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Background independence of closed superstring field theory

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Abstract: Given a family of world-sheet superconformal field theories related by marginal deformation, we can formulate superstring field theory based on any of these world-sheet theories. Background independence is the statement that these different superstring field theories are related to each other by field redefinition. We prove background independence of closed superstring field theory.

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1 Introduction

Dealing with the problem of mass renormalization and vacuum shift in superstring perturbation theory requires formulation of quantum superstring field theory. We now have such a field theory for heterotic and type II string theories (for a review see [1]) arising out of a generalization of the bosonic closed string field theory constructed in [2–5] and tree level Neveu-Schwarz (NS) sector superstring field theory described in [6]. However this field theory is apparently background dependent. One starts with a specific world-sheet superconformal field theory (SCFT) describing a specific string compactification, and uses the correlation functions of vertex operators in this world-sheet theory to construct the interaction vertices of the superstring field theory. Therefore two different world-sheet theories describing different string compactifications will lead to apparently different superstring field theories. Background independence is the statement that these apparently different superstring field theories are related to each other by field redefinition. Our goal in this paper will be to prove background independence for the case where the different world-sheet

theories are related by marginal deformation. Besides being of theoretical interest this also has 'practical' consequences — for example background independence of superstring field theory is used implicitly in the proof of soft theorem (see e.g. [7]) where it is assumed that the response of string field theory to an infinitesimal on-shell graviton field is identical to that under a change in the target space metric in the underlying world-sheet theory.

Background independence of bosonic string field theory was proved in [8, 9]. As in those papers, we shall restrict our analysis to the case where the pair of world-sheet theories, for which we want to establish the equivalence of the corresponding string field theories, are related to each other by infinitesimal marginal deformation. Once the result is proved for infinitesimal deformations, it also establishes the result for finite marginal deformations since a finite marginal deformation can be built from successive application of infinitesimal deformations. Of course, beyond a critical distance between the world-sheet theories the required field redefinition may diverge, but this is simply a reflection of the fact that the coordinate system in the space of string fields formulated around one world-sheet theory may break down beyond a certain distance.

Superstring field theory enjoys infinite parameter gauge invariance. Due to this gauge invariance the field redefinition that relates a pair of superstring field theories is not unique — given one such field redefinition we can find infinite number of other field redefinitions that differ from the first one by gauge transformation with possibly field dependent gauge transformation parameters. This will be seen explicitly in our analysis.

Our analysis will differ from that of [9] for bosonic string field theory in one important way. The analysis of [9] focussed on the Batalin-Vilkovisky (BV) master action. For this reason establishing equivalence of two superstring field theories required taking into account possible change in the integration measure under the field redefinition. Here we shall work with the one particle irreducible (1PI) effective action. Since the tree amplitudes computed from the 1PI effective action give the full amplitudes of string theory, we do not need to worry about the change in the integration measure. In fact it is also not necessary to prove the equivalence of the actions — it is sufficient to show that the equations of motion of the two theories get related to each other by field redefinition. This is useful due to the fact that superstring field theory, as formulated in [1], contains a set of additional free fields besides the interacting string field. While these additional fields are needed for writing down the action, the interacting field equations can be written in terms of the physical string fields, and contains full information about the S-matrix. Therefore it will be enough to show that the interacting field equations in the two theories are related to each other by field redefinition. This is the strategy we shall follow.

Rest of the paper is organized as follows. In section 2 we review some of the details of the superstring field theory and also introduce some new notations that will simplify our analysis. In section 3 we discuss two ways of describing string field theory around a new background. The first corresponds to deforming the world-sheet SCFT by an infinitesimal marginal deformation and formulating string field theory around the new background. The second approach is to take the original string field theory action and expand it around an infinitesimal classical solution to the linearized equations of motion corresponding to the same marginal deformation. Background independence is the statement that these two

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string field theories are related by a field redefinition. In section 4 we give geometric interpretations of the kinetic and interaction terms of the superstring field theory around marginally deformed background that is needed for our proof of background independence. Using these results, we give an explicit proof of the background independence in section 5 by describing a systematic algorithm for constructing the field redefinition that relates the two versions of superstring field theory. In section 6 we describe extension of our analysis to type II string theories and also possible applications of our method to other versions of open and closed superstring field theories.

2 Review of superstring field theory and some notations

In this section we shall first review the construction of superstring field theory described in [1] and then introduce some new notations that will simplify our analysis.

2.1 Superstring field theory

We begin with a very brief review of superstring field theory — more details can be found in [1]. For simplicity we shall consider heterotic string theory, but the analysis can be easily generalized to type II string theory. Our starting point is the matter-ghost world-sheet superconformal field theory, describing string theory in a specific background. We denote by b_n, c_n, L_n and $\bar{b}_n, \bar{c}_n, \bar{L}_n$ the usual modes of the holomorphic and anti-holomorphic parts of the anti-commuting ghost fields and the total stress tensor. We also denote by β, γ the superconformal ghosts and by ξ, η, ϕ the fields obtained via bosonization of these fields:

$$\beta = \partial\xi\, e^{-\phi}, \quad \gamma = \eta\, e^{\phi} \tag{2.1}$$

satisfying the operator product expansion

$$\begin{aligned} \xi(z)\,\eta(w) &= (z-w)^{-1} + \text{non-singular terms}, \\ e^{q\phi}(z)\,e^{q'\phi}(w) &= (z-w)^{-q\,q'} e^{(q+q')\phi}(w) + \text{less singular terms}\,. \end{aligned} \tag{2.2}$$

ξ_n, η_n will denote the modes of the ξ, η fields. We define the picture number of a state such that ξ carries picture number 1, η carries picture number -1, $e^{q\phi}$ carries picture number q and other fields carry picture number 0. We shall denote by $\mathcal{H}_p$ the Hilbert space of the world-sheet theory carrying picture number p and subject to the following constraints on the states

$$\begin{aligned} &b_0^-|\Psi\rangle = 0, && L_0^-|\Psi\rangle = 0, && \eta_0|\Psi\rangle = 0, \\ &b_0^\pm \equiv b_0 \pm \bar{b}_0, && L_0^\pm \equiv L_0 \pm \bar{L}_0, && c_0^\pm \equiv \frac{1}{2}\,(c_0 \pm \bar{c}_0)\,. \end{aligned} \tag{2.3}$$

States in $\mathcal{H}_p$ for $p \in \mathbb{Z}$ are NS sector states and for $p \in \mathbb{Z}+\frac{1}{2}$ are Ramond (R) sector states.

Q_B will denote the nilpotent BRST operator

$$Q_B = \oint dz\, \jmath_B(z) + \oint d\bar{z}\, \bar{\jmath}_B(\bar{z})\,, \tag{2.4}$$

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where

$$\bar{\jmath}_B(\bar{z}) = \bar{c}(\bar{z})\bar{T}_m(\bar{z}) + \bar{b}(\bar{z})\bar{c}(\bar{z})\bar{\partial}\bar{c}(\bar{z})\,, \tag{2.5}$$

$$\jmath_B(z) = c(z)(T_m(z) + T_{\beta,\gamma}(z)) + \gamma(z)T_F(z) + b(z)c(z)\partial c(z) - \frac{1}{4}\gamma(z)^2 b(z)\,, \tag{2.6}$$

and $\oint$ is normalized so that $\oint dz/z = 1$, $\oint d\bar{z}/\bar{z} = 1$. T_m denotes the stress tensor of the matter SCFT, $T_{\beta,\gamma}$ denotes the stress tensor of the β,γ system or equivalently the ξ,η,ϕ system and T_F denotes the supercurrent of the matter SCFT.

The picture changing operator (PCO) $\mathcal{X}$ is defined as [10, 11]

$$\mathcal{X}(z) = \{Q_B, \xi(z)\} = c\,\partial\xi + e^{\phi}T_F - \frac{1}{4}\partial\eta\, e^{2\phi}\, b - \frac{1}{4}\partial\left(\eta\, e^{2\phi}\, b\right)\,. \tag{2.7}$$

This is a BRST invariant primary operator of dimension zero carrying picture number 1. $\mathcal{X}_0$ will denote the zero mode of $\mathcal{X}(z)$:

$$\mathcal{X}_0 = \int \frac{dz}{z}\,\mathcal{X}(z)\,. \tag{2.8}$$

Also we define the operator $\mathcal{G}$ such that, acting on a state in $\mathcal{H}_p$,

$$\mathcal{G} = \begin{cases} 1 & \text{if} \quad p \in \mathbb{Z}, \\ \mathcal{X}_0 & \text{if} \quad p \in \mathbb{Z} + \frac{1}{2}\,. \end{cases} \tag{2.9}$$

Heterotic string field theory contains a pair of fields $|\Psi\rangle$ and $|\widetilde{\Psi}\rangle$ which are taken to be arbitrary states in

$$\widehat{\mathcal{H}} \equiv \mathcal{H}_{-1} + \mathcal{H}_{-1/2}, \quad \text{and} \quad \widetilde{\mathcal{H}} \equiv \mathcal{H}_{-1} + \mathcal{H}_{-3/2}\,, \tag{2.10}$$

respectively. If $\{|\hat{\varphi}_r\rangle\}$ denote the basis states in $\widehat{\mathcal{H}}$ then we expand the string field $|\Psi\rangle$ as a linear combination $\sum_r \psi_r|\hat{\varphi}_r\rangle$ and identify the coefficients of expansion ψ_r as the dynamical variables of the theory. One of the key features of closed string field is that the coefficient ψ_r has the same grassmann parity as the vertex operator $\hat{\varphi}_r$ of the SCFT so that Ψ is always grassmann even. Similar remarks hold for $|\widetilde{\Psi}\rangle$.

The string field action S is a function of these variables ψ_r and $\widetilde{\psi}_r$. For our purpose it will be most convenient to work with the one particle irreducible (1PI) effective action which takes the form:[1]

$$S = g_s^{-2}\left[-\frac{1}{2}\langle\widetilde{\Psi}|c_0^-\mathcal{G}\,Q_B|\widetilde{\Psi}\rangle + \langle\widetilde{\Psi}|c_0^- Q_B|\Psi\rangle + \sum_{N\geq 1}\frac{1}{N!}\{\Psi^N\}\right]\,. \tag{2.11}$$

Here g_s is the string coupling. The last term involving $\{\Psi^N\}$ describes interaction term and is defined as follows.

1. We denote by $\mathcal{M}_{g,m,n}$ the moduli space of genus g Riemann surfaces carrying m NS punctures and n R punctures. We include in the definition of $\mathcal{M}_{g,m,n}$ the choice of spin structure.

[1] The genus zero contribution to $\{\Psi^N\}$ vanishes for $N = 1, 2$.

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2. We define $\widetilde{\mathcal{P}}_{g,m,n}$ to be a fiber bundle with base $\mathcal{M}_{g,m,n}$ and fiber containing information on the choice of local coordinates at each puncture and the locations of $2g-2+m+n/2$ PCO's on the Riemann surface. Therefore specifying a section of $\widetilde{\mathcal{P}}_{g,m,n}$ corresponds to making a specific choice of local coordinates at the punctures and choice of PCO locations for every Riemann surface of genus g and $m+n$ punctures.

3. We shall use the name generalized subspace to denote a formal weighed sum of subspaces of $\widetilde{\mathcal{P}}_{g,m,n}$ with the understanding that integration over such weighted subspaces mean weighted sum of integrals over each of the subspaces that are being summed. The generalized subspaces can include vertical segments along which the PCO locations change from an initial configuration to a final configuration one at a time, keeping the moduli and local coordinates at the punctures fixed. These vertical segments can pass through spurious poles [11–13] where the conformal field theory correlation functions on the corresponding Riemann surfaces have singularities. However all the relevant integrals along these vertical segments will be well defined [14–16], and as a result we can continue to treat the vertical segments as regular subspaces of $\widetilde{\mathcal{P}}_{g,m,n}$ for various manipulations.

4. We shall use the name 'section segment' of $\widetilde{\mathcal{P}}_{g,m,n}$ to denote a $6g-6+2m+2n$ dimensional generalized subspace of $\widetilde{\mathcal{P}}_{g,m,n}$, such that the projection of its boundary on the base of $\widetilde{\mathcal{P}}_{g,m,n}$ encloses a proper subspace of the moduli space $\mathcal{M}_{g,m,n}$.

5. For given g,m,n and a set of m different NS states $K_1,\cdots K_m$ and n different R states $L_1,\cdots L_n$, one can define a natural p-form $\Omega_p^{(g,m,n)}(K_1,\cdots K_m,L_1,\cdots,L_n)$ on $\widetilde{\mathcal{P}}_{g,,m,n}$ for any integer p. At a given point σ in $\widetilde{\mathcal{P}}_{g,,m,n}$, the expression for $\Omega_p^{(g,m,n)}(K_1,\cdots K_m,L_1,\cdots,L_n)$ involves correlation function of the vertex operators $\{K_i\},\{L_i\}$ inserted using the appropriate local coordinate system at the punctures corresponding to the point σ, PCO's inserted at appropriate locations corresponding to the point σ, and other insertions of ghost fields.

6. We now define

$$\begin{aligned}\{K_1\cdots K_m\,L_1\cdots L_n\}_g &= \int_{\mathcal{R}_{g,m,n}} \Omega^{(g,m,n)}_{6g-6+2m+2n}(K_1,\cdots K_m,L_1,\cdots,L_n),\\ \{K_1\cdots K_m\,L_1\cdots L_n\} &= \sum_{g=0}^{\infty}(g_s)^{2g}\,\{K_1\cdots K_m\,L_1\cdots L_n\}_g\,,\end{aligned} \tag{2.12}$$

where $\mathcal{R}_{g,m,n}$ is a section segment of $\widetilde{\mathcal{P}}_{g,m,n}$ that never includes singular Riemann surfaces corresponding to separating type degeneration but includes singular Riemann surfaces corresponding to non-separating type degenerations. $\mathcal{R}_{g,m,n}$ is taken to be symmetric under the exchange of any pair of NS punctures or any pair of R punctures, and satisfies some other constraints that we shall describe shortly.

7. Given the definition of $\{\cdots\}$ in (2.12), the interaction term $\{\Psi^N\}$ in (2.11) just means that the state Ψ is inserted N times inside the curly bracket.

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$\Omega_p^{(g,m,n)}(K_1,\cdots K_m,L_1,\cdots,L_n)$ satisfies the useful identity

$$\begin{aligned}&\Omega_p^{(g,m,n)}\left(Q_B\,K_1,\cdots K_m,L_1,\cdots,L_n\right)+\cdots+\Omega_p^{(g,m,n)}\left(K_1,\cdots K_m,L_1,\cdots,Q_BL_n\right)\\&=(-1)^p d\Omega_{p-1}^{(g,m,n)}\left(K_1,\cdots K_m,L_1,\cdots,L_n\right)\,.\end{aligned}\tag{2.13}$$

In writing this identity we have assumed that if any of the vertex operators K_i or L_j are grassmann odd, they have been multiplied by a grassmann odd c-number to make them grassmann even. We can recover the correct sign of each term by pulling these grassmann odd c-numbers in each term to the extreme right[2] and then removing a common factor involving their product. This is the convention we shall follow throughout.

While we shall not need the explicit form of $\Omega_p^{(g,m,n)}$ for our analysis, the following information will be useful. The grassmann parity of $\Omega_p^{(g,m,n)}(K_1,\cdots K_m,L_1,\cdots,L_n)$ is $(-1)^p$ if the states $\{K_i\}$ and $\{L_j\}$ are all grassmann even. This arises from the p number of additional grassmann odd operators inserted in the correlator that defines Ω_p. These are inserted on the *left* of all the states $\{K_i\}$ and $\{L_j\}$. For even states it does not make any difference, but for example this is needed for the correct interpretation of (2.13) where the states Q_BK_i and Q_BL_j are grassmann odd.

Given a pair of points $a\in\widetilde{\mathcal{P}}_{g_1,m_1,n_1}$ and $b\in\widetilde{\mathcal{P}}_{g_2,m_2,n_2}$, we define by $\{a,b\}$ a one parameter family of points in $\widetilde{\mathcal{P}}_{g_1+g_2,m_1+m_2-2,n_1+n_2}$ as follows. Let w_1 be the local coordinate at the last NS puncture of the Riemann surface a and $\widetilde{w}_1$ be the local coordinate at the first NS puncture of the Riemann surface b. Then we construct a one parameter family of Riemann surfaces by making the identification[3]

$$w_1\,\widetilde{w}_1=e^{i\theta},\qquad 0\le\theta<2\pi\,.\tag{2.14}$$

The resulting Riemann surfaces have genus g_1+g_2, m_1+m_2-2 NS punctures and n_1+n_2 R punctures. The operation (2.14) will be called twist sewing. Similarly given $a\in\widetilde{\mathcal{P}}_{g_1,m_1,n_1}$ and $b\in\widetilde{\mathcal{P}}_{g_2,m_2,n_2}$, we define by $\{a;b\}$ a one parameter family of points in $\widetilde{\mathcal{P}}_{g_1+g_2,m_1+m_2,n_1+n_2-2}$ obtained by twist sewing at R punctures. This means that we identify the local coordinate z_1 around the last R puncture of a and the local coordinate $\widetilde{z}_1$ around the first R-puncture of b via

$$z_1\,\widetilde{z}_1=e^{i\theta},\qquad 0\le\theta<2\pi\,.\tag{2.15}$$

This time we also need to insert an extra PCO on the resulting Riemann surface. This is done via the insertion of

$$\mathcal{X}_0=\oint_{|z_1|=1}\frac{dz_1}{z_1}\mathcal{X}(z_1)=\oint_{|\widetilde{z}_1|=1}\frac{d\widetilde{z}_1}{\widetilde{z}_1}\mathcal{X}\left(\widetilde{z}_1\right)\,.\tag{2.16}$$

[2]The reason that it is more convenient to move the grassmann odd parameters to the extreme right instead of extreme left is that in the definition of $\Omega_p^{(g,m,n)}$ there are additional insertions of b-ghosts and other grassmann odd operators besides the operators A_i, and we choose the convention in which they are inserted to the left of the A_i's in a correlation function. There are precisely p insertions of such grassmann odd operators in the definition on Ω_p. Therefore if we want to move the grassmann odd c-numbers to the extreme left, we have to take into account an extra factor of $(-1)^p$ for each such grassmann odd c-number.

[3]This differs from the convention used in [1] where we used $e^{-i\theta}$ instead of $e^{i\theta}$. Our θ parameter is related to that of [1] by a minus sign.

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Therefore $\{a;b\}$ should viewed as a generalized subspace of $\widehat{\mathcal{P}}_{g_1+g_2,m_1+m_2,n_1+n_2-2}$ in the sense described below (2.11) since we are averaging over many subspaces of $\widehat{\mathcal{P}}_{g_1+g_2,m_1+m_2,n_1+n_2-2}$, differing by the location of the PCO.

We are now ready to state the consistency requirement on $\mathcal{R}_{g,m,n}$. It takes the form

$$\begin{aligned}\partial\mathcal{R}_{g,m,n} = &-\frac{1}{2}\sum_{\substack{g_1,g_2\\ g_1+g_2=g}}\sum_{\substack{m_1,m_2\\ m_1+m_2=m+2}}\sum_{\substack{n_1,n_2\\ n_1+n_2=n}}\mathbf{S}[\{\mathcal{R}_{g_1,m_1,n_1},\mathcal{R}_{g_2,m_2,n_2}\}]\\ &-\frac{1}{2}\sum_{\substack{g_1,g_2\\ g_1+g_2=g}}\sum_{\substack{m_1,m_2\\ m_1+m_2=m}}\sum_{\substack{n_1,n_2\\ n_1+n_2=n+2}}\mathbf{S}[\{\mathcal{R}_{g_1,m_1,n_1};\mathcal{R}_{g_2,m_2,n_2}\}]\,,\end{aligned} \tag{2.17}$$

where for example $\{\mathcal{R}_{g_1,m_1,n_1},\mathcal{R}_{g_2,m_2,n_2}\}$ is the result of twist sewing every Riemann surface of $\mathcal{R}_{g_1,m_1,n_1}$ with every Riemann surface of $\mathcal{R}_{g_2,m_2,n_2}$. $\mathbf{S}$ denotes sum over all inequivalent permutation of the external punctures so that the right hand side becomes symmetric under the exchange of any pair of NS punctures and any pair of R punctures. $\partial\mathcal{R}_{g,m,n}$ denotes the boundary of $\mathcal{R}_{g,m,n}$, excluding the boundaries associated with non-separating type degeneration of Riemann surfaces. One important feature of the subspaces $\mathcal{R}_{g,m,n}$ is that they do not include separating type degenerations.[4]

The amplitudes of string field theory are given by sum of tree level Feynman diagrams computed from the 1PI effective action (2.11). For any amplitude with external states $K_1,\cdots K_m$, $L_1,\cdots L_n$ there is one diagram that has a single vertex and no internal propagator — its contribution is given by $\{K_1\cdots K_m\,L_1\cdots L_n\}$ (up to factor of i). This involves integration over the section segments $\mathcal{R}_{g,m,n}$ which cover only part of the moduli space $\mathcal{M}_{g,m,n}$. The contributions from other Feynman diagrams, involving one or more internal propagators, are given by integration over other section segments. Eq. (2.17) guarantees that together these section segments constitute sections of $\widetilde{\mathcal{P}}_{g,m,n}$ over the full moduli space $\mathcal{M}_{g,m,n}$. In particular the regions of the moduli space containing separating type degenerations arise from Feynman diagrams with one or more propagators.

We end this review by describing the structure of classical equations of motion derived from the action — this encodes information on the tree level amplitude computed from (2.11) which is turn gives the loop amplitudes of string theory. The classical equations of motion take the form

$$\begin{aligned}Q_B(|\Psi\rangle-\mathcal{G}|\widetilde{\Psi}\rangle) &= 0,\\ Q_B|\widetilde{\Psi}\rangle+\sum_{N\geq 0}\frac{1}{N!}\left[\Psi^N\right] &= 0\,,\end{aligned} \tag{2.18}$$

where for $A_1,\cdots A_N\in\widehat{\mathcal{H}}$, we define $[A_1\cdots A_N]\in\widetilde{\mathcal{H}}$ via

$$\langle\phi|c_0^-|[A_1\cdots A_N]\rangle=\{\phi\,A_1\cdots A_N\}\,,\quad\forall\,|\phi\rangle\in\widehat{\mathcal{H}}\,. \tag{2.19}$$

Multiplying the second equation in (2.18) by $\mathcal{G}$ and adding it to the first equation we get

$$Q_B|\Psi\rangle+\sum_N\frac{1}{N!}\mathcal{G}\left[\Psi^N\right]=0\,. \tag{2.20}$$

[4]Possible choices of local coordinate systems satisfying these requirements can be found in [5, 17].

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This can be regarded as the equation of motion of the interacting field $|\Psi\rangle$. For given $|\Psi\rangle$ satisfying (2.20), we can find $|\widetilde{\Psi}\rangle$ by solving the second equation in (2.18). Given a particular solution to this equation, the only left over freedom in the choice of $|\widetilde{\Psi}\rangle$ is the addition of a BRST invariant state to the solution. Therefore degrees of freedom associated with $|\widetilde{\Psi}\rangle$ describe non-interacting free fields. This can also be seen from the analysis of the Feynman rules of the theory [1].

Note that in (2.20) we have dropped the summation range over N. In carrying out our manipulations we shall often not specify the ranges in various sums with the understanding that the sum ranges over the full set for which the summand does not vanish.

2.2 Some new notations

So far we have introduced interaction vertices $\{\cdots\}$ where each external state is an off-shell string field represented by a state in $\widehat{\mathcal{H}}$. However for our analysis we shall also need more general interaction vertices where not all external states are in $\widehat{\mathcal{H}}$. For this we introduce the space $\widetilde{\mathcal{P}}_{g,m,n;r,s}$ whose base is the moduli space of genus g Riemann surface with m NS and n R punctures in $\widehat{\mathcal{H}}$, and r NS and s R punctures in $\widetilde{\mathcal{H}}$. The fiber of $\widetilde{\mathcal{P}}_{g,m,n;r,s}$ contains information on the local coordinates at all the $m+n+r+s$ punctures and the locations of the $2g-2+m+n/2+r+3s/2$ PCO's that are needed to get a non-vanishing correlation function. We can introduce the p forms $\Omega_p^{(g,m,n;r,s)}$ on these spaces by simple generalization of the definition of $\Omega_p^{(g,m,n)}$ reviewed in [1]. This p-form satisfies the analog of (2.13):

$$\begin{aligned}
&\Omega_p^{(g,m,n;r,s)}\left(Q_B\, K_1,\cdots K_m, L_1,\cdots, L_n; \widetilde{K}_1,\cdots \widetilde{K}_r, \widetilde{L}_1,\cdots \widetilde{L}_s\right)+\cdots \\
&+\Omega_p^{(g,m,n;r,s)}\left(K_1,\cdots K_m, L_1,\cdots, L_n; \widetilde{K}_1,\cdots \widetilde{K}_r, \widetilde{L}_1,\cdots Q_B\widetilde{L}_s\right) \\
&=(-1)^p d\Omega_{p-1}^{(g,m,n;r,s)}\left(K_1,\cdots K_m, L_1,\cdots, L_n; \widetilde{K}_1,\cdots \widetilde{K}_r, \widetilde{L}_1,\cdots \widetilde{L}_s\right) .
\end{aligned} \tag{2.21}$$

In order to avoid writing too many indices we shall now introduce a short-hand notation

$$\mathcal{Q}_{g,M,R}=\sum_{m+n=M}\sum_{r+s=R}\widetilde{\mathcal{P}}_{g,m,n;r,s}\,, \tag{2.22}$$

where the sum simply denotes union. Therefore $Q_{g,M,R}$ is the union of all the spaces $\widetilde{\mathcal{P}}_{g,m,n;r,s}$ with $m+n=M$ and $r+s=R$. Consequently we also define $\omega_p^{(g,M;R)}$ to be a p-form on $Q_{g,M,R}$ such that

$$\omega_p^{(g,M;R)}=(g_s)^{2g}\,\Omega_p^{(g,m,n;r,s)} \quad \text{when restricted to } \widetilde{\mathcal{P}}_{g,m,n;r,s}\,. \tag{2.23}$$

Now let us suppose that $\mathcal{A}_{g_1,M_1,N_1}$ and $\mathcal{B}_{g_2,M_2,N_2}$ are subspaces of $\mathcal{Q}_{g_1,M_1,N_1}$ and $\mathcal{Q}_{g_2,M_2,N_2}$ respectively. We define

$$\mathcal{A}_{g_1,M_1,N_1}*\mathcal{B}_{g_2,M_2,N_2}\,, \tag{2.24}$$

as the result of twist sewing $\mathcal{A}_{g_1,M_1,N_1}$ and $\mathcal{B}_{g_2,M_2,N_2}$ similar to the manner in which we defined it in the paragraphs containing (2.14)–(2.17), but with some difference:

1. The sewing is done by picking one of the M_1 punctures of $\mathcal{A}_{g_1,M_1,N_1}$ carrying states in $\widehat{\mathcal{H}}$ and one of the M_2 punctures of $\mathcal{B}_{g_2,M_2,N_2}$ carrying states in $\widehat{\mathcal{H}}$. Therefore the

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sewed surfaces form a subspace of $\mathcal{Q}_{g_1+g_2,M_1+M_2-2,N_1+N_2}$. Since $\mathcal{A}$ and $\mathcal{B}$ are not necessarily symmetric under the exchange of all the punctures, we need to specify which of the punctures of $\mathcal{A}$ and $\mathcal{B}$ are sewed to each other. In what follows we shall see that some of the subspaces $\mathcal{A}$ and/or $\mathcal{B}$ carry special punctures where we always insert a marginal operator. The special punctures are never sewed. We shall follow the convention that leaving aside the special puncture, in $\mathcal{A}*\mathcal{B}$ we sew the last $\widehat{\mathcal{H}}$ puncture in $\mathcal{A}$ and first $\widehat{\mathcal{H}}$ puncture in $\mathcal{B}$. Most subspaces of interest will be symmetric under the exchange of all the punctures other than the special puncture; so it will not make a difference which puncture we use for sewing.

2. Since $\mathcal{A}$ and $\mathcal{B}$ are not necessarily even dimensional subspaces, we need to specify the orientation of the space after gluing. We use the convention that the volume form on the sewed space is given by the wedge product of the volume form on $\mathcal{A}$, $d\theta$ where θ is the sewing parameter and the volume form on $\mathcal{B}$ in this order. This gives

$$\partial(\mathcal{A}*\mathcal{B}) = (\partial\mathcal{A})*\mathcal{B} + (-1)^{\dim\mathcal{A}+1}\mathcal{A}*\partial\mathcal{B}\,. \tag{2.25}$$

The symbol $*$ describes the effect of sewing two punctures, each in $\widehat{\mathcal{H}}$. As described below (2.15), it involves insertion of the operator $\mathcal{G}$. We shall now introduce two new symbols — $\mathcal{A} \twoheadrightarrow \mathcal{B}$ that sews the last puncture in $\widehat{\mathcal{H}}$ from $\mathcal{A}$ to the first puncture in $\widetilde{\mathcal{H}}$ in $\mathcal{B}$ and $\mathcal{A} \twoheadleftarrow \mathcal{B}$ that sews the last puncture in $\widetilde{\mathcal{H}}$ from $\mathcal{A}$ to the first puncture in $\widehat{\mathcal{H}}$ in $\mathcal{B}$, without any insertion of the operator $\mathcal{G}$. The quick way to remember the notation is that the arrow points towards the subspace that contributes the puncture in $\widetilde{\mathcal{H}}$. Both $\twoheadrightarrow$ and $\twoheadleftarrow$ satisfy identities similar to that in (2.25).

Given a p-dimensional subspace $\mathcal{A}_{g,M,R}$ of $\mathcal{Q}_{g,M,R}$, we shall define

$$\mathcal{A}_{g,M,R}\left(A_1,\cdots A_M;\widetilde{A}_1,\cdots\widetilde{A}_R\right) \equiv \int_{\mathcal{A}_{g,M,R}} \omega_p^{(g,M;R)}\left(A_1,\cdots A_M;\widetilde{A}_1,\cdots\widetilde{A}_R\right)$$
$$\text{for}\quad A_i\in\widehat{\mathcal{H}},\ \widetilde{A}_i\in\widetilde{\mathcal{H}}\,. \tag{2.26}$$

Furthermore if the k of the states $A_1,\cdots A_M$ are equal to some state A, then we shall express the argument as A^k instead of k copies of A. Similar notation is follows for states in $\widetilde{\mathcal{H}}$. In this notation (2.21) can be stated as:

$$\begin{aligned}&\mathcal{A}_{g,M,R}\left(Q_BA_1,\cdots A_M;\widetilde{A}_1,\cdots\widetilde{A}_R\right)+\cdots\mathcal{A}_{g,M,R}\left(A_1,\cdots A_M;\widetilde{A}_1,\cdots Q_B\widetilde{A}_R\right)\\ &=(-1)^{\dim(\mathcal{A}_{g,M,R})}\,\partial\mathcal{A}_{g,M,R}\left(A_1,\cdots A_M;\widetilde{A}_1,\cdots\widetilde{A}_R\right)\,.\end{aligned} \tag{2.27}$$

Let us now consider the quantity

$$\mathcal{A}_{g_1,M_1,N_1}*\mathcal{B}_{g_2,M_2,N_2}\left(A_1,\cdots A_{M_1-1};\widetilde{A}_1,\cdots\widetilde{A}_{N_1}|B_1,\cdots B_{M_2-1};\widetilde{B}_1,\cdots\widetilde{B}_{N_2}\right)\,. \tag{2.28}$$

In writing the arguments on the right hand side we have used the convention that all the states inserted into the punctures of $\mathcal{A}$ are written first, followed by the states that are inserted at the punctures of $\mathcal{B}$, and within each group, the states that are in $\widehat{\mathcal{H}}$ are written

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first followed by the states in $\widetilde{\mathcal{H}}$. We shall furthermore assume that A_i's, B_i's, $\widetilde{A}_i$'s and $\widetilde{B}_i$'s have been made even, possibly via multiplication by a grassmann odd c-number. In that case we can write down an expression for (2.28) in terms of the corresponding quantities for component Riemann surfaces as follows. Let $|\varphi_r\rangle$ and $|\varphi_r^c\rangle$ be a conjugate pair of basis states in $\widehat{\mathcal{H}} \oplus \widetilde{\mathcal{H}}$ satisfying

$$\langle\varphi_r^c|c_0^-|\varphi_s\rangle = \delta_{rs} = \langle\varphi_s|c_0^-|\varphi_r^c\rangle, \quad \sum_s |\varphi_s\rangle\langle\varphi_s^c| = b_0^- = \sum_s |\varphi_s^c\rangle\langle\varphi_s| \,. \tag{2.29}$$

It is easy to check that φ_r and φ_r^c have opposite grassmann parities. We shall denote by $(-1)^{n_r}$ the grassmann parity of φ_r with n_r taking values 0 or 1. Then using the completeness relation given in the last equation of (2.29) we have

$$\begin{aligned}
&\mathcal{A}_{g_1,M_1,N_1} * \mathcal{B}_{g_2,M_2,N_2}\Big(A_1,\cdots A_{M_1-1};\widetilde{A}_1,\cdots\widetilde{A}_{N_1}|B_1,\cdots B_{M_2-1};\widetilde{B}_1,\cdots\widetilde{B}_{N_2}\Big)\\
&= \sum_{r,s}\mathcal{A}_{g_1,M_1,N_1}\Big(A_1,\cdots A_{M_1-1},\varphi_r;\widetilde{A}_1,\cdots\widetilde{A}_{N_1}\Big)\ \mathcal{B}_{g_2,M_2,N_2}\Big(\varphi_s,B_1,\cdots B_{M_2-1};\widetilde{B}_1,\cdots\widetilde{B}_{N_2}\Big)\\
&\quad\times\langle\varphi_r^c|c_0^-\mathcal{G}|\varphi_s^c\rangle\times(-1)^{n_s\dim\mathcal{B}}\,,
\end{aligned}\tag{2.30}$$

where the last factor arises from having to move φ_s inside the argument of $\mathcal{B}_{g_2,M_2,N_2}$ to the extreme left through the $\dim\mathcal{B}\equiv\dim\mathcal{B}_{g_2,M_2,N_2}$ number of anti-commuting ghost insertions present in the correlation function that defines $\mathcal{B}_{g_2,M_2,N_2}$. Similarly we have

$$\begin{aligned}
&\mathcal{A}_{g_1,M_1,N_1} \twoheadrightarrow \mathcal{B}_{g_2,M_2,N_2}\Big(A_1,\cdots A_{M_1-1};\widetilde{A}_1,\cdots\widetilde{A}_{N_1}|B_1,\cdots B_{M_2};\widetilde{B}_1,\cdots\widetilde{B}_{N_2-1}\Big)\\
&= \sum_{r,s}\mathcal{A}_{g_1,M_1,N_1}\Big(A_1,\cdots A_{M_1-1},\varphi_r;\widetilde{A}_1,\cdots\widetilde{A}_{N_1}\Big)\ \mathcal{B}_{g_2,M_2,N_2}\Big(B_1,\cdots B_{M_2};\varphi_s,\widetilde{B}_1,\cdots\widetilde{B}_{N_2-1}\Big)\\
&\quad\times\langle\varphi_r^c|c_0^-|\varphi_s^c\rangle\times(-1)^{n_s\dim\mathcal{B}}\,,
\end{aligned}\tag{2.31}$$

and

$$\begin{aligned}
&\mathcal{A}_{g_1,M_1,N_1} \twoheadleftarrow \mathcal{B}_{g_2,M_2,N_2}\Big(A_1,\cdots A_{M_1};\widetilde{A}_1,\cdots\widetilde{A}_{N_1-1}|B_1,\cdots B_{M_2-1};\widetilde{B}_1,\cdots\widetilde{B}_{N_2}\Big)\\
&= \sum_{r,s}\mathcal{A}_{g_1,M_1,N_1}\Big(A_1,\cdots A_{M_1};\widetilde{A}_1,\cdots\widetilde{A}_{N_1-1},\varphi_r\Big)\ \mathcal{B}_{g_2,M_2,N_2}\Big(\varphi_s,B_1,\cdots B_{M_2-1};\widetilde{B}_1,\cdots\widetilde{B}_{N_2}\Big)\\
&\quad\times\langle\varphi_r^c|c_0^-|\varphi_s^c\rangle\times(-1)^{n_s\dim\mathcal{B}}\,.
\end{aligned}\tag{2.32}$$

Using these relations and ghost number conservation rules one can show that if $\mathcal{A}_{g_1,M_1,N_1}$ and $\mathcal{B}_{g_2,M_2,N_2}$ are symmetric under the exchange of the punctures in $\widehat{\mathcal{H}}$ and the exchange of the punctures in $\widetilde{\mathcal{H}}$, then,

$$\begin{aligned}
&\mathcal{A}_{g_1,M_1,N_1} * \mathcal{B}_{g_2,M_2,N_2}\Big(A_1,\cdots A_{M_1-1};\widetilde{A}_1,\cdots\widetilde{A}_{N_1}|B_1,\cdots B_{M_2-1};\widetilde{B}_1,\cdots\widetilde{B}_{N_2}\Big)\\
&= \mathcal{B}_{g_2,M_2,N_2} * \mathcal{A}_{g_1,M_1,N_1}\Big(B_1,\cdots B_{M_2-1};\widetilde{B}_1,\cdots\widetilde{B}_{N_2}|A_1,\cdots A_{M_1-1};\widetilde{A}_1,\cdots\widetilde{A}_{N_1}\Big)\\
&\quad\times(-1)^{\dim\mathcal{A}\dim\mathcal{B}+\dim\mathcal{A}+\dim\mathcal{B}}\,,
\end{aligned}\tag{2.33}$$

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for even states $\{A_i, B_i, \widetilde{A}_i, \widetilde{B}_i\}$. Similarly we have

$$\begin{aligned}
&\mathcal{A}_{g_1,M_1,N_1} \leftrightarrow\!\!\!* \, \mathcal{B}_{g_2,M_2,N_2} \left(A_1, \cdots A_{M_1}; \widetilde{A}_1, \cdots \widetilde{A}_{N_1-1} | B_1, \cdots B_{M_2-1}; \widetilde{B}_1, \cdots \widetilde{B}_{N_2}\right) \\
&= \mathcal{B}_{g_2,M_2,N_2} \twoheadrightarrow\!\!\!* \, \mathcal{A}_{g_1,M_1,N_1} \left(B_1, \cdots B_{M_2-1}; \widetilde{B}_1, \cdots \widetilde{B}_{N_2} | A_1, \cdots A_{M_1}; \widetilde{A}_1, \cdots \widetilde{A}_{N_1-1}\right) \\
&\quad \times (-1)^{\dim\mathcal{A}\,\dim\mathcal{B}+\dim\mathcal{A}+\dim\mathcal{B}} \,. \qquad (2.34)
\end{aligned}$$

These relations can also be derived using the definition of the volume form on $\mathcal{A} * \mathcal{B}$ (and similar definitions for $\mathcal{A} \leftrightarrow\!\!\!* \, \mathcal{B}$ and $\mathcal{A} \twoheadrightarrow\!\!\!* \, \mathcal{B}$) given above (2.25).

We shall streamline the notations even further by defining the formal sum

$$\mathcal{Q}_{M,N} \equiv \sum_{g\geq 0} \mathcal{Q}_{g,M,N} \,. \qquad (2.35)$$

Similarly given a family of subspaces $\mathcal{A}_{g.M,N}$ of $\mathcal{Q}_{g,M,N}$ we shall define

$$\mathcal{A}_{M,N} \equiv \sum_{g\geq 0} \mathcal{A}_{g,M,N} \,, \qquad (2.36)$$

so that $\mathcal{A}_{M,N} \subset \mathcal{Q}_{M,N}$. Therefore when we have a symbol with two subscripts it will be understood that we have summed over g.

Let us now rewrite some of the old identities in the new notation. If we define

$$\mathcal{V}_{g,M,0} \equiv \sum_{\substack{m,n \\ m+n=M}} \mathcal{R}_{g,m,n} \,, \qquad (2.37)$$

describing a subspace of $\mathcal{Q}_{g,M,0}$, and

$$\mathcal{V}_{M,0} = \sum_{g\geq 0} \mathcal{V}_{g,M,0} \,, \qquad (2.38)$$

then we have

$$(g_s)^{2g} \{A_1 \cdots A_M\}_g = \mathcal{V}_{g,M,0}(A_1, \cdots A_M), \qquad \{A_1 \cdots A_M\} = \mathcal{V}_{M,0}(A_1, \cdots A_M) \,. \qquad (2.39)$$

In this notation, eq. (2.17) may be expressed as

$$\partial\mathcal{V}_{M,0}\left(\Phi^M\right) = -\frac{1}{2} \sum_{\substack{M_1,M_2 \\ M_1+M_2=M}} \frac{M!}{M_1!M_2!} \mathcal{V}_{M_1+1,0} * \mathcal{V}_{M_2+1,0}\left(\Phi^{M_1}|\Phi^{M_2}\right) \,, \qquad (2.40)$$

where the combinatorial factor arises from symmetrization operation **S** in (2.17). To recover (2.17) from here we need to take $\Phi = A_1 + \cdots + A_M$ and pick the coefficient of $A_1 \cdots A_M$ from both sides. Using (2.27) we can also express (2.40) as

$$M\,\mathcal{V}_{M,0}\left(Q_B\Phi, \Phi^{M-1}\right) = -\frac{1}{2} \sum_{\substack{M_1,M_2 \\ M_1+M_2=M}} \frac{M!}{M_1!M_2!} \mathcal{V}_{M_1+1,0} * \mathcal{V}_{M_2+1,0}\left(\Phi^{M_1}|\Phi^{M_2}\right) \,. \qquad (2.41)$$

This is the form of the relation that is directly used to prove the invariance of the 1PI effective action (2.11) under the infinitesimal gauge transformation:

$$\delta|\widetilde{\Psi}\rangle = Q_B|\widetilde{\Lambda}\rangle + \sum_M \frac{1}{M!}\left[\Lambda\Psi^M\right] \,, \qquad \delta|\Psi\rangle = Q_B|\Lambda\rangle + \sum_M \frac{1}{M!}\mathcal{G}\left[\Lambda\Psi^M\right] \,, \qquad (2.42)$$

where $\Lambda \in \widehat{\mathcal{H}}$ and $\widetilde{\Lambda} \in \widetilde{\mathcal{H}}$ are infinitesimal grassmann odd gauge transformation parameters.

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3 Statement of background independence

In this section we shall discuss the precise statement of background independence of superstring field theory that we shall attempt to prove.

3.1 Marginal deformation of the superconformal field theory

Let us suppose that the matter sector of the world-sheet SCFT admits a supersymmetry preserving marginal deformation. In the heterotic world-sheet theory this implies the existence of a dimension $(1, 1/2)$ superconformal primary operator $\mathcal{O}$ of the matter theory.[5] We also have an associated dimension $(1, 1)$ matter primary $\widetilde{\mathcal{O}}$ defined through the operator product expansion

$$\begin{aligned} T_F(z)\,\mathcal{O}(w,\bar{w}) &\simeq (z-w)^{-1}\,\widetilde{\mathcal{O}}(w,\bar{w}) + \text{less singular terms}\,, \\ T_F(z)\,\widetilde{\mathcal{O}}(w,\bar{w}) &\simeq \frac{1}{4}(z-w)^{-2}\mathcal{O}(w,\bar{w}) + \frac{1}{4}(z-w)^{-1}\partial\mathcal{O}(w,\bar{w}) \ + \text{less singular terms}\,. \end{aligned} \tag{3.1}$$

$\mathcal{O}$ and $\widetilde{\mathcal{O}}$ may be considered as the lower and the upper components of a superfield. We can now consider a neighboring SCFT that is related to the original theory by marginal deformation by the operator $\widetilde{\mathcal{O}}$. Without loss of generality we shall restrict our discussion to first order in the deformation parameter since we can build a finite deformation by successive application of infinitesimal deformations. In this case the correlation functions of the new theory on any Riemann surface can be computed in terms of correlation functions of the original theory on the same Riemann surface by making an additional insertion of the marginal operator and integrating over the location of its insertion point. More precisely we deform every correlation function by inserting into the correlation function an operator

$$-\frac{\lambda}{2\pi i}\int dz\wedge d\bar{z}\,\widetilde{\mathcal{O}}(z,\bar{z})\,, \tag{3.2}$$

for some infinitesimal operator λ. However this apparently suffers from divergences when the locations of the marginal operator approach the locations of the other operators inserted in the correlation function. It also suffers from an ambiguity since in principle the states of the undeformed theory and the states of the deformed theory belong to different Hilbert spaces and there is no canonical isomorphism between these Hilbert spaces. Therefore by state operator correspondence, there is also no canonical isomorphism between the local operators in the two theories, and there is no absolute notion of comparing correlation functions in the two theories.

It turns out that both problems can be resolved by introducing the notion of a connection on the Hilbert space of the family of SCFT's related by marginal deformation. This establishes a (non-canonical) isomorphism between the Hilbert spaces of the original theory and the deformed theory. Once such an isomorphism is established, it becomes meaningful to express the correlation function of one theory in terms of the correlation function of the

[5] We use the convention that left-movers are anti-holomorphic and right-movers are holomorphic fields on the world-sheet.

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other theory. It turns out that when we choose a non-singular connection it also eliminates the divergence problems. We shall now state the result for the deformed correlation function for a particular choice of connection [18–21]. In that case the correlation function of a set of operators $A_1, \cdots A_N$ on a Riemann surface Σ, inserted using local coordinates $w_1, \cdots w_N$, changes by

$$\delta\langle A_1 \cdots A_N\rangle_\Sigma = -\frac{\lambda}{2\pi i}\int_{\Sigma-\cup_i D_i} dz\wedge d\bar{z}\,\langle\,\widetilde{\mathcal{O}}(z,\bar{z})\,A_1\cdots A_N\rangle_\Sigma \tag{3.3}$$

where $\Sigma - \cup_i D_i$ denotes the whole Riemann surface sans unit disks $|w_i| \leq 1$ around each puncture. Since in (3.3) the location z of the additional operator $\widetilde{\mathcal{O}}$ never approaches any of the punctures, there is no divergence of the type mentioned earlier. The fact that we have to exclude disks of unit radius from around each puncture is dictated by the special choice of connection we have taken. Excluding disks of radius a for any other number a will correspond to a different choice of connection. $a \to 0$ limit corresponds to the original prescription of integrating over the whole Riemann surface. In this limit we get back the divergences mentioned earlier but this can now be attributed to the fact that the corresponding choice of connection is singular. The particular choice of connection described in (3.3) preserves the BPZ inner product.

Since the PCO's are typically inserted on $\Sigma - \cup_i D_i$, we can get divergences when the location z of $\widetilde{\mathcal{O}}$ approaches the location w of the PCO. However since the divergent piece is proportional to $(z-w)^{-2}$, we can perform the integral by restricting the z integral to $|z-w| > \epsilon$ for some small number ϵ and then taking the $\epsilon \to 0$ limit. This gives a finite integral and furthermore the result of integration is not affected under a non-singular change of coordinates i.e. we get the same result by performing the integral over the region $|f(z) - f(w)| > \epsilon$ and then taking the $\epsilon \to 0$ limit. For this reason we shall not worry about collision of $\widetilde{\mathcal{O}}$ with the PCO's.

For the particular choice of connection that excludes disks of unit radius, we can calculate the deformations of the standard SCFT operators like the Virasoro generators and the modes of the supercurrent. The modes of the ghost fields remain unchanged since the deformation is only in the matter sector. The Virasoro generators L_n and $\bar{L}_n$ change by [18–21]

$$\delta L_n = \lambda \oint_{|z|=1} d\bar{z}\, z^{n+1}\,\widetilde{\mathcal{O}}(z,\bar{z}), \qquad \delta\bar{L}_n = \lambda \oint_{|z|=1} dz\, \bar{z}^{n+1}\,\widetilde{\mathcal{O}}(z,\bar{z})\,, \tag{3.4}$$

where the integral is performed over a circle at $|z| = 1$. $\oint$ includes appropriate factors of $\pm(2\pi i)^{-1}$ so that $\oint dz/z = 1$, $\oint d\bar{z}/\bar{z} = 1$. The modes G_n of the matter supercurrent change by [19][6]

$$\delta G_n = \frac{1}{4}\lambda \oint_{|z|=1} d\bar{z}\, z^{n+1/2}\,\mathcal{O}(z,\bar{z})\,. \tag{3.5}$$

[6]The sign and normalization of the right hand side of (3.5) differs from that in [19] due to a difference in convention. The correct normalization can be derived using the procedure described in [18] using (3.1).

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As a result the BRST operator Q_B and the zero mode of the PCO $\mathcal{X}_0$ change by

$$\delta Q_B = \lambda \oint_{|z|=1} d\bar{z}\, c(z)\, \widetilde{\mathcal{O}}(z,\bar{z}) + \lambda \oint_{|z|=1} dz\, \bar{c}(\bar{z})\, \widetilde{\mathcal{O}}(z,\bar{z}) + \frac{1}{4}\lambda \oint_{|z|=1} d\bar{z}\, \gamma(z)\, \mathcal{O}(z,\bar{z})\,, \tag{3.6}$$

and

$$\delta \mathcal{X}_0 = \frac{1}{4}\lambda \oint_{|z|=1} d\bar{z}\, z^{-1}\, e^{\phi}(z)\, \mathcal{O}(z,\bar{z})\,. \tag{3.7}$$

It is straightforward (although somewhat tedious) to verify that

$$\{Q_B, \delta Q_B\} = \mathcal{O}\big(\lambda^2\big)\,, \qquad [Q_B, \delta\mathcal{X}_0] + [\delta Q_B, \mathcal{X}_0] = \mathcal{O}\big(\lambda^2\big)\,, \tag{3.8}$$

so that $Q_B + \delta Q_B$ is nilpotent and commutes with $\mathcal{X}_0 + \delta\mathcal{X}_0$ to first order in the perturbation parameter λ. The following argument also shows that the connection preserves the projection into the $L_0^- = 0$ states, i.e. $\delta L_0 - \delta\bar{L}_0$ vanishes while acting on a state with $L_0 = \bar{L}_0$. Let us suppose that ϕ is a vertex operator with $L_0 = \bar{L}_0 = h$. Then we have the operator product expansion

$$\widetilde{\mathcal{O}}(z,\bar{z})\, \phi(0) = \sum_{p,q} \bar{z}^{p-1}\, z^{q-1}\, \phi_{p,q}(0)\,, \tag{3.9}$$

where $\phi_{p,q}$ have dimension $(p+h, q+h)$. Therefore

$$\big(\delta L_0 - \delta\bar{L}_0\big)\, |\phi\rangle = \lambda \oint_{|z|=1} d\bar{z}\, z \sum_{p,q} \bar{z}^{p-1}\, z^{q-1}\, |\phi_{p,q}\rangle - \lambda \oint_{|z|=1} dz\, \bar{z} \sum_{p,q} \bar{z}^{p-1}\, z^{q-1}\, |\phi_{p,q}\rangle \tag{3.10}$$

Since the contour integrals are performed over circles of unit radii around the origin, each of these terms vanish unless $p = q$. On the other hand for $p = q$ the two terms gives identical result $|\phi_{p,q}\rangle$ and cancel. This shows that $\delta L_0 - \delta\bar{L}_0$ vanishes while acting on a state with $L_0 = \bar{L}_0$. Since the b_n's are not deformed at all, the $b_0 - \bar{b}_0 = 0$ condition is also preserved by this connection.

We can now consider the string field theory action $S'(\Pi, \widetilde{\Pi})$ formulated in the new background. It is given by

$$S'\left(\Pi, \widetilde{\Pi}\right) = S\left(\Pi, \widetilde{\Pi}\right) + \delta S\left(\Pi, \widetilde{\Pi}\right)\,, \tag{3.11}$$

where

$$\delta S = g_s^{-2}\left[-\frac{1}{2}\langle\widetilde{\Pi}|c_0^-\,(\delta\mathcal{G}\, Q_B + \mathcal{G}\,\delta Q_B)|\widetilde{\Pi}\rangle + \langle\widetilde{\Pi}|c_0^-\,\delta Q_B|\Pi\rangle + \sum_N \frac{1}{N!}\delta\{\Pi^N\}\right]\,. \tag{3.12}$$

Here $\delta\{A_1 \cdots A_N\}$ denotes the change in $\{A_1 \cdots A_N\}$ due to the change in the correlation function given in (3.3).

The equations of motion for the interacting part of the theory is obtained by replacing the kinetic and interaction terms in (2.20) by their counterpart in the new background. This gives

$$Q_B|\Pi\rangle + \sum_N \frac{1}{N!}\,\mathcal{G}\left[\Pi^N\right] + \delta Q_B\,|\Pi\rangle + \sum_N \frac{1}{N!}\,\delta\,\mathcal{G}\left[\Pi^N\right] + \sum_N \frac{1}{N!}\,\mathcal{G}\,\delta\left[\Pi^N\right] = 0\,, \tag{3.13}$$

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where $\delta\left[\Pi^N\right]$ is defined via the equation

$$\langle A|c_0^-|\delta\left[\Pi^N\right]\rangle = \delta\left\{A\,\Pi^n\right\} \tag{3.14}$$

for any state A.

3.2 Superstring field theory around shifted background

Given the superconformal primary operator $\mathcal{O}(z,\bar{z})$ of dimension (1,1/2), we can construct a BRST invariant operator $\bar{c}ce^{-\phi}\mathcal{O}$. Since the genus zero contribution $\{A_1\cdots A_N\}_0$ vanishes for $N\leq 3$, the interaction terms begin at cubic order. Therefore

$$|\Psi\rangle = |\widetilde{\Psi}\rangle = \lambda\,|\Psi_0\rangle, \qquad |\Psi_0\rangle \equiv \bar{c}_1\,c_1\,e^{-\phi}(0)|\mathcal{O}\rangle\,, \tag{3.15}$$

gives a solution to the *classical* equations of motion (2.18) of the undeformed theory to order λ. Note that since the solution carries picture number -1 and therefore describes an NS sector state, $\mathcal{G}=1$ in this sector and we can take $\widetilde{\Psi}=\Psi$.[7] We shall not demand that $\lambda|\Psi_0\rangle$ remains a solution to the quantum corrected equations of motion, just as we do not assume that $|\Pi\rangle=0$ is a solution to the deformed equation of motion (3.13) after including the quantum effects in the 1PI effective action.

We now define shifted fields

$$|\Phi\rangle = |\Psi\rangle - \lambda\,|\Psi_0\rangle, \quad |\widetilde{\Phi}\rangle = |\widetilde{\Psi}\rangle - \lambda\,|\Psi_0\rangle\,, \tag{3.16}$$

and expand the action in a power series expansion in $|\Phi\rangle$. The action takes the form

$$S\left(\Psi,\widetilde{\Psi}\right) = S\left(\Psi_0,\Psi_0\right) + S''\left(\Phi,\widetilde{\Phi}\right)\,, \tag{3.17}$$

where

$$S''\left(\Phi,\widetilde{\Phi}\right) = g_s^{-2}\left[-\frac{1}{2}\langle\widetilde{\Phi}|c_0^-\mathcal{G}\,Q_B|\widetilde{\Phi}\rangle + \langle\widetilde{\Phi}|c_0^-Q_B|\Phi\rangle + \sum_N\frac{1}{N!}\{\Phi^N\} + \lambda\sum_N\frac{1}{N!}\{\Psi_0\,\Phi^N\}\right] + \mathcal{O}(\lambda^2). \tag{3.18}$$

The equations of motion derived from (3.18) takes the form

$$\begin{aligned} &Q_B(|\Phi\rangle - \mathcal{G}|\widetilde{\Phi}\rangle) = 0 \\ &Q_B|\widetilde{\Phi}\rangle + \sum_N\frac{1}{N!}\left[\Phi^N\right] + \lambda\sum_N\frac{1}{N!}\left[\Psi_0\,\Phi^N\right] = 0\,. \end{aligned} \tag{3.19}$$

Multiplying the second equation in (3.19) by $\mathcal{G}$ and adding it to the first equation we get the equation of motion for the interacting field $|\Phi\rangle$:

$$Q_B|\Phi\rangle + \sum_N\frac{1}{N!}\,\mathcal{G}\left(\left[\Phi^N\right] + \lambda\left[\Psi_0\,\Phi^N\right]\right) = 0\,. \tag{3.20}$$

For given $|\Phi\rangle$ satisfying (3.20), the second equation of (3.19) determines $|\widetilde{\Phi}\rangle$ up to addition of BRST invariant states. Therefore as before, the degrees of freedom of $\widetilde{\Phi}$ are free fields, and (3.20) contains full information about the S-matrix of the interacting part of the theory.

[7] Since the equations of motion allow us to shift $\widetilde{\Psi}$ by a BRST invariant state without affecting Ψ, we could set $\widetilde{\Psi}=0$.

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3.3 Statement of the problem

We now have two potential descriptions of quantum corrected interacting field equations of superstring field theory in a background related to the original background via marginal deformation. The first one, described by (3.13), uses the formulation of superstring field theory around the deformed world-sheet SCFT. The second one, described by (3.20) is the equation of motion of superstring field theory formulated around the original background, but expanded around a new solution to the equations of motion representing the marginal deformation. The statement of background independence is that these two sets of equations are equivalent (to order λ). Therefore there must be a field redefinition relating Π to Φ that makes the first set of equations of motion into linear combinations of the second set (with possibly field dependent coefficients).[8] Since the two sets of equations of motion differ by order λ, we can assume that the (field dependent) matrix relating the two sets of equations differ from the identity matrix by terms of order λ. Denoting the original equations of motion by $|E_0\rangle = 0$, and the equations (3.13) and (3.20) by $|E_0\rangle + \lambda|E_1\rangle = 0$ and $|E_0\rangle + \lambda|E_2\rangle = 0$ respectively, we can state the requirement as

$$|E_0\rangle + \lambda|E_1\rangle = (1 + \lambda\, M)(|E_0\rangle + \lambda|E_2\rangle) \tag{3.21}$$

where M is some linear operator on $\widehat{\mathcal{H}}$ that could be field dependent. To order λ this gives

$$|E_1\rangle - |E_2\rangle = M\,|E_0\rangle\,. \tag{3.22}$$

We shall look for a field redefinition of the form

$$|\Pi\rangle = |\Phi\rangle + \delta\,|\Phi\rangle\,, \tag{3.23}$$

for some state $\delta\,|\Phi\rangle$ of order λ, and analyze the difference between the equations of motion (3.13) and (3.20) to order λ. The inner product between an arbitrary state $\widetilde{A} \in \widetilde{\mathcal{H}}$ and this difference may then be written as

$$\begin{aligned}\Delta \equiv \langle\widetilde{A}|c_0^-\Bigg(&Q_B\,|\delta\Phi\rangle + \sum_N \frac{1}{N!}\mathcal{G}\left[\Phi^N\,\delta\Phi\right] + \delta Q_B|\Phi\rangle + \sum_N \frac{1}{N!}\delta\mathcal{G}\,\left[\Phi^N\right] + \sum_N \frac{1}{N!}\mathcal{G}\,\delta\left[\Phi^N\right] \\ &-\lambda\sum_N \frac{1}{N!}\mathcal{G}\,\left[\Psi_0\,\Phi^N\right]\Bigg)\,.\end{aligned} \tag{3.24}$$

For definiteness we shall take $\widetilde{A}$ to be grassmann even — this can always be achieved by multiplying the state by a grassmann odd c-number if needed. In that case (3.24) may be written as

$$\begin{aligned}\Delta = \langle Q_B\widetilde{A}|c_0^-|\delta\Phi\rangle &+ \sum_N \frac{1}{N!}\{\left(\mathcal{G}\widetilde{A}\right)\Phi^N\,\delta\Phi\} + \langle\widetilde{A}|c_0^-\delta Q_B|\Phi\rangle + \sum_N \frac{1}{N!}\{\left(\delta\mathcal{G}\widetilde{A}\right)\,\Phi^N\} \\ &+\sum_N \frac{1}{N!}\delta\{\left(\mathcal{G}\,\widetilde{A}\right)\,\Phi^N\} - \lambda\sum_N \frac{1}{N!}\{\left(\mathcal{G}\widetilde{A}\right)\,\Psi_0\Phi^N\}\,.\end{aligned} \tag{3.25}$$

[8] The advantage of working with the equations of motion derived from the 1PI effective action is that we do not need to worry about the change in the integration measure over fields under a change in background, since the effect of integration measure is included in the definition of the 1PI effective action. This avoids some of the complications encountered in [9].

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eq. (3.22) now translates to the requirement that Δ vanishes to order λ when $|\Phi\rangle$ satisfies the zeroth order equation of motion

$$Q_B|\Phi\rangle + \sum_N \frac{1}{N!}\,\mathcal{G}\,[\Phi^N] = 0\,. \tag{3.26}$$

Our goal will be to prove the existence of $\delta\Phi$ satisfying this requirement.

One could have asked for more — demanding that the two actions $S'(\Pi,\widetilde{\Pi})$ and $S''(\Phi,\widetilde{\Phi})$ are equivalent. However it is easy to see that this cannot hold. Since the equations of motion of $\Pi,\widetilde{\Pi}$ use the deformed BRST operator Q'_B and deformed interaction terms, it follows that the free field degrees of freedom encoded in $\widetilde{\Pi}$ correspond to states annihilated by Q'_B. On the other hand it follows from (3.19) that the free field degrees of freedom encoded in $\widetilde{\Phi}$ correspond to states annihilated by Q_B. Since Q_B and Q'_B have different cohomologies (in particular the mass spectrum computed from the two operators are different) there cannot be a field redefinition that relates the two actions. However this is not necessary for the proof of background dependence — it is enough to show that the 1PI equations of motion for the interacting fields are related by field redefinition. This is what we shall attempt to do.

4 Geometric interpretation of the kinetic and interaction terms of the deformed theory

In this section we shall give geometric interpretation of δQ_B, $\delta\mathcal{G}$ and $\delta\{\Phi^N\}$ by associating to each of them appropriate subspaces of $\mathcal{Q}_{g,M,N}$. This will be used in the next section to convert (3.25) into a geometric form.

4.1 Deformation of the BRST operator

Let us begin with δQ_B. For this we introduce a zero dimensional subspace $\mathcal{V}'_{0,2,1}\subset\mathcal{Q}_{0,2,1}$ of the following form. If z denotes the standard coordinate on the complex plane used to parametrize a genus 0 surface, then the second puncture in $\widehat{\mathcal{H}}$ and the puncture in $\widetilde{\mathcal{H}}$ are taken to be at $z=0$ and $z=\infty$ respectively, with z and $1/z$ as local coordinates around the two punctures. The first puncture in $\widehat{\mathcal{H}}$ is inserted at $z=1$. This is a special puncture where we always insert the vertex operator Ψ_0 described in (3.15)[8, 9]. Since this is a dimension zero primary we do not need to specify the choice of local coordinates at this puncture — all choices are equivalent. Since the total picture number carried by the vertex operators Ψ_0, and the two vertex operators in $\widehat{\mathcal{H}}$ and $\widetilde{\mathcal{H}}$ is -3, we need one PCO insertion. This is inserted at the special puncture at $z=1$, converting Ψ_0 to a zero picture vertex operator

$$\lim_{w\to z}\mathcal{X}(w)\Psi_0(z) = c\,\bar{c}\,\widetilde{\mathcal{O}} - \frac{1}{4}\,\gamma\,\bar{c}\,\mathcal{O}\,, \tag{4.1}$$

without generating any singular term. In this case in the notation of (2.26),

$$\mathcal{V}'_{0,2,1}\left(\Psi_0, B; \widetilde{A}\right) = \left\langle \widetilde{A}\,\middle|\left[c\,\bar{c}\,\widetilde{\mathcal{O}}(1) - \frac{1}{4}\,\gamma\,\bar{c}\,\mathcal{O}(1)\right]\middle|\,B\right\rangle \tag{4.2}$$

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for grassmann even vertex operators A and $\widetilde{B}$. Using the invariance of states in $\widehat{\mathcal{H}}$ and $\widetilde{\mathcal{H}}$ under L_0^- and b_0^-, and standard identities for sphere three point function in conformal field theory, we can bring this to the form

$$\begin{aligned}\mathcal{V}'_{0,2,1}\left(\Psi_0, B; \widetilde{A}\right) &= \left\langle \widetilde{A}\middle| c_0^- \left[\oint_{|z|=1} d\bar{z}\, c(z)\, \widetilde{\mathcal{O}}(z,\bar{z}) + \oint_{|z|=1} dz\, \bar{c}(\bar{z})\, \widetilde{\mathcal{O}}(z,\bar{z}) \right.\right. \\ &\qquad \left.\left. + \frac{1}{4} \oint_{|z|=1} d\bar{z}\, \gamma(z)\, \mathcal{O}(z,\bar{z}) \right] \middle| B \right\rangle \\ &= \lambda^{-1} \langle \widetilde{A}| c_0^- \delta Q_B |B\rangle \,. \end{aligned} \tag{4.3}$$

To see how this works, let us examine the second term on the right hand side of (4.2). We first write

$$\bar{c}(1) = \sum_n \bar{c}_n = -\frac{1}{2}\sum_n (c_n - \bar{c}_n) + \frac{1}{2}(c_n + \bar{c}_n)\,, \tag{4.4}$$

and note that since $|\widetilde{A}\rangle$ and $|B\rangle$ are annihilated by b_0^-, the only term in (4.4) that contributes to (4.2) is the $-(c_0 - \bar{c}_0)/2 = -c_0^-$ term. This reduces the second term on the right hand side of (4.2) to

$$\frac{1}{4} \langle \widetilde{A}| c_0^- \gamma(1)\, \mathcal{O}(1)|B\rangle \,. \tag{4.5}$$

Denoting by (h_A, h_A) and (h_B, h_B) the conformal weights of $|\widetilde{A}\rangle$ and $|B\rangle$, and recalling that $\gamma\,\mathcal{O}$ has conformal weight $(1,0)$, we get

$$\begin{aligned}\langle \widetilde{A}| c_0^- \gamma(z)\, \mathcal{O}(z,\bar{z})|B\rangle &= \bar{z}^{h_A - h_B - 1} z^{h_A - h_B} \langle \widetilde{A}| c_0^- \gamma(1)\, \mathcal{O}(1)|B\rangle = \bar{z}^{-1} \langle \widetilde{A}| c_0^- \gamma(1)\, \mathcal{O}(1)|B\rangle\,, \\ &\qquad \text{at } |z| = 1\,. \end{aligned} \tag{4.6}$$

Therefore we get

$$\frac{1}{4} \oint_{|z|=1} d\bar{z}\, \langle \widetilde{A}| c_0^- \gamma(z)\, \mathcal{O}(z,\bar{z})|B\rangle = \frac{1}{4} \langle \widetilde{A}| c_0^- \gamma(1)\, \mathcal{O}(1)|B\rangle \,. \tag{4.7}$$

This establishes the equality of the second term on the right hand side of (4.2) and the term in the second line of (4.3). Similar manipulations can be carried out for the other terms [8, 9].

From (4.3) we have

$$\langle \widetilde{A}| c_0^- \delta Q_B |B\rangle = \lambda\, \mathcal{V}'_{0,2,1}\left(\Psi_0, B; \widetilde{A}\right) \,. \tag{4.8}$$

Note that since $\mathcal{V}'_{0,2,1}$ is a zero dimensional subspace, the right hand side of (4.8) corresponds to simply evaluating $\omega_0^{(0,2;1)}(\Psi_0, B; \widetilde{A})$ on a specific point in $\mathcal{Q}_{0,2,1}$ that represents $\mathcal{V}'_{0,2,1}$. Also $\partial\mathcal{V}'_{0,2,1} = 0$. When $\widetilde{A}$ and B have general grassmann parities $(-1)^{\widetilde{A}}$ and $(-1)^B$, then this equation will have an additional sign of $(-1)^{\widetilde{A}\,B}$. This can be seen by multiplying $\widetilde{A}$ and B by grassmann odd c-numbers if needed to make them grassmann even, applying (4.8) and finally stripping off the grassmann odd c-numbers by moving them to the extreme right or extreme left.

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4.2 Deformation of the picture changing operator

Next we turn to $\delta\mathcal{G}$. For this we define a one dimensional subspace $\mathcal{V}''_{0,1,2}$ of $\mathcal{Q}_{0,1,2}$ as follows. If any of the punctures in $\widetilde{\mathcal{H}}$ is NS puncture, we declare $\mathcal{V}''_{0,1,2}$ to be zero, reflecting the fact that $\delta\mathcal{G}$ vanishes on NS sector state. When the two punctures in $\widetilde{\mathcal{H}}$ are Ramond punctures, $\mathcal{V}''_{0,1,2}$ describes a one dimensional subspace of $\mathcal{Q}_{0,1,2}$ such that for each element of $\mathcal{V}''_{0,1,2}$ the choice of local coordinates at the punctures are fixed in the same way as for $\mathcal{V}'_{0,2,1}$ at all the punctures. In particular the puncture in $\widehat{\mathcal{H}}$ is a special puncture at $z = 1$ where Ψ_0 is inserted, the first puncture in $\widetilde{\mathcal{H}}$ is at $z = 0$ with local coordinate z and the second puncture in $\widetilde{\mathcal{H}}$ is at $z = \infty$ with local coordinate $1/z$. Therefore the different elements of $\mathcal{V}''_{0,1,2}$ differ only in the choice of PCO locations, making this a 'vertical segment' in the language of [14, 15]. This one dimensional vertical segment interpolates between the following pair of PCO configurations. In both configurations one PCO is inserted at the special puncture, converting Ψ_0 into a zero picture vertex operator given in (4.1). The second PCO is inserted as $\mathcal{X}_0$ around the puncture at ∞ in the initial configuration and $\mathcal{X}_0$ around the puncture at 0 in the final configuration. In the correlation function that defines $\mathcal{V}''_{0,1,2}(\Psi_0, \widetilde{A}, \widetilde{B})$, this corresponds to the difference between two terms — an insertion of ξ_0 around the puncture at ∞ and the insertion of ξ_0 at the puncture at 0 [14].[9] Therefore we have, for grassmann even $\widetilde{A}$, $\widetilde{B}$:

$$\mathcal{V}''_{0,1,2}\left(\Psi_0, \widetilde{A}, \widetilde{B}\right) = \left\langle \widetilde{B} \middle| \left[\xi_0, \left(c\,\bar{c}\,\widetilde{\mathcal{O}}(1) - \frac{1}{4}\,\gamma\,\bar{c}\,\mathcal{O}(1)\right)\right] \middle| \widetilde{A} \right\rangle = -\frac{1}{4}\,\langle \widetilde{B}|e^{\phi}\,\bar{c}\,\mathcal{O}(1)|\widetilde{A}\rangle\,. \quad (4.9)$$

Following the same logic described below (4.3) we can express this as

$$\mathcal{V}''_{0,1,2}\left(\Psi_0, \widetilde{A}, \widetilde{B}\right) = -\frac{1}{4}\left\langle \widetilde{B} \middle| c_0^- \oint_{|z|=1} \frac{d\bar{z}}{z} e^{\phi}\mathcal{O}(z,\bar{z}) \middle| \widetilde{A} \right\rangle = -\lambda^{-1}\,\langle \widetilde{B}|c_0^-\,\delta\mathcal{X}_0|\widetilde{A}\rangle\,. \quad (4.10)$$

Therefore we may write, for grassmann even states $\widetilde{A}, \widetilde{B} \in \widetilde{\mathcal{H}}$,

$$\langle \widetilde{B}|c_0^-\,\delta\mathcal{G}|\widetilde{A}\rangle = -\lambda\,\mathcal{V}''_{0,1,2}\left(\Psi_0; \widetilde{A}, \widetilde{B}\right)\,. \quad (4.11)$$

When $\widetilde{A}$ and $\widetilde{B}$ are NS sector states then both sides vanish and the equation holds identically. When $\widetilde{A}$ and $\widetilde{B}$ are R sector states then this equation follows from the equality of (4.9) and (4.10). When $\widetilde{A}$, $\widetilde{B}$ have general grassmann parities then this equation will have an additional factor of $(-1)^{\widetilde{B}+\widetilde{B}\widetilde{A}}$. This extra sign arises from the fact that when we convert $\widetilde{A}$ and $\widetilde{B}$ to grassmann even operators by multiplying them by (possibly) grassmann odd c-numbers ζ_A and ζ_B from the right so that (4.11) holds, and try to strip off the ζ's by moving them to the extreme right on both sides in the combination $\zeta_B\zeta_A$, we get a factor of $(-1)^{\widetilde{B}+\widetilde{B}\widetilde{A}}$ on the left hand side arising from the effect of passing ζ_B through c_0^- and $\widetilde{A}$. An identical result is obtained by moving the ζ's to the extreme left.

[9] Even though ξ_0 is not a good operator in the small Hilbert space in which we are working, the difference between two insertions of ξ_0 is in the small Hilbert space.

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Since $\mathcal{V}''_{0,1,2}$ interpolates between two configurations: $\mathcal{V}'_{0,2,1}$ with an extra factor of $\mathcal{G}$ inserted around ∞ and $\mathcal{V}'_{0,2,1}$ with an extra factor of $\mathcal{G}$ inserted around 0, we have

$$\partial\mathcal{V}''_{0,1,2}\left(\Psi_0;\widetilde{A},\widetilde{B}\right)=\mathcal{V}'_{0,2,1}\left(\Psi_0,\mathcal{G}\widetilde{A};\widetilde{B}\right)-\mathcal{V}'_{0,2,1}\left(\Psi_0,\mathcal{G}\widetilde{B};\widetilde{A}\right)\,, \tag{4.12}$$

for grassmann even $\widetilde{A}$ and $\widetilde{B}$. Another useful property of $\mathcal{V}''_{0,1,2}(\Psi_0;\widetilde{A},\widetilde{B})$ is that it is antisymmetric under $\widetilde{A}\leftrightarrow\widetilde{B}$ for grassmann even $\widetilde{A}$ and $\widetilde{B}$,

$$\mathcal{V}''_{0,1,2}\left(\Psi_0;\widetilde{A},\widetilde{B}\right)=-\mathcal{V}''_{0,1,2}\left(\Psi_0;\widetilde{B},\widetilde{A}\right)\,. \tag{4.13}$$

For this reason when we sew a $\widetilde{\mathcal{H}}$ puncture of $\mathcal{V}''$ to another puncture belonging to another Riemann surface, it is important to specify which of the two $\widetilde{\mathcal{H}}$ punctures of $\mathcal{V}''_{0,1,2}$ is being used for sewing. Following the conventions given below (2.24), if the first $\widetilde{\mathcal{H}}$ puncture of $\mathcal{V}''_{0,1,2}$ takes part in the sewing, then we shall write $\mathcal{V}''_{0,1,2}$ to the right of the $\twoheadrightarrow$ symbol, whereas if the second $\widetilde{\mathcal{H}}$ puncture of $\mathcal{V}''_{0,1,2}$ takes part in the sewing, then we shall write $\mathcal{V}''_{0,1,2}$ to the left of the $\twoheadleftarrow$ symbol. Due to this antisymmetry property, the relations of the form (2.34) acquire extra minus sign when one of the subspaces is $\mathcal{V}''_{0,1,2}$.

4.3 Deformation of the interaction terms

Finally we turn to the interpretation of $\delta\{\Phi^N\}$. This may be expressed as[10]

$$\delta\{\Phi^N\}=\lambda\sum_{g\geq 0}\mathcal{V}'_{g,N+1,0}\left(\Psi_0,\,\Phi^N\right)\,, \tag{4.14}$$

where $\mathcal{V}'_{g,N+1,0}$ is a $(6g+2N-4)$ dimensional subspace of $\mathcal{Q}_{g,N+1,0}$ defined as follows. We begin with the Riemann surfaces associated with the subspace $\mathcal{V}_{g,N,0}\subset\mathcal{Q}_{g,N,0}$ and insert a special puncture at any point on the Riemann surface outside the unit disks: $|w_i|\geq 1$ for $1\leq i\leq N$, where w_i is the local coordinate around the i-th puncture on the Riemann surfaces associated with $\mathcal{V}_{g,N,0}$. Since we shall always insert the conformally invariant vertex operator Ψ_0 at the special puncture, we do not need to specify the local coordinate at this puncture. Therefore $\mathcal{V}'_{g,N+1,0}$ corresponds to a subspace of $\mathcal{Q}_{g,N+1,0}$ modulo the choice of local coordinates at the special puncture, and all relations involving $\mathcal{V}'_{g,N+1,0}$ that we shall write below will be modulo this choice. Note that for $g=0$ this definition of $\mathcal{V}'_{g,N+1,0}$ is valid for $N\geq 3$. We define $\mathcal{V}'_{0,3,0}$ to be zero. We shall use the convention that the first puncture of $\mathcal{V}'_{g,N+1,0}$ will be the special puncture. From the symmetry of $\mathcal{V}_{g,N,0}$ under the permutations of the punctures, it follows that $\mathcal{V}'_{g,N+1,0}$ is symmetric under the permutations of all the punctures other than the special puncture. Since $\mathcal{V}_{g,N,0}$ avoids regions of the moduli space with separating type degenerations, and since in the definition of $\mathcal{V}'_{g,N+1,0}$ we always keep the special puncture away from the other punctures, $\mathcal{V}'_{g,N+1,0}$ also avoids separating type degenerations.

Let us define

$$\mathcal{V}'_{N+1,0}\equiv\sum_{g\geq 0}\mathcal{V}'_{g,N+1,0}\,. \tag{4.15}$$

[10] The $(-2\pi i)^{-1}$ factor in (3.2) arises in (4.14) from a $(-2\pi i)^{-1}$ factor included in the definition of $\Omega_p^{(g,m,n)}$ for every additional puncture (see e.g. [1]).

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Then (4.14) may be expressed as

$$\delta\{\Phi^N\} = \lambda\, \mathcal{V}'_{N+1,0}\left(\Psi_0,\, \Phi^N\right) . \tag{4.16}$$

The result for $\delta\{A_1 \cdots A_N\}$ can be found from (4.16) by taking $\Phi = A_1 + \cdots + A_N$ and keeping terms proportional to $A_1 \cdots A_N$ on both sides. For example replacing Φ by $A + \Phi$ and keeping terms linear in A on both sides of (4.16) we get

$$\delta\{A\Phi^{N-1}\} = \lambda\, \mathcal{V}'_{N+1,0}\left(\Psi_0, A, \Phi^{N-1}\right) . \tag{4.17}$$

In the following analysis we shall make use of various relations involving the $*$, $\leftrightarrow\!\!*$ and $\twoheadrightarrow$ product of the subspaces $\mathcal{V}'_{0,2,1}$, $\mathcal{V}''_{0,1,2}$ and $\mathcal{V}'_{g,N+1,0}$. Since these vertices are not symmetric in all the punctures, we need to carefully specify which of the punctures are sewed. In this we shall use the convention that the special puncture carried by the corresponding Riemann surfaces never takes part in sewing. Therefore for example when we have a subspace of the form $\mathcal{A} * \mathcal{V}'_{g,N+1,0}$, it is the left-most puncture of $\mathcal{V}'$ *other than the special puncture* that takes part in the sewing. Similar convention will be used for $\mathcal{V}'_{0,2,1}$ and $\mathcal{V}''_{0,1,2}$.

The boundary of $\mathcal{V}'_{N+1,0}$ will play a special role in the subsequent analysis. This can be determined as follows. Since

$$\mathcal{V}^{\text{deformed}}_{N,0}\left(\Phi^N\right) \equiv \mathcal{V}_{N,0}\left(\Phi^N\right) + \lambda \mathcal{V}'_{N+1,0}\left(\Psi_0, \Phi^N\right) \tag{4.18}$$

gives the interaction vertex of the deformed theory, it satisfies an identity similar to (2.41):

$$M\, \mathcal{V}^{\text{deformed}}_{M,0}\big((Q_B+\delta Q_B)\Phi, \Phi^{M-1}\big) = -\frac{1}{2} \sum_{\substack{M_1,M_2\\ M_1+M_2=M}} \frac{M!}{M_1!M_2!}\, \mathcal{V}^{\text{deformed}}_{M_1+1,0} *' \mathcal{V}^{\text{deformed}}_{M_2+1,0}\left(\Phi^{M_1}|\Phi^{M_2}\right), \tag{4.19}$$

where $*'$ is defined in the same way as $*$ except that for Ramond sector sewing we insert $\mathcal{G}+\delta\mathcal{G}$ instead of $\mathcal{G}$ around one of the punctures that are sewed. Now, using (4.11), (2.30)–(2.32) and the anti-symmetry property (4.13), we can write

$$\begin{aligned} &\mathcal{V}_{M_1+1,0} *' \mathcal{V}_{M_2+1,0}\left(\Phi^{M_1}|\Phi^{M_2}\right) \\ &= \mathcal{V}_{M_1+1,0} * \mathcal{V}_{M_2+1,0}\left(\Phi^{M_1}|\Phi^{M_2}\right) + \lambda\, \mathcal{V}_{M_1+1,0} \twoheadrightarrow \mathcal{V}''_{0,1,2} \leftrightarrow\!\!* \mathcal{V}_{M_2+1,0}\left(\Phi^{M_1}|\Psi_0|\Phi^{M_2}\right) . \end{aligned} \tag{4.20}$$

While applying (2.30)–(2.32), (4.11) and (4.13) to arrive at (4.20), we have to keep in mind that not all the arguments that appear in the intermediate steps of the analysis are grassmann even, and therefore there will be extra signs that have to be computed carefully using the procedure explained earlier. Expanding (4.19) in powers of λ using (4.18), (4.20), and collecting the coefficients of the order λ term, we get

$$\begin{aligned} M\, \mathcal{V}'_{M+1,0}\left(\Psi_0, Q_B\Phi, \Phi^{M-1}\right) = &-\sum_{\substack{M_1,M_2\\ M_1+M_2=M}} \frac{M!}{M_1!M_2!}\, \mathcal{V}_{M_1+1,0} * \mathcal{V}'_{M_2+2,0}\left(\Phi^{M_1}|\Psi_0, \Phi^{M_2}\right) \\ &-\frac{1}{2} \sum_{\substack{M_1,M_2\\ M_1+M_2=M}} \frac{M!}{M_1!M_2!}\, \mathcal{V}_{M_1+1,0} \twoheadrightarrow \mathcal{V}''_{0,1,2} \leftrightarrow\!\!* \mathcal{V}_{M_2+1,0}\left(\Phi^{M_1}|\Psi_0|\Phi^{M_2}\right) \\ &-M\, \lambda^{-1}\, \mathcal{V}_{M,0}\left(\delta Q_B\Phi, \Phi^{M-1}\right) . \end{aligned} \tag{4.21}$$

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As already mentioned, this relation holds up to the choice of local coordinates at the special puncture. Using (2.32) and (4.8) we can express this as

$$\begin{aligned}\partial\mathcal{V}'_{M+1,0}(\Psi_0,\Phi^M) = &-\sum_{\substack{M_1,M_2\\M_1+M_2=M}}\frac{M!}{M_1!M_2!}\,\mathcal{V}_{M_1+1,0}*\mathcal{V}'_{M_2+2,0}\left(\Phi^{M_1}|\Psi_0,\Phi^{M_2}\right)\\ &-\frac{1}{2}\sum_{\substack{M_1,M_2\\M_1+M_2=M}}\frac{M!}{M_1!M_2!}\,\mathcal{V}_{M_1+1,0}\twoheadrightarrow\mathcal{V}''_{0,1,2}\leftarrow\!\!*\,\mathcal{V}_{M_2+1,0}\left(\Phi^{M_1}|\Psi_0|\Phi^{M_2}\right)\\ &-M\,\mathcal{V}'_{0,2,1}\leftarrow\!\!*\,\mathcal{V}_{M,0}\left(\Psi_0,\Phi|\Phi^{M-1}\right)\,.\end{aligned}\tag{4.22}$$

An equivalent relation, that will be useful later, is obtained by replacing Φ by $\Phi+A$ and keeping terms linear in A on both sides:

$$\begin{aligned}&M\,\partial\mathcal{V}'_{M+1,0}\left(\Psi_0,A,\Phi^{M-1}\right)\\ =&-\sum_{\substack{M_1,M_2\\M_1+M_2=M}}\frac{M!}{M_1!(M_2-1)!}\,\mathcal{V}_{M_1+1,0}*\mathcal{V}'_{M_2+2,0}\left(\Phi^{M_1}|\Psi_0,A,\Phi^{M_2-1}\right)\\ &-\sum_{\substack{M_1,M_2\\M_1+M_2=M}}\frac{M!}{(M_1-1)!M_2!}\,\mathcal{V}_{M_1+1,0}*\mathcal{V}'_{M_2+2,0}\left(A,\Phi^{M_1-1}|\Psi_0,\Phi^{M_2}\right)\\ &-\sum_{\substack{M_1,M_2\\M_1+M_2=M}}\frac{M!}{(M_1-1)!M_2!}\mathcal{V}_{M_1+1,0}\twoheadrightarrow\mathcal{V}''_{0,1,2}\leftarrow\!\!*\,\mathcal{V}_{M_2+1,0}\left(A,\Phi^{M_1-1}|\Psi_0|\Phi^{M_2}\right)\\ &-M\,\mathcal{V}'_{0,2,1}\leftarrow\!\!*\,\mathcal{V}_{M,0}\left(\Psi_0,A|\Phi^{M-1}\right)-M(M-1)\,\mathcal{V}'_{0,2,1}\leftarrow\!\!*\,\mathcal{V}_{M,0}\left(\Psi_0,\Phi|A,\Phi^{M-2}\right).\end{aligned}\tag{4.23}$$

5 Proof of background independence

In this section we shall show the existence of $\delta\Phi$ satisfying the $\Delta\simeq 0$ equation, where $\simeq$ denotes equality up to terms that vanish by leading order equations of motion (3.26). We shall seek a solution for $\delta\Phi$ such that for any grassmann even state $\widetilde{B}\in\widetilde{\mathcal{H}}$:

$$\langle\widetilde{B}|c_0^-|\delta\Phi\rangle = \lambda\sum_{N\geq 0}\frac{1}{N!}\,\mathcal{B}_{N+1,1}\left(\Psi_0,\Phi^N;\widetilde{B}\right)\,,\tag{5.1}$$

$$\mathcal{B}_{N+1,1}\equiv\sum_{g\geq 0}\mathcal{B}_{g,N+1,1}\,,\tag{5.2}$$

where $\mathcal{B}_{g,N+1,1}$ is a $6g+2N-1$ dimensional subspace of $\mathcal{Q}_{g,N+1,1}$ that is to be determined. The dimension of $\mathcal{B}_{g,N+1,1}$ is fixed by the requirement that the total ghost number carried by the arguments minus the dimension of $\mathcal{B}_{g,N+1,1}$ should be equal to $6-6g$ due to ghost number conservation. The first $\widehat{\mathcal{H}}$ puncture of $\mathcal{B}_{g,N+1,1}$ is taken to be the special puncture where we insert the state Ψ_0. This puncture never takes part in the sewing operation, and we leave unspecified the choice of local coordinate at this puncture. $\mathcal{B}_{g,N+1,1}$ is taken to be symmetric under the permutations of the rest of the N punctures in $\widehat{\mathcal{H}}$ where Φ's are

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inserted. $\widetilde{B}$ is inserted at the only $\widetilde{\mathcal{H}}$ puncture of $\mathcal{B}_{g,N+1,1}$. We shall look for solutions for $\mathcal{B}_{g,N+1,1}$ that avoid separating type degenerations.

If $\widetilde{B}$ is grassmann odd then there is an addition minus sign in (5.1) arising as follows. When we convert a grassmann odd operator to a grassmann even operator by multiplication by a grassmann odd c-number ζ, and try to strip off ζ by moving it to the extreme right or left on both sides, we get an extra minus sign arising from the effect of passing ζ through c_0^- on the left hand side of (5.1) when we move ζ to the right, or from having to pass ζ through the $6g+2N-1$ grassmann odd operators on the right hand side of (5.1), implicit in the definition of $\mathcal{B}_{g,N+1,1}(\Psi_0,\Phi^N;\widetilde{B})$, when we move ζ to the left.

5.1 Geometrization of the problem of background independence

In this section we shall describe the geometric interpretation of different terms appearing in (3.25).

First term: we begin with the first term on the right hand side of (3.25). Since $Q_B\widetilde{A}$ is grassmann odd, we have, from (5.1), (2.27) and the BRST invariance of $|\Psi_0\rangle$, that

$$\begin{aligned}
&\langle Q_B\widetilde{A}|c_0^-|\delta\Phi\rangle = -\lambda\sum_N \frac{1}{N!}\,\mathcal{B}_{N+1,1}\left(\Psi_0,\Phi^N;Q_B\widetilde{A}\right)\\
&= \lambda\sum_N \frac{1}{N!}\,\partial\mathcal{B}_{N+1,1}\left(\Psi_0,\Phi^N;\widetilde{A}\right) + \lambda\sum_N \frac{1}{(N-1)!}\,\mathcal{B}_{N+1,1}\left(\Psi_0,\Phi^{N-1},Q_B\Phi;\widetilde{A}\right)\,.
\end{aligned} \tag{5.3}$$

Using the leading order equations of motion (3.26) this may be rewritten as

$$\begin{aligned}
&\lambda\sum_N \frac{1}{N!}\,\partial\mathcal{B}_{N+1,1}\left(\Psi_0,\Phi^N;\widetilde{A}\right)\\
&-\lambda\sum_{N,M} \frac{1}{(N-1)!M!}\,\mathcal{B}_{N+1,1}\left(\Psi_0,\Phi^{N-1},\varphi_r;\widetilde{A}\right)\langle\varphi_r^c|c_0^-\mathcal{G}|\varphi_s^c\rangle\langle\varphi_s|c_0^-\left[\Phi^M\right]\rangle\\
&= \lambda\sum_N \frac{1}{N!}\,\partial\mathcal{B}_{N+1,1}\left(\Psi_0,\Phi^N;\widetilde{A}\right) - \lambda\sum_{N,M}\frac{1}{N!M!}\,\mathcal{B}_{N+2,1}*\mathcal{V}_{M+1,0}\left(\Psi_0,\Phi^N;\widetilde{A}|\Phi^M\right)\,,
\end{aligned} \tag{5.4}$$

where in the second step we have used (2.19), (2.30) and (2.39) .

Second term: in order to give a geometric interpretation of the second term on the right hand side of (3.25), we express this as

$$\begin{aligned}
&\sum_N \frac{1}{N!}\{\left(\mathcal{G}\widetilde{A}\right)\Phi^N\,\varphi_r\}\langle\varphi_r^c|c_0^-|\delta\Phi\rangle\\
&= \lambda\sum_N\sum_M \frac{1}{N!}\,\frac{1}{M!}\,(-1)^{n_s}\,\mathcal{V}_{N+2,0}\left(\left(\mathcal{G}\widetilde{A}\right),\Phi^N,\varphi_r\right)\langle\varphi_r^c|c_0^-|\varphi_s^c\rangle\,\mathcal{B}_{M+1,1}\left(\Psi_0,\Phi^M;\varphi_s\right)\,.
\end{aligned} \tag{5.5}$$

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Using (2.31) and the fact that $\dim \mathcal{B}_{g,M+1,1} = 6g + 2M - 1$, we can express this as

$$\lambda \sum_N \sum_M \frac{1}{N!} \frac{1}{M!} \mathcal{V}_{N+2,0} \twoheadrightarrow \mathcal{B}_{M+1,1} \left(\left(\mathcal{G} \widetilde{A} \right), \Phi^N | \Psi_0, \Phi^M \right) . \tag{5.6}$$

Note that since $\mathcal{V}_{N+2,0}$ is fully symmetric in all the punctures we need not specify the puncture that participates in the sewing. $\mathcal{B}_{M+2,1}$ is not fully symmetric, but since it has only one $\widetilde{\mathcal{H}}$ puncture, there is no ambiguity in which puncture takes part in the sewing.

Third term: it follows from (4.8) that the third term on the right hand side of (3.25) may be written as

$$\langle \widetilde{A} | c_0^- \delta Q_B | \Phi \rangle = \lambda \, \mathcal{V}'_{0,2,1} \left(\Psi_0, \Phi; \widetilde{A} \right) . \tag{5.7}$$

Fourth term: using (4.11) the fourth term on the right hand side of (3.25) may be expressed as

$$\begin{aligned} & \sum_N \frac{1}{N!} \{ \Phi^N \varphi_r \} \langle \varphi_r^c | c_0^- | \varphi_s^c \rangle \langle \varphi_s | c_0^- \delta \mathcal{G} | \widetilde{A} \rangle \\ &= \lambda \sum_N \frac{1}{N!} (-1)^{n_s+1} \mathcal{V}_{N+1,0} \left(\Phi^N, \varphi_r \right) \langle \varphi_r^c | c_0^- | \varphi_s^c \rangle \mathcal{V}''_{0,1,2} \left(\Psi_0; \widetilde{A}, \varphi_s \right) \\ &= \lambda \sum_N \frac{1}{N!} \mathcal{V}_{N+1,0} \twoheadrightarrow \mathcal{V}''_{0,1,2} \left(\Phi^N | \Psi_0; \widetilde{A} \right) \end{aligned} \tag{5.8}$$

Fifth term: we shall now consider the fifth term on the right hand side of (3.25). Using (4.17) this may be expressed as

$$\lambda \sum_N \frac{1}{N!} \mathcal{V}'_{N+2,0} \left(\Psi_0, \left(\mathcal{G} \, \widetilde{A} \right), \Phi^N \right) . \tag{5.9}$$

Sixth term: finally the sixth term on the right hand side of (3.25) may be written as

$$- \lambda \sum_N \frac{1}{N!} \mathcal{V}_{N+2,0} \left(\Psi_0, \left(\mathcal{G} \widetilde{A} \right), \Phi^N \right) . \tag{5.10}$$

Using (5.4), (5.6), (5.7), (5.8), (5.9) and (5.10) we can now express (3.25) as

$$\begin{aligned} \Delta = & \lambda \sum_N \frac{1}{N!} \partial \mathcal{B}_{N+1,1} \left(\Psi_0, \Phi^N; \widetilde{A} \right) - \lambda \sum_{N,M} \frac{1}{N! M!} \mathcal{B}_{N+2,1} * \mathcal{V}_{M+1,0} \left(\Psi_0, \Phi^N; \widetilde{A} | \Phi^M \right) \\ & + \lambda \sum_N \sum_M \frac{1}{N!} \frac{1}{M!} \mathcal{V}_{N+2,0} \twoheadrightarrow \mathcal{B}_{M+1,1} \left(\left(\mathcal{G} \widetilde{A} \right), \Phi^N | \Psi_0, \Phi^M \right) + \lambda \, \mathcal{V}'_{0,2,1} \left(\Psi_0, \Phi, \widetilde{A} \right) \\ & + \lambda \sum_N \frac{1}{N!} \mathcal{V}_{N+1,0} \twoheadrightarrow \mathcal{V}''_{0,1,2} \left(\Phi^N | \Psi_0; \widetilde{A} \right) + \lambda \sum_N \frac{1}{N!} \mathcal{V}'_{N+2,0} \left(\Psi_0, \left(\mathcal{G} \, \widetilde{A} \right), \Phi^N \right) \\ & - \lambda \sum_N \frac{1}{N!} \mathcal{V}_{N+2,0} \left(\Psi_0, \left(\mathcal{G} \widetilde{A} \right), \Phi^N \right) . \end{aligned} \tag{5.11}$$

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Demanding the vanishing of Δ now gives

$$
\begin{aligned}
&\partial\mathcal{B}_{N+1,1}\left(\Psi_0,\Phi^N;\widetilde{A}\right)\\
=&\sum_{\substack{M_1,M_2\\M_1+M_2=N}}\frac{N!}{M_1!M_2!}\mathcal{B}_{M_1+2,1}*\mathcal{V}_{M_2+1,0}\left(\Psi_0,\Phi^{M_1};\widetilde{A}|\Phi^{M_2}\right)\\
&-\sum_{\substack{M_1,M_2\\M_1+M_2=N}}\frac{N!}{M_1!M_2!}\mathcal{V}_{M_1+2,0}\twoheadrightarrow\mathcal{B}_{M_2+1,1}\left(\left(\mathcal{G}\widetilde{A}\right),\Phi^{M_1}|\Psi_0,\Phi^{M_2}\right)-\delta_{N,1}\mathcal{V}'_{0,2,1}\left(\Psi_0,\Phi;\widetilde{A}\right)\\
&-\mathcal{V}_{N+1,0}\twoheadrightarrow\mathcal{V}''_{0,1,2}\left(\Phi^N|\Psi_0,\widetilde{A}\right)-\mathcal{V}'_{N+2,0}\left(\Psi_0,\left(\mathcal{G}\,\widetilde{A}\right),\Phi^N\right)+\mathcal{V}_{N+2,0}\left(\Psi_0,\left(\mathcal{G}\widetilde{A}\right),\Phi^N\right).
\end{aligned}
\tag{5.12}
$$

Since Ψ_0 is inserted at the special puncture, this equation needs to hold as a relation between subspaces of $\mathcal{Q}_{N+1,1}$ up to choice of local coordinates at the special puncture.

5.2 Absence of obstruction

We can solve eq. (5.12) for $\mathcal{B}_{N+1,1}\equiv\sum_{g\geq 0}\mathcal{B}_{g,N+1,1}$ iteratively by carrying out genus expansion on both sides. Using that fact that $\mathcal{V}_{0,N,0}$ vanishes for $N\leq 2$, it is easy to verify that the expression for $\partial\mathcal{B}_{g,N+1,1}$ obtained from (5.12) contains on the right hand side $\mathcal{B}_{g',N'+1,1}$ for $g'<g$, and / or $\mathcal{B}_{g,N'+1,1}$ for $N'<N$. In particular the equation for $\mathcal{B}_{0,2,1}$ does not involve any $\mathcal{B}_{g,N,1}$ on the right hand side. Once this is determined we can solve for $\mathcal{B}_{0,3,1}$ since its equation involves only $\mathcal{B}_{0,2,1}$ on the right hand side. Proceeding this was we can first determine all the $\mathcal{B}_{0,N,1}$ iteratively, then determine all the $\mathcal{B}_{1,N,1}$ and so on.

There is however a possible obstruction to solving these equations. Since $\partial(\partial\mathcal{B}_{g,N+1,1})=0$, in order that the equation for $\mathcal{B}_{g,N+1,1}$ obtained from (5.12) has a solution, we need to show that the ∂ annihilates the right hand side of the equation. This gives

$$
\begin{aligned}
0=&\sum_{\substack{M_1,M_2\\M_1+M_2=N}}\frac{N!}{M_1!M_2!}\partial\mathcal{B}_{M_1+2,1}*\mathcal{V}_{M_2+1,0}\left(\Psi_0,\Phi^{M_1};\widetilde{A}|\Phi^{M_2}\right)\\
&+\sum_{\substack{M_1,M_2\\M_1+M_2=N}}\frac{N!}{M_1!M_2!}\mathcal{B}_{M_1+2,1}*\partial\mathcal{V}_{M_2+1,0}\left(\Psi_0,\Phi^{M_1};\widetilde{A}|\Phi^{M_2}\right)\\
&-\sum_{\substack{M_1,M_2\\M_1+M_2=N}}\frac{N!}{M_1!M_2!}\partial\mathcal{V}_{M_1+2,0}\twoheadrightarrow\mathcal{B}_{M_2+1,1}\left(\left(\mathcal{G}\widetilde{A}\right),\Phi^{M_1}|\Psi_0,\Phi^{M_2}\right)\\
&+\sum_{\substack{M_1,M_2\\M_1+M_2=N}}\frac{N!}{M_1!M_2!}\mathcal{V}_{M_1+2,0}\twoheadrightarrow\partial\mathcal{B}_{M_2+1,1}\left(\left(\mathcal{G}\widetilde{A}\right),\Phi^{M_1}|\Psi_0,\Phi^{M_2}\right)\\
&-\partial\mathcal{V}_{N+1,0}\twoheadrightarrow\mathcal{V}''_{0,1,2}\left(\Phi^N|\Psi_0;\widetilde{A}\right)+\mathcal{V}_{N+1,0}\twoheadrightarrow\partial\mathcal{V}''_{0,1,2}\left(\Phi^N|\Psi_0;\widetilde{A}\right)\\
&-\partial\mathcal{V}'_{N+2,0}\left(\Psi_0,\left(\mathcal{G}\,\widetilde{A}\right),\Phi^N\right)+\partial\mathcal{V}_{N+2,0}\left(\Psi_0,\left(\mathcal{G}\widetilde{A}\right),\Phi^N\right).
\end{aligned}
\tag{5.13}
$$

Since in the iterative scheme described above the right hand side involves $\mathcal{B}_{g',N',1}$ for lower values of g' or N' which already satisfy (5.12), we need to prove (5.13) for $\mathcal{B}_{N'+1,1}$'s appearing on the right hand side satisfying (5.12). This allows us to simplify the different terms on the right hand side of (5.13) as follows.

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The first term on the right hand side of (5.13) is given by

$$
\begin{aligned}
I_1 \equiv & \sum_{\substack{M_1,M_2\\ M_1+M_2=N}} \frac{N!}{M_1!M_2!}\,\partial\mathcal{B}_{M_1+2,1} * \mathcal{V}_{M_2+1,0}\left(\Psi_0,\Phi^{M_1};\widetilde{A}|\Phi^{M_2}\right)\\
= & \sum_{\substack{M_1,M_2\\ M_1+M_2=N}} \frac{N!}{M_1!M_2!} \sum_{\substack{M_3,M_4\\ M_3+M_4=M_1}} \frac{M_1!}{M_3!M_4!}\mathcal{B}_{M_3+2,1} * \mathcal{V}_{M_4+2,0} * \mathcal{V}_{M_2+1,0}\left(\Psi_0,\Phi^{M_3};\widetilde{A}|\Phi^{M_4}|\Phi^{M_2}\right)\\
& - \sum_{\substack{M_1,M_2\\ M_1+M_2=N}} \frac{N!}{M_1!M_2!} \sum_{\substack{M_3,M_4\\ M_3+M_4=M_1}} \frac{M_1!}{M_3!M_4!}\mathcal{V}_{M_4+1,0} * \mathcal{B}_{M_3+3,1} * \mathcal{V}_{M_2+1,0}\left(\Phi^{M_4}|\Psi_0,\Phi^{M_3};\widetilde{A}|\Phi^{M_2}\right)\\
& - \sum_{\substack{M_1,M_2\\ M_1+M_2=N}} \frac{N!}{M_1!M_2!} \sum_{\substack{M_3,M_4\\ M_3+M_4=M_1}} \frac{M_1!}{M_3!M_4!}\mathcal{V}_{M_3+2,0} \twoheadrightarrow \mathcal{B}_{M_4+2,1} * \mathcal{V}_{M_2+1,0}\left(\left(\mathcal{G}\widetilde{A}\right),\Phi^{M_3}|\Psi_0,\Phi^{M_4}|\Phi^{M_2}\right)\\
& + \sum_{\substack{M_1,M_2\\ M_1+M_2=N}} \frac{N!}{M_1!M_2!} \sum_{\substack{M_3,M_4\\ M_3+M_4=M_1}} \frac{M_1!}{M_3!M_4!}\mathcal{B}_{M_4+1,1} \leftrightarrow\!* \mathcal{V}_{M_3+3,0} * \mathcal{V}_{M_2+1,0}\left(\Psi_0,\Phi^{M_4}|\left(\mathcal{G}\widetilde{A}\right),\Phi^{M_3}|\Phi^{M_2}\right)\\
& - \mathcal{V}'_{0,2,1} * \mathcal{V}_{N+1,0}\left(\Psi_0;\widetilde{A}|\Phi^{N}\right)\\
& - \sum_{\substack{M_1,M_2\\ M_1+M_2=N}} \frac{N!}{M_1!M_2!}\mathcal{V}''_{0,1,2} \leftrightarrow\!* \mathcal{V}_{M_1+2,0} * \mathcal{V}_{M_2+1,0}\left(\Psi_0;\widetilde{A}|\Phi^{M_1}|\Phi^{M_2}\right)\\
& - \sum_{\substack{M_1,M_2\\ M_1+M_2=N}} \frac{N!}{M_1!M_2!}\mathcal{V}'_{M_1+3,0} * \mathcal{V}_{M_2+1,0}\left(\Psi_0,\mathcal{G}\widetilde{A},\Phi^{M_1}|\Phi^{M_2}\right)\\
& + \sum_{\substack{M_1,M_2\\ M_1+M_2=N}} \frac{N!}{M_1!M_2!}\mathcal{V}_{M_1+3,0} * \mathcal{V}_{M_2+1,0}\left(\Psi_0,\mathcal{G}\widetilde{A},\Phi^{M_1}|\Phi^{M_2}\right) .
\end{aligned}
\tag{5.14}
$$

Since this manipulation is somewhat involved, we shall illustrate how the first two terms on the right hand side of (5.14) arise by picking, in the expression for $\partial\mathcal{B}_{M_1+2,1}(\Psi_0,\Phi^{M_1+1};\widetilde{A})$, the first term on the right hand side of (5.12):

$$
\sum_{\substack{M_3,M_4\\ M_3+M_4=M_1+1}} \frac{(M_1+1)!}{M_3!M_4!}\,\mathcal{B}_{M_3+2,1} * \mathcal{V}_{M_4+1,0}\left(\Psi_0,\Phi^{M_3};\widetilde{A}|\Phi^{M_4}\right) . \tag{5.15}
$$

We need to sew this to $\mathcal{V}_{M_2+1,0}$. This is done by picking a puncture other than the special puncture in $\partial\mathcal{B}_{M_1+2,1}$ and a puncture of $\mathcal{V}_{M_2+1,0}$ and sewing them. Now since $\partial\mathcal{B}_{M_1+2,1}$ and $\mathcal{V}_{M_2+1,0}$ are symmetric under the permutations of the punctures in $\widehat{\mathcal{H}}$ (other than the special puncture), it does not matter which puncture we choose for sewing. However when we pick the term given in (5.15) in the expression for $\partial\mathcal{B}_{M_1+2,1}$, then we have to allow for the puncture to come either from $\mathcal{B}_{M_3+2,1}$ or from $\mathcal{V}_{M_4+1,0}$ with appropriate weight factors given by $M_3/(M_3+M_4)$ and $M_4/(M_3+M_4)$ respectively. Thus for example the net contribution to I_1 from the term where we choose the sewing puncture of $\partial\mathcal{B}_{M_1+2,0}$ from $\mathcal{V}_{M_4+1,0}$ will be given by

$$
\begin{aligned}
& \sum_{\substack{M_1,M_2\\ M_1+M_2=N}} \frac{N!}{M_1!M_2!} \sum_{\substack{M_3,M_4\\ M_3+M_4=M_1+1}} \frac{(M_1+1)!}{M_3!M_4!}\,\frac{M_4}{M_3+M_4}\times\\
& \times\mathcal{B}_{M_3+2,1} * \mathcal{V}_{M_4+1,0} * \mathcal{V}_{M_2+1,0}\left(\Psi_0,\Phi^{M_3};\widetilde{A}|\Phi^{M_4-1}|\Phi^{M_2}\right) .
\end{aligned}
\tag{5.16}
$$

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By making a change of variable $M_4 \to M_4+1$ we recover the first term on the right hand side of (5.14). Similarly the result of choosing the sewing puncture in $\partial\mathcal{B}_{M_1+2,1}$ from $\mathcal{B}_{M_3+2,1}$ is given by the second term on the right hand side of (5.14), with the extra minus sign arising from switching the order of $\mathcal{B}_{M_3+2,1}$ and $\mathcal{V}_{M_4+1,0}$ (see (2.33)).

The second term on the right hand side of (5.13) is given by

$$\begin{aligned} I_2 &\equiv \sum_{\substack{M_1,M_2\\ M_1+M_2=N}} \frac{N!}{M_1!M_2!}\mathcal{B}_{M_1+2,1} * \partial\mathcal{V}_{M_2+1,0}\left(\Psi_0,\Phi^{M_1};\widetilde{A}|\Phi^{M_2}\right) \\ &= -\sum_{\substack{M_1,M_2\\ M_1+M_2=N}} \frac{N!}{M_1!M_2!}\sum_{\substack{M_3,M_4\\ M_3+M_4=M_2}} \frac{M_2!}{M_3!M_4!}\mathcal{B}_{M_1+2,1}*\mathcal{V}_{M_3+2,0}*\mathcal{V}_{M_4+1,0}\left(\Psi_0,\Phi^{M_1};\widetilde{A}|\Phi^{M_3}|\Phi^{M_4}\right). \end{aligned} \tag{5.17}$$

The third term on the right hand side of (5.13) is given by

$$\begin{aligned} I_3 &\equiv -\sum_{\substack{M_1,M_2\\ M_1+M_2=N}} \frac{N!}{M_1!M_2!}\partial\mathcal{V}_{M_1+2,0} \twoheadrightarrow \mathcal{B}_{M_2+1,1}\left(\left(\mathcal{G}\widetilde{A}\right),\Phi^{M_1}|\Psi_0,\Phi^{M_2}\right) \\ &= \sum_{\substack{M_1,M_2\\ M_1+M_2=N}} \frac{N!}{M_1!M_2!}\sum_{\substack{M_3,M_4\\ M_3+M_4=M_1}} \frac{M_1!}{M_3!M_4!}\mathcal{V}_{M_3+2,0}*\mathcal{V}_{M_4+2,0} \twoheadrightarrow \mathcal{B}_{M_2+1,1}\left(\left(\mathcal{G}\widetilde{A}\right),\Phi^{M_3}|\Phi^{M_4}|\Psi_0,\Phi^{M_2}\right) \\ &\quad+\sum_{\substack{M_1,M_2\\ M_1+M_2=N}} \frac{N!}{M_1!M_2!}\sum_{\substack{M_3,M_4\\ M_3+M_4=M_1}} \frac{M_1!}{M_3!M_4!}\mathcal{V}_{M_3+1,0}*\mathcal{V}_{M_4+3,0} \twoheadrightarrow \mathcal{B}_{M_2+1,1}\left(\Phi^{M_3}|\left(\mathcal{G}\widetilde{A}\right),\Phi^{M_4}|\Psi_0,\Phi^{M_2}\right). \end{aligned} \tag{5.18}$$

The fourth term on the right hand side of (5.13) is given by

$$\begin{aligned} I_4 &\equiv \sum_{\substack{M_1,M_2\\ M_1+M_2=N}} \frac{N!}{M_1!M_2!}\mathcal{V}_{M_1+2,0} \twoheadrightarrow \partial\mathcal{B}_{M_2+1,1}\left(\left(\mathcal{G}\widetilde{A}\right),\Phi^{M_1}|\Psi_0,\Phi^{M_2}\right) \\ &= \sum_{\substack{M_1,M_2\\ M_1+M_2=N}} \frac{N!}{M_1!M_2!}\sum_{\substack{M_3,M_4\\ M_3+M_4=M_2}} \frac{M_2!}{M_3!M_4!}\mathcal{V}_{M_1+2,0} \twoheadrightarrow \mathcal{B}_{M_3+2,1}*\mathcal{V}_{M_4+1,0}\left(\left(\mathcal{G}\widetilde{A}\right),\Phi^{M_1}|\Psi_0,\Phi^{M_3}|\Phi^{M_4}\right) \\ &\quad-\sum_{\substack{M_1,M_2\\ M_1+M_2=N}} \frac{N!}{M_1!M_2!}\sum_{\substack{M_3,M_4\\ M_3+M_4=M_2}} \frac{M_2!}{M_3!M_4!}\mathcal{V}_{M_1+2,0}*\mathcal{V}_{M_3+2,0} \twoheadrightarrow \mathcal{B}_{M_4+1,1}\left(\left(\mathcal{G}\widetilde{A}\right),\Phi^{M_1}|\Phi^{M_3}|\Psi_0,\Phi^{M_4}\right) \\ &\quad-N\,\mathcal{V}_{N+1,0} \twoheadrightarrow \mathcal{V}'_{0,2,1}\left(\left(\mathcal{G}\widetilde{A}\right),\Phi^{N-1}|\Psi_0,\Phi\right) \\ &\quad-\sum_{\substack{M_1,M_2\\ M_1+M_2=N}} \frac{N!}{M_1!M_2!}\mathcal{V}_{M_1+2,0} \twoheadrightarrow \mathcal{V}''_{0,1,2} \leftarrow\!\!* \mathcal{V}_{M_2+1,0}\left(\left(\mathcal{G}\widetilde{A}\right),\Phi^{M_1}|\Psi_0|\Phi^{M_2}\right) \\ &\quad-\sum_{\substack{M_1,M_2\\ M_1+M_2=N}} \frac{N!}{M_1!M_2!}\mathcal{V}_{M_1+2,0}*\mathcal{V}'_{M_2+2,0}\left(\left(\mathcal{G}\widetilde{A}\right),\Phi^{M_1}|\Psi_0,\Phi^{M_2}\right) \\ &\quad+\sum_{\substack{M_1,M_2\\ M_1+M_2=N}} \frac{N!}{M_1!M_2!}\mathcal{V}_{M_1+2,0}*\mathcal{V}_{M_2+2,0}\left(\left(\mathcal{G}\widetilde{A}\right),\Phi^{M_1}|\Psi_0,\Phi^{M_2}\right). \end{aligned} \tag{5.19}$$

The fifth term on the right hand side of (5.13) is given by

$$\begin{aligned} I_5 &\equiv -\partial\mathcal{V}_{N+1,0} \twoheadrightarrow \mathcal{V}''_{0,1,2}\left(\Phi^N|\Psi_0;\widetilde{A}\right) \\ &= \sum_{\substack{M_1,M_2\\ M_1+M_2=N}} \frac{N!}{M_1!M_2!}\mathcal{V}_{M_1+1,0}*\mathcal{V}_{M_2+2,0} \twoheadrightarrow \mathcal{V}''_{0,1,2}\left(\Phi^{M_1}|\Phi^{M_2}|\Psi_0;\widetilde{A}\right). \end{aligned} \tag{5.20}$$

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Using (4.12), the sixth term on the right hand side of (5.13) is given by

$$
\begin{aligned}
I_6 &\equiv \mathcal{V}_{N+1,0} \twoheadrightarrow \partial \mathcal{V}''_{0,1,2}\left(\Phi^N|\Psi_0;\widetilde{A}\right) \\
&= \mathcal{V}_{N+1,0} * \mathcal{V}'_{0,2,1}\left(\Phi^N|\Psi_0;\widetilde{A}\right) - \mathcal{V}'_{0,2,1} \leftrightarrow\!\!\!* \, \mathcal{V}_{N+1,0}\left(\Psi_0,\mathcal{G}\widetilde{A}|\Phi^N\right) .
\end{aligned} \tag{5.21}
$$

Using (4.23), the seventh term on the right hand side of (5.13) is given by

$$
\begin{aligned}
I_7 &\equiv -\partial \mathcal{V}'_{N+2,0}\left(\Psi_0,\left(\mathcal{G}\,\widetilde{A}\right),\Phi^N\right) \\
= &\sum_{\substack{M_1,M_2\\ M_1+M_2=N}} \frac{N!}{M_1!M_2!}\mathcal{V}_{M_1+1,0} * \mathcal{V}'_{M_2+3,0}\left(\Phi^{M_1}|\Psi_0,\mathcal{G}\widetilde{A},\Phi^{M_2}\right) \\
&+ \sum_{\substack{M_1,M_2\\ M_1+M_2=N}} \frac{N!}{M_1!M_2!}\mathcal{V}_{M_1+2,0} * \mathcal{V}'_{M_2+2,0}\left(\mathcal{G}\widetilde{A},\Phi^{M_1}|\Psi_0,\Phi^{M_2}\right) \\
&+ \sum_{\substack{M_1,M_2\\ M_1+M_2=N}} \frac{N!}{M_1!M_2!}\mathcal{V}_{M_1+2,0} \twoheadrightarrow \mathcal{V}''_{0,1,2} \leftrightarrow\!\!\!* \, \mathcal{V}_{M_2+1,0}\left(\mathcal{G}\widetilde{A},\Phi^{M_1}|\Psi_0|\Phi^{M_2}\right) \\
&+\mathcal{V}'_{0,2,1} \leftrightarrow\!\!\!* \, \mathcal{V}_{N+1,0}\left(\Psi_0,\mathcal{G}\widetilde{A}|\Phi^N\right) + N\,\mathcal{V}'_{0,2,1} \leftrightarrow\!\!\!* \, \mathcal{V}_{N+1,0}\left(\Psi_0,\Phi|\mathcal{G}\widetilde{A},\Phi^{N-1}\right) .
\end{aligned} \tag{5.22}
$$

The eighth term on the right hand side of (5.13) is given by

$$
\begin{aligned}
I_8 &\equiv \partial \mathcal{V}_{N+2,0}\left(\Psi_0,\left(\mathcal{G}\widetilde{A}\right),\Phi^N\right) \\
= &- \sum_{\substack{M_1,M_2\\ M_1+M_2=N}} \frac{N!}{M_1!M_2!}\mathcal{V}_{M_1+3,0} * \mathcal{V}_{M_2+1,0}\left(\Psi_0,\mathcal{G}\widetilde{A},\Phi^{M_1}|\Phi^{M_2}\right) \\
&- \sum_{\substack{M_1,M_2\\ M_1+M_2=N}} \frac{N!}{M_1!M_2!}\mathcal{V}_{M_1+2,0} * \mathcal{V}_{M_2+2,0}\left(\Psi_0,\Phi^{M_1}|\mathcal{G}\widetilde{A},\Phi^{M_2}\right) .
\end{aligned} \tag{5.23}
$$

We can simplify some of the expressions by using

$$
\begin{aligned}
&\sum_{\substack{M_1,M_2\\ M_1+M_2=N}} \frac{N!}{M_1!M_2!} \sum_{\substack{M_3,M_4\\ M_3+M_4=M_1}} \frac{M_1!}{M_3!M_4!} f(M_1,M_2,M_3,M_4) \\
&= \sum_{\substack{M_2,M_3,M_4\\ M_2+M_3+M_4=N}} \frac{N!}{M_2!M_3!M_4!} f(M_3+M_4,M_2,M_3,M_4) ,
\end{aligned} \tag{5.24}
$$

etc. With such rearrangements we can easily see that the terms on the right hand side of (5.13) cancel pairwise. In particular:

1. The first term in I_1 cancels I_2.

2. The second term in I_1 is anti-symmetric under the exchange $M_2 \leftrightarrow M_4$ due to (2.33) and vanishes after we sum over M_2, M_4.

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3. The third term in I_1 cancels the first term in I_4.
4. The fourth term in I_1 cancels the second term in I_3 after using (2.33) and (2.34).
5. The fifth term in I_1 cancels the first term in I_6 after using (2.33).
6. The sixth term in I_1 cancels I_5 after using (2.33), (2.34) and (4.13).
7. The seventh term in I_1 cancels the first term in I_7 after using (2.33).
8. The eighth term in I_1 cancels the first term in I_8.
9. The first term in I_3 cancels the second term in I_4.
10. The third term in I_4 cancels the fifth term in I_7 after using (2.34).
11. The fourth term in I_4 cancels the third term in I_7.
12. The fifth term in I_4 cancels the second term in I_7.
13. The sixth term in I_4 cancels the second term in I_8 after using (2.33).
14. The second term in I_6 cancels the fourth term in I_7.

This shows that the right hand side of (5.12) has no boundary.

5.3 Proof of background independence

Once we have shown that the right hand side of (5.12) has no boundary, one can argue for the existence of $\mathcal{B}_{N+1,1}$ satisfying (5.12) as follows. Let us write (5.12) as

$$\partial\mathcal{B}_{N+1,1}\left(\Psi_0,\Phi^N;\widetilde{A}\right)=\bar{\mathcal{V}}_{N+1,1}\left(\Psi_0,\Phi^N;\widetilde{A}\right)-\bar{\mathcal{V}}'_{N+1,1}\left(\Psi_0,\Phi^N;\widetilde{A}\right)\,, \tag{5.25}$$

where

$$\begin{aligned}
\bar{\mathcal{V}}_{N+1,1}\left(\Psi_0,\Phi^N;\widetilde{A}\right) &= \mathcal{V}_{N+2,0}\left(\Psi_0,\mathcal{G}\widetilde{A},\Phi^N\right)\,,\\
\bar{\mathcal{V}}'_{N+1,1}\left(\Psi_0,\Phi^N;\widetilde{A}\right) &= -\sum_{\substack{M_1,M_2\\M_1+M_2=N}}\frac{N!}{M_1!M_2!}\mathcal{B}_{M_1+2,1}*\mathcal{V}_{M_2+1,0}\left(\Psi_0,\Phi^{M_1};\widetilde{A}|\Phi^{M_2}\right)\\
&\quad+\sum_{\substack{M_1,M_2\\M_1+M_2=N}}\frac{N!}{M_1!M_2!}\mathcal{V}_{M_1+2,0}\twoheadrightarrow\mathcal{B}_{M_2+1,1}\left(\left(\mathcal{G}\widetilde{A}\right),\Phi^{M_1}|\Psi_0,\Phi^{M_2}\right)\\
&\quad+\delta_{N,1}\mathcal{V}'_{0,2,1}\left(\Psi_0,\Phi;\widetilde{A}\right)+\mathcal{V}_{N+1,0}\twoheadrightarrow\mathcal{V}''_{0,1,2}\left(\Phi^N|\Psi_0,\widetilde{A}\right)\\
&\quad+\mathcal{V}'_{N+2,0}\left(\Psi_0,\left(\mathcal{G}\,\widetilde{A}\right),\Phi^N\right)\,.
\end{aligned} \tag{5.26}$$

Both $\bar{\mathcal{V}}_{N+1,1}$ and $\bar{\mathcal{V}}'_{N+1,1}$ are appropriate subspaces of $\mathcal{Q}_{N+1,1}$ modulo the choice of local coordinates at the special puncture where Ψ_0 is inserted. Vanishing of the right hand side of (5.13) implies that $\bar{\mathcal{V}}_{N+1,1}$ and $\bar{\mathcal{V}}'_{N+1,1}$ have common boundary. Furthermore since by

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assumption the $\mathcal{B}_{g,M,1}$'s appearing on the right hand side of (5.26) do not contain any separating type degenerations, $\bar{\mathcal{V}}'_{N+1,0}$ also does not contain any separating type degeneration. Now noting that the dimensions of $\bar{\mathcal{V}}_{g,N+1,1}$ and $\bar{\mathcal{V}}'_{g,N+1,1}$ are both given by $6g-2+2N$, their projections on to the base of $\mathcal{Q}_{g,N+1,1}$ will generically be $6g-2+2N$ dimensional — the same as the dimension of the base given by the moduli space of Riemann surfaces of genus g with $N+2$ punctures. Therefore the interiors of $\bar{\mathcal{V}}_{g,N+1,1}$ and $\bar{\mathcal{V}}'_{g,N+1,1}$ either both have the same projection into the moduli space, or they are complements of each other and have opposite orientation. It is easy to argue that the former is true — since neither $\bar{\mathcal{V}}_{g,N+1,1}$ nor $\bar{\mathcal{V}}'_{g,N+1,1}$ contain any Riemann surface with separating type degenerations, they cannot be complements of each other. This shows that we can regard $\bar{\mathcal{V}}_{N+1,1}$ and $\bar{\mathcal{V}}'_{N+1,1}$ as two different section segments with the same boundary. Therefore if we can show that $\bar{\mathcal{V}}_{N+1,1}$ and $\bar{\mathcal{V}}'_{N+1,1}$ are in the same 'homology class' of $\mathcal{Q}_{g,N+1}$, it will establish the existence of a $\mathcal{B}_{N+1,1}$ that has $\bar{\mathcal{V}}_{N+1,1}-\bar{\mathcal{V}}'_{N+1,1}$ as its boundary. If we forget the fiber coordinates associated with PCO locations, then it follows from an analysis described in [8] that $\bar{\mathcal{V}}_{N+1,1}$ and $\bar{\mathcal{V}}'_{N+1,1}$ are deformable to each other and are therefore in the same homology class. Therefore we need to focus on the part of the fiber parametrized by the choice of PCO locations. Strictly speaking this does not describe a regular manifold since we need to exclude codimension two subspaces to avoid spurious poles. Nevertheless the rules for integration along these fiber directions — called vertical integration [14, 15] — are such that they behave *as if* these fiber directions describe a homologically trivial space — any closed submanifold may be regarded as the boundary of another manifold. This may seem counterintuitive — for example given two possible locations of a PCO on a Riemann surface, one could have multiple curves that interpolate between these two points which differ by non-contractible cycles of the Riemann surface and are therefore not deformable to each other. However the integrals of Ω_p along all of these curves give the same result since the result of vertical integration only depends on the end-points and not on the details of the path. Therefore these paths are equivalent. This may give the impression that the integral of Ω_p along any closed vertical subspace vanishes, but this is not quite true. For example if we have a pair of PCO's and consider the one dimensional vertical cycle in which their locations are moved as

$$(z_1,z_2)\to(z_1',z_2)\to(z_1',z_2')\to(z_1,z_2')\to(z_1,z_2)\,, \tag{5.27}$$

then integral of Ω_1 along this cycle does not vanish. Nevertheless we can effectively reexpress this as an integral of $d\Omega_1$ along a two dimensional vertical cycle (or more precisely a similar integral where $d\Omega_1$ is replaced by Ω_2 with modified argument using (2.21)). The net result is that one can regard the fiber directions of $\widetilde{\mathcal{P}}_{g,m,n}$ labelled by the PCO locations to be homologically trivial [15]. This in turn establishes the existence of $\mathcal{B}_{N+1,1}$ satisfying (5.25).

This analysis also gives an iterative proof that the subspaces $\mathcal{B}_{g,N+1,1}$ do not include separating type degenerations. Assuming that the $\mathcal{B}_{g,M+1,1}$'s appearing on the right hand side of (5.26) do not involve separating type degenerations, and using the knowledge that none of the subspaces $\mathcal{V}_{M,0}$, $\mathcal{V}'_{M,0}$, $\mathcal{V}'_{0,2,1}$ and $\mathcal{V}''_{0,1,2}$ contain separating type degenerations, we see that $\bar{\mathcal{V}}_{N+1,1}$ and $\bar{\mathcal{V}}'_{N+1,1}$ defined in (5.26) do not contain separating type degenera-

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tions. Therefore $\mathcal{B}_{N+1,1}$ computed from (5.25) can also be chosen to avoid separating type degenerations.

Finally, we note that the choice of $\mathcal{B}_{N+1,1}$ satisfying (5.25) is not unique. Since $\mathcal{Q}_{g,N+1,1}$ is an infinite dimensional space while $\mathcal{B}_{g,N+1,1}$ is finite dimensional, there are in general infinite families of subspaces of $\mathcal{Q}_{N+1,1}$ with the same boundary given by the right hand side of (5.25). Therefore the field redefinition relating the two string field theories is not unique. This is simply a reflection of the infinite parameter gauge invariance of the theory as mentioned in the introduction.

6 Generalizations to other theories

The extension of the analysis given above to type II superstring field theory is straightforward. The 1PI equations of motion of the interacting fields still take the form given in (2.20), with the only difference that now the string field contains four different sectors: NSNS, NSR, RNS and RR. The operator $\mathcal{G}$ in these four sectors take the form 1, $\mathcal{X}_0$, $\bar{\mathcal{X}}_0$ and $\mathcal{X}_0\bar{\mathcal{X}}_0$ respectively, where now $\bar{\mathcal{X}}_0$ represents the zero mode of the PCO in the left-moving sector of the world-sheet theory. The analog of (3.1) now takes the form

$$
\begin{aligned}
&T_F(z)\,\mathcal{O}(w,\bar{w}) \simeq -(z-w)^{-1}\,\overline{\mathcal{O}}(w,\bar{w}) + \text{less singular terms}\\
&\bar{T}_F(\bar{z})\,\mathcal{O}(w,\bar{w}) \simeq (\bar{z}-\bar{w})^{-1}\,\widehat{\mathcal{O}}(w,\bar{w}) + \text{less singular terms}\\
&T_F(z)\,\widehat{\mathcal{O}}(w,\bar{w}) \simeq (z-w)^{-1}\,\widetilde{\mathcal{O}}(w,\bar{w}) + \text{less singular terms}\\
&\bar{T}_F(\bar{z})\,\overline{\mathcal{O}}(w,\bar{w}) \simeq (\bar{z}-\bar{w})^{-1}\,\widetilde{\mathcal{O}}(w,\bar{w}) + \text{less singular terms}\\
&T_F(z)\,\overline{\mathcal{O}}(w,\bar{w}) \simeq -\frac{1}{4}(z-w)^{-2}\mathcal{O}(w,\bar{w}) - \frac{1}{4}(z-w)^{-1}\partial\mathcal{O}(w,\bar{w}) \ + \text{less singular terms}\\
&\bar{T}_F(z)\,\widehat{\mathcal{O}}(w,\bar{w}) \simeq \frac{1}{4}(\bar{z}-\bar{w})^{-2}\mathcal{O}(w,\bar{w}) + \frac{1}{4}(\bar{z}-\bar{w})^{-1}\bar{\partial}\mathcal{O}(w,\bar{w}) \ + \text{less singular terms}\\
&T_F(z)\,\widetilde{\mathcal{O}}(w,\bar{w}) \simeq \frac{1}{4}(z-w)^{-2}\widehat{\mathcal{O}}(w,\bar{w}) + \frac{1}{4}(z-w)^{-1}\partial\widehat{\mathcal{O}}(w,\bar{w}) \ + \text{less singular terms}\\
&\bar{T}_F(z)\,\widetilde{\mathcal{O}}(w,\bar{w}) \simeq \frac{1}{4}(\bar{z}-\bar{w})^{-2}\overline{\mathcal{O}}(w,\bar{w}) + \frac{1}{4}(\bar{z}-\bar{w})^{-1}\bar{\partial}\overline{\mathcal{O}}(w,\bar{w}) \ + \text{less singular terms}\,,
\end{aligned}
\tag{6.1}
$$

where $\mathcal{O}$ is a dimension $(1/2,1/2)$ operator. The perturbation (3.2) retains the same form

$$
-\frac{\lambda}{2\pi i}\int dz\wedge d\bar{z}\,\widetilde{\mathcal{O}}(z,\bar{z})\,. \tag{6.2}
$$

eqs. (3.6) and (3.7) are replaced by

$$
\delta Q_B = \lambda\oint d\bar{z}\,c(z)\,\widetilde{\mathcal{O}}(z,\bar{z}) + \lambda\oint dz\,\bar{c}(\bar{z})\,\widetilde{\mathcal{O}}(z,\bar{z}) + \frac{1}{4}\lambda\oint d\bar{z}\,\gamma(z)\,\widehat{\mathcal{O}}(z,\bar{z}) + \frac{1}{4}\lambda\oint dz\,\bar{\gamma}(\bar{z})\,\overline{\mathcal{O}}(z,\bar{z})\,, \tag{6.3}
$$

and

$$
\begin{aligned}
\delta\mathcal{X}_0 &= \frac{1}{4}\lambda\oint d\bar{z}\,z^{-1}\,e^{\phi}(z)\,\widehat{\mathcal{O}}(z,\bar{z})\,,\\
\delta\bar{\mathcal{X}}_0 &= \frac{1}{4}\lambda\oint dz\,\bar{z}^{-1}\,e^{\bar{\phi}}(\bar{z})\,\overline{\mathcal{O}}(z,\bar{z})\ .
\end{aligned}
\tag{6.4}
$$

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Besides the identities (3.8), we now also have

$$\big[Q_B, \delta\bar{\mathcal{X}}_0\big] + \big[\delta Q_B, \bar{\mathcal{X}}_0\big] = \mathcal{O}\big(\lambda^2\big)\,, \quad \big[\mathcal{X}_0, \delta\bar{\mathcal{X}}_0\big] = \mathcal{O}\big(\lambda^2\big)\,, \quad \big[\bar{\mathcal{X}}_0, \delta\mathcal{X}_0\big] = \mathcal{O}\big(\lambda^2\big)\,. \tag{6.5}$$

The proof of background independence of the equations of motion proceeds as in the case of heterotic string theory with the only difference that we need to pay a little more attention to the definition of $\mathcal{V}''_{0,1,2}$. It is defined to be zero in the NSNS sector as for the NS sector of the heterotic string theory, whereas its definition in the NSR and RNS sectors is similar to that for the R sector of the heterotic string theory. The definition of $\mathcal{V}''_{0,1,2} \subset \mathcal{Q}_{0,1,2}$ in the RR sector involves a vertical segment in which the PCO arrangement moves from $\mathcal{X}_0\bar{\mathcal{X}}_0$ around ∞ to $\mathcal{X}_0\bar{\mathcal{X}}_0$ around 0. We can do this either by first moving $\bar{\mathcal{X}}_0$ from ∞ to 0 and then $\mathcal{X}_0$ from ∞ to 0 or vice versa. Since $\mathcal{X}_0$ and $\delta\bar{\mathcal{X}}_0$ commute and $\bar{\mathcal{X}}_0$ and $\delta\mathcal{X}_0$ commute, the two ways of moving the PCO's from ∞ to 0 give identical results for $\mathcal{V}''_{0,1,2}$. The rest of the analysis is identical to that for the heterotic string theory.

One could also explore the possibility of extending the analysis to other versions of superstring field theory. There are various other versions of open and closed superstring field theories at tree level. These theories may be broadly divided into two classes — those based on A_∞ algebra (for open string) or L_∞ algebra (for closed string), and those that do not have manifest underlying A_∞ or L_∞ algebra. The former class of theories have the property that the sum of the Feynman diagrams of the theory produces in a straightforward manner the string amplitudes computed from the first quantized formalism. The tree level open and closed superstring field theories constructed in [22–27]. provide sample examples of theories in this class. In the second class of theories, which include for example the theories described in [28–31], the Feynman diagrams do not produce the expected string amplitudes in a straightforward manner, and so far this has been checked either by explicit computation in a case by case basis (see e.g. [32–34]), or by showing its equivalence to another version of the theory that uses A_∞/L_∞ algebra after suitable gauge fixing and field redefinition [35–37]. We expect that the method described in this paper can be used to prove the background independence of the first class of theories in a straightforward manner. However proving background independence of other versions of string field theory may require developing new techniques.

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