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Universality of Toda equation in $\mathcal{N} = 2$ superconformal field theories

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Abstract: We show that extremal correlators in all Lagrangian $\mathcal{N} = 2$ superconformal field theories with a simple gauge group, when suitably defined the $\mathbb{S}^4$, are governed by the same universal Toda system of equations, which dictates the structure of extremal correlators to all orders in the perturbation series. A key point is the construction of a convenient orthogonal basis for the chiral ring, by arranging towers of operators in order of increasing dimension, which has the property that the associated two-point functions satisfy decoupled Toda chain equations. We explicitly verify this in all known SCFTs based on $\mathrm{SU}(N)$ gauge groups as well as in superconformal QCD based on orthogonal and symplectic groups. As a by-product, we find a surprising non-renormalization property for the $\mathcal{N} = 2$ $\mathrm{SU}(N)$ SCFT with one hypermultiplet in the rank-2 symmetric representation and one hypermultiplet in the rank-2 antisymmetric representation, where the two-loop terms of a large class of supersymmetric observables identically vanish.

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1 Introduction and summary

Correlation functions are central quantities in Quantum Field Theory. While in general one has to resort to sophisticated perturbative methods to compute them at multiloop level, in supersymmetric theories some correlation functions can be computed exactly, thus offering a unique window into the structure of gauge field theories. This is, in particular, the case of extremal correlation functions of chiral primary operators (CPOs) in four-dimensional superconformal field theories (SCFTs) with $\mathcal{N}=2$ supersymmetry and freely generated chiral rings [1–5].

CPOs are operators annihilated by all the Poincaré supercharges of one chirality. It has been argued that CPOs are all scalar operators. Moreover, in SCFTs defined on $\mathbb{R}^4$, because of R-charge conservation, the structure constants of the OPE are true constants and hence the operators are endowed with a ring structure, the so-called chiral ring. Because of this, 2- and 3-point functions contain all the information about the ring. In the following we will be interested on 2-point functions,

$$\langle O_I(x)\overline{O}_{\overline{J}}(0)\rangle = \frac{g_{I\overline{J}}}{|x|^{\Delta_I+\Delta_{\overline{J}}}}\,\delta_{\Delta_I,\Delta_{\overline{J}}}\,, \tag{1.1}$$

where $g_{I\overline{J}}$ is in general a non-trivial function of the Yang-Mills coupling constant.

The chiral ring is an interesting object *per se* with possibly a very rich geometry. Indeed, very recently it has been found that such rings need not be freely generated [6–8]. In this paper, we will consider theories with freely generated chiral rings, where there are no constraints on the CPOs.

Given an SCFT, the space of all possible marginal deformations defines the conformal manifold. It turns out that exact marginal deformations preserving $\mathcal{N}=2$ supersymmetry are CPOs of dimension 2. Thus, the matrix $g_{I\overline{J}}$ restricted to the I, $\overline{J}$ corresponding to dimension 2 CPOs, known as Zamolodchikov metric, has the interpretation of a metric on the conformal manifold. It is possible to show that the Kähler potential for this metric coincides, up to a numerical factor, with the logarithm of the $\mathbb{S}^4$ partition function [9], which in turn can be exactly computed through localization [10]. This implies an important connection between the conformal manifold, the sphere partition function and the correlation functions of CPOs, which was beautifully explored in a series of papers starting with [1]. For the purpose of this note, it will suffice to recall that extremal correlators of CPOs can be exactly computed through localization. A crucial insight in [5] is that the $\mathbb{R}^4 \leftrightarrow \mathbb{S}^4$ mapping induces, through the conformal anomaly, a mixing among operators of any chosen Δ with lower dimensional ones. This mixing can be disentangled through C (GS) orthogonalization with respect to all lower dimensional operators.

Elaborating on this, there has recently been conceptual and technical progress in understanding different properties of correlation functions of CPOs. This includes, in particular, exact results in the large N, planar limit [11–14], where instanton contributions are exponentially suppressed, comparison between localization and field theory computations [15] and correlation functions involving Wilson loop operators [12, 16, 17]. More recently, extremal correlators have been computed in a novel large R charge, double-scaling limit [18], further investigated in [19] by exploiting the Toda equation. In some cases, the large R charge limit was shown to admit a precise effective field theory description [20, 21].

In [1] it was shown that, when regarding the 2-point correlators as functions of the marginal couplings, a beautiful structure analogous to the 2d tt^* equations emerges. While these equations hold on general grounds, in the following we will concentrate on Lagrangian theories based on simple gauge groups (with freely generated chiral ring), for which there is only one marginal coupling, namely the Yang-Mills coupling itself $\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{g^2}$. In the case of SU(2) superconformal QCD (SQCD), it was shown in [1, 4] that the underlying structure is that of a semi-infinite Toda chain. This structure naturally emerges from the exact computation of the correlation functions through localization as a consequence of the algebraic structure of the Gram-Schmidt orthogonalization.

The decoupled Toda chain structure was suggested to extend to SU(N) SQCD based on a two-loop computation in [4]. In this paper, this was conjectured to hold true to arbitrary loops provided a sufficient condition, which amounts to a certain no-mixing ansatz, is satisfied. This condition was in turn proved not to hold beyond two loops by explicit calculation in the SU(3), SU(4) cases in [5], concluding that the decoupled Toda chain structure would fail already at three loops. However, very recently, it was shown in [18] that, at least for the subsector of operators of the form $(\mathrm{Tr}\phi^2)^n$, the extremal correlators do satisfy a Toda equation. This fact was further explored and extended to the next family

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of operators of the form $(\mathrm{Tr}\phi^2)^n \mathrm{Tr}\phi^3$ in [19], where it was argued that also for the next one $(\mathrm{Tr}\phi^2)^n \mathrm{Tr}\phi^4$ a decoupled Toda chain would be obtained upon full orthogonalization.

In this paper we show that, when the theory is suitably defined on the sphere, all extremal correlators of CPOs do satisfy decoupled Toda chain equations. More explicitly, it is possible to define a version of the theory on the $\mathbb{S}^4$ such that CPO's with different quantum numbers are orthogonal and, at the same time, their correlators satisfy decoupled Toda chains. The key in establishing this fact is that there is a natural order for orthogonalization which, to best of our knowledge, has not been considered before in the literature on multivariate orthogonal polynomials. To implement it, we consider as seeds the set of all CPOs O_I not including those of the form $(\mathrm{Tr}\phi^2)^n$. The latter play a special role allowing to construct towers starting from the seeds O_I as $T_I = \{O_I,\, O_I\,(\mathrm{Tr}\phi^2),\, O_I\,(\mathrm{Tr}\phi^2)^2,\, \cdots\}$. The prescription is then to orthogonalize each tower with respect to all lower dimensional towers, and, within any given tower, orthogonalize operators sequentially from left to right. Note that this does not fix the order whenever two towers have the same seed dimension. However it turns out that the order chosen to orthogonalize these does not matter, as any order gives rise to the same decoupled Toda chain. We refer to section 3 below for the detailed prescription. Once this is done, unnormalized extremal correlators of CPOs satisfy the decoupled Toda chain equations[1]

$$\partial_\tau \partial_{\bar\tau} \log \widetilde{G}_I = \frac{\widetilde{G}_{I+\mathbf{1}}}{\widetilde{G}_I} - \frac{\widetilde{G}_I}{\widetilde{G}_{I-\mathbf{1}}}\,. \tag{1.2}$$

These equations in fact encapsulate an infinite number of semi-infinite Toda chains and, consequently, they need to be supplemented with appropriate boundary conditions. We argue that *the same decoupled Toda system* (1.2) *dictates the structure of the perturbation series of extremal correlators in any Lagrangian CFT based on a simple gauge group.*

We have explicitly checked this fact in all SCFTs based on $\mathrm{SU}(N)$ gauge groups (as classified in [23]) as well as in SQCD and $\mathcal{N}=4$ SYM based on SO and Sp groups at least to the first term non-linear in Riemann ζ functions. When computationally possible, we have checked the decoupled Toda chain equations beyond 10 loops. The decoupled Toda chains that we find are precisely those in eq. (1.2) in all cases.

A connection between the Toda equation and matrix models has been known from very early times [22]. In this paper we adapt this connection to the present case of extremal correlators in $\mathcal{N}=2$ SCFTs to provide an analytic proof of our findings.

Finally, in section 4.1 we point out an intriguing non-renormalization property for the class of $\mathrm{SU}(N)$ $\mathcal{N}=2$ SCFTs with matter consisting in one hypermultiplet in the rank-2 symmetric representation and one hypermultiplet in the rank-2 antisymmetric representation: the two-loop terms of all extremal correlators exactly vanish. The corresponding perturbative expansions do not contain terms with $\zeta(3)$ coefficients.

On purely mathematical grounds, we are finding that for any matrix model corresponding to an arbitrary $\mathcal{N}=2$ CFT it is possible to introduce multivariate orthogonal polynomials such that their correlators satisfy decoupled Toda chains. To the best of our knowledge,

[1] We use a multi-index notation explained in section 3. In particular $\mathbf{1} = (1, 0, \ldots, 0)$.

this remarkable fact has not appeared in the literature on multivariate orthogonal polynomials. As discussed in section 2.3, the relevant orthogonal basis involves mixing with higher dimensional operators. Since higher dimensional couplings amount to couplings with positive powers of the sphere radius, the flat space limit of our operators is not a priori well-defined. In physical terms, this implies that strictly speaking we are computing extremal correlators for a basis of operators for the theory defined on the four-sphere. It is nevertheless interesting that the final extremal correlators satisfying decoupled Toda equations do not depend on the radius of the sphere and thus have a smooth flat space limit. This may suggest that there could be an underlying Toda structure also for the theory on $\mathbb{R}^4$. In the simplest case of $\mathcal{N} = 4$ theory, all higher dimensional couplings identically vanish and the orthogonal towers introduced in [1, 4, 5] naturally emerge from our orthogonalization order.

Our main result, namely that for any CFT on $\mathbb{S}^4$ the same system of decoupled Toda chains governs the perturbative expansion of correlators, opens the door for many future investigations. We leave these issues open for future studies.

2 Introductory examples

To set the stage, recall that we are interested in extremal correlators of chiral primary operators (CPOs) in four-dimensional SCFTs on $\mathbb{R}^4$ in perturbation theory (that is, in the sector with zero instanton number). Moreover, we will concentrate on Lagrangian $\mathcal{N} = 2$ theories with freely generated chiral rings based on a simple gauge group. The latter contains a scalar ϕ. It is convenient to introduce the notation:

$$\phi_k \equiv \mathrm{Tr}\phi^k \,. \tag{2.1}$$

Note that the range of possible k's is set by the gauge algebra, for instance, in SU(N), $k = 2, \cdots, N$ (the operators ϕ_j with $j > N$ can be expressed in terms of the fundamental invariants).

Due to superconformal invariance, the computation of correlation functions of CPOs on $\mathbb{R}^4$ can be mapped to the computation of correlation functions in the $\mathbb{S}^4$ matrix model. However one must disentangle the additional mixing of operators when going to the $\mathbb{S}^4$ due to the conformal anomaly. As argued in [5], this can be done by a Gram-Schmidt orthogonalization. Let us first study this procedure in the simplest SU(2) and SU(3) cases, which already exhibit some key features.

2.1 The SU(2) case

Let us consider, for definiteness, superconformal QCD (SQCD) with gauge group SU(2) (that is, SU(2) theory with 4 fundamental massless hypermultiplets). Note that results for SU(2) $\mathcal{N} = 4$ SYM can be easily extracted by simply considering the leading term in the perturbative expansion in powers of g (this is because $\mathcal{N} = 4$ correlators are not renormalized).

For SU(2) we have a single Casimir invariant ϕ_2. Hence one would expect that the CPOs form a one-dimensional tower $O_n = \phi_2^n$ starting with the identity. However, when going to the $\mathbb{S}^4$, due to the conformal anomaly, O_n, O_m for $n \neq m$ are not anymore orthogonal. The mixture can be disentangled by a Gram-Schmidt procedure, which in this case amounts to writing

$$O_0 = 1\,, \qquad O_1 = \phi_2 - \alpha_1^0\, O_0\,, \qquad O_2 = \phi_2^2 - \alpha_2^1\, O_1 - \alpha_2^0\, O_0\,, \qquad \cdots \tag{2.2}$$

where the α_i^j are easily fixed by demanding orthogonality of the O_i's with all the other O_j for $j < i$ with respect to the inner product defined by the $\mathbb{S}^4$ matrix model. Then, by construction $\langle O_n | O_m \rangle = \widetilde{G}_n\, \delta_{nm}$, so that the desired correlators on $\mathbb{R}^4$ are obtained by simply normalizing $G_n = \frac{\widetilde{G}_n}{\widetilde{G}_0}$. As a consequence of this structure, and as first anticipated in [3], these correlators satisfy a Toda equation of the form

$$\partial_\tau \partial_{\bar{\tau}} \log \widetilde{G}_n = \frac{\widetilde{G}_{n+1}}{\widetilde{G}_n} - \frac{\widetilde{G}_n}{\widetilde{G}_{n-1}}\,. \tag{2.3}$$

This can be put in a more familiar form by a defining $G_n = e^{q_n}$. Then one has a system of equations representing a Toda chain [3]. The emergence of this Toda chain can be deduced from the tt^* equations of [1]. An alternative approach, relying on localization, was presented in [5]. As discussed in that reference, the Gram-Schmidt procedure can be efficiently encoded as a ratio of certain determinants, which leads to the emergence of the Toda equation described above (we refer to [5] for the complete argument). The Toda equation holds for any SCFT with gauge group SU(2).

2.2 The SU(3) case

Let us now consider the case of SQCD with SU(3) gauge group, again restricting to perturbation theory. Just as in the SU(2) case, the $\mathcal{N} = 4$ correlators can be easily extracted as the leading terms in the $g \to 0$ limit.

Results for the first few operators in this case can be found in [4, 5]. In this case the Casimir invariants are ϕ_2 and ϕ_3. Hence the operators resulting from the diagonalization process starting with a monomial $\phi_2^n \phi_3^m$ can be labelled as $O_{(n,m)}$. The SU(3) case exhibits a new feature with respect to the SU(2) case. Since the Gram-Schmidt procedure is designed to remove mixtures with lower-dimensional operators arising from the anomalous $\mathbb{S}^4 \leftrightarrow \mathbb{R}^4$ conformal mapping, when there is degeneracy (i.e. more than one operator at a given dimension), the matrix of correlators is typically non-diagonal. Indeed, for SU(3) SQCD this happens starting at dimension 6, where one has the operators ϕ_2^3 and ϕ_3^2. Let us first focus on the correlators for the lowest dimensional operators where this issue does not arise, in particular on $\widetilde{G}_{(0,0)}$, $\widetilde{G}_{(1,0)}$ and $\widetilde{G}_{(2,0)}$. It is straightforward to check that these satisfy the Toda equation[2]

$$\partial_\tau \partial_{\bar{\tau}} \log \widetilde{G}_{(1,0)} = \frac{\widetilde{G}_{(2,0)}}{\widetilde{G}_{(1,0)}} - \frac{\widetilde{G}_{(1,0)}}{\widetilde{G}_{(0,0)}}\,. \tag{2.4}$$

In fact, it is easy to compute these correlators to very high loop orders and check that the Toda equation holds (we have done this up to tenth loop order g^{20}). Motivated by this, it is reasonable to wonder whether the Toda equation should hold generically. However, one immediately encounters the problem alluded above, namely, that correlators are not diagonal

[2] There is a factor of 4^{Δ} with respect to [5] reabsorbed in the normalization of the operators as in [4].

Operator/Δ	0	1	2	3	4	5	6	7	8	9	10	$\dots$
$(n,0)$	1		ϕ_2		ϕ_2^2		ϕ_2^3		ϕ_2^4		ϕ_2^5	$\dots$
$(n,1)$				ϕ_3		$\phi_2\phi_3$		$\phi_2^2\phi_3$		$\phi_2^3\phi_3$		$\dots$
$(n,2)$							ϕ_3^2		$\phi_2\phi_3^2$		$\phi_2^2\phi_3^2$	$\dots$
$(n,3)$										ϕ_3^3		$\dots$
						$\vdots$	$\vdots$					

Table 1. Ordering the operators in SU(3) SQCD.

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since the diagonalization has only been performed with respect to lower dimensional operators. In order to remedy this, we could add one further step on top of the procedure of [5] and iteratively construct the operators by orthogonalizing with respect to all others with lower dimension, and as well those of the same dimension.[3] Note that, as in any orthogonalization procedure, this bears a certain ambiguity, since one has to choose an ordering to run the Gram-Schmidt algorithm. In order to devise the optimal strategy, and with an eye on the Toda equation, note that given an operator it gives rise to a full tower upon insertions of ϕ_2, which in turn is slightly special since its insertions, at the perturbative level, can be traded by derivatives with respect to τ (which is, together with the algebra of the Gram-Schmidt procedure, why the Toda equation emerged for SU(2)). This suggests to momentarily step back and consider instead organizing the operators as in table 1.

The ordering in table 1 suggests an strategy for such orthogonalization: choose a $\Delta_{\max}$ and orthogonalize the operators starting from those on the first row, from left to right, until the dimension is higher than $\Delta_{\max}$. Then move to the next row and iterate, including previous towers, until exhausting all operators of dimension smaller or equal than the chosen $\Delta_{\max}$.

Below we quote the first few correlators (up to $\Delta_{\max} = 6$) computed in this way (recall that, by construction, the matrix of correlators is diagonal). We introduce $F_{(m,n)}$ defined by

$$G_{(m,n)} = \left(\frac{g^2}{16\pi}\right)^{\Delta_{(m,n)}} F_{(m,n)} \,. \tag{2.5}$$

SU(3) superconformal theory with 6 fundamental hypermultiplets.

$$\begin{aligned}
F_{(0,0)} &= 1,\\
F_{(1,0)} &= 16-\frac{45\zeta(3)g^4}{2\pi^4}+\frac{425\zeta(5)g^6}{8\pi^6}+\frac{25\left(1188\zeta(3)^2-3577\zeta(7)\right)g^8}{768\pi^8}+\mathcal{O}(g^{10}),\\
F_{(2,0)} &= 640-\frac{2160\zeta(3)g^4}{\pi^4}+\frac{6375\zeta(5)g^6}{\pi^6}+\frac{25\left(24516\zeta(3)^2-67963\zeta(7)\right)g^8}{96\pi^8}+\mathcal{O}(g^{10}),\\
F_{(3,0)} &= 46080-\frac{272160\zeta(3)g^4}{\pi^4}+\frac{969000\zeta(5)g^6}{\pi^6}+\frac{15\left(325296\zeta(3)^2-876365\zeta(7)\right)g^8}{4\pi^8}+\mathcal{O}(g^{10}),\\
F_{(0,1)} &= 40-\frac{135g^4\zeta(3)}{2\pi^4}+\frac{6275g^6\zeta(5)}{48\pi^6}+\frac{25g^8\left(7452\zeta(3)^2-15533\zeta(7)\right)}{1536\pi^8}+\mathcal{O}(g^{10}),
\end{aligned}$$

[3]This strategy was adopted in [19] for the operators $(\mathrm{Tr}\phi^2)^n\mathrm{Tr}\phi^3$ in SU(3) and $(\mathrm{Tr}\phi^2)^n\mathrm{Tr}\phi^4$ in SU(4).

$$F_{(1,1)} = 1120-\frac{4410g^4\zeta(3)}{\pi^4}+\frac{144725g^6\zeta(5)}{12\pi^6}+\frac{35g^8\left(157356\zeta(3)^2-350665\zeta(7)\right)}{384\pi^8}+\mathcal{O}(g^{10}),$$
$$F_{(0,2)} = 6720-\frac{28350\zeta(3)g^4}{\pi^4}+\frac{139125\zeta(5)g^6}{2\pi^6}+\frac{1575\left(477\zeta(3)^2-854\zeta(7)\right)g^8}{8\pi^8}+\mathcal{O}(g^{10}).$$

It is straightforward to compute these correlators to an arbitrary order. Note that already at this 4-loop order there is non-linear dependence on the Riemann ζ function coefficients. We also note that $F_{(1,1)}$ correctly reproduces the correlator previously computed in (3.16) in [19].

One can now check that the Toda equations[4]

$$\partial_\tau\partial_{\bar\tau}\log\widetilde{G}_{(n,m)} = \frac{\widetilde{G}_{(n+1,m)}}{\widetilde{G}_{(n,m)}}-\frac{\widetilde{G}_{(n,m)}}{\widetilde{G}_{(n-1,m)}}\,, \tag{2.6}$$

are satisfied. We have checked this system of equations up to, and including 12-loop order $O(g^{24})$.

The above decoupled system of Toda chains hold in any SCFT with gauge group SU(3). Following the classification of [23], reviewed in section 4.1, there are three such theories, namely the SQCD case which we have just studied, the $\mathcal{N}=4$ case (where the result holds trivially, since it is akin to keeping only the leading terms in g in the SQCD case) and SU(3) with one rank-2 symmetric and one fundamental hypermultiplet (see next subsection). For this latter case, proceeding as described, one finds the correlators

$$\begin{aligned}
F_{(0,0)} &= 1\,,\\
F_{(1,0)} &= 16-\frac{25\zeta(5)g^6}{4\pi^6}+\frac{6125\zeta(7)g^8}{192\pi^8}-\frac{33075\zeta(9)g^{10}}{256\pi^{10}}\\
&\quad+\frac{175\left(740\zeta(5)^2+35497\zeta(11)\right)g^{12}}{12288\pi^{12}}+\mathcal{O}(g^{14})\,,\\
F_{(2,0)} &= 640-\frac{750\zeta(5)g^6}{\pi^6}+\frac{116375\zeta(7)g^8}{24\pi^8}-\frac{99225\zeta(9)g^{10}}{4\pi^{10}}+\\
&\quad+\frac{125\left(5160\zeta(5)^2+248479\zeta(11)\right)g^{12}}{256\pi^{12}}+\mathcal{O}(g^{14})\,,\\
F_{(3,0)} &= 46080-\frac{114000\zeta(5)g^6}{\pi^6}+\frac{900375\zeta(7)g^8}{\pi^8}-\frac{22524075\zeta(9)g^{10}}{4\pi^{10}}+\\
&\quad+\frac{375\left(116740\zeta(5)^2+5715017\zeta(11)\right)g^{12}}{64\pi^{12}}+\mathcal{O}(g^{14})\,,\\
F_{(0,1)} &= 40-\frac{925g^6\zeta(5)}{24\pi^6}+\frac{67375g^8\zeta(7)}{384\pi^8}-\frac{341775g^{10}\zeta(9)}{512\pi^{10}}\\
&\quad+g^{12}\left(\frac{649375\zeta(5)^2}{9216\pi^{12}}+\frac{23321375\zeta(11)}{9216\pi^{12}}\right)+O\left(g^{14}\right)\,,\\
F_{(1,1)} &= 1120-\frac{19075g^6\zeta(5)}{6\pi^6}+\frac{1941625g^8\zeta(7)}{96\pi^8}-\frac{13307175g^{10}\zeta(9)}{128\pi^{10}}\\
&\quad+g^{12}\left(\frac{30265375\zeta(5)^2}{2304\pi^{12}}+\frac{2378674375\zeta(11)}{4608\pi^{12}}\right)+O\left(g^{14}\right)\,,
\end{aligned}$$

[4] Recall that the $\tilde{G}_{(n,m)}$'s represent the unnormalized correlators, with $G_{(n,m)}=\tilde{G}_{(n,m)}/\tilde{G}_{(0,0)}$.

$$F_{(0,2)} = 6720 - \frac{18375\zeta(5)g^6}{\pi^6} + \frac{1708875\zeta(7)g^8}{16\pi^8} - \frac{32645025\zeta(9)g^{10}}{64\pi^{10}} + \\ + \frac{6125\left(2405\zeta(5)^2 + 100089\zeta(11)\right)g^{12}}{256\pi^{12}} + \mathcal{O}(g^{14})\,. \tag{2.7}$$

Again, here we explicitly quote results up to the first non-linear order in the ζ's. It is nevertheless straightforward to go to any desired higher order, checking that indeed, the Toda equations are satisfied for the corresponding $\tilde{G}_{(n,m)}$.

Remarkably, (2.7) does not contain terms with $\zeta(3)$. This arises due to a surprising cancellation in the two-loop contribution to the partition function of the coefficient of $\zeta(3)$. Hence, any supersymmetric observable in this theory that can be computed from insertions in the localized (matrix model) partition function will not have, in perturbation theory, any contribution proportional to any power of $\zeta(3)$.

2.3 A lesson

It is instructive to come back to the ordering prescription to run the Gram-Schmidt orthogonalization. For instance, up to dimension $\Delta_{\max} = 8$, in the SU(3) case such sequence is

$$(0,0) \to (1,0) \to (2,0) \to (3,0) \to (4,0) \to (0,2) \to (1,2)\,. \tag{2.8}$$

To begin with, one may wonder what would happen if one chose a different ordering. Consider, for example,

$$(0,0) \to (1,0) \to (2,0) \to (3,0) \to (0,2) \to (4,0) \to (1,2)\,. \tag{2.9}$$

In this case, one would find that the decoupled Toda equations fail to hold. Here operators have been arranged in order of increasing dimension, but this is not the correct order that leads to decoupled Toda chains. Explicit calculations will be given in section 3.4.

It should be noted that our sequence of orthogonalization implies that a given operator $O_I\phi_2^n$ may mix with a higher dimensional component of the upper tower.[5] When this happens, identifying the flat-theory counterpart of the resulting orthogonal operators seems to be problematic, particularly in considering the cutoff $\Delta_{\max} \to \infty$. This is despite the fact that the corresponding correlation functions are well defined in the infinite radius limit and satisfy Toda equations (1.2).

An important open problem is to understand the extent to which the universal Toda structure discovered here is a property which is inherent to the theory on $\mathbb{S}^4$, or it also implies a Toda structure on $\mathbb{R}^4$ correlation functions of suitably defined operators. Irrespective of this, it is nonetheless remarkable that the same infinite system of decoupled Toda chain equations dictate the structure of extremal correlators on $\mathbb{S}^4$ obtained with the Toda order, for all lagrangian CFTs.

Interestingly, the universal Toda structure can already be exhibited with no need of a full orthogonalization of the CPOs belonging to different towers. The main point is that a Toda chain exists in each tower of operators of the form $O_I\phi_2^n$. Once these operators are orthogonalized through GS procedure by arranging them in order of increasing dimension,

[5]We thank Bruno Le Floch for useful comments on that point.

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the resulting correlation functions satisfy the same Toda chain equation (1.2) independently of the seed O_I. In this way, one can exhibit the Toda structure with no need of mixing with higher dimensional operators. Of course, the complete orthogonal basis is eventually needed in order to determine the complete set of $\mathbb{R}^4$ correlation functions of CPOs.[6]

A proof is as follows. Consider the first tower $\{\phi_2^n\}$. In [18], in the context of SU(N) SQCD, $N \leq 5$, it was shown that orthogonalizing these operators by arranging them in order of increasing dimension leads to the same Toda equation (1.2). The underlying reason can be understood from the form of the partition function in the zero-instanton sector,

$$Z = \int [da] e^{-2\pi \mathrm{Im}\tau \phi_2} f(\phi_2, \phi_3, \phi_4, \dots) \,. \tag{2.10}$$

Two-point correlation functions of ϕ_2^n are obtained by differentiation with respect to τ. As a result, the final diagonal correlators are given by the determinant formula, which is known to satisfy (1.2). Next, consider a tower with seed O_I, i.e. $\{O_I \phi_2^n\}$. Suppose we orthogonalize only operators belonging to this tower. So we consider correlators with a single insertion of $O_I O_I$ and insertions $\phi_2^n\ \phi_2^m$. Since our proof for the first tower $\{\phi_2^n\}$ does not rely on the specific form of f, clearly the same proof applies now, replacing f by $g(\phi_2, \phi_3, \dots) \equiv O_I O_I f(\phi_2, \phi_3, \dots)$.

Having shown that each orthogonalized tower satisfies Toda equations (1.2), the next problem is to complete the orthogonalization among operators belonging to different towers *without spoiling the Toda structure of the correlators*. This is highly non-trivial, because correlators will change once each operator gets mixed with operators of other towers. We claim that the Toda structure is maintained by our ordering described above (see general definition in section 3.1). On the contrary, a sequence of orthogonalization where one orthogonalizes a given operator with respect to *all* operators of lower dimensions (including those belonging to different towers) fails already at three loops, as shown in [5]. Orthogonalizing with respect to operators of *lower or equal* dimensions, as in [19], also fails (see section 3.4). This justifies our choice of ordering given above.

In general, there may be more than one tower at a given dimension. For the cases with SU(2) and SU(3) gauge groups discussed above this degeneration does not occur. A simple example is given by the towers corresponding respectively to ϕ_3^4 and to ϕ_4^3 in SU(N) for $N \geq 4$. As described below, we find that, when these degenerate towers occur, either ordering between them leads to the same decoupled Toda chains structure. Our claim is that the (non-normalized) correlators $\widetilde{G}_I = \langle O_I \, O_I \rangle$ obtained using this ordering of operators satisfy the infinite set of Toda equations (2.6) and that these can be packaged in the compact form (1.2).

It is also important to stress that the algorithm is an *orthogonalization* and not an *orthonormalization*. The normalization of the operators is already fixed by the convenient choice of (coupling independent) three-point functions, which take values 1 or 0.

[6]In general, different sequences in GS orthogonalization may give different correlation functions. This seems to reflect the ambiguity in the normal ordering prescription in defining $\mathbb{R}^4$ composite operators; see e.g. [15] for calculations in the SQCD context.

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Note that the above prescription is independent of the gauge group. In the next section, we show that the algorithm generalizes for arbitrary Lagrangian SCFTs based on a simple gauge group.

3 Extremal correlators and generalized Toda equation

3.1 The Toda orders and the main equation

Let us consider any four-dimensional (Lagrangian) $\mathcal{N}=2$ CFT with a simple gauge group (therefore, we consider a connected Lie group). We refer to appendix A for notations. The chiral ring is freely generated, which means that we can choose a (linear) basis of monomial operators that we call (ϕ_I) labeled by a multi-index $I \in \mathcal{I} := \mathbb{N}^r$, defined by

$$\phi_I = \prod_{k \in D(\mathfrak{g})} \phi_k^{I_k} \,. \tag{3.1}$$

This last equation crucially uses the fact that the ring is freely generated: any monomial operator admits a *unique* expression in terms of the ϕ_k, so the family of operators $(\phi_I)_{I\in\mathcal{I}}$ is indeed a linear basis. The chiral ring structure constants are defined by the OPE

$$\phi_I(x)\phi_J(0) = \sum_{K\in\mathcal{I}} C_{IJ}^K \phi_K(0) + \dots \,. \tag{3.2}$$

Thanks to the monoid structure of $\mathcal{I}$, the structure constants are trivial,

$$C_{IJ}^K = \delta_{I+J,K} \,. \tag{3.3}$$

We define a family of total orders, that we call the *Toda orders*, on the set of indices $\mathcal{I}$ as follows. Let us denote by $\mathcal{S}$ the map that associates to each $I = (n_1, \dots, n_r) \in \mathcal{I}$ with its seed,

$$\mathcal{S}(I) = (0, n_2, \dots, n_r) \,. \tag{3.4}$$

Then the order is defined by $I < I'$ if

- $\Delta(\mathcal{S}(I)) < \Delta(\mathcal{S}(I'))$ or
- $\Delta(\mathcal{S}(I)) = \Delta(\mathcal{S}(I'))$ and $\mathcal{S}(I) \prec \mathcal{S}(I')$ or
- $\mathcal{S}(I) = \mathcal{S}(I')$ and $n_1 < n_1'$.

In this definition, $\prec$ is any total order on the $(r-1)$-tuples, and this is why there are several Toda orders when the rank of the gauge group is ≥ 3. The smallest non-vanishing element is $(1, 0, \dots, 0)$ which we denote by the shorthand $\mathbf{1}$. These Toda orders formalize what we described in the previous section: operators are organized in towers of the form

$$T_I = \{O_I,\, O_I\, \phi_2,\, O_I\, \phi_2^2,\, \cdots\} \,, \tag{3.5}$$

and the towers are ordered by the conformal dimension of their seed. When various seeds have the same dimension, we order them arbitrarily using $\prec$ (one can choose e.g. the lexicographic order).

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We can deform the theory on the sphere $\mathbb{S}^4$, introducing parameters τ_k for $k \in D(\mathfrak{g})$. Here $D(\mathfrak{g}) \in \mathbb{N}^r$ is the set of degrees of fundamental invariants of $\mathfrak{g}$, and we identify $\tau \equiv \tau_2$. The partition function is (compare with the undeformed partition function in appendix A)

$$Z^{\mathcal{T}}[\tau_k, \bar{\tau}_k] = \int_{\mathfrak{h}} [\mathrm{d}a] \Delta(a) Z^{\mathcal{T}}_{1-\mathrm{loop}}(a) \exp\left(- \sum_{k \in D(\mathfrak{g})} 2\pi^{k/2} \mathrm{Im}\, \tau_k\, \phi_k \right) . \tag{3.6}$$

Likewise, we introduce the differential operators

$$\partial_I = \prod_{k \in D(\mathfrak{g})} \left(\frac{\partial}{\partial \tau_k} \right)^{I_k} , \qquad \bar{\partial}_I = \prod_{k \in D(\mathfrak{g})} \left(\frac{\partial}{\partial \bar{\tau}_k} \right)^{I_k} . \tag{3.7}$$

Then one can compute the (infinite) matrix of *unnormalized* correlators of operators on $\mathbb{S}^4$, which is given by derivatives of the sphere partition function $Z^{\mathcal{T}}[\tau_k, \bar{\tau}_k]$ with respect to the couplings and setting $\tau_{k'} = \bar{\tau}_{k'} = 0$ for $k' \in D(\mathfrak{g}) - \{2\}$. We call this matrix $\widetilde{M}$, so that

$$\widetilde{M}_{IJ} = \left(\partial_I \bar{\partial}_J Z^{\mathcal{T}} \right) |_{\tau_{k'>2} = \bar{\tau}_{k'>2} = 0} \,. \tag{3.8}$$

We then apply an unnormalized Gram-Schmidt orthogonalization process to $\widetilde{M}$, in the order defined above on $\mathcal{I}$. The eigenvalues of the matrix obtained in that way are denoted $\widetilde{G}_I = \widetilde{G}_I(\tau, \bar{\tau})$. In the next subsection, we prove that these obey the generalized Toda equation

$$\partial_\tau \partial_{\bar{\tau}} \log \widetilde{G}_I = \frac{\widetilde{G}_{I+\mathbf{1}}}{\widetilde{G}_I} - \frac{\widetilde{G}_I}{\widetilde{G}_{I-\mathbf{1}}} \,. \tag{1.2}$$

Note that this equation is valid for all $I \in \mathcal{I}$ such that $I - \mathbf{1}$ exists (i.e. which is not a seed $\mathcal{S}(I')$).[7] Therefore, it must be supplemented with boundary conditions, and extremal correlators of CPOs depend on the theory under study only through these boundary conditions and the degrees of fundamental invariants.

3.2 Analytic proof of the decoupled Toda chain equations

In this subsection, we give a proof of (1.2), following a similar strategy as in [22]. We introduce the notation $q_I = \log \widetilde{G}_I$ for any index I. After orthogonalization following a Toda order, the operators O_I can be written as

$$O_I = \phi_I + \sum_{J<I} \alpha_{I,J}(t) \phi_J \,, \qquad \text{with} \qquad \langle O_I O_{I'} \rangle = e^{q_I} \delta_{I,I'} \,. \tag{3.9}$$

Here the correlators are defined by

$$\langle X \rangle = \int [\mathrm{d}a] \Delta(a)\, Z_{1-\mathrm{loop}}(\vec{a})\, e^{-t\vec{a}^2}\, X \,, \tag{3.10}$$

[7] The Toda equation presented here can be seen as a generalization of the usual semi-infinite Toda chain, the difference lying in the structure of the ordered set of indices $\mathcal{I}$. While the usual semi-infinite Toda chain is labeled by the integers $\mathbb{N}$, which as a totally ordered set form the infinite ordinal ω, equation (1.2) is defined on $\mathcal{I}$, which is the ordinal ω^2.

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so that taking a derivative with respect to t gives

$$\partial_t\langle X\rangle = \langle\partial_t X\rangle - \langle\phi_{\mathbf{1}} X\rangle\,. \tag{3.11}$$

Taking a derivative of the normalization (3.9) with respect to t, and using the rule above, we find

$$\langle\partial_t O_I O_{I'}\rangle + \langle O_I\partial_t O_{I'}\rangle - \langle O_I\phi_{\mathbf{1}} O_{I'}\rangle = e^{q_I}\partial_t q_I\delta_{I,I'}\,. \tag{3.12}$$

From that relation, we obtain

$$\partial_t O_I = \sum_{J<I}\frac{\langle O_I\phi_{\mathbf{1}} O_J\rangle}{\langle O_J O_J\rangle}O_J\,. \tag{3.13}$$

Now we investigate the operator $O_I\phi_{\mathbf{1}}$. We can decompose it as[8]

$$O_I\phi_{\mathbf{1}} = O_{I+\mathbf{1}} + \sum_{J\leq I}\frac{\langle O_I(\phi_{\mathbf{1}} O_J)\rangle}{\langle O_J O_J\rangle}O_J \tag{3.14}$$

and, similarly, the term within brackets in the above equation can be expanded as

$$\phi_{\mathbf{1}} O_J = O_{J+\mathbf{1}} + \sum_{K\leq J}\frac{\langle O_J\phi_{\mathbf{1}} O_K\rangle}{\langle O_K O_K\rangle}O_K\,. \tag{3.15}$$

Substituting this into (3.14) and using the orthogonality relation we obtain

$$\begin{aligned}
O_I\phi_{\mathbf{1}} &= O_{I+\mathbf{1}} + \sum_{J\leq I}\frac{O_J}{\langle O_J O_J\rangle}\left(\langle O_I O_{J+\mathbf{1}}\rangle + \sum_{K\leq J}\frac{\langle O_J\phi_{\mathbf{1}} O_K\rangle}{\langle O_K O_K\rangle}\langle O_I O_K\rangle\right)\\
&= O_{I+\mathbf{1}} + \sum_{J\leq I}\frac{\langle O_I O_{J+\mathbf{1}}\rangle}{\langle O_J O_J\rangle}O_J + \sum_{K\leq J\leq I}\frac{\langle O_I O_K\rangle\langle O_J\phi_{\mathbf{1}} O_K\rangle}{\langle O_J O_J\rangle\langle O_K O_K\rangle}O_J\\
&= O_{I+\mathbf{1}} + \frac{\langle O_I O_I\rangle}{\langle O_{I-\mathbf{1}} O_{I-\mathbf{1}}\rangle}O_{I-\mathbf{1}} + \frac{\langle O_I\phi_{\mathbf{1}} O_I\rangle}{\langle O_I O_I\rangle}O_I\\
&= O_{I+\mathbf{1}} + e^{q_I - q_{I-\mathbf{1}}}O_{I-\mathbf{1}} + e^{-q_I}\langle O_I\phi_{\mathbf{1}} O_I\rangle O_I\,.
\end{aligned} \tag{3.16}$$

Next, we substitute (3.16) back into (3.13) and get

$$\partial_t O_I = e^{q_I - q_{I-\mathbf{1}}}O_{I-\mathbf{1}}\,. \tag{3.17}$$

In addition, (3.12) for $I = I'$ gives

$$\partial_t q_I = -e^{-q_I}\langle O_I\phi_{\mathbf{1}} O_I\rangle\,. \tag{3.18}$$

Thus, (3.16) finally becomes

$$O_I\phi_{\mathbf{1}} = O_{I+\mathbf{1}} + e^{q_I - q_{I-\mathbf{1}}}O_{I-\mathbf{1}} - \partial_t q_I O_I\,. \tag{3.19}$$

Taking one more derivative, we obtain:

[8] Note that this is the crucial equation. In particular, this is the only place where we use the fact that our order is Toda.

- On the l.h.s.

$$\partial_t O_I \phi_{\mathbf{1}} = e^{q_I - q_{I-\mathbf{1}}} \left(O_I + e^{q_{I-\mathbf{1}} - q_{I-\mathbf{1}-\mathbf{1}}} O_{I-\mathbf{1}-\mathbf{1}} - \partial_t q_{I-\mathbf{1}} O_{I-\mathbf{1}} \right) \tag{3.20}$$

- On the r.h.s.

$$e^{q_{I+\mathbf{1}} - q_I} O_I + \partial_t e^{q_I - q_{I-\mathbf{1}}} O_{I-\mathbf{1}} + e^{q_I - q_{I-\mathbf{2}}} O_{I-\mathbf{2}} - \partial_t^2 q_I O_I - \partial_t q_I e^{q_I - q_{I-\mathbf{1}}} O_{I-\mathbf{1}} \tag{3.21}$$

Identifying the coefficients of O_I in (3.20) and (3.21), we obtain

$$\partial_t^2 q_I = e^{q_{I+\mathbf{1}} - q_I} - e^{q_I - q_{I-\mathbf{1}}}\,, \tag{3.22}$$

Q.E.D.

3.3 Finite-dimensional implementation

In practice, for concrete computations in perturbation theory, one has to implement orthogonalization up to some maximal conformal dimension Δ_{max}. Because of the structure of the Toda orders, Δ_{max} has to be chosen carefully depending on the order of expansion in perturbation theory: if the sphere partition function is computed to precision $O(g^{2d})$, then one needs

$$\Delta_{\text{max}} \geq d\,. \tag{3.23}$$

Then the algorithm can be formalized as follows:

1. List all the operators ϕ_I with conformal dimension $\Delta \leq \Delta_{\text{max}}$.

2. Order these operators following a Toda order.

3. Compute the matrix $\tilde{M}$ in that basis.

4. Perform a Gram-Schmidt orthogonalization algorithm (without normalization) on $\widetilde{M}$.

The diagonal elements of the matrix thus obtained are the correlators $\widetilde{G}_I$.

3.4 Comparison with other orderings

We stress that the algorithm presented here is not only sufficient to obtain the Toda equation, but it is also necessary, in the sense that any ordering that is not a Toda order will fail to give extremal correlators obeying the decoupled Toda Chains (1.2). In the cases where no two seeds have the same conformal dimension, there is a unique Toda order and we claim that this unique Toda order is the only ordering that gives correlators satisfying (1.2).[9]

To conclude this section, we will now carry out some precision tests that allow to compare the correlators obtained with different orderings. In order to perform very high loop order calculations, one trick is to formally replace some of the $\zeta(2n-1)$ in equation (A.5)

[9] On the other hand, we have checked on one example the degree of freedom left by the choice of $\prec$. Namely, we computed the first terms of the two chains at dimension $\Delta = 12$ in the SU(4) theory (see table 2) of type **(A5)** — defined in section 4.1 — and checked, in a 16-loop computation, that the two orderings of $(0,4,0)$ and $(0,0,3)$ are both compatible with (1.2).

Operator /Δ	0	1	2	3	4	5	6	7	8	9	10	11	12
$(n,0,0)$	1		ϕ_2		ϕ_2^2		ϕ_2^3		ϕ_2^4		ϕ_2^5		ϕ_2^6
$(n,1,0)$				ϕ_3		$\phi_2\phi_3$		$\phi_2^2\phi_3$		$\phi_2^3\phi_3$		$\phi_2^4\phi_3$	
$(n,0,1)$					ϕ_4		$\phi_2\phi_4$		$\phi_2^2\phi_4$		$\phi_2^3\phi_4$		$\phi_2^4\phi_4$
$(n,2,0)$							ϕ_3^2		$\phi_2\phi_3^2$		$\phi_2^2\phi_3^2$		$\phi_2^3\phi_3^2$
$(n,1,1)$								$\phi_3\phi_4$		$\phi_2\phi_3\phi_4$		$\phi_2^2\phi_3\phi_4$	
$(n,0,2)$									ϕ_4^2		$\phi_2\phi_4^2$		$\phi_2^2\phi_4^2$
$(n,3,0)$										ϕ_3^3		$\phi_3^3\phi_2$	
$(n,2,1)$											$\phi_3^2\phi_4$		$\phi_3^3\phi_4$
$(n,4,0)$													ϕ_3^4
$(n,0,3)$													ϕ_4^3

Table 2. Schematic representation of the operators ϕ_I for SU(4) theory (rank 3). The ordering of the operators is obtained by going through the (infinite) lines one after another. In practical computations, one introduces a cut-off $\Delta_{\max}$ on the right; then the number of operators is finite, and the orthogonalization should be done line after line (and not column after column). While immaterial to obtain the Toda structure, we have chosen an ordering for ϕ_3^4 and ϕ_4^3.

by 0; as a by-product, the results have a more manageable size and can be reported here to high enough precision so that the effects of the orderings appear. Here we consider SU(3) SQCD theory and focus on the terms in the perturbation series involving only $\zeta(5)$ coefficients, which can be achieved by formally replacing $\zeta(2n-1)$ by 0 for any $n \neq 3$. We analyze in this theory the influence of the order for implementing the GS procedure on the correlators labeled $(4,0)$, $(0,2)$, $(1,2)$, $(2,2)$. We compare

1. The (unique) Toda order

$$\{(n,0)|n \in \mathbb{N}\} \cup \{(0,2),(1,2),(2,2)\}\,. \tag{3.24}$$

   The correlators computed using this order will be denoted with the letter F, following the convention (2.5).

2. Arranging operators in order of increasing conformal dimension

$$\{(0,0),(1,0),(2,0),(3,0),(0,2),(4,0),(1,2),(5,0),(2,2)\}\,. \tag{3.25}$$

   The correlators computed using this order will be denoted F'.

We find

$$\begin{aligned}
F_{(4,0)} &= 5160960 + \frac{194208000 g^6 \zeta(5)}{\pi^6} + \frac{6686216250 g^{12} \zeta(5)^2}{\pi^{12}} + O\left(g^{15}\right)\\
F'_{(4,0)} &= 5160960 + \frac{194208000 g^6 \zeta(5)}{\pi^6} + \frac{6685625625 g^{12} \zeta(5)^2}{\pi^{12}} + O\left(g^{15}\right)\\
F_{(0,2)} &= 6720 + \frac{139125 g^6 \zeta(5)}{2\pi^6} + \frac{39265625 g^{12} \zeta(5)^2}{48\pi^{12}} + O\left(g^{18}\right)\\
F'_{(0,2)} &= 6720 + \frac{139125 g^6 \zeta(5)}{2\pi^6} + \frac{838534375 g^{12} \zeta(5)^2}{1024\pi^{12}} + O\left(g^{18}\right)
\end{aligned}$$

$$
\begin{aligned}
F_{(1,2)} &= 268800 + \frac{6478500 g^6 \zeta(5)}{\pi^6} + \frac{900878125 g^{12} \zeta(5)^2}{6\pi^{12}} + O\left(g^{18}\right) \\
F'_{(1,2)} &= 268800 + \frac{6478500 g^6 \zeta(5)}{\pi^6} + \frac{9623009375 g^{12} \zeta(5)^2}{64\pi^{12}} + O\left(g^{18}\right) \\
F_{(2,2)} &= 23654400 + \frac{1043196000 g^6 \zeta(5)}{\pi^6} + \frac{119937702500 g^{12} \zeta(5)^2}{3\pi^{12}} + O\left(g^{18}\right) \\
F'_{(2,2)} &= 23654400 + \frac{1043196000 g^6 \zeta(5)}{\pi^6} + \frac{640773739375 g^{12} \zeta(5)^2}{16\pi^{12}} + O\left(g^{18}\right)
\end{aligned}
\tag{3.26}
$$

We see that the correlators obtained by the two orderings differ in the 6 loop term. Using these, we can compute the degree of violation of the decoupled Toda equations for both orders:

$$
\begin{aligned}
\partial_\tau \partial_{\bar\tau} \log \widetilde{G}_I - \left.\left(\frac{\widetilde{G}_{I+\mathbf{1}}}{\widetilde{G}_I} - \frac{\widetilde{G}_I}{\widetilde{G}_{I-\mathbf{1}}}\right)\right|_{I=(3,0)} &= O\left(g^{18}\right) \\
\partial_\tau \partial_{\bar\tau} \log \widetilde{G}'_I - \left.\left(\frac{\widetilde{G}'_{I+\mathbf{1}}}{\widetilde{G}'_I} - \frac{\widetilde{G}'_I}{\widetilde{G}'_{I-\mathbf{1}}}\right)\right|_{I=(3,0)} &= \frac{13125 g^{16} \zeta(5)^2}{262144\pi^{14}} + O\left(g^{18}\right) \\
\partial_\tau \partial_{\bar\tau} \log \widetilde{G}_I - \left.\left(\frac{\widetilde{G}_{I+\mathbf{1}}}{\widetilde{G}_I} - \frac{\widetilde{G}_I}{\widetilde{G}_{I-\mathbf{1}}}\right)\right|_{I=(1,2)} &= O\left(g^{18}\right) \\
\partial_\tau \partial_{\bar\tau} \log \widetilde{G}'_I - \left.\left(\frac{\widetilde{G}'_{I+\mathbf{1}}}{\widetilde{G}'_I} - \frac{\widetilde{G}'_I}{\widetilde{G}'_{I-\mathbf{1}}}\right)\right|_{I=(1,2)} &= -\frac{28125 g^{16} \zeta(5)^2}{262144\pi^{14}} + O\left(g^{18}\right)
\end{aligned}
\tag{3.27}
$$

Therefore, we find that (1.2) is satisfied for a Toda order only.

4 Further examples

4.1 General $\mathcal{N} = 2$ CFTs with gauge group SU(N)

As discussed in [23], for theories based on an SU(N) gauge group there are essentially 8 cases which we list below, explicitly constructing the 1-loop factor for each case. We first recall that the contribution to the 1-loop partition function of the (adjoint) vector multiplet is

$$
Z^{(\mathrm{VM})}_{\text{1-loop}} = \prod_{i<j} H(a_i - a_j)^2 \,, \tag{4.1}
$$

where $H(x)$ is defined in (A.4). In addition we will need the contributions of hypermultiplets in various representations: the fundamental (of dimension N and Dynkin label $[1, 0, \cdots, 0]$), the rank-2 symmetric representation (of dimension $N(N+1)/2$ and Dynkin label $[2, 0, \cdots, 0]$), the rank-2 antisymmetric representation (of dimension $N(N-1)/2$ and Dynkin label $[0, 1, \cdots, 0]$) and the rank-3 antisymmetric representation (of dimension $\frac{1}{3!}N(N-1)N-2)$ and Dynkin label $[0, 0, 1, \cdots, 0]$). These read, respectively,

$$
\begin{aligned}
Z_{\text{1-loop}}^{(\text{fund})} &= \prod_i \frac{1}{H(a_i)} \,, \\
Z_{\text{1-loop}}^{(2-\text{symm})} &= \frac{1}{\prod_i H(2a_i) \prod_{i<j} H(a_i+a_j)} \,, \\
Z_{\text{1-loop}}^{(2-\text{antisymm})} &= \frac{1}{\prod_{i<j} H(a_i+a_j)} \,, \\
Z_{\text{1-loop}}^{(3-\text{antisymm})} &= \frac{1}{\prod_{i<j<k} H(a_i+a_j+a_k)} \,.
\end{aligned} \tag{4.2}
$$

(A1). $\mathcal{N}=4$ SYM. This case is familiar enough and we will refer to the literature for the explicit formulas.

(A2). $2N$ fundamental representations (SQCD) [24]. This case has been extensively studied and we will refer to the literature for the explicit formulas.

(A3). $N-2$ fundamental representations and one rank-2 symmetric representation. This case occurs for all $N \geq 3$. The complete one-loop factor is given by

$$
Z_{\text{1-loop}} = \frac{\prod_{i<j} H(a_i-a_j)^2}{\left[\prod_i H(a_i)^{N-2} H(2a_i)\right] \prod_{i<j} H(a_i+a_j)} \,. \tag{4.3}
$$

One can check that the exponential factor $e^{-x^2/n}$ in the function $H(x)$, defined in (A.4), cancels out, as it should be for a conformal field theory. This factor will cancel, as expected, in all examples below. In theories where this factor does not cancel out (e.g. $\mathcal{N}=2$ SU(N) theory with $N_f < 2N$ fundamentals), there is a logarithmic UV divergence in the original one-loop determinants that has to be absorbed into a renormalization of the coupling constant, leading to a non-zero β function [10].

(A4).

a) $N+2$ fundamental representations and one rank-2 antisymmetric representation. This case occurs for all $N \geq 4$. The complete one-loop factor in the localization partition function is given by

$$
Z_{\text{1-loop}} = \frac{\prod_{i<j} H(a_i-a_j)^2}{\left(\prod_i H(a_i)\right)^{N+2} \prod_{i<j} H(a_i+a_j)} \,. \tag{4.4}
$$

A discussion on large N properties of this theory can be found in [25].

b) CFT with two rank-2 antisymmetric and 4 fundamental representations, occurring for all $N \geq 5$.

$$
Z_{\text{1-loop}} = \frac{\prod_{i<j} H(a_i-a_j)^2}{\left(\prod_i H(a_i)\right)^{4} \prod_{i<j} H(a_i+a_j)^2} \,. \tag{4.5}
$$

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(A5). The CFT with one rank-2 symmetric and one rank-2 antisymmetric, for $N \geq 4$. This gives the one-loop factor

$$Z_{\text{1-loop}} = \frac{\prod_{i<j} H(a_i - a_j)^2}{\left(\prod_i H(2a_i)\right) \prod_{i<j} H(a_i + a_j)^2}\,. \tag{4.6}$$

Remarkably, for these theories, the two-loop term, proportional to $\zeta(3)$, exactly cancels out. Hence we find a "non-renormalization" theorem for all SU(N) CFTs with one rank-2 symmetric and one rank-2 antisymmetric representations: the two-loop contribution to any extremal correlator of CPOs vanishes. This can be seen by using equation (A.5) of the appendix and noting that the $\zeta(3)$ term cancels between numerator and denominator. It also has implications for any supersymmetric observable that can be computed from the localized partition function, including, in particular, the VEV of the 1/2 BPS circular Wilson loop and correlation functions between the Wilson loop operator and CPOs [12, 16]: the corresponding perturbation series do not contain $\zeta(3)$ coefficients and the two loop terms are the same as in $\mathcal{N} = 4$ super Yang-Mills. In the perturbation theory computed with ordinary methods, this requires a massive cancellation of Feynman diagrams. This is a surprise, since the theory should not have any additional supersymmetry. We do not understand the underlying reason for this two-loop cancellation. Presumably, it could be due to the fact that the group-theoretic factors in some combined Feynman diagrams accidentally coincide with the case of the hypermultiplet in the adjoint representation (note that the matter content of this theory is similar to that of the $\mathcal{N}$=4 SU(N) $\times$ U(1) theory, since both theories have the same number of hypermultiplets, $\frac{1}{2}N(N+1) + \frac{1}{2}N(N+1) = N^2$). Clearly, it would be interesting to understand the origin of this cancellation.

Note that the case (A3) with $N = 3$ is also included in this family, since for $N = 3$ the antisymmetric representation is equivalent to the fundamental representation.

(A6). The CFT with two rank-3 antisymmetric, occurring only for SU(6). We obtain

$$Z^{\text{SU}(6)}_{\text{1-loop}} = \frac{\prod_{i<j} H(a_i - a_j)^2}{\prod_{i<j<k} H(a_i + a_j + a_k)^2}\,. \tag{4.7}$$

(A7). A CFT with one rank-3 antisymmetric and $N_f = \frac{1}{2}(9N - N^2 - 6)$, appearing for $N = 6, 7, 8$ (for lower N, it becomes equivalent to one of the above CFTs). The one-loop factor is

$$Z^{\text{SU}(N)}_{\text{1-loop}} = \frac{\prod_{i<j} H(a_i - a_j)^2}{\left(\prod_i H(a_i)\right)^{N_f} \prod_{i<j<k} H(a_i + a_j + a_k)}\,, \qquad (N_f, N) = (6,6), (4,7), (1,8)\,. \tag{4.8}$$

(A8). An SU(6) gauge theory with two fundamental representations, one rank-2 antisymmetric and one rank-3 antisymmetric. We have

$$Z^{\text{SU}(6)}_{\text{1-loop}} = \frac{\prod_{i<j} H(a_i - a_j)^2}{\left(\prod_i H(a_i)\right)^2 \prod_{i<j} H(a_i + a_j) \prod_{i<j<k} H(a_i + a_j + a_k)}\,. \tag{4.9}$$

Once again, we check that the exponential factor $e^{-x^2/n}$ in the $H(x)$ functions cancel in all cases, as expected.

Toda equation. We have checked that, within the limits allowed by the computational capabilities, upon implementing our algorithm the correlation functions of all the SU(N) theories listed above satisfy the universal Toda equations (1.2). Our checks include up to $N = 5$ and beyond ten-loop order.[10]

In the following subsections, we provide some explicit examples for other cases: symplectic and orthogonal gauge groups.

4.2 Symplectic gauge group

As outlined above, we expect our procedure to hold independently of the theory, and in particular, for any gauge group.

As an example, we will consider USp(4) SQCD, that is $\mathcal{N} = 2$ SYM with gauge group USp(4) and 6 hypermultiplets in the fundamental representation. Using the general formula (A.3) in appendix, we find that the 1-loop determinant for $\mathcal{N} = 2$ super Yang-Mills with gauge group USp($2N$) and $2N + 2$ fundamental hypermultiplets is given by

$$Z_{\text{1-loop}} = \frac{\prod_{1\leq i\leq N} H(2a_i)^2 \prod_{1\leq i<j\leq N} H(a_i+a_j)^2 H(a_i-a_j)^2}{\prod_{1\leq i\leq N} H(a_i)^{2(2N+2)}} \,. \tag{4.10}$$

The group USp(4), of type C_2, has a chiral ring generated by operators with degrees 2 and 4, so that our operators will be labelled as $O_{(n,m)}$. Using the orthogonalization algorithm, we can compute the correlation functions. For illustrative purposes, here we show the first three non-trivial terms of the Toda chain with seed $(0, 0)$ (here and below, we use again functions F defined analogously to (2.5)):

$$\begin{aligned}
F_{(1,0)} &= 20 - \frac{945 g^4 \zeta(3)}{32\pi^4} + \frac{17325 g^6 \zeta(5)}{256\pi^6} + \frac{36225 g^8 \left(48\zeta(3)^2 - 133\zeta(7)\right)}{32768\pi^8} + \mathcal{O}(g^{10}) \,, \\
F_{(2,0)} &= 960 - \frac{6615 g^4 \zeta(3)}{2\pi^4} + \frac{294525 g^6 \zeta(5)}{32\pi^6} + \frac{6615 g^8 \left(6168\zeta(3)^2 - 15295\zeta(7)\right)}{4096\pi^8} + \mathcal{O}(g^{10}) \,, \\
F_{(3,0)} &= 80640 - \frac{476280 g^4 \zeta(3)}{\pi^4} + \frac{12525975 g^6 \zeta(5)}{8\pi^6} + \\
&\quad + \frac{19845 g^8 \left(110136\zeta(3)^2 - 260015\zeta(7)\right)}{1024\pi^8} + \mathcal{O}(g^{10}) \,,
\end{aligned}$$

and the first three terms of the Toda chain with seed $(0, 1)$:

$$\begin{aligned}
F_{(0,1)} &= 105 - \frac{33075 g^4 \zeta(3)}{128\pi^4} + \frac{1126125 g^6 \zeta(5)}{2048\pi^6} \\
&\quad + \frac{33075 g^8 \left(4614\zeta(3)^2 - 9443\zeta(7)\right)}{262144\pi^8} + \mathcal{O}(g^{10}) \,, \\
F_{(1,1)} &= 3780 - \frac{297675 g^4 \zeta(3)}{16\pi^4} + \frac{27286875 g^6 \zeta(5)}{512\pi^6} + \\
&\quad + \frac{99225 g^8 \left(46602\zeta(3)^2 - 101479\zeta(7)\right)}{65536\pi^8} + \mathcal{O}(g^{10}) \,.
\end{aligned}$$

[10] We omit the long formulas for the correlators in each case, which are kindly available upon request.

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$$
\begin{aligned}
F_{(2,1)} &= 302400 - \frac{2381400 g^4 \zeta(3)}{\pi^4} + \frac{66268125 g^6 \zeta(5)}{8\pi^6} \\
&\quad + \frac{496125 g^8 \left(54423\zeta(3)^2 - 120232\zeta(7)\right)}{2048\pi^8} + O\left(g^{10}\right) .
\end{aligned}
$$

Once again, one can check that the corresponding unnormalized correlation functions $\tilde{G}_{(n,m)}$ satisfy the universal Toda equation (1.2).

On the other hand, implementing orthogonalization by arranging operators in order of increasing conformal dimension would again lead to a failure of the Toda equation at four loops. In particular, one would find $F_{(3,0)} - F'_{(3,0)} = \frac{8505 g^8 \zeta(3)^2}{16\pi^8} + \ldots$, and (1.2) would be violated by terms of order $\zeta(3)^2 g^{12}$ for $I = (2,0)$.

4.3 Orthogonal gauge group

In this final subsection, we give one example for an $\mathcal{N} = 2$ SCFT with an orthogonal gauge group of type B_N. From (A.3), we find that the one-loop determinant for SQCD (where there are $2N - 1$ hypermultiplets in the fundamental representation) is

$$
Z_{\text{1-loop}} = \frac{\prod_{1 \le i \le N} H(a_i)^2 \prod_{1 \le i < j \le N} H(a_i + a_j)^2 H(a_i - a_j)^2}{\prod_{1 \le i \le N} H(a_i)^{2(2N-1)}} \tag{4.11}
$$

We choose *SQCD* with gauge group SO(7) — a rank 3 case. We only quote the first few orders for the first two Toda chains. Again, one can check that (1.2) is satisfied in all cases.

$$
\begin{aligned}
F_{(0,0,0)} &= 1 , \\
F_{(1,0,0)} &= 42 - \frac{945 g^4 \zeta(3)}{32\pi^4} + \frac{17325 g^6 \zeta(5)}{256\pi^6} + \frac{4725 g^8 \left(192\zeta(3)^2 - 917\zeta(7)\right)}{32768\pi^8} + O\left(g^{10}\right) , \\
F_{(2,0,0)} &= 3864 - \frac{23625 g^4 \zeta(3)}{4\pi^4} + \frac{121275 g^6 \zeta(5)}{8\pi^6} \\
&\quad + \frac{4725 g^8 \left(921\zeta(3)^2 - 3668\zeta(7)\right)}{512\pi^8} + O\left(g^{10}\right) , \\
F_{(3,0,0)} &= 579600 - \frac{5740875 g^4 \zeta(3)}{4\pi^4} + \frac{130717125 g^6 \zeta(5)}{32\pi^6} \\
&\quad + \frac{42525 g^8 \left(6383856\zeta(3)^2 - 22799371\zeta(7)\right)}{94208\pi^8} + O\left(g^{10}\right) ,
\end{aligned} \tag{4.12}
$$

$$
\begin{aligned}
F_{(0,1,0)} &= \frac{15120}{23} - \frac{1068795 g^4 \zeta(3)}{2116\pi^4} + \frac{17307675 g^6 \zeta(5)}{16928\pi^6} + O\left(g^8\right) , \\
F_{(1,1,0)} &= \frac{876960}{23} - \frac{32180085 g^4 \zeta(3)}{529\pi^4} + \frac{646932825 g^6 \zeta(5)}{4232\pi^6} + O\left(g^8\right) , \\
F_{(2,1,0)} &= \frac{108743040}{23} - \frac{6325888590 g^4 \zeta(3)}{529\pi^4} + \frac{72579501225 g^6 \zeta(5)}{2116\pi^6} + O\left(g^8\right) .
\end{aligned} \tag{4.13}
$$

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A Notations and conventions

We consider a Lagrangian theory $\mathcal{T}$ with a simple[11] gauge group with Lie algebra $\mathfrak{g}$ of rank r and a matter content that makes it a CFT. The sphere partition function of the theory $\mathcal{T}$ on $\mathbb{S}^4$ is given by the localization formula [10]

$$Z^{\mathcal{T}}_{S^4}[\tau,\bar{\tau}] = \int_{\mathfrak{h}} [\mathrm{d}a]\Delta(a) Z^{\mathcal{T}}_{1-\text{loop}}(a) \exp\left(-2\pi \mathrm{Im}\,\tau\, \phi_2\right) Z_{\text{inst}} . \tag{A.1}$$

where ϕ_2 is the generator of the chiral ring at degree 2 (see the normalization (2.1)), $\mathfrak{h}$ is the Cartan subalgebra of $\mathfrak{g}$, $\Delta(a)$ is the Vandermonde determinant

$$\Delta(a) = \prod_{\beta \in \text{Roots}^+(\mathfrak{g})} (\beta \cdot a)^2 , \tag{A.2}$$

and $Z^{\mathcal{T}}_{1-\text{loop}}(a)$ is the one-loop determinant defined by

$$Z^{\mathcal{T}}_{1-\text{loop}}(a) = \frac{\prod_{\beta \in \text{Roots}(\mathfrak{g})} H(\beta \cdot a)}{\prod_{w \in \text{Weights}(\mathcal{T})} H(w \cdot a)} \tag{A.3}$$

with

$$H(x) \equiv \prod_{n=1}^{\infty} \left(1 + \frac{x^2}{n^2}\right)^n e^{-\frac{x^2}{n}} . \tag{A.4}$$

In this paper we study the sector with zero instanton number, so we set $Z_{\text{inst}} = 1$. Perturbation theory is generated by using the expansion

$$\log H(x) = -\sum_{n=2}^{\infty} (-1)^n \frac{\zeta(2n-1)}{n} x^{2n} . \tag{A.5}$$

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