, \quad (2.5)$$

where the coefficients $c_{\mathbf{s}}^{(\ell)}(N)$ have been computed at order $\mathcal{O}(\lambda^{10})$ in [7]. The first cases are

$$\begin{aligned} c_3^{(2)}(N) &= -18, \\ c_9^{(5)}(N) &= \frac{10584(26N^8 + 28N^6 - 3N^4 + 6N^2 - 9)}{N^3(N^2 + 1)(N^2 + 3)(N^2 + 5)(N^2 + 7)}, \\ c_5^{(3)}(N) &= \frac{200(2N^2 - 1)}{N(N^2 + 3)}, \\ c_{5^2}^{(6)}(N) &= \frac{184800(N^2 - 4)(N^6 - N^4 - 43N^2 - 37)}{(N^2 + 1)(N^2 + 3)^2(N^2 + 5)(N^2 + 7)(N^2 + 9)}, \\ c_7^{(4)}(N) &= -\frac{1225(8N^6 + 4N^4 - 3N^2 + 3)}{N^2(N^2 + 1)(N^2 + 3)(N^2 + 5)}, \\ c_{11}^{(6)}(N) &= -\frac{71148(122N^{10} + 280N^8 + 48N^6 - 15N^4 + 45)}{N^4(N^2 + 1)(N^2 + 3)(N^2 + 5)(N^2 + 7)(N^2 + 9)}. \end{aligned} \quad (2.6)$$

It has been conjectured that all terms associated with multiple products of zeta functions do vanish for $N = 2$. For this value, i.e. for the $\text{SU}(2)$ theory, the expansion (2.5) reduces to

$$\begin{aligned} \log F(\kappa; 2) &= -\frac{9\zeta(3)}{32\pi^4} \kappa^2 + \frac{25\zeta(5)}{128\pi^6} \kappa^3 - \frac{2205\zeta(7)}{16384\pi^8} \kappa^4 + \frac{3213\zeta(9)}{32768\pi^{10}} \kappa^5 - \frac{78771\zeta(11)}{1048576\pi^{12}} \kappa^6 \\ &+ \frac{250965\zeta(13)}{4194304\pi^{14}} \kappa^7 - \frac{105424605\zeta(15)}{2147483648\pi^{16}} \kappa^8 \\ &+ \frac{265525975\zeta(17)}{6442450944\pi^{18}} \kappa^9 - \frac{12108123027\zeta(19)}{343597383680\pi^{20}} \kappa^{10} + \mathcal{O}(\kappa^{11}). \end{aligned} \quad (2.7)$$

³Contrary to [3], here we denote by κ the large n fixed coupling to avoid confusion with the more conventional 't Hooft coupling λ .

This simple exponentiation property is not true for $N > 2$. For instance, for $N = 3$ one finds

$$\begin{aligned}
 \log F(\kappa; 3) = & -\frac{9\zeta(3)}{32\pi^4} \kappa^2 + \frac{425\zeta(5)}{2304\pi^6} \kappa^3 - \frac{17885\zeta(7)}{147456\pi^8} \kappa^4 + \frac{5565\zeta(9)}{65536\pi^{10}} \kappa^5 \\
 & + \left(\frac{1925\zeta(5)^2}{14155776\pi^{12}} - \frac{2668897\zeta(11)}{42467328\pi^{12}} \right) \kappa^6 + \left(\frac{32984237\zeta(13)}{679477248\pi^{14}} - \frac{5005\zeta(5)\zeta(7)}{14155776\pi^{14}} \right) \kappa^7 \\
 & + \left(\frac{35035\zeta(7)^2}{150994944\pi^{16}} + \frac{146575\zeta(5)\zeta(9)}{402653184\pi^{16}} - \frac{2245755655\zeta(15)}{57982058496\pi^{16}} \right) \kappa^8 \\
 & + \left(-\frac{1519375\zeta(5)^3}{1174136684544\pi^{18}} - \frac{3488485\zeta(7)\zeta(9)}{7247757312\pi^{18}} - \frac{546184925\zeta(5)\zeta(11)}{1565515579392\pi^{18}} \right. \\
 & + \left. \frac{669686057755\zeta(17)}{21134460321792\pi^{18}} \right) \kappa^9 + \left(\frac{8083075\zeta(5)^2\zeta(7)}{1565515579392\pi^{20}} + \frac{77643709\zeta(9)^2}{309237645312\pi^{20}} \right. \\
 & + \left. \frac{2905703801\zeta(7)\zeta(11)}{6262062317568\pi^{20}} + \frac{4074100745\zeta(5)\zeta(13)}{12524124635136\pi^{20}} - \frac{29805018472801\zeta(19)}{1127171217162240\pi^{20}} \right) \kappa^{10} \\
 & + \mathcal{O}(\kappa^{11}),
 \end{aligned} \tag{2.8}$$

and, starting at order κ^6 , products of zeta functions appear. This seemingly technical or accidental feature will have a role in the following discussion. A natural issue is then to explore the possibility of non-trivial scaling functions in other partially protected sectors. To this aim, we shall consider here chiral correlators of one primary with a Maldacena-Wilson loop.

3 Chiral 1-point function with $\frac{1}{2}$ -BPS Wilson loop

We consider the $\frac{1}{2}$ -BPS Maldacena-Wilson loop defined by [9, 10]

$$W = \frac{1}{N} \text{Tr} \, \text{P} \exp \left[g \int_C dt \left(i A \cdot \dot{x} + \frac{R}{\sqrt{2}} (\varphi + \bar{\varphi}) \right) \right], \tag{3.1}$$

where g is the gauge coupling, C is a circle of radius R , φ is the complex scalar field in the vector multiplet, and the trace is taken in the fundamental representation.

Conformal invariance fully constrains the position dependence of the 1-point function of the chiral primary operator $\mathcal{O}_{\mathbf{n}}$ with W . For a loop placed at the origin, one has [11]

$$\langle W \mathcal{O}_{\mathbf{n}}(x) \rangle = \frac{A_{\mathbf{n}}(g)}{(2\pi |x|_C)^{|\mathbf{n}|}}, \tag{3.2}$$

where $|x|_C$ is a suitable $\text{SO}(1, 2) \times \text{SO}(3)$ distance between x and the loop respecting the conformal subgroup unbroken by the loop. All the remaining information about the 1-point function is encoded in the coupling dependent normalization $A_{\mathbf{n}}(g)$ in (3.2).

3.1 Localization results

As discussed in [11], the function $A_{\mathbf{n}}(g)$ may be computed by the same matrix model that appears in the partition function and encoding the localization solution of the $\mathcal{N} = 2$ theory on S^4 with a specific (finite) Ω -deformation [16–18]. Since we are interested in a weak-coupling expansion, we shall drop the instanton contribution. After this simplification, the

sphere partition function is associated with a perturbed Gaussian matrix model. Up to a g -independent normalization it reads

$$Z_{S^4} = \left(\frac{g^2}{8\pi^2} \right)^{\frac{N^2-1}{2}} \int Da e^{-\text{Tra}^2 - S_{\text{int}}(a)}, \quad (3.3)$$

where $a = a^\ell T^\ell$ is a traceless $N \times N$ matrix, $Da = \prod_{\ell=1}^{N^2-1} \frac{da^\ell}{\sqrt{2\pi}}$, and the non-Gaussian *interacting action* $S_{\text{int}}(a)$ is an infinite series

$$S_{\text{int}}(a) = \sum_{n=0}^{\infty} s_n(a) \zeta(2n+3) g^{2n+4}, \quad (3.4)$$

where $s_n(a)$ are invariant functions of a . The first terms read explicitly

$$S_{\text{int}}(a) = \frac{3\zeta(3)g^4}{(8\pi^2)^2} (\text{Tra}^2)^2 - \frac{10\zeta(5)g^6}{3(8\pi^2)^3} \left[3\text{Tra}^4 \text{Tra}^2 - 2(\text{Tra}^3)^2 \right] + \dots, \quad (3.5)$$

and higher powers of g are associated with higher transcendentality terms. As usual, correlation functions are computed by

$$\langle F(a) \rangle = \frac{\int Da F(a) e^{-\text{Tra}^2 - S_{\text{int}}(a)}}{\int Da e^{-\text{Tra}^2 - S_{\text{int}}(a)}}. \quad (3.6)$$

The $\mathcal{N} = 4$ SYM theory is obtained as a special limit where the interacting action is dropped and the instanton contribution — already discarded at weak-coupling — is also removed. In this limit, the partition function is computed by a very simple Gaussian model and correlators are obtained after full Wick contraction where the basic contraction is $\langle a^\ell a^{\ell'} \rangle = \delta^{\ell\ell'}$. In the following we shall be interested in the multi-trace expectation values, cf. (2.1),

$$\langle \mathcal{O}_{\mathbf{n}}(a) \rangle, \quad \text{where } \mathcal{O}_{\mathbf{n}}(a) \stackrel{\text{def}}{=} \prod_{k=1}^{\ell} \text{Tra}^{n_k}. \quad (3.7)$$

As discussed in [11] it is possible to compute the function $A_{\mathbf{n}}(g)$ in (3.2) by the following prescription

$$A_{\mathbf{n}}(g) = \langle W(a) : \mathcal{O}_{\mathbf{n}}(a) : \rangle, \quad W(a) = \frac{1}{N} \text{Tr} \exp \left(\frac{g}{\sqrt{2}} a \right), \quad (3.8)$$

where $: \mathcal{O}_{\mathbf{n}}(a) :$ is obtained by Gram-Schmidt orthogonalization with respect to all operators with dimension smaller than $|\mathbf{n}|$. In [11], several interesting results are obtained for the function $A_{\mathbf{n}}(g)$ at specific values of the multi-index $\mathbf{n}$, both at finite N as well as in the planar limit $N \rightarrow \infty$ with fixed Ng^2 . This is achieved by exploiting recursion relations with respect to $\mathbf{n}$ [25]. In general, a potential unavoidable problem is that it may be difficult to obtain results parametric in $\mathbf{n}$, whereas this is essential for our purposes. This is a difficulty already showing up in the $\mathcal{N} = 4$ theory due to the complication of the Gram-Schmidt procedure required to construct normal ordering. In the $\mathcal{N} = 2$ theory, complications are worse due to the g -dependence introduced by the interaction term (3.4).

In the next section we shall consider the special class of operators Ω_n in (2.3), show how to solve the above problems, and derive a set of results depending parametrically in n . This will be the starting point to discuss the large R-charge limit $n \rightarrow \infty$ in section 5.

Remark. Concerning notation, in the following we shall need to tell between quantities evaluated in the $\mathcal{N} = 2$ or $\mathcal{N} = 4$ theories. In such cases, we shall denote by $\langle \mathcal{O} \rangle$ and $:\mathcal{O}:$ the expectation values and associated normal ordering in the $\mathcal{N} = 4$ matrix model. Instead, we shall denote by $\langle\langle \mathcal{O} \rangle\rangle$ and $\circ \mathcal{O} \circ$ the same quantities in the $\mathcal{N} = 2$ theory.

4 The correlators $\langle W \Omega_n \rangle$

As we explained in the Introduction, we are interested in the functions

$$A_n^{\mathcal{N}=2}(g) = \langle\langle W \circ \Omega_n \circ \rangle\rangle. \quad (4.1)$$

As we discussed, to some extent it is straightforward to evaluate the perturbative expansion of $A_n^{\mathcal{N}=2}$ at fixed n . However, our main concern is the limit (2.4) and the associated asymptotic ratio

$$\lim_{n \rightarrow \infty} \frac{A_n^{\mathcal{N}=2}}{A_n^{\mathcal{N}=4}} = \lim_{n \rightarrow \infty} \frac{\langle\langle W \circ \Omega_n \circ \rangle\rangle}{\langle W : \Omega_n : \rangle}, \quad \kappa = n g^2 = \text{fixed}. \quad (4.2)$$

For this quantity a different approach is required. Notice that in the case of the chiral 2-point function, the existence of a Toda equation — equivalent to the tt^* equations — has been crucial in this respect [7]. In this section, we shall present results for $A_n(g)$ in the $\mathcal{N} = 4$ theory. This is a piece entering the ratio (4.2). Next, we move to the $\mathcal{N} = 2$ theory and compute the first non-trivial correction to $A_n^{\mathcal{N}=4}$, i.e. the term proportional to $\zeta(3)$, cf. (3.5). Although we are ultimately interested in the finite N case, we shall check our results by matching general expressions valid in the planar limit that have been conjectured in [11].

4.1 Results in the $\mathcal{N} = 4$ theory

Normal ordering. Let $\mathcal{O}^\Delta$ be a basis for all multi-trace matrix model operators with dimension Δ . A possible choice is $\mathcal{O}^\Delta = \{\mathcal{O}_n(a), |n| = \Delta\}$, cf. (3.7). The Gram-Schmidt orthogonalization process amounts to write

$$:\Omega: = \Omega + \sum_{\Delta' < \Delta} \sum_{\Omega' \in \mathcal{O}^{\Delta'}} c_{\Omega'} \Omega', \quad (4.3)$$

where the constants c are determined by the condition $\langle : \Omega : \Omega' \rangle = 0$ for all $\Omega' \in \mathcal{O}^{\Delta'}$ with $\Delta' < \Delta$. A useful remark is that (4.3) is equivalent to the subtraction of all possible (partial) Wick pairings inside Ω [11, 25]. This is why the expansion (4.3) is dubbed normal ordering and denoted by the usual double colon notation. To prove this, we notice that by linearity we can reduce the problem to field monomial in the higher order traces $\text{Tr}(a^n)$. Let $W(\Omega)$ be the operator constructed by subtraction of Wick pairings. The correlator $\langle W(\Omega) \Omega' \rangle$ is obtained after full Wick pairing. If $\dim \Omega' < \dim \Omega$, some pairing is necessarily inside $W(\Omega)$ and we get zero. By uniqueness, $:\Omega: \equiv W(\Omega)$.

The case of Ω_n . We now consider the special case of the operators $\Omega_n = (\text{Tra}^2)^n$. Their expectation value is

$$\langle \Omega_n \rangle = \frac{\Gamma(\alpha + n)}{\Gamma(\alpha)}, \quad \alpha = \frac{N^2 - 1}{2}, \quad (4.4)$$

as follows easily from (3.3). According the remarks in the previous paragraph we can write the following specialized version of (4.3)⁴

$$:\Omega_n: = \Omega_n + \sum_{k=0}^{n-1} c_k^{(n)} \Omega_k. \quad (4.5)$$

We can give the explicit form of the coefficients $c_k^{(n)}$ as well as the inverse change of basis from $:\Omega^{(n)}: = \{:\Omega_k:\}_{0 \leq k \leq n}$ to $\Omega^{(n)} = \{\Omega_k\}_{0 \leq k \leq n}$. Indeed, we now show that

$$:\Omega_n: = \sum_{k=0}^n (-1)^{n+k} \frac{\Gamma(\alpha + n)}{\Gamma(\alpha + k)} \binom{n}{k} \Omega_k, \quad \Omega_n = \sum_{k=0}^n \frac{\Gamma(\alpha + n)}{\Gamma(\alpha + k)} \binom{n}{k} : \Omega_k :, \quad (4.6)$$

where α has been defined in (4.4). In the following it will be sometimes convenient to adopt an explicit matrix vector notation for (4.6) and write

$$:\Omega: = M \Omega, \quad M_{n,k} = c_k^{(n)} = (-1)^{n+k} \frac{\Gamma(\alpha + n)}{\Gamma(\alpha + k)} \binom{n}{k}. \quad (4.7)$$

To prove (4.6), we start from the orthogonality condition $\langle : \Omega_n : \Omega_m \rangle = 0$, with $m = 0, 1, \dots, n-1$, that we write as

$$\langle \Omega_{n+m} + \sum_{k=0}^{n-1} c_k^{(n)} \Omega_{k+m} \rangle = 0. \quad (4.8)$$

Using (4.4), these conditions may be written

$$\Gamma(\alpha + n + m) + \sum_{k=0}^{n-1} c_k^{(n)} \Gamma(\alpha + k + m) = 0, \quad 0 \leq m \leq n-1. \quad (4.9)$$

The solution is unique and reads⁵

$$c_k^{(n)} = (-1)^{n+k} \frac{\Gamma(\alpha + n)}{\Gamma(\alpha + k)} \binom{n}{k}. \quad (4.10)$$

⁴As we recalled in the Introduction, we are interested in $\mathbb{R}^4$ correlation functions. In general, these are computed by evaluating the matrix model correlators of normal ordered operators, cf. (4.3), that are orthogonal to *all* lower dimensional chiral operators. As we discussed after (4.3), in the $\mathcal{N} = 4$ theory, the Gram orthogonalization amounts to subtraction of Wick pairings. This very convenient feature implies that Ω_n mixes only with $\Omega_{n' < n}$ operators as in (4.5). We shall see that this feature extends to some corrections in the $\mathcal{N} = 2$ theory, but in general it is violated outside the $\mathcal{N} = 4$ theory and a more complicated mixing has to be considered, for instance between $(\text{Tra}^2)^3$ and Tra^4 .

⁵We can replace (4.10) into (4.9). This gives the quantity $\Gamma(\alpha + m) \prod_{k=0}^{n-1} (m - k)$ that vanishes when the integer m is such that $0 \leq m \leq n-1$.

Notice that since $c_n^{(n)} = 1$, we can write (4.5) in the simpler form (4.6). Finally, we can prove that

$$(M^{-1})_{n,k} = (-1)^{n+k} M_{n,k} = \frac{\Gamma(\alpha + n)}{\Gamma(\alpha + k)} \binom{n}{k}, \quad (4.11)$$

by exploiting the algebraic relation

$$\sum_{k=0}^n M_{n,k} (M^{-1})_{k,m} = \sum_{k=0}^n (-1)^{n+k} \frac{\Gamma(\alpha + n)}{\Gamma(\alpha + m)} \binom{n}{k} \binom{k}{m} = \frac{(-1)^{n+1} \sin(\pi m)}{\pi(n-m)} \frac{\Gamma(\alpha + n)}{\Gamma(\alpha + m)}, \quad (4.12)$$

and noting that the r.h.s. is zero for integer $m \neq n$ and one for $m \rightarrow n$.

A useful elementary identity. The specific explicit coefficients (4.9) allow to prove the following elementary identity. For any function $F(ga)$ we can write

$$\langle F : \Omega_n : \rangle = g^{2n} \left(\frac{1}{2g} \frac{d}{dg} \right)^n \langle F \rangle. \quad (4.13)$$

Proof. It is convenient to assume that $F(ga)$ may be expanded in a series of operators of the form (3.7).⁶ Thus, we write for some set $\mathcal{M}$ of multi-indices

$$F(ga) = \sum_{\mathbf{m} \in \mathcal{M}} c_{\mathbf{m}} \mathcal{O}_{\mathbf{m}}(ga) = \sum_{\mathbf{m} \in \mathcal{M}} c_{\mathbf{m}} g^{|\mathbf{m}|} \mathcal{O}_{\mathbf{m}}(a). \quad (4.14)$$

We compute $\langle \mathcal{O}_{\mathbf{m}} \Omega_n \rangle$ by iterating the recursion relation proved in [25]

$$\langle \mathcal{O}_{\mathbf{m}} \Omega_1 \rangle = \frac{N^2 - 1 + |\mathbf{m}|}{2} \langle \mathcal{O}_{\mathbf{m}} \rangle. \quad (4.15)$$

This gives

$$\langle \mathcal{O}_{\mathbf{m}} \Omega_n \rangle = \left(\prod_{k=1}^n \frac{N^2 - 1 + |\mathbf{m}| + 2(k-1)}{2} \right) \langle \mathcal{O}_{\mathbf{m}} \rangle = \frac{\Gamma(\frac{|\mathbf{m}|}{2} + n + \alpha)}{\Gamma(\frac{|\mathbf{m}|}{2} + \alpha)} \langle \mathcal{O}_{\mathbf{m}} \rangle. \quad (4.16)$$

Finally, using the mixing solution (4.6), we have (for $|\mathbf{m}| > 2n$ otherwise we get zero by construction of $:\Omega_n:$)

$$\langle \mathcal{O}_{\mathbf{m}} : \Omega_n : \rangle = \sum_{k=0}^n (-1)^{n+k} \frac{\Gamma(\alpha + n)}{\Gamma(\alpha + k)} \binom{n}{k} \frac{\Gamma(\frac{|\mathbf{m}|}{2} + k + \alpha)}{\Gamma(\frac{|\mathbf{m}|}{2} + \alpha)} \langle \mathcal{O}_{\mathbf{m}} \rangle = \frac{(\frac{1}{2}|\mathbf{m}|)!}{(\frac{1}{2}|\mathbf{m}| - n)!} \langle \mathcal{O}_{\mathbf{m}} \rangle, \quad (4.17)$$

Replacing this in (4.14) gives

$$\langle F : \Omega_n : \rangle = \sum_{\mathbf{m} \in \mathcal{M}} c_{\mathbf{m}} g^{|\mathbf{m}|} \frac{(\frac{1}{2}|\mathbf{m}|)!}{(\frac{1}{2}|\mathbf{m}| - n)!} \langle \mathcal{O}_{\mathbf{m}} \rangle = g^{2n} \left(\frac{d}{d(g^2)} \right)^n \langle F(ga) \rangle, \quad (4.18)$$

which is the same as (4.13). □

⁶This is already enough for our later use. A more general proof is given in appendix B.

Remark 1. It may be interesting to compare (4.13) with what is obtained by taking derivatives with respect to $1/g^2$, *i.e.* essentially with respect to $\text{Im}\tau$ where τ is the complexified gauge coupling. In this case we would have obtained Gaussian averages with various insertions of powers of Ω_1 . Instead, derivatives with respect to g^2 automatically build up the normal ordered operators $:\Omega_n:$. In particular, the N dependence of the normal ordering expansion (4.6) is fully taken into account by the N dependence of the basic expectation $\langle F \rangle$.

Remark 2. The relation (4.13) may be written in explicit form by simply rearranging derivatives. This gives

$$\langle F : \Omega_n : \rangle = \frac{(-1)^n}{2^{2n}} \sum_{p=1}^n (-2)^p \frac{(2n-1-p)!}{(n-p)!(p-1)!} g^p \frac{d^p}{dg^p} \langle F \rangle. \quad (4.19)$$

A useful application of formula (4.13) is to the case when F is the Wilson loop (3.8)

$$W = \frac{1}{N} \sum_{k=0}^{\infty} \frac{g^k}{2^{\frac{k}{2}} k!} \text{Tra}^k, \quad (4.20)$$

with expectation value in the $\text{SU}(N)$ theory given by [11]⁷

$$\langle W \rangle = \frac{1}{N} L_{N-1}^1 \left(-\frac{g^2}{4} \right) \exp \left[\frac{g^2}{8} \left(1 - \frac{1}{N} \right) \right]. \quad (4.21)$$

Formula (4.13) simply predicts

$$\langle W : \Omega_n : \rangle = g^{2n} \left(\frac{1}{2g} \frac{d}{dg} \right)^n \langle W \rangle. \quad (4.22)$$

Special cases are conveniently obtained by applying (4.19)

$$\begin{aligned} \langle W : \Omega_1 : \rangle &= \frac{g}{2} \partial_g \langle W \rangle, \\ \langle W : \Omega_2 : \rangle &= \frac{g}{4} \partial_g (-1 + g \partial_g) \langle W \rangle, \\ \langle W : \Omega_3 : \rangle &= \frac{g}{8} \partial_g (3 - 3g \partial_g + g^2 \partial_g^2) \langle W \rangle, \\ \langle W : \Omega_4 : \rangle &= \frac{g}{16} \partial_g (-15 + 15g \partial_g - 6g^2 \partial_g^2 + g^3 \partial_g^3) \langle W \rangle, \end{aligned} \quad (4.23)$$

and so on. As a check, the first two lines are in agreement with eqs. (4.7, 4.11) of [11].

Thanks to the special structure of $\langle W \rangle$ in (4.21), it is also simple to determine the closed expression of (4.22) for any given N , a kind of formula that will be useful in the

⁷For the $\text{U}(N)$ theory, the center term $-g^2/(8N)$ in the exponent is absent [26].

following. The first cases are

$$\begin{aligned}
 \langle W : \Omega_n : \rangle|_{\text{SU}(2)} &= \left(\frac{g^2}{16}\right)^n e^{\frac{g^2}{16}} \left(1 + 2n + \frac{1}{8}g^2\right), \\
 \langle W : \Omega_n : \rangle|_{\text{SU}(3)} &= \left(\frac{g^2}{12}\right)^n e^{\frac{g^2}{12}} \left[1 + \frac{3}{2}n + \frac{3}{2}n^2 + \frac{1}{4}(1+n)g^2 + \frac{1}{96}g^4\right], \\
 \langle W : \Omega_n : \rangle|_{\text{SU}(4)} &= \left(\frac{3g^2}{32}\right)^n e^{\frac{3g^2}{32}} \left[\frac{64n^3 + 96n^2 + 164n + 81}{81} + \frac{16n^2 + 32n + 27}{72}g^2 \right. \\
 &\quad \left. + \frac{2n + 3}{96}g^4 + \frac{g^6}{1536}\right]. \tag{4.24}
 \end{aligned}$$

The SU(2) computation is particularly simple and is easily checked by direct evaluation of the partition function and correlators in the Gaussian matrix model. This is briefly reviewed in appendix A. The advantage of the relations proved in this section is that they are parametric in N , i.e. in the gauge group rank.

4.2 $\zeta(3)$ corrections in the $\mathcal{N} = 2$ theory

The correction factor arising in $\mathcal{N} = 2$ correlators due to the first term in (3.4) is

$$1 - \frac{3g^4}{64\pi^4} \zeta(3) \Omega_2 + \dots = 1 - \varepsilon \Omega_2 + \dots \tag{4.25}$$

where $\varepsilon = \frac{3g^4}{64\pi^4} \zeta(3)$ is the term we want to control at linear order, and dots denote higher transcendentality structures. To compute its effects in correlators, we first need to determine how it affects normal ordering. To this aim, we need a simple relation valid for the $\mathcal{N} = 4$ normal ordered operators and expressing the decomposition of $:\Omega_n : \Omega_2$ in terms of normal ordered operators

$$\begin{aligned}
 :\Omega_n : \Omega_2 &= :\Omega_{n+2} : + 2(\alpha + 2n + 1) : \Omega_{n+1} : + (\alpha^2 + (6n + 1)\alpha + 6n^2) : \Omega_n : \\
 &\quad + 2n(\alpha + 2n - 1)(\alpha + n - 1) : \Omega_{n-1} : + n(n - 1)(\alpha + n - 1)(\alpha + n - 2) : \Omega_{n-2} :. \tag{4.26}
 \end{aligned}$$

This is obtained by using the special form of the matrix M in (4.7) and writing

$$:\Omega_n : \Omega_2 = \sum_{m=0}^n M_{n,m} \Omega_m \Omega_2 = \sum_{m=0}^n \sum_{\ell=0}^{m+2} M_{n,m} (M^{-1})_{m+2,\ell} : \Omega_\ell :. \tag{4.27}$$

Replacing the explicit form of M and M^{-1} from (4.6), (4.7) and doing the finite sums we get the above expansion (4.26). We can now determine the ε perturbation appearing in the $\mathcal{N} = 2$ normal ordering

$$\begin{aligned}
 \circ \Omega_n \circ &= \sum_{k=0}^n (c_k^{(n)} + \varepsilon d_k^{(n)} + \dots) \Omega_k, \\
 d_k^{(n)} &= (-1)^{k+n+1} (k+1) (2\alpha - 1 + 3n + k) \frac{\Gamma(\alpha + n)}{\Gamma(\alpha + k)} \binom{n}{k+1}. \tag{4.28}
 \end{aligned}$$

Proof. Let us start with the obvious expansion⁸

$$\langle\langle A \rangle\rangle = \frac{\langle A (1 - \varepsilon \Omega_2 + \dots) \rangle}{\langle 1 - \varepsilon \Omega_2 + \dots \rangle} = \langle A \rangle + \varepsilon \left[-\langle A \Omega_2 \rangle + \langle A \rangle \langle \Omega_2 \rangle \right] + \dots \quad (4.29)$$

Now, let us consider $A = {}^\circ\Omega_n {}^\circ X$ with $\dim X < 2n$. If we write ${}^\circ\Omega_n {}^\circ = : \Omega_n : + \delta \Omega_n$, we find

$$\langle (\delta \Omega_n - : \Omega_n : \Omega_2) X \rangle = 0. \quad (4.30)$$

Due to the defining property of normal ordering, the explicit solution is

$$\delta \Omega_n = : \Omega_n : \Omega_2 - : \Omega_{n+2} : - k : \Omega_{n+1} : - k' : \Omega_n :, \quad (4.31)$$

where the constant k, k' are chosen such that $\delta \Omega_n$ does not contain Ω_{n+1} and Ω_n . From (4.26) we read

$$k = 2(2n + \alpha + 1), \quad k' = 6n^2 + 6n\alpha + \alpha(\alpha + 1). \quad (4.32)$$

Rearranging (4.31) proves formula (4.28). $\square$

Remark. Notice that in the r.h.s. of (4.28) we find only operators of the form Ω_n . This is a special property of the $\zeta(3)$ correction and is false for corrections associated with higher transcendentality terms.

Finally, we can compute the $\zeta(3) \sim \varepsilon$ correction to the 1-point function of the chiral primary ${}^\circ\Omega_n {}^\circ$ with the $\frac{1}{2}$ -BPS Wilson loop. The result is

$$\langle\langle W {}^\circ\Omega_n {}^\circ \rangle\rangle = \langle W : \Omega_n : \rangle + \varepsilon X_n + \dots, \quad (4.33)$$

with

$$\begin{aligned} X_n &= -f^{(n+2)} - 2(\alpha + 2n + 1)f^{(n+1)} - 6n(\alpha + n)f^{(n)}, \\ f^{(n)} &= g^{2n} \left(\frac{1}{2g} \frac{d}{dg} \right)^n \langle W \rangle. \end{aligned} \quad (4.34)$$

Proof. We have

$$\begin{aligned} \langle\langle W {}^\circ\Omega_n {}^\circ \rangle\rangle &= \frac{\langle W {}^\circ\Omega_n {}^\circ (1 - \varepsilon \Omega_2 + \dots) \rangle}{\langle 1 - \varepsilon \Omega_2 + \dots \rangle} = \langle W {}^\circ\Omega_n {}^\circ \rangle \\ &+ \varepsilon \left[-\langle W {}^\circ\Omega_n {}^\circ \Omega_2 \rangle + \langle W {}^\circ\Omega_n {}^\circ \rangle \langle \Omega_2 \rangle \right] + \dots \\ &= \langle W : \Omega_n : \rangle + \varepsilon \left[\sum_{k=0}^{n-1} d_k^{(n)} \langle W \Omega_k \rangle - \langle W : \Omega_n : \Omega_2 \rangle + \langle W : \Omega_n : \rangle \langle \Omega_2 \rangle \right] + \dots \end{aligned} \quad (4.35)$$

Now we can use $\langle \Omega_2 \rangle = \alpha(\alpha + 1)$, and the decomposition (4.26). Finally, the first term in the square bracket in (4.35) is manipulated using

$$\begin{aligned} \sum_{k=0}^{n-1} d_k^{(n)} \Omega_k &= \sum_{k=0}^{n-1} d_k^{(n)} \sum_{m=0}^k (M_{k,m})^{-1} : \Omega_m : \\ &= 2n(\alpha + 2n)(\alpha + n - 1) : \Omega_{n-1} : + n(n-1)(\alpha + n - 1)(\alpha + n - 2) : \Omega_{n-2} :. \end{aligned} \quad (4.36)$$

Using (4.22) proves (4.34). $\square$

⁸We remind that $\langle\langle \dots \rangle\rangle$ denotes the expectation value in the $\mathcal{N} = 2$ theory.

Expanding at weak coupling, the leading term in the X_n correction reads

$$\begin{aligned} X_1 &= -\frac{3(N^4 - 1)}{8N} g^2 + \dots, \\ X_2 &= -\frac{(N^2 - 1)(N^2 + 3)(2N^2 - 3)}{32N^2} g^4 + \dots, \\ X_3 &= -\frac{3(N^2 - 1)(N^2 + 5)(N^4 - 3N^2 + 3)}{512N^3} g^6 + \dots, \\ X_4 &= -\frac{(N^2 - 1)(N^2 + 7)(2N^6 - 8N^4 + 15N^2 - 15)}{5120N^4} g^8 + \dots, \end{aligned} \quad (4.37)$$

and so on. One can check that, after multiplication by $\frac{3\zeta(3)}{(8\pi^2)^2} g^4$, the first two lines give the entries labeled (2) and (2,2) in table (2) in [11].

4.2.1 Checks in the planar limit

From the result (4.34) it is easy to work out the planar limit $N \rightarrow \infty$ with fixed $\lambda = N g^2$. Indeed we can expand

$$f^{(n)} = \lambda^n \frac{d^n}{d\lambda^n} \langle W \rangle = f_0^{(n)} + \mathcal{O}(1/N^2), \quad f_0^{(n)} = \lambda^n \frac{d^n}{d\lambda^n} \left[\frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda}) \right]. \quad (4.38)$$

In (4.34), we have at leading order

$$X_n = -N^2 (f_0^{(n+1)} + 3n f_0^{(n)}) + \mathcal{O}(N^0), \quad (4.39)$$

and therefore

$$\langle\langle W^\circ \Omega_n^\circ \rangle\rangle = \left\{ \lambda^n \frac{d^n}{d\lambda^n} - \frac{3g^4 \zeta(3)}{64\pi^4} N^2 \left[\lambda^{n+1} \frac{d^{n+1}}{d\lambda^{n+1}} + 3n \lambda^n \frac{d^n}{d\lambda^n} \right] + \dots \right\} \frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda}). \quad (4.40)$$

Rearranging derivatives, we can write (4.40) in the form

$$\langle\langle W^\circ \Omega_n^\circ \rangle\rangle = \left\{ 1 - \frac{3\zeta(3)}{64\pi^4} \lambda^2 \left[\lambda \frac{d}{d\lambda} + 2n \right] + \dots \right\} \lambda^n \frac{d^n}{d\lambda^n} \left[\frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda}) \right]. \quad (4.41)$$

Sample cases are

$$\begin{aligned} \langle\langle W \rangle\rangle &= \frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda}) - \frac{3\zeta(3)}{(8\pi^2)^2} \lambda^2 I_2(\sqrt{\lambda}) + \dots, \\ \langle\langle W^\circ \Omega_1^\circ \rangle\rangle &= I_2(\sqrt{\lambda}) - \frac{3\zeta(3)}{(8\pi^2)^2} \left[\lambda^2 I_0(\sqrt{\lambda}) + \frac{1}{2} \lambda^{3/2} (\lambda - 4) I_1(\sqrt{\lambda}) \right], \\ \langle\langle W^\circ \Omega_2^\circ \rangle\rangle &= -2I_0(\sqrt{\lambda}) + \frac{8 + \lambda}{2\sqrt{\lambda}} I_1(\sqrt{\lambda}) \\ &\quad - \frac{3\zeta(3)}{(8\pi^2)^2} \left[\frac{\lambda - 24}{4} \lambda^2 I_0(\sqrt{\lambda}) + \lambda^{3/2} (\lambda + 12) I_1(\sqrt{\lambda}) \right] + \dots \end{aligned} \quad (4.42)$$

These may be compared with the general formula conjectured in [11] that reads

$$\langle\langle W^\circ \Omega_n^\circ \rangle\rangle = \frac{\lambda^{\frac{n-1}{2}}}{2^{n-1}} I_{n+1}(\sqrt{\lambda}) - \frac{3\zeta(3)}{(8\pi^2)^2} \frac{\lambda^{\frac{n+4}{2}}}{2^n} \left[I_n(\sqrt{\lambda}) + \frac{2(2n-1)}{\sqrt{\lambda}} I_{n+1}(\sqrt{\lambda}) \right] + \dots \quad (4.43)$$

and, of course, there is agreement after some rearrangement of the Bessel functions.

5 Large R-charge limit and universality

We now make full use of the results presented in the previous section to discuss the large R-charge limit of the normalized one-point function of the chiral primary $\Omega_n(\varphi)$ with the $\frac{1}{2}$ -BPS Wilson loop in the $\mathcal{N} = 2$ theory. In particular, the result (4.34) controls exactly the contribution $\sim \zeta(3)$. When combined with explicit low N calculations, it provides a guide to understand the behaviour of higher transcendentality contributions, as we are going to illustrate.

5.1 Analysis of the $\zeta(3)$ correction

We define the asymptotic ratio of one-point functions, cf. (4.2),

$$G(\kappa; N) = \lim_{n \rightarrow \infty} \frac{\langle\langle W \circ \Omega_n \circ \rangle\rangle}{\langle W : \Omega_n : \rangle} \Big|_{\kappa = n g^2}. \quad (5.1)$$

The very existence of the large n limit in (5.1) is something that deserves investigation. To begin our analysis, let us consider the simplest $SU(2)$ case. Using (4.41) and the first equation in (4.24) we obtain the ratio

$$\begin{aligned} \frac{\langle\langle W \circ \Omega_n \circ \rangle\rangle}{\langle W : \Omega_n : \rangle} &= 1 - \frac{3g^4}{16384\pi^4(8+16n+g^2)} \\ &\times \left[6144n(2n+1)(2n+3) + 640g^2(2n+1)(2n+3) + 40g^4(2n+3) + g^6 \right] \zeta(3) \\ &+ \text{h.z.}, \end{aligned} \quad (5.2)$$

where “h.z.” stands for higher transcendentality terms $\zeta(5)$, $\zeta(3)^2$, and so on. The expression (5.2) is exact in the gauge coupling g , i.e. it resums all contributions $\sim \zeta(3)g^{4+2k}$, with $k = 0, 1, 2, \dots$. For large n and fixed $\kappa = n g^2$ the limit (5.1) reads

$$G(\kappa; 2) = 1 - \frac{9\zeta(3)}{32\pi^4} \kappa^2 + \mathcal{O}(\kappa^3), \quad (5.3)$$

and comes entirely from the $\zeta(3)g^4$ term in (5.2). The same analysis may be repeated for higher N , see (4.24). In all cases one finds the same leading behaviour as in (5.3). In general, one can show that for $SU(N)$ the $\zeta(3)$ correction reads

$$\frac{\langle\langle W \circ \Omega_n \circ \rangle\rangle}{\langle W : \Omega_n : \rangle} = 1 - \frac{9n(2n+N^2-1)\zeta(3)}{64\pi^4} \left[g^4 + \mathcal{O}(g^6) \right] + \text{h.z.}, \quad (5.4)$$

and thus (5.3) is valid for any N . Indeed, our results allow to check that the higher order terms in coupling g are always accompanied by lower powers of n and do not contribute in the $n \rightarrow \infty$ limit. Notice however, that the complexity of the detailed dependence on n increases with N . For instance, even for the simplest $\zeta(3)$ correction, the $\mathcal{O}(g^6)$ term

in (5.4) reads for $SU(N)$ with $N = 2, 3, 4$

$$\begin{aligned}
 \frac{\langle\langle W \circ \Omega_n \circ \rangle\rangle}{\langle W : \Omega_n : \rangle} \Big|_{SU(2)} &= 1 - \frac{9n(2n+3)\zeta(3)g^4}{64\pi^4} \left[1 + \frac{g^2(4n+5)}{48n(2n+1)} + \mathcal{O}(g^4) \right] + \text{h.z.}, \\
 \frac{\langle\langle W \circ \Omega_n \circ \rangle\rangle}{\langle W : \Omega_n : \rangle} \Big|_{SU(3)} &= 1 - \frac{9n(2n+8)\zeta(3)g^4}{64\pi^4} \left[1 + \frac{g^2(2n+5)(3n^2+9n+8)}{36n(n+4)(3n^2+3n+2)} + \mathcal{O}(g^4) \right] \\
 &\quad + \text{h.z.}, \\
 \frac{\langle\langle W \circ \Omega_n \circ \rangle\rangle}{\langle W : \Omega_n : \rangle} \Big|_{SU(4)} &= 1 - \frac{9n(2n+15)\zeta(3)g^4}{64\pi^4} \\
 &\quad \times \left[1 + \frac{g^2(4n+17)(64n^3+288n^2+548n+405)}{32n(2n+15)(64n^3+96n^2+164n+81)} + \mathcal{O}(g^4) \right] + \text{h.z.}
 \end{aligned} \tag{5.5}$$

We cannot give a general formula parametric in N , but in all cases, the correction term in the square brackets goes like $g^2/n = \kappa/n^2$ and is negligible at large n .

5.2 Higher transcendentality terms

The agreement of (5.3) with the analogous universal $\zeta(3)$ term in the chiral 2-point scaling function $F(\kappa; N)$, cf. (2.7) and (2.8), is a strong motivation to look at higher transcendentality terms. We can work out the first values of the ratio $\langle\langle W \circ \Omega_n \circ \rangle\rangle / \langle W : \Omega_n : \rangle$ looking at the first appearance of a certain transcendentality term $\zeta(s)$, cf. (2.5), working at fixed small N . In other words, we shall assume that the $\zeta(3)$ pattern repeats for higher transcendentality structures. In particular, for a given structure, higher powers of g will be associated to subleading terms for $n \rightarrow \infty$, as in (5.5).⁹

SU(2) theory. Working out the mixing algorithm for $\mathcal{N} = 2$ correlators and evaluating the correction to the 1-point function with the Wilson loop with $n = 1, \dots, 20$, we find the following $\zeta(5)$ correction in SU(2) case (extending (5.2) that was for the $\zeta(3)$ term)

$$\begin{aligned}
 \frac{\langle\langle W \circ \Omega_n \circ \rangle\rangle}{\langle W : \Omega_n : \rangle} \Big|_{SU(2), \zeta(5) \text{ term}} &= \frac{5g^6\zeta(5)}{2097152(8+16n+g^2)\pi^6} \left[163840n(2n+1)(4n^2+9n+8) \right. \\
 &\quad + 17920g^2(2n+1)(4n^2+10n+9) + 2688g^4(2n^2+6n+5) \\
 &\quad \left. + 112g^6(n+2) + g^8 \right].
 \end{aligned} \tag{5.6}$$

As in (5.3), the lowest power of g is the only one surviving the $n \rightarrow \infty$ limit at fixed ng^2 . Extending the analysis to higher transcendentality structures and writing only the terms

⁹These assumptions may be examined in details for the SU(2) and SU(3) theories, where it is straightforward to compute the explicit integral definition of the partition function, as we did for the $\mathcal{N} = 4$ SU(2) theory in appendix A.

that give contribution in the $n \rightarrow \infty$ limit, we obtain,

$$\begin{aligned}
 \left. \frac{\langle\langle W \circ \Omega_n \circ \rangle\rangle}{\langle W : \Omega_n : \rangle} \right|_{\text{SU}(2)} &= 1 - 9n(2n+3)\zeta(3) \left[\left(\frac{g^2}{8\pi^2} \right)^2 + \dots \right] \\
 &+ 25n(4n^2+9n+8)\zeta(5) \left[\left(\frac{g^2}{8\pi^2} \right)^3 + \dots \right] \\
 &+ \left[\frac{27}{2}n(12n^3+60n^2+81n+47)\zeta(3)^2 \right. \\
 &\quad \left. - \frac{2205}{16}n(2n+3)(2n^2+3n+4)\zeta(7) \right] \left[\left(\frac{g^2}{8\pi^2} \right)^4 + \dots \right] \\
 &+ \left[-\frac{225}{2}n(2n+3)(8n^3+42n^2+52n+45)\zeta(3)\zeta(5) \right. \\
 &\quad \left. + \frac{3213}{32}n(32n^4+120n^3+240n^2+270n+163)\zeta(9) \right] \\
 &\times \left[\left(\frac{g^2}{8\pi^2} \right)^5 + \dots \right] \\
 &+ \left[-\frac{243}{2}n(2n+3)(4n^4+36n^3+103n^2+107n+72)\zeta(3)^3 \right. \\
 &\quad + \frac{125}{16}n(640n^5+5760n^4+16600n^3+26400n^2+24700n+12777)\zeta(5)^2 \\
 &\quad + \frac{19845}{16}n(8n^5+68n^4+190n^3+307n^2+288n+147)\zeta(3)\zeta(7) \\
 &\quad \left. - \frac{78771}{256}n(2n+3)(4n^2+6n+11)(8n^2+12n+19)\zeta(11) \right] \\
 &\times \left[\left(\frac{g^2}{8\pi^2} \right)^6 + \dots \right] + \text{h.z.}, \tag{5.7}
 \end{aligned}$$

where dots stand for higher order rational corrections in g . In other words, (5.7) gives the exact n dependence of the lowest order correction (in g) to the structures

$$\zeta(3), \zeta(5), \zeta(3)^2, \zeta(7), \zeta(3)\zeta(5), \zeta(3)^3, \zeta(5)^2, \zeta(3)\zeta(7), \zeta(11). \tag{5.8}$$

Taking the $n \rightarrow \infty$ limit, one obtains

$$\begin{aligned}
 G(\kappa; 2) &= \lim_{n \rightarrow \infty} \left. \frac{\langle\langle W \circ \Omega_n \circ \rangle\rangle}{\langle W : \Omega_n : \rangle} \right|_{\kappa=ng^2} = 1 - \frac{9\zeta(3)}{32\pi^4}\kappa^2 + \frac{25\zeta(5)}{128\pi^6}\kappa^3 + \left(\frac{81\zeta(3)^2}{2048\pi^8} - \frac{2205\zeta(7)}{16384\pi^8} \right)\kappa^4 \\
 &+ \left(\frac{3213\zeta(9)}{32768\pi^{10}} - \frac{225\zeta(3)\zeta(5)}{4096\pi^{10}} \right)\kappa^5 + \left(-\frac{243\zeta(3)^3}{65536\pi^{12}} + \frac{625\zeta(5)^2}{32768\pi^{12}} \right. \\
 &\quad \left. + \frac{19845\zeta(3)\zeta(7)}{524288\pi^{12}} - \frac{78771\zeta(11)}{1048576\pi^{12}} \right)\kappa^6 + \mathcal{O}(\kappa^7), \tag{5.9}
 \end{aligned}$$

and this nicely exponentiates to

$$\log G(\kappa; 2) = -\frac{9\zeta(3)}{32\pi^4}\kappa^2 + \frac{25\zeta(5)}{128\pi^6}\kappa^3 - \frac{2205\zeta(7)}{16384\pi^8}\kappa^4 + \frac{3213\zeta(9)}{32768\pi^{10}}\kappa^5 - \frac{78771\zeta(11)}{1048576\pi^{12}}\kappa^6 + \mathcal{O}(\kappa^7), \tag{5.10}$$

with only simple zeta functions. Comparing with (2.7) we see that $G(\kappa; 2)$ coincides with $F(\kappa; 2)$ at this order. We are thus led to the following

$$\text{conjecture: } G(\kappa; 2) = F(\kappa; 2) \equiv \mathcal{F}(\kappa), \quad (5.11)$$

that our calculation supports at order κ^6 included. The importance of (5.11) as a guiding remark will be manifest in the next section. Before elaborating on this, it is interesting to explore what happens for higher rank gauge groups.

SU(3) theory. Already the study of the first case $N = 3$ reveals some interesting feature of the G function. We may check that again, higher transcendentality terms are dominated by the lowest order contribution in g as soon as $n \rightarrow \infty$. For instance, the equation corresponding to (5.6) reads in SU(3) case

$$\begin{aligned} \frac{\langle\langle W \circ \Omega_n \circ \rangle\rangle}{\langle W : \Omega_n : \rangle} \Big|_{\text{SU(3), } \zeta(5) \text{ term}} &= \frac{5 g^6 \zeta(5)}{7962624 [48(3n^2 + 3n + 2) + 24g^2(n + 1) + g^4] \pi^6} \\ &\times \left[829440n(42n^4 + 375n^3 + 850n^2 + 927n + 254) \right. \\ &\quad + 103680g^2(77n^4 + 695n^3 + 1746n^2 + 1562n + 408) \\ &\quad + 1728g^4(392n^3 + 3213n^2 + 6971n + 4080) \\ &\quad + 48g^6(588n^2 + 3801n + 5288) \\ &\quad \left. + 12g^8(56n + 209) + 7g^{10} \right]. \end{aligned} \quad (5.12)$$

Keeping only the relevant terms in large n limit and working out higher transcendentality structures up to $\zeta(11)$ the SU(3) expansion corresponding to (5.7) has a similar structure

$$\begin{aligned} \frac{\langle\langle W \circ \Omega_n \circ \rangle\rangle}{\langle W : \Omega_n : \rangle} \Big|_{\text{SU(3)}} &= 1 - 18n(n+4)\zeta(3) \left[\left(\frac{g^2}{8\pi^2} \right)^2 + \dots \right] \\ &+ \frac{50n(42n^4 + 375n^3 + 850n^2 + 927n + 254)}{9(3n^2 + 3n + 2)} \zeta(5) \left[\left(\frac{g^2}{8\pi^2} \right)^3 + \dots \right] \\ &+ \left[18n(9n^3 + 90n^2 + 252n + 199)\zeta(3)^2 \right. \\ &\quad \left. - \frac{245n(147n^5 + 1971n^4 + 6815n^3 + 12817n^2 + 11122n + 2168)}{36(3n^2 + 3n + 2)} \zeta(7) \right] \left[\left(\frac{g^2}{8\pi^2} \right)^4 + \dots \right] \\ &+ \left[- \frac{100n(42n^6 + 669n^5 + 3754n^4 + 9073n^3 + 12914n^2 + 7772n + 2496)}{3n^2 + 3n + 2} \zeta(3)\zeta(5) \right. \\ &\quad \left. + \frac{7n(3834n^6 + 72063n^5 + 354855n^4 + 885855n^3 + 1388231n^2 + 916002n + 195160)}{6(3n^2 + 3n + 2)} \zeta(9) \right] \left[\left(\frac{g^2}{8\pi^2} \right)^5 + \dots \right] \\ &+ \left[-108n(n+4)(9n^4 + 126n^3 + 558n^2 + 957n + 632)\zeta(3)^3 \right. \\ &\quad \left. + \frac{250n(2940n^7 + 62790n^6 + 518341n^5 + 1817865n^4 + 3833881n^3 + 4405191n^2 + 2647324n + 758148)}{81(3n^2 + 3n + 2)} \zeta(5)^2 \right. \\ &\quad \left. + \frac{245n(441n^7 + 9441n^6 + 71925n^5 + 250551n^4 + 511026n^3 + 625772n^2 + 342468n + 101560)}{6(3n^2 + 3n + 2)} \zeta(3)\zeta(7) \right] \end{aligned}$$

$$\begin{aligned}
 & -\frac{77}{324(3n^2+3n+2)}n(87549n^7+2205555n^6+14721010n^5+47885868n^4+101153611n^3 \\
 & +122829225n^2+76171270n+16090872)\zeta(11)\Bigg]\left[\left(\frac{g^2}{8\pi^2}\right)^6+\dots\right]+\text{h.z.} \quad (5.13)
 \end{aligned}$$

In the $n \rightarrow \infty$ limit we find

$$\begin{aligned}
 G(\kappa; 3) = & 1 - \frac{9\zeta(3)}{32\pi^4}\kappa^2 + \frac{175\zeta(5)}{1152\pi^6}\kappa^3 + \left(\frac{81\zeta(3)^2}{2048\pi^8} - \frac{12005\zeta(7)}{147456\pi^8}\right)\kappa^4 + \left(\frac{1491\zeta(9)}{32768\pi^{10}} - \frac{175\zeta(3)\zeta(5)}{4096\pi^{10}}\right)\kappa^5 \\
 & + \left(-\frac{243\zeta(3)^3}{65536\pi^{12}} + \frac{30625\zeta(5)^2}{2654208\pi^{12}} + \frac{12005\zeta(3)\zeta(7)}{524288\pi^{12}} - \frac{2247091\zeta(11)}{84934656\pi^{12}}\right)\kappa^6 + \mathcal{O}(\kappa^7). \quad (5.14)
 \end{aligned}$$

Taking the logarithm we obtain again a neat exponentiation involving only terms linear in ζ functions

$$\log G(\kappa; 3) = -\frac{9\zeta(3)}{32\pi^4}\kappa^2 + \frac{175\zeta(5)}{1152\pi^6}\kappa^3 - \frac{12005\zeta(7)}{147456\pi^8}\kappa^4 + \frac{1491\zeta(9)}{32768\pi^{10}}\kappa^5 - \frac{2247091\zeta(11)}{84934656\pi^{12}}\kappa^6 + \mathcal{O}(\kappa^7). \quad (5.15)$$

Comparing with (2.8), we see that $G(\kappa; 3) \neq F(\kappa; 3)$. In particular, the scaling function G is simpler since exponentiates as in $N = 2$ case.

6 The all-order SU(2) large R-charge scaling function: a conjecture

Given the striking universal presence of the scaling function $F(\kappa; 2) \equiv \mathcal{F}(\kappa)$, we now present a puzzling conjecture about its all-order expansion. In (2.7) and (5.10) we already presented the expansion of its logarithm. For the reader's advantage, let us repeat here the first four terms

$$\log \mathcal{F}(\kappa) = -\frac{9\zeta(3)}{32\pi^4}\kappa^2 + \frac{25\zeta(5)}{128\pi^6}\kappa^3 - \frac{2205\zeta(7)}{16384\pi^8}\kappa^4 + \frac{3213\zeta(9)}{32768\pi^{10}}\kappa^5 + \dots \quad (6.1)$$

As discussed in [3] the special feature of SU(2) is the linear dependence on the ζ functions.¹⁰ In the $\mathcal{N} = 2$ matrix model there is another natural and universal object with this property, i.e. the interaction action S_{int} in (3.4). Evaluating the expectation value $\langle S_{\text{int}} \rangle$ in the Gaussian matrix model associated with the SU(N) theory gives

$$\begin{aligned}
 \langle S_{\text{int}} \rangle = & \frac{3(N^4 - 1)\zeta(3)}{256\pi^4}g^4 - \frac{5(2N^2 - 1)(N^4 - 1)\zeta(5)}{2048\pi^6N}g^6 \\
 & + \frac{35(N^2 - 1)(8N^6 + 4N^4 - 3N^2 + 3)\zeta(7)}{131072\pi^8N^2}g^8 \\
 & - \frac{21(N^2 - 1)(26N^8 + 28N^6 - 3N^4 + 6N^2 - 9)\zeta(9)}{524288\pi^{10}N^3}g^{10} \\
 & + \frac{77(N^2 - 1)(122N^{10} + 280N^8 + 48N^6 - 15N^4 + 45)\zeta(11)}{16777216\pi^{12}N^4}g^{12} + \mathcal{O}(g^{14}). \quad (6.2)
 \end{aligned}$$

The $N = 2$ specialization of (6.2) is

$$\langle S_{\text{int}} \rangle = \frac{45\zeta(3)}{256\pi^4}g^4 - \frac{525\zeta(5)}{4096\pi^6}g^6 + \frac{59535\zeta(7)}{524288\pi^8}g^8 - \frac{530145\zeta(9)}{4194304\pi^{10}}g^{10} + \frac{46081035\zeta(11)}{268435456\pi^{12}}g^{12} + \mathcal{O}(g^{14}). \quad (6.3)$$

¹⁰As we have discussed, this is a special feature of the $F(\kappa; 2)$ function that is not valid for $N > 2$. However, it seems not to be a characterization of $G(\kappa; N)$ for $N > 2$.

Comparing with (6.1) we observe the following remarkably simple relation

$$\log \mathcal{F}(\kappa) = \langle S_{\text{int}} \rangle \quad \text{with the replacement} \quad g^{2n} \rightarrow -\frac{4^n}{(1+2n)n!} \kappa^n, \quad (6.4)$$

where the replacement has to be done after expansion in powers of g^2 . Assuming this relation to be correct at all orders we can evaluate $\langle S_{\text{int}} \rangle$ using the formulas of appendix (A). This gives

$$\log \mathcal{F}(\kappa) = \frac{4\sqrt{2}}{\sqrt{\pi}} \int_{\mathbb{R}} da \, a^2 e^{-2a^2} \sum_{n=2}^{\infty} (-1)^n \frac{2(4^n - 4)}{n} \zeta(2n-1) \left(\frac{1}{8\pi^2}\right)^n a^{2n} \frac{-4^n}{(1+2n)n!} \kappa^n. \quad (6.5)$$

Doing the trivial integrals, we obtain a closed form for all perturbative coefficients of $\log \mathcal{F}(\kappa)$

$$\log \mathcal{F}(\kappa) = 4\sqrt{\pi} \sum_{n=2}^{\infty} (-1)^{n+1} \frac{4^n - 4}{n n! (2\pi)^{2n+1}} \Gamma\left(n + \frac{1}{2}\right) \zeta(2n-1) \kappa^n. \quad (6.6)$$

One can easily check that the additional terms beyond those in (6.1) fully reproduce the $\mathcal{O}(\kappa^{10})$ result (2.7) giving strong support to the validity of (6.6), i.e. of the replacement rule (6.4).

The series (6.6) has a finite convergence radius, being convergent for $|\kappa| < \pi^2$. A simple representation of its analytic continuation that may be used for any $\kappa > 0$ is

$$\log \mathcal{F}(\kappa) = \frac{1}{\pi^2} \int_0^{\infty} \frac{dt}{t^2} \frac{4\pi^2 J_0\left(\frac{t\sqrt{\kappa}}{\pi}\right) + 4\pi t \sqrt{\kappa} J_1\left(\frac{t\sqrt{\kappa}}{\pi}\right) - \kappa t^2 - 4\pi^2}{e^t + 1}, \quad (6.7)$$

where J_p are standard Bessel functions. For $\kappa \gg 1$, one has $\log \mathcal{F}(\kappa) = -\frac{\log 2}{\pi^2} \kappa + \mathcal{O}(\log \kappa)$, see appendix C for more information. Of course, this large κ expansion must be taken with some caution because $\mathcal{F}(\kappa)$ does not include the instanton contribution. Instantons are expected not to contribute at finite κ — and in particular in the weak coupling expansion — but may be important when attempting to reach the large κ regime.

A Direct evaluation of $\mathcal{N} = 4$ correlators in the SU(2) theory

The analysis of the low gauge group rank cases may be carried on explicitly by direct evaluation of the partition function, at least in the simple Gaussian case, i.e. for the $\mathcal{N} = 4$ theory. Let us briefly discuss the SU(2) case as an illustration and to write down some expressions that are used in the main text. The partition function (3.3) may be written in terms of a_k , $k = 1, \dots, N$, the eigenvalues of a ¹¹

$$Z_{S^4} = \mathcal{C} \int \prod_{k=1}^N da_k \, \delta\left(\sum_k a_k\right) \prod_{k < \ell} (a_k - a_\ell)^2 e^{-S_{\text{int}}} e^{-\sum a^2}, \quad (\text{A.1})$$

¹¹The normalization $\mathcal{C}$ depends on g but drops in correlation functions.

where

$$S_{\text{int}} = -\log \frac{\prod_{k < \ell=1}^N h(a_k - a_\ell, g)^2}{\prod_{k=1}^N h(a_k, g)^{2N}},$$

$$\log h(x, g) = (1 + \gamma_E) \frac{g^2}{8\pi^2} x^2 + \sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n} \zeta(2n-1) \left(\frac{g^2}{8\pi^2} \right)^n x^{2n} \quad (\text{A.2})$$

In the special case $N = 2$ we have (with a suitable redefinition of $\mathcal{C}$),

$$Z_{S^4} = \mathcal{C} \int_{\mathbb{R}} da \, a^2 \, e^{-S_{\text{int}}} \, e^{-2a^2}, \quad S_{\text{int}} = \sum_{n=2}^{\infty} (-1)^n \frac{2(4^n - 4)}{n} \zeta(2n-1) \left(\frac{g^2}{8\pi^2} \right)^n a^{2n}. \quad (\text{A.3})$$

The Wilson loop operator is simply

$$W(a) = \cosh \left(\frac{g}{\sqrt{2}} a \right), \quad (\text{A.4})$$

and the normal ordered operator Ω_n in $\mathcal{N} = 4$ may be written, cf. (4.6),

$$:\Omega_n: = (-1)^n n! L_n^{1/2}(2a^2) = \frac{1}{2^{2n+1}} \frac{H_{2n+1}(a\sqrt{2})}{a\sqrt{2}}, \quad (\text{A.5})$$

where $L_n^{1/2}$ is the generalized Laguerre polynomial L_n^q with $q = 1/2$, and $H_n(z)$ are Hermite polynomials. As a check one can compute

$$\begin{aligned} \langle W : \Omega_n : \rangle &= \frac{1}{2^{2n+1}} \frac{4\sqrt{2}}{\sqrt{\pi}} \int_{\mathbb{R}} da \, a^2 \cosh \left(\frac{g}{\sqrt{2}} a \right) \frac{H_{2n+1}(a\sqrt{2})}{a\sqrt{2}} e^{-2a^2} \\ &= \left(\frac{g^2}{16} \right)^n e^{\frac{g^2}{16}} \left(1 + 2n + \frac{1}{8} g^2 \right), \end{aligned} \quad (\text{A.6})$$

in agreement with (4.24).

B General proof of eq. (4.13)

A general proof of the relation (4.13) that does not assume F to be represented in terms of $\mathcal{O}_m$ operators is as follows. For any g -independent function $F(a)$, let us denote by $\langle \cdot \rangle_0$ the Gaussian model expectation value after scaling $a \rightarrow a/g$

$$\langle F(a) \rangle_0 = \frac{\int Da \, F(a) \exp \left(-\frac{1}{g^2} \Omega_1 \right)}{\int Da \exp \left(-\frac{1}{g^2} \Omega_1 \right)}, \quad (\text{B.1})$$

Clearly, we have

$$\langle F(a) \Omega_n \rangle_0 = g^{2n} \langle F(ga) \Omega_n \rangle. \quad (\text{B.2})$$

Now, let us assume that (4.13) holds, with the expansion (4.7) in the l.h.s. We find

$$\begin{aligned}
 \langle F : \Omega_{n+1} : \rangle &= g^{2(n+1)} \frac{d}{dg^2} \left[g^{-2n} \langle F : \Omega_n : \rangle \right] = g^{2(n+1)} \frac{d}{dg^2} \left[g^{-2n} \langle F \sum_{k=0}^n c_k^{(n)} \Omega_k \rangle \right] \\
 &= g^{2(n+1)} \sum_{k=0}^n c_k^{(n)} \frac{d}{dg^2} \left[g^{-2n-2k} \langle F \Omega_k \rangle_0 \right] \\
 &= - \sum_{k=0}^n c_k^{(n)} (n+k) \langle F \Omega_k \rangle + g^{2(n+1)} \sum_{k=0}^n c_k^{(n)} g^{-2n-2k} \frac{1}{g^4} \left[\langle F \Omega_{k+1} \rangle_0 - \langle F \Omega_k \rangle_0 \langle \Omega_1 \rangle_0 \right] \\
 &= - \sum_{k=0}^n c_k^{(n)} (n+k) \langle F \Omega_k \rangle + \sum_{k=0}^n c_k^{(n)} \left[\langle F \Omega_{k+1} \rangle - \langle F \Omega_k \rangle \langle \Omega_1 \rangle \right]. \tag{B.3}
 \end{aligned}$$

This is true if

$$: \Omega_{n+1} : = \sum_{k=0}^n c_k^{(n)} \left[\Omega_{k+1} - (n+k+\alpha) \Omega_k \right]. \tag{B.4}$$

Indeed, replacing in the r.h.s. the expression of $c_k^{(n)}$ one finds the correct expression $: \Omega_{n+1} : = \sum_{k=0}^{n+1} c_k^{(n+1)} \Omega_k$. By induction, (4.13) is proved.

C Asymptotic expansion of the function $\log \mathcal{F}(\kappa)$

Let us expand the integral representation (6.7) at large $\kappa > 0$. We can split the leading piece by adding and subtracting it. This gives

$$\log \mathcal{F}(\kappa) = -\frac{\log 2}{\pi^2} \kappa + 4 \int_0^\infty \frac{dt}{t^2} \frac{J_0\left(\frac{t\sqrt{\kappa}}{\pi}\right) + t \frac{\sqrt{\kappa}}{\pi} J_1\left(\frac{t\sqrt{\kappa}}{\pi}\right) - 1}{e^t + 1}. \tag{C.1}$$

The second piece is a Mellin convolution $\int_0^\infty dt h(\lambda t) f(t)$, where $\lambda = \frac{\sqrt{\kappa}}{\pi}$ and we are interested in the asymptotic expansion for $\lambda \gg 1$. From the elementary Mellin transforms¹²

$$\begin{aligned}
 \mathcal{M}\left[\frac{1}{e^t + 1}\right] &= 2^{-s} (2^s - 2) \zeta(s) \Gamma(s), \\
 \mathcal{M}\left[J_0(t) - 1\right] &= \frac{2^{s-1} \Gamma(\frac{s}{2})}{\Gamma(1 - \frac{s}{2})}, \quad \mathcal{M}\left[J_1(t)\right] = \frac{2^{s-1} \Gamma(\frac{s}{2} + \frac{1}{2})}{\Gamma(\frac{3}{2} - \frac{s}{2})}, \tag{C.2}
 \end{aligned}$$

we derive by Mellin inversion the asymptotic expansion of the integral in (C.1) by computing the residue

$$\text{Res}_{s=0} \left[\lambda^{-s} \frac{(2^{s+2} - 1) \pi^{-s} (s+1) \zeta(s+2)}{(\cos(\pi s) - 1) \Gamma(1 - \frac{s}{2})^2} \right] = \log \lambda - \frac{7 \log 2}{3} - 12 \zeta'(-1). \tag{C.3}$$

Adding the leading term, this gives

$$\log \mathcal{F}(\kappa) = -\frac{\log 2}{\pi^2} \kappa + \frac{1}{2} \log \left(\frac{\kappa}{\pi^2} \right) - \frac{7 \log 2}{3} - 12 \zeta'(-1) + \dots, \tag{C.4}$$

where dots stand for contributions that vanish with $\kappa \rightarrow \infty$ faster than any inverse power of κ .¹³

¹²As usual, $\mathcal{M}[f(t)] = \int_0^\infty dt t^{s-1} f(t)$.

¹³To give an idea of how fast they vanish, their weight relative to the leading term is already about 10^{-15} at $\kappa = 1000$.

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