Published for SISSA by Springer

Received: November 3, 2015
Accepted: January 21, 2016
Published: February 8, 2016

Heterotic string on the CHL orbifold of K3

Shouvik Datta,[a,b] Justin R. David[b] and Dieter Lüst[c,d]

[a] *Institut für Theoretische Physik, ETH Zürich, Wolfgang Pauli Strasse, CH-8093 Zürich, Switzerland*
[b] *Centre for High Energy Physics, Indian Institute of Science, C.V. Raman Avenue, Bangalore 560012, India*
[c] *Arnold Sommerfeld Center for Theoretical Physics, Theresienstrasse 37, 80333 München, Germany*
[d] *Max-Planck-Institut für Physik, Föhringer Ring 6, 80805 München, Germany*

E-mail: shouvik@itp.phys.ethz.ch, justin@cts.iisc.ernet.in, dieter.luest@lmu.de

Abstract: We study $\mathcal{N} = 2$ compactifications of heterotic string theory on the CHL orbifold $(K3 \times T^2)/\mathbb{Z}_N$ with $N = 2, 3, 5, 7$. $\mathbb{Z}_N$ acts as an automorphism on $K3$ together with a shift of $1/N$ along one of the circles of T^2. These compactifications generalize the example of the heterotic string on $K3 \times T^2$ studied in the context of dualities in $\mathcal{N} = 2$ string theories. We evaluate the new supersymmetric index for these theories and show that their expansion can be written in terms of the McKay-Thompson series associated with the $\mathbb{Z}_N$ automorphism embedded in the Mathieu group M_{24}. We then evaluate the difference in one-loop threshold corrections to the non-Abelian gauge couplings with Wilson lines and show that their moduli dependence is captured by Siegel modular forms related to dyon partition functions of $\mathcal{N} = 4$ string theories.

Keywords: Superstrings and Heterotic Strings, Conformal Field Models in String Theory, Superstring Vacua

ArXiv ePrint: 1510.05425

doi:10.1007/JHEP02(2016)056

JHEP02(2016)056

Contents

1 Introduction

$\mathcal{N} = 2$ compactifications of heterotic string theory have proved to be good testing ground to explore duality symmetries of string theory. One of the main motivations to explore these compactifications is that these vacua have dual realization in terms of type II compactifications on Calabi-Yau. Identifying dual pairs on the heterotic and type II side enables highly non-trivial tests of dualities with $\mathcal{N} = 2$ symmetry [1]. The simplest example of such theories is the heterotic string theory compactified on $K3 \times T^2$. This theory was first constructed in $d = 6$ in [2, 3]. An important observable for the test of duality in this theory is the dependence of the one-loop corrections of gauge and gravitational coupling constants on the vector multiplet moduli of the theory. The moduli dependence of these threshold corrections are encoded in automorphic forms of the heterotic duality group [4–9].

Our goal in this paper is to first consider more general compactifications of the heterotic string on $(K3 \times T^2)/\mathbb{Z}_N$, with $N = 2, 3, 5, 7$. $\mathbb{Z}_N$ acts by a $1/N$ shift on one of the circles of T^2 together with an action on the internal CFT describing the heterotic string theory on $K3$. This freely acting orbifold of $K3 \times T^2$ was first studied on the type II side first as duals of CHL compactifications [10, 11] of the heterotic string [12–14]. We will call this orbifold, the CHL orbifold of $K3$. These compactifications of the heterotic string on the CHL orbifold of $K3$ preserve $\mathcal{N}=2$ supersymmetry and the number of vector multiplets, but reduce the the number of charged and un-charged hypermultiplets in the theory. They also affect the vector multiplet moduli dependence of the one-loop corrections. The two main aspects of these compactifications we study in this paper are the new supersymmetric index and the gauge threshold corrections. We summarize the results obtained in the next few paragraphs.

The basic quantity from which one-loop thresholds of heterotic string on $K3\times T^2$ are obtained is the new supersymmetric index [7, 9, 15–18] which is defined as

$$\mathcal{Z}_{\rm new}(q,\bar{q}) = \frac{1}{\eta^2(\tau)}{\rm Tr}_R\left(Fe^{i\pi F}q^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{\bar{c}}{24}}\right). \tag{1.1}$$

The trace in the above expression is taken over the Ramond sector in the internal CFT with central charges $(c,\bar{c})=(22,9)$. Here F is the world sheet fermion number of the right moving $\mathcal{N}=2$ supersymmetric internal CFT. For the standard embedding of the spin connection into a SU(2) of one of the E_8's of the heterotic string, it was shown [7, 9] that this index decomposes as

$$\begin{aligned}\mathcal{Z}_{\rm new}(q,\bar{q}) &= \frac{8}{\eta^{12}}\Gamma_{2,2}(q,\bar{q})E_4(q)\times\frac{E_6(q)}{\eta^{12}}, &(1.2)\\ &= \frac{8}{\eta^{12}}\Gamma_{2,2}(q,\bar{q})E_4(q)\left[\frac{\theta_2(\tau)^6}{\eta(\tau)^6}Z_{K3}(q,-1)+q^{\frac{1}{4}}\frac{\theta_3(\tau)^6}{\eta(\tau)^6}Z_{K3}(q,-q^{1/2})\right.\\ &\qquad\left.-q^{\frac{1}{4}}\frac{\theta_4(\tau)^6}{\eta(\tau)^6}Z_{K3}(q,q^{1/2})\right]. &(1.3)\end{aligned}$$

Here E_4,E_6 refer to Eisenstein series of weight $4,6$ respectively, $Z_{K3}(q,z)$ is the elliptic genus of the $\mathcal{N}=4$ conformal field theory of $K3$ and

$$\frac{\Gamma_{10,2}}{\eta^{10}} = \frac{1}{\eta^{10}}\Gamma_{2,2}(q,\bar{q})E_4(q), \tag{1.4}$$

is the partition function for the second E_8 lattice along with the lattice from T^2. In [19], it was shown that due to the factorization of the new supersymmetric index as given in second equation of (1.3), the BPS states of the heterotic compactifications on $K3\times T^2$ have a decomposition in terms of representation of the Mathieu group M_{24}. We will evaluate the new supersymmetric index for heterotic compactifications of the CHL orbifolds of $K3$ and show that new supersymmetric index is given by the same form as in (1.3) but now with $Z_{K3}(q,z)$ replaced by the twisted elliptic genus of the CHL orbifolds of $K3$. We will evaluate the new supersymmetric index explicitly for the $N=2$ CHL orbifold $(K3\times T^2)/\mathbb{Z}_2$ and then generalize this for the other values of N using results of [20]. We then generalize the observation of [19] and show that the BPS states for heterotic compactifications of the CHL orbifolds of $K3$ have a decomposition in terms of representations of the Mathieu group M_{24}.

Threshold corrections are important observables in string compactifications and there has been a recent revival in studying properties of these observables mainly due to the work of [21–24]. Let us examine the threshold corrections evaluated in $K3\times T^2$ compactifications which we will generalize in this work to CHL orbifolds of $K3$. For concreteness consider the standard embedding in which the spin connection connection of $K3$ is equated to the gauge connection. Starting from the $E_8 \times E_8$ theory compactifying on $K3 \times T^2$ at generic points of the moduli space of T^2 results in $E_7 \times E_8 \times \mathrm{U}(1)^4$. Let the E_8 which is broken to E_7 be referred to G' and the second E_8 be called as G. Let $\Delta_{G'}(T,U,V)$ and $\Delta_G(T,U,V)$ be the corresponding one-loop corrections to gauge coupling corrections. T, U refer to the Kähler and complex structure moduli of the torus T^2 and V is the Wilson line modulus in T^2. Then it was shown [25] that the difference in the thresholds is given by

$$\Delta_{G'}(T,U,V) - \Delta_G(T,U,V) = -48 \log \left[(\det \mathrm{Im}\Omega)^{10} \left| \Phi_{10}(T,U,V) \right|^2 \right], \tag{1.5}$$

where

$$\Omega = \begin{pmatrix} U & V \\ V & T \end{pmatrix}, \tag{1.6}$$

and $\Phi_{10}(T,U,V)$ is the unique cusp form of weight 10 transforming under the duality group $\mathrm{Sp}(2,\mathbb{Z}) \simeq \mathrm{SO}(3,2,\mathbb{Z})$. In [25], it was also shown that this difference in thresholds was independent of the way $K3$ was realized and is also holds for non-standard embeddings. In this paper, we evaluate the difference for heterotic compactifications on CHL orbifolds of $K3$ and show that the difference in the threshold corrections for the two gauge groups G, G' is given by

$$\Delta(G,T,U,V) - \Delta(G',T,U,V) = -48 \log \left[(\det \mathrm{Im}\Omega)^{k} \left| \Phi_{k}(T,U,V) \right|^2 \right], \tag{1.7}$$

where Ω^k is a weight k modular form transforming under subgroups of $\mathrm{Sp}(2,\mathbb{Z})$ with k

$$k = \frac{24}{N+1} - 2, \tag{1.8}$$

where $N = 2, 3, 5, 7$ labels the various CHL orbifolds. This generalizes the observation in [25]. Thus the gauge threshold corrections are automorphic forms under sub-groups of the duality group of the parent un-orbifolded theory.

The cusp form Φ_{10} also makes its appearance in partition function of dyons in heterotic on T^6, a theory which has $\mathcal{N} = 4$ supersymmetry [26–29].[1] This theory is related to type II on $K3 \times T^2$ by string-string duality. In [20, 31, 32], it was shown that the partition function of dyons for the CHL orbifolds of the heterotic preserving $\mathcal{N} = 4$ supersymmetry are captured by Siegel modular forms of weight k transforming under subgroups of $\mathrm{Sp}(2,\mathbb{Z})$ with k given by (1.8) for the various CHL orbifolds of the heterotic theory. These theories are related to type II on the CHL orbifold of $K3$ which has $\mathcal{N} = 4$ supersymmetry. We show that the modular forms Φ_k obtained for the difference of the thresholds in (1.7) are related by a $\mathrm{Sp}(2,\mathbb{Z})$ transformation to the dyon partition function in CHL orbifolds. The relationship between the difference in the thresholds of the non-abelian gauge groups of the $\mathcal{N} = 2$ heterotic compactification to the dyon partition functions in the $\mathcal{N} = 4$ heterotic is certainly interesting and worth exploring further. We will comment on this relation in section 6.

[1] It was recently shown that certain BPS saturated amplitude in type II on $K3 \times T^2$ also depends on Φ_{10} [30].

This paper is organized as follows. In section 2, we discuss the spectrum of heterotic compactifications on the CHL orbifold $(K3\times T^2)/\mathbb{Z}_N$ and show that the orbifold preserves the number of vectors but reduces the number of hypers. In section 3, we evaluate the new supersymmetric index for compactifications on the CHL orbifold of $K3$. We will discuss the case of $N=2$ in detail for which we realize $K3$ as a $\mathbb{Z}_2$ orbifold. We then generalize the results for the other values of N. In section 4, we show that the the new supersymmetric index for these orbifolds contains representations of the Mathieu group M_{24}. In section 5, we evaluate the difference in the gauge corrections between the groups G and G' and show that it is captured by a modular form Φ_k transforming under subgroups of $\mathrm{Sp}(2,\mathbb{Z})$. Section 6 contains our conclusions and discussions. Appendix A contains various identities involving modular forms used to obtain our results. Appendix B contains details regarding lattice sums and finally appendix C has the details of the calculations for the $\mathbb{Z}_2$ CHL orbifold of $K3$.

2 Spectrum of heterotic on CHL orbifolds of $K3$

In this section we derive the spectrum on $(K3\times T^2)/\mathbb{Z}_N$ compactifications. Before we go ahead, let us recall how these manifolds are constructed. The non-zero hodge numbers of $K3$ are given by

$$h_{(0,0)}=h_{(2,2)}=h_{(0,2)}=h_{(2,0)}=1, \qquad h_{(1,1)}=20. \tag{2.1}$$

The Hodge numbers of T^2 are given by

$$h'_{00}=h'_{(1,0)}=h'_{(0,1)}=h'_{(1,1)}=1. \tag{2.2}$$

To ensure $\mathcal{N}=2$ supersymmetry we need to preserve SU(2) holonomy. This implies that the $\mathbb{Z}_N$ acts freely [12]. The orbifold action must also preserve the holomorphic 2-forms on $K3$ and the holomorphic 1-form on T^2. It is known that the $\mathbb{Z}_N$ symmetry action on $K3$ always involves fixed points on $K3$ [33], therefore it should freely act on T^2. This action is just a shift by a unit $1/N$ on one of the circles of T^2. Since the orbifold action involves both $K3$ and T^2 the compactifications on the CHL orbifold of $K3$ can not be thought of as obtained from a $\mathcal{N}=1$ vacuum in $d=6$. Thus $(0,0)$ and $(2,2)$ form are just the scalar form and the volume form on $K3$ which are preserved under the action of $\mathbb{Z}_N$. Also the $1/N$ shift on the circle does not project out any of the forms on T^2. Thus the orbifold acts only on the $(1,1)$-forms of $K3$. The number of such forms on $K3$ which are invariant are given by $2k$ with [20]

$$h_{(1,1)}=2k, \qquad k=\frac{24}{N+1}-2, \qquad \text{for } N=2,3,5,7. \tag{2.3}$$

Among the $(1,1)$ forms which are not projected out is the Kähler form $g_{k\bar{l}}$. The Kähler form, the $(0,2)$ and $(2,0)$ forms are self dual while the $2k-1$ forms are anti-self dual. Thus the Euler number of the orbifold along the $K3$ directions reduces to $2k+4$. This information of the CHL orbifold $(K3\times T^2)/\mathbb{Z}_N$ is sufficient to obtain the spectrum of massless modes in $d=4$. We generalize the method developed in [3] for $K3$ compactification of the heterotic string. We will first discuss the states arising from compactifying the $d=10$ graviton multiplet and then we will examine the spectrum from the $d=10$ Yang-Mills multiplet.

Universal sector

We call the spectrum from the $d = 10$ graviton multiplet the universal sector. This multiplet consists of the following fields

$$R(10) = \{G_{MN}, \Psi_M^{(-)}, B_{MN}, \Psi^{(+)}, \varphi\}. \tag{2.4}$$

Here G_{MN} is the graviton, $\Psi^{(-1)}$ is a negative-chirality Majorana-Weyl gravitino, B_{MN} the anti-symmetric tensor and $\Psi^{(+)}$ is a positive-chirality Majorana-Weyl spinor. On dimensional reduction these fields should organize themselves to a $\mathcal{N} = 2$ graviton multiplet, vector multiplets and hypermultiplets in $d = 4$. The field content of these multiplets are given by

$$\begin{aligned} R(4) &= \{g_{\mu\nu}, \psi_\mu^i, a_\mu\}, & i &= 1, 2, \\ V(4) &= \{A_\mu, \psi'^i, \phi^i\}, \\ H(4) &= \{\chi^i, \varphi^a\}, & a &= 1, \cdots 4. \end{aligned} \tag{2.5}$$

The $\mathcal{N} = 2$ graviton multiplet in $d = 4$ consists of a graviton $g_{\mu\nu}$, two Majorana gravitinos ψ_μ^i, $i = 1, 2$, and the graviphoton a_μ. The vector multiplet consists of the gauge field A_μ, two Majorana spinors ψ'^i and two real scalars ϕ^i. The hypermultiplet consists of two Majorana spinors χ^i and 4 real scalars φ^a with $a = 1 \cdots 4$. We will label the 4 non-compact direction by $\mu, \nu \in \{0, 1, 2, 3\}$. The directions of the T^2 by $r, s \in \{4, 5\}$ and the directions of the K3 by $m, n \in \{6, 7, 8, 9\}$.

Let us first examine the bosonic fields under dimensional reduction. The $d = 10$ graviton reduces as $G_{\mu\nu} = g_{\mu\nu}(x) \otimes 1 \otimes 1$ where 1 refers to the constant scalar form on $(K3 \times T^2)/\mathbb{Z}_N$. There are 2 vectors from $G_{\mu r} = A_\mu(x) \otimes f_r \otimes 1$ where f_r refers to the 2 holomorphic 1-forms on T^2 which are unprojected by the orbifold. Similarly there are 2 vectors $B_{\mu r} = A_\mu(x) \otimes f_r \otimes 1$. These 4 vectors arrange themselves into the single graviton multiplet and 3 vector multiplets. Let us now count the total number of scalars, this will determine the number of hypers. There are totally 4 scalars from the following components of the metric in 10 dimensions $G_{44}, G_{55}, G_{45}, B_{45}$. Now consider the scalars arising from the metric and the anti-symmetric tensor with indices along the $K3$ directions. The anti-symmetric tensor reduces as $B_{mn} = \phi(x) \otimes 1 \otimes f_{mn}$ where f_{mn} are the harmonic 2-forms on the CHL orbifold of $K3$. This results in $2k + 2$ scalars. To obtain massless scalars from the metric we require solutions of the Lichnerowicz equation on the CHL orbifold of $K3$. These are constructed as follows, let us use $a, \bar{b} \in \{1, 2\}$ to refer to the two complex directions along the CHL orbifold of $K3$. Then the zero modes from the metric are constructed as follows [3]

$$\begin{aligned} h_{a\bar{b}} &= f'_{a\bar{b}}, \\ h_{ab} &= (\epsilon_{ac} f'_{b\bar{d}} + \epsilon_{bc} f'_{a\bar{d}}) g^{\bar{d}c}, \\ h_{\bar{a}\bar{b}} &= h^*_{ab}. \end{aligned} \tag{2.6}$$

Here $f'_{a\bar{b}}$ refer to the $2k$ harmonic $(1, 1)$-forms on the CHL orbifold of $K3$. Note that $h_{a,b}$ and $h_{\bar{a}\bar{b}}$ vanish when $f'_{a\bar{b}}$ is the Kähler form. Therefore there are $3 \times 2k - 2$ solutions of

JHEP02(2016)056

the Lichnerowicz equation on the CHL orbifold of $K3$. This leads to $6k-2$ scalars from the dimensional reduction of the metric with indices along the CHL orbifold of $K3$. The 10 dimensional dilaton reduces as $\varphi = \varphi(x)\otimes 1\otimes 1$ to give rise to a single scalar. Finally the anti-symmetric tensor reduces as $B_{\mu\nu} = b_{\mu\nu}(x)\times 1\times 1$, but a anti-symmetric tensor in $d=4$ is equivalent to a scalar by hodge-duality. Adding all the scalars we get $8k+6$ scalars. Among these 6 scalars are needed to complete the 3 vector muliplets. The rest of the scalars arrange themselves in to $2k$ hyper multiplets. To summarize we have the following dimensional reduction of the graviton multiplet in $d=10$.

$$R(10) \to R(4) + 3V(4) + 2kH(4)\,. \tag{2.7}$$

To complete the analysis let us verify that the fermions also arrange themselves into these multiplets. Before we go ahead we need to recall some facts about index theory. There is a one to one correspondence of solution of the massless Dirac equation on a 4 dimensional complex manifold and the number of harmonic $(0,p)$ forms [34, 35]. The $(0,0)$ form and a $(0,2)$ form on the CHL orbifold of $K3$ results in two real Dirac zero modes which have negative internal chirality [3]. Let us call these spinors Ω and ω. Consider the gravitino in $d=10$ it reduces to a Rarita-Schwinger field in $d=4$ as the following 4 real gravitinos

$$\begin{aligned}
\Psi^{(-)}_\mu &= \psi^{(+)1}_\mu(x)\otimes\xi^{(+)}\otimes\Omega^{(-)},\\
\Psi^{(-)}_\mu &= \psi^{(-)1}_\mu(x)\otimes\xi^{(-)}\otimes\Omega^{(-)},\\
\Psi^{(-)}_\mu &= \psi^{(+)2}_\mu(x)\otimes\xi^{(+)}\otimes\omega^{(-)},\\
\Psi^{(-)}_\mu &= \psi^{(-)2}_\mu(x)\otimes\xi^{(+)}\otimes\omega^{(-)},
\end{aligned} \tag{2.8}$$

where $\xi^{(\pm)}$ are the constant spinors on T^2. The superscripts refer to the chirality. These 4 real spinors organize themselves as 2 Majorana Rarita-Schwinger fields ψ^i_μ in $d=4$. These form the superpartners in the graviton multiplet $R(4)$. Now consider again the gravitino in 10 dimensions and reduce it with the vector index along the T^2 directions, these result in spinors in $d=4$. Using the similar reduction as in (2.8) we can conclude that there are $2\times 2=4$ Majorana spinors in $d=4$. Finally reduce the $d=10$ spinor $\Psi^{(+)}$ again on similar lines as in (2.8) and we obtain 2 Majorana spinors in $d=4$. Thus totally we have 6 Majorana spinors which form the superpartners of the 3 vectors multiplets. Now let us move to the situation when the gravitino has indices along the CHL orbifold of $K3$. Now given a harmonic $(1,1)$ form we can construct the following solutions to the Rarita-Schwinger equations on the CHL orbifold of $K3$ [3].

$$\zeta_a = f'_{a\bar b}\Gamma^{\bar b}\Omega^{(-)}, \qquad \zeta_{\bar b} = f'_{a\bar b}\Gamma^{a}\omega^{(-)}. \tag{2.9}$$

Here Γ's are the internal γ-matrices and f' refer to the $2k$ $(1,1)$ forms. Again by reducing the $d=10$ gravitinos with a similar construction as in (2.8) but with the vector indices of the gravitino along the CHL orbifold of $K3$ we obtain $2\times 2k = 4k$ Majorana spinors in $d=4$ which form the fermionic content in the $2k$ hyper multiplets. This completes the analysis of the dimensional reduction of the graviton multiplet in 10 dimensions which results in the fields given in (2.7). Thus we see that it is only the number of hypers in the universal sector which is sensitive to the orbifolding.

Gauge sector

Now let us examine the spectrum that arise from dimensional reduction of the Yang-Mills multiplet in $d = 10$. The field content of this multiplet is given by

$$Y(10) = \{A_M, \Lambda^{(-)}\}. \tag{2.10}$$

The negative chirality Majorana fermions as well as the gauge bosons are in the adjoint representation of $E_8 \otimes E_8$ transforming as $(\mathbf{248}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{248})$. This multiplet must decompose to $\mathcal{N} = 2$ vectors and hypers in $d = 4$. To obtain the number of vectors and hypers we will use index theory to find the number of zero modes of fermions in the CHL orbifold of $K3$. To preserve supersymmetry in $d = 4$ the spin connection must be set to equal to the gauge connection. Let us consider the standard embedding in which the we take an SU(2) out of the first E_8 and set it equal to the spin connection on the CHL orbifold of $K3$. As mentioned earlier the SU(2) holonomy of the spin connection is preserved by the orbifolding procedure. This procedure breaks the E_8 to a subgroup, let us consider the maximal subgroup $E_7 \otimes \mathrm{SU}(2)$, in which the SU(2) of the gauge connection is set equal to the SU(2) spin connection. Under the maximal subgroup $E_7 \otimes \mathrm{SU}(2) \otimes E_8$, the Yang-Mills multiplet decomposes as follows.

$$(\mathbf{248}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{248}) = (\mathbf{133}, \mathbf{1}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{3}, \mathbf{1}) \oplus (\mathbf{56}, \mathbf{2}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{1}, \mathbf{248}). \tag{2.11}$$

On the left hand side of the above equation we have kept track of the quantum numbers of $E_7, \mathrm{SU}(2)$ and the second E_8. Dimensional reduction of the $d = 10$ gauge bosons in the $(\mathbf{133}, \mathbf{1}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{1}, \mathbf{248})$ representation to $d = 4$ gives rise to gauge bosons in the $(\mathbf{133}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{248})$ representation of $E_7 \otimes E_8$. The corresponding scalars in these vector multiplets also arise in the dimensional reduction from the $d = 10$ gauge bosons with vector indices along the T^2 directions. Now the fermionic super partners of these fields in the vector multiplets arise as follows. Consider the fermions of Yang-Mills multiplet in $d = 10$ in the representation $(\mathbf{133}, \mathbf{1}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{1}, \mathbf{248})$, they are uncharged respect to the SU(2) and therefore behave conventionally. That is for these fermions, we can use the two spin 1/2 zero modes on the CHL orbifold of $K3$ of negative chirality denoted by Ω, ω earlier to to construct two Majorana fermions in $d = 4$ in the same representations. These are the fermionic partners in the vector multiplets. Let us state the existence of the two spin 1/2 zeros modes as an index theorem. Essentially we have

$$I_{\gamma \cdot \nabla} = n_{1/2}^{(-1)} - n_{1/2}^{(+1)} = \frac{1}{(2k+4)(8\pi^2)} \int \mathrm{Tr}(R \wedge R) = 2. \tag{2.12}$$

Note that, we have normalized the integral by the Euler number of the CHL orbifold and the integral is performed over the orbifold. $n_{1/2}^{(\pm 1)}$ counts the number of massless spin 1/2 zero modes of the appropriate chirality.

Let us examine the fermions which are charged under the SU(2) in the decomposition (2.11). Since the corresponding gauge connection is identified to be the spin connection, these fermions must arrange themselves into $\mathcal{N} = 2$ hypers. First consider the fermions which transform non-trivially under the SU(2). To obtain the number of fermions in $d=4$ we need to use the index theorem of the Dirac operator on the of the CHL orbifold of $K3$. Since these fermions are charged under the SU(2) we need the expression for the twisted index, which is given by [36]

$$\begin{aligned} I^{\mathbf{r}}_{\gamma\cdot\nabla} &= n^{(-1)}_{1/2}(\mathbf{r}) - n^{(+1)}_{1/2}(\mathbf{r}), \\ &= \frac{1}{8\pi^2}\int\left(\frac{r}{(2k+4)}\mathrm{Tr}(R\wedge R) - \mathrm{Tr}_{\mathbf{r}}(F\wedge F)\right). \end{aligned} \tag{2.13}$$

We label the representation of the fermions by its dimension, this is denoted by $\mathbf{r}$ and the dimension of this representation is denoted by r. Note that just as in (2.12), we have normalized the integral of the curvature term by the Euler number of the CHL orbifold of $K3$. For $k=10$, the expression reduces to that for $K3$. Setting the gauge connection equal to the spin connection we obtain

$$\mathrm{Tr}_{\mathbf{2}}(F\wedge F) = \frac{1}{2}\mathrm{Tr}(R\wedge R)\,. \tag{2.14}$$

The $1/2$ is because the trace in the $\mathrm{Tr}(R\wedge R)$ is taken in the $\mathbf{4}$ of SU(4) which are two doublets of SU(2). Now one can relate the trace in representation $\mathbf{r}$ to the trace in the doublet by

$$\mathrm{Tr}_{\mathbf{r}}(F\wedge F) = \frac{1}{6}r(r^2-1)\mathrm{Tr}_{\mathbf{2}}(F\wedge F)\,. \tag{2.15}$$

Substituting this relation in (2.13) and using the last equality in (2.12) we obtain

$$n^{(-1)}_{1/2}(\mathbf{r}) - n^{(+1)}_{1/2}(\mathbf{r}) = 2r - \frac{1}{3}(k+2)r(r^2-1)\,. \tag{2.16}$$

Note that for the singlet $r=1$, the expression shows that there exist two negative chirality modes which was known by explicit construction as the spinors $\Omega^{(-1)}, \omega^{(-1)}$. Now each pair of spin 1/2 zero modes given by the index (2.16) gives rise to a pair of Majorana fermions in $d=4$ which form the fermions in a single hypermultiplet. Thus the number of hypers in the representation $\mathbf{r}$ of SU(2) in $d=4$ from the gauge sector is given by

$$N^{\mathbf{r}}_H = \frac{1}{6}(k+2)r(r^2-1) - r\,. \tag{2.17}$$

Note that this is always an integer. Let us apply this formula to the fermions which transform non-trivially under SU(2). Consider the doublets transforming as $(\mathbf{56},\mathbf{2},\mathbf{1})$. Using (2.17) we can conclude that there are k charged hypers in the $(\mathbf{56},\mathbf{1})$ representation of $E_7\times E_8$. Similarly consider the triplets $(\mathbf{1},\mathbf{3},\mathbf{1})$ which lead to $4(k+2)-3$ hypers uncharged under the gauge group. From the above discussion we see that the Yang-Mills multiplet in $d=10$ results in the following multiplets in $d=4$

$$\begin{aligned} Y(10) \to\ & V(4)[(\mathbf{133},\mathbf{1}) + (\mathbf{1},\mathbf{248})] \\ & + H(4)[k(\mathbf{56},\mathbf{1}) + (4(k+2)-3)(\mathbf{1},\mathbf{1})]\,. \end{aligned} \tag{2.18}$$

Here we have also indicated the representations of $E_7\otimes E_8$. As a simple check note that for $K3$ we have $k=10$ which results in the well known 10 charged hypers and 65 uncharged hypers [1]. The complete spectrum in $d = 4$ is given by

$$\begin{aligned} R(10) + Y(10) \to R(4) &+ V(4)[3(\mathbf{1},\mathbf{1}) + (\mathbf{133},\mathbf{1}) + (\mathbf{1},\mathbf{128})] \\ &+ H(4)[k(\mathbf{56},\mathbf{1}) + (6k+5)(\mathbf{1},\mathbf{1})]\,. \end{aligned} \tag{2.19}$$

Thus, compactifications on the CHL orbifold of $K3$ change the number of the hypers. It is important to note that these orbifolds involve the shift on S^1 together with the automorphism in $K3$ which reduces the number of $(1,1)$ forms. Therefore, they cannot be thought of as a four manifold which implies this compactification cannot be lifted to 6 dimensions. Thus, the difference in the number of hypers and vectors is not constrained by anomaly cancellation in $d = 6$.

Let us now discuss the generic spectrum of these models. The generic spectrum is labeled by the number of uncharged hypers M and number of commuting U(1) denoted by N. For the embedding of SU(2) we have considered the model is given by

$$(M, N) = (6k + 5, 19). \tag{2.20}$$

We have listed this for the various (M, N) values of k corresponding to the CHL orbifold.

$$\begin{aligned} &k = 10, && (65, 19), \\ &k = 6, && (41, 19), \\ &k = 4, && (29, 19), \\ &k = 2, && (17, 19), \\ &k = 1, && (11, 19). \end{aligned} \tag{2.21}$$

For all of these models the unbroken gauge group is $E_7 \otimes E_8$. In the dual type II theory these models arise from Calabi-Yau compactifications with Hodge numbers $(h_{(1,1)}, h_{(2,1)}) = (N-1, M-1) = (18, 6k+4)$. CHL orbifolding of $K3$ just reduces the number of hypers.

Let us now consider compactifications in which a SU(n) with $n = 3, 4, 5$ of one of the E_8 is embedded in the spin connection. Doing so, breaks the E_8 to E_6, SO(10) and SU(5) respectively. The number of uncharged hypers from the gravition multiplet remains invariant and is given by $2k$. A similar analysis shows that the number of uncharged hypers from the Yang-Mills multiplet is given by the index

$$N_H(\text{singlets}) = (2k + 4)n - (n^2 - 1). \tag{2.22}$$

Note that this expression reduces to $4(k+2)-3$ for $n = 2$ as seen earlier in detail. Therefore adding the $2k$ uncharged hypers from the universal sector, the total number of uncharged hypers for these compactifications is given by $2k(n + 1) - (n^2 - 4n - 1)$. Thus the (M, N) values for these models are

$$(M, N) = (2k[n + 1] - [n^2 - 4n - 1], 21 - n). \tag{2.23}$$

Again we see that it is only the number of hypers that are affected by k. These models are the generalization of the ones considered in [1] for $k = 10$. Though the number of vectors are not affected by these compactifications, it will be clear from our analysis of the threshold corrections that the duality group under which these models are invariant are subgroups of the parent theory. For the rest of the paper we will restrict our study to the case of the standard embedding when one of the E_8 is broken to E_7. However we expect our results for the new supersymmetric index as well as the threshold corrections for the CHL orbifolds will generalise with slight modifications to other gauge groups.

3 New supersymmetric index for CHL orbifolds of $K3$

In this section we evaluate the new supersymmetric index for the CHL orbifold of $K3$. This index forms the basic ingredient for both gauge and gravitational threshold corrections for the heterotic compactifications we considered in the previous section. The new supersymmetric index is defined as[2]

$$\mathcal{Z}_{\rm new}(q,\bar{q}) = \frac{1}{\eta^2(\tau)}\mathrm{Tr}_R\left(Fe^{i\pi F}q^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{\bar{c}}{24}}\right). \tag{3.1}$$

Here, the trace is taken over the internal CFT with central charge $(c,\tilde{c}) = (22,9)$. Note that the left movers are bosonic while the right movers are supersymmetric. The right moving internal CFT has a $\mathcal{N}=2$ superconformal symmetry. It admits a U(1) current which can serve as the world sheet fermion number, we denote this as F. The subscript R refers to the fact that we take the trace in the Ramond sector for the right movers. For the $K3\times T^2$ compactifications, this index was evaluated in [7] using the $\mathbb{Z}_2$ orbifold realization of $K3$. We will first generalise this computation for the CHL orbifold $(K3\times T^2)/\mathbb{Z}_2$. Then using observations from the explicit calculations done for the $\mathbb{Z}_2$ orbifold, we will generalise and obtain the expression of the new supersymmetric index for the CHL orbifolds $(K3\times T^2)/\mathbb{Z}_N$ with $N=3,5,7$.

3.1 The $\mathbb{Z}_2$ orbifold

The $N=2$ CHL orbifold of $K3$ admits the following simple orbifold realization. First, $K3$ is realized as a $\mathbb{Z}_2$ orbifold by the action g on a torus T^4, and then, the CHL orbifold of $K3$ is obtained by the action of g' given below.

$$\begin{aligned} g: &\quad (y^4,y^5,y^6,y^7,y^8,y^9) \to (y^4,y^5,-y^6,-y^7,-y^8,-y^9), \\ g': &\quad (y^4,y^5,y^6,y^7,y^8,y^9) \to (y^4+\pi,y^5,y^6+\pi,y^7,y^8,y^9). \end{aligned} \tag{3.2}$$

Here, the directions $4,5$ label the T^2 and the $6,7,8,9$ directions are the $K3$ directions. Note that, the g' action involves as shift of π along one of the circle of T^2. This is embedded in the heterotic string by performing a shift of π along 2 of the directions of the E_8' lattice[3] i.e. there is a shift given by

$$X^I \to X^I + (\pi,\pi,0,0,0,0,0,0), \tag{3.3}$$

[2] We will use q,τ to refer to the modular parameter of the worldsheet, they are related by $q=e^{2\pi i\tau}$ and similarly $\bar{q}=e^{-2\pi i\bar{\tau}}$.

[3] The lattice in which the spin connection is embedded will be denoted by E_8' or G'.

where X^I refer to the bosonic co-ordinates of the E_8' lattice. If the action g' is not implemented the action of g together with the shift in (3.3) breaks E_8' to E_7. The presence of g' ensures the CHL orbifolding. This shift in (3.3) is coupled to the g, g' action as follows.

$$\mathcal{Z}_{\rm new}(q,\bar{q}) = \left(\frac{1}{\eta^2(\tau)} \sum_{a,b=0,1} \mathcal{Z}_{(a,b)}[E_8'; q] \times \mathcal{Z}_{(a,b)}[\mathrm{CHL}; q, \bar{q}] \right) \times \mathcal{Z}[E_8; q]. \tag{3.4}$$

Here, $\mathcal{Z}[E_8; q]$ is the partition function of the second E_8 lattice which is given by

$$\mathcal{Z}[E_8; q] = \frac{E_4}{\eta^8}. \tag{3.5}$$

The Eisenstein series, E_4, admits the following decomposition in terms of theta functions.

$$E_4 = \frac{1}{2}\left(\theta_2^8 + \theta_3^8 + \theta_4^8\right). \tag{3.6}$$

The partition function of the E_8' which involves the following shifted lattice sum.

$$\mathcal{Z}_{(a,b)}[E_8'; q] = 2\frac{1}{\eta^8} e^{-2\pi i \frac{ab}{n^2}\gamma^2} \sum_{\lambda \in \Gamma^8 + \frac{a}{2}\gamma} e^{2\pi i \frac{b}{n}\lambda\cdot\gamma} q^{\frac{1}{2}\lambda^2}. \tag{3.7}$$

The sum runs over all the lattice vectors λ of E_8. The lattice shift γ for the $\mathbb{Z}_2$ case is given by

$$\gamma = (1,1,0,0,0,0,0,0), \qquad n = 2. \tag{3.8}$$

In appendix B we have evaluated the shifted lattice sum for various values of (a, b). This result is given by

$$\begin{aligned} \mathcal{Z}_{(0,0)}[E_8'; q] &= \frac{\theta_2^8 + \theta_3^8 + \theta_4^8}{\eta^8}, & \mathcal{Z}_{(0,1)}[E_8'; q] &= \frac{\theta_3^6\theta_4^2 + \theta_4^6\theta_3^2}{\eta^8}, \\ \mathcal{Z}_{(1,0)}[E_8'; q] &= \frac{\theta_2^6\theta_3^2 + \theta_3^6\theta_2^2}{\eta^8}, & \mathcal{Z}_{(0,1)}[E_8'; q] &= -\frac{\theta_2^6\theta_4^2 - \theta_4^6\theta_2^2}{\eta^8}. \end{aligned} \tag{3.9}$$

What is now left, is to define the partition function over $(K3 \times T^2)/\mathbb{Z}_2$ referred as $\mathcal{Z}[\mathrm{CHL}; q, \bar{q}]$ in (3.4). For this we first define the lattice momenta on the T^2 which is given by

$$\begin{aligned} \frac{1}{2}p_R^2 &= \frac{1}{2T_2U_2}|-m_1U + m_2 + n_1T + n_2TU|^2, \\ \frac{1}{2}p_L^2 &= \frac{1}{2}p_R^2 + m_1n_1 + m_2n_2. \end{aligned} \tag{3.10}$$

The variables T, U refer to the complex structure and the Kähler moduli of the torus T^2. Then the partition function can be written as

$$\mathcal{Z}_{(a,b)}[\mathrm{CHL}; q, \bar{q}] = \frac{1}{\eta^2} \sum_{m_1,m_2,n_1,n_2} q^{\frac{1}{2}p_L^2} \bar{q}^{\frac{1}{2}p_R^2} \mathcal{F}_{m_1,m_2,n_1,n_2}(a, b; q), \tag{3.11}$$

where the $1/\eta^2$ factor arises due to the left moving bosonic oscillators where $\mathcal{F}_{m_1,m_2,n_1,n_2}(a,b;q)$ is independent of T,U and is given by

$$\mathcal{F}_{m_1,m_2,n_1,n_2}(a,b;q) = \frac{1}{2}\sum_{r,s=0}^{1} F_{m_1,m_2,n_1,n_2}(a,r,b,s;q), \tag{3.12}$$
$$F_{m_1,m_2,n_1,n_2}(a,r,b,s;q) = \mathrm{Tr}_{m_1,m_2,n_1,n_2;g^a,g'^r;RR}\left(g^b g'^s e^{i\pi(F^{T^4}+F^{T^2})}(F^{T^4}+F^{T^2})q^{L_0'}\bar{q}^{\bar{L}_0'}\right).$$

Here

$$L_0' = L_0 - \frac{p_L^2}{2}, \qquad \bar{L}_0' = L_0 - \frac{p_R^2}{2}. \tag{3.13}$$

The trace is taken over the subspace of Hilbert space carrying momentum (m_1,m_2) and winding (n_1,n_2). The subscripts g,g' in the trace indicates that the trace should be taken in the twisted section. The definition of $L_0',\bar{L}_0'$ ensures that the partition function $\mathcal{F}_{m_1,m_2,n_1,n_2}$ is independent of the T^2 moduli. Since the left moving bosonic oscillators on T^2 has been taken into account in (3.11), the trace does not involve these oscillators. Note that if one does not have the presence the insertions of the action of the $\mathbb{Z}_2$ element g' which is responsible for orbifolding $K3\times T^2$, the coupling of the shifts in the E_8' reduces to the coupling of $K3$ realized as a involution of T^4 by the action of g. F^{T^4} is right moving world sheet fermion number of the $(0,4)$ superconformal algebra of T^4. This U(1) is twice the U(1) of the SU(2) present in the $(0,4)$ superconformal algebra. Finally F^{T^2} is the right moving world sheet fermion number of the $(0,2)$ superconformal algebra of T^2. It can be seen that among that unless the fermionic zero modes on T^2 are saturated the trace given in the last line of (3.12) vanishes. Therefore we obtain

$$F_{m_1,m_2,n_1,n_2}(a,r,b,s;q) = \mathrm{Tr}_{m_1,m_2,n_1,n_2;g^a,g'^r;RR}\left(g^b g'^s e^{i\pi(F^{T^4}+F^{T^2})}F^{T^2}q^{L_0'}\bar{q}^{\bar{L}_0'}\right). \tag{3.14}$$

The detailed evaluation of the trace is provided in the appendix C. The result for the various sectors are given by

$$\mathcal{F}_{m_1,m_2,n_1,n_2}(0,0;q) = 0, \tag{3.15}$$
$$\mathcal{F}_{m_1,m_2,n_1,n_2}(0,1;q) = \begin{cases} -2\left(1+(-1)^{m_1}\right)\frac{\theta_3^2\theta_4^2}{\eta^4} & \text{for } \{m_1,m_2,n_1,n_2\}\in\mathbb{Z}, \\ 0 & \text{for } \{m_1,m_2,n_2\}\in\mathbb{Z},\ \ \{n_1\}\in\mathbb{Z}+\frac{1}{2}, \end{cases}$$
$$\mathcal{F}_{m_1,m_2,n_1,n_2}(1,0;q) = \begin{cases} 2\frac{\theta_2^2\theta_3^2}{\eta^4} & \text{for } \{m_1,m_2,n_1,n_2\}\in\mathbb{Z}, \\ 2\frac{\theta_2^2\theta_3^2}{\eta^4} & \text{for } \{m_1,m_2,n_2\}\in\mathbb{Z}, \{n_1\}\in\mathbb{Z}+\frac{1}{2}, \end{cases}$$
$$\mathcal{F}_{m_1,m_2,n_1,n_2}(1,1;q) = \begin{cases} -2\frac{\theta_2^2\theta_4^2}{\eta^4} & \text{for } \{m_1,m_2,n_1,n_2\}\in\mathbb{Z}, \\ -2(-1)^{m_1}\frac{\theta_2^2\theta_4^2}{\eta^4} & \text{for } \{m_1,m_2,n_2\}\in\mathbb{Z},\ \ \{n_1\}\in\mathbb{Z}+\frac{1}{2}. \end{cases}$$

The contributions in which the winding n_1 takes half integer values arise due to the twisted sectors in the element g'. The contributions proportional to $(-1)^{m_1}$ arise due to the insertions of the element g' in the trace. Note that if one ignores the contributions where n_1 takes half integer values and the ones proportional to $(-1)^{m_1}$, the result for the various sectors is proportional to that for $K3$ realized as a $\mathbb{Z}_2$ orbifold of T^2. The expressions in (3.15) can be then be substituted in (3.11) to obtain the partition function on the CHL orbifold of $K3$.

Let us now use the results in (3.9) and (3.15) to obtain the new supersymmetric index given in (3.4). Note that the dependence of the traces in (3.15) over the winding and momenta is mild. One just needs to consider the case when $n_1 \in \mathbb{Z}$ and $n_1 \in \mathbb{Z} + \frac{1}{2}$ separately. Multiplying the various sectors and summing over the sectors we obtain

$$\mathcal{Z}^{(2)}_{\text{new}}(q,\bar{q}) = \frac{2E_4}{\eta^{12}} \times \Bigg[\sum_{m_1,m_2,n_1n_2\in\mathbb{Z}} q^{\frac{p_L^2}{2}} \bar{q}^{\frac{p_R^2}{2}} \left(-2\frac{E_6}{\eta^{12}} - (-1)^{m_1} \frac{\theta_4^4\theta_3^4(\theta_4^4+\theta_3^4)}{\eta^{12}} \right) \tag{3.16}$$
$$+ \sum_{m_1,m_2,n_2\in\mathbb{Z},n_1\in\mathbb{Z}+\frac{1}{2}} q^{\frac{p_L^2}{2}} \bar{q}^{\frac{p_R^2}{2}} \left(\frac{\theta_2^4}{\eta^{12}} \left\{ \theta_3^4(\theta_2^4+\theta_3^4) + (-1)^{m_1}\theta_4^4(\theta_2^4-\theta_4^4) \right\} \right) \Bigg].$$

The superscript $^{(2)}$ refers to the fact that this is the index for the orbifold $(K3\times T^2)/\mathbb{Z}_2$. Here we have used the decomposition of E_6 in terms of θ-functions which is given by

$$2E_6 = -\theta_2^6(\theta_3^4+\theta_4^4)\theta_2^2 + \theta_3^6(\theta_4^4-\theta_2^4)\theta_3^2 + \theta_4^6(\theta_2^4+\theta_3^4)\theta_4^2. \tag{3.17}$$

Note that this is the generalization of the new supersymmetric index obtained for the standard embedding in $K3\times T^2$ compactifications given in (1.3) for which we obtain the just the term involving E_6 in the first line (3.16). The result we have in (3.16) is the expression for the new supersymmetric index for the compactifications on $(K3\times T^2)/\mathbb{Z}_2$.

We will now discuss two equivalent ways of rewriting the expression in (3.16) which are useful for the questions addressed in this paper.

Decomposition in terms of characters of D_6. From the general arguments in [7], we expect that the new supersymmetric index for $K3\times T^2$ decomposes in terms of characters of the sub-lattice D_6 of E_8'. The coefficients in this decomposition can be written in terms of the elliptic genus of the $\mathcal{N}=4$ superconformal field theory of the $d=4$ compact manifold. For $K3\times T^2$ compactifications, this decomposition of the new supersymmetric index is given in (1.3). We will show that the new supersymmetric index for the $(K3\times T^2)/\mathbb{Z}_2$ also can be decomposed in terms of characters of D_6 with coefficients as the twisted elliptic genus of $K3$. Let us first define the twisted elliptic genus for the CHL orbifolds of $K3$. Let g' be the generator of the $\mathbb{Z}_N$ action on $K3$ which results in the CHL orbifold. We define the twisted elliptic genus of $K3$ as

$$F^{(r,s)}(\tau,z) = \frac{1}{N}\mathrm{Tr}^{K3}_{RR;g'^r}\left((-1)^{F^{K3}+\bar{F}^{K3}} g'^s e^{2\pi i z F^{K3}} q^{L_0-c/24}\bar{q}^{\bar{L}_0-c/24} \right),$$
$$0 \le r, s, \le (N-1). \tag{3.18}$$

where the trace is taken in the $\mathcal{N}=4$ super conformal field theory associated with $K3$ in the g'^r twisted Ramond sector. F^{K3} and $\bar{F}^{K3}$ denote the left and right world sheet fermion number which can be written as the U(1) charges corresponding to the SU(2) R-symmetry in this theory. The twisted elliptic genus for the various CHL orbifolds were provided in [20]. The results for the $N = 2$ CHL orbifold are given by

$$F^{(0,0)}(\tau, z) = 4\left[\frac{\theta_2(\tau, z)^2}{\theta_2(\tau, 0)^2} + \frac{\theta_3(\tau, z)^2}{\theta_3(\tau, 0)^2} + \frac{\theta_4(\tau, z)^2}{\theta_4(\tau, 0)^2}\right], \tag{3.19}$$
$$F^{(0,1)}(\tau, z) = 4\frac{\theta_2(\tau, z)^2}{\theta_2(\tau, 0)^2}, \quad F^{(1,0)}(\tau, z) = 4\frac{\theta_4(\tau, z)^2}{\theta_4(\tau, 0)^2}, \quad F^{(1,0)}(\tau, z) = 4\frac{\theta_3(\tau, z)^2}{\theta_3(\tau, 0)^2}.$$

Using these expressions for the twisted elliptic genus we can see that the new supersymmetric index in (3.16) can be written as

$$\begin{aligned}
\mathcal{Z}_{\rm new}(q, \bar q)^{(2)} = \frac{2E_4}{\eta^{12}} \times \Bigg[\sum_{m_1, m_2, n_1 n_2 \in \mathbb{Z}} q^{\frac{p_L^2}{2}} \bar q^{\frac{p_R^2}{2}} \Bigg\{ \frac{\theta_2^6}{\eta^6}\left(F^{(0,0)}\left(\tau, \frac{1}{2}\right) + (-)^{m_1} F^{(0,1)}\left(\tau, \frac{1}{2}\right)\right) \\
+ q^{1/4}\frac{\theta_3^6}{\eta^6}\left(F^{(0,0)}\left(\tau, \frac{1+\tau}{2}\right) + (-)^{m_1} F^{(0,1)}\left(\tau, \frac{1+\tau}{2}\right)\right) \\
- q^{1/4}\frac{\theta_4^6}{\eta^6}\left(F^{(0,0)}\left(\tau, \frac{\tau}{2}\right) + (-)^{m_1} F^{(0,1)}\left(\tau, \frac{\tau}{2}\right)\right)\Bigg\} \\
+ \sum_{m_1, m_2, n_2 \in \mathbb{Z}, n_1 \in \mathbb{Z}+1/2} q^{\frac{p_L^2}{2}} \bar q^{\frac{p_R^2}{2}} \Bigg\{ \frac{\theta_2^6}{\eta^6}\left(F^{(1,0)}\left(\tau, \frac{1}{2}\right) + (-)^{m_1} F^{(1,1)}\left(\tau, \frac{1}{2}\right)\right) \\
+ q^{1/4}\frac{\theta_3^6}{\eta^6}\left(F^{(1,0)}\left(\tau, \frac{1+\tau}{2}\right) + (-)^{m_1} F^{(1,1)}\left(\tau, \frac{1+\tau}{2}\right)\right) \\
- q^{1/4}\frac{\theta_4^6}{\eta^6}\left(F^{(1,0)}\left(\tau, \frac{\tau}{2}\right) + (-)^{m_1} F^{(1,1)}\left(\tau, \frac{\tau}{2}\right)\right)\Bigg\}\Bigg].
\end{aligned} \tag{3.20}$$

Though the above expression is lengthy, the structure of the index is quite easy to decipher. To see this, let us list the characters of the the D_6 lattice. Consider the lattice in the fermionic representation. Then we have the following partition functions for the various sectors.

$$\mathcal{Z}(D6; NS^+; q) = \frac{\theta_3^6}{\eta^6}, \quad \mathcal{Z}(D6; NS^-, R; q) = \frac{\theta_4^6}{\eta^6}, \quad \mathcal{Z}(D6; R; q) = \frac{\theta_2^6}{\eta^6}. \tag{3.21}$$

Here NS^- refers to the Neveu-Schwarz sector with $(-1)^F$ inserted in the trace. F is the worldsheet fermion number of these left moving fermions of the D_6 lattice. R refers to the Ramond sector. From (3.20) we note that the coefficients of these D_6 partitions functions are the twisted elliptic genus of $\mathbb{Z}_2$ CHL orbifold of $K3$. The contribution of $\mathcal{Z}(D6; NS^-, R; q)$ is weighted with -1. It is important to note that the new supersymmetric index given in (3.16) was obtained by an explict calculation and it admitted a decomposition in the form given in (3.20). It is interesting that the structure seen for $K3 \times T^2$ by [7, 9] in which the elliptic genus of the internal CFT plays the role in determining the new supersymmetric index is generalized to the twisted elliptic genus for the CHL compactification.

Decomposition in terms of Eisenstein series. It is also useful to rewrite the new supersymmetric index in (3.16) in another form to obtain the gauge threshold corrections. For this, note that we have the following identities between modular forms.

$$-(\theta_3^8\theta_4^4 + \theta_4^8\theta_3^4) = -\frac{2}{3}\left(E_6 + 2\mathcal{E}_2(\tau)E_4\right), \tag{3.22}$$

$$\theta_3^8\theta_2^4+\theta_2^8\theta_3^4=-\frac{2}{3}\left(E_6-\mathcal{E}_2\left(\frac{\tau}{2}\right)E_4\right),$$
$$\theta_2^8\theta_4^4-\theta_2^8\theta_4^4=-\frac{2}{3}\left(E_6-\mathcal{E}_2\left(\frac{\tau+1}{2}\right)E_4\right),$$

where[4]

$$\mathcal{E}_N(\tau)=\frac{12i}{\pi(N-1)}\partial_\tau\log\frac{\eta(\tau)}{\eta(N\tau)}. \tag{3.23}$$

The identities in (3.22) have been verified by performing a q-expansion which is detailed in the appendix A. Substituting these identities in (3.16) we obtain the form

$$\mathcal{Z}^{(2)}_{\rm new}(q,\bar q)=-\frac{2E_4}{\eta^{12}}\times\Bigg[\sum_{m_1,m_2,n_1n_2\in\mathbb{Z}}q^{\frac{p_L^2}{2}}\bar q^{\frac{p_R^2}{2}}\frac{1}{\eta^{12}}\left\{2E_6+(-1)^{m_1}\frac{2}{3}\left(E_6+2\mathcal{E}_2(\tau)E_4\right)\right\} \tag{3.24}$$
$$+\sum_{\substack{m_1,m_2,n_2\in\mathbb{Z},\\ n_1\in\mathbb{Z}+\frac{1}{2}}}q^{\frac{p_L^2}{2}}\bar q^{\frac{p_R^2}{2}}\frac{2}{3\eta^{12}}\left\{\left(E_6-\mathcal{E}_2\left(\frac{\tau}{2}\right)E_4\right)+(-1)^{m_1}\left(E_6-\mathcal{E}_2\left(\frac{\tau+1}{2}\right)E_4\right)\right\}\Bigg].$$

It is also instructive to derive the the expression in (3.24) for the new supersymmetric index directly from from (3.20). For this we use the more general form for the twisted elliptic genus of the $N=2$ CHL orbifold of $K3$ from [20].

$$F^{(0,0)}(\tau,z)=4A(\tau,z),\qquad F^{(0,1)}(\tau,z)=\frac{4}{3}A(\tau,z)-\frac{2}{3}B(\tau,z)\mathcal{E}(\tau), \tag{3.25}$$
$$F^{(1,0)}(\tau,z)=\frac{4}{3}A(\tau,z)+\frac{1}{3}B(\tau,z)\mathcal{E}_2\left(\frac{\tau}{2}\right),\quad F^{(1,1)}(\tau,z)=\frac{4}{3}A(\tau,z)+\frac{1}{3}B(\tau,z)\mathcal{E}_2\left(\frac{\tau+1}{2}\right),$$

where

$$A(\tau,z)=\frac{\theta_2(\tau,z)^2}{\theta_2(\tau,0)^2}+\frac{\theta_3(\tau,z)^2}{\theta_3(\tau,0)^2}+\frac{\theta_4(\tau,z)^2}{\theta_4(\tau,0)^2},\qquad B(\tau,z)=\frac{\theta_1(\tau,z)^2}{\eta^6}. \tag{3.26}$$

Substituting these forms for the twisted Elliptic genus in (3.20) it is easy to see that it organizes into the form (3.24). To show this it is convenient to use the identities

$$A\left(\tau,\frac{1}{2}\right)=\frac{(\theta_4^4\theta_2^2+\theta_3^4\theta_2^2)}{4\eta^6},\qquad B\left(\tau,\frac{1}{2}\right)=\frac{\theta_2^2}{\eta^6}, \tag{3.27}$$
$$A\left(\tau,\frac{\tau}{2}\right)=\frac{q^{-1/4}\left(\theta_3^4\theta_4^2+\theta_2^4\theta_4^2\right)}{4\eta^6},\qquad B\left(\tau,\frac{\tau}{2}\right)=-\frac{q^{-1/4}\theta_4^2}{\eta^6},$$
$$A\left(\tau,\frac{\tau+1}{2}\right)=\frac{q^{-1/4}\left(-\theta_4^4\theta_3^2+\theta_2^4\theta_3^2\right)}{4\eta^6},\qquad B\left(\tau,\frac{\tau+1}{2}\right)=\frac{q^{-1/4}\theta_3^2}{\eta^6}.$$

Using these identities in (3.20) we obtain (3.24).

[4]The modular function $\mathcal{E}_N$ was introduced in [20] where it was called E_N.

Modular invariance. The new supersymmetric index has the property that $\tau_2 \mathcal{Z}_{\rm new}(\tau, \bar\tau)$ has to be an $SL(2,\mathbb{Z})$ non-holomorphic modular form of weight -2. This is essentially because it occurs in threshold integrals along with modular forms of weight 2[5] and the integrand in any threshold integral has to be modular invariant. Let us now verify that $\tau_2 \mathcal{Z}_{\rm new}$ indeed transforms as a weight -2 modular form. For this, we need the following transformation property of $\mathcal{E}_N$

$$\mathcal{E}_N(\tau+1) = \mathcal{E}_N(\tau), \qquad \mathcal{E}_N(-1/\tau) = -\tau^2 \frac{1}{N}\mathcal{E}_N(\tau/N). \tag{3.28}$$

Using this property, it is easy to see that for the special case of $N=2$ we have

$$\mathcal{E}_2\left(-\frac{1}{2\tau}\right) = -2\tau^2 \mathcal{E}_2(\tau), \qquad \mathcal{E}_2\left(-\frac{1}{2\tau}+\frac{1}{2}\right) = \tau^2 \mathcal{E}_2\left(\frac{\tau+1}{2}\right). \tag{3.29}$$

Let us define the following lattice sums over T^2

$$\begin{aligned}
\Gamma_{2,2}^{(0,0)}(\tau,\bar\tau) &= \sum_{m_1,m_2,n_1,n_2\in\mathbb{Z}} q^{\frac{p_L^2}{2}} \bar q^{\frac{p_R^2}{2}}, \\
\Gamma_{2,2}^{(0,1)}(\tau,\bar\tau) &= \sum_{m_1,m_2,n_1,n_2\in\mathbb{Z}} q^{\frac{p_L^2}{2}} \bar q^{\frac{p_R^2}{2}} (-1)^{m_1}, \\
\Gamma_{2,2}^{(1,0)}(\tau,\bar\tau) &= \sum_{\substack{m_1,m_2,n_2\in\mathbb{Z}, \\ n_1\in\mathbb{Z}+\frac{1}{2}}} q^{\frac{p_L^2}{2}} \bar q^{\frac{p_R^2}{2}}, \\
\Gamma_{2,2}^{(1,1)}(\tau,\bar\tau) &= \sum_{\substack{m_1,m_2,n_2\in\mathbb{Z}, \\ n_1\in\mathbb{Z}+\frac{1}{2}}} \tilde q^{\frac{p_L^2}{2}} \bar q^{\frac{p_R^2}{2}} (-1)^{m_1}.
\end{aligned} \tag{3.30}$$

From the expression for p_L, p_R given in (3.10) it is easy to see that under the shift $\tau \to \tau+1$, we obtain the following relations between the lattice sums

$$\begin{aligned}
\tau_2 \Gamma_{2,2}^{(0,0)}(\tau+1,\bar\tau+1) &= \tau_2 \Gamma_{2,2}^{(0,0)}(\tau,\bar\tau), \\
\tau_2 \Gamma_{2,2}^{(0,1)}(\tau+1,\bar\tau+1) &= \tau_2 \Gamma_{2,2}^{(0,1)}(\tau,\bar\tau), \\
\tau_2 \Gamma_{2,2}^{(1,0)}(\tau+1,\bar\tau+1) &= \tau_2 \Gamma_{2,2}^{(1,1)}(\tau,\bar\tau), \\
\tau_2 \Gamma_{2,2}^{(1,1)}(\tau+1,\bar\tau+1) &= \tau_2 \Gamma_{2,2}^{(1,0)}(\tau,\bar\tau).
\end{aligned} \tag{3.31}$$

Using Poisson resummation one can show that under the transformation $\tau \to -1/\tau$ the following relations hold

$$\begin{aligned}
(-1/\tau)_2\, \Gamma_{2,2}^{(0,0)}(-1/\tau,-1/\bar\tau) &= \tau_2 \Gamma_{2,2}^{(0,0)}(\tau,\bar\tau), \\
(-1/\tau)_2\, \Gamma_{2,2}^{(0,1)}(-1/\tau,-1/\bar\tau) &= \tau_2 \Gamma_{2,2}^{(1,0)}(\tau,\bar\tau), \\
(-1/\tau)_2\, \Gamma_{2,2}^{(1,0)}(-1/\tau,-1/\bar\tau) &= \tau_2 \Gamma_{2,2}^{(0,1)}(\tau,\bar\tau), \\
(-1/\tau)_2\, \Gamma_{2,2}^{(1,1)}(-1/\tau,-1/\bar\tau) &= \tau_2 \Gamma_{2,2}^{(1,1)}(\tau,\bar\tau).
\end{aligned} \tag{3.32}$$

[5] This will be seen in section 5.

Using the equations (3.28), (3.29), (3.31) and (3.32) it is easy to see that $\tau_2 \mathcal{Z}^{(2)}_{\rm new}$ where the new supersymmetric index given in the form (3.24) is a modular form of weight -2. To demonstrate this we have to also use the fact that η, E_4, E_6 are modular forms of weight $1/2, 4, 6$ respectively. This result ensures that the result for the integrand in the threshold corrections is modular invariant.

3.2 The $\mathbb{Z}_N$ orbifold

From the explicit calculation and the discussions in the earlier section for the $N = 2$ CHL orbifold of $K3$ it is easy to arrive at the expression for the new supersymmetric index for the other values of N. To write down the expression for the index it is useful to define the following

$$
\begin{aligned}
\mathcal{I}^{(r,s)}_{RR}(q,\bar q) &= \sum_{\substack{m_1,m_2,n_2\in\mathbb{Z},\\ n_1=\mathbb{Z}+\frac{r}{N}}} q^{\frac{p_L^2}{2}} \bar q^{\frac{p_R^2}{2}} e^{2\pi i m_1 s/N} F^{(r,s)}\left(\tau, \frac{1}{2}\right), \\
\mathcal{I}^{(r,s)}_{(NS+)}(q,\bar q) &= q^{1/4} \sum_{\substack{m_1,m_2,n_2\in\mathbb{Z},\\ n_1=\mathbb{Z}+\frac{r}{N}}} q^{\frac{p_L^2}{2}} \bar q^{\frac{p_R^2}{2}} e^{2\pi i m_1 s/N} F^{(r,s)}\left(\tau, \frac{\tau+1}{2}\right), \\
\mathcal{I}^{(r,s)}_{(NS-)}(q,\bar q) &= -q^{1/4} \sum_{\substack{m_1,m_2,n_2\in\mathbb{Z},\\ n_1=\mathbb{Z}+\frac{r}{N}}} q^{\frac{p_L^2}{2}} \bar q^{\frac{p_R^2}{2}} e^{2\pi i m_1 s/N} F^{(r,s)}\left(\tau, \frac{\tau}{2}\right), \\
&\qquad\qquad\qquad\qquad \text{for } 0 \le r, s, \le N-1.
\end{aligned} \tag{3.33}
$$

Here $F^{(r,s)}(\tau, z)$ is the twisted elliptic genus of the CHL orbifold of $K3$ which is given by [20]

$$
\begin{aligned}
F^{(0,0)}(\tau,z) &= \frac{8}{N} A(\tau,z), \\
F^{(0,s)}(\tau,z) &= \frac{8}{N(N+1)} A(\tau,z) - \frac{2}{N+1}\mathcal{E}_N(\tau) B(\tau,z), \quad \text{for } 1 \le s \le N-1, \\
F^{(r,rk)}(\tau,z) &= \frac{8}{N(N+1)} A(\tau,z) + \frac{2}{N(N+1)}\mathcal{E}_N\left(\frac{\tau+k}{N}\right) B(\tau,z), \\
&\qquad \text{for } \ 1 \le r \le N-1, \ 0 \le k \le N-1,
\end{aligned} \tag{3.34}
$$

where $A(\tau,z), B(\tau,z)$ are defined in (3.26). Using these definitions, the new supersymmetric index for the $\mathbb{Z}_N$ CHL orbifold of $K3$ is given by

$$
\mathcal{Z}^{(N)}_{\rm new}(q,\bar q) = \frac{2E_4}{\eta^{12}} \sum_{r,s=0}^{N-1} \left[\frac{\theta_2^6}{\eta^6}\mathcal{I}^{(r,s)}_{R} + \frac{\theta_3^6}{\eta^6}\mathcal{I}^{(r,s)}_{NS+} + \frac{\theta_4^6}{\eta^6}\mathcal{I}^{(r,s)}_{NS-} \right]. \tag{3.35}
$$

Substituting the expressions for the twisted elliptic genus from (3.34) and using the relations in (3.27) we obtain the following expression for the new supersymmetric index in terms of Eisenstein functions

$$
\mathcal{Z}^{(N)}_{\rm new}(q,\bar q) = -\frac{2E_4}{\eta^{12}} \times \tag{3.36}
$$

$$\left[\sum_{m_1,m_2,n_2n_2\in\mathbb{Z}} q^{\frac{p_L^2}{2}}\bar{q}^{\frac{p_R^2}{2}} \frac{1}{\eta^{12}} \left\{ \frac{4}{N}E_6 + \left(\sum_{s=1}^{N-1} e^{\frac{2\pi i s m_1}{N}} \right) \left(\frac{4}{N(N+1)}E_6 + \frac{4}{N+1}\mathcal{E}_N(\tau)E_4 \right) \right\} \right.$$

$$\left. + \sum_{r=1}^{N-1} \sum_{\substack{m_1,m_2,n_2\in\mathbb{Z},\\ n_1=\mathbb{Z}+\frac{r}{N}}} q^{\frac{p_L^2}{2}}\bar{q}^{\frac{p_R^2}{2}} \sum_{k=0}^{N-1} \frac{e^{\frac{2\pi i r k m_1}{N}}}{\eta^{12}} \left\{ \frac{4}{N(N+1)}E_6 - \frac{2}{N(N+1)}\mathcal{E}_N\left(\frac{\tau+k}{N}\right)E_4 \right\} \right].$$

A simple check of the above formula is that it reduces to (3.24) for the $N = 2$ case. One can re-write this expression by performing the sum over the phases wherever possible, but it is convenient to keep the expression as it is. It can be shown that $\tau_2\mathcal{Z}_{\text{new}}^{(N)}(q,\bar{q})$ is a modular form of weight -2 by generalizing the method discussed for the $N = 2$ case in detail. Therefore the structure of the new elliptic index for CHL orbifolds of $K3$ is such that the Eisenstein function E_6 which occurs for the $K3$ is modified to the form given in the curly brackets of the expression in (3.36).

Our analysis of the new supersymmetric index for heterotic compactification on the CHL orbifolds of $K3$ was restricted to the case of the standard embedding when one of the gauge groups of the heterotic is broken to E_7. However we expect our observation that the new supersymmetric index decomposes to sum over twisted elliptic genera of $K3$ will be true for other embeddings and gauge groups. For the unorbifolded case, that fact the elliptic genus of $K3$ determines the new supersymmetric index was explicitly shown by the study of various cases in [9, 25]. We expect similar results to hold for the compactifications considered in this paper and it will interesting to perform explicit checks for the various gauge groups.

4 Mathieu moonshine

From the analysis of the new supersymmetric index for CHL orbifolds of $K3$ we have seen that it is essentially determined by the twisted elliptic index of $K3$. This property is seen in the expressions (3.24) for the $N = 2$ orbifold and (3.36) for other values of N. It is known [37–40] that the twisted elliptic genus of $K3$ admits M_{24} symmetry. Therefore, it must be possible to discover the M_{24} representations in the new supersymmetric index for the CHL orbifolds of $K3$, just as it was done for the new supersymmetric index for $K3$ compactifications in [19].

Let us first recall how Mathieu moonshine — i.e. M_{24} representations — is seen in the elliptic genus of $K3$. It is given by

$$Z_{K3}(\tau,z) = 8A(\tau,z). \tag{4.1}$$

Let us decompose the elliptic genus into the elliptic genera of the short and the long representations of the $\mathcal{N} = 4$ super conformal algebra. These are given by [41]

$$\mathrm{ch}_{h=\frac{1}{4},l=0}(\tau,z) = -ie^{\pi i z}\frac{\theta_1(\tau,z)}{\eta(\tau)^3}\sum_{n=-\infty}^{\infty}\frac{1}{1-e^{2\pi i(n\tau+z)}}e^{\pi i\tau n(n+1)}e^{2\pi i(n+\frac{1}{2})}, \tag{4.2}$$

$$\mathrm{ch}_{h=n+\frac{1}{4},l=\frac{1}{2}}(\tau,z) = e^{2\pi i\tau(n-\frac{1}{8})}\frac{\theta_1(\tau,z)^2}{\eta(\tau)^2}.$$

Then we have

$$Z_{K3}(\tau,z) = 24\mathrm{ch}_{h=\frac{1}{4},l=0}(\tau,z) + \sum_{n=0}^{\infty} A_n^{(1)}\mathrm{ch}_{h=n+\frac{1}{4},l=\frac{1}{2}}(\tau,z). \tag{4.3}$$

where the first few values of $A_n^{(1)}$ are given by

$$A_n^{(1)} = -2,\ 90,\ 462,\ 1540,\ 4554,\ 11592, \dots \tag{4.4}$$

These coefficients are either the dimensions or the sums of dimensions of the irreducible representations of the group M_{24} [42]. The generalization of this observation to the twisted elliptic genus of $K3$ was done in [37, 38, 40]. Let us first discuss the $N = 2$ CHL orbifold of $K3$. Consider the twisted elliptic index

$$2F^{(0,1)}(\tau,z) = \frac{8}{3}A(\tau,z) - \frac{4}{3}B(\tau,z)\mathcal{E}_2(\tau). \tag{4.5}$$

This admits the following decomposition in terms of $\mathcal{N} = 4$ Virasoro characters

$$2F^{(0,1)}(\tau,z) = 8\mathrm{ch}_{h=\frac{1}{4},l=0}(\tau,z) + \sum_{n=0}^{\infty} A_n^{(2)}\mathrm{ch}_{h=n+\frac{1}{4},l=\frac{1}{2}}(\tau,z). \tag{4.6}$$

Where the coefficient 8 is the twisted Euler number of K which is given by

$$\chi_N = \frac{24}{N+1}, \qquad N = 2,3,5,7. \tag{4.7}$$

In (4.6) the first few values of $A_n^{(2)}$ are given by

$$A_n^{(2)} = -2,\ -6,\ 14,\ -28,\ 42,\ -56,\ 86,\ -138,\ \dots \tag{4.8}$$

These coefficients can be identified with McKay-Thompson series constructed out of trace of the element g corresponding to the $\mathbb{Z}_2$ involution of $K3$ embedded in M_{24}. From the structure of the new supersymmetric index in (3.20) and (3.24) the new supersymmetric index in the $(0,1)$ sector given by

$$G^{(2)}(q) = -\frac{4}{3}\left[\frac{E_6 + 2\mathcal{E}_2(\tau)E_4}{\eta^{12}}\right]. \tag{4.9}$$

We have multiplied by a factor of 2 to agree with the normalizations of the twisted elliptic genus of $K3$ used in [37]. Then the new supersymmetric index in the $(0,1)$ sector admits the following decomposition

$$G^{(2)}(q) = 8g_{h=\frac{1}{4},l=0}(\tau) + \sum_{n=0}^{\infty} A_n^{(2)} g_{h=n+\frac{1}{4},l=\frac{1}{2}}(\tau), \tag{4.10}$$

where

$$g_{h=\frac{1}{4},l=0}(\tau)=\frac{\theta_2^6}{\eta^6}\mathrm{ch}_{h=\frac{1}{4},l=0}\left(\tau,\frac{1}{2}\right)+q^{1/4}\frac{\theta_3^6}{\eta^6}\mathrm{ch}_{h=\frac{1}{4},l=0}\left(\tau,\frac{1+\tau}{2}\right)-q^{1/4}\frac{\theta_4^6}{\eta^6}\mathrm{ch}_{h=\frac{1}{4},l=0}\left(\tau,\frac{\tau}{2}\right),$$

$$g_{h=\frac{1}{4},l=0}(\tau)=\frac{\theta_2^6}{\eta^6}\mathrm{ch}_{h=\frac{1}{4},l=\frac{l}{2}}\left(\tau,\frac{1}{2}\right)+q^{1/4}\frac{\theta_3^6}{\eta^6}\mathrm{ch}_{h=\frac{1}{4},l=0}\left(\tau,\frac{1+\tau}{2}\right)-q^{1/4}\frac{\theta_4^6}{\eta^6}\mathrm{ch}_{h=\frac{1}{4},l=0}\left(\tau,\frac{\tau}{2}\right). \quad (4.11)$$

The g's are products of characters of D_6 and $\mathcal{N}=4$ Virasoro characters. $G^{(2)}$ given in (4.9) is the generalization of

$$G^{(1)}(q)=-2\frac{E_6}{\eta^{12}} \quad (4.12)$$

which is the new supersymmetric index for $K3$ compactifications. Substituting the expressions for g's from (4.11) into (4.10) and using (4.9) we can solve for the coefficients $A_n^{(2)}$. We have checked using Mathematica that the first 8 coefficients fall into the McKay-Thompson series for the $\mathbb{Z}_2$ involution embedded in M_{24} given in (4.8).

Let us now proceed with the analysis for other values of N. From (3.36) we see that the new supersymmetric index in the $(0,1)$ sector is given by

$$G^{(N)}(q)=\frac{-N}{\eta^{12}}\left[\frac{4}{N(N+1)}E_6+\frac{4}{N+1}\mathcal{E}_N(\tau)E_4\right]. \quad (4.13)$$

Here we have multiplied a factor of N to agree with the normalizations of the twisted elliptic genus of $K3$ in [37]. Let us write $G^{(N)}$ as

$$G^{(N)}(q)=\chi_N g_{h=\frac{1}{4},l=0}(\tau)+\sum_{n=0}^{\infty}A_n^{(N)}g_{h=n+\frac{1}{4},l=\frac{1}{2}}(\tau). \quad (4.14)$$

By equating (4.14) and (4.13) we can solve for the coefficients $A_n^{(N)}$.[6] The first few coefficients are given by

$$\begin{aligned}
A_n^{(3)} &= -2,\ 0,\ -6,\ 10,\ 0,\ -18,\ 20,\ 0,\ \ldots,\\
A_n^{(5)} &= -2,\ 0,\ 2,\ 0,\ -6,\ 2,\ 0,\ 6\,,\ldots,\\
A_n^{(7)} &= -2,\ -1,\ 0,\ 0,\ 4,\ 0,\ -2,\ 2,\ldots.
\end{aligned} \quad (4.15)$$

These are the coefficients of the McKay-Thompson series for the $\mathbb{Z}_N$ automorphism of $K3$ embedded in M_{24}.[7] The appearance of this series is expected once we have demonstrated that the new supersymmetric index can be decomposed into the twisted elliptic genus of $K3$. The explicit evaluation of the coefficients serves as a simple consistency check of our calculations. It is also presented to establish the identity in (4.14) independently which can be of use for future reference. Thus the BPS states in these compactifications have a decomposition in terms of the coefficients of the McKay-Thompson series.

As we have seen explicitly, for the $N=2$ case, the new supersymmetric index in the $(1,0)$ twisted sector is related to that of the $(0,1)$ sector by the modular transformation $\tau\to$

[6] A Mathematica routine was used for this.

[7] Compare table 1 of [37].

$-1/\tau$. This is also true for other values of N. This implies that the new supersymmetric index in these sectors must also contain the modular transformed version of the McKay-Thompson series. It will be interesting to show this explicitly. There are 26 McKay-Thompson series corresponding to the 26 conjugacy classes of M_{24}. It will be interesting to to construct and study the properties of the the new supersymmetric index corresponding to remaining classes. The twisted elliptic genera of $K3$ for each of these classes have been constructed in [37–40][8] which will be a good starting point for this study.

5 Gauge threshold corrections

In this section, we will evaluate the one-loop threshold corrections for each of the two unbroken gauge groups E_7 and E_8 as a function of the Kähler and complex structure moduli and the Wilson line modulus on T^2 for the heterotic compactifications on CHL orbifolds of $K3$. To begin we will recall the evaluation of the threshold integrals for the gauge couplings of heterotic on $K3 \times T^2$. We then proceed to generalize to the case of the $\mathbb{Z}_2$ CHL orbifold and then present the results for the $\mathbb{Z}_N$ orbifold with $N = 3, 5, 7$. We will show that the difference in the threshold integrals of the two unbroken gauge groups reduces to Siegel modular forms associated with dyon partition functions in $\mathcal{N} = 4$ string compactifications studied in [20].

5.1 Thresholds in $K3 \times T^2$

Let us first discuss the situation without the Wilson line turned on. The moduli dependence of the one-loop running of the gauge group is given by

$$\Delta_G(T,U) = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \left(\mathcal{B}_G - b(G)\right), \tag{5.1}$$

where $\mathcal{B}$ is a trace over the internal Hilbert space which is defined as

$$\mathcal{B}_G(\tau,\bar\tau) = \frac{1}{\eta^2}\mathrm{Tr}_R \left\{ F e^{i\pi F} q^{L_0 - \frac{c}{24}} \tilde{q}^{\tilde{L}_0 - \frac{\tilde{c}}{24}} \left(Q^2(G) - \frac{1}{8\pi\tau_2}\right)\right\}, \tag{5.2}$$

where Q is the charge of the lattice vectors. The coefficient $b(G)$ is the one-loop beta function which is present to ensure that the integral is well-defined in the limit $\tau_2 \to \infty$. Since we will be interested only in the moduli dependence, this coefficient will not play a crucial role in our analysis. Note that $\mathcal{B}$ is closely related to the new supersymmetric index. In fact the term proportional to $1/8\pi\tau_2$ is the new supersymmetric index. The easiest way to determine the term with the charge insertion $Q^2(G)$ is to consider the action of $q\partial_q$ on the partition function of the appropriate lattice sum so that $\tau_2\mathcal{B}$ is modular invariant. The integral in (5.1) is carried out over the fundamental domain.

Let us recall how to evaluate the one-loop threshold integrands for the groups E_7 and E_8 for the $K3 \times T^2$ compactifications. For group E_8, the integrand is given by

$$\mathcal{B}^{(1)}_{E_8}(\tau\bar\tau) = -2\Gamma_{2,2}(q,\bar q)\frac{1}{\eta^{24}}\left(\alpha_G q\partial_q E_4 - \frac{1}{8\pi\tau_2}\right) 4E_6. \tag{5.3}$$

[8] See [43] for an earlier explicit construction of the twisted elliptic genus for the $N = 4$ orbifold.

Here we have supressed the moduli dependence of $\mathcal{B}$ which arises due to the lattice sum on T^2 given by $\Gamma_{2,2}$. Note that, this is essentially an operation on the new supersymmetric index for these compactifications which is given in (1.2). The charge insertion of the E_8 lattice is obtained by the action of $q\partial_q$ on the lattice sum $E_4(q)$. The coefficient α_G is determined by demanding $\tau_2\mathcal{B}$ is modular invariant. To determine this coefficient consider the following identity due to Ramanujan

$$q\partial_q E_4 = \frac{1}{3}(E_2E_4 - E_6). \tag{5.4}$$

Substituting this identity in (5.3) we obtain

$$\mathcal{B}^{(1)}_{E_8}(\tau\bar{\tau}) = -8\Gamma_{2,2}(q,\bar{q})\frac{1}{\eta^{24}}\left\{\left(\frac{\alpha_G}{3}E_2 - \frac{1}{8\pi\tau_2}\right)E_4E_6 - \frac{\alpha_G}{3}E_6^2\right\}. \tag{5.5}$$

It is now clear that choosing $\alpha_G = \frac{1}{8}$ ensures the the quasi-modular form E_2 occurs in the combination

$$\tilde{E}_2 = E_2 - \frac{3}{\pi\tau_2}. \tag{5.6}$$

which transforms as a good modular form of weight 2. Therefore the threshold integrand for the gauge group E_8 is given by

$$\mathcal{B}^{(1)}_{E_8}(\tau,\bar{\tau}) = -\frac{1}{3}\Gamma_{2,2}(q,\bar{q})\frac{1}{\eta^{24}}\left\{\left(E_2 - \frac{3}{\pi\tau_2}\right)E_4E_6 - E_6^2\right\}. \tag{5.7}$$

Similarly the threshold integrand for the group E_7 is obtained by evaluating

$$\mathcal{B}^{(1)}_{E_7}(\tau,\bar{\tau}) = -8\Gamma_{2,2}(q,\bar{q})\frac{1}{\eta^{24}}\left(\alpha_{G'}q\partial_q E_6 - \frac{1}{8\pi\tau_2}\right)E_4. \tag{5.8}$$

Now we have the Ramanujan identity

$$q\partial_q E_6 = \frac{1}{2}(E_2E_6 - E_4^2). \tag{5.9}$$

This identity together with modular invariance determines $\alpha_{G'} = 1/12$. Thus the threshold integrand for the gauge group E_7 is given by

$$\mathcal{B}^{(1)}_{E_7}(\tau,\bar{\tau}) = -\frac{1}{3}\Gamma_{2,2}(q,\bar{q})\frac{1}{\eta^{24}}\left\{\left(E_2 - \frac{3}{\pi\tau_2}\right)E_4E_6 - E_4^3\right\}. \tag{5.10}$$

Finally consider the difference in the threshold integrands for the gauge groups in (5.7) and (5.10). We obtain

$$\begin{aligned}\mathcal{B}^{(1)}_{E_7} - \mathcal{B}^{(1)}_{E_8} &= \frac{1}{3\eta^{24}}\Gamma_{2,2}\left(E_4^3 - E_6^2\right), \\ &= 576\Gamma_{2,2}.\end{aligned} \tag{5.11}$$

To obtain the second line we have used the identity

$$E_4^3 - E_6^2 = 1728\eta^{24}. \tag{5.12}$$

Therefore the threshold integral reduces to the trivial integral over the fundamental domain of just the lattice sum which is given by

$$\Delta^{(1)}_{E_7}(T,U) - \Delta^{(1)}_{E_8}(T,U) = 576 \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2}(\Gamma_{2,2} - 1). \tag{5.13}$$

The constant (-1) can be obtained by carefully keeping track of the constants $b_(G)$ in the threshold integrand (5.1). Essentially the (-1) serves to regulate the integral as $\tau_2 \to \infty$. This integral was done by [4] and the result reduces to the product of the Dedekind η functions.

$$\Delta^{(1)}_{E_7}(T,U) - \Delta^{(1)}_{E_8}(T,U) = -48 \log(T_2^{12} U_2^{12} |\eta(T)\eta(U)|^{48}). \tag{5.14}$$

Here we are ignoring moduli independent constants. T_2, U_2 are the imaginary parts of the the T, U moduli of the torus T^2. Note that the normalization of the thresholds used in this paper involves a division by the beta function compared to standard normalizations in the literature. This is keep uniformity in the discussion when we evaluate the difference in thresholds as well as when we turn to the CHL orbifolds.

Wilson line $V \neq 0$. Let us now repeat this exercise with the Wilson line V on the torus T^2 turned on. The Wilson line can be embedded either in the gauge group E_8 or E_7. We will take the Wilson line to be embedded in E_8.[9] The procedure to evaluate gauge thresholds with the Wilson line was given in [9]. Here we out line the steps. Due to the presence of the Wilson line, the lattice sum over T^2 is enhanced to $\Gamma_{3,2}$ which is given by

$$\Gamma_{3,2} = \sum_{m_1,m_2,n_1,n_2,b} q^{\frac{p_L^2}{2}} \bar{q}^{\frac{p_R^2}{2}}. \tag{5.15}$$

where

$$\begin{aligned} \frac{p_R^2}{2} &= \frac{1}{4\,\mathrm{det}\mathrm{Im}\Omega} \left| -m_1 U + m_2 + n_1 T + n_2(TU - V^2) + bV \right|^2, \\ \frac{p_L^2}{2} &= \frac{p_R^2}{2} + m_1 n_1 + m_2 n_2 + \frac{1}{4} b^2. \end{aligned} \tag{5.16}$$

and

$$\Omega = \begin{pmatrix} U & V \\ V & T \end{pmatrix}. \tag{5.17}$$

Thus the lattice sum over T^2 is characterized by the five charges (m_1, m_2, n_1, n_2, b). The new supersymmetric index with the Wilson line is then determined by first re-writing the lattice sum over E_8 in terms of a Jacobi form of index 1 given by

$$E_{4,1}(\tau, z) = \frac{1}{2}\left[\theta_2(\tau,z)^2\theta_2^6 + \theta_3(\tau,z)^2\theta_3^6 + \theta_4(\tau,z)^2\theta_4^6\right]. \tag{5.18}$$

Note that $E_{4,1}(\tau, 0) = E_4(q)$, essentially we have decomposed the E_8 lattice into D_6 and D_2 and introduced a chemical potential for the charges in the D_2 sub-lattice. This breaks

[9]The discussion can be generalized when the Wilson line is embedded in E_7, with the same results.

the gauge group E_8 down to $\mathrm{SO}(12)\times\mathrm{U}(1)$ we will refer to this group as G. We then decompose this Jacobi form of index one into SU(2) characters as follows

$$E_{4,1}(\tau,z)=E_{4,1}^{\mathrm{even}}(q)\theta_{\mathrm{even}}(\tau,z)+E_{4,1}^{\mathrm{odd}}(q)\theta_{\mathrm{odd}}(\tau,z). \tag{5.19}$$

where

$$\theta_{\mathrm{even}}(\tau,z)=\theta_3(2\tau,2z), \qquad \theta_{\mathrm{odd}}(\tau,z)=\theta_2(2\tau,2z). \tag{5.20}$$

This decomposition can be performed using the relations

$$\begin{aligned}\theta_1^2(\tau,z) &= \theta_2(2\tau,0)\theta_3(2\tau,2z)-\theta_3(2\tau,0)\theta_2(2\tau,2z),\\ \theta_2^2(\tau,z) &= \theta_2(2\tau,0)\theta_3(2\tau,2z)+\theta_3(2\tau,0)\theta_2(2\tau,2z),\\ \theta_3^2(\tau,z) &= \theta_3(2\tau,0)\theta_3(2\tau,2z)+\theta_2(2\tau,0)\theta_2(2\tau,2z),\\ \theta_4^2(\tau,z) &= \theta_3(2\tau,0)\theta_3(2\tau,2z)-\theta_2(2\tau,0)\theta_2(2\tau,2z).\end{aligned} \tag{5.21}$$

Using these relations we get

$$\begin{aligned}E_{4,1}^{\mathrm{even}}(q) &= \frac{1}{2}\left(\theta_2(2\tau,0)\theta_2^6+\theta_3(2\tau,0)\theta_3^6+\theta_3(2\tau,0)\theta_4^6\right),\\ E_{4,1}^{\mathrm{odd}}(q) &= \frac{1}{2}\left(\theta_3(2\tau,0)\theta_2^6+\theta_2(2\tau,0)\theta_3^6-\theta_2(2\tau,0)\theta_4^6\right).\end{aligned} \tag{5.22}$$

Note that the even and odd parts depend only on the modular parameter τ. Finally the modified new supersymmetric index in the presence of the Wilson line is written as

$$\mathcal{Z}_{\mathrm{new}}^{(1)}(q,\bar q)=-8\frac{E_6}{\eta^{24}}\left(\sum_{\substack{m_1,m_2,n_1,n_2\in\mathbb{Z},\\ b\in 2\mathbb{Z}}} q^{\frac{p_L^2}{2}}\bar q^{\frac{p_R^2}{2}}E_{4,1}^{\mathrm{even}}(q)+\sum_{\substack{m_1,m_2,n_1,n_2\in\mathbb{Z},\\ b\in 2\mathbb{Z}+1}} q^{\frac{p_L^2}{2}}\bar q^{\frac{p_R^2}{2}}E_{4,1}^{\mathrm{odd}}(q)\right). \tag{5.23}$$

Here p_L,p_R contain the Kähler, complex structure and the Wilson line moduli dependence of the T^2. A similar procedure can be carried out when the Wilson line is embedded in the unbroken group E_7. In this situation the Jacobi form $E_{6,1}$ given by

$$E_{6,1}(\tau,z)=\frac{1}{2}\left(-\theta_2^6(\theta_3^4+\theta_4^4)\theta_2^2(\tau,z)+\theta_3^6(\theta_4^4-\theta_2^4)\theta_3^2(\tau,z)+\theta_4^6(\theta_2^4+\theta_3^4)\theta_4^2(\tau,z)\right). \tag{5.24}$$

must be decomposed into its even and odd parts. The coupling of the lattice sum $\Gamma_{3,2}$ to the even and odd parts of $E_{4,1}$ in (5.23) is compactly denoted as

$$\mathcal{Z}_{\mathrm{new}}^{(1)}(q,\bar q)=-8\frac{E_6}{\eta^{24}}E_{4,1}\otimes\Gamma_{3,2}(q,\bar q). \tag{5.25}$$

Now we move to evaluating the integrand $\mathcal{B}_G$ in the gauge thresholds with the Wilson line. Let us evaluate the threshold integrand for the group E_8 first. To determine the coefficient of the α_G in the action $q\partial_q$ we need the following identity analogous to (5.4) which is given in [44, 45]

$$q\partial_q E_{4,1}^{\mathrm{even,odd}}=\frac{7}{24}\left(E_2E_{4,1}^{\mathrm{even,odd}}-E_{6,1}^{\mathrm{even,odd}}\right). \tag{5.26}$$

For completeness we also provide the identity which is required if the Wilson line is embedded in E_7

$$q\partial_q E_{6,1}^{\rm even,odd} = \frac{11}{24}\left(E_2 E_{6,1}^{\rm even,odd} - E_{4,1}^{\rm even,odd} E_4\right). \tag{5.27}$$

From (5.26) it is easy to see that to preserve modular invariance we need $\alpha_G = 1/7$. Therefore we obtain

$$\mathcal{B}_G^{(1)}(\tau,\bar\tau) = -\frac{1}{3}\frac{1}{\eta^{24}}\left\{\left(E_2 - \frac{3}{\pi\tau_2}\right)E_{4,1}E_6 - E_{6,1}E_6\right\}\otimes\Gamma_{3,2}(q,\bar q). \tag{5.28}$$

The threshold integrand $\mathcal{B}_{G'}$ for group E_7 is given by

$$\mathcal{B}_{G'}^{(1)}(\tau,\bar\tau) = -\frac{1}{3}\frac{1}{\eta^{24}}\left\{\left(E_2 - \frac{3}{\pi\tau_2}\right)E_{4,1}E_6 - E_4^2 E_{4,1}\right\}\otimes\Gamma_{3,2}(q,\bar q). \tag{5.29}$$

Let us now take the difference between threshold corrections corresponding to the two gauge groups. We obtain

$$\Delta_{G'}^{(1)}(T,U,V) - \Delta_G^{(1)}(T,U,V) = \int_{\mathcal{F}}\frac{d^2\tau}{\tau_2}\frac{1}{3\eta^{24}}\left(E_4^2E_{4,1} - E_6E_{6,1}\right)\otimes\Gamma_{3,2}(q,\bar q). \tag{5.30}$$

Here we have ignored the constant term in the integrand which can be determined by examining the behaviour of the integrand as $\tau_2\to\infty$. The combination of the Eisenstein series which occurs in the (5.30) can be identified with the elliptic genus of $K3$ due to the following identities

$$\begin{aligned}\frac{1}{\eta^{24}}\left[E_4^2E_{4,1}(\tau,z) - E_6E_{6,1}(\tau,z)\right] &= 72Z_{K3}(\tau,z) = 576A(\tau,z), \\ \frac{1}{\eta^{24}}\left[E_4^2E_{4,1}^{\rm even,odd} - E_6E_{6,1}^{\rm even,odd}\right] &= 72Z_{K3}^{\rm even,odd} = 576A^{\rm even,odd}.\end{aligned} \tag{5.31}$$

where $Z_{K3}(\tau,z) = 8A(\tau,z)$ is the elliptic genus of $K3$. The integral in (5.30) can be performed [46] and it results in

$$\Delta_{G'}^{(1)}(T,U,V) - \Delta_G^{(1)}(T,U,V) = -48\log\left[(\det{\rm Im}\Omega)^{10}|\Phi_{10}(T,U,V)|^2\right]. \tag{5.32}$$

where $\Phi_{10}(T,U,V)$ is the unique cusp modular form of weight 10 under $\mathrm{Sp}(2,\mathbb{Z})$ which is also known as the Igusa cusp form. The observation that the difference in thresholds of the two gauge groups results in the Igusa cusp form was made in [25]. It is also important to note that the duality symmetry $\mathrm{SO}(3,2)$ present classically in heterotic on $K3\times T^2$ is broken to $\mathrm{Sp}(2,\mathbb{Z})$ due to this quantum correction.

The modular form $\Phi_{10}(T,U,V)$ also determines the degeneracies of 1/4 BPS dyons in heterotic string theories compactified on T^6 or equivalently type II theories on $K3\times T^2$. Note that these theories are $\mathcal{N}=4$ string vacua while we have evaluated the threshold correction (5.32), in heterotic compactified on $K3\times T^2$ which has $\mathcal{N}=2$ supersymmetry. It is also interesting that the difference in thresholds is in fact sensitive only the elliptic genus of $K3$. In the next subsections we will generalize this property of the gauge thresholds to heterotic compactified on the CHL orbifolds of $K3$.

5.2 Thresholds in the $\mathbb{Z}_2$ orbifold

Let us first evaluate the threshold integrands without the Wilson line turned on for the $\mathbb{Z}_2$ orbifold of K3. As we have seen in the previous subsection, the most suitable form of the new supersymmetric index for this task is the expression in (3.24) in terms of the Eisenstein series. Let us write in a compact form using the lattice sums defined in (3.30).

$$\mathcal{Z}^{(2)}_{\rm new}(q,\bar q) = -2\frac{E_4}{\eta^{24}}\Big[\Gamma^{(0,0)}_{2,2}2E_6 + \Gamma^{(0,1)}_{2,2}\frac{2}{3}(E_6+2\mathcal{E}_2(\tau))E_4) \qquad (5.33)$$
$$+\Gamma^{(1,0)}_{2,2}\frac{2}{3}\left(E_6-\mathcal{E}_2\left(\frac{\tau}{2}\right)E_4\right)+\Gamma^{(1,1)}_{2,2}\frac{2}{3}\left(E_6-\mathcal{E}_2\left(\frac{\tau+1}{2}\right)E_4\right)\Big].$$

As discussed in the earlier subsection, the insertion of Q^2 in the construction of the integrand $\mathcal{B}$ in (5.2) is done by the action of $\alpha_G q\partial_q$ with $\alpha_G = 1/8$ and $\alpha_{G'} = 1/12$ when the derivative acts on the lattice partition function E_4 and E_6 respectively. This ensures modular invariance of the resulting integrand. Let us first evaluate the threshold integral for the gauge group E_8. For this, $\alpha_G q\partial_q$ acts only on the first E_4 in (5.33). This results in

$$\mathcal{B}^{(2)}_{E_8}(q,\bar q) = -\frac{2}{24\eta^{24}}(\tilde E_2E_4-E_6)\Big[\Gamma^{(0,0)}_{2,2}2E_6+\Gamma^{(0,1)}_{2,2}\frac{2}{3}(E_6+2\mathcal{E}_2(\tau))E_4) \qquad (5.34)$$
$$+\Gamma^{(1,0)}_{2,2}\frac{2}{3}\left(E_6-\mathcal{E}_2\left(\frac{\tau}{2}\right)E_4\right)+\Gamma^{(1,1)}_{2,2}\frac{2}{3}\left(E_6-\mathcal{E}_2\left(\frac{\tau+1}{2}\right)E_4\right)\Big].$$

where $\tilde E_2$ is given by (5.6). Similarly the gauge threshold integrand for the E_7 gauge group is given by

$$\mathcal{B}^{(2)}_{E_7}(q,\bar q) = -\frac{2E_4}{24\eta^{24}}\Big[\Gamma^{(0,0)}_{2,2}2(\hat E_2E_6-E_4^2)+\Gamma^{(0,1)}_{2,2}\frac{2}{3}(\hat E_2E_6-E_4^2+2\mathcal{E}_2(\tau))(\hat E_2E_4-E_6)$$
$$+\Gamma^{(1,0)}_{2,2}\frac{2}{3}\left(\hat E_2E_6-E_4^2-\mathcal{E}_2\left(\frac{\tau}{2}\right)(\hat E_2E_4-E_6)\right)$$
$$+\Gamma^{(1,1)}_{2,2}\frac{2}{3}\left(\hat E_2E_6-E_4^2-\mathcal{E}_2\left(\frac{\tau+1}{2}\right)(\hat E_2E_4-E_6)\right)\Big]. \qquad (5.35)$$

Now upon taking the difference in the threshold integrands we obtain

$$\mathcal{B}^{(2)}_{E_7}-\mathcal{B}^{(2)}_{E_8} = 144\left[2\Gamma^{(0,0)}_{2,2}+\frac{2}{3}\Gamma^{(0,1)}_{2,2}+\frac{2}{3}\Gamma^{(1,0)}_{2,2}+\frac{2}{3}\Gamma^{(1,1)}_{2,2}\right]. \qquad (5.36)$$

The modular integral with these difference can be performed using the methods in [20]. The difference in the gauge thresholds is given by

$$\Delta^{(2)}_{E_7}-\Delta^{(2)}_{E_8} = -48\log\left\{T_2^8U_2^8|\eta(T)\eta(2T)|^{16}|\eta(U)\eta(2U)|^{16}\right\}. \qquad (5.37)$$

Wilson line $V \neq 0$. Let us turn on the Wilson line with values in the gauge group E_8. To write down the modification in the new supersymmetric index it is convenient to introduce the Lattice sums with the Wilson lines. Let us define

$$\Gamma^{(0,0)\rm even}_{3,2} = \sum_{\substack{m_1,m_2,n_1,n_2\in\mathbb{Z},\\ b\in 2\mathbb{Z}}} q^{\frac{p_L^2}{2}}\bar q^{\frac{p_R^2}{2}}, \qquad \Gamma^{(0,0)\rm odd}_{3,2} = \sum_{\substack{m_1,m_2,n_1,n_2\in\mathbb{Z},\\ b\in 2\mathbb{Z}+1}} q^{\frac{p_L^2}{2}}\bar q^{\frac{p_R^2}{2}}, \qquad (5.38)$$

$$\Gamma_{3,2}^{(0,1)\text{even}} = \sum_{\substack{m_1,m_2,n_1,n_2\in\mathbb{Z},\\ b\in 2\mathbb{Z}}} q^{\frac{p_L^2}{2}}\bar{q}^{\frac{p_R^2}{2}}(-1)^{m_1}, \quad \Gamma_{3,2}^{(0,1)\text{odd}} = \sum_{\substack{m_1,m_2,n_1,n_2\in\mathbb{Z},\\ b\in 2\mathbb{Z}}} q^{\frac{p_L^2}{2}}\bar{q}^{\frac{p_R^2}{2}}(-1)^{m_1},$$

$$\Gamma_{3,2}^{(1,0)\text{even}} = \sum_{\substack{m_1,m_2,n_2\in\mathbb{Z},\\ n_1\in\mathbb{Z}+\frac{1}{2},\ b\in 2\mathbb{Z}}} q^{\frac{p_L^2}{2}}\bar{q}^{\frac{p_R^2}{2}}, \quad \Gamma_{3,2}^{(1,0)\text{odd}} = \sum_{\substack{m_1,m_2,n_2\in\mathbb{Z},\\ n_1\in\mathbb{Z}+\frac{1}{2},\ b\in 2\mathbb{Z}+1}} q^{\frac{p_L^2}{2}}\bar{q}^{\frac{p_R^2}{2}},$$

$$\Gamma_{3,2}^{(1,1)\text{even}} = \sum_{\substack{m_1,m_2,n_2\in\mathbb{Z},\\ n_1\in\mathbb{Z}+\frac{1}{2},\ b\in 2\mathbb{Z}}} q^{\frac{p_L^2}{2}}\bar{q}^{\frac{p_R^2}{2}}(-1)^{m_1}, \quad \Gamma_{3,2}^{(1,1)\text{odd}} = \sum_{\substack{m_1,m_2,n_2\in\mathbb{Z},\\ n_1\in\mathbb{Z}+\frac{1}{2},\ b\in 2\mathbb{Z}+1}} q^{\frac{p_L^2}{2}}\bar{q}^{\frac{p_R^2}{2}}(-1)^{m_1},$$

where p_R, p_L are the lattice momenta with the Wilson line given in (5.16). The new supersymmetric index with Wilson line in the E_8 gauge group is given by

$$\begin{aligned}\mathcal{Z}_{\text{new}}^{(2)} &= -2\frac{E_{4,1}}{\eta^{24}} \otimes \left[\Gamma_{3,2}^{(0,0)} 2E_6 + \Gamma_{3,2}^{(0,1)}\frac{2}{3}(E_6 + 2\mathcal{E}_2(\tau))E_4)\right. \\ &\quad \left. +\Gamma_{3,2}^{(1,0)}\frac{2}{3}\left(E_6 - \mathcal{E}_2\left(\frac{\tau}{2}\right)E_4\right) + \Gamma_{3,2}^{(1,1)}\frac{2}{3}\left(E_6 - \mathcal{E}_2\left(\frac{\tau+1}{2}\right)E_4\right)\right].\end{aligned} \tag{5.39}$$

Note here the product $\otimes$ refers to the fact that the even/odd part of the $E_{4,1}$ multiplies the even/odd part of the various lattice sums as explained in the earlier subsection. The threshold integrand for the gauge group E_8 broken down to G is given by

$$\begin{aligned}\mathcal{B}_G^{(2)}(q,\bar{q}) &= -\frac{2}{24\eta^{24}}(\tilde{E}_2 E_{4,1} - E_{6,1}) \otimes \left[\Gamma_{3,2}^{(0,0)} 2E_6 + \Gamma_{3,2}^{(0,1)}\frac{2}{3}(E_6 + 2\mathcal{E}_2(\tau))E_4)\right. \\ &\quad \left. +\Gamma_{3,2}^{(1,0)}\frac{2}{3}\left(E_6 - \mathcal{E}_2\left(\frac{\tau}{2}\right)E_4\right) + \Gamma_{3,2}^{(1,1)}\frac{2}{3}\left(E_6 - \mathcal{E}_2\left(\frac{\tau+1}{2}\right)E_4\right)\right].\end{aligned} \tag{5.40}$$

To obtain this note that the insertion of Q^2 to obtain the threshold integrand is realized by $\alpha_G q\partial_q$ acting on $E_{4,1}$ with $\alpha_G = \frac{1}{7}$. The threshold integrand for the gauge group E_7 is given by

$$\begin{aligned}\mathcal{B}_{G'}^{(2)}(q,\bar{q}) &= -\frac{2E_{4,1}}{24\eta^{24}} \otimes \left[\Gamma_{3,2}^{(0,0)} 2(\hat{E}_2 E_6 - E_4^2) + \Gamma_{3,2}^{(0,1)}\frac{2}{3}(\hat{E}_2 E_6 - E_4^2 + 2\mathcal{E}_2(\tau))(\hat{E}_2 E_4 - E_6)\right. \\ &\quad +\Gamma_{3,2}^{(1,0)}\frac{2}{3}\left(\hat{E}_2 E_6 - E_4^2 - \mathcal{E}_2\left(\frac{\tau}{2}\right)(\hat{E}_2 E_4 - E_6)\right) \\ &\quad \left. +\Gamma_{3,2}^{(1,1)}\frac{2}{3}\left(\hat{E}_2 E_6 - E_4^2 - \mathcal{E}_2\left(\frac{\tau+1}{2}\right)(\hat{E}_2 E_4 - E_6)\right)\right].\end{aligned} \tag{5.41}$$

Taking the difference in the threshold integrands given in (5.40) and (5.41) we obtain

$$\begin{aligned}\mathcal{B}_{G'}^{(2)} - \mathcal{B}_G^{(2)} &= \frac{1}{12\eta^{12}}\left\{2\Gamma_{3,2}^{(0,0)} \otimes (E_{4,1}E_4^2 - E_{6,1}E_6)\right. \\ &\quad + \frac{2}{3}\Gamma_{3,2}^{(0,1)} \otimes \left[(E_{4,1}E_4^2 - E_{6,1}E_6) + 2\mathcal{E}_2(\tau)(E_{4,1}E_6 - E_{6,1}E_4)\right] \\ &\quad + \frac{2}{3}\Gamma_{3,2}^{(1,0)} \otimes \left[(E_{4,1}E_4^2 - E_{6,1}E_6) - \mathcal{E}_2\left(\frac{\tau}{2}\right)(E_{4,1}E_6 - E_{6,1}E_4)\right] \\ &\quad \left. +\frac{2}{3}\Gamma_{3,2}^{(1,1)} \otimes \left[(E_{4,1}E_4^2 - E_{6,1}E_6) - \mathcal{E}_2\left(\frac{\tau+1}{2}\right)(E_{4,1}E_6 - E_{6,1}E_4)\right]\right\}.\end{aligned} \tag{5.42}$$

We now use the identity in (5.31) as well as the following identity verified in appendix A

$$\frac{1}{\eta^{24}}\left(E_{4,1}(\tau,z)E_6 - E_{6,1}(\tau,z)E_4\right) = -144\frac{\theta_1(\tau,z)^2}{\eta^6} = -144B(\tau,z), \tag{5.43}$$
$$\frac{1}{\eta^{24}}\left(E_{4,1}^{\text{even,odd}}E_6 - E_{6,1}^{\text{even,odd}}E_4\right) = -144\frac{(\theta_1^2)^{\text{even,odd}}}{\eta^6} = -144B^{\text{even,odd}}.$$

Substituting the identities (5.31) and (5.43) we obtain

$$\mathcal{B}^{(2)}_{G'} - \mathcal{B}^{(2)}_{G} = 24\left\{\Gamma^{(0,0)}_{3,2}\otimes 4A + \Gamma^{(0,1)}_{3,2}\otimes\left[\frac{4}{3}A - \frac{2}{3}B\mathcal{E}_2(\tau)\right]\right. \tag{5.44}$$
$$\left. +\Gamma^{(1,0)}_{3,2}\otimes\left[\frac{4}{3}A + \frac{1}{3}B\mathcal{E}_2\left(\frac{\tau}{2}\right)\right] + \Gamma^{(1,1)}_{3,2}\otimes\left[\frac{4}{3}A + \frac{1}{3}B\mathcal{E}_2\left(\frac{\tau+1}{2}\right)\right]\right\}.$$

On comparing the twisted elliptic genus for the $N=2$ CHL orbifold of $K3$ given in (3.25) we can rewrite the above equation as

$$\mathcal{B}^{(2)}_{G'} - \mathcal{B}^{(2)}_{G} = 24\left\{\Gamma^{(0,0)}_{3,2}\otimes F^{(0,0)} + \Gamma^{(0,1)}_{3,2}\otimes F^{(0,1)} + \Gamma^{(1,0)}_{3,2}\otimes F^{(1,0)} + \Gamma^{(1,1)}_{3,2}\otimes F^{(1,1)}\right\}. \tag{5.45}$$

This is precisely the integrand in the modular integral to obtain the Siegel modular form $\Phi_6(\Omega)$ of weight 6. Using the result of the integration in [20], we obtain

$$\Delta^{(2)}_{G'}(U,T,V) - \Delta^{(2)}_{G}(U,T,V) = -48\log\left[(\det \mathrm{Im}\Omega)^6|\Phi_6(U,T,V)|^2\right]. \tag{5.46}$$

The Siegel modular form, $\Phi_6(T,U,V)$, transforms as a weight 6 form under a subgroup of $\mathrm{Sp}(2,\mathbb{Z})$. This subgroup is explicitly discussed in [20].[10] The appearance of the Φ_6 in the threshold calculation here shows that the duality group of this compactification is a subgroup of $\mathrm{Sp}(2,\mathbb{Z})$. Just as in the case of heterotic string on $K3\times T^2$, the modular form Φ_6 is also related to the partition function of 1/4 BPS dyons in on type II theory on the CHL orbifold of $K3$. This theory has $\mathcal{N}=4$ supersymmetry, it is dual to the original CHL compactifications of heterotic studied in [10]. Let $\tilde{\Phi}_6$ be the generating function of dyons in this theory, then the modular form Φ_6 is related to $\tilde{\Phi}_6$ in (5.46) by the following $\mathrm{Sp}(2,\mathbb{Z})$ transformation.

$$\Phi_6(U,T,V) = T^{-6}\tilde{\Phi}_6\left(U - \frac{V^2}{T}, -\frac{1}{T}, \frac{V}{T}\right). \tag{5.47}$$

5.3 Thresholds in the $\mathbb{Z}_N$ orbifold

In this subsection we generalize the calculation of the gauge one loop thresholds to the $\mathbb{Z}_N$ orbifold for $N=3,5,7$. Since we have discussed the case for $N=2$ in detail we will directly present the results for the threshold with Wilson line embedded in the unbroken gauge group E_8. Again to present the results it is convenient to define the following lattice sums.

$$\Gamma^{(0,s)\text{even}}_{3,2} = \sum_{\substack{m_1,m_2,n_1,n_2\in\mathbb{Z},\\ b\in 2\mathbb{Z}}} q^{\frac{p_L^2}{2}}\bar{q}^{\frac{p_R^2}{2}}e^{\frac{2\pi i s m_1}{N}}, \qquad \Gamma^{(0,s)\text{odd}}_{3,2} = \sum_{\substack{m_1,m_2,n_1,n_2\in\mathbb{Z},\\ b\in 2\mathbb{Z}+1}} q^{\frac{p_L^2}{2}}\bar{q}^{\frac{p_R^2}{2}}e^{\frac{2\pi i s m_1}{N}}, \tag{5.48}$$

[10] See below equation (3.20) of [20].

$$\Gamma_{3,2}^{(r,rk)\text{even}} = \sum_{\substack{m_1,m_2,n_2\in\mathbb{Z},\\ n_1\in\mathbb{Z}+\frac{r}{N},\ b\in 2\mathbb{Z}}} q^{\frac{p_L^2}{2}} \bar{q}^{\frac{p_R^2}{2}} e^{\frac{2\pi i r k m_1}{N}}, \quad \Gamma_{3,2}^{(r,rk)\text{odd}} = \sum_{\substack{m_1,m_2,n_2,\in\mathbb{Z},\\ n_1\in\mathbb{Z}+\frac{r}{N},\ b\in 2\mathbb{Z}+1}} q^{\frac{p_L^2}{2}} \bar{q}^{\frac{p_R^2}{2}} e^{\frac{2\pi i r k m_1}{N}}.$$

From the expression for the new supersymmetric index in (3.36), it is easy to generalize for the situation with the Wilson line embedded in the E_8 gauge group. This is given by

$$\begin{aligned}\mathcal{Z}_{\text{new}}^{(N)} = -2\frac{E_{4,1}}{\eta^{24}} \otimes \Bigg\{ \Gamma_{3,2}^{(0,0)} \frac{4}{N} E_6 + \sum_{s=1}^{N-1} \Gamma_{3,2}^{(0,s)} \left[\frac{4}{N(N+1)} E_6 + \frac{4}{N+1} \mathcal{E}_N(\tau) E_4 \right] \\ + \sum_{r=1,k=0}^{N-1} \Gamma_{3,2}^{(r,rk)} \left[\frac{4}{N(N+1)} E_6 - \frac{2}{N(N+1)} \mathcal{E}_N\left(\frac{\tau+k}{N}\right) E_4 \right] \Bigg\}. \quad (5.49)\end{aligned}$$

Again, using the same manipulations to evaluate the difference in the threshold integrands for the two gauge groups, we obtain

$$\begin{aligned}\mathcal{B}_{G'}^{(N)} - \mathcal{B}_G^{(N)} = 24 \Bigg\{ \Gamma_{3,2}^{(0,0)} \otimes \frac{8}{N} A + \sum_{s=1}^{N-1} \Gamma_{3,2}^{(0,s)} \otimes \left[\frac{8}{N(N+1)} A - \frac{2}{N(N+1)} \mathcal{E}_N(\tau) B \right] \\ + \sum_{r=1,k=0}^{N-1} \Gamma_{3,2}^{(r,rk)} \otimes \left[\frac{8}{N(N+1)} A + \frac{2}{N(N+1)} \mathcal{E}_N\left(\frac{\tau+k}{N}\right) B \right] \Bigg\}. \quad (5.50)\end{aligned}$$

Now using the expressions for the twisted elliptic genus for the CHL orbifold of $K3$ given in (3.34) we can recast the above expression as

$$\mathcal{B}_{G'}^{(N)} - \mathcal{B}_G^{(N)} = 24 \sum_{r,s=0}^{N-1} \Gamma^{(r,s} \otimes F^{(r,s)}. \quad (5.51)$$

The integral of this function over the fundamental domain has been performed in [20]. The result of this integral is

$$\Delta_{G'}^{(N)}(U,T,V) - \Delta_G^{(N)}(U,T,V) = -48 \log[(\det \operatorname{Im}\Omega)^k |\Phi_k(U,T,V)|^2]. \quad (5.52)$$

Here Φ_k is the Siegel modular form of weight k transforming according to a subgroup of $\mathrm{Sp}(2,\mathbb{Z})$. This modular form is related to $\tilde{\Phi}_k$ the generating function for 1/4 BPS dyons in type II theory compactified on the CHL orbifold of $K3$ by the $\mathrm{Sp}(2,\mathbb{Z})$ transformation

$$\Phi_k(U,T,V) = T^{-k} \tilde{\Phi}_k\left(U - \frac{V^2}{T}, -\frac{1}{T}, \frac{V}{T} \right). \quad (5.53)$$

We have thus demonstrated that the moduli dependence in the difference in the gauge thresholds for heterotic string compactified on the CHL orbifold of $K3$ are captured by Siegel modular forms Φ_k of weight $k = \frac{24}{(N+1)} - 2$. These are related to the modular forms which are generating functions for 1/4 BPS states in $\mathcal{N} = 4$ string theories obtained by compactifying type II theories on the CHL orbifold of $K3$. We would like to again emphasise that our analysis was done only for the standard embedding in which one of the E_8 of the heterotic was broken to E_7. However we expect the difference in gauge thresholds will still be determined by the Siegel modular form Φ_k. For the unorbifolded case this was explicitly demonstrated in [25] by considering various embeddings and gauge groups.

6 Conclusions

We have introduced $\mathcal{N} = 2$ string theories constructed by compactifying heterotic string theories on CHL orbifolds of $K3$. These generalize the well studied example of the heterotic string compactified on $K3 \times T^2$. The CHL orbifolding reduces the number of hypers in the resulting $\mathcal{N} = 2$ theory and preserves the vectors in the theory. These models do not have a lift to 6 dimensions since the orbifolding involves a shift on one of the circles of T^2. We evaluated the new supersymmetric index for these compactifications and showed that it admits an expansion in terms of the McKay-Thompson series of the group M_{24} associated with the $\mathbb{Z}_N$ automorphism used to construct the CHL orbifold.

We then studied the moduli dependence of one-loop corrections to the gauge couplings in the CHL orbifolds of $K3$. We showed that the moduli dependence of the difference in the gauge thresholds is captured by Siegel modular forms closely related to partition function of 1/4 BPS dyons in $\mathcal{N} = 4$ string theories. These Siegel modular forms transform under sub-groups of $\mathrm{Sp}(2, \mathbb{Z})$ which shows that the CHL orfbifolding reduces the duality symmetry of the original $K3$ compactification to a subgroup of $\mathrm{Sp}(2, \mathbb{Z})$.

It will be interesting to evaluate gravitational thresholds in these theories to see if these also admit a nice structure seen for the gauge thresholds. Another direction to explore is generalize the observations of this paper to examples involving different embeddings with other gauge groups. A simple example to study is the compactification in the heterotic string which will lead to the Siegel modular which captured degeneracies of dyons in type II $\mathcal{N} = 4$ constructed in [47]. Another generalization is to consider compactifications in heterotic based on the new classes of twisted elliptic genera of $K3$ constructed in [37, 38, 40].

We observed that the difference in integrands of the gauge thresholds reduces to the twisted elliptic genus of $K3$ for the CHL orbifold. This points to the fact that the difference in the thresholds is essentially sensitive only to a supersymmetric index of the internal CFT. It will be interesting to prove this in general. A similar phenomenon was observed by [48, 49], in which the authors evaluated the difference in thresholds in compactifications of heterotic which completely break supersymmetry. They noticed that the difference in thresholds is purely a holomorphic function in the modular parameter indicative of a supersymmetric index.

Another direction worth exploring is the $\mathcal{N} = 2$ string duality between heterotic string theory compactified on these CHL orbifolds of $K3$ and the appropriate Calabi-Yau on the type II side. Since the CHL orbifolds reduce the number of hypers, the appropriate Calabi-Yau should have the reduced Hodge number $h_{2,1} = 6k + 4$. It is interesting to study what symmetry action on the Calabi-Yau reproduces this Hodge number. In this context it will be also important to study the one-loop threshold corrections to gravitational couplings in these models. Note that the modular forms Φ_k obtained in the difference of thresholds of the CHL compactifications in this paper factorize in the $V \to 0$ limit as [20]

$$\lim_{V\to 0} \Phi_k(U, T, V) \sim V^2 (\eta(T)\eta(NT))^{k+2} (\eta(U)\eta(NU))^{k+2}. \qquad (6.1)$$

It is also interesting to investigate if the difference in thresholds have other degeneration limits for discrete values of V as seen in [48, 49].[11] This degeneration should correspond

[11]We thank Ioannis Florakis for raising this point.

to charged states becoming massless since it corresponds to a logarithmic singularity in the one-loop threshold. It will be interesting to explore this phenomenon on the dual Calabi-Yau compactification in type II.

Acknowledgments

J.R.D thanks Edi Gava, Dileep Jatkar, K.S. Narain and Ashoke Sen for discussions related to this problem at several instances over the last few years. He thanks Shailesh Lal and Dileep Jatkar for initially collaborating on this project. He also thanks the hospitality of the string group at MPI, München during which this collaboration was initiated. We are very grateful to Ioannis Florakis for a careful reading of the manuscript and suggestions which helped to improve it. Furthermore we thank Stephan Stieberger for discussions, encouragement to pursue and complete this project and also for reading the manuscript. A major part of this work was completed when S.D was supported by a research associateship of the Indian Institute of Science. The research of S.D is presently supported by the NCCR SwissMAP, funded by the Swiss National Science Foundation. The work of D.L. is partially supported by the ERC Advanced Grant "Strings and Gravity" (Grant.No. 32004) and by the DFG cluster of excellence "Origin and Structure of the Universe".

A Theta functions and Eisenstein series

Our notations for the the Jacobi theta functions are summarized in the following expansion

$$\begin{aligned}
\theta_1(\tau,z) &= \sum_{n\in\mathbb{Z}} \exp\left[i\pi\left(n+\frac{1}{2}\right)^2\tau + 2\pi i\left(n+\frac{1}{2}\right)\left(z+\frac{1}{2}\right)\right], \qquad (A.1)\\
\theta_2(\tau,z) &= \sum_{n\in\mathbb{Z}} \exp\left[i\pi\left(n+\frac{1}{2}\right)^2\tau + 2\pi i\left(n+\frac{1}{2}\right)z\right],\\
\theta_3(\tau,z) &= \sum_{n\in\mathbb{Z}} \exp\left[i\pi n^2\tau + 2\pi i n z\right],\\
\theta_4(\tau,z) &= \sum_{n\in\mathbb{Z}} \exp\left[i\pi n^2\tau + 2\pi i n\left(z+\frac{1}{2}\right)\right].
\end{aligned}$$

When there is no ambiguity we will use the following notation for theta functions at the origin

$$\theta_2(\tau,0)=\theta_2(q)=\theta_2, \quad \theta_3(\tau,0)=\theta_3(q)=\theta_3, \quad \theta_4(\tau,0)=\theta_4(q)=\theta_4. \qquad (A.2)$$

The Dedekind η function is defined by the product

$$\eta(\tau) = q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n), \qquad (A.3)$$

where $q=e^{2\pi i\tau}$. It is also useful to present the infinite product representation of the theta functions at the origin

$$\frac{\theta_2(\tau)}{\eta(\tau)} = q^{\frac{1}{12}}\prod_{n=1}^{\infty}(1+q^n)(1+q^{n-1}), \qquad (A.4)$$

$$\frac{\theta_3(\tau)}{\eta(\tau)} = q^{-\frac{1}{24}} \prod_{n=1}^{\infty} (1 + q^{n-\frac{1}{2}})(1 + q^{n-\frac{1}{2}}),$$
$$\frac{\theta_2(\tau)}{\eta(\tau)} = q^{-\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^{n-\frac{1}{2}})(1 - q^{n-\frac{1}{2}}).$$

One identity of theta functions which we repeatedly use is triple product identity

$$\theta_2\theta_3\theta_4 = 2\eta^3. \tag{A.5}$$

Finally we will also use the following shift properties of the theta functions.

$$\begin{aligned}
\theta_4\left(\tau, z + \frac{1}{2}\right) &= \theta_3(\tau, z), & \theta_1\left(\tau, z + \frac{\pi}{2}\right) &= \theta_2(\tau, z), \\
\theta_2\left(\tau, z + \frac{1}{2}\right) &= -\theta_1(\tau, z), & \theta_3\left(\tau, z + \frac{1}{2}\right) &= \theta_4(\tau, z), \\
\theta_4\left(\tau, z + \frac{\tau}{2}\right) &= ie^{-\frac{\pi i\tau}{4} - i\pi z}\theta_1(\tau, z), & \theta_1\left(\tau, z + \frac{\tau}{2}\right) &= ie^{-\frac{\pi i\tau}{4} - i\pi z}\theta_4(\tau, z), \\
\theta_2\left(\tau, z + \frac{\tau}{2}\right) &= e^{-\frac{\pi i\tau}{4} - i\pi z}\theta_3(\tau, z), & \theta_3\left(\tau, z + \frac{\tau}{2}\right) &= e^{-\frac{\pi i\tau}{4} - i\pi z}\theta_4(\tau, z).
\end{aligned} \tag{A.6}$$

The Eisenstein series E_2 is a quasi-modular form whose series expansion is given by

$$E_2(q) = 1 - 24\sum_{n=1}^{\infty} \sigma_1(n) q^n. \tag{A.7}$$

where $\sigma_1(n)$ is the sum of positive integral divisors of n. The combination

$$\tilde{E}_2 = E_2 - \frac{3}{\pi\tau_2}, \tag{A.8}$$

transforms as a good modular form of weight 2. The Eisenstein series E_4 and E_6 are related to the theta functions by the well known identities

$$\begin{aligned}
E_4 &= \frac{1}{2}\left(\theta_2^8 + \theta_3^8 + \theta_4^8\right), \\
E_6 &= \frac{1}{2}\left[-\theta_2^6(\theta_3^4 + \theta_4^4)\theta_2^2 + \theta_3^6(\theta_4^4 - \theta_2^4)\theta_3^2 + \theta_4^6(\theta_2^4 + \theta_3^4)\theta_4^2\right].
\end{aligned} \tag{A.9}$$

We will also require the modular form $\mathcal{E}_N$ by

$$\mathcal{E}_N(\tau) = \frac{12i}{\pi(N-1)}\partial_\tau \log\frac{\eta(\tau)}{\eta(N\tau)}. \tag{A.10}$$

Under modular transformations it behaves as

$$\mathcal{E}_N(\tau + 1) = \mathcal{E}_N, \qquad \mathcal{E}_N(-1/\tau) = -\tau^2\frac{1}{N}\mathcal{E}_N(\tau/N). \tag{A.11}$$

The relations given in (3.22) involving θ functions Eisenstein series and the $\mathcal{E}$ function is used to write the new supersymmetric in terms of the Eisenstein series. We have established these identities by performing q expansions in Mathematica, we have listed out the first few terms

$$
\begin{aligned}
-(\theta_3^8\theta_4^4+\theta_4^8\theta_3^4) &= -\frac{2}{3}\left(E_6+2\mathcal{E}_2(\tau)E_4\right) \qquad (A.12)\\
&= 2+16q-496q^2+3904q^3-16880q^4+50016q^5-121024q^6+\cdots\,,\\
\theta_3^8\theta_2^4+\theta_2^8\theta_3^4 &= -\frac{2}{3}\left(E_6-\mathcal{E}_2\left(\frac{\tau}{2}\right)E_4\right) \qquad (A.13)\\
&= 16q^{1/2}+512q+3904q^{3/2}+16384q^2+50016q^{5/2}+124928q^3+\cdots\,,\\
\theta_2^8\theta_4^4-\theta_2^8\theta_4^4 &= -\frac{2}{3}\left(E_6-\mathcal{E}_2\left(\frac{\tau+1}{2}\right)E_4\right) \qquad (A.14)\\
&= 16q^{1/2}-512q+3904q^{3/2}-16384q^2+50016q^{5/2}-124928q^3+\cdots\,.
\end{aligned}
$$

Finally we establish the identities in (5.31) and (5.43). First recall that the Jacobi forms of index 1 admit a even odd decomposition given by

$$f(\tau,z)=f^{\text{even}}(\tau)\theta_{\text{even}}+f^{\text{odd}}(\tau)\theta_{\text{odd}}, \qquad (A.15)$$

where

$$f^{\text{even}}=\sum_{N\equiv 0(4)}c(N)q^{N/4} \qquad f^{\text{odd}}=\sum_{N\equiv -1(4)}c(N)q^{N/4}, \qquad (A.16)$$

and

$$\theta_{\text{even}}(\tau,z)=\theta_3(2\tau,2z), \qquad \theta_{\text{odd}}(\tau,z)=\theta_2(2\tau,2z). \qquad (A.17)$$

It is clear that $E_{4,1}(\tau,z)$ and $E_{6,1}(\tau,z)$ defined in (5.18) and (5.24) admit this decomposition using the identities given in (5.21). From these we find that

$$
\begin{aligned}
E_{4,1}^{\text{even}} &= \theta_3^7(2\tau)+7\theta_3^3(2\tau)\theta_2^4(2\tau),\\
E_{4,1}^{\text{odd}} &= \theta_2^7(2\tau)+7\theta_2^3(2\tau)\theta_3^4(2\tau). \qquad (A.18)
\end{aligned}
$$

while for $E_{6,1}$ it is

$$
\begin{aligned}
E_{6,1}^{\text{even}} &= \theta_3^{11}(2\tau)-11\,\theta_2^8(2\tau)\theta_3^3(2\tau)-22\,\theta_2^4(2\tau)\theta_3^7(2\tau),\\
E_{6,1}^{\text{odd}} &= \theta_2^{11}(2\tau)-11\,\theta_3^8(2\tau)\theta_2^3(2\tau)-22\,\theta_3^4(2\tau)\theta_2^7(2\tau). \qquad (A.19)
\end{aligned}
$$

From (A.16) we see that q expansions of the 'even' and 'odd' parts of the Jacobi forms are different therefore we can introduce the notation [9] in which we combine these expansions

$$\widehat{f}(\tau)=f^{\text{even}}(\tau)+f^{\text{odd}}(\tau). \qquad (A.20)$$

Then we establish (5.31) and (5.43 by performing the q expansions in Mathematica which are given by

$$
\begin{aligned}
&8\left(\frac{\widehat{\theta_2^2}}{\theta_2^2}+\frac{\widehat{\theta_3^2}}{\theta_3^2}+\frac{\widehat{\theta_4^2}}{\theta_4^2}\right)=\frac{1}{72}\,\frac{E_4^2\widehat{E_{4,1}}-E_6\widehat{E_{6,1}}}{\eta^{24}} \qquad (A.21)\\
&=\frac{2}{q^{1/4}}+20-128q^{3/4}+216q-1026q^{7/4}+1616q^2-5504q^{11/4}+8032q^3+\cdots\,,
\end{aligned}
$$

and

$$-2\,\frac{\widehat{\theta}_1^2}{\eta^6} = \frac{1}{72}\,\frac{E_6\widehat{E_{4,1}} - E_4\widehat{E_{6,1}}}{\eta^{24}}$$
$$= \frac{2}{q^{1/4}} - 4 + 16q^{3/4} - 24q + 78q^{7/4} - 112q^2 + 304q^{11/4} - 416q^3 + \cdots\,. \tag{A.22}$$

B Lattice sums

In this appendix we provide the details of evaluating the lattice sum over the shifted lattice E_8' defined by

$$P_{(a,b)} = e^{-2\pi i\frac{ab}{n^2}\gamma^2} \sum_{\lambda\in\Gamma^8+\frac{a}{2}\gamma} e^{2\pi i\frac{b}{n}\lambda\cdot\gamma} q^{\frac{1}{2}\lambda^2}. \tag{B.1}$$

The sum runs over all the lattice vectors λ of E_8. The lattice shift γ of the $\mathbb{Z}_2$ orbifold is given by

$$\gamma = (1,1,0^6). \tag{B.2}$$

Before we proceed let us recall that the roots of E_8 are given by

$$\begin{aligned} &112 \text{ root vectors of } D_8 : (\dots,\pm1,\dots,\pm1,\dots),\\ &128 \text{ 8-dimensional vectors } : \left(\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2},\dots\right). \end{aligned}$$

Here the '…' in the 112 root vectors of D_8 represent zeros. The lattice vectors are then of two types.

$$\begin{aligned} \lambda_A &= (n_1, n_2, \cdots, n_8),\\ \lambda_B &= \left(n_1+\frac{1}{2}, \cdots, n_8+\frac{1}{2}\right), \end{aligned} \tag{B.3}$$

with the constraint

$$\sum_{i=1}^{8} n_i = \text{even integer}. \tag{B.4}$$

Let us now perform the lattice sum without any shifts. This is the $(0,0)$ sector.

$$P_{(0,0)} = \sum_{\lambda_A} q^{\frac{1}{2}\lambda_A\cdot\lambda_A} + \sum_{\lambda_B} q^{\frac{1}{2}\lambda_B\cdot\lambda_B}. \tag{B.5}$$

We can impose the constraint via an extra factor

$$\frac{1}{2}(1+e^{i\pi\sum n_i}) = \begin{cases} 1 & \text{if } \sum_i n_i = \text{even integer}\\ 0 & \text{otherwise.}\end{cases}$$

This results in

$$P_{(0,0)} = \frac{1}{2}\prod_{i=1}^{8}\sum_{n_i\in\mathbb{Z}} e^{i\pi\tau n_i^2} + \frac{1}{2}\prod_{i=1}^{8}\sum_{n_i\in\mathbb{Z}} e^{i\pi n_i^2} e^{i\pi\tau n_i}$$

$$+\frac{1}{2}\prod_{i=1}^{8}\sum_{n_i\in\mathbb{Z}} e^{i\pi\tau(n_i+\frac{1}{2})^2}+\frac{1}{2}\prod_{i=1}^{8}\sum_{n_i\in\mathbb{Z}} e^{i\pi\tau(n_i+\frac{1}{2})^2}e^{i\pi(n_i+\frac{1}{2})}, \tag{B.6}$$

which can be written in terms of the Jacobi θ-functions as

$$P_{(0,0)}=\frac{1}{2}\left[\theta_3^8+\theta_4^8+\theta_2^8+\theta_1^8\right]. \tag{B.7}$$

The last term is zero. Hence the final expression for the lattice sum is

$$P_{(0,0)}=\frac{1}{2}\left[\theta_2^8+\theta_3^8+\theta_4^8\right]. \tag{B.8}$$

For the case $(a,b)=(0,1)$. The weight vectors are the same as (B.3). To evaluate the phase we use the shift in (B.2) and the weight vectors to get

$$\lambda_A\cdot\gamma=n_1+n_2, \qquad \lambda_B\cdot\gamma=\left(n_1+\frac{1}{2}\right)+\left(n_2+\frac{1}{2}\right). \tag{B.9}$$

The lattice sum is

$$\begin{aligned}
P_{(0,1)}=&\frac{1}{2}\prod_{i=1}^{6}\sum_{n_i\in\mathbb{Z}} e^{\pi i\tau n_i^2}\prod_{i=1}^{2}\sum_{n_i\in\mathbb{Z}} e^{\pi i\tau n_i^2}e^{\pi i n_i}+\frac{1}{2}\prod_{i=1}^{6}\sum_{n_i\in\mathbb{Z}} e^{\pi i\tau n_i^2}\prod_{i=1}^{2}\sum_{n_i\in\mathbb{Z}} e^{\pi i\tau n_i^2}e^{2\pi i n_i}\\
&+\frac{1}{2}\prod_{i=1}^{6}\sum_{n_i\in\mathbb{Z}} e^{\pi i\tau(n_i+\frac{1}{2})^2}\prod_{i=1}^{2}\sum_{n_i\in\mathbb{Z}} e^{\pi i\tau(n_i+\frac{1}{2})^2}e^{\pi i(n_i+\frac{1}{2})}\\
&+\frac{1}{2}\prod_{i=1}^{6}\sum_{n_i\in\mathbb{Z}} e^{\pi i\tau(n_i+\frac{1}{2})^2}\prod_{i=1}^{2}\sum_{n_i\in\mathbb{Z}} e^{\pi i\tau(n_i+\frac{1}{2})^2}e^{2\pi i(n_i+\frac{1}{2})},\\
&=\frac{1}{2}\left[\theta_3^6\theta_4^2+\theta_4^6\theta_3^2+\theta_2^6\theta_1^2-\theta_2^2\theta_1^6\right]\;=\;\frac{1}{2}\left[\theta_3^6\theta_4^2+\theta_4^6\theta_3^2\right].
\end{aligned} \tag{B.10}$$

For $(a,b)=(1,0)$, the weight vectors are

$$\begin{aligned}
\lambda'_A&=(n_1+\frac{1}{2},n_2+\frac{1}{2},n_3,n_4\cdots,n_8),\\
\lambda'_B&=\left(n_1+1,n_2+1,n_3+\frac{1}{2},\cdots,n_8+\frac{1}{2}\right).
\end{aligned} \tag{B.11}$$

The lattice sum is then

$$\begin{aligned}
P_{(1,0)}=&\frac{1}{2}\prod_{i=1}^{2}\sum_{n_i\in\mathbb{Z}} e^{i\pi\tau(n_i+\frac{1}{2})^2}\prod_{i=1}^{6}\sum_{n_i\in\mathbb{Z}} e^{i\pi\tau n_i^2}+\frac{1}{2}\prod_{i=1}^{2}\sum_{n_i\in\mathbb{Z}} e^{i\pi\tau(n_i+\frac{1}{2})^2}e^{i\pi n_i}\prod_{i=1}^{6}\sum_{n_i\in\mathbb{Z}} e^{i\pi\tau n_i^2}e^{i\pi n_i}\\
&+\frac{1}{2}\prod_{i=1}^{2}\sum_{n_i\in\mathbb{Z}} e^{i\pi\tau n_i^2}\prod_{i=1}^{6}\sum_{n_i\in\mathbb{Z}} e^{i\pi\tau(n_i+\frac{1}{2})^2}+\frac{1}{2}\prod_{i=1}^{2}\sum_{n_i\in\mathbb{Z}} e^{i\pi\tau n_i^2}e^{i\pi n_i}\prod_{i=1}^{6}\sum_{n_i\in\mathbb{Z}} e^{i\pi\tau(n_i+\frac{1}{2})^2}e^{i\pi n_i},\\
&=\frac{1}{2}\left[\theta_3^6\theta_2^2+\theta_2^6\theta_3^2-\theta_4^6\theta_1^2-\theta_1^6\theta_4^2\right]=\frac{1}{2}\left[\theta_3^6\theta_2^2+\theta_2^6\theta_3^2\right].
\end{aligned} \tag{B.12}$$

Finally for $(a,b)=(1,1)$ the weight vectors are same as the ones in equation (B.11). In addition we also have the extra phase since $b\neq 0$. Here

$$\lambda'_A\cdot\gamma=\left(n_1+\frac{1}{2}\right)+\left(n_2+\frac{1}{2}\right) \qquad \lambda'_B\cdot\gamma=n_1+n_2 \tag{B.13}$$

So the lattice sum in this case is given by

$$
\begin{aligned}
-P_{(1,1)} = & \frac{1}{2}\prod_{i=1}^{2}\sum_{n_i\in\mathbb{Z}} e^{\pi i\tau(n_i+\frac{1}{2})^2+\pi i(n_i+\frac{1}{2})}\prod_{i=1}^{6}\sum_{n_i\in\mathbb{Z}} e^{\pi i n_i^2} \\
& +\frac{1}{2}\prod_{i=1}^{2}\sum_{n_i\in\mathbb{Z}} e^{\pi i\tau(n_i+\frac{1}{2})^2+\pi i n_i+\pi i(n_i+\frac{1}{2})}\prod_{i=1}^{6}\sum_{n_i\in\mathbb{Z}} e^{\pi i n_i^2+\pi i n_i} \\
& +\frac{1}{2}\prod_{i=1}^{2}\sum_{n_i\in\mathbb{Z}} e^{\pi i\tau n_i^2} e^{\pi i n_i}\prod_{i=1}^{6}\sum_{n_i\in\mathbb{Z}} e^{\pi i\tau(n_i+\frac{1}{2})^2} \\
& +\frac{1}{2}\prod_{i=1}^{2}\sum_{n_i\in\mathbb{Z}} e^{\pi i\tau n_i^2+\pi i n_i+\pi i n_i}\prod_{i=1}^{6}\sum_{n_i\in\mathbb{Z}} e^{\pi i\tau(n_i+\frac{1}{2})^2+\pi i(n_i+\frac{1}{2})}, \\
= & \frac{1}{2}\left[\theta_3^6\theta_1^2-\theta_4^6\theta_2^2+\theta_2^6\theta_4^2-\theta_1^6\theta_3^2\right] \;=\; \frac{1}{2}\left[\theta_2^6\theta_4^2-\theta_4^6\theta_2^2\right].
\end{aligned} \tag{B.14}
$$

Note that is the case where there are corrections due to the shift and factors present due to the even integer constraint and the extra phase. The overall negative sign is due to the overall phase in the definition (B.1).

C Details for the $\mathbb{Z}_2$ orbifold

This appendix provides the details of the evaluation of the following trace

$$
F_{m_1,m_2,n_1,n_2}(a,r,b,s;q) = \mathrm{Tr}_{m_1,m_2,n_1,n_2;g^a g'^r;RR}\left(g^b g'^s e^{i\pi(F^{T^4}+F^{T^2})}F^{T^2} q^{L_0'}\bar{q}^{\bar{L}_0'}\right). \tag{C.1}
$$

The orbifold action g and g' is defined in (3.2). We label the various sectors in terms of only the action of g. The action of g' is summed over in each of these sectors. Also in the above trace the bosonic oscillators in the holomorphic direction of the T^2 is not included since it is has already been included in (3.11). Due to the presence of the fermionic zero modes associated with T^4 which is along the $6,7,8,9$ direction the trace vanishes for $a=0, b=0$ irrespective of the values of r and s. Therefore we have

$$
F_{m_1,m_2,n_1,n_2}(0,r,0,s;q) = 0. \tag{C.2}
$$

Therefore from the definition of $\mathcal{F}$ in (3.12) we see that this implies

$$
\mathcal{F}_{m_1,m_2,n_1,n_2}(0,0;q) = 0. \tag{C.3}
$$

Now let us move to the $(0,1)$ sector. We have the following

$$
\begin{aligned}
F_{m_1,m_2,n_1,n_2}(0,0,1,0;q) &= -4\frac{1}{[q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1+q^n)]^4} \\
&= -4\frac{\theta_3^2\theta_4^2}{\eta^4}.
\end{aligned} \tag{C.4}
$$

In the above equation the factor 4 arises from the anti-holomorphic fermion zero modes associated with the T^4. The bosonic and the fermionic oscillators in the anti-holomorphic sector cancel. What is left behind are the 4 bosonic oscillators in the holomorphic sector. The action of g on these oscillators reverses the sign. We have used the product representation of the θ_2 and then triple product identity in (A.5) to arrive at (C.4). The over all negative sign is associated with the action of$g$ on the vacuum. This choice of the action of g on the vacuum ensures the final result is modular invariant. Next we have

$$F_{m_1,m_2,n_1,n_2}(0,0,1,1;q) = -(-1)^{m_1} 4\frac{\theta_3^2\theta_4^2}{\eta^4}. \tag{C.5}$$

The only difference in this trace from that of (C.4) is the insertion of g' in the trace. This picks up the factor $(-1)^{m_1}$ on the state carrying m_1 units of momentum along the circle y^4. Now the following traces vanish

$$F_{m_1,m_2,n_1,n_2}(0,1,1,0;q) = F_{m_1,m_2,n_1,n_2}(0,1,1,1;q) = 0. \tag{C.6}$$

This is because in this sector the winding numbers along y^6 is half integer moded and the action of g as well as gg' reverses the sign of these modes and therefore they do not contribute in the trace. Thus we have

$$\mathcal{F}_{m_1,m_2,n_1,n_2}(0,1;q) = \begin{cases} -2\left(1+(-1)^{m_1}\right)\frac{\theta_3^2\theta_4^2}{\eta^4} & \text{for } \{m_1,m_2,n_1,n_2\}\in\mathbb{Z}, \\ 0 & \text{for } \{m_1,m_2,n_2\}\in\mathbb{Z},\ \ \{n_1\}\in\mathbb{Z}+\frac{1}{2}. \end{cases} \tag{C.7}$$

For the twisted $(1,0)$ sector we have

$$\begin{aligned} F_{m_1,m_2,n_1,n_2}(1,0,0,0;q) &= 16\frac{1}{[q^{-\frac{1}{48}}\prod_{n=1}^{\infty}(1-q^{n-1/2})]^4}, \\ &= 4\frac{\theta_2^2\theta_3^2}{\eta^4}. \end{aligned} \tag{C.8}$$

Here the factor of 16 in the first line is due to the 16 twisted sectors localized at the 16 fixed points of T^4 at $y^m = 0,\pi$ for $m = 6,7,8,9$. To arrive at the second line we have used the product representation of θ_4 and the identity (A.5). Again the bosonic and fermionic oscillators in the anti-holomorphic sector cancel leaving behind the bosonic oscillators in the holomorphic sector. These oscillators are half integer modded since they belong to the twisted sector. Now

$$F_{m_1,m_2,n_1,n_2}(1,0,0,1;q) = 0. \tag{C.9}$$

This is because the action of g' exchanges the fixed points pairwise, the twisted sector states are off diagonal and therefore the trace vanishes.

$$F_{m_1,m_2,n_1,n_2}(1,1,0,0;q) = 4\frac{\theta_2^2\theta_3^2}{\eta^4}. \tag{C.10}$$

Here the states twisted by gg' are now labelled by the fixed points $y^6 = \frac{\pi}{2}, \frac{3\pi}{2}, y^m = 0,\pi$ for $m = 7,8,9$. The rest of the analysis to obtain the above equation is same as that in (C.8), but note that here the winding $n_1 \in \mathbb{Z}+\frac{1}{2}$ due to the twisting by g'. Finally

$$F_{m_1,m_2,n_1,n_2}(1,1,0,1;q) = 0. \tag{C.11}$$

This is because the action of the g' insertion exchanges the fixed points and the elements are off diagonal in the trace. In summary the contributions in this sector are

$$\mathcal{F}_{m_1,m_2,n_1,n_2}(1,0;q) = \begin{cases} 2\frac{\theta_2^2\theta_3^2}{\eta^4} & \text{for } \{m_1,m_2,n_1,n_2\} \in \mathbb{Z}, \\ 2\frac{\theta_2^2\theta_3^2}{\eta^4} & \text{for } \{m_1,m_2,n_2\} \in \mathbb{Z}, \{n_1\} \in \mathbb{Z}+\frac{1}{2}. \end{cases} \tag{C.12}$$

Lets now look at the $(1,1)$ sector. We have

$$\begin{aligned} F_{m_1,m_2,n_1,n_2}(1,0,1,0;q) &= -16\frac{1}{[q^{-\frac{1}{48}}\prod_{n=1}^{\infty}(1-q^{n-1/2})]^4}, \\ &= -4\frac{\theta_2^2\theta_4^2}{\eta^4}. \end{aligned} \tag{C.13}$$

Again due to the arguments mentioned earlier, it is only the bosonic oscillators in the T^4 directions which contribute. The 16 in the first line is due to the presence of the 16 fixed point and the negative sign is because the action of g on the vacuum gives a negative sign. Now

$$F_{m_1,m_2,n_1,n_2}(1,0,1,1;q) = 0. \tag{C.14}$$

This is because the insertion of gg' in the trace exchanges the fixed points pair wise and therefore the elements are off diagonal in the trace. Again due to the same reason of the elements being off diagonal we have

$$F_{m_1,m_2,n_1,n_2}(1,1,1,0;q) = 0. \tag{C.15}$$

Note here the due to twisted by g' the states are at $y^6 = \frac{\pi}{2}, \frac{3\pi}{2}, y^m = 0, \pi$ for $m = 7,8,9$. Finally

$$F_{m_1,m_2,n_1,n_2}(1,1,1,1;q) = -4(-1)^{m_1}\frac{\theta_2^2\theta_4^2}{\eta^4}. \tag{C.16}$$

Here the analysis is same as in the case of (C.13). The $(-1)^{m_1}$ occurs due to the presence g' in the trace. Also $n_1 \in \mathbb{Z}+\frac{1}{2}$ since the states are twisted by g'. To summarize this sector results in

$$\mathcal{F}_{m_1,m_2,n_1,n_2}(1,1;q) = \begin{cases} -2\frac{\theta_2^2\theta_4^2}{\eta^4} & \text{for } \{m_1,m_2,n_1,n_2\} \in \mathbb{Z}, \\ -2(-1)^{m_1}\frac{\theta_2^2\theta_4^2}{\eta^4} & \text{for } \{m_1,m_2,n_2\} \in \mathbb{Z}, \;\; \{n_1\} \in \mathbb{Z}+\frac{1}{2}. \end{cases} \tag{C.17}$$

References

[1] S. Kachru and C. Vafa, *Exact results for $N = 2$ compactifications of heterotic strings*, *Nucl. Phys.* **B 450** (1995) 69 [hep-th/9505105] [INSPIRE].

[2] M.B. Green, J.H. Schwarz and P.C. West, *Anomaly free chiral theories in six-dimensions*, *Nucl. Phys.* **B 254** (1985) 327 [INSPIRE].

[3] M.A. Walton, *The heterotic string on the simplest Calabi-Yau manifold and its orbifold limits*, Phys. Rev. **D 37** (1988) 377 [INSPIRE].

[4] L.J. Dixon, V. Kaplunovsky and J. Louis, *Moduli dependence of string loop corrections to gauge coupling constants*, Nucl. Phys. **B 355** (1991) 649 [INSPIRE].

[5] P. Mayr and S. Stieberger, *Moduli dependence of one loop gauge couplings in* $(0,2)$ *compactifications*, Phys. Lett. **B 355** (1995) 107 [hep-th/9504129] [INSPIRE].

[6] B. de Wit, V. Kaplunovsky, J. Louis and D. Lüst, *Perturbative couplings of vector multiplets in* $N = 2$ *heterotic string vacua*, Nucl. Phys. **B 451** (1995) 53 [hep-th/9504006] [INSPIRE].

[7] J.A. Harvey and G.W. Moore, *Algebras, BPS states and strings*, Nucl. Phys. **B 463** (1996) 315 [hep-th/9510182] [INSPIRE].

[8] B. de Wit, G. Lopes Cardoso, D. Lüst, T. Mohaupt and S.-J. Rey, *Higher order gravitational couplings and modular forms in* $N = 2$, $D = 4$ *heterotic string compactifications*, Nucl. Phys. **B 481** (1996) 353 [hep-th/9607184] [INSPIRE].

[9] G. Lopes Cardoso, G. Curio and D. Lüst, *Perturbative couplings and modular forms in* $N = 2$ *string models with a Wilson line*, Nucl. Phys. **B 491** (1997) 147 [hep-th/9608154] [INSPIRE].

[10] S. Chaudhuri, G. Hockney and J.D. Lykken, *Maximally supersymmetric string theories in* $D < 10$, Phys. Rev. Lett. **75** (1995) 2264 [hep-th/9505054] [INSPIRE].

[11] S. Chaudhuri and J. Polchinski, *Moduli space of CHL strings*, Phys. Rev. **D 52** (1995) 7168 [hep-th/9506048] [INSPIRE].

[12] P.S. Aspinwall, *Some relationships between dualities in string theory*, Nucl. Phys. Proc. Suppl. **46** (1996) 30 [hep-th/9508154] [INSPIRE].

[13] S. Ferrara and C. Kounnas, *Extended supersymmetry in four-dimensional type II strings*, Nucl. Phys. **B 328** (1989) 406 [INSPIRE].

[14] J.H. Schwarz and A. Sen, *Type IIA dual of the six-dimensional CHL compactification*, Phys. Lett. **B 357** (1995) 323 [hep-th/9507027] [INSPIRE].

[15] I. Antoniadis, E. Gava and K.S. Narain, *Moduli corrections to gravitational couplings from string loops*, Phys. Lett. **B 283** (1992) 209 [hep-th/9203071] [INSPIRE].

[16] I. Antoniadis, E. Gava and K.S. Narain, *Moduli corrections to gauge and gravitational couplings in four-dimensional superstrings*, Nucl. Phys. **B 383** (1992) 93 [hep-th/9204030] [INSPIRE].

[17] S. Cecotti, P. Fendley, K.A. Intriligator and C. Vafa, *A new supersymmetric index*, Nucl. Phys. **B 386** (1992) 405 [hep-th/9204102] [INSPIRE].

[18] S. Cecotti and C. Vafa, *Ising model and* $N = 2$ *supersymmetric theories*, Commun. Math. Phys. **157** (1993) 139 [hep-th/9209085] [INSPIRE].

[19] M.C.N. Cheng, X. Dong, J. Duncan, J. Harvey, S. Kachru and T. Wrase, *Mathieu moonshine and* $N = 2$ *string compactifications*, JHEP **09** (2013) 030 [arXiv:1306.4981] [INSPIRE].

[20] J.R. David, D.P. Jatkar and A. Sen, *Product representation of dyon partition function in CHL models*, JHEP **06** (2006) 064 [hep-th/0602254] [INSPIRE].

[21] C. Angelantonj, I. Florakis and B. Pioline, *A new look at one-loop integrals in string theory*, Commun. Num. Theor. Phys. **6** (2012) 159 [arXiv:1110.5318] [INSPIRE].

JHEP02(2016)056

[22] C. Angelantonj, I. Florakis and B. Pioline, *One-loop BPS amplitudes as BPS-state sums*, *JHEP* **06** (2012) 070 [arXiv:1203.0566] [INSPIRE].

[23] C. Angelantonj, I. Florakis and B. Pioline, *Rankin-Selberg methods for closed strings on orbifolds*, *JHEP* **07** (2013) 181 [arXiv:1304.4271] [INSPIRE].

[24] C. Angelantonj, I. Florakis and B. Pioline, *Threshold corrections, generalised prepotentials and Eichler integrals*, *Nucl. Phys.* **B 897** (2015) 781 [arXiv:1502.00007] [INSPIRE].

[25] S. Stieberger, $(0,2)$ *heterotic gauge couplings and their M-theory origin*, *Nucl. Phys.* **B 541** (1999) 109 [hep-th/9807124] [INSPIRE].

[26] R. Dijkgraaf, E.P. Verlinde and H.L. Verlinde, *Counting dyons in* $N = 4$ *string theory*, *Nucl. Phys.* **B 484** (1997) 543 [hep-th/9607026] [INSPIRE].

[27] G. Lopes Cardoso, B. de Wit, J. Kappeli and T. Mohaupt, *Asymptotic degeneracy of dyonic* $N = 4$ *string states and black hole entropy*, *JHEP* **12** (2004) 075 [hep-th/0412287] [INSPIRE].

[28] D. Shih, A. Strominger and X. Yin, *Recounting dyons in* $N = 4$ *string theory*, *JHEP* **10** (2006) 087 [hep-th/0505094] [INSPIRE].

[29] D. Gaiotto, *Re-recounting dyons in* $N = 4$ *string theory*, hep-th/0506249 [INSPIRE].

[30] S. Hohenegger and S. Stieberger, *BPS saturated string amplitudes:* $K3$ *elliptic genus and Igusa cusp form*, *Nucl. Phys.* **B 856** (2012) 413 [arXiv:1108.0323] [INSPIRE].

[31] D.P. Jatkar and A. Sen, *Dyon spectrum in CHL models*, *JHEP* **04** (2006) 018 [hep-th/0510147] [INSPIRE].

[32] J.R. David and A. Sen, *CHL dyons and statistical entropy function from D1-D5 system*, *JHEP* **11** (2006) 072 [hep-th/0605210] [INSPIRE].

[33] V.V. Nikulin, *Finite automorphism groups of Kähler* $K3$ *surfaces*, *Trans. Moscow. Math. Soc.* **38** (1979) 71.

[34] P. Candelas, G.T. Horowitz, A. Strominger and E. Witten, *Vacuum configurations for superstrings*, *Nucl. Phys.* **B 258** (1985) 46 [INSPIRE].

[35] E. Witten, *Symmetry breaking patterns in superstring models*, *Nucl. Phys.* **B 258** (1985) 75 [INSPIRE].

[36] T. Eguchi, P.B. Gilkey and A.J. Hanson, *Gravitation, gauge theories and differential geometry*, *Phys. Rept.* **66** (1980) 213 [INSPIRE].

[37] M.C.N. Cheng, $K3$ *surfaces,* $N = 4$ *dyons and the Mathieu group* M_{24}, *Commun. Num. Theor. Phys.* **4** (2010) 623 [arXiv:1005.5415] [INSPIRE].

[38] M.R. Gaberdiel, S. Hohenegger and R. Volpato, *Mathieu twining characters for* $K3$, *JHEP* **09** (2010) 058 [arXiv:1006.0221] [INSPIRE].

[39] M.R. Gaberdiel, S. Hohenegger and R. Volpato, *Mathieu moonshine in the elliptic genus of* $K3$, *JHEP* **10** (2010) 062 [arXiv:1008.3778] [INSPIRE].

[40] T. Eguchi and K. Hikami, *Note on twisted elliptic genus of* $K3$ *surface*, *Phys. Lett.* **B 694** (2011) 446 [arXiv:1008.4924] [INSPIRE].

[41] T. Eguchi, H. Ooguri, A. Taormina and S.-K. Yang, *Superconformal algebras and string compactification on manifolds with* $\mathrm{SU}(N)$ *holonomy*, *Nucl. Phys.* **B 315** (1989) 193 [INSPIRE].

[42] T. Eguchi, H. Ooguri and Y. Tachikawa, *Notes on the $K3$ surface and the Mathieu group M_{24}*, Exper. Math. **20** (2011) 91 [arXiv:1004.0956] [INSPIRE].

[43] S. Govindarajan and K. Gopala Krishna, *BKM Lie superalgebras from dyon spectra in Z_N CHL orbifolds for composite N*, JHEP **05** (2010) 014 [arXiv:0907.1410] [INSPIRE].

[44] T. Kawai, *String duality and modular forms*, Phys. Lett. **B 397** (1997) 51 [hep-th/9607078] [INSPIRE].

[45] G. Lopes Cardoso, *Perturbative gravitational couplings and Siegel modular forms in $D = 4$, $N = 2$ string models*, Nucl. Phys. Proc. Suppl. **56B** (1997) 94 [hep-th/9612200] [INSPIRE].

[46] T. Kawai, *$N = 2$ heterotic string threshold correction, $K3$ surface and generalized Kac-Moody superalgebra*, Phys. Lett. **B 372** (1996) 59 [hep-th/9512046] [INSPIRE].

[47] J.R. David, D.P. Jatkar and A. Sen, *Dyon spectrum in $N = 4$ supersymmetric type II string theories*, JHEP **11** (2006) 073 [hep-th/0607155] [INSPIRE].

[48] C. Angelantonj, I. Florakis and M. Tsulaia, *Universality of gauge thresholds in non-supersymmetric heterotic vacua*, Phys. Lett. **B 736** (2014) 365 [arXiv:1407.8023] [INSPIRE].

[49] C. Angelantonj, I. Florakis and M. Tsulaia, *Generalised universality of gauge thresholds in heterotic vacua with and without supersymmetry*, Nucl. Phys. **B 900** (2015) 170 [arXiv:1509.00027] [INSPIRE].

JHEP02(2016)056