On the large R-charge $\mathcal{N} = 2$ chiral correlators and the Toda equation

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ABSTRACT: We consider $\mathcal{N} = 2$ $SU(N)$ SQCD in four dimensions and a weak-coupling regime with large R-charge recently discussed in [arXiv:1803.00580](https://arxiv.org/abs/1803.00580). If φ denotes the adjoint scalar in the $\mathcal{N} = 2$ vector multiplet, it has been shown that the 2-point functions in the sector of chiral primaries $(\text{Tr}\varphi^2)^n$ admit a finite limit when $g_{\text{YM}} \rightarrow 0$ with large R-charge growing like $\sim 1/g_{\text{YM}}^2$. The correction with respect to $\mathcal{N} = 4$ correlators is a non-trivial function $F(\lambda; N)$ of the fixed coupling $\lambda = n g_{\text{YM}}^2$ and the gauge algebra rank N . We show how to exploit the Toda equation following from the tt^* equations in order to control the R-charge dependence. This allows to determine $F(\lambda; N)$ at order $\mathcal{O}(\lambda^{10})$ for generic N , greatly extending previous results and placing on a firmer ground a conjecture proposed for the $SU(2)$ case. We show that a similar Toda equation, discussed in the past, may indeed be used for the additional sector $(\text{Tr}\varphi^2)^n \text{Tr}\varphi^3$ due to the special mixing properties of these composite operators on the 4-sphere. We discuss the large R-limit in this second case and compute the associated scaling function F at order $\mathcal{O}(\lambda^7)$ and generic N . Large N factorization is also illustrated as a check of the computation.

KEYWORDS: Extended Supersymmetry, Matrix Models

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1 Introduction

The search for explicit tests of AdS/CFT correspondence has often induced new applications of old non-perturbative theoretical tools aimed at connecting the two sides of the correspondence.

From the point of view of the dual gauge theory, superconformal invariance implies that natural objects to be addressed are 2- and 3-point functions of primary fields. In the early studies of type IIB superstring on $\text{AdS}_5 \times S^5$ dual to $\mathcal{N} = 4$ SYM in 4d, particular attention has been devoted to the protected chiral sector where the supergravity limit is exact due to superconformal non-renormalization theorems [1–5]. Instead, more general unprotected operators have anomalous dimensions depending non-trivially on the 't Hooft coupling. Currently, they are efficiently computed by the Quantum Spectral Curve approach [6–8].

Wilson loops are a (non-local) different class of gauge theory observables which are also particularly suited from the point of view of testing AdS/CFT. In $\mathcal{N} = 4$ SYM, the relevant observable is the Maldacena-Wilson loop $W(C)$ where the six scalar fields are coupled to the loop C in a locally supersymmetric way [9]. At strong coupling, $W(C)$ has a clean dual representation as a disc string amplitude. In some cases, exact results for $W(C)$ are available using supersymmetric localization [10]. Indeed, it was first conjectured that the $\frac{1}{2}$ -BPS circular loop could have been captured by a matrix model [11, 12]. Pestun proved this claim in [13] by showing that the path-integral computing $W(C)$ in S^4 can be obtained from the contributions at a finite set of critical points. This remarkable feature, due to supersymmetry, reduces the calculation to a finite dimensional (Gaussian) matrix model as first suggested in [11]. These exact results give strong coupling predictions that open the way to very non-trivial tests of the AdS/CFT correspondence, as in [14].

If supersymmetry is reduced to $\mathcal{N} = 2$, where the low energy effective action is captured by Seiberg-Witten theory [15, 16], further exact results are available by exploiting once again localization [17]. Indeed, Pestun's analysis in [13] already includes the case of pure $\mathcal{N} = 2$ and mass-deformed $\mathcal{N} = 2^*$ SYM on the 4-sphere S^4 . With respect to the $\mathcal{N} = 4$ theory, there is an additional perturbative 1-loop contribution and instanton corrections encoded in Nekrasov partition function [18, 19]. The associated matrix model is no more Gaussian, but nevertheless may be treated efficiently.¹

Here we are interested in high order perturbative expansion of correlators of chiral primaries in $\mathcal{N} = 2$ superconformal theories in 4d. Although not protected as in $\mathcal{N} = 4$, chiral primaries belong to short multiplets and are annihilated by all right-chiral supercharges. This implies that, under operator product expansion (OPE), they have a (chiral) ring structure whose dependence on the marginal gauge couplings is non-trivial, and encoded in the coefficients appearing in the 2- and 3-point functions. In particular, it has been shown how to compute the full set of chiral two-point functions in $\mathcal{N} = 2$ superconformal $SU(N)$ Yang-Mills theory with $N_f = 2N$ fundamental hypermultiplets by localization [24–27].

In this context, the authors of [28] have recently reconsidered the study of particular chiral correlators of the form

$$g_{2n}(\tau, \bar{\tau}) = \langle (\text{Tr} \varphi^2)^n (\text{Tr} \bar{\varphi}^2)^n \rangle, \quad \varphi \equiv \varphi(\infty), \quad \bar{\varphi} \equiv \bar{\varphi}(0), \quad (1.1)$$

where $\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g_{\text{YM}}^2}$ is the complexified gauge coupling and φ is the complex adjoint scalar in the $\mathcal{N} = 2$ vector multiplet. The 2-point function $g_{2n}(\tau, \bar{\tau})$ may be computed in the large R-charge limit where²

$$g \rightarrow 0, \quad \text{with fixed } \lambda = n g^2. \quad (1.2)$$

In the large n limit (1.2), it is possible to show that the following ratio is finite

$$F(\lambda; N) = \lim_{n \rightarrow \infty} \frac{g_{2n}|_{\text{SQCD}}}{g_{2n}|_{\mathcal{N}=4}}. \quad (1.3)$$

¹For applications of localization to Wilson loops in $\mathcal{N} = 2$ superconformal theories see [20–23].

²The fixed large n coupling λ in (1.2) follows the notation in [28] and should not be confused with the large N 't Hooft coupling.

The function $F(\lambda; N)$ has certain special features discussed in [28], like universality and exponentiation in the $SU(2)$ case, and seems an interesting non-trivial object by itself. It has been computed at order $\mathcal{O}(\lambda^3)$ for generic gauge group rank N , and at order $\mathcal{O}(\lambda^5)$ for the specific values $N = 2, 3, 4, 5$.

From a general perspective, it is interesting that non-trivial correlators are found in the zero coupling limit, as soon as the R-charge is large $\sim 1/g^2$. Further motivation for the study of the limit (1.2) is that it falls into the general framework of semiclassical effective field theory description of strongly-coupled conformal theories in sectors of large global charge [29, 30].³

As shown in [24], the correlator (1.1) obeys a Toda equation equivalent to the 4d version of the two-dimensional topological-anti-topological fusion tt^* equations [43, 44]. In the analysis of [28], this fact is used as check of the localization computations. In this paper, we consider the higher order calculation of the function $F(\lambda; N)$ for generic N . To this aim, we shall exploit the Toda equation as a strong constraint to control the exact R-charge dependence order by order in the weak coupling expansion. This approach will turn out to be quite effective for the calculation of the ratio (1.3) and its large R-charge limit. In particular, we shall present the expression of $F(\lambda; N)$ at order $\mathcal{O}(\lambda^{10})$, valid for generic N . As a byproduct, our results support the exponentiation property observed for the $SU(2)$ theory in [28].

A further important issue concerns which type of correlators may generalize those in (1.1), while still admitting a sensible large R-charge limit in the regime (1.2). As we explained, it is quite important to control the R-charge dependence by a (possibly modified) Toda equation. For $SU(N)$ theories this simple decoupling of the tt^* equations has been discussed in [25, 26] for the correlators

$$g_{2n}^\Phi(\tau, \bar{\tau}) = \langle (\text{Tr} \varphi^2)^n \Phi (\text{Tr} \bar{\varphi}^2)^n \bar{\Phi} \rangle, \quad (1.4)$$

and expected to be valid for suitable chiral primaries Φ at 2-loops, but violated at 3-loops and higher order in general [26]. This is related to the mixing pattern that the conformal anomaly induces when chiral primaries are mapped from $\mathbb{R}^4$ to the 4-sphere S^4 — where localization is naturally worked out. Here, we show that the case $\Phi = \text{Tr} \varphi^3$ is special by exploiting the normal ordering properties of the composites $(\text{Tr} \varphi^2)^n \text{Tr} \varphi^3$ and their relation with mixing [27]. We prove by explicit localization computations that the correlators in (1.4) obey in this case a modified decoupled Toda equation, first suggested in [25]. Besides, we exploit it to compute at order $\mathcal{O}(\lambda^7)$ and generic N the function F^Φ expressing a ratio analogous to (1.3). Large N factorization is also illustrated as a check of the computation.

It would be interesting to extend these calculations to other gauge algebras and to more complicated correlators like the one-point functions of chiral primaries in presence of Wilson loops, recently treated by localization in [45]. Besides, it would be desirable to understand what is the meaning of the proposed large R-charge limit from the AdS/CFT perspective, i.e. whether it has a useful dual gravity interpretation.

³Additional references about the large charge expansion of general conformal field theories and in AdS/CFT context may be found in [31–42].

The detailed plan of the paper is the following. In section 2 we briefly recall the main definitions, previous results on chiral 2-point functions, and their large R-charge limit. In section 3 we introduce the basic tools needed to compute chiral correlators by localization, the map to the matrix model observables, and also give some explicit examples at 5-loop order. In section 4 we explain how to use the Toda equation to control the R-charge dependence in the $(\text{Tr}\varphi^2)^n$ sector and derive our main result, i.e. the function $F(\lambda; N)$ at order $\mathcal{O}(\lambda^{10})$ and generic N . Finally, in section 5 we discuss the sector $(\text{Tr}\varphi^2)^n \text{Tr}\varphi^3$, illustrate a modified Toda equation and test it against low level explicit correlators, and obtain the associated large R-charge scaling function at order $\mathcal{O}(\lambda^7)$ and generic N . Section 6 is devoted to a brief discussion of the large N expansion of our results. Various appendices collect technical details and further discussions.

2 Chiral correlators in $\mathcal{N} = 2$ superconformal theories

Four dimensional superconformal $\mathcal{N} = 2$ theories admit chiral primary operators Φ_I which are primary fields annihilated by half of the supersymmetry charges, $[\overline{Q}, \Phi_I] = 0$. In terms of the R-symmetry group $\text{SU}(2)_R \times \text{U}(1)_r$, and dilatation eigenvalue Δ , they have vanishing $\text{SU}(2)_R$ isospin and abelian R-charge r saturating the unitarity bound $\Delta = \frac{r}{2}$.⁴ The 2-point function between chiral primaries is

$$\langle \Phi_I(x) \bar{\Phi}_{\bar{J}}(0) \rangle = \frac{g_{I\bar{J}}}{|x|^{2\Delta}}, \quad (2.1)$$

where the coefficient $g_{I\bar{J}}$ depends on the marginal couplings. The operator product expansion is non singular and reads [48]

$$\Phi_I(x) \Phi_J(0) = C^K_{IJ} \Phi_K(0) + \dots, \quad (2.2)$$

where the chiral ring coefficients C^K_{IJ} enter the 3-point functions, as usual. If there are exactly marginal couplings, the space of marginal deformations has complex coordinates $(z_i, \bar{z}_i)$ and the infinitesimal change of the action $\delta S = \delta z \int d^4x \mathcal{O}_i(x)$ (plus a similar antiholomorphic part) preserves $\mathcal{N} = 2$ supersymmetry if $\mathcal{O}_i$ are dimension 4 descendants of chiral primaries φ_i with $r = 4$. Their Zamolodchikov metric $g_{i\bar{j}}$ appearing in (2.1) is Kähler with a potential given by the logarithm of the partition function on S^4 regulated in order to preserve the massive supersymmetry algebra $\text{OSp}(2|4)$ [49]

$$g_{i\bar{j}} = \partial_i \partial_{\bar{j}} \log Z_{S^4}. \quad (2.3)$$

If the partition function Z_{S^4} may be computed by localization, this provides the Zamolodchikov metric and the 2-point function of the $\Delta = 2$ chiral primaries.

This approach has been exploited in [24, 50] discussing mainly the example of $\text{SU}(2)$ SQCD, i.e. YM with 4 fundamental hypermultiplets. In that case, the chiral ring has one

⁴Chiral primaries are usually assumed to be scalar fields with Lorentz spins $(j_1, j_2) = (0, 0)$. Actually, one of the two Lorentz spin is automatically zero, while the other is zero in Lagrangian theories and expected to be zero in general, see [46]. In general, they can be detected from the superconformal index, see for instance the review paper [47].

primary $\varphi_{2n} \sim (\text{Tr} \varphi^2)^n$ for each dimension $\Delta = 2n$, where φ is the complex adjoint scalar in the $\mathcal{N} = 2$ vector multiplet. Formula (2.3) may be fully *extended* to compute the 2-point function $g_{2n}(\tau, \bar{\tau})$ in

$$\langle \varphi_{2n}(x) \bar{\varphi}_{2n}(0) \rangle = \frac{g_{2n}(\tau, \bar{\tau})}{|x|^{4n}}, \quad (2.4)$$

where τ is the Yang-Mills complexified coupling

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}. \quad (2.5)$$

To this aim, the $n = 1$ input, taken from (2.3), is fed into the following (semi-infinite) Toda equation⁵

$$\partial_\tau \partial_{\bar{\tau}} \log g_{2n} = \frac{g_{2n+2}}{g_{2n}} - \frac{g_{2n}}{g_{2n-2}} - g_2. \quad (2.6)$$

This equation is a special case of the 4d analogue of the two-dimensional tt^* topological fusion equations [43, 44] for the $\mathcal{N} = (2, 2)$ chiral ring and derived in 4d using superconformal Ward identities in [51].⁶

These results have been extended in [25] to $\text{SU}(N)$ SQCD, with $N_f = 2N$ massless fundamental hypermultiplets. The extension is not trivial because, although there is again only one marginal coupling, the structure of the chiral ring is more complicated. In the $\text{SU}(N)$ theory, a generic chiral primary takes the form

$$\varphi_{\{n_1, \dots, n_{N-1}\}} \sim \prod_{\ell=1}^{N-1} \left(\text{Tr} \varphi^{\ell+1} \right)^{n_\ell}. \quad (2.7)$$

The tt^* equations are now an infinite set of coupled, non-linear equations for matrix-valued objects whose dimension increases with the conformal dimension. Due to the special nature of $\varphi_2 = \text{Tr} \varphi^2$, which is the only one with $\Delta = 2$, it has been proposed that the tt^* equations may be decoupled into separate Toda equations [25]. In particular this has been conjectured to happen for special primaries Φ — called C_2 -primaries — constructed such that the two point functions

$$g_{2n}^\Phi = \langle \Phi^{(n)}(1) \bar{\Phi}^{(n)}(0) \rangle, \quad \Phi^{(n)} = (\text{Tr} \varphi^2)^n \Phi, \quad (2.8)$$

obey the modified Toda equation, cf. (2.6),

$$\partial_\tau \partial_{\bar{\tau}} \log g_{2n}^\Phi = \frac{g_{2n+2}^\Phi}{g_{2n}^\Phi} - \frac{g_{2n}^\Phi}{g_{2n-2}^\Phi} - g_2. \quad (2.9)$$

Relation (2.9) was tested at 2 loops⁷ in $\text{SU}(3)$ and $\text{SU}(4)$ theories by using conventional Feynman diagrams, see for instance [52–54]. In order to push the test to higher order, in [26] it has been shown how to use localization for the computation of rather general chiral 2-point functions (and related extremal 3-point functions).⁸ For each chiral primary φ_i

⁵To be precise, (2.6) may be mapped to the Toda equation by a simple change of variable.

⁶Notice that the 2-point functions g_{2n} are enough to compute the *extremal* 3-point function $\langle \varphi_{2m}(x_1) \varphi_{2n}(x_2) \bar{\varphi}_{2m+2n}(y) \rangle$ due to the associativity properties of the chiral ring algebra.

⁷To be clear, this means the correction $\mathcal{O}(g_{\text{YM}}^4)$ with respect to the tree level value.

⁸See also [55] for a discussion in terms of Alday-Gaiotto-Tachikawa duality.

(marginal or not) it is possible to deform the partition function on S^4 by a term associated with the complex coupling τ_i such that

$$\langle \varphi_i(N) \bar{\varphi}_j(S) \rangle_{S^4} = \partial_{\tau_i} \partial_{\bar{\tau}_j} \log Z_{S^4}(\boldsymbol{\tau}, \bar{\boldsymbol{\tau}}), \quad (2.10)$$

where N, S are the north and south poles of S^4 . Since $\mathcal{N} = 2$ supersymmetry is preserved, it is possible to compute $\log Z_{S^4}(\boldsymbol{\tau}, \bar{\boldsymbol{\tau}})$ by minor modifications of the usual localization formulas at weak-coupling, especially when instanton corrections are not a concern as in our weak-coupling expansions.

A well known complication is that we are interested in flat space correlators, while (2.10) is computed on the 4-sphere and, due to conformal anomaly, there are important differences. This is expressed concisely in terms of a mixing

$$\varphi_{\mathbb{R}^4} \rightarrow \varphi_{S^4}^{\Delta} + c_1 R \varphi_{S^4}^{\Delta-2} + c_2 R^2 \varphi_{S^4}^{\Delta-4} + \dots, \quad (2.11)$$

where R is the Ricci scalar. The explicit mixing coefficients may be found by a Gram-Schmidt orthogonalization of the operators on S^4 .⁹ In general, this procedure leads to far from explicit results. One important exception is the case of g_{2n} in (2.4). In this case, in [26] it is shown that ($Z \equiv Z_{S^4}$)

$$g_{2n}(\tau, \bar{\tau}) = \frac{1}{Z} \frac{\det M^{(n)}}{\det M^{(n-1)}}, \quad M_{ab}^{(p)} = \partial_{\tau}^a \partial_{\bar{\tau}}^b Z, \quad a, b = 0, \dots, p. \quad (2.12)$$

The ultimate reason for this remarkable determinant representation is the simple mixing pattern of the $(\text{Tr} \varphi^2)^n$ operators for which (2.11) reads

$$(\text{Tr} \varphi^2)^n \rightarrow (\text{Tr} \varphi^2)^n + c_1 R (\text{Tr} \varphi^2)^{n-1} + c_2 R^2 (\text{Tr} \varphi^2)^{n-2} + \dots + c_n R^n \mathbb{I}, \quad (2.13)$$

and Gram-Schmidt orthogonalization is a triangular matrix in the one-dimensional vector space of chiral primaries. For our purposes, an important remark is that (2.12) implies the Toda equation (2.6) [26]. Explicit 3 loop calculations carried on in SU(3) and SU(4) theories in [26] suggests that the generalized relation (2.9) is not valid at all orders for the C_2 primaries introduced in [25], i.e. the tt^* equations do not decouple.¹⁰

2.1 Large R-charge limit of $\mathcal{N} = 2$ correlators

As we mentioned in the Introduction, the authors of [28] have recently proposed to study the chiral correlators g_{2n} in (2.4) by considering the zero gauge coupling limit $g \rightarrow 0$ while

⁹In general, multiple trace operators are mixed with single trace operators, at finite rank N . A special solution at large N is discussed in [56] and applied to check the chiral primary correlator with a Wilson loop computed in [57] with a 2-matrix model proposal. The large N limit of correlators is also studied in [58–60].

¹⁰To clarify what happens, let us recall that the construction in [25] is based on the chiral primary tower $(\text{Tr} \varphi^2)^n \mathcal{O}_I$, where $\{\mathcal{O}_I\}$ is a basis for fixed dimension chiral primaries with definite orthogonality properties. Its construction requires a rotation in field space that happens to be coupling-dependent starting at three loops. This extra $\tau, \bar{\tau}$ dependence spoils the decoupling property. An alternative point of view will be illustrated in section 5, see also appendix C.

simultaneously increasing large R-charge $\sim 1/g^2$, see (1.2) Working in $SU(N)$ SQCD, they proved that in this limit one has

$$g_{2n} = F(\lambda; N) \left(\frac{\lambda}{2\pi e} \right)^{2n} n^{\frac{N^2-1}{2}} + \dots, \quad (2.14)$$

where $F(\lambda; N)$ is the asymptotic ratio to the $\mathcal{N} = 4$ correlators defined in (1.3). In the following discussion, it will be convenient to define the fixed R-charge ratios

$$F(g, n; N) = \frac{g_{2n}|_{\text{SQCD}}}{g_{2n}|_{\mathcal{N}=4}}, \quad F(\lambda; N) = \lim_{n \rightarrow \infty} F(\sqrt{\lambda/n}, n; N). \quad (2.15)$$

The perturbative expansion of $F(g, n; N)$ is in integer inverse powers of

$$\text{Im } \tau = \frac{4\pi}{g^2}. \quad (2.16)$$

One of the results of [28] are the explicit first two corrections

$$F(g, n; N) = 1 - \frac{9n(N^2 + 2n - 1)}{4\pi^2 (\text{Im } \tau)^2} \zeta(3) + \frac{5n(2N^2 - 1)(3N^4 + (15n - 3)N^2 + 20n^2 - 15n + 4)}{4\pi^3 N(N^2 + 3)(\text{Im } \tau)^3} \zeta(5) + \dots \quad (2.17)$$

From this expression, we obtain in the limit (1.2), see (2.15),

$$F(\lambda; N) = 1 - \frac{9\zeta(3)}{32\pi^4} \lambda^2 + \frac{25(2N^2 - 1)\zeta(5)}{64\pi^6 N(N^2 + 3)} \lambda^3 + \dots \quad (2.18)$$

Explicit results at order $\mathcal{O}(\lambda^5)$ are presented in [28] for the specialized cases $N = 2, 3, 4, 5$. At higher orders, the function F contains products of ζ -functions. Remarkably, for the $SU(2)$ theory, one can write

$$F(\lambda; 2) = \exp \left(-\frac{9\zeta(3)}{32\pi^4} \lambda^2 + \frac{25\zeta(5)}{128\pi^6} \lambda^3 - \frac{2205\zeta(7)}{16384\pi^8} \lambda^4 + \frac{3213\zeta(9)}{32768\pi^{10}} \lambda^5 + \dots \right), \quad (2.19)$$

with only simple ζ -functions in the exponent. This feature has been conjectured to hold at all orders in the small λ expansion. For $N > 2$, only terms involving $\zeta(3)$ are supposed to exponentiate, i.e. all dependence on the transcendental $\zeta(3)$ is captured by the first term in (2.19) and is independent on N , as follows from (2.17).

3 Localization computation of the chiral correlators

In this section, we briefly recall the available tools to evaluate chiral correlators by localization, the map to the matrix model observables, and also give some explicit examples for later purposes.

3.1 Matrix model $\mathcal{N} = 2$ partition function

Let us consider the $\mathcal{N} = 2$ $SU(N)$ Yang-Mills theory with $N_f = 2N$ fundamental hypermultiplets (SQCD). The celebrated partition function resulting from localization reads [10]

$$Z_{\text{SQCD}} = \int d^N a \delta\left(\sum_n a_n\right) \Delta(a) e^{-2\pi \text{Im} \tau \sum a_i^2} |Z_{1\text{-loop}}|^2 |Z_{\text{inst}}|^2, \quad (3.1)$$

where a_n are the eigenvalues of the traceless $N \times N$ matrix a related to the vacuum expectation value of the adjoint scalar φ in the $\mathcal{N} = 2$ vector multiplet. The function $\Delta(a) = \prod_{n < m} (a_n - a_m)^2$ is the Vandermonde determinant. The 1-loop contribution to the partition function is

$$|Z_{1\text{-loop}}|^2 = \frac{\prod_{n < m} H(a_n - a_m)^2}{\prod_n H(a_n)^{2N}}, \quad H(x) = \prod_{n=1}^{\infty} \left(1 + \frac{x^2}{n^2}\right)^{n^2} e^{-\frac{x^2}{n}}. \quad (3.2)$$

Finally, Z_{inst} is the Nekrasov partition function [18, 19] evaluated with equivariant Ω -deformation $\varepsilon_{1,2}$ parameters equal to the inverse radius of S^4 . In the limit (1.2) this contribution may be neglected and we shall not discuss it further.

3.2 The $\mathcal{N} = 4$ case and Gaussian correlators

The partition function of $\mathcal{N} = 4$ SYM is obtained by removing the 1-loop and instanton terms and evaluates to the simple expression

$$Z_{\mathcal{N}=4} = \frac{(2\pi)^{\frac{N-1}{2}}}{\sqrt{N}} \frac{G(N+2)}{(4\pi \text{Im} \tau)^{\frac{N^2-1}{2}}}, \quad (3.3)$$

where $G(N)$ is the Barnes G-function.¹¹ For our applications, closely following the neat analysis in [27, 45], it is convenient to rescale the matrix a in order to put the classical part of the partition function in standard Gaussian form $e^{-\text{Tr} a^2}$. This requires

$$a \rightarrow \left(\frac{g^2}{8\pi^2}\right)^{1/2} a. \quad (3.4)$$

Going to the basis $a = \sum_{\ell=1}^{N^2-1} a^\ell t^\ell$, with $\text{Tr} t^a = 0$ and $\text{Tr}(t^a t^b) = \frac{1}{2} \delta^{ab}$, one can show that the relevant measure is flat $\prod_{n=1}^N da_n \Delta(a) = c_N \prod_{\ell=1}^{N^2-1} da^\ell$. Thus, matrix model expectation values are computed as Gaussian averages $\langle F(a) \rangle = \int da e^{-\text{Tr} a^2} F(a)$ where the normalization $da = \prod_{\ell=1}^{N^2-1} \frac{da^\ell}{\sqrt{2\pi}}$ is chosen in order to have $\langle 1 \rangle = 1$. Explicit calculations are performed by first applying Wick contractions to pair all a 's. Then, contractions are computed by using the Gaussian matrix model relation $\langle a^p a^q \rangle = \delta^{pq}$. This makes a pair of t^ℓ appear and reduction is achieved by using standard $SU(N)$ identities [27]. With the definition

$$t_{n_1, n_2, \dots} = \langle \text{Tr}(a^{n_1}) \text{Tr}(a^{n_2}) \dots \rangle, \quad (3.5)$$

¹¹We remind that for integer N one has $G(N+2) = \prod_{n=1}^N n!$.

one finds up to level $\sum_i n_i = 8$ the expressions

$$\begin{aligned}
 t_2 &= \frac{N^2 - 1}{2}, & t_{2,2} &= \frac{N^4 - 1}{4} \\
 t_4 &= \frac{(N^2 - 1)(2N^2 - 3)}{4N}, & t_6 &= \frac{5(N^2 - 1)(N^4 - 3N^2 + 3)}{8N^2}, \\
 t_{3,3} &= \frac{3(N^2 - 1)(N^2 - 4)}{8N}, & t_{4,2} &= \frac{(N^2 - 1)(N^2 + 3)(2N^2 - 3)}{8N}, \\
 t_{2,2,2} &= \frac{(N^4 - 1)(N^2 + 3)}{8}, & t_8 &= \frac{7(N^2 - 1)(2N^6 - 8N^4 + 15N^2 - 15)}{16N^3}, \\
 t_{2,6} &= \frac{5(N^2 - 1)(N^2 + 5)(N^4 - 3N^2 + 3)}{16N^2}, & t_{3,5} &= \frac{15(N^2 - 4)(N^2 - 2)(N^2 - 1)}{16N^2}, \\
 t_{4,4} &= \frac{(N^2 - 1)(4N^6 + 20N^4 - 99N^2 + 135)}{16N^2}, & t_{2,2,4} &= \frac{(N^2 - 1)(N^2 + 3)(N^2 + 5)(2N^2 - 3)}{16N}, \\
 t_{2,3,3} &= \frac{3(N^2 - 4)(N^2 - 1)(N^2 + 5)}{16N}, & t_{2,2,2,2} &= \frac{(N^4 - 1)(N^2 + 3)(N^2 + 5)}{16}. \tag{3.6}
 \end{aligned}$$

3.3 $\mathcal{N} = 2$ as a perturbation around $\mathcal{N} = 4$ SYM

As is well known, see for instance [56], it is very convenient to obtain the $\mathcal{N} = 2$ correlators from the $\mathcal{N} = 4$ ones by simply expanding (3.2) using

$$\log H(x) = -(1 + \gamma_E) x^2 - \sum_{\ell=2}^{\infty} (-1)^\ell \frac{\zeta(2\ell - 1)}{\ell} x^{2\ell}, \tag{3.7}$$

and reconstructing products of higher order traces Tra^n in the expansion. This procedure is straightforward and we find $|Z_{1\text{-loop}}|^2 = e^{-S_{\text{int}}}$ with (see (3.5))

$$\begin{aligned}
 S_{\text{int}} &= 3\zeta(3)t_{2,2} - \frac{10}{3}\zeta(5)(3t_{2,4} - 2t_{3,3}) + \frac{7}{2}\zeta(7)(4t_{2,6} - 8t_{3,5} + 5t_{4,4}) \\
 &\quad - \frac{6}{5}\zeta(9)(15t_{2,8} - 40t_{3,7} + 70t_{4,6} - 42t_{5,5}) + \dots \tag{3.8}
 \end{aligned}$$

Taking into account relations like (3.6) and the rescaling in (3.4), this gives immediately the relation between partition functions

$$\begin{aligned}
 Z_{\text{SQCD}} &= Z_{\mathcal{N}=4} \left[1 - \frac{3(N^4 - 1)\zeta(3)}{16\pi^2 (\text{Im } \tau)^2} + \frac{5(N^4 - 1)(2N^2 - 1)\zeta(5)}{32\pi^3 N (\text{Im } \tau)^3} \right. \\
 &\quad + \left(9(N^2 + 1)(N^2 + 3)(N^2 + 5)\zeta(3)^2 - \frac{35}{N^2}(8N^6 + 4N^4 - 3N^2 + 3)\zeta(7) \right) \frac{N^2 - 1}{512\pi^4 (\text{Im } \tau)^4} \\
 &\quad + \left(-15(N^2 + 1)(N^2 + 5)(N^2 + 7)(2N^2 - 1)\zeta(3)\zeta(5) \right. \\
 &\quad \left. + \frac{21}{N^2}(26N^8 + 28N^6 - 3N^4 + 6N^2 - 9)\zeta(9) \right) \frac{N^2 - 1}{512\pi^5 N (\text{Im } \tau)^5} + \dots \left. \right]. \tag{3.9}
 \end{aligned}$$

3.4 Sample 5-loops chiral two-point functions

Chiral primaries may be computed by taking into account (3.8) and the normalization (3.9). Let us give some examples at 5-loop order. This will be useful in the following to explicitly

test certain differential equations of Toda type. In this section, we consider the following chiral primaries on $\mathbb{R}^4$

$$\mathcal{O}_2 = \text{Tr}\varphi^2, \quad \mathcal{O}_{2,2} = (\text{Tr}\varphi^2)^2, \quad \mathcal{O}_{2,3} = (\text{Tr}\varphi^2) \text{Tr}\varphi^3, \dots \quad (3.10)$$

These are related to mixed operators on S^4 that we can express conveniently in terms of the matrix model multiple traces, see (2.11),

$$\begin{aligned} \mathcal{O}_2 &\rightarrow t_2 + c_{2,0}(g) \mathbb{I}, & \mathcal{O}_{2,2} &\rightarrow t_{2,2} + c_{2,2;2}(g) t_2 + c_{2,2;0} \mathbb{I}, \\ \mathcal{O}_{2,3} &\rightarrow t_{2,3} + c_{2,3;3}(g) t_3. \end{aligned} \quad (3.11)$$

Here t_n are as in (3.5) but we have made explicit the rescaling (3.4) to make g appear explicitly in the mixing coefficients. These are determined by Gram-Schmidt orthogonalization. For instance, one has in the above

$$\begin{aligned} c_{2,0}(g) = & -\frac{N^2-1}{2} \left[1 - \frac{3g^4(N^2+1)\zeta(3)}{64\pi^4} + \frac{15g^6(N^2+1)(2N^2-1)\zeta(5)}{1024\pi^6 N} \right. \\ & + g^8 \left(\frac{9(N^2+1)(N^2+2)\zeta(3)^2}{2048\pi^8} - \frac{35(8N^6+4N^4-3N^2+3)\zeta(7)}{16384\pi^8 N^2} \right) \\ & + g^{10} \left(\frac{105(26N^8+28N^6-3N^4+6N^2-9)\zeta(9)}{262144\pi^{10} N^3} \right. \\ & \left. \left. - \frac{225(N^2+1)(N^2+3)(2N^2-1)\zeta(3)\zeta(5)}{65536\pi^{10} N} \right) + \mathcal{O}(g^{12}) \right], \end{aligned} \quad (3.12)$$

and similar expansions for the other coefficients, as in

$$\begin{aligned} c_{2,2;2}(g) = & -(N^2+1) \left[1 - \frac{3g^4(N^2+5)\zeta(3)}{64\pi^4} + \frac{15g^6(N^2+11)(2N^2-1)\zeta(5)}{1024\pi^6 N} \right. \\ & + g^8 \left(\frac{9(N^4+12N^2+29)\zeta(3)^2}{2048\pi^8} - \frac{35(N^2+19)(8N^6+4N^4-3N^2+3)\zeta(7)}{16384\pi^8 N^2(N^2+1)} \right) \\ & + g^{10} \left(\frac{105(N^2+29)(26N^8+28N^6-3N^4+6N^2-9)\zeta(9)}{262144\pi^{10} N^3(N^2+1)} \right. \\ & \left. \left. - \frac{225(N^2+5)(N^2+15)(2N^2-1)\zeta(3)\zeta(5)}{65536\pi^{10} N} \right) + \mathcal{O}(g^{12}) \right], \\ c_{2,3;3}(g) = & -\frac{N^2+5}{2} \left[1 - \frac{3g^4(N^2+7)\zeta(3)}{64\pi^4} + \frac{15g^6(N^2-1)(2N^4+45N^2+105)\zeta(5)}{1024\pi^6 N(N^2+5)} \right. \\ & + g^8 \left(\frac{9(N^2+7)(N^2+8)\zeta(3)^2}{2048\pi^8} - \frac{35(8N^8+260N^6+281N^4-378N^2+693)\zeta(7)}{16384\pi^8 N^2(N^2+5)} \right) \\ & + g^{10} \left(\frac{105(N^2-1)(26N^8+1180N^6+3309N^4+3096N^2+3861)\zeta(9)}{262144\pi^{10} N^3(N^2+5)} \right. \\ & \left. \left. - \frac{225(N^2-1)(N^2+9)(2N^4+45N^2+105)\zeta(3)\zeta(5)}{65536\pi^{10} N(N^2+5)} \right) + \mathcal{O}(g^{12}) \right]. \end{aligned} \quad (3.13)$$

After resolution of mixing, correlators can be computed by simply replacing the r.h.s. in (3.11) and using the Gaussian correlators, cf. (3.6). Since we want to match the normalization in [26] for the 2-point functions of the form $\langle \mathcal{O}_n(\infty) \bar{\mathcal{O}}_n(0) \rangle$, we also have to multiply the matrix model expression by $2^{|\mathbf{n}|} = 2^{n_1+n_2+\dots}$ including also the factor $(\frac{g^2}{4\pi})^{|\mathbf{n}|}$.

In particular, let us present the complete five-loop expressions (with generic N) of two correlators computed at 3-loops in [26] for $N = 3$ and $N = 4$. These are

$$\begin{aligned} \langle \mathcal{O}_{2,2} \overline{\mathcal{O}}_{2,2} \rangle = & 8(N^4 - 1) \left(\frac{g^2}{4\pi} \right)^4 \left[1 - \frac{9g^4(N^2 + 3)\zeta(3)}{32\pi^4} + \frac{15g^6(N^2 + 6)(2N^2 - 1)\zeta(5)}{128\pi^6 N} \right. \\ & + g^8 \left(\frac{9(29N^4 + 204N^2 + 355)\zeta(3)^2}{4096\pi^8} - \frac{175(N^2 + 10)(8N^6 + 4N^4 - 3N^2 + 3)\zeta(7)}{8192\pi^8 N^2(N^2 + 1)} \right) \\ & + g^{10} \left(\frac{315(N^2 + 15)(26N^8 + 28N^6 - 3N^4 + 6N^2 - 9)\zeta(9)}{65536\pi^{10} N^3(N^2 + 1)} \right. \\ & \left. \left. - \frac{135(2N^2 - 1)(7N^4 + 78N^2 + 211)\zeta(3)\zeta(5)}{16384\pi^{10} N} \right) + \mathcal{O}(g^{12}) \right], \end{aligned} \quad (3.14)$$

and

$$\begin{aligned} \langle \mathcal{O}_{2,3} \overline{\mathcal{O}}_{2,3} \rangle = & 6 \frac{(N^2 - 1)(N^2 - 4)(N^2 + 5)}{N} \left(\frac{g^2}{4\pi} \right)^5 \left[1 - \frac{9g^4(N^2 + 5)\zeta(3)}{32\pi^4} \right. \\ & + \frac{5g^6(N^2 - 1)(46N^4 + 703N^2 + 1525)\zeta(5)}{1024\pi^6 N(N^2 + 5)} + g^8 \left(\frac{9(7N^4 + 75N^2 + 215)\zeta(3)^2}{1024\pi^8} \right. \\ & \left. - \frac{35(73N^8 + 1501N^6 + 1537N^4 - 2043N^2 + 3900)\zeta(7)}{16384\pi^8 N^2(N^2 + 5)} \right) \\ & + g^{10} \left(\frac{63(N - 1)(N + 1)(452N^8 + 13312N^6 + 36365N^4 + 34180N^2 + 41835)\zeta(9)}{262144\pi^{10} N^3(N^2 + 5)} \right. \\ & \left. - \frac{45(N - 1)(N + 1)(78N^6 + 1721N^4 + 11042N^2 + 18575)\zeta(3)\zeta(5)}{32768\pi^{10} N(N^2 + 5)} \right) + \mathcal{O}(g^{12}) \right]. \end{aligned} \quad (3.15)$$

Specialization to $SU(3)$ gives

$$\begin{aligned} \langle \mathcal{O}_{2,2} \overline{\mathcal{O}}_{2,2} \rangle_{N=3} = & \left(\frac{g^2}{4\pi} \right)^4 \left[640 - \frac{2160g^4\zeta(3)}{\pi^4} + \frac{6375g^6\zeta(5)}{\pi^6} + g^8 \left(\frac{51075\zeta(3)^2}{8\pi^8} - \frac{1699075\zeta(7)}{96\pi^8} \right) \right. \\ & \left. + g^{10} \left(\frac{417375\zeta(9)}{8\pi^{10}} - \frac{707625\zeta(3)\zeta(5)}{16\pi^{10}} \right) + \mathcal{O}(g^{12}) \right], \\ \langle \mathcal{O}_{2,3} \overline{\mathcal{O}}_{2,3} \rangle_{N=3} = & \left(\frac{g^2}{4\pi} \right)^5 \left[1120 - \frac{4410g^4\zeta(3)}{\pi^4} + \frac{144725g^6\zeta(5)}{12\pi^6} + g^8 \left(\frac{458955\zeta(3)^2}{32\pi^8} - \frac{12273275\zeta(7)}{384\pi^8} \right) \right. \\ & \left. + g^{10} \left(\frac{11641175\zeta(9)}{128\pi^{10}} - \frac{2945775\zeta(3)\zeta(5)}{32\pi^{10}} \right) + \mathcal{O}(g^{12}) \right], \end{aligned} \quad (3.16)$$

extending the 3-loops results in eq. (3.45) and eq. (3.56, first line) of [26]. Similarly, specialization to $SU(4)$ gives

$$\begin{aligned} \langle \mathcal{O}_{2,2} \overline{\mathcal{O}}_{2,2} \rangle_{N=4} = & \left(\frac{g^2}{4\pi} \right)^4 \left[2040 - \frac{43605g^4\zeta(3)}{4\pi^4} + \frac{1304325g^6\zeta(5)}{32\pi^6} \right. \\ & + g^8 \left(\frac{25343685\zeta(3)^2}{512\pi^8} - \frac{1151616375\zeta(7)}{8192\pi^8} \right) \\ & \left. + g^{10} \left(\frac{266283200925\zeta(9)}{524288\pi^{10}} - \frac{3469385925\zeta(3)\zeta(5)}{8192\pi^{10}} \right) + \mathcal{O}(g^{12}) \right], \end{aligned}$$

$$\begin{aligned} \langle \mathcal{O}_{2,3} \overline{\mathcal{O}}_{2,3} \rangle_{N=4} = & \left(\frac{g^2}{4\pi} \right)^5 \left[5670 - \frac{535815g^4\zeta(3)}{16\pi^4} + \frac{248558625g^6\zeta(5)}{2048\pi^6} \right. \\ & + g^8 \left(\frac{81826605\zeta(3)^2}{512\pi^8} - \frac{13344472575\zeta(7)}{32768\pi^8} \right) \\ & \left. + g^{10} \left(\frac{11997966800925\zeta(9)}{8388608\pi^{10}} - \frac{87052714875\zeta(3)\zeta(5)}{65536\pi^{10}} \right) + \mathcal{O}(g^{12}) \right], \quad (3.17) \end{aligned}$$

extending the 3-loops results in eq. (3.46) and eq. (3.57) of [26].

4 Toda equation as a constraint: the function $F(\lambda; N)$ at $\mathcal{O}(\lambda^{10})$

The explicit results derived in the previous sections are not useful for the study of the large n limit of g_{2n} . Indeed, the computational complexity grows quickly with n , and an analytic treatment of the n -dependence is mandatory. We now show that this is fully provided by the Toda equation. The tree level value of g_{2n} is equal to its $\mathcal{N} = 4$ limit and reads [56]

$$g_{2n}|_{\mathcal{N}=4} = \frac{n! 2^{2n}}{(\text{Im } \tau)^{2n}} \frac{\Gamma\left(\frac{N^2-1}{2} + n\right)}{\Gamma\left(\frac{N^2-1}{2}\right)}. \quad (4.1)$$

Explicit results, as in (2.17), suggest that the perturbative corrections takes the following form

$$\begin{aligned} g_{2n}|_{\text{SQCD}} = & \frac{n! 2^{2n}}{(\text{Im } \tau)^{2n}} \frac{\Gamma\left(\frac{N^2-1}{2} + n\right)}{\Gamma\left(\frac{N^2-1}{2}\right)} \left[1 + \mathcal{A}_2 n \frac{n + \xi_{2,0}}{(\text{Im } \tau)^2} + \mathcal{A}_3 n \frac{n^2 + \xi_{3,1} n + \xi_{3,0}}{(\text{Im } \tau)^3} + \right. \\ & \left. + \mathcal{A}_4 n \frac{n^3 + \xi_{4,2} n^2 + \xi_{4,1} n + \xi_{4,0}}{(\text{Im } \tau)^4} + \dots \right], \quad (4.2) \end{aligned}$$

where the constants $\mathcal{A}_k$ and $\xi_{p,q}$ are functions of N , but not of n or the coupling. This simple structure may be plugged in the Toda equation (2.6). Remarkably, the Toda equation is strong enough to fix all these constants $\xi_{p,q}$ while the normalization $\mathcal{A}_k$ remains free. For instance, the first two corrections in (4.2) take the form

$$\begin{aligned} g_{2n}|_{\text{SQCD}} = & \frac{n! 2^{2n}}{(\text{Im } \tau)^{2n}} \frac{\Gamma\left(\frac{N^2-1}{2} + n\right)}{\Gamma\left(\frac{N^2-1}{2}\right)} \left[1 + \mathcal{A}_2 n \frac{n + \frac{N^2-1}{2}}{(\text{Im } \tau)^2} \right. \\ & \left. + \mathcal{A}_3 n \frac{n^2 + \frac{3}{4}(N^2-1)n + \frac{1}{20}(3N^4-3N^2+4)}{(\text{Im } \tau)^3} + \dots \right]. \quad (4.3) \end{aligned}$$

The Toda equation may be imposed at high order in the perturbative expansion. In general, the coefficients $\xi_{p,q}$ have also a dependence on the constants $\mathcal{A}_k$. For instance, at the next order, one finds

$$\begin{aligned} \xi_{4,2} &= N^2 - 1 + \frac{\mathcal{A}_2^2}{\mathcal{A}_4}, \\ \xi_{4,1} &= \frac{1}{28} \left(23 - 12N^2 + 9N^4 - (20 - 20N^2 + N^4) \frac{\mathcal{A}_2^2}{\mathcal{A}_4} \right), \\ \xi_{4,0} &= \frac{1}{504} \left(18(-8 + 7N^2 + N^6) + (163 - 147N^2 + 77N^4 - 9N^6) \frac{\mathcal{A}_2^2}{\mathcal{A}_4} \right). \quad (4.4) \end{aligned}$$

The final expression may be evaluated at $n = 1$ in terms of the correlator g_2 which is related to the double derivative of the partition function. This allows to fix the constants $\mathcal{A}_k$. Explicitly, one finds the following first five values¹²

$$\begin{aligned}
 \mathcal{A}_2 &= -\frac{9\zeta(3)}{2\pi^2}, \quad \mathcal{A}_3 = \frac{25(2N^2-1)\zeta(5)}{\pi^3 N(N^2+3)}, \\
 \mathcal{A}_4 &= \frac{81\zeta(3)^2}{8\pi^4} - \frac{1225(8N^6+4N^4-3N^2+3)\zeta(7)}{16\pi^4 N^2(N^2+1)(N^2+3)(N^2+5)}, \\
 \mathcal{A}_5 &= \frac{1323(26N^8+28N^6-3N^4+6N^2-9)\zeta(9)}{4\pi^5 N^3(N^2+1)(N^2+3)(N^2+5)(N^2+7)} - \frac{225(2N^2-1)\zeta(3)\zeta(5)}{2\pi^5 N(N^2+3)}, \\
 \mathcal{A}_6 &= \frac{11025(8N^6+4N^4-3N^2+3)\zeta(3)\zeta(7)}{32\pi^6 N^2(N^2+1)(N^2+3)(N^2+5)} \\
 &\quad - \frac{17787(122N^{10}+280N^8+48N^6-15N^4+45)\zeta(11)}{16\pi^6 N^4(N^2+1)(N^2+3)(N^2+5)(N^2+7)(N^2+9)} \\
 &\quad + \frac{25(100N^{12}+2331N^{10}+13070N^8+20941N^6+20985N^4+14138N^2+7875)\zeta(5)^2}{2\pi^6 N^2(N^2+1)(N^2+3)^2(N^2+5)(N^2+7)(N^2+9)} \\
 &\quad - \frac{243\zeta(3)^3}{16\pi^6}.
 \end{aligned} \tag{4.5}$$

The full expression of $F(g, n; N)$ is clearly unwieldy. With the definition

$$F(g, n; N) = 1 + \sum_{k=2}^{\infty} \frac{f_k(n; N)}{(\text{Im } \tau)^k}, \tag{4.6}$$

the $k = 2, 3$ terms have been written in (2.17). The next two are

$$\begin{aligned}
 f_4(n; N) &= \frac{9n(13-45n+36n^2+36n^3+12(-1+3n+3n^2)N^2+(11+9n)N^4)\zeta(3)^2}{32\pi^4} \\
 &\quad - \frac{175n(8-7n+14n^2+(1+7n)N^2+N^4)(3-3N^2+4N^4+8N^6)\zeta(7)}{64\pi^4 N^2(N^2+1)(N^2+3)(N^2+5)}, \\
 f_5(n; N) &= -\frac{45n(-1+2n+N^2)(-1+2N^2)}{32\pi^5 N(N^2+3)} [57-52n+90n^2+40n^3 \\
 &\quad + 6(1+9n+5n^2)N^2+3(3+2n)N^4] \zeta(3)\zeta(5) \\
 &\quad - \frac{21n}{128\pi^5 N^3(N^2+1)(N^2+3)(N^2+5)(N^2+7)} \\
 &\quad \times [7(67-330n+560n^2-360n^3+288n^4)+30(-17+77n-56n^2+84n^3)N^2 \\
 &\quad + 10(38-21n+112n^2)N^4+30(1+7n)N^6+15N^8] \zeta(9).
 \end{aligned} \tag{4.7}$$

Additional terms are increasingly more involved, but with roughly the same structure. At this level, we did not identify any simple regularity.

¹²Although algorithmic, the procedure requires some careful coding when pushed to high orders. The complete solutions at order $\mathcal{O}(g^{20})$ is available under request.

4.1 Large R-charge limit

From our results for the functions f_k in (4.6), we can take the limit (1.2). It is convenient to present the result in the following logarithmic form¹³ in order to emphasize possible exponentiation properties

$$\log F(\lambda; N) = \sum_{\ell=2}^{\infty} \left(\frac{\lambda}{8\pi^2} \right)^{\ell} \sum_{\mathbf{s}} F_{\mathbf{s}}^{(\ell)}(N) \zeta(\mathbf{s}), \quad \zeta(\mathbf{s}) = \zeta(s_1) \zeta(s_2) \dots \quad (4.8)$$

At order $\mathcal{O}(\lambda^{10})$, the functions $F_{\mathbf{s}}^{(\ell)}(N)$ are

$$\begin{aligned} F_3^{(2)}(N) &= -18, \\ F_5^{(3)}(N) &= \frac{200(2N^2 - 1)}{N(N^2 + 3)}, \\ F_7^{(4)}(N) &= -\frac{1225(8N^6 + 4N^4 - 3N^2 + 3)}{N^2(N^2 + 1)(N^2 + 3)(N^2 + 5)}, \\ F_9^{(5)}(N) &= \frac{10584(26N^8 + 28N^6 - 3N^4 + 6N^2 - 9)}{N^3(N^2 + 1)(N^2 + 3)(N^2 + 5)(N^2 + 7)}, \\ F_{5^2}^{(6)}(N) &= \frac{184800(N^2 - 4)(N^6 - N^4 - 43N^2 - 37)}{(N^2 + 1)(N^2 + 3)^2(N^2 + 5)(N^2 + 7)(N^2 + 9)}, \\ F_{11}^{(6)}(N) &= -\frac{71148(122N^{10} + 280N^8 + 48N^6 - 15N^4 + 45)}{N^4(N^2 + 1)(N^2 + 3)(N^2 + 5)(N^2 + 7)(N^2 + 9)}, \\ F_{5,7}^{(7)}(N) &= -\frac{960960(N^2 - 4)(22N^8 + 11N^6 - 1167N^4 - 531N^2 + 705)}{N(N^2 + 1)(N^2 + 3)^2(N^2 + 5)(N^2 + 7)(N^2 + 9)(N^2 + 11)}, \\ F_{13}^{(7)}(N) &= \frac{8833968(34N^{10} + 110N^8 - 29N^6 + 20N^4 - 15)}{N^5(N^2 + 3)(N^2 + 5)(N^2 + 7)(N^2 + 9)(N^2 + 11)}, \end{aligned} \quad (4.9)$$

with the $\ell = 8, 9, 10$ functions being written in appendix A. One can check that all terms associated with multiple products of zeta functions vanish for $N = 2$. For this value, our expression reduces to

$$\begin{aligned} \log F(\lambda; 2) &= -\frac{9\lambda^2\zeta(3)}{32\pi^4} + \frac{25\lambda^3\zeta(5)}{128\pi^6} - \frac{2205\lambda^4\zeta(7)}{16384\pi^8} + \frac{3213\lambda^5\zeta(9)}{32768\pi^{10}} \\ &\quad - \frac{78771\lambda^6\zeta(11)}{1048576\pi^{12}} + \frac{250965\lambda^7\zeta(13)}{4194304\pi^{14}} - \frac{105424605\lambda^8\zeta(15)}{2147483648\pi^{16}} \\ &\quad + \frac{265525975\lambda^9\zeta(17)}{6442450944\pi^{18}} - \frac{12108123027\lambda^{10}\zeta(19)}{343597383680\pi^{20}} + \mathcal{O}(\lambda^{11}). \end{aligned} \quad (4.10)$$

Extending the $\mathcal{O}(\lambda^5)$ results of [28], see (2.19), and confirming their conjecture for the exponentiation in terms of simple ζ -functions, at least at this order. Other instances have

¹³Here $\mathbf{s} = (s_1, s_2, \dots)$ is a multi-index with non-negative integer values. We shall also adopt the notation s_k^p to denote repetitions of p instances of s_k . So, for instance, $\zeta(3^2, 5^3) = \zeta(3, 3, 5, 5, 5) = \zeta(3)^2 \zeta(5)^3$ and so on.

a more complicated structure with products of ζ -functions. For instance, in $SU(3)$ one has

$$\begin{aligned}
 \log F(\lambda; 3) = & -\frac{9\lambda^2\zeta(3)}{32\pi^4} + \frac{425\lambda^3\zeta(5)}{2304\pi^6} - \frac{17885\lambda^4\zeta(7)}{147456\pi^8} + \frac{5565\lambda^5\zeta(9)}{65536\pi^{10}} \\
 & + \lambda^6 \left(\frac{1925\zeta(5)^2}{14155776\pi^{12}} - \frac{2668897\zeta(11)}{42467328\pi^{12}} \right) + \lambda^7 \left(\frac{32984237\zeta(13)}{679477248\pi^{14}} - \frac{5005\zeta(5)\zeta(7)}{14155776\pi^{14}} \right) \\
 & + \lambda^8 \left(\frac{35035\zeta(7)^2}{150994944\pi^{16}} + \frac{146575\zeta(5)\zeta(9)}{402653184\pi^{16}} - \frac{2245755655\zeta(15)}{57982058496\pi^{16}} \right) \\
 & + \lambda^9 \left(-\frac{1519375\zeta(5)^3}{1174136684544\pi^{18}} - \frac{3488485\zeta(7)\zeta(9)}{7247757312\pi^{18}} - \frac{546184925\zeta(5)\zeta(11)}{1565515579392\pi^{18}} \right. \\
 & \left. + \frac{669686057755\zeta(17)}{21134460321792\pi^{18}} \right) + \lambda^{10} \left(\frac{8083075\zeta(5)^2\zeta(7)}{1565515579392\pi^{20}} + \frac{77643709\zeta(9)^2}{309237645312\pi^{20}} \right. \\
 & \left. + \frac{2905703801\zeta(7)\zeta(11)}{6262062317568\pi^{20}} + \frac{4074100745\zeta(5)\zeta(13)}{12524124635136\pi^{20}} - \frac{29805018472801\zeta(19)}{1127171217162240\pi^{20}} \right) + \mathcal{O}(\lambda^{11}).
 \end{aligned} \tag{4.11}$$

5 Decoupled Toda equation for the $\text{Tr}\varphi^3$ tower

Given the effectiveness of Toda equation in the high order calculation of g_{2n} , it is natural to ask whether similar cases may be treated with the same approach. As we explained, the Toda equation is a consequence of the determinant representation (2.12). On the other hand, this follows from the one-dimensional mixing structure of $(\text{Tr}\varphi^2)^n$ in (2.13). From this point of view, a special case is that of g_{2n}^Φ , see (2.8), where $\Phi = \text{Tr}\varphi^3$. To simplify notation, we shall denote this 2-point function by the special notation $\widehat{g}_{2n}$.

The important (easy) remark is that $(\text{Tr}\varphi^2)^n \text{Tr}\varphi^3$ can only mix with similar operators with $n' < n$. This is clear from the normal-ordering interpretation of mixing discussed in [27] and relations like (pairing stands for Wick contraction)

$$\text{Tr}\varphi^2 \overline{\text{Tr}\varphi^2}^3 \sim \text{Tr}\varphi^3, \quad \text{Tr}(\varphi \overline{\varphi}) \sim \text{Tr}\varphi = 0, \text{ etc.} \tag{5.1}$$

This simple mixing pattern implies the validity of the Toda equation (2.9), i.e.

$$\partial_\tau \partial_{\bar\tau} \log \widehat{g}_{2n} = \frac{\widehat{g}_{2n+2}}{\widehat{g}_{2n}} - \frac{\widehat{g}_{2n}}{\widehat{g}_{2n-2}} - g_2. \tag{5.2}$$

Before showing how to use (5.2) to constrain the R-charge dependence, let us begin with some explicit check of (5.2) at low values of n . The correlator $\widehat{g}_0 = \langle \text{Tr}\varphi^3 \text{Tr}\bar\varphi^3 \rangle$ is the easiest since there is no mixing to account for. One finds, at order $\mathcal{O}(g^{10})$,

$$\begin{aligned}
 \widehat{g}_0 = & \frac{3(N^2 - 4)(N^2 - 1)}{N} \left(\frac{g^2}{4\pi} \right)^3 \left[1 - \frac{9g^4(N^2 + 3)\zeta(3)}{64\pi^4} + \frac{5g^6(N^2 - 1)(22N^2 + 53)\zeta(5)}{1024\pi^6 N} \right. \\
 & + g^8 \left(\frac{27(3N^4 + 20N^2 + 37)\zeta(3)^2}{4096\pi^8} - \frac{105(11N^6 + 12N^4 - 16N^2 + 29)\zeta(7)}{16384\pi^8 N^2} \right) \\
 & + g^{10} \left(\frac{63(N^2 - 1)(192N^6 + 552N^4 + 515N^2 + 645)\zeta(9)}{262144\pi^{10} N^3} \right. \\
 & \left. \left. - \frac{45(N^2 - 1)(50N^4 + 373N^2 + 633)\zeta(3)\zeta(5)}{65536\pi^{10} N} \right) \right] + \mathcal{O}(g^{12}).
 \end{aligned} \tag{5.3}$$

The next correlator is

$$\hat{g}_2 = \langle \text{Tr} \varphi^2 \text{Tr} \varphi^3 \text{Tr} \bar{\varphi}^2 \text{Tr} \bar{\varphi}^3 \rangle, \quad (5.4)$$

and has been computed in (3.15). Working out the mixing coefficients appearing in the correlators for $n = 2, 3$ one finds the explicit results collected in appendix B. Plugging these results in (5.2) and taking g_2 from (4.1) and (4.6), one checks that the Toda equation (5.2) for $\hat{g}_{2n}$ is indeed satisfied at this order.

The case of more complicated towers has roused some debate in the past. A detailed account of what happens to decoupling is discussed and clarified in a simple example in appendix C.

5.1 Using the Toda equation as a constraint for the $\hat{g}_{2n}$

We can propose an Ansatz for the perturbative corrections to $\hat{g}_{2n}$ correlators similar to (4.2). It reads

$$\begin{aligned} \hat{g}_{2n} = & \frac{c_N n! 2^{2n}}{(\text{Im } \tau)^{2n+3}} \frac{\Gamma\left(\frac{N^2-1}{2} + 3 + n\right)}{\Gamma\left(\frac{N^2-1}{2} + 3\right)} \left[1 + \mathcal{B}_2 \frac{n^2 + \eta_{2,1}n + \eta_{2,0}}{(\text{Im } \tau)^2} + \mathcal{B}_3 \frac{n^3 + \eta_{3,2}n^2 + \eta_{3,1}n + \eta_{3,0}}{(\text{Im } \tau)^3} \right. \\ & \left. + \mathcal{B}_4 \frac{n^4 + \eta_{4,3}n^3 + \eta_{4,2}n^2 + \eta_{4,1}n + \eta_{4,0}}{(\text{Im } \tau)^4} + \dots \right], \end{aligned} \quad (5.5)$$

where $c_N = \frac{3(N^2-4)(N^2-1)}{N}$. Compared with (4.2), we see that in this case the corrections do not vanish for $n = 0$ because $\hat{g}_0 = \langle \text{Tr} \varphi^3 \text{Tr} \bar{\varphi}^3 \rangle$ is clearly non-trivial. Again, imposing the Toda equation, we can fix all the η -coefficients but not the $\mathcal{B}_k$ ones. For instance, one finds at first order

$$\eta_{2,1} = \frac{1}{2}(N^2 + 5), \quad \eta_{2,0} = \frac{1}{48} \left[2(35 + 12N^2 + N^4) + \frac{9(N^4 - 1)\zeta(3)}{\pi^2 \mathcal{B}_2} \right]. \quad (5.6)$$

Matching the Ansatz (5.5) to the $n = 0$ case gives then

$$\mathcal{B}_2 = -\frac{9\zeta(3)}{2\pi^2}. \quad (5.7)$$

We carried on this procedure up to $\mathcal{O}(g^{14})$. The ratio $\text{SQCD}/\mathcal{N} = 4$ is now expressed by

$$\hat{F}(g, n; N) = \frac{\hat{g}_{2n}|_{\text{SQCD}}}{\hat{g}_{2n}|_{\mathcal{N}=4}} = 1 + \sum_{k=2}^{\infty} \frac{\hat{f}_k(n; N)}{(\text{Im } \tau)^k}, \quad (5.8)$$

where

$$\begin{aligned} f_2(n; N) = & -\frac{9(n+1)(N^2+2n+3)\zeta(3)}{4\pi^2}, \\ f_3(n; N) = & \frac{5(N^2-1)}{16\pi^3 N(N^2+5)(N^2+7)(N^2+9)} [105(3+2n)(53+90n+40n^2) \\ & + (14509+30780n+19800n^2+3600n^3)N^2 + (4259+7952n+3300n^2+160n^3)N^4 \\ & + (515+804n+120n^2)N^6 + 2(11+12n)N^8] \zeta(5), \end{aligned}$$

$$\begin{aligned}
 f_4(n; N) = & \frac{9}{32\pi^4} [(3+2n)(74+153n+99n^2+18n^3) + 12(1+n)(10+12n+3n^2)N^2 \\
 & + (18+29n+9n^2)N^4] \zeta(3)^2 - \frac{35}{64\pi^4 N^2(N^2+5)(N^2+7)(N^2+9)(N^2+11)} \\
 & [3465(1+n)(3+2n)(29+35n+14n^2) \\
 & - 3(688+24675n+62370n^2+55860n^3+17640n^4)N^2 \\
 & + 2(33327+98135n+118060n^2+71890n^3+19670n^4)N^4 \\
 & + 5(33429+91204n+95074n^2+44268n^3+7280n^4)N^6 \\
 & + (74319+182900n+148445n^2+42000n^3+1120n^4)N^8 \\
 & + (13446+29405n+15540n^2+1120n^3)N^{10} \\
 & + 4(273+505n+90n^2)N^{12} + (33+40n)N^{14}] \zeta(7), \tag{5.9}
 \end{aligned}$$

and so on.¹⁴

5.2 Large R-charge limit

In the large R-charge limit (1.2) we have the following expansion of the logarithm of the function $\widehat{F}(\lambda, N) = \lim_{n \rightarrow \infty} \widehat{F}(\sqrt{\lambda/n}, n; N)$

$$\begin{aligned}
 \log \widehat{F}(\lambda; N) = & -\frac{9\lambda^2 \zeta(3)}{32\pi^4} + \frac{25\lambda^3(N^2-1)(2N^4+45N^2+105)\zeta(5)}{64\pi^6 N(N^2+5)(N^2+7)(N^2+9)} \\
 & - \frac{1225\lambda^4(8N^8+260N^6+281N^4-378N^2+693)\zeta(7)}{4096\pi^8 N^2(N^2+5)(N^2+7)(N^2+9)(N^2+11)} \\
 & + \frac{1323\lambda^5(N^2-1)(26N^8+1180N^6+3309N^4+3096N^2+3861)\zeta(9)}{4096\pi^{10} N^3(N^2+5)(N^2+7)(N^2+9)(N^2+11)(N^2+13)} \\
 & + \lambda^6 \left(\frac{5775}{8192\pi^{12}(N^2+5)^2(N^2+7)^2(N^2+9)^2(N^2+11)(N^2+13)(N^2+15)} \right. \\
 & (N^{14}+88N^{12}+15N^{10}-18088N^8-39661N^6+1053540N^4+4281405N^2+4399500)\zeta(5)^2 \\
 & \left. - \frac{17787(122N^{12}+6950N^{10}+24848N^8+8085N^6-12645N^4+15345N^2+32175)\zeta(11)}{65536\pi^{12}N^4(N^2+5)(N^2+7)(N^2+9)(N^2+11)(N^2+13)(N^2+15)} \right) \\
 & + \lambda^7 \left(\frac{552123(N^2-1)}{131072\pi^{14}N^5(N^2+5)(N^2+7)(N^2+9)(N^2+11)(N^2+13)(N^2+15)(N^2+17)} \right. \\
 & (34N^{12}+2424N^{10}+17285N^8+29655N^6+24450N^4+41145N^2+16575)\zeta(13) \\
 & \left. - \frac{15015(N^2-1)}{32768\pi^{14}N(N^2+5)^2(N^2+7)^2(N^2+9)^2(N^2+11)(N^2+13)(N^2+15)(N^2+17)} \right. \\
 & (22N^{14}+2323N^{12}+6951N^{10}-473938N^8-1641088N^6+30589515N^4 \\
 & \left. + 136118115N^2+144364500)\zeta(5)\zeta(7) \right) + \mathcal{O}(\lambda^8), \tag{5.10}
 \end{aligned}$$

where we have emphasized the ζ -functions. Compared to $F(\lambda; N)$, the $\zeta(3)$ -dependent terms again exponentiate and are independent on N . Apart from this, there are no further

¹⁴Again, the next functions are available under request.

special simplifications for low values of N . The first non-trivial specialized cases are $N = 3, 4$ that give¹⁵

$$\begin{aligned}
 \log \widehat{F}(\lambda; 3) &= -\frac{9\lambda^2\zeta(3)}{32\pi^4} + \frac{25\lambda^3\zeta(5)}{144\pi^6} - \frac{15925\lambda^4\zeta(7)}{147456\pi^8} + \frac{147\lambda^5\zeta(9)}{2048\pi^{10}} \\
 &\quad + \lambda^6 \left(\frac{1925\zeta(5)^2}{14155776\pi^{12}} - \frac{8599591\zeta(11)}{169869312\pi^{12}} \right) + \lambda^7 \left(\frac{3177031\zeta(13)}{84934656\pi^{14}} - \frac{5005\zeta(5)\zeta(7)}{14155776\pi^{14}} \right) + \mathcal{O}(\lambda^8), \\
 \log \widehat{F}(\lambda; 4) &= -\frac{9\lambda^2\zeta(3)}{32\pi^4} + \frac{955\lambda^3\zeta(5)}{5888\pi^6} - \frac{429289\lambda^4\zeta(7)}{4521984\pi^8} + \frac{78057\lambda^5\zeta(9)}{1310720\pi^{10}} \\
 &\quad + \lambda^6 \left(\frac{146531\zeta(5)^2}{973969408\pi^{12}} - \frac{68971014343\zeta(11)}{1734512476160\pi^{12}} \right) \\
 &\quad + \lambda^7 \left(\frac{387146868537\zeta(13)}{13876099809280\pi^{14}} - \frac{28131103\zeta(5)\zeta(7)}{77917552640\pi^{14}} \right) + \mathcal{O}(\lambda^8). \tag{5.11}
 \end{aligned}$$

6 Relation with large N factorization

Given our results that hold for all values of the R-charge n and gauge rank N , it is interesting to consider their large N expansion, see also [58–60]. To this aim, let us define in this section the conventional 't Hooft coupling $\lambda = Ng^2$, not to be confused with the large R-charge coupling in (1.2). We are interested in

$$\mathcal{F}_n(\lambda) = \lim_{N \rightarrow \infty} F(\sqrt{\lambda/N}, n; N), \tag{6.1}$$

where $F(g, n; N)$ is in (2.15).¹⁶ Using the results in (4.6), we find

$$\log \mathcal{F}_n(\lambda) = n \log \mathcal{F}_1(\lambda), \tag{6.2}$$

with (we write only the first 7 orders, but we checked (6.2) at order $\mathcal{O}(\lambda^{10})$)

$$\begin{aligned}
 \log \mathcal{F}_1(\lambda) &= -\frac{9\lambda^2\zeta(3)}{64\pi^4} + \frac{15\lambda^3\zeta(5)}{128\pi^6} + \lambda^4 \left(\frac{99\zeta(3)^2}{8192\pi^8} - \frac{175\zeta(7)}{2048\pi^8} \right) + \lambda^5 \left(\frac{4095\zeta(9)}{65536\pi^{10}} - \frac{405\zeta(3)\zeta(5)}{16384\pi^{10}} \right) \\
 &\quad + \lambda^6 \left(-\frac{189\zeta(3)^3}{131072\pi^{12}} + \frac{6375\zeta(5)^2}{524288\pi^{12}} + \frac{2835\zeta(3)\zeta(7)}{131072\pi^{12}} - \frac{98637\zeta(11)}{2097152\pi^{12}} \right) \\
 &\quad + \lambda^7 \left(\frac{4995\zeta(3)^2\zeta(5)}{1048576\pi^{14}} - \frac{21595\zeta(5)\zeta(7)}{1048576\pi^{14}} - \frac{77805\zeta(3)\zeta(9)}{4194304\pi^{14}} + \frac{153153\zeta(13)}{4194304\pi^{14}} \right) \\
 &\quad + \lambda^8 \left(\frac{13203\zeta(3)^4}{67108864\pi^{16}} - \frac{169425\zeta(3)\zeta(5)^2}{33554432\pi^{16}} - \frac{38115\zeta(3)^2\zeta(7)}{8388608\pi^{16}} + \frac{568645\zeta(7)^2}{67108864\pi^{16}} \right. \\
 &\quad \left. + \frac{144585\zeta(5)\zeta(9)}{8388608\pi^{16}} + \frac{2155923\zeta(3)\zeta(11)}{134217728\pi^{16}} - \frac{7818525\zeta(15)}{268435456\pi^{16}} \right) + \dots \tag{6.3}
 \end{aligned}$$

¹⁵Here it does not make sense to set $N = 2$ since the correlators $\widehat{g}_{2n}$ are zero due to $\text{Tr} \varphi^3 = 0$ in $\text{SU}(2)$. This is consistent with $c_2 = 0$ in (5.5).

¹⁶It is important to remark that absence of instanton corrections at large N is based on the assumption that the instanton moduli integration does not spoil the instanton action exponential suppression as $g \rightarrow 0$. In general, this may be a non-trivial issues, and instantons may lead to large N phase transitions, see for instance [61]. A test of this assumption in $\mathcal{N} = 2$ SQCD is discussed in [21] where it is shown that the one-instanton corrections to the partition function leads to a $\sqrt{N}$ enhancement of the exponentially small instanton action factor. Thus, it is negligible at large N .

Relation (6.2) expresses factorization

$$\langle (\text{Tr} \varphi^2)^n (\text{Tr} \bar{\varphi}^2)^n \rangle \sim \langle \text{Tr} \varphi^2 \text{Tr} \bar{\varphi}^2 \rangle^n, \quad (6.4)$$

at leading large N . The finite N corrections to (6.2) are $\mathcal{O}\left(\frac{n}{N^2}, \frac{n^2}{N^2}\right)$, where we explicitly denoted that the dependence on n is the sum of linear and quadratic pieces.

A similar analysis for the function $\hat{\mathcal{F}}$ which is the large N limit of the function $\hat{F}$ in (5.8) shows that (6.2) reads now

$$\log \frac{\hat{\mathcal{F}}_n(\lambda)}{\hat{\mathcal{F}}_0(\lambda)} = n \log \mathcal{F}_1(\lambda), \quad (6.5)$$

where $\log \mathcal{F}_1(\lambda)$ is the expression in (6.3). Again, this implies the following large N factorization

$$\langle (\text{Tr} \varphi^2)^n \text{Tr} \varphi^3 (\text{Tr} \bar{\varphi}^2)^n \text{Tr} \bar{\varphi}^3 \rangle \sim \langle \text{Tr} \varphi^2 \text{Tr} \bar{\varphi}^2 \rangle^n \langle \text{Tr} \varphi^3 \text{Tr} \bar{\varphi}^3 \rangle. \quad (6.6)$$

Subleading corrections to large N may of course be extracted easily from the general expressions valid for generic N .

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A Functions $F^{(\ell)}(N)$ for $\ell = 8, 9, 10$

Let us define

$$P_k = \left(\frac{N^2 + 1}{2} \right)_k, \quad (x)_k = \frac{\Gamma(x+k)}{\Gamma(x)}. \quad (\text{A.1})$$

The functions $F^{(\ell)}(N)$ defined in section 4 for $\ell = 8, 9, 10$ are given by the following expressions.

$$\begin{aligned} F_{7^2}^{(8)}(N) &= \frac{1576575(N^2-4)}{256N^2P_3P_7} (98N^{14} + 928N^{12} - 4151N^{10} - 44359N^8 - 42036N^6 \\ &\quad + 26754N^4 + 14553N^2 - 16299), \\ F_{5,9}^{(8)}(N) &= \frac{675675(N^2-4)}{2N^2(N^2+3)P_7} (23N^{10} + 70N^8 - 1455N^6 - 1335N^4 + 192N^2 - 855) \\ F_{15}^{(8)}(N) &= -\frac{41409225}{256N^6P_7} (540N^{14} + 3780N^{12} + 4676N^{10} + 440N^8 + 329N^6 \\ &\quad - 735N^4 + 735N^2 + 315), \\ F_{5^3}^{(9)}(N) &= \frac{3038750(N^2-4)}{3N(N^2+3)^2P_8} (2N^{12} - 28N^{10} - 511N^8 + 2187N^6 + 24479N^4 + 8701N^2 - 7950), \end{aligned}$$

$$\begin{aligned}
 F_{7,9}^{(9)}(N) &= -\frac{1276275(N^2-4)}{32N^3P_3P_8}(724N^{16}+9884N^{14}-19378N^{12}-455219N^{10} \\
 &\quad -693063N^8-78342N^6+24156N^4-48195N^2+162729), \\
 F_{5,11}^{(9)}(N) &= -\frac{10027875(N^2-4)}{8N^3(N^2+3)P_8}(142N^{12}+926N^{10}-9807N^8 \\
 &\quad -23606N^6-3754N^4+2184N^2+7035), \\
 F_{17}^{(9)}(N) &= \frac{29548805}{32N^7P_8}(1866N^{16}+19976N^{14}+45374N^{12} \\
 &\quad +17804N^{10}+345N^8-1260N^6+4410N^4-6300N^2-1575), \\
 F_{52,7}^{(10)}(N) &= -\frac{121246125(N^2-4)(N^2-1)}{32N^2(N^2+3)P_3P_9}(7N^{16}-16N^{14}-2593N^{12}-11454N^{10} \\
 &\quad +132421N^8+888024N^6+1182001N^4+282422N^2-148380), \\
 F_{92}^{(10)}(N) &= \frac{2909907(N^2-4)}{64N^4(N^2+7)P_3P_9}(7697N^{20}+205301N^{18}+1156794N^{16} \\
 &\quad -6284951N^{14}-65595428N^{12}-119443263N^{10}-65064156N^8 \\
 &\quad -37063809N^6-36848169N^4-7387038N^2-18683298), \\
 F_{7,11}^{(10)}(N) &= \frac{53348295(N^2-4)}{256N^4P_3P_9}(3218N^{18}+60754N^{16}+34194N^{14}-2724467N^{12} \\
 &\quad -7509568N^{10}-3899769N^8+2216346N^6+1637547N^4-330750N^2-1099665), \\
 F_{5,13}^{(10)}(N) &= \frac{297226215(N^2-4)}{8N^4(N^2+3)P_9}(111N^{14}+1214N^{12}-6968N^{10}-42034N^8-20548N^6 \\
 &\quad +5225N^4-9555N^2-8085), \\
 F_{19}^{(10)}(N) &= -\frac{2133423721}{640N^8P_9}(10688N^{18}+165378N^{16}+616452N^{14}+541822N^{12} \\
 &\quad +49410N^{10}+8235N^8-11340N^6-28350N^4+85050N^2+14175). \tag{A.2}
 \end{aligned}$$

B The correlators $\hat{g}_{2n}$ for $n = 2, 3$

The explicit expressions for the correlators $\hat{g}_{2n}$ for $n = 2, 3$ are

$$\begin{aligned}
 \hat{g}_4 &= \frac{24(N^2-1)(N^2-4)(N^2+5)(N^2+7)}{N} \left(\frac{g^2}{4\pi} \right)^7 \left[1 - \frac{27g^4(N^2+7)\zeta(3)}{64\pi^4} \right. \\
 &\quad + \frac{5g^6(N^2-1)(70N^6+1973N^4+16886N^2+32095)\zeta(5)}{1024\pi^6N(N^2+5)(N^2+7)} \\
 &\quad + g^8 \left(\frac{9(14N^4+207N^2+805)\zeta(3)^2}{1024\pi^8} \right. \\
 &\quad \left. \left. - \frac{35(113N^{10}+4312N^8+45949N^6+41951N^4-54186N^2+113925)\zeta(7)}{16384\pi^8N^2(N^2+5)(N^2+7)} \right) \right. \\
 &\quad + g^{10} \left(\frac{63(N^2-1)(712N^{10}+37376N^8+575639N^6+1477845N^4+1403305N^2+1644195)\zeta(9)}{262144\pi^{10}N^3(N^2+5)(N^2+7)} \right. \\
 &\quad \left. \left. - \frac{45(N^2-1)(310N^8+11471N^6+154469N^4+833209N^2+1319325)\zeta(3)\zeta(5)}{65536\pi^{10}N(N^2+5)(N^2+7)} \right) \right] + \mathcal{O}(g^{12}),
 \end{aligned}$$

$$\begin{aligned}
 \widehat{g}_6 = & \frac{144(N^2-1)(N^2-4)(N^2+5)(N^2+7)(N^2+9)}{N} \left(\frac{g^2}{4\pi} \right)^9 \left[1 - \frac{9g^4(N^2+9)\zeta(3)}{16\pi^4} \right. \\
 & + \frac{5g^6(N^2-1)(94N^8+4007N^6+62135N^4+382249N^2+645435)\zeta(5)}{1024\pi^6 N(N^2+5)(N^2+7)(N^2+9)} \\
 & + g^8 \left(\frac{27(31N^4+584N^2+2865)\zeta(3)^2}{4096\pi^8} \right. \\
 & \left. - \frac{105(51N^{12}+2903N^{10}+58654N^8+416054N^6+336111N^4-414173N^2+982800)\zeta(7)}{16384\pi^8 N^2(N^2+5)(N^2+7)(N^2+9)} \right) \\
 & + g^{10} \left(\frac{63(N^2-1)}{262144\pi^{10} N^3(N^2+5)(N^2+7)(N^2+9)} (972N^{12}+74304N^{10}+2058613N^8 \right. \\
 & + 21049036N^6+50557990N^4+48478260N^2+53986905) \zeta(9) \\
 & \left. - \frac{45(N^2-1)(128N^{10}+6850N^8+145489N^6+1476329N^4+6789279N^2+9999045)\zeta(3)\zeta(5)}{16384\pi^{10} N(N^2+5)(N^2+7)(N^2+9)} \right) \\
 & \left. + \mathcal{O}(g^{12}) \right]. \tag{B.1}
 \end{aligned}$$

C Decoupling violations and Toda equation for general towers

Let us clarify to what extent we may have decoupling and reduction to a single semi-infinite Toda equation for more general towers of the form $\Phi^{(n)} = (\text{Tr}\varphi^2)^n \Phi$ with $\dim \Phi > 3$. Although general discussions already appeared in [25, 26], we consider here in full details the example of $\Phi = \text{Tr}\varphi^4$ and consider in somewhat more details what is the fate of the would-be decoupled Toda equation.

Compared with the case $\Phi = \text{Tr}\varphi^3$, an important difference is that Wick contraction inside $\text{Tr}\varphi^4$ brings back to the tower of the identity $\Phi = \mathbb{I}$. Let us work at the level of matrix model traces t_n , cf. (3.5) and denote by $:T:$, the operator T with resolved mixing, i.e. after subtraction of the suitable contraction ordering contributions, i.e. as in the r.h.s. of (3.11).¹⁷ At dimension $4 + 2n$, we can consider the two operators

$$\mathcal{O}_n =: \underbrace{t_{2,\dots,2,4}}_n :, \quad \mathcal{O}'_n =: \underbrace{t_{2,\dots,2}}_{n+2} :. \tag{C.1}$$

These are *natural* in the sense that they map to the simple flat space operators

$$\mathcal{O}_n^{\mathbb{R}^4} = (\text{Tr}\varphi^2)^n \text{Tr}\varphi^4, \quad \mathcal{O}'_n{}^{\mathbb{R}^4} = (\text{Tr}\varphi^2)^{n+2}. \tag{C.2}$$

Besides, we can construct the operator

$$\widetilde{\mathcal{O}}_n = \mathcal{O}_n - \kappa_n(g) \mathcal{O}'_n, \tag{C.3}$$

where $\kappa_n(g)$ is chosen such that $\langle \widetilde{\mathcal{O}}_n \overline{\mathcal{O}}'_n \rangle = 0$. The second term in (C.3) is a further subtraction unrelated to S^4 mixing, since the two operators have the same dimension. Instead, it is a loop corrected orthogonalization similar in spirit to the tree level diagonalization discussed in [25], and shown to be problematic due to three loop corrections in [26].

¹⁷Notice that in the notation of [45], this is denoted $:X:_g$ to emphasize the difference with respect to the $\mathcal{N} = 4$ limit.

Now, it should be clearly stated that the 2-point function of $\tilde{\mathcal{O}}_n$ do satisfy the decoupled Toda equation (2.9), contrary to $\mathcal{O}_n$. So, decoupling is possible, but the point is that it is not achieved in a natural way because the coefficients $\kappa_n(g)$ in (C.3) depend on n . This means that, for instance, the large n limit has to take into account the extra n dependence in $\kappa_n(g)$.

To see this in details, let us work out the explicit expressions of the above operators at level $n = 0$, i.e. dimension 4. We find

$$\mathcal{O}_0 = t_4 - \frac{(2N^2 - 3)}{N(N^2 + 1)} t_{2,2} - \frac{25(N^2 - 9)(N^2 - 4)\zeta(5)}{256\pi^6(N^2 + 1)} \left[t_{2,2} + \frac{3}{20} (N^4 - 1) - \frac{3}{4} (N^2 + 1)t_2 \right] g^6 + \mathcal{O}(g^8), \quad (\text{C.4})$$

$$\mathcal{O}'_0 = t_{2,2} + \frac{1}{4} (N^4 - 1) - (N^2 + 1)t_2 \frac{3(N^2 + 1)(N^2 + 5)\zeta(3)}{64\pi^4} \left[t_2 - \frac{(N^2 - 1)(N^2 + 4)}{2(N^2 + 5)} \right] g^4 - \frac{15(N^2 + 1)(N^2 + 11)(2N^2 - 1)\zeta(5)}{1024\pi^6 N} \left[t_2 - \frac{(N^2 - 1)(N^2 + 9)}{2(N^2 + 11)} \right] g^6 + \mathcal{O}(g^8). \quad (\text{C.5})$$

The condition $\langle \tilde{\mathcal{O}}_0 \overline{\mathcal{O}}'_0 \rangle = \langle [\mathcal{O}_0 - \kappa_0(g) \mathcal{O}'_0] \overline{\mathcal{O}}'_0 \rangle = 0$ gives

$$\kappa_0(g) = \frac{2N^2 - 3}{N(N^2 + 1)} + \frac{25(N^2 - 9)(N^2 - 4)\zeta(5)}{256\pi^6(N^2 + 1)} g^6 + \mathcal{O}(g^8). \quad (\text{C.6})$$

Repeating the calculation for $n = 1, 2, 3$ gives a similar result but with a modified $\mathcal{O}(g^6)$ term

$$\kappa_n(g) = \frac{2N^2 - 3}{N(N^2 + 1)} + \frac{25(N^2 - 9)(N^2 - 4)\zeta(5)}{256\pi^6(N^2 + 1)} \alpha_n g^6 + \mathcal{O}(g^8), \quad (\text{C.7})$$

where

$$\alpha_1 = \frac{N^2 + 11}{N^2 + 3}, \quad \alpha_2 = \frac{(N^2 + 9)(N^2 + 15)}{(N^2 + 3)(N^2 + 5)}, \quad \alpha_3 = \frac{(N^2 + 9)(N^2 + 11)(N^2 + 19)}{(N^2 + 3)(N^2 + 5)(N^2 + 7)}. \quad (\text{C.8})$$

As anticipated, the coefficient $\kappa_n(g)$ depends on n starting at three loops.

Let us now show that the 2-point function $\tilde{g}_{2n}$ of the $\tilde{\mathcal{O}}_n$ operators is still captured by the Toda equation. We find (the tree level is known, see for instance [26])

$$\tilde{g}_{2n} = \frac{4(N^2 - 1)(N^2 - 4)(N^2 - 9)}{N^2 + 1} \frac{n! 2^{2n}}{(\text{Im } \tau)^{2n+4}} \frac{\Gamma(\frac{N^2 - 1}{2} + 4 + n)}{\Gamma(\frac{N^2 - 1}{2} + 4)} (1 + \delta \tilde{g}_{2n}), \quad (\text{C.9})$$

with

$$\begin{aligned} \delta \tilde{g}_0 &= -\frac{3g^4(N^2 + 4)\zeta(3)}{16\pi^4} + \frac{5g^6(N^2 - 1)(8N^2 + 3)\zeta(5)}{256\pi^6 N} + \mathcal{O}(g^8), \\ \delta \tilde{g}_2 &= -\frac{3g^4(7N^2 + 43)\zeta(3)}{64\pi^4} + \frac{5g^6(N^2 - 1)(7N^4 + 127N^2 + 45)\zeta(5)}{128\pi^6 N(N^2 + 7)} + \mathcal{O}(g^8), \\ \delta \tilde{g}_4 &= -\frac{3g^4(5N^2 + 41)\zeta(3)}{32\pi^4} + \frac{5g^6(N^2 - 1)(20N^6 + 689N^4 + 6150N^2 + 2121)\zeta(5)}{256\pi^6 N(N^2 + 7)(N^2 + 9)} + \mathcal{O}(g^8) \\ \delta \tilde{g}_6 &= -\frac{3g^4(13N^2 + 133)\zeta(3)}{64\pi^4} \\ &\quad + \frac{5g^6(N^2 - 1)(13N^8 + 672N^6 + 11876N^4 + 71268N^2 + 24051)\zeta(5)}{128\pi^6 N(N^2 + 7)(N^2 + 9)(N^2 + 11)} + \mathcal{O}(g^8). \end{aligned} \quad (\text{C.10})$$

Using (C.9) and (C.10), one can check that the first instances of the Toda equation (2.9) are satisfied. This includes the effect of the 3-loop $\zeta(5)$ terms which are responsible for the g -dependent mixing discussed in [26]. This means that it is easy to obtain closed expressions for the expansions in (C.10) by the methods discussed in the main text. For instance, the first correction in (C.10) is

$$\delta\tilde{g}_{2n} = -\frac{3[16 + 21n + 6n^2 + (4 + 3n)N^2]}{64\pi^4}\zeta(3)g^4 + \dots, \quad (\text{C.11})$$

while higher order corrections may be treated as we discussed the $\mathbb{I}$ and $\text{Tr}\varphi^3$ towers.

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