JHEP

Published for SISSA by Springer

Received: January 7, 2016
Accepted: January 26, 2016
Published: February 22, 2016

The kinematical $AdS_5 \times S^5$ Neumann coefficient

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Abstract: For the case of two particles a solution of the string field theory vertex axioms can be factorized into a standard form factor and a kinematical piece which includes the dependence on the size of the third string. In this paper we construct an exact solution of the kinematical axioms for $AdS_5 \times S^5$ which includes all order wrapping corrections w.r.t. the size of the third string. This solution is expressed in terms of elliptic Gamma functions and ordinary elliptic functions. The solution is valid at any coupling and we analyze its weak coupling, pp-wave and large L limit.

Keywords: AdS-CFT Correspondence, Integrable Field Theories, String Field Theory

ArXiv ePrint: 1512.01471

doi:10.1007/JHEP02(2016)138

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1 Introduction

Recently there has been significant progress in our understanding of string interactions for string theories in curved backgrounds which exhibit integrability. In our previous paper [1] we formulated a set of functional equations for the (light-cone) String Field Theory (SFT) three-string vertex for the case when the worldsheet theory is integrable. The axioms *per-se* apply to the case when two of the strings are large (more precisely they are decompactified) while the third string can be of an arbitrary finite size L. The axioms depend in a nontrivial way on the size L. The decompactification limit corresponds to cutting the string pants diagram (see figure 1) along one edge. Since the third string has a finite size, the decompactification limit includes arbitrary number of wrapping corrections w.r.t. L. This can be explicitly seen in the case of the pp-wave background geometry where we have at our disposal an exact explicit solution for any value of L. Unfortunately we do not have, for the moment, a solution in the most interesting case of the $AdS_5 \times S^5$ geometry. This paper is a step in that direction.

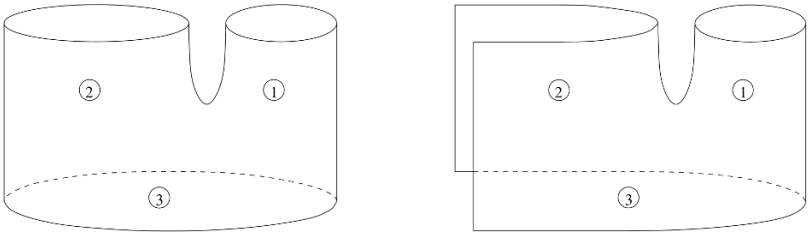

Figure 1. The SFT vertex and its decmpactified version.

In [2] a different approach was developed explicitly geared towards the computation of OPE coefficients in $\mathcal{N} = 4$ SYM. Here the string vertex was cut along *three* edges into two hexagons. This corresponds to the decompactification limit of all three strings. In this context, functional equations for the hexagon in $AdS_5 \times S^5$ have been solved exactly. The passage to finite volume incorporating wrapping effects involves, however, an iterative prescription for gluing the hexagons together through integrating over an arbitrary number of particles on the edges being glued. Thus wrapping effects are build on iteratively. Recently there appeared some further nontrivial checks of this proposal [3, 4] and it was even related [5] in the HHL ($L = 0$) case to diagonal finite volume form factors. This is the structure which was conjectured in [6] and checked at weak coupling in [7].

In contrast, the finite L solution of the SFT vertex axioms should at once resum an infinite set of wrapping corrections and thus should provide some helpful information for the hexagon gluing procedure.

In this paper we would like to find the simplest possible solutions of the SFT vertex axioms concentrating on exactly treating the L dependence. Of course any solution is given up to some analogs of CDD factors which *a-priori* can also be L dependent (although the equations that they satisfy do not contain L). So what we are aiming at is providing a 'minimal' L dependent solution. It will then remain an important problem whether this solution is physical or whether it has to be suplemented by some additional CDD-like factors. A similar question will arise for solutions for relativistic interacting integrable QFT's (e.g. sinh-Gordon or the O(N) model on the decompactified pants diagram), which we will briefly also mention. It would be very interesting to cross-check these simplest relativistic solutions in some other way and to understand whether in that case any additional CDD-like factors are in fact necessary. This would be important for our understanding of the required analytical structure. Perhaps some integrable lattice realizations of these integrable relativistic QFT's might shed light on these issues.

The plan of this paper is as follows. In section 2 we will briefly review the String Field Theory vertex axioms proposed in [1] and concentrate on the case of two particles relevant for the present paper. Then we will review the structure of the pp-wave Neumann coefficient in section 3 and consider the trivial relativistic solutions for sinh-Gordon and O(N) in section 4. In the following section we will review the $AdS_5 \times S^5$ elliptic curve and proceed to analyze and solve the relevant functional equations on the $AdS_5 \times S^5$ torus. Finally we will describe the pp-wave, weak coupling and large L limits of the obtained solutions. We close the paper with a discussion and outlook.

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2 String field theory vertex axioms

The universal exponential part of the light cone string field theory vertex both in flat spacetime and in the pp-wave geometry has the form

$$|V\rangle = \exp\left\{\frac{1}{2}\sum_{r,s=1}^{3}\sum_{n,m} N_{nm}^{rs}\, a_n^{+(r)} a_m^{+(s)}\right\}|0\rangle\ . \tag{2.1}$$

Here r and s labels the three strings in figure 1, $a_n^{+(r)}$ are the corresponding creation operators for excitations of mode number n on string $\#r$, while the numerical coefficients N_{nm}^{rs} are the so-called Neumann coefficients. Physically they represent matrix elements of the three string vertex with just two particles distributed among the three strings.

In the case of *interacting* worldsheet theory, we no longer expect the exponential form (2.1) to hold, and *a-priori* we will expect to have independent amplitudes for any number of particles:[1]

$$\mathbf{N}^{3|2;1}_{L_3|L_2;L_1}\left(\theta_1,\ldots,\theta_n \,\middle|\, \theta'_1,\ldots,\theta'_m\,;\,\theta''_1,\ldots,\theta''_l\right). \tag{2.2}$$

As argued in [1], we will consider the decompactified vertex with the strings #2 and #3 being infinite, and the string #1 being of size L (see figure 1).

$$\mathbf{N}^{3|2;1}_{\infty|\infty;L}\left(\theta_1,\ldots,\theta_n \,\middle|\, \theta'_1,\ldots,\theta'_m\,;\,\theta''_1,\ldots,\theta''_l\right). \tag{2.3}$$

In this case the functional equations will only depend explicitly on the particles in strings #2 and #3, so we can use a shorthand notation

$$N^{3|2}_{\bullet,L}\left(\theta_1,\ldots,\theta_n \,\middle|\, \theta'_1,\ldots,\theta'_m\right) \tag{2.4}$$

where the $\bullet$ stands for a specific state on string #1: $\bullet \equiv \{\theta''_1,\ldots,\theta''_l\}$.

In this paper we will restrict ourselves to amplitudes with just two particles. In analogy to the Minkowski and pp-wave case we will use the term Neumann coefficients for them. Without loss of generality we can take the two particles to be in the incoming string #3. In the notation of [1], we have

$$N_{\bullet,L}(\theta_1,\theta_2)_{i_1,i_2} = N^{3|2}_{\bullet,L}(\theta_1,\theta_2|\varnothing)_{i_1,i_2}\ . \tag{2.5}$$

Also on string #1 we will put the vacuum state[2] $\bullet = \varnothing$.

[1] Of course, there are some relations between the amplitudes with various numbers of particles, but we do not expect them to be as simple as following from an exponential form of the vertex.

[2] The equations for a generic state on string #1 are identical but we expect a much more complicated analytical structure with nontrivial additional CDD factors. We leave the investigation of these interesting and important issues for future work.

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Two particle SFT vertex axioms. The axioms from [1] in the case of two particles reduce to

$$N_{\bullet,L}(\theta_1,\theta_2)_{i_1,i_2} = S^{kl}_{i_1 i_2}(\theta_1,\theta_2) N_{\bullet,L}(\theta_2,\theta_1)_{l,k} \tag{2.6}$$

$$N_{\bullet,L}(\theta_1,\theta_2)_{i_1,i_2} = e^{-ip(\theta_1)L} N_{\bullet,L}(\theta_2,\theta_1-2i\pi)_{i_2,i_1} \tag{2.7}$$

$$-i\mathrm{Res}_{\theta'=\theta} N_{\bullet,L}(\theta'+i\pi,\theta)_{\bar{i},i} = \left(1-e^{ip(\theta)L}\right) N_{\bullet,L}\,. \tag{2.8}$$

From now on, we will normalize our formulas by setting $N_{\bullet,L}=1$.

Solving these axioms with nontrivial nondiagonal S-matrix does not seem *a-priori* simple, however in the special case of two particles we can look for a solution of the form[3]

$$N_{\bullet,L}(\theta_1,\theta_2)_{i_1,i_2} \equiv N(\theta_1,\theta_2)\cdot F(\theta_1,\theta_2)_{i_1,i_2} \tag{2.9}$$

where $F(\theta_1,\theta_2)_{i_1,i_2}$ satisfies the standard L-independent form-factor axioms[4]

$$F(\theta_1,\theta_2)_{i_1,i_2} = S^{kl}_{i_1 i_2}(\theta_1,\theta_2) F(\theta_2,\theta_1)_{l,k} \tag{2.10}$$

$$F(\theta_1,\theta_2)_{i_1,i_2} = F(\theta_2,\theta_1-2i\pi)_{i_2,i_1} \tag{2.11}$$

supplemented with the condition

$$F(\theta+i\pi,\theta)_{k,i} = \delta_{k\bar{i}}\,. \tag{2.12}$$

Then it is easy to show that the two particle SFT axioms (2.6)–(2.8) will be satisfied provided that the scalar $N(\theta_1,\theta_2)$ satisfies the SFT vertex axioms for a noninteracting theory i.e. with $S=1$:

$$N(\theta_1,\theta_2) = N(\theta_2,\theta_1) \tag{2.13}$$

$$N(\theta_1,\theta_2) = e^{-ip(\theta_1)L} N(\theta_2,\theta_1-2\pi i) \tag{2.14}$$

$$-i\mathrm{Res}_{\theta'=\theta} N(\theta'+i\pi,\theta) = \left(1-e^{ip(\theta)L}\right). \tag{2.15}$$

For a relativistic theory, these are exactly the axioms satisfied by the (decompactified) pp-wave Neumann coefficients which are explicitly known. Hence in the relativistic case the problem of finding a solution of the vertex axioms with two particles only reduces to finding ordinary form factors satisfying the additional condition (2.12).

The remaining freedom is a multiplication by a SFT analog of a CDD factor $f(\theta_1,\theta_2)$ which satisfies the simple equations

$$f(\theta_1,\theta_2)=f(\theta_2,\theta_1) \qquad f(\theta_1,\theta_2)=f(\theta_2,\theta_1-2\pi i) \qquad f(\theta'+i\pi,\theta)=1\,. \tag{2.16}$$

[3] We assume this form for all particle types i including possible boundstates. However for solving for $N(\theta_1,\theta_2)$ one uses the explicit form of the dispersion relation. In this paper we only consider the case of the same dispersion relation for the two particles and, in the case of $AdS_5\times S^5$, the dispersion relation of the fundamental magnons.

[4] In the case when the scattering matrix has poles corresponding to boundstates the two particle form factor F_{i_1,i_2} satisfies the dynamical singularity axiom. This axiom expresses the one particle boundstate form factor in terms of the two particle fundamental form factor and does not restrict the two particle form factor itself. These boundstates have no effect on N.

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The goal of this paper is to solve the counterpart of (2.13)–(2.15) in the case of $AdS_5 \times S^5$ kinematics, where the rapidities live on the appropriate covering space of the torus [8], and the counterpart of the shift by $i\pi$ is a shift by a half-period of the corresponding elliptic curve. For obvious reasons we will call the resulting functions *kinematical* Neumann coefficients.

Before we proceed, we will discuss the generality of the decomposition (2.9). Suppose that we have completed our goal and found the kinematical Neumann coefficient $N(\theta_1, \theta_2)$ satisfying (2.13)–(2.15). Now let us consider a completely general solution $N_{\bullet,L}(\theta_1, \theta_2)_{i_1,i_2}$ of the two particle SFT axioms (2.6)–(2.8). We can form the ratio

$$F(\theta_1, \theta_2)_{i_1,i_2} \equiv \frac{N_{\bullet,L}(\theta_1, \theta_2)_{i_1,i_2}}{N(\theta_1, \theta_2)} \tag{2.17}$$

where the denominator is the (assumed to be known) kinematical Neumann coefficient. Then it is easy to see that the above ratio $F(\theta_1, \theta_2)_{i_1,i_2}$ solves the classical form-factor axioms which do not have any explicit dependence on L. Thus the decomposition (2.9) is completely general.

However one has to keep in mind the possibility that the concrete form-factor solution of equations (2.10)–(2.11) may also depend on L. This potential L dependence is not, however, forced upon us by the equations. Thus our solution will provide a minimal L dependent solution of the SFT axioms.

3 The pp-wave Neumann coefficient

Before addressing the case of the $AdS_5 \times S^5$ kinematics, let us describe in some detail the (decompactified) pp-wave Neumann coefficients. Their general structure will also form a guiding principle for seeking a generalization to the full $AdS_5 \times S^5$ kinematics, as of course the pp-wave relativistic limit can be understood as a very specific corner in the full $AdS_5 \times S^5$ moduli space at strong coupling.

We are interested here in the $N^{33}(\theta_1, \theta_2)$ Neumann coefficient which we will denote from now on as $N_{\text{pp-wave}}(\theta_1, \theta_2)$. It's decompactified limit can be written in the following form[5] [1]

$$N_{\text{pp-wave}}(\theta_1, \theta_2) \equiv N^{33}(\theta_1, \theta_2) = \frac{2\pi^2}{L} \cdot \underbrace{\frac{1 + \tanh\frac{\theta_1}{2} \tanh\frac{\theta_2}{2}}{M \cosh\theta_1 + M \cosh\theta_2}}_{P(\theta_1,\theta_2)} n(\theta_1) n(\theta_2) \tag{3.1}$$

which will be convenient for generalization to the $AdS_5 \times S^5$ case. Let us first analyze the $P(\theta_1, \theta_2)$ factor. It implements for us the kinematical singularity (2.15). The denominator has a very simple interpretation as a sum of the energies of the two particles. This will have an obvious generalization to the full $AdS_5 \times S^5$ context, however the drawback of such an expression is that there is an additional spurious singularity at $\theta_1 = -\theta_2 + i\pi$ in addition

[5] Here we extracted a numerical factor for later convenience.

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to the correct kinematical singularity at $\theta_1 = \theta_2 + i\pi$. The role of the tanh functions in the numerator is exactly to cancel this spurious singularity in a minimal way:

$$P(\theta_1,\theta_2) = \frac{1+\tanh\frac{\theta_1}{2}\tanh\frac{\theta_2}{2}}{M\cosh\theta_1 + M\cosh\theta_2} = \frac{\frac{1}{\cosh\frac{\theta_1}{2}}\frac{1}{\cosh\frac{\theta_2}{2}}\cdot\cosh\frac{\theta_1+\theta_2}{2}}{2M\cosh\frac{\theta_1-\theta_2}{2}\cosh\frac{\theta_1+\theta_2}{2}}\,. \tag{3.2}$$

Since the residue of P at the kinematical pole is

$$-i\mathrm{Res}_{\theta'=\theta}P(\theta'+i\pi,\theta) = \frac{2i}{M\sinh\theta} \tag{3.3}$$

and $P(\theta_1,\theta_2)$ is symmetric and $2\pi i$-periodic, the remaining axioms (2.13)–(2.15) become

$$n(\theta+2\pi i) = e^{-ip(\theta)L}\,n(\theta) \tag{3.4}$$

$$n(\theta)\,n(\theta+i\pi) = \frac{1}{2i}\frac{L}{2\pi^2}M\sinh\theta\left(1-e^{ip(\theta)L}\right). \tag{3.5}$$

The monodromy relation (3.4) in fact follows from (3.5), but it is convenient to first extract a simple solution of (3.4) and then deal with a $2\pi i$-periodic function satisfying a modified version of (3.5). Namely we introduce

$$n(\theta) = e^{-\frac{\theta}{2\pi}p(\theta)L}\tilde{n}(\theta)\,. \tag{3.6}$$

Then $\tilde{n}(\theta)$ is $2\pi i$-periodic and satisfies

$$\tilde{n}(\theta)\,\tilde{n}(\theta+i\pi) = -\frac{L}{2\pi^2}M\sinh\theta\,\sin\frac{p(\theta)L}{2}\,. \tag{3.7}$$

There are many solutions to this equation, but once we require that the zeros lie on the line $\Re e(\theta)=0$, the solution is given by

$$\tilde{n}(\theta) = \frac{1}{\tilde{\Gamma}_{\frac{ML}{2\pi}}(\theta+i\pi)} \equiv -\frac{L}{2\pi^2}M\sinh\theta\,\sin\frac{p(\theta)L}{2}\cdot\tilde{\Gamma}_{\frac{ML}{2\pi}}(\theta) \tag{3.8}$$

where $\tilde{\Gamma}_{\frac{ML}{2\pi}}(\theta)$ is a new special functions introduced in [9] and slightly redefined in [1]. Let us write directly a product representation for $\tilde{n}(\theta)$ denoting $\mu = ML/(2\pi)$

$$\tilde{n}(\theta) = e^{-\mu\cosh\theta\left(\gamma+\log\frac{\mu}{2e}\right)}\cdot\mu\sinh\theta\cdot\prod_{n=1}^{\infty}\frac{\sqrt{n^2+\mu^2}-\mu\cosh\theta}{n}e^{\frac{\mu\cosh\theta}{n}}\,. \tag{3.9}$$

The product factors in the numerator ensure that all the nontrivial zeroes required by the r.h.s. of (3.7) lie on the real line and that there are no zeroes on the line $\Re e(\theta)=\pi$. The prefactor, which does not have any pole or zero can be understood from the large L asymptotics. Since for large μ

$$\mu\sinh\theta\cdot\prod_{n=1}^{\infty}\frac{\sqrt{n^2+\mu^2}-\mu\cosh\theta}{n}e^{\frac{\mu\cosh\theta}{n}} = -2e^{\mu\cosh\theta\left(\gamma+\log\frac{\mu}{2e}\right)}e^{\frac{\theta}{2\pi}pL}\sqrt{\frac{\mu}{\pi}}\sin\frac{pL}{2}\cosh\frac{\theta}{2}+O(e^{-\mu}) \tag{3.10}$$

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the prefactor simply kills the exponentially large growth of $n(\theta)$. Observe also that in this limit the monodromy of $n(\theta)$ is cancelled due to the appearance of the $e^{\frac{\theta}{2\pi}pL}$ factor. Note, however, that this asymptotics is only valid in the open interval $\Im m(\theta) \in (0, 2\pi)$ so for any finite L, $\tilde{n}(\theta)$ remains a periodic function. We will return to this point later in section 9.

In the next section we will write the solutions for sinh-Gordon and O(N) model and continue in the following section to introduce the covering space of the $AdS_5 \times S^5$ torus, describe some general features of function theory on the elliptic curve and then we will proceed to generalize the structures and formulas encountered in the present section to the fully general $AdS_5 \times S^5$ case.

4 Interacting relativistic integrable QFT's

Before we quote the relevant formulas let us first comment on the meaning of the solutions of the SFT vertex axioms in the case of such relativistic integrable field theories like sinh-Gordon or O(N) model which clearly do not form a consistent string theory. Indeed it is important to note that the SFT vertex axioms from [1] do not require that. They just describe the behaviour of an integrable quantum field theory on a two-dimensional spacetime which has the geometry of the decompactified pants diagram as in figure 1 (right). Clearly we may put *any* quantum field theory on such a geometry and investigate its properties. This is similar to the question of the spectrum of a QFT on a cylinder which can be formulated for any QFT without any requirement of a string theory interpretation.

Let us note in passing that the question of determining what are the properties of an integrable QFT which ensure that it can arise as a consistent string theory in some gauge-fixing is currently completely unexplored.

From the discussion in section 2 it is clear that the minimal two-particle solutions of the SFT vertex axioms of any relativistic integrable QFT will have its volume dependence given by the pp-wave Neumann coefficient $N_{\text{pp-wave}}(\theta_1, \theta_2)$ given by equations (3.1), (3.6) and (3.8). The remaining ingredient is an appropriately normalized minimal form factor solution.

Thus for sinh-Gordon we have

$$N^{\text{shG}}_{\text{min},L}(\theta_1, \theta_2) = N_{\text{pp-wave}}(\theta_1, \theta_2) \cdot \frac{f^{\text{shG}}_{\text{min}}(\theta_1 - \theta_2)}{f^{\text{shG}}_{\text{min}}(i\pi)} \tag{4.1}$$

where $f^{\text{shG}}_{\text{min}}(\theta)$ is the standard sinh-Gordon minimal form factor [10]

$$f^{\text{shG}}_{\text{min}}(\theta) = \exp\left\{4\int \frac{dt}{t}\frac{\sinh(tp)\sinh(t(1-p))}{\cosh(t)\,\sinh(2t)}\sin^2\left(\frac{t}{\pi}(i\pi-\theta)\right)\right\} \tag{4.2}$$

where p is related to the sinh-Gordon coupling constant.

For the O(N) model we have to be slightly more careful and choose the minimal form factor in the singlet channel. Thus we get

$$N^{O(N)}_{\text{min},L}(\theta_1, \theta_2)_{i_1 i_2} = N_{\text{pp-wave}}(\theta_1, \theta_2) \cdot \frac{f^{\text{singlet}}_{\text{min}}(\theta_1 - \theta_2)}{f^{\text{singlet}}_{\text{min}}(i\pi)}\delta_{i_1 i_2} \tag{4.3}$$

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where [11]

$$f_{\rm min}^{\rm singlet}(\theta) = \frac{\sinh\theta}{i\pi-\theta}\exp\left\{2\int\frac{dt}{t\sinh(t)}\frac{1-e^{-t\nu}}{1+e^{-t}}\sin^2\left(\frac{t}{2\pi}(i\pi-\theta)\right)\right\} \tag{4.4}$$

and $\nu = \frac{2}{N-2}$.

We give these formulas here explicitly as it would be very interesting to cross-check them with some direct construction of these relativistic integrable QFT's e.g. through some integrable lattice discretization. This would be important as it would shed light on whether such a minimal solution is indeed the physical one or whether one should also include some more complicated CDD factors possibly with some additional L dependence.

5 The $AdS_5 \times S^5$ elliptic curve

In [8] it was argued that a natural parametrization of the kinematics of a single excitation of the $AdS_5 \times S^5$ string is given by the universal covering of an appropriate, coupling constant dependent elliptic curve (equivalently a torus).

Here we will review the relevant formulas as given in [12], modyfing their definition of g by a factor of 2 in order to agree with

$$g^2 = \frac{\lambda}{16\pi^2} \tag{5.1}$$

so that the dispersion relation is given by

$$E = \sqrt{1+16g^2\sin^2\frac{p}{2}}\,. \tag{5.2}$$

The key quantities are $x^\pm$ satisfying

$$x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{i}{g} \qquad \frac{x^+}{x^-} = e^{ip}\,. \tag{5.3}$$

The modulus of the elliptic curve is $k = -16g^2$, and we have

$$2\omega_1 = 4K(k) \qquad 2\omega_2 = 4iK(1-k) - 4K(k) \tag{5.4}$$

where ω_1 is related to the periodicity of momentum, while ω_2 is the crossing half-period. Let us also denote by w the relevant complex variable on the universal covering space of the torus. Then we have[6]

$$E = \mathrm{dn}(w,k) \qquad \sin\frac{p}{2} = \mathrm{sn}(w,k) \qquad p = 2\,\mathrm{am}\,w \tag{5.5}$$

and

$$x^\pm = \frac{1}{4g}\left(\frac{\mathrm{cn}\,w}{\mathrm{sn}\,w} \pm i\right)\cdot(1+\mathrm{dn}\,w)\,. \tag{5.6}$$

[6] k is given in the conventions of *Mathematica*. From now on we will often suppress giving k explicitly.

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Note that the worldsheet momentum p is not globally well defined on the complex plane. This will lead to significant complications in solving the SFT vertex axioms which we will discuss in the next section.

The definitions given above are very concise, however they partly obscure the natural periodicity as $p \to p + 2\pi$ when $w \to w + \omega_1$. Hence we expect that the physics should be described by a torus with periods ω_1 and $2\omega_2$.

To make this explicit, and also to use θ functions we will often work with the rescaled complex variable

$$z = \frac{w}{\omega_1} \tag{5.7}$$

and the elliptic curve will have the modular parameter

$$\tau = \frac{2\omega_2}{\omega_1}\,. \tag{5.8}$$

For compatibility with the mathematical definitions that we will be using later, we define

$$q = e^{i\pi\tau}\,. \tag{5.9}$$

Let us now review the weak coupling and pp-wave limits of the above parametrization.

The weak coupling limit. In the weak coupling limit, the period $\omega_1 \to \pi$, while $\omega_2 \to i\infty$. The z coordinate becomes simply related to the worldsheet momentum

$$p(z) \sim 2\pi z \tag{5.10}$$

while the energy becomes

$$E(z) \sim 1 + 8g^2 \sin^2 \pi z\,. \tag{5.11}$$

The pp-wave limit. At strong coupling the periods ω_1, ω_2 have the following expansion

$$\omega_1 \sim \frac{\log g + 4\log 2}{2g} \qquad \omega_2 \sim \frac{i\pi}{4g}\,. \tag{5.12}$$

The second formula strongly suggests identifying the relativistic rapidity θ in the pp-wave limit with

$$\theta = 4gw = 4g\omega_1 z\,. \tag{5.13}$$

Then the crossing transformation is $\theta \to \theta + i\pi$. One subtlety that one has to keep in mind is that the pp-wave definition of the momentum $\tilde{p}$ differs from the standard one by an appropriate rescaling

$$\tilde{p} \equiv 2gp\,. \tag{5.14}$$

Then indeed $E \to \sqrt{1+\tilde{p}^2}$ and $\tilde{p} = \sinh\theta$. Let us note that due to the behaviour of ω_1, after the rescaling (5.13) the edge of the torus related to momentum periodicity gets pushed to infinity.

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6 Functional equations on the AdS torus

The functional equations for the kinematical Neumann coefficients for $AdS_5 \times S^5$ are given by

$$N(z_1, z_2) = N(z_2, z_1) \tag{6.1}$$

$$N(z_1, z_2) = e^{-ip(z_1)L} N(z_2, z_1 - \tau) \tag{6.2}$$

$$-i\mathrm{Res}_{z'=z}\, N\Big(z' + \frac{\tau}{2}, z\Big) = \Big(1 - e^{ip(z)L}\Big)\,. \tag{6.3}$$

We will supplement these equations with the requirement that the zeroes of $N(z_1, z_2)$ lie on the physical line ($\Im m(z) = 0$).

Despite their structural similarity with the relativistic equations, the highly rigid function theory on a torus leads to various stringent restrictions and puzzles. In particular the worldsheet momentum p is not globally well defined on the complex plane. This has two consequences. Firstly, the exponential factors e^{ipL} are much more heavily constrained than in the relativistic case. Indeed they are well defined meromorphic functions only for integer L (for half integer L they are also meromorphic but on a larger torus with periodicity $z \to z + 2$). This property is indeed very natural from the gauge theory point of view as the size of the string is always integer (or half-integer) as it is identified with the discrete J charge.

This new feature of the AdS kinematics will also severly complicate solving the SFT vertex monodromy axiom. Indeed a function of the form

$$e^{\mathrm{const}\cdot z\cdot p(z)} \tag{6.4}$$

similar to the function $e^{-\frac{\theta}{2\pi}p(\theta)L}$ which was used in the relativistic case in (3.6) does not make sense on the elliptic curve (or on its covering space) as it has branch cuts and is not meromorphic.

Let us now turn to finding a solution of (6.1)–(6.3). Instead of directly attacking the functional relations (6.1)–(6.3), we will try to follow the steps employed when solving the functional relations in the pp-wave case, and decompose $N(z_1, z_2)$ into some simpler structures. Recall (3.1):

$$N^{33}(\theta_1, \theta_2) = \frac{2\pi^2}{L} \cdot \underbrace{\frac{1 + \tanh\frac{\theta_1}{2}\tanh\frac{\theta_2}{2}}{M\cosh\theta_1 + M\cosh\theta_2}}_{P(\theta_1,\theta_2)} n(\theta_1) n(\theta_2)\,. \tag{6.5}$$

We will look for a similar decomposition

$$N(z_1, z_2) = \frac{2\pi^2}{L} \cdot \underbrace{\frac{1 + f(z_1)f(z_2)}{E(z_1) + E(z_2)}}_{P(z_1,z_2)} n(z_1) n(z_2) \tag{6.6}$$

with the functions $f(z)$ and $n(z)$ to be determined.

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The function $f(z)$. The key role of the numerator in $P(z_1, z_2)$ is to cancel the unwanted pole at $z_1 = -z_2 + \tau/2$ in the denominator. Since we want the solution to reduce to the pp-wave solution in the appropriate limit, we will make a shortcut and try to find a natural elliptic generalization of $\tanh\theta/2$. The key properties of $\tanh\theta/2$ which are also necessary to cancel that spurious pole amount to

$$f(z+\tau/2) = \frac{1}{f(z)} \qquad f(-z) = -f(z)\,. \tag{6.7}$$

Elliptic functions can be constructed in diverse ways. For later convenience we will use the q-theta function $\theta_0(z)$ defined through

$$\theta_0(z) = -ie^{i\pi\left(z-\frac{\tau}{4}\right)+i\pi\frac{\tau}{12}} \cdot \frac{\theta_1(\pi z, e^{i\pi\tau})}{\eta(\tau)} \tag{6.8}$$

as a basic building block. This function obeys the properties:

$$\theta_0(z+1) = \theta_0(z) \tag{6.9}$$
$$\theta_0(z+\tau) = -e^{-2\pi i z}\theta_0(z) \tag{6.10}$$
$$\theta_0(\tau - z) = \theta_0(z) \tag{6.11}$$
$$\theta_0(-z) = -e^{-2\pi i z}\theta_0(z)\,. \tag{6.12}$$

Its main property is that it has a single zero at $z = 0$. Thus it may be used to construct elliptic functions by specifying the positions of their zeroes and poles. Indeed any elliptic function can be written as

$$\text{const} \cdot \frac{\theta_0(z-a_1)\theta_0(z-a_2)\cdot\ldots\cdot\theta_0(z-a_n)}{\theta_0(z-b_1)\theta_0(z-b_2)\cdot\ldots\cdot\theta_0(z-b_n)} \tag{6.13}$$

with the constraint $\sum_{i=1}^n a_i = \sum_{i=1}^n b_i$ for double periodicity. It is well known that the elliptic functions have to have $n \geq 2$.

The function $f(z)$ thus has to have the following form

$$f(z) = C\frac{\theta_0(z)\,\theta_0(z-z_0)}{\theta_0\left(z-\frac{\tau}{2}\right)\theta_0\left(z-z_0+\frac{\tau}{2}\right)}\,. \tag{6.14}$$

In order for $f(z)f(z+\tau/2) = 1$ to hold, C can be calculated to be

$$C = \mp e^{i\pi\left(z_0-\frac{\tau}{2}\right)}\,. \tag{6.15}$$

In the following we will pick the upper sign. In order for this function to be odd, z_0 has to be a half period. We have two possibilities:

$$z_0 = \frac{1}{2} \qquad \text{or} \qquad z_0 = \frac{1+\tau}{2}\,. \tag{6.16}$$

Provisionally we will use the function with the first choice of z_0 as it has no pole on the physical line. Thus we set

$$f(z) = -iq^{-\frac{1}{2}} \cdot \frac{\theta_0(z)\,\theta_0\left(z-\frac{1}{2}\right)}{\theta_0\left(z-\frac{\tau}{2}\right)\theta_0\left(z-\frac{1}{2}+\frac{\tau}{2}\right)}\,. \tag{6.17}$$

Both choices of z_0 in (6.14), however, lead to functions which go over to $\tanh\frac{\theta}{2}$ in the pp-wave limit as can be seen in figure 2.

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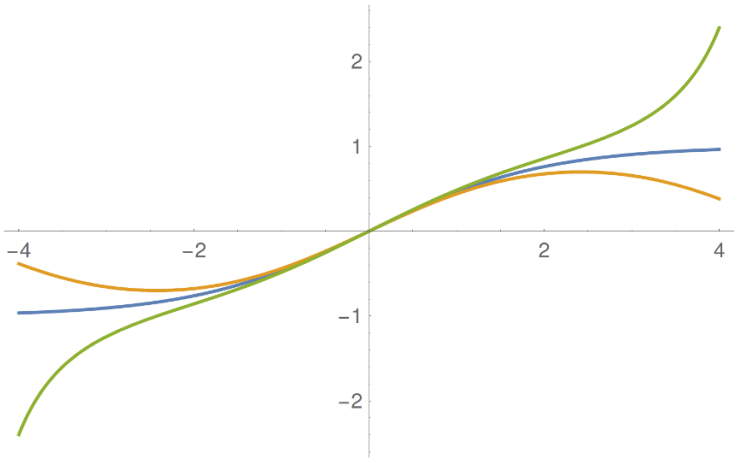

Figure 2. The functions $f(z)$ with the two choices of z_0 and $\tanh\frac{\theta}{2}$ for $\lambda = 10000$.

The function $n(z)$. Let us now consider the analog of (3.3) which for the reader's convenience we repeat here

$$-i\text{Res}_{\theta'=\theta}P(\theta'+i\pi,\theta) = \frac{2i}{M\sinh\theta}\,. \tag{6.18}$$

It would be tempting to identify the expression $M\sinh\theta$ in the residue with the momentum, but on the elliptic curve this would be problematic, as the momentum is not a well defined function. In fact it can be equivalently understood as $E'(\theta)$, which in contrast has a well defined elliptic generalization. Since in the previous subsection we have already explicitly defined $P(z_1,z_2)$, of course we do not have any freedom here but we just have to compute the appropriate residue. It is quite encouraging that $E'(z)$ indeed appears in the exact answer:

$$-i\text{Res}_{z'=z}P(z'+\tau/2,z) = \frac{2i}{E'(z)}\,. \tag{6.19}$$

We are now left with the following functional equations for $n(z)$:

$$n(z+\tau) = e^{-ip(z)L}n(z) \tag{6.20}$$

$$n(z)n(z+\tau/2) = \frac{LE'(z)}{4\pi^2 i}\left(1-e^{ip(z)L}\right). \tag{6.21}$$

Again the first equation is a direct consequence of the second one. For later convenience let us give an expression for $E'(z)$ in terms of the momenta:

$$E'(z) = -4g^2\omega_1 i\left(e^{ip}-e^{-ip}\right) = 8g^2\sin p\,. \tag{6.22}$$

The relevant crossing equation for $n(z)$ becomes then

$$n(z)n(z+\tau/2) = -\frac{4g^2L}{\pi^2}\sin p\sin\frac{pL}{2}e^{\frac{ipL}{2}}\,. \tag{6.23}$$

In this paper we will concentrate on the case of even $L=2n$ which is simpler than the general case. Let us first construct an *elliptic function* $G_L^{ev}(z)$ which has the correct location

of zeroes following from (6.21). Then we will concentrate on solving the monodromy equation in the simplest possible setting. Similarly as in the pp-wave limit, we will require all the zeroes (in the fundamental domain) to lie on the physical real axis.

It is natural to implement this condition by defining

$$G^{ev}_{L=2n}(z) = \sqrt{\frac{L}{2}} \prod_{k=1}^{n-1} \frac{\sqrt{1+16g^2\sin^2\frac{\pi k}{L}} - E(z)}{4g\sin\frac{\pi k}{L}}\,. \tag{6.24}$$

This function satisfies the following functional equation

$$G^{ev}_L(z) G^{ev}_L(z+\tau/2) = \frac{\sin\frac{pL}{2}}{\sin p}\,. \tag{6.25}$$

Let us now write $n(z)$ as

$$n(z) = \frac{2g\sqrt{L}}{\pi}\sin p\, G^{ev}_L(z) h^{ev}_L(z)\,. \tag{6.26}$$

Then the remaining function $h^{ev}_L(z)$ will satisfy a very simple equation

$$h^{ev}_L(z) h^{ev}_L(z+\tau/2) = e^{\frac{ipL}{2}} \tag{6.27}$$

leading to

$$h^{ev}_L(z+\tau) = e^{-ipL} h^{ev}_L(z)\,. \tag{6.28}$$

This function will be the direct elliptic counterpart of $e^{-\frac{\theta}{2\pi}p(\theta)L}$ in the relativistic case, however the analyticity properties in the 'elliptic' rapidity plane force the solution to be much more complicated.

6.1 Elliptic Gamma function and the monodromy condition

In order to solve the monodromy functional equations we will need to use the so-called elliptic Gamma function $\Gamma(z,\tau,\sigma)$. Its definition and main properties are discussed in [13]. It is the unique meromorphic solution of the difference equation

$$\Gamma(z+\sigma,\tau,\sigma) = \theta_0(z,\tau)\Gamma(z,\tau,\sigma) \tag{6.29}$$

such that i) $\Gamma(z+1,\tau,\sigma) = \Gamma(z,\tau,\sigma)$, ii) $\Gamma(z,\tau,\sigma)$ is holomorphic on the upper half plane, and it is normalized by iii) $\Gamma((\tau+\sigma)/2,\tau,\sigma) = 1$. It is given by an explicit product representation

$$\Gamma(z,\tau,\sigma) = \prod_{j,k=0}^{\infty} \frac{1-e^{2\pi i((j+1)\tau+(k+1)\sigma-z)}}{1-e^{2\pi i(j\tau+k\sigma+z)}}\,. \tag{6.30}$$

In the fundamental domain there are no zeroes and the only poles are on the real line at integer values of z. All other poles occur in the lower half plane. In the present paper we will need just the special case with $\sigma=\tau$, which we will denote by the shorthand notation

$$\Gamma_{\rm ell}(z) \equiv \Gamma(z,\tau,\tau) = \prod_{k=0}^{\infty}\left(\frac{1-e^{2\pi i\tau(k+2)}e^{-2\pi iz}}{1-e^{2\pi i\tau k}e^{2\pi iz}}\right)^{k+1}\,. \tag{6.31}$$

It satisfies

$$\Gamma_{\rm ell}(z+1) = \Gamma_{\rm ell}(z) \qquad \Gamma_{\rm ell}(z+\tau) = \theta_0(z)\Gamma_{\rm ell}(z)\,. \tag{6.32}$$

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The monodromy condition. The function $h_L^{ev}(z)$ satisfies the following monodromy condition

$$h_L^{ev}(z+\tau) = e^{-ipL} h_L^{ev}(z)\,. \tag{6.33}$$

Let us first investigate the more elementary equation

$$H(z+\tau) = e^{-ip} H(z)\,. \tag{6.34}$$

We can readily construct such a function using the elliptic Gamma functions $\Gamma_{\rm ell}(z)$ once we express e^{-ip} in terms of the elementary θ_0 functions:

$$e^{-ip} = q^{\frac{1}{2}} \cdot \frac{\theta_0\left(z-\frac{1}{2}+\frac{\tau}{4}\right)\theta_0\left(z-\frac{1}{2}-\frac{3\tau}{4}\right)}{\theta_0^2\left(z-\frac{1}{2}-\frac{\tau}{4}\right)}\,. \tag{6.35}$$

Thus the function $H(z)$ satisfying (6.34) can be given by

$$H(z) = e^{i\frac{\pi}{2}z} \cdot \frac{\Gamma_{\rm ell}\left(z-\frac{1}{2}+\frac{\tau}{4}\right)\Gamma_{\rm ell}\left(z-\frac{1}{2}-\frac{3\tau}{4}\right)}{\Gamma_{\rm ell}^2\left(z-\frac{1}{2}-\frac{\tau}{4}\right)}\,. \tag{6.36}$$

However due to the innocous looking leftover constant $q^{\frac{1}{2}}$ appearing in the expression for e^{-ip}, we are forced to include the exponential factor $e^{i\frac{\pi}{2}z}$ which violates the $z \to z+1$ periodicity. Indeed $H(z)$ satisfies

$$H(z+1) = iH(z)\,. \tag{6.37}$$

Nevertheless for the case of even L which we are considering in the present paper we may easily obtain a $z \to z+1$ periodic solution to (6.33). Let us take first $L=2$. Then the solution is

$$C \cdot e^{-i\frac{p(z)}{2}} H^2(z) e^{-2ip(z)}\,. \tag{6.38}$$

The term $e^{-ip/2}$ restores $z \to z+1$ periodicity, the other factor of e^{-2ip} ensures that the expression is real on the physical line $\Im m(z)=0$, while the constant $C = 1/(H(0)H(\tau/2))$ is enough to satisfy the remaining equation (6.27) for $L=2$. The generalization to any even $L=2n$ is now trivial:

$$h_{L=2n}^{ev}(z) = \frac{1}{H^n(0)H^n(\tau/2)} \cdot e^{-i\frac{p}{2}n} e^{-ipL} H(z)^L\,. \tag{6.39}$$

This solves all the required equations and is real on the real axis. In the next section we will put all these partial formulas together and explore some of the properties of the AdS kinematical Neumann coefficient.

7 The kinematical $AdS_5 \times S^5$ Neumann coefficient

Let us now collect together the relevant formulas. The resulting expression is an exact solution of the AdS axioms for the kinematical Neumann coefficient (6.1)–(6.3) for any even value of $L=2n$. The solution is of course valid for any value of the gauge theory coupling constant. We have

$$N(z_1,z_2) = \frac{2\pi^2}{L} \cdot \frac{1+f(z_1)f(z_2)}{E(z_1)+E(z_2)} n(z_1)n(z_2) \tag{7.1}$$

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where

$$f(z) = -iq^{-\frac{1}{2}} \cdot \frac{\theta_0(z)\,\theta_0\left(z-\frac{1}{2}\right)}{\theta_0\left(z-\frac{\tau}{2}\right)\theta_0\left(z-\frac{1}{2}+\frac{\tau}{2}\right)} \tag{7.2}$$

while $n(z)$ is composed of two pieces

$$n(z) = \frac{2g\sqrt{L}}{\pi}\sin p\; G_L^{ev}(z) h_L^{ev}(z) \tag{7.3}$$

with $G_L^{ev}(z)$ being an elliptic function ensuring the correct positions of zeroes as required by the kinematical singularity axiom

$$G_{L=2n}^{ev}(z) = \sqrt{\frac{L}{2}} \prod_{k=1}^{n-1} \frac{\sqrt{1+16g^2\sin^2\frac{\pi k}{L}} - E(z)}{4g\sin\frac{\pi k}{L}} \tag{7.4}$$

while $h_L^{ev}(z)$ implements the correct monodromy under the shift $z \to z+\tau$

$$h_{L=2n}^{ev}(z) = \frac{1}{H^n(0)H^n(\tau/2)} \cdot e^{-i\frac{p}{2}n} e^{-ipL} H(z)^L \tag{7.5}$$

with

$$H(z) = e^{i\frac{\pi}{2}z} \cdot \frac{\Gamma_{\text{ell}}\left(z-\frac{1}{2}+\frac{\tau}{4}\right)\Gamma_{\text{ell}}\left(z-\frac{1}{2}-\frac{3\tau}{4}\right)}{\Gamma_{\text{ell}}^2\left(z-\frac{1}{2}-\frac{\tau}{4}\right)}\,. \tag{7.6}$$

An extension of this construction to odd L (and possibly half-integer ones) would require a modification of (7.4). A further complication is that the functions $e^{ipL/2}$ would no longer be periodic under $z \to z+1$ but would pick up a minus sign. It is not completely clear whether a similar violation of periodicity would be physically acceptable or not. We leave therefore these issues for future work. In the following we will discuss the singularitites of the kinematical Neumann coefficient and its pp-wave, weak coupling and large L limits.

7.1 Singularity structure

Let us now analyze the singularity structure of the solution $N(z,z')$ as a function of z keeping z' fixed.

From the definition of the elliptic Gamma function we see that the potential zeroes and poles of $h_{L=2n}^{ev}(z)$ can occur only for the points

$$z_1 = \frac{1}{2} + \frac{\tau}{4} \qquad z_2 = \frac{1}{2} + \frac{3\tau}{4} \tag{7.7}$$

in the 'fundamental' domain (which we define here as the set $0 \le \Re e(z) < 1$, $0 \le \Im m(z) < \tau$). These points represent the poles and zeroes of e^{-ip} and thus represent infinite (complex) momentum thus having singularities there is quite natural.

The function $h_{L=2n}^{ev}(z)$ has a pole of order n at z_1 and a zero of order $3n$ at z_2. The fact that the number of poles and zeroes does not balance is not a contradiction as this function has nontrivial monodromy in the τ direction and thus is not elliptic. The poles of $G_L^{ev}(z)$ just follow from the poles of the energy $E(z)$ which has first order poles both at z_1 and z_2. Consequently $G_L^{ev}(z)$ has poles of order $n-1$ both at z_1 and at z_2. It also has

$2n-2$ zeroes on the real axis (within the 'fundamental' domain). Thus the product of the two functions has a pole of order $2n-1$ at z_1 and a zero of order $2n+1$ at z_2. Finally $\sin p$ has poles of order 2 both at z_1 and at z_2 and zeroes at $z = 0, 1/2, \tau/2, 1/2+\tau/2$. Of these zeroes the first two on the real axis are expected, while we will have to track the ones at $z = \tau/2, 1/2+\tau/2$.

Therefore $n(z)$ has a pole of order $2n+1$ at z_1, a zero of order $2n-1$ at z_2 and two single zeroes at $z = \tau/2, 1/2+\tau/2$ apart from the expected set of real zeroes.

It remains to analyze the singularities of the L independent piece

$$\frac{1+f(z)f(z')}{E(z)+E(z')}$$

as a function of z (keeping z' fixed). Generically we would expect this function to be an elliptic function of order 4, but since by construction $f(z)$ was choosen to cancel the unphysical pole at $z = -z'+\tau/2$ it is a function of order 3. This function has the kinematical pole at $z = z' + \tau/2$, the remaining two first order poles are at $z = \tau/2, 1/2+\tau/2$ which exactly cancel with the complex zeros of $n(z)$. This cancellation is a nice consistency check of this solution. All the zeroes are of first order. Two of them are at z_1 and z_2, while the last one is at $z = z' + 1/2 + \tau/2$.

Putting all these considerations together, we see that the solution $N(z, z')$ has a pole of order L at z_1, a zero of order L at z_2, a first order pole at $z = z' + \tau/2$ (the kinematical pole), a set of zeroes on the real axis and an additional zero at $z = z' + 1/2 + \tau/2$. It would be interesting to understand the meaning of this additional zero.

Let us just mention in passing that if we define

$$N_{\text{reg}}(z, z') \equiv N(z, z')\, e^{ip(z)L/2} e^{ip(z')L/2} \tag{7.8}$$

we can get rid of any zeroes and poles at z_1 and z_2 altogether.

7.2 The pp-wave limit

In order to study the pp-wave limit, we have to take $g \to \infty$ together with $L \to \infty$ keeping fixed

$$\tilde{p} = 2gp \qquad \tilde{L} = \frac{L}{2g}\,. \tag{7.9}$$

Then, as mentioned earlier, the dispersion relation becomes $E = \sqrt{1+\tilde{p}^2}$ and the relativistic rapidity is linked with the z coordinate on the torus through

$$\theta = 4g\omega_1 z\,. \tag{7.10}$$

Taking this limit analytically on the kinematical Neumann coefficient is rather involved and we did not carry it out in full but we performed a numerical check. However let us comment first on some partial analytical results which indicate that the various functions which we introduced like the elliptic function $G_L^{ev}(z)$ and the $h_{L=2n}^{ev}(z)$ containing the elliptic Gamma function are in fact strongly interrelated.

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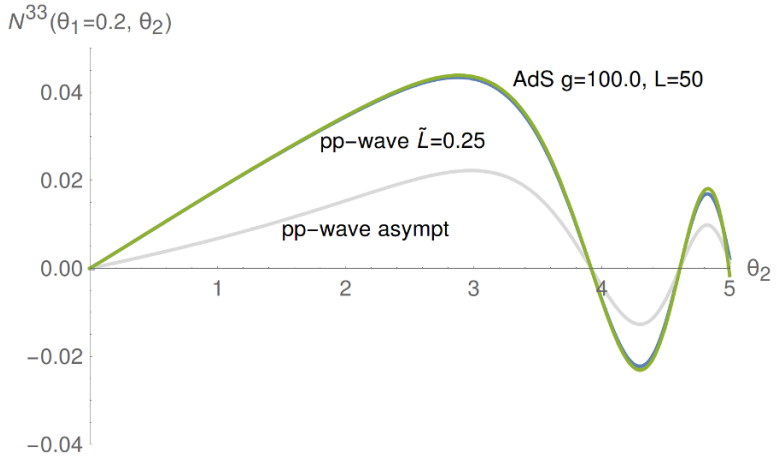

Figure 3. The pp-wave Neumann coefficient $N_{\text{pp-wave}}(\theta_1 = 0.2, \theta_2)$ for $\tilde{L} = 0.25$ together with its asymptotic part (neglecting wrapping) and the full $AdS_5 \times S^5$ kinematical Neumann coefficient for $g = 100$ and $L = 50$ (which corresponds to $\tilde{L} = 0.25$).

Using the properties of the elliptic Gamma function in [13] one can obtain the pp-wave limit of $h^{ev}_{L=2n}(z)$:

$$h^{ev}_{L}(z) \to e^{-\frac{1}{2\pi}\tilde{L}\,\theta\,\sinh\theta} \cdot e^{\frac{1+4\log 2+\log g}{2\pi}\,\tilde{L}\,\cosh\theta}\,. \tag{7.11}$$

The first term is exactly the relativistic monodromy function used in (3.6). The second term, however, involves already a part of the exponential factor in (3.9), but due to the $\log g$ in the exponent, this function does not really have a pp-wave limit. It turns out that only when multiplied by $G^{ev}_{L}(z)$, the $\log g$ term apparently cancels and we have a well defined pp-wave limit of the complete expression. We checked this numerically (see figure 3). There we compare the full $AdS_5 \times S^5$ answer with the pp-wave expression in the far from asymptotic regime where wrapping is important and the full pp-wave exact expression (3.9) is needed.

8 Weak coupling limit

In this section we analyze the weak coupling limit ($g \to 0$) of the kinematical Neumann coefficient and connect it to a decompactified spin chain calculation. In this limit the real period of the torus, ω_1 goes to π, while the imaginary one diverges as $\omega_2 \to i\infty$. This makes the domains, related for finite g by crossing, disconnected at weak coupling. Since the spin chain calculation gives nonvanishing result only for the kinematics when there is one incoming particle in string #3 and one outgoing particle in string #2 we have to continue analytically the kinematical Neumann coefficient to describe this process

$$N_{23}(z', z) \equiv e^{-ip(z')\frac{L}{2}}\, N\left(z' + \frac{\tau}{2}, z\right) \tag{8.1}$$

before taking the weak coupling limit [1]. Using the functional relations

$$n(z) = \frac{1}{n\left(z+\frac{\tau}{2}\right)} \frac{LE'(z)}{4\pi^2 i}(1 - e^{ip(z)L})\,; \qquad f\left(z+\frac{\tau}{2}\right) = \frac{1}{f(z)} \tag{8.2}$$

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we transform the required quantity into:

$$N\left(z'+\frac{\tau}{2},z\right)=\frac{E'(z)}{2i}\frac{1+\frac{f\left(z'+\frac{\tau}{2}\right)}{f\left(z+\frac{\tau}{2}\right)}}{E(z)-E(z')}\frac{n\left(z'+\frac{\tau}{2}\right)}{n\left(z+\frac{\tau}{2}\right)}(1-e^{ipL})\,. \tag{8.3}$$

In evaluating its weak coupling limit we note that the elliptic nome goes to zero, $q\to 0$, and the theta functions simplify to trigonometric functions. In particular we find that

$$\frac{f\left(z'+\frac{\tau}{2}\right)}{f\left(z+\frac{\tau}{2}\right)}=\frac{\sin 2\pi z}{\sin 2\pi z'}=\frac{\sin p}{\sin p'} \tag{8.4}$$

where in the second equality we used that at weak coupling $p=2\pi z$. Consequently in this limit we also have

$$E(z)=1+8g^2\sin^2\pi z+\ldots \qquad \text{and} \qquad E'(z)=8g^2\sin p+\ldots \tag{8.5}$$

which allows us to evaluate the weak coupling limit of the L-independent prefactor.

Let us now turn to analyze

$$\frac{n\left(z'+\frac{\tau}{2}\right)}{n\left(z+\frac{\tau}{2}\right)}=\frac{\sin p'}{\sin p}\frac{G_L^{ev}\left(z'+\frac{\tau}{2}\right)}{G_L^{ev}\left(z+\frac{\tau}{2}\right)}\frac{h_L^{ev}\left(z'+\frac{\tau}{2}\right)}{h_L^{ev}\left(z+\frac{\tau}{2}\right)}\,. \tag{8.6}$$

Firstly we see that

$$\frac{G_L^{ev}\left(z'+\frac{\tau}{2}\right)}{G_L^{ev}\left(z+\frac{\tau}{2}\right)}=\prod_{k=1}^{n-1}\frac{\sqrt{1+16g^2\sin^2\frac{\pi k}{L}}+E(z')}{\sqrt{1+16g^2\sin^2\frac{\pi k}{L}}+E(z)}=1+\ldots \tag{8.7}$$

The small q limit of the elliptic gamma function comes from the first factor in the product

$$\Gamma_{\rm ell}(z)=\prod_{k=0}^{\infty}\left(\frac{1-q^{2(k+2)}e^{-2i\pi z}}{1-q^{2k}e^{2i\pi z}}\right)^{k+1}=\frac{1}{(1-e^{2i\pi z})}+\ldots \tag{8.8}$$

whenever $\Im m(z)>-\tau$. This implies

$$H\left(z+\frac{\tau}{2}\right)=q^{\frac{1}{4}}e^{i\frac{\pi}{2}z}\frac{\Gamma_{\rm ell}\left(z-\frac{1}{2}+\frac{3\tau}{4}\right)\Gamma_{\rm ell}\left(z-\frac{1}{2}-\frac{\tau}{4}\right)}{\Gamma_{\rm ell}\left(z-\frac{1}{2}+\frac{\tau}{4}\right)^2}=q^{\frac{3}{4}}e^{-i\frac{3\pi}{2}z}+\ldots \tag{8.9}$$

and leads to

$$\frac{n\left(z'+\frac{\tau}{2}\right)}{n\left(z+\frac{\tau}{2}\right)}=\frac{\sin p'}{\sin p}\frac{e^{2i\pi p'n}}{e^{2i\pi pn}}=\frac{\sin p'}{\sin p}e^{i\frac{L}{2}(p'-p)}\,. \tag{8.10}$$

Putting everything together we obtain

$$N\left(z'+\frac{\tau}{2},z\right)=\frac{\pi}{i}\cot\frac{p'-p}{2}e^{i\frac{L}{2}(p'-p)}(1-e^{ipL})\,. \tag{8.11}$$

This implies for the weak coupling limit of the amplitude $N_{23}(z',z)$:

$$N_{23}(z',z)=\pi\frac{1+e^{i(p'-p)}}{1-e^{i(p'-p)}}\cdot(1-e^{ipL})\cdot e^{-i\frac{pL}{2}}\,. \tag{8.12}$$

In the following we compare this result to an infinite volume spin-chain calculation.

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8.1 Decompactifed spin chain calculation

In the weak coupling limit the Neumann coefficients are expected to be related to the tree level 3pt functions of the dual gauge theory. To calculate these 3pt functions one has to diagonalize the 1-loop dilatation operator and evaluate the overlap of its eigenstates in the decompactified geometry shown on figure 4. This is equivalent to a decompactified spin chain calculation. Figure 4 depicts the geometry in which the decompactified string #3 splits into the decompactified string #2 and the finite string #1. We assume that we have one particle for string #3 with momentum p and one particle for string #2 with momentum p', and the vacuum for string #1. In the language of gauge theory this setting translates into three operators, $\mathcal{O}_i$, for $i = 1, 2, 3$ as follows: for $\mathcal{O}_3$ we take an infinitely long operator built up from one single X and infinitely many Z scalar operators. The coordinate space eigenstate of the dilatation operator can be parameterized by its momentum p:

$$|\mathcal{O}_3\rangle = \sum_{n\in Z} e^{ipn}|n\rangle, \tag{8.13}$$

where $|n\rangle$ is of the form $\dots ZZZXZZZ\dots$ and the operator X is located at position n.[7] The operator we take for $\mathcal{O}_2$ is in the conjugate sector to $\mathcal{O}_3$, it contains infinitely many $\bar{Z}$ and one single $\bar{X}$. Finally for the third operator we take $\mathcal{O}_1 = \mathrm{Tr}(\bar{Z}^L)$, whose state is $\langle\mathcal{O}_1| = {}_L\langle 0|$. To implement the right geometry we split $\mathcal{O}_2$ as

$$\langle\mathcal{O}_2| = \sum_{n'\le -\frac{L}{2}} e^{-ip'\left(n'+\frac{L}{2}\right)}\langle n'| + \sum_{n'>\frac{L}{2}} e^{-ip'\left(n'-\frac{L}{2}\right)}\langle n'| \tag{8.14}$$

and insert $\mathcal{O}_1$ in the middle. This basis is very similar to the one, which was used to calculate the decompactified Neumann coefficients in [1]. In calculating the overlap $(\langle\mathcal{O}_2|\otimes\langle\mathcal{O}_1|)|\mathcal{O}_3\rangle$ we note that at tree level the nontrivial contractions are $\langle n'|n\rangle = \delta_{n,n'}$. This implies

$$\begin{aligned}(\langle\mathcal{O}_2|\otimes\langle\mathcal{O}_1|)|\mathcal{O}_3\rangle &= e^{-ip'\frac{L}{2}}\sum_{n\le -\frac{L}{2}} e^{i(p-p')n} + e^{ip'\frac{L}{2}}\sum_{n>\frac{L}{2}} e^{i(p-p')n} \\ &= \frac{1}{1-e^{i(p'-p)}}\cdot(1-e^{ipL})\cdot e^{-ip\frac{L}{2}}\,.\end{aligned} \tag{8.15}$$

The above equation is very similar to the one obtained from the weak coupling limit of the $AdS_5\times S^5$ kinematical Neumann coefficient (8.12), except the factor $1+e^{i(p'-p)}$ which, however, satisfies the AdS version of the CDD axioms[8] (2.16). The appearance of such an additional factor is very natural as we factored out the S-matrix dependent ordinary two particle form factor (2.9), which varies from sector to sector. Here, however we calculated only the one related to the $su(2)$ sector.

[7] This state is normalized to δ function in p.

[8] The CDD equations were written for two incoming particles so again we have to cross back. In particular this will change $1+e^{i(p'-p)}$ to $1+e^{-i(p'+p)}$.

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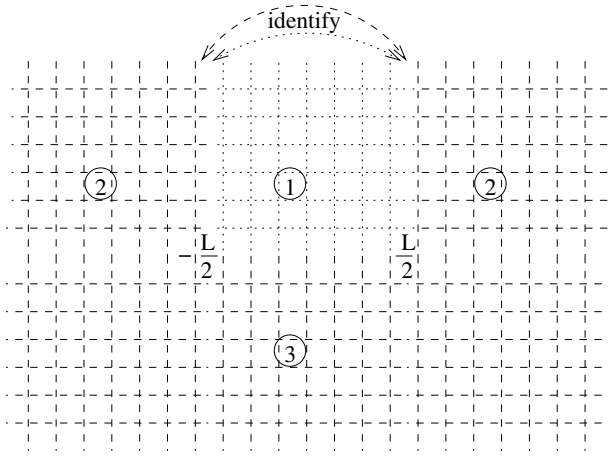

Figure 4. The geometry of the spin chain calculation. The world sheet of string #3 is replaced with an infinite spin chain, which splits into the infinite spin chain of string #2 and the periodic spin chain of size L replacing string #1.

9 The large L limit

It is also interesting to analyze the kinematical Neumann coefficient in the limit of large L keeping the remaining variables like the gauge coupling or the momenta at generic values. A distinctive feature of the exact pp-wave solution which was emphasized in [1] is that in the large L limit the Neumann coefficient $N_{33}(\theta, \theta')$ looses monodromy. This has to be understood in the following sense. The exact function $n(\theta)$ satisfies

$$n(\theta + 2\pi i) = e^{-ip(\theta)L}\, n(\theta)\,. \tag{9.1}$$

However if we first take the large L asymptotics (cf. (3.10), which is valid for $\Im m(\theta) \in (0, 2\pi)$), we get

$$n_{\rm as}(\theta + 2\pi i) = -n_{\rm as}(\theta)\,. \tag{9.2}$$

This can be reformulated as the statement that

$$\lim_{\varepsilon\to 0^+} \lim_{L\to\infty} \frac{n(\theta + 2\pi i - i\varepsilon)}{n(\theta + i\varepsilon)} = -1 \tag{9.3}$$

while if we take the limit $\varepsilon \to 0^+$ first, we get of course

$$\lim_{\varepsilon\to 0^+} \frac{n(\theta + 2\pi i - i\varepsilon)}{n(\theta + i\varepsilon)} = e^{-ip(\theta)L} \tag{9.4}$$

for real θ.

We will now establish that a similar property holds for the AdS kinematical Neumann coefficient. Recall the structure of the corresponding quantity

$$n(z) = \frac{2g\sqrt{L}}{\pi} \sin p\; G_L^{ev}(z) h_L^{ev}(z)\,. \tag{9.5}$$

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Since $h_L^{ev}(z)$ is essentially just the L^{th} power of a combination of the L-independent elliptic Gamma functions, its asymptotic limit is trivial and its asymptotic monodromy coincides with the normal one i.e.

$$\frac{h_L^{ev}(z+\tau-\varepsilon\tau)}{h_L^{ev}(z+\varepsilon\tau)} \to e^{-ipL} \tag{9.6}$$

for real z. We thus have to show that the corresponding ratio of $G_L^{ev}(z)$ has the opposite monodromy

$$\frac{G_L^{ev}(z+\tau-\varepsilon\tau)}{G_L^{ev}(z+\varepsilon\tau)} \to e^{ipL} \tag{9.7}$$

when we first take L large.

Let us focus on checking this condition when z is real and belongs to the interval $z \in (0, 1/2)$. Then for points slightly above the real axis we have $E(z+\tilde{\varepsilon}\tau) = E(z) + i\varepsilon$ with both $\tilde{\varepsilon}$ and ε positive. Similarly we will have $E(z+\tau-\tilde{\varepsilon}\tau) = E(z) - i\varepsilon$. Thus the ratio (9.7) becomes

$$\prod_{k=1}^{n-1} \frac{\sqrt{1+16g^2\sin^2\frac{\pi k}{L}} - E + i\varepsilon}{\sqrt{1+16g^2\sin^2\frac{\pi k}{L}} - E - i\varepsilon} \tag{9.8}$$

and $L = 2n$. Let us transform the product into an exponent of a sum of logarithms and take the large L limit by rewriting the discrete sum as an integral.[9] We get

$$\exp\left\{\frac{L}{2}\int_0^1 \log\left(\sqrt{1+16g^2\sin^2\frac{\pi y}{2}} - E + i\varepsilon\right) - \log\left(\sqrt{1+16g^2\sin^2\frac{\pi y}{2}} - E - i\varepsilon\right) dy\right\}. \tag{9.9}$$

Now we have to be careful concerning the sign of the argument of the logarithm. When

$$y < y_* \equiv \frac{2}{\pi}\arcsin\frac{1}{4g}\sqrt{E^2-1} \tag{9.10}$$

the argument is negative and we deal essentially with the discontinuity across the branch cut which is $2i\pi$. Thus we get

$$\exp\left\{\frac{L}{2}\cdot 2i\pi\cdot y_*\right\}. \tag{9.11}$$

However using the dispersion relation $E = \sqrt{1+16g^2\sin^2 p/2}$ we see that $y_* = p/\pi$. This gives finally

$$e^{ipL} \tag{9.12}$$

which exactly cancels (9.6).

10 Conclusions

The String Field Theory (SFT) vertex axioms for $AdS_5 \times S^5$ are very challenging to solve for two reasons. Firstly, they incorporate a nondiagonal S-matrix which even for the case of ordinary relativistic form factors severly complicates their solution. Secondly, they involve

[9] This would be the leading term in a finite Poisson resummation. We will not control subleading terms so we will not determine order 1 terms in the monodromy (like the -1 in the asymptotic pp-wave case).

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the dependence on the size of the third closed string L which leads to the incorporation of all order multiple wrapping effects w.r.t. this parameter. This is in contrast to the hexagon approach where the question of wrapping is dealt with iteratively on top of an exact asymptotic solution. On the other hand, the possibility of handling analytically an infinite set of wrapping corrections at once is very appealing.

In this paper we have analyzed the two-particle SFT vertex axioms and found that one can factor out the complicated dependence on the dynamical S-matrix into an L-independent form factor[10] and a piece that satisfies the *L-dependent* axioms but with $S = 1$. We refer to this solution as the 'kinematical Neumann coefficient'. By definition this solution includes an infinite set of wrapping corrections w.r.t. L which are necessary to solve the SFT vertex axioms. Of course, one can conceive adding some additional L dependence in a form of an analog of a CDD factor, but since L does not appear in the CDD equations this is not enforced by any equations. So in this sense the 'kinematical Neumann coefficient' is a minimal solution as far as wrapping is concerned.

In this paper we have constructed explicitly the kinematical Neumann coefficient in the case of $AdS_5 \times S^5$ kinematics for any even L and any value of the gauge theory coupling constant. We have verified that this expression has the correct pp-wave limit and that it reduces to the spin chain answer (up to an L-independent CDD factor). In addition we have analyzed the large L limit and verified that it obeys analogous properties to the known pp-wave solution namely the apparent cancelation of monodromy in the physical strip.

It would be very interesting to solve the form factor equations in $AdS_5 \times S^5$ in order to obtain a complete expression. Also it is important to understand whether any CDD factors are necessary for a physical solution. Especially whether any additional wrapping effects would have to be included in the CDD factors. We hope that this question could also be addressed in the simpler relativistic setting where there exist integrable lattice realizations of some theories. To facilitate that we quote the complete solutions of the two-particle SFT vertex axioms for the sinh-Gordon and O(N) model.

The weak coupling analogue of the string vertex is the spin vertex. In [17, 18] the authors constructed the finite volume (size) spin vertex for all sectors at leading order. It would be interesting to investigate the decompactification limit of the vertex, which corresponds to our geometry and see how the vertex factorizes into a kinematical and a form factor part. This leading order calculation could be extended to higher loops by relating the higher order long-ranged spin chains to inhomogenous spin chains [19] or by using the separation of variables basis [16]. However, to directly compare the spin vertex of [17] with the string vertex one would have to include in the former the contribution of Bethe wave functions of the external states. A comparision has so far been done only in the pp-wave limit [20].

Finally we hope that the exact treatment of multiple wrapping corrections in the kinematical Neumann coefficient formula will be helpful for the treatment of wrapping in the hexagon approach.

[10] Its determination still remains as an outstanding open problem. See [14, 15] for details.

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Acknowledgments

RJ was supported by NCN grant 2012/06/A/ST2/00396 and ZB by a Lendület Grant. This work was supported by a Polish-Hungarian Academy of Sciences cooperation.

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