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More on Bogomol'nyi equations of three-dimensional generalized Maxwell-Higgs model using on-shell method

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Abstract: We use a recent *on-shell* method, developed in [1], to construct Bogomol'nyi equations of the three-dimensional generalized Maxwell-Higgs model [2]. The resulting Bogomol'nyi equations are parametrized by a constant C_0 and they can be classified into two types determined by the value of $C_0 = 0$ and $C_0 \neq 0$. We identify that the Bogomol'nyi equations obtained by Bazeia et al. [2] are of the ($C_0 = 0$)-type Bogomol'nyi equations. We show that the Bogomol'nyi equations of this type do not admit the Prasad-Sommerfield limit in its spectrum. As a resolution, the vacuum energy must be lifted up by adding some constant to the potential. Some possible solutions whose energy equal to the vacuum are discussed briefly. The *on-shell* method also reveals a new ($C_0 \neq 0$)-type Bogomol'nyi equations. This non-zero C_0 is related to a non-trivial function f_{C_0} defined as a difference between energy density of the scalar potential term and of the gauge kinetic term. It turns out that these Bogomol'nyi equations correspond to vortices with locally non-zero pressures, while their *average pressure* $\mathcal{P}$ remain zero globally by the finite energy constraint.

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Contents

1 Introduction

Bogomolnyi method is a smart trick to reduce the second-order Euler-Lagrange equations into the first-order, whose solitonic solutions possess minimum energies. For topologically nontrivial field's vacuum manifold the solutions are stable since at the boundary they map each point in coordinate space with different global minimum of the potential.

So far the Bogomolnyi equations were derived by saturating the lower bound of the corresponding static energy (the so-called *off-shell* method, or known as Bomolnyi's trick) [3]. This method may not always give the Bogomolnyi equations easily, especially when the Lagrangian contains noncanonical terms, as in the case of k-defects [4–10]. Recently, two of us proposed an alternative in obtaining the first-order equations by directly evaluating the Euler-Lagrange equations, later dubbed the *on-shell* method [1]. This formalism reproduces the known Bogomolnyi equations for kinks, vortices, and monopoles, as well as Dirac-Born-Infeld (DBI) kinks and vortices. This is a novel result though still preliminary, since it might enable us in constructing BPS (Bogomonlyi-Prasad-Sommerfield) states for general defects. Not only it is interesting in its own right, but also these least-energy solitonic solutions might have different properties from their canonical BPS counterparts. In the context of cosmology this might shed a new light on the dynamics of defects.

Not long time ago one of us [2] studied topological vortices in the generalized Maxwell-Higgs theory, whose dynamics are controlled by two positive functions in the Lagrangian, $G(|\phi|)$ and $w(|\phi|)$. It was shown that, for several choices of $G-$ and $w-$functions, there exist BPS solutions with various topology and energies (that can be greater than the canonical BPS tensions). Soon it was followed by the discovery of prescription for obtaining their analytical BPS vortex solutions [11]. The similar study was also done on generalized BPS monopoles [12–14].[1]

Here in this paper we look for something more modest by following a different route. Our aim is twofold. First, we wish to improve the on-shell method so that it includes noncanonical Lagrangian. Second, by applying it to the generalized Maxwell-Higgs theory we try to construct set of auxiliary functions, along with their constraint equations, that generate the corresponding Bogomol'nyi equations. It is expected that for arbitrarily positive functions $G(|\phi|)$ and $w(|\phi|)$ a large class of first-order Bogomolnyi equations (and their solutions) can be obtained.

The effective one-dimensional Euler-Lagrange equations, equation (6) in the on-shell method of [1], are difficult to get since the right hand side of the equations is only allowed to depend on the parameter r and the fields ϕ^a. It was very fortunate that examples given in [1] for the non-standard theory, which were the DBI defects, have not suffered from this difficulty. However, it should not happen in general for any theory with non-standard kinetic terms, such as the Generalized Maxwell-Higgs theory discussed in this article. Here, we need to improve the on-shell method such that the right hand side of the effective Euler-Lagrange equations are allowed to depend on first derivative of the fields ϕ^a. As a simple case, let us consider a theory with the effective degree of freedom is given by ϕ, in which the effective one dimensional Lagrangian $\mathcal{L} = \mathcal{L}(r, \phi, \phi')$ and the Euler-Lagrange equation are given by

$$\begin{aligned}
0 &= \frac{\partial \mathcal{L}}{\partial \phi} - \frac{d}{dr}\left(\frac{\partial \mathcal{L}}{\partial \phi'}\right) \\
0 &= \mathcal{A}(r,\phi,\phi') - \mathcal{B}_r(r,\phi,\phi') - \mathcal{B}_\phi(r,\phi,\phi')\phi' - \mathcal{B}_{\phi'}(r,\phi,\phi')\phi'' \\
\phi'' &= \frac{1}{\mathcal{B}_{\phi'}(r,\phi,\phi')}\left(\mathcal{A}(r,\phi,\phi') - \mathcal{B}_r(r,\phi,\phi') - \mathcal{B}_\phi(r,\phi,\phi')\phi'\right), \qquad (1.1)
\end{aligned}$$

where

$$\mathcal{A} = \frac{\partial \mathcal{L}}{\partial \phi}, \qquad \mathcal{B}_x = \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial \phi'}\right), \qquad x \equiv (r, \phi, \phi'). \qquad (1.2)$$

The Euler-Lagrange equation can be arranged into

$$\phi'' + f(r,\phi,\phi')\phi' = 0. \qquad (1.3)$$

We then need to determine what would be the expected function of $f(r,\phi,\phi')$ provided that the left hand side of (1.3) can be rewritten as

$$\phi'' + f(r,\phi,\phi')\phi' = \frac{1}{h}(h\phi')' + \dots, \qquad (1.4)$$

where $h \equiv h(r,\phi)$. Now, since $h' = \frac{\partial h}{\partial r} + \frac{\partial h}{\partial \phi}\phi'$, it yields that the function f must be of the form

$$f(r,\phi,\phi') = \frac{1}{h}\frac{\partial h}{\partial r} + \frac{1}{h}\frac{\partial h}{\partial \phi}\phi' + (\text{non-linear terms in } \phi'). \qquad (1.5)$$

[1] These are rather remarkable results, since the search for analytic BPS vortex solutions has been notoriously difficult and so far has been futile, while the finding of BPS monopole solutions by Prasad and Sommerfield was achieved only after several trials and errors [15]. It is the appearance of G and w functions that, in spite of making the EoM appear more complicated, actually helps in obtaining the suitable solutions that satisfy the boundary conditions.

We keep the linear terms of f, in ϕ', in the left hand side of (1.3) and move the non-linear terms to the right hand side of (1.3). The Bogomol'nyi equation is then given by $h(r,\phi)\phi' = X(\phi)$, while the constraint equation is now

$$\frac{X'}{h} = g(r,\phi,\phi'), \tag{1.6}$$

where g contains all remaining non-linear terms coming from f. Notice that upon substituting the Bogomol'nyi equation into (1.3), we can get back the form of effective Euler-Lagrange equation as in the equation (6) of [1].

For multiple fields theory,[2] generalization of the above procedures are more involved. As such, for each field ϕ^a, the effective one dimensional Euler-Lagrange equations are

$$0 = \mathcal{A}^a(r,\phi,\phi') - \mathcal{B}^a_r(r,\phi,\phi') - \sum_b \mathcal{B}^a_{\phi^b}(r,\phi,\phi')\phi^{b\prime} - \sum_b \mathcal{B}^a_{\phi^{b\prime}}(r,\phi,\phi')\phi^{b\prime\prime}, \tag{1.7}$$

$$\phi^{a\prime\prime} = \frac{1}{\mathcal{B}^a_{\phi^{a\prime}}(r,\phi,\phi')}\left(\mathcal{A}^a(r,\phi,\phi') - \mathcal{B}^a_r(r,\phi,\phi') - \sum_{b\neq a} \mathcal{B}^a_{\phi^b}(r,\phi,\phi')\phi^{b\prime} - \sum_{b\neq a} \mathcal{B}^a_{\phi^{b\prime}}(r,\phi,\phi')\phi^{b\prime\prime}\right),$$

where

$$\mathcal{A}^a = \frac{\partial\mathcal{L}}{\partial\phi^a}, \qquad \mathcal{B}^a_x = \frac{\partial}{\partial x}\left(\frac{\partial\mathcal{L}}{\partial\phi^{a\prime}}\right), \qquad x \equiv \left(r,\phi^b,\phi^{b\prime}\right), \qquad b = 1,\ldots,N_\phi. \tag{1.8}$$

One should notice that the Euler-Lagrange equations are linear in ϕ''. Taking the same procedures as in the case of a single field theory, we may write the Euler-Lagrange equation, for each ϕ^a, as

$$\phi^{a\prime\prime} + f^a(r,\phi,\phi')\phi^{a\prime} = g^a(r,\phi,\phi') + \sum_{b\neq a} k_{ab}(r,\phi,\phi')\left[\phi^{b\prime\prime} + f^b(r,\phi,\phi')\phi^{b\prime}\right], \tag{1.9}$$

where f is linear function in ϕ'. To have the Bogomol'nyi equations, the function f^b must be of the form

$$f^b(r,\phi,\phi') = \frac{1}{h^b}\frac{\partial h^b}{\partial r} + \frac{1}{h^b}\sum_c \frac{\partial h^b}{\partial\phi^c}\phi^{c\prime}, \qquad c = 1,\ldots,N_\phi\,, \tag{1.10}$$

where $h^b \equiv h^b(r,\phi)$. The Bogomol'nyi equations then are given by

$$h^b(r,\phi)\phi^{b\prime} = X^b(\phi) \tag{1.11}$$

and the constraint equations are

$$\frac{X^{a\prime}}{h^a} = g^a(r,\phi,\phi') + \sum_{b\neq a} k_{ab}(r,\phi,\phi')\frac{X^{b\prime}}{h^b}. \tag{1.12}$$

As in [1], the topological charge can directly be obtained by inserting the Bogomol'nyi equations into the energy functional. We shall obtain, in general,

$$dQ = \sum_a F[X^a(\phi)]\phi^{a\prime}, \tag{1.13}$$

[2]Here, we follow the conventions in [1] for $N_\phi-$fields theory.

where $F[X^a(\phi)]$ is a general functional of $X^a(\phi)$ whose form depends on the actual kinetic form of the Lagrangian. In particular, for canonical case $F[X^a(\phi)] = X^a(\phi)$. Its integral becomes

$$\begin{aligned} E_{\mathrm{BPS}} &= \int dQ, \\ &= Q(r=\infty) - Q(r=0). \end{aligned} \tag{1.14}$$

2 Generalized Maxwell-Higgs model

As an example of application of the prescription above, let us now consider a generalized Maxwell-Higgs theory described by the following (1+2)-dimensional Lagrangian density [2]

$$\mathcal{L}_G = -\frac{1}{4} G(|\phi|) F_{\mu\nu} F^{\mu\nu} + w(|\phi|) |D_\mu \phi|^2 - V(|\phi|)\,, \tag{2.1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $D_\mu \phi = \partial_\mu + ieA_\mu \phi$, and the Minkowskian metric is $\eta^{\mu\nu} \equiv diag(+,-,-)$. Here, we take the gauge coupling e and the vacuum expectation value v of the scalar field to be real and positive. The functions $G(|\phi|)$ and $w(|\phi|)$ are constrained to be positive and depend explicitly only on the Higgs field amplitude, $|\phi|$, but not on its derivative.[3] In this article, we will consider a static solitonic object, in particular topological vortices, in which all the fields are static. Furthermore, we will consider the spatial part of the action and write it in terms of the spherical coordinates.

We chose a temporal gauge $A^0 = 0$ and the static fields ansatz

$$\phi = v\ g(r) e^{in\theta}, \qquad\qquad \mathbf{A} = -\frac{a(r) - n}{e\ r} \hat{\theta}\,, \tag{2.2}$$

where (r, θ) is the polar coordinates and $n = \pm 1, \pm 2, \dots$ is an integer winding number. Notice that the Lagrangian is invariant under two-dimensional rotation and an abelian gauge transformation, SO(2)×U(1). The ansatz for the Higgs field is chosen to be invariant under subgroup of this symmetry which is the SO(2) rotational transformation with a particular choice of U(1) gauge transformation, that cancels the two-dimensional rotation. It is guaranteed that the solutions of the effective equations of motion, derived by using this ansatz, are also the solutions of the full equation of motions [17].

Using these ansatz, the static energy, proportional to the static action, can be simply written as

$$E = 2\pi \int dr\ r \left(\frac{G}{2e^2} \left(\frac{1}{r} \frac{da}{dr} \right)^2 + v^2 w \left(\left(\frac{dg}{dr} \right)^2 + \frac{g^2 a^2}{r^2} \right) + V \right). \tag{2.3}$$

The Euler-Lagrange equations, or equations of motion, derived from the above static energy are

$$G \frac{d^2 a}{dr^2} + \left(\frac{dG}{dr} - \frac{G}{r} \right) \frac{da}{dr} = 2e^2 v^2 g^2 a w, \tag{2.4}$$

[3]The case for field-derivative-dependent functions will be addressed in the forthcoming publication.

and

$$w\left(\frac{d^2g}{dr^2}+\frac{1}{r}\frac{dg}{dr}-\frac{a^2g}{r^2}\right)-\frac{1}{4v^2}\left(\frac{1}{er}\frac{da}{dr}\right)^2\frac{dG}{dg}=\frac{1}{2v^2}\frac{dV}{dg}-\frac{1}{2}\left(\left(\frac{dg}{dr}\right)^2-\frac{g^2a^2}{r^2}\right)\frac{dw}{dg}. \tag{2.5}$$

The vacuum solution of the above theory (2.1) is related to the solution in which $A_\mu = 0$ and $\phi = v$. For the case of topological vortex, we consider the case in which $v \neq 0$. For topological vortex solutions, we require the fields a and g to behave asymptotically, near the origin and the boundary, as follows

$$\begin{aligned} a(r\to 0) &= n, \quad g(r\to 0) = 0, \\ a(r\to\infty) &= 0, \quad g(r\to\infty) = 1. \end{aligned} \tag{2.6}$$

How fast the functions a and g approaching their asymptotic values, namely the next leading order terms, is determined by the Bogomol'nyi equations and the explicit form of G, w, and V, with a condition the static energy (2.3) is finite.

3 Bogomol'nyi equations

In order to obtain the Bogomol'nyi equations, following the prescription in section 1, we rewrite the Euler-Lagrange equations into

$$\frac{r}{G}\frac{d}{dr}\left(\frac{G}{r}\frac{da}{dr}\right)=\frac{2}{G}e^2v^2g^2aw, \tag{3.1}$$

and

$$\frac{1}{rw^{1/2}}\frac{d}{dr}\left(rw^{1/2}\frac{dg}{dr}\right)=\frac{1}{4wv^2e^2G^2}\left(\frac{G}{r}\frac{da}{dr}\right)^2+\frac{a^2g}{r^2}+\frac{1}{2v^2}\frac{dV}{dg}+\frac{g^2a^2}{2r^2w}\frac{dw}{dg}. \tag{3.2}$$

The first term on the right hand side of equation (3.2) contains first derivative of field a, $a'(r)$. It can be turned into a non-derivative fields dependence by using the Bogomol'nyi equations as we will show later in detail. Now, let us introduce some auxiliary fields into the Euler-Lagrange equations as follows

$$r\frac{d}{dr}\left(\frac{G}{r}\frac{da}{dr}-X\right)+r\frac{dX}{dr}=2e^2v^2g^2aw, \tag{3.3}$$

and

$$\frac{w^{1/2}}{r}\frac{d}{dr}\left(rw^{1/2}\frac{dg}{dr}-Y\right)+\frac{w^{1/2}}{r}\frac{dY}{dr}=\frac{1}{4v^2e^2G^2}\left(\frac{G}{r}\frac{da}{dr}\right)^2\frac{dG}{dg}+\frac{a^2wg}{r^2}+\frac{1}{2v^2}\frac{dV}{dg}+\frac{g^2a^2}{2r^2}\frac{dw}{dg}, \tag{3.4}$$

where X and Y are the auxiliary functions that depend only on the fields a and g, but not their derivatives, and do not depend explicitly on r. From these equations, we can extract the Bogomol'nyi equations which are

$$\frac{G}{r}\frac{da}{dr}-X=0 \tag{3.5}$$

and

$$rw^{1/2}\frac{dg}{dr} - Y = 0. \tag{3.6}$$

The Bogomol'nyi equations are complimented by the constraint equations

$$r\frac{dX}{dr} = 2e^2v^2g^2aw, \tag{3.7}$$

and

$$\frac{w^{1/2}}{r}\frac{dY}{dr} = \frac{X^2}{4v^2e^2G^2}\frac{dG}{dg} + \frac{a^2wg}{r^2} + \frac{1}{2v^2}\frac{dV}{dg} + \frac{g^2a^2}{2r^2}\frac{dw}{dg}. \tag{3.8}$$

Notice that we have substituted the first term on the right hand side of the constraint equation (3.8) by using the Bogomol'nyi equation (3.5). Substituting further the Bogomol'nyi equations into the constraint equations yields

$$\frac{\partial X}{\partial g}\frac{Y}{rw^{1/2}} + \frac{\partial X}{\partial a}\frac{rX}{G} = \frac{2}{r}e^2v^2g^2aw, \tag{3.9}$$

and

$$\frac{\partial Y}{\partial g}\frac{Y}{rw^{1/2}} + \frac{\partial Y}{\partial a}\frac{rX}{G} = \frac{r}{w^{1/2}}\left(\frac{X^2}{4v^2e^2G^2}\frac{dG}{dg} + \frac{a^2wg}{r^2} + \frac{1}{2v^2}\frac{dV}{dg} + \frac{g^2a^2}{2r^2}\frac{dw}{dg}\right). \tag{3.10}$$

Next, we solve those constraint equations by dividing each of them into terms that depend on the explicit power of r. Solving those terms independently, this process yields several equations:

$$\frac{\partial X}{\partial a} = 0, \qquad \frac{\partial X}{\partial g}\frac{Y}{w^{1/2}} = 2e^2v^2g^2aw, \tag{3.11}$$

$$\frac{\partial Y}{\partial g}Y = a^2wg + \frac{g^2a^2}{2}\frac{dw}{dg}, \qquad \frac{\partial Y}{\partial a}\frac{X}{G} = \frac{X^2}{4v^2e^2G^2w^{1/2}}\frac{dG}{dg} + \frac{1}{2v^2w^{1/2}}\frac{dV}{dg}. \tag{3.12}$$

The problem is now reduced to finding the auxiliary functions, X and Y, which solve the above (constraint) equations. The first equation in (3.11) implies that X is independent of a. The general solution for Y can be obtained by solving the first equation in (3.12) which is given by $Y^2(g,a) = a^2g^2w + C_0(a)$, where C_0 is an arbitrary function of a. However, for nontrivial solutions, the second equation in (3.11) restricts the function $C_0 \propto a^2$. In general, we may write the solution for Y to be $Y^2(g,a) = a^2\left(g^2w + C_0\right)$, where now C_0 is just a constant. Since the first equation in (3.11) gives $X \equiv X(g)$, all the auxiliary functions are essentially separable functions. Writing all the auxiliary functions to be separable,

$$X(g,a) = X_g(g)X_a(a), \qquad Y(g,a) = Y_g(g)Y_a(a), \tag{3.13}$$

without loss of generality we can take $X_a = 1$, $Y_a = a$, and $Y_g^2 = g^2w + C_0$. Using $Y_g = \pm\sqrt{g^2w + C_0}$, we obtain from the second equation in (3.11)

$$X_g = \pm e^2v^2\left(2\int dg\frac{g^2w^{3/2}}{\sqrt{g^2w+C_0}} + C_1\right), \tag{3.14}$$

where C_1 is an integration constant. Therefore we obtain that the Bogomol'nyi equation (3.5) depends on functions w and G, while the Bogomol'nyi equation (3.6) depends only on function w.

It will be useful later to define functions

$$R(g) = \frac{X_g}{G}, \qquad S(g) = \frac{Y_g}{w^{1/2}}. \tag{3.15}$$

Using the previously obtained functions: X_g and Y_g, we are left with only one constraint equation, the second equation in (3.6), which in terms of functions S and R is simply written as

$$V' = 2v^2 wRS - \frac{R^2}{2e^2} G'. \tag{3.16}$$

From now on, we will use $' \equiv \frac{\partial}{\partial g}$ if it is not defined explicitly. The Bogomol'nyi equations can simply be rewritten as follows:

$$r\frac{dg}{dr} = a\ S, \qquad \frac{1}{r}\frac{da}{dr} = R. \tag{3.17}$$

So, we can say that the equations in (3.15) generate the Bogomol'nyi equations in (3.17) for the generalized Maxwell-Higgs model (2.1) once we fix the functions: w and G, and the constants: C_0 and C_1, while the constraint equation (3.16) determines the form of potential V once we know all these functions and constants. At first sight, the constraint equation (3.16) is different from the standard one obtained in [2] which, in our conventions, can be written as

$$\left(\sqrt{\frac{G\ V}{2}}\right)' = ev^2 wg. \tag{3.18}$$

However, we will show later in the next section that the constraint equation (3.18) of [2] is a particular case of our constraint equation (3.16).

3.1 Bogomol'nyi equations for $C_0 = 0$

In this subsection, we consider a particular type of equations, which we call $(C_0 = 0)$-type Bogomol'nyi equations. This type of equations is provided by taking $C_0 = 0$, for which we obtain $S = \pm q$ and

$$X_g = \pm e^2 v^2 \left(\int d(g^2)\ w + C_1\right). \tag{3.19}$$

It is tempted to expect from the above integral that $w \equiv w(g^2)$ which happens to be the case in all Bogomol'nyi equations of [2]. One can also check that all functions of w, G, and V in each Bogomol'nyi equations of [2] are solutions to the constraint equation (3.16). In this case, the Bogomol'nyi equations can be simply written as

$$r\frac{dg}{dr} = \pm a\ g, \tag{3.20}$$

$$\frac{1}{r}\frac{da}{dr} = \pm\frac{e^2 v^2}{G}\left(\int d(g^2)\ w + C_1\right), \tag{3.21}$$

and the constraint equation (3.16) now becomes

$$V' = \pm 2v^2 wRg - \frac{R^2}{2e^2}G'. \tag{3.22}$$

Using the fact that $X_g' = (RG)' = \pm 2e^2v^2 gw$, the constraint equation can be rewritten as

$$V' = \frac{1}{e^2}R(GR)' - \frac{R^2}{2e^2}G' = \frac{R^2}{2e^2}G' + \frac{RG}{e^2}R'. \tag{3.23}$$

The solution to this differential equation is

$$V = \frac{1}{2e^2}R^2G + \text{constant} = \frac{e^2v^4}{2G}\left(\int d(g^2)w + C_1\right)^2 + \text{constant}. \tag{3.24}$$

Here, this constant can actually be set to zero by shifting the potential V in the action. Furthermore, we will see later that by imposing a condition that the energy of the vortex to be finite, this constant is forced to be zero. In this case, it turns out that the potential (3.24) also solves the constraint equation (3.18), and thus it is the same as the constraint equation in [2]. The potentials obtained in [2] can be derived simply by using the constraint (3.24) with a particular choice of the functions and parameters:

(a). Standard Maxwell-Higgs model
$G = 1; \quad w = 1; \quad C_1 = -1 \quad \longrightarrow \quad V = \frac{e^2v^4}{2}(1-g^2)^2.$

(b). $G = \frac{(g^2+3)^2}{g^2}; \quad w = 2(g^2+1); \quad C_1 = -3 \quad \longrightarrow \quad V = g^2\frac{e^2v^4}{2}(1-g^2)^2.$

(c). $G = (g^2+1)^2; \quad w = 2g^2; \quad C_1 = -1 \quad \longrightarrow \quad V = \frac{e^2v^4}{2}(1-g^2)^2.$

(d). $G = \frac{k^2}{2e^2v^2g^2}; \quad w = 1; \quad C_1 = -1 \quad \longrightarrow \quad V = \frac{e^4v^6}{k^2}g^2(1-g^2)^2.$

Restricting to the vanishing constant in the constraint (3.24), the BPS equations can be simply written as

$$r\frac{dg}{dr} = \pm a\ g, \tag{3.25}$$

$$\frac{1}{r}\frac{da}{dr} = \pm e\sqrt{\frac{2V}{G}}. \tag{3.26}$$

These Bogomol'nyi equations are also called BPS equations in which the solutions to these equations correspond to BPS vortices.

Flat potential. A slight advantage of our constraint equation (3.23) is that the potential V can be safely taken to be zero. Unlike the one in [2], or equation (3.18), setting $V = 0$ will not give us a solution. In the limit of the coupling at which the potential $V = 0$, also known as Prasad-Sommerfield limit [15], the solution for G is given by

$$G = C_2^2 e^4 v^4 \left(\int d(g^2)\ w + C_1\right)^2 \longrightarrow C_2^2 R^2 G = 1, \tag{3.27}$$

where C_2 is an non-zero integration constant related to the non-zero constant in (3.24). Although the constraint (3.18) is not suitable for the case of $V = 0$, the solution (3.27) can actually be obtained from it by setting the potential to be constant $V = \frac{1}{2e^2 C_2^2}$. This is related to the fact, as we will discuss in the next section, that the finiteness energy requires a shift in the potential by a constant. Nevertheless, the Bogomol'nyi, or to be precise BPS, equations now become

$$\begin{aligned} \frac{dg}{dr} &= \pm \frac{ag}{r}, \\ \frac{da}{dr} &= \pm \frac{r}{C_2 \sqrt{G}}. \end{aligned} \tag{3.28}$$

Here, the function G depends on the function w and the constants (C_1 and C_2). We present some of the examples, with $C_2 = 1$, as follows

- $w = 1; \quad C_1 = -1 \quad \longrightarrow \quad G = e^4 v^4 (g^2 - 1)^2.$
- $w = 2g^2; \quad C_1 = -1 \quad \longrightarrow \quad G = e^4 v^4 (g^4 - 1)^2.$
- $w = 2(g^2 + 1); \quad C_1 = -3 \quad \longrightarrow \quad G = e^4 v^4 (g^2 - 1)^2 (g^2 + 3)^2.$

Later, we will find that all of the above examples turn out to give infinite energy. This can be seen due to the presence of singularity of the corresponding BPS equations near the boundary. As an example, consider the configuration (c) above in which $w = 2(g^2 + 1)$ and $C_1 = -3$ gives $G = e^4 v^4 (g^2 - 1)^2 (g^2 + 3)^2$. The BPS equations are

$$\begin{aligned} g' &= \pm \frac{ag}{r}, \\ a' &= \pm \frac{r}{e^2 v^2 (g^2 - 1)(g^2 + 3)}. \end{aligned} \tag{3.29}$$

The second equation blows up at infinity, since $g(r \to \infty) \to 1$. On the other hand, there should be many possibilities of $G(g)$ such that it satisfies the boundary conditions. For example, we can take

$$G = \frac{e^4 v^4}{(1 - g^2)^2}. \tag{3.30}$$

This can be obtained by taking[4]

$$w = \frac{1}{(1 - g^2)^2}, \qquad C_1 = 0. \tag{3.31}$$

It is amusing that the combination of G and w above, when inserted into the equations (3.28), produces precisely the equations for ordinary BPS Maxwell-Higgs vortices (up to some overall constants),

$$\begin{aligned} g' &= \pm \frac{ag}{r}, \\ \frac{a}{r} &= \pm \frac{(1 - g^2)}{e^2 \nu^2}. \end{aligned} \tag{3.32}$$

[4]This choice opens up a possibility that G and w can take up rational-form functions.

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Since we know that BPS vortices exist, so do these *flat potential* BPS generalized Maxwell-Higgs vortices. However, there is a subtlety here that the functions w and G now can be singular near the boundary, or $g \to 1$.

3.2 Bogomol'nyi equations for $C_0 \neq 0$

For a general case, we can rewrite the constraint equation (3.16) in terms of R and S as follows

$$V' = \frac{R^2}{e^2}\left(\frac{S^2}{g^2} - \frac{1}{2}\right)G' + \frac{RG}{e^2}\frac{S^2}{g^2}R'. \tag{3.33}$$

Unlike the ($C_0 = 0$)-type case, the right hand side of the constraint equation above is more complicated and it is very difficult to write it as a total derivative of some functions and hence difficult to find the solution. However, we may try to follow what we did as in the ($C_0 = 0$)-type case and write the constraint (3.33) simply as

$$2e^2V' = \left(R^2G^2\right)\left(\frac{1}{G}\right)' + \frac{S^2}{g^2G}\left(R^2G^2\right)'. \tag{3.34}$$

To have a total derivative, we are tempted to identify

$$\frac{1}{G} + C_3 = \frac{S^2}{g^2}\frac{1}{G}, \tag{3.35}$$

where C_3 is just a constant which we can just add to the constraint equation above by shifting $\left(\frac{1}{G}\right)' \to \left(\frac{1}{G} + C_3\right)'$. The value of C_3 needs to be non-zero otherwise it would not be consistent with $C_0 \neq 0$ since $S^2 = g^2$. With this identification, we obtain that

$$G = \frac{1}{g^2 w}\frac{C_0}{C_3}. \tag{3.36}$$

This is consistent with the ($C_0 = 0$)-type Bogomol'nyi equations, in which we have to take $C_3 = 0$ in order for G to be non-trivial. However, this is a little bit peculiar because G-dependence of w is in contradiction with the ($C_0 = 0$)-type Bogomol'nyi equations. We might expect that G is still independent of w, or arbitrary, for more general case in which the constant C_0 can be non-zero. It turns out that this solution can not lead to the finite energy solution as discussed in the next section.

Although the constraint equation (3.33) does not seem to have a solution, let us write explicitly the Bogomol'nyi equations:

$$\frac{dg}{dr} = \pm\frac{a}{r}\sqrt{\frac{g^2w + C_0}{w}}, \tag{3.37}$$

$$\frac{da}{dr} = \pm e^2v^2\frac{r}{G}\left(2\int dg\frac{g^2w^{3/2}}{\sqrt{g^2w + C_0}} + C_1\right). \tag{3.38}$$

The solutions to these Bogomol'nyi equations correspond to, what we call, the ($C_0 \neq 0$)-type BPS vortices. Even if we are able to find solutions for the constraint equation (3.33), it is not guaranteed that those solutions will have finite energy. We will see later that there are some possibilities in which the solutions to the constraint equation (3.33) would give a finite energy.

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4 Static energy

The static energy can be rewritten into a nicer form by substituting Bogomol'nyi equations (3.17) into the energy density (2.3),

$$E_{\mathrm{BPS}} = 2\pi \int \left(\left(\frac{GR}{2e^2} + \frac{V}{R} \right) da + \frac{v^2 wa}{S} \left(S^2 + g^2\right) dg \right). \tag{4.1}$$

The terms inside the parenthesis of the integral formula of (4.1) depends on functions a and g. Therefore we may define a function $Q \equiv 2\pi a \left(\frac{GR}{2e^2} + \frac{V}{R}\right)$ such that $E_{Sol} = \int dQ$, with a condition

$$V' = v^2 w(S^2 + g^2)\frac{R}{S} - \frac{R^2}{2e^2}G' + \left(\frac{V}{R} - \frac{RG}{2e^2}\right) R'. \tag{4.2}$$

Substituting this equation into the constraint (3.33) yields

$$\left(\frac{RG}{2} - \frac{e^2 V}{R}\right) R' = e^2 v^2 \frac{wR}{S}(g^2 - S^2), \tag{4.3}$$

or it can also be written as

$$\left(\frac{e^2 V}{G} - \frac{R^2}{2}\right) G' = 2e^2 v^2 \frac{g^2 w}{RS} \left(\frac{e^2 V}{G} - \frac{R^2}{2}\frac{S^2}{g^2}\right). \tag{4.4}$$

Now let us see if the vortices have finite energy using the Derrick's Theorem [16, 17]. We can write the scaled static energy of (2.3) to be

$$\begin{aligned}
E(\lambda) &= \lambda^2 E_{\mathrm{gauge}} + E_{\mathrm{scalar}} + \frac{1}{\lambda^2} E_{\mathrm{pot}}, \\
E_{\mathrm{gauge}} &= \int d^2x \, \frac{G}{2e^2 r^2} \left(\frac{da}{dr}\right)^2, \qquad E_{\mathrm{pot}} = \int d^2x \; V, \\
E_{\mathrm{scalar}} &= \int d^2x \left(v^2 w \left(\frac{dg}{dr}\right)^2 + v^2 a^2 g^2 \frac{w}{r^2} \right),
\end{aligned} \tag{4.5}$$

where $0 < \lambda < \infty$ is the scale factor. There is a stationary point if we vary the $E(\lambda)$ over λ at which is finite and positive. It means there are some vortices with finite energy. Furthermore, the virial theorem requires $E_{\mathrm{gauge}} = E_{\mathrm{pot}}$. Consider a simple case which both energy densities are equal pointwise. After substituting the Bogomol'nyi equations (3.17), it yields that

$$V = \frac{R^2 G}{2e^2}, \tag{4.6}$$

Substituting this into the equation (4.3), or (4.4), implies that $S^2 = g^2$, or it means $C_0 = 0$. Therefore if we assume that the energy can be written as an integral over a form dQ then the ($C_0 \neq 0$)-type vortices will have infinite energy. Even if we do not use this assumption and just use the equation (4.6), we can show that

$$V' = -\frac{R^2}{2e^2}G' + 2v^2 g^2 w \frac{R}{S} \tag{4.7}$$

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by taking a first derivative of the equation (4.6) over g and using the Bogomol'nyi equations (3.17). This is equal to the constraint (3.33) providing that $S^2 = g^2$, which also concludes that $C_0 = 0$. Therefore we may ignore the ($C_0 \neq 0$)-type Bogomol'nyi equations as they are not physical since their energy is infinite.

4.1 Finite energy for $C_0 = 0$

Notice that the requirement for the energy of the solution to be finite, for the point wise case, forces us to set the constant in (3.24) to be zero. In this case, the static energy (2.3) can be simplified to

$$\begin{aligned} E_{\mathrm{BPS}} &= 2\pi \int dr\, r \left(\frac{GR}{e^2} \frac{1}{r} \frac{da}{dr} \pm 2v^2 w \frac{ga}{r} \frac{dg}{dr} \right), \\ &= 2\pi \int \left(\frac{GR}{e^2} da \pm 2v^2 wag\; dg \right). \end{aligned} \tag{4.8}$$

Recalling that $(RG)' = \pm 2e^2v^2gw$, we can obtain the aforementioned function $Q = 2\pi \frac{GR}{e^2} a$. Now, the static energy is simply written as

$$E_{\mathrm{BPS}} = Q(r \to \infty) - Q(r \to 0). \tag{4.9}$$

Using formula (3.19), it yields[5]

$$E_{\mathrm{BPS}} = \left| 2\pi v^2 n \left(\lim_{r\to 0} \int w\; d(g^2) + C_1 \right) \right|. \tag{4.12}$$

Here, we have assumed that $Q(r \to \infty) = 0$ and hence $GR(r \to \infty) = O(r^0)$. In another words, we assume that GR is not singular near the boundary. This can be shown to be satisfied in general by writing $GR = \sqrt{2e^2VG}$, using the equation (4.6). Recalling that near the boundary, the potential V approaches the vacuum solution, in which $V = 0$, then it only requires that $G(r \to \infty) = O(r^0)$. The finiteness of energy also requires $\lim_{r\to 0} \int w\; d(g^2)$, or $\sqrt{VG}(r \to 0)$, to be finite. Since all these functions (w, G and V) are functions of g, we may rewrite it as $\int w\; d(g^2)\big|_{g=0}$, or $\sqrt{VG}(g = 0)$, to be finite.

The static energy can be proportional to the topological charge $Q_{Top} = 2\pi v^2 |n|$ as such $E_{\mathrm{BPS}} = C_Q Q_{Top}$, where $C_Q \geq 0$. In the case of $C_Q = 1$, we obtain that

$$\lim_{r\to 0} \int w\; d(g^2) + C_1 = \pm 1, \tag{4.13}$$

[5]Notice that for any polynomial w-function, $w(g) \sim g^m$ with $m \geq 0$, the $\lim_{r\to 0} \int w\; d(g^2)$ always yields zero. In this case the topological charge is solely determined by the constant C_1. On the other hand, we can also easily construct a rational w-function, say

$$w = \frac{1}{(g^2+1)^2}. \tag{4.10}$$

This function is positive and regular at the origin, whose (indefinite) integral gives

$$\int w\; d(g^2) = -\frac{1}{(g^2+1)}. \tag{4.11}$$

The limit then yields -1. In this particular case, the charge would depend on $(C_1 - 1)$.

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which means the static energy equal to the standard vortex. For example for $w = 1$, we have $C_1 = \pm 1$. In the list of examples in 3.1, the (a), (c), and (d) are of this type. If $C_Q > 1$ then the static energy is higher than the standard vortex, $E_{\mathrm{BPS}} > Q_{Top}$, and they are determined by

$$\left| \lim_{r \to 0} \int w \; d(g^2) + C_1 \right| > 1. \tag{4.14}$$

The example (b) in 3.1 is in this type in which the static energy $E_{\mathrm{BPS}} = 3Q_{Top}$. There are also some interesting Bogomol'nyi equations in which $C_Q < 1$, or $E_{\mathrm{BPS}} < Q_{Top}$, and the condition is given by

$$\left| \lim_{r \to 0} \int w \; d(g^2) + C_1 \right| < 1. \tag{4.15}$$

The most interesting of one is when $C_Q = 0$, or $E_{\mathrm{BPS}} = 0$, with a condition

$$C_1 = - \lim_{r \to 0} \int w \; d(g^2). \tag{4.16}$$

This raises a question, do Bogomol'nyi solutions with zero energy exist? A rigorous answer needs a rigorous proof. In this paper we do not attempt to answer it. We just note that if we choose the following set of functions and parameter

$$w = 2g^2 - 1, \quad G = 1, \quad C_1 = 0, \tag{4.17}$$

we can end up with the following Bogomolnyi equation

$$\frac{a'}{r} = \pm e^2 \nu^2 g^2 \left(1 - g^2\right), \tag{4.18}$$

whose potential is $V = \frac{e^2 \nu^4}{2} g^4 \left(1 - g^2\right)^2$, an S^0 surrounded by an S^1 vacuum topology. The equation satisfies both regularity at the origin and finiteness of energy. Due to the vacuum manifold, this is an example of nontopological soliton discussed in [2, 18].[6]

Flat potential. As we mentioned previously, finiteness in the static energy requires the constant in the constraint equation (3.24) to be zero, or $2e^2 V = R^2 G$. Taking $V = 0$ is not possible in this case and might cause the resulting energy to be infinite. Nevertheless, let us just ignore the requirement for finite energy and allow the potential $V = 0$. The static energy in this case can be written as

$$\begin{aligned} E_{\mathrm{BPS}} &= 2\pi \int \left(\frac{RG}{2e^2} da \pm 2v^2 wag \; dg \right), \\ &= \int dQ - \frac{\pi}{e^2} \int RG \; da. \end{aligned} \tag{4.20}$$

[6] For Bogomol'nyi topological solitons we need potential whose vacuum manifold is nontrivial. For example if $w = 1$ then it yields $C_1 = 0$. Now we can set $G = g^4/(1 - g^2)^2$ such that the theory still has the standard symmetry breaking Higgs potential $V = \frac{1}{2} e^2 v^4 (1 - g^2)^2$. However these functions do not satisfy the near origin condition for the Bogomol'nyi equation;

$$\frac{a'}{r} = \pm e^2 \nu^2 \frac{\left(1 - g^2\right)^2}{g^2} \tag{4.19}$$

is singular at the origin.

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Substituting the BPS equations (3.28), we obtain

$$E_{\mathrm{BPS}} = \frac{2\pi}{e^2\,|C_2|}\; a\sqrt{G}\Big|_{r=0}^{r\to\infty} - \lim_{r\to\infty}\frac{\pi}{2e^2C_2^2}r^2. \tag{4.21}$$

Indeed, we find that the static energy is infinite which comes from the last term on the right hand of equation (4.21). This infinity can be removed by adding a positive constant potential to the action.[7] The positive constant potential needed to cancel this infinity is equal to the potential computed using equation (4.6) with a given solution for G is (3.27). Therefore if we take the potential to be non-zero constant in the first place, we will have no problem in taking the finite energy equation (4.6), and thus the static energy will be finite.

The first term on the right hand side of equation (4.21)depends on $a\sqrt{G}$ at the boundaries. As we mentioned previously, there is a subtlety in function G if we impose regularity on the Bogomol'nyi equations (3.28). To have Bogomol'nyi equations (3.28) that respect appropriate boundary conditions, $G \sim \left(1-g^2\right)^{-2m}$, for some positive integer m. Although G is infinite near the boundary, by taking appropriate leading order of function a as such it is going to zero faster than $1/\sqrt{G}$, we could obtain the static energy which is

$$E_{\mathrm{BPS}} = \left|2\pi v^2 n \lim_{r\to 0} a\sqrt{G}\right|. \tag{4.22}$$

For our case in equation (3.32), it yields

$$E_{\mathrm{BPS}} = \left|\frac{2\pi v^2 n}{C_2}\right|. \tag{4.23}$$

It is interesting that the arbitrary choice of C_2 results in different value of E_{BPS}.

4.2 Finite energy for $C_0 \neq 0$

From the previous discussion, it is clear that the finite energy equation (4.6) strongly restricts the constant $C_0 = 0$. Therefore the Bogomol'nyi equations for $C_0 \neq 0$ would not give a finite static energy of the vortex. However, we should recall that the equation (4.6) is not a general result of the Derrick's theorem, followed by the virial theorem. There is more general result of the virial theorem in which the finiteness of energy requires

$$\int_0^\infty dr\; r\left(\frac{R^2G}{2e^2} - V\right) = 0, \tag{4.24}$$

such that the integrand is non-zero pointwise. Up to now, we do not know how to substitute the definite integral equation (4.24) into the constraint equation (3.16). What we can do is we can try to rewrite the constraint equation (3.16) to be the following

$$\left(V - \frac{R^2G}{2e^2}\right)' = \frac{C_0R}{e^2g^2w}\,(RG)'. \tag{4.25}$$

[7] Since we do not, at the moment, couple the theory with gravity, adding a constant potential does not change the physics.

Using $(RG)' = 2e^2v^2g^2w/S$, it can be simplified further to

$$\left(V - \frac{R^2 G}{2e^2}\right)' = 2C_0 v^2 \frac{R}{S}. \tag{4.26}$$

One can see that if $C_0 = 0$ then the left hand side of equation (4.26) must be some constant. However, if this constant is non-zero then the integral equation (4.24) can not be satisfied. Therefore the constant must be zero and indeed it is consistent with the finite energy equation (4.6). Now, if $C_0 \neq 0$ then the left hand side of equation (4.26) must be some function. Suppose we define

$$f_{C_0}(g) \equiv V - \frac{R^2 G}{2e^2} \tag{4.27}$$

is a function solely depends on g, with $f_{(C_0=0)} \equiv f_0 = 0$. To have a finite energy, using equation (4.24), this function must satisfy

$$\int_0^\infty dr\; r f_{C_0}(g(r)) = 0. \tag{4.28}$$

In general, the function f_{C_0} can only be zero or a non-trivial function satisfied (4.28) which corresponds to the $(C_0 = 0)$- or $(C_0 \neq 0)$-type Bogomol'nyi equations, respectively.

There are many solutions for f_{C_0}, in terms of variabel r, that satisfy the condition (4.28). As an example is given by the special Laguerre functions with the following integral [19]

$$\int_0^\infty dr\; r\; e^{-r} L_n(r) = 0, \qquad L_n(r) = e^r \frac{d^n}{dr^n}\left(r^n e^{-r}\right), \tag{4.29}$$

where L_n is the Laguerre functions for $n > 1$. Substituting the function f_{C_0}, in terms of r, into the constrain equation (4.26), and exploiting the Bogomol'nyi equations (3.17), it yields solution for a as follows

$$a^2 = \frac{1}{C_0 v^2} \int dr\; r^2 f'_{C_0}(r) + C_a, \tag{4.30}$$

where now $' \equiv \frac{d}{dr}$ and C_a is an integration constant. However, it is not obvious that any function of f_{C_0}, which satisfies (4.28), would be a good solution for a with boundary conditions (2.6). Finding a suitable function for $f_{C_0}(r)$, that satisfy the boundary conditions (2.6), might give us the explicit form of functions $w(g)$ and later also $G(g)$. This will be investigated further for the future work.

Using the BPS equations (3.17) and equation (4.27), the static energy (4.1) can be written as

$$E_{\text{BPS}} = 2\pi \int \left(\frac{GR}{e^2} da + \frac{v^2 wa}{S} \left(2g^2 + \frac{C_0}{w}\right) dg + \frac{f_{C_0}}{R} da \right). \tag{4.31}$$

We can neglect the last term since it gives zero contribution due to integral equation (4.28). Using $(GR)' = 2e^2v^2g^2w/S$, the first two terms can be combined, and so the static energy is simplified to[8]

$$E_{\rm BPS} = \int dQ + 2\pi C_0 \int dg \frac{v^2 a}{S} = \int dQ + 2\pi C_0 v^2 \int dr\ \frac{a^2}{r}. \tag{4.32}$$

It shows that the static energy can not be written simply in terms of function Q. One can see that the static energy of $(C_0 \neq 0)$-type vortices above has an additional term in its formula compared to the static energy of $(C_0 = 0)$-type vortices. The boundary conditions (2.6) would lead the last term of (4.32) to (logarithmic) infinity since the integrand is singular at the origin. To attain finite energy vortices, for $C_0 > 0$, then one may need to modify the boundary conditions (2.6), in particular to allow a boundary condition $a(0) = 0$, since the first term of (4.32) is always positive,

$$\int dQ = 2\pi \int dr \left(G\ R^2 + 2v^2 a^2 g^2 w\right) r > 0. \tag{4.33}$$

However, we are not in favor of changing the topology of the solutions, and thus we avoid changing the boundary conditions (2.6). On the other hand taking $C_0 < 0$ would lead to negative infinity, but one may expect that there is a positive infinity contribution coming from the first term of (4.32) that could cancel it.[9] However, there is a subtlety here. One must carefully chose the function w and the value of $C_0 < 0$ such that $g^2 w + C_0 \geq 0$ every where. Nevertheless, the finiteness and positiveness of static energy of $(C_0 \neq 0)$-type vortices demand a more profound analysis and they are beyond the scope of this article.

Energy-Momentum Tensor of the Generalized Maxwell-Higgs model is given by

$$T_{\mu\nu} = -G\ F_{\mu\alpha} F^{\alpha}_{\nu} + w\left(D_\mu\phi\overline{D_\nu\phi} + \overline{D_\mu\phi}D_\nu\phi\right) - \eta_{\mu\nu}\mathcal{L}. \tag{4.34}$$

In Cartesian coordinates, the energy-momentum tensor components are

$$T_{xx} = \frac{G}{e^2 r^2}\left(\frac{da}{dr}\right)^2 + 2v^2 w\left(\left(\frac{dg}{dr}\right)^2 \cos^2\theta + \frac{a^2 g^2}{r^2}\sin^2\theta\right) + \mathcal{L}, \tag{4.35}$$

$$T_{yy} = \frac{G}{e^2 r^2}\left(\frac{da}{dr}\right)^2 + 2v^2 w\left(\left(\frac{dg}{dr}\right)^2 \sin^2\theta + \frac{a^2 g^2}{r^2}\cos^2\theta\right) + \mathcal{L}, \tag{4.36}$$

$$T_{xy} = v^2 w \sin(2\theta)\left(\left(\frac{dg}{dr}\right)^2 - \frac{a^2 g^2}{r^2}\right). \tag{4.37}$$

It is easy to check that the $(C_0 = 0)$-type Bogomol'nyi equations leads to BPS vortices with zero shear stress, $T_{xy} = 0$, while the $(C_0 \neq 0)$-type Bogomol'nyi equations leads to BPS vortices with non-zero shear stress. In polar coordinates, the energy-momentum tensor

[8] The presence of additional term in the energy which can not be written as a total derivative, and furthermore it diverges, make it difficult to realize the supersymmetric extension of these $(C_0 \neq 0)$-type BPS vortices.

[9] Recall that the function R here depends on C_0 which may give infinity proportional to C_0 to the first term of (4.32).

components are

$$T_{rr} = \cos^2\theta \; T_{xx} + \sin^2\theta \; T_{yy} + \sin(2\theta) \; T_{xy}, \tag{4.38}$$

$$T_{\theta\theta} = r^2 \left(\sin^2\theta \; T_{xx} + \cos^2\theta \; T_{yy} - \sin(2\theta) \; T_{xy}\right), \tag{4.39}$$

$$T_{r\theta} = \frac{r}{2} \sin(2\theta) \left(T_{yy} - T_{xx}\right) + r\cos(2\theta) \; T_{xy}. \tag{4.40}$$

One can check that $T_{r\theta}$ is automatically zero, $T_{r\theta} = 0$. The pressures in polar coordinates can be explicitly written as

$$T_{rr} = v^2 w \left(\left(\frac{dg}{dr}\right)^2 - \frac{a^2 g^2}{r^2}\right) + \frac{G}{2e^2 r^2}\left(\frac{da}{dr}\right)^2 - V, \tag{4.41}$$

$$T_{\theta\theta} = r^2 \left(-v^2 w \left(\left(\frac{dg}{dr}\right)^2 - \frac{a^2 g^2}{r^2}\right) + \frac{G}{2e^2 r^2}\left(\frac{da}{dr}\right)^2 - V\right). \tag{4.42}$$

For the case of ($C_0 = 0$)-type, using BPS equations (3.25) and (3.26), the pressures are simply zero, while the ($C_0 \neq 0$)-type case, the pressures are non-zero:

$$T_{rr} = v^2 C_0 \frac{a^2}{r^2} - f_{C_0}, \tag{4.43}$$

$$T_{\theta\theta} = -r^2 \left(v^2 C_0 \frac{a^2}{r^2} + f_{C_0}\right). \tag{4.44}$$

Moreover we define the *average pressure*

$$\mathcal{P} = \frac{\mathcal{P}_x + \mathcal{P}_y}{2} \equiv \frac{T_{xx} + T_{yy}}{2} = \frac{G}{2e^2 r^2}\left(\frac{da}{dr}\right)^2 - V. \tag{4.45}$$

In what follows, we implement the "stability condition" [2], $\mathcal{P}_x = \mathcal{P}_y = 0$, from which we get

$$\mathcal{P}_x - \mathcal{P}_y = 0 \to \frac{dg}{dr} = \pm \frac{ag}{r}, \tag{4.46}$$

$$\mathcal{P}_x + \mathcal{P}_y = 0 \to B \equiv \frac{1}{e\,r}\frac{da}{dr} = \pm\sqrt{\frac{2V}{G}}, \tag{4.47}$$

i.e., the BPS equations of the general model (2.1). Here, once $\mathcal{P}_x = \mathcal{P}_y = 0$, one concludes that also $\mathcal{P} = 0$, see (4.45). One can also simply extract these BPS equations by imposing the "stability condition" in polar coordinates, $T_{rr} = T_{\theta\theta} = 0$. Suppose we do not take the "stability condition" in the first place and rewrite the *average pressure* as

$$\mathcal{P} = -f_{C_0}(r). \tag{4.48}$$

As we mentioned before, the function f_{C_0} is related to the parameter C_0 through the equation (4.26). Thus we may conclude that this "stability condition", $f_0 = 0$, corresponds to the ($C_0 = 0$)-type Bogomol'nyi equations. On the other hand the non-trivial function of f_{C_0} satisfied (4.28), or non-zero *average pressure* $\mathcal{P}$, corresponds to the new ($C_0 \neq 0$)-type Bogomol'nyi equations.

5 Summary

The main purpose of this article is to show how the *on-shell* method, developed in [1], can be used to find the Bogomol'nyi equations of the generalized Maxwell-Higgs theory in three-dimensional spacetime [2]. In particular, we improved the *on-shell* method to allow the terms in the equations of motion, that would later be identified as the constraint equations, to depend on derivative of the fields. The improvement is necessary to tackle a particular type of theory such as the one considered in this article. This might opens some possibilities to further improvements and modifications of the *on-shell* method in obtaining the Bogomol'nyi equations of the other non-standard theories.

In the case of the generalized Maxwell-Higgs theory considered here, we found that the Bogomol'nyi equations can be classified into two types which are parametrized by a constant C_0. The first type is for $C_0 = 0$ in which we obtained the standard Bogomol'nyi equations as in [2, 11]. An advantage of using the *on-shell* method is that we obtained the constraint equation (3.23) that can be applied for the case of zero potential. Although it turns out that the resulting energy is infinite, we were able to show that the static energy could be finite by adding an appropriate non-zero constant to the potential. We also discussed possibilities for the existence of vortices with the energy is equal to the vacuum. From what we know, this has not been discussed in the literature so far and it might be interesting to study the physical properties of this vortices compared to the vacuum.

The second new type Bogomol'nyi equations, that we found here, is when we take $C_0 \neq 0$. These equations are relatively new, although they can be obtained non-trivially using the standard *off-shell*, or Hamiltonian, method or sometimes called Bogomol'nyi's trick. It turned out that these equations are related to the difference between the energy density of potential term of the scalar field and kinetic term of the gauge field which is given by a non-trivial function f_{C_0}. If the function f_{C_0} is a constant then the requirement for finite energy vortex forces this constant to be zero, $f_0 = 0$, and hence gives us back the Bogomol'nyi equations of the first type, $C_0 = 0$. The requirement for finite energy vortex on the Bogomol'nyi equations of the second type, $C_0 \neq 0$, restrict further the non-trivial function f_{C_0} such that its integral over whole two-dimensional space is zero, see (4.28). Here, we do not attempt to find the explicit expressions of the Bogomol'nyi equations of the second type since they will be discussed in the future work. However, we were able to show the solution for a in terms of f_{C_0} given by equation (4.30). It was also shown that the finite energy of the ($C_0 \neq 0$)-type vortices, without changing the topology of solutions near the boundary, can only be achieved for $C_0 < 0$ and requires some fine tunning on parameters of the theory in order for the energy (4.32) to be finite. This rises a doubt whether the finite energy configurations do exist or not. To resolve this, one may try to couple the theory with gravity as in the case of BPS Skyrmions [21] in order for the energy to be finite.

A new feature of the ($C_0 \neq 0$)-type Bogomol'nyi equations is that they correspond to BPS vortices with non-zero pressures, and even non-zero shear stress. It would be interesting to study the thermodynamics properties of these non-BPS vortices.[10] With

[10]The thermodynamical properties of Skyrmion have been studied in [20] for $(3+1)$-dimension theory.

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non-zero energy- and pressures-density, one can study further about the equation of state in the generalized Maxwell-Higss model and this may have important applications in physics. As an example is the applications of BPS Skyrmion in neutron star and QCD [21, 22].[11]

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