

# (Non)-renormalization of the chiral vortical effect coefficient

---

**Siavash Golkar and Dam T. Son**

*Kadanoff Center for Theoretical Physics and Enrico Fermi Institute, University of Chicago,  
5640 South Ellis Ave., Chicago, IL 60637, U.S.A.*

*E-mail:* [golkar@uchicago.edu](mailto:golkar@uchicago.edu), [dtson@uchicago.edu](mailto:dtson@uchicago.edu)

**ABSTRACT:** We show using diagrammatic arguments that in some (but not all) cases, the temperature dependent part of the chiral vortical effect coefficient is independent of the coupling constant. An interpretation of this result in terms of quantization in the effective 3 dimensional Chern-Simons theory is also given. In the language of 3D, dimensionally reduced theory, the value of the chiral vortical coefficient is related to the formula  $\sum_{n=1}^{\infty} n = -1/12$ . We also show that in the presence of dynamical gauge fields, the CVE coefficient is not protected from renormalization, even in the large  $N$  limit.

**KEYWORDS:** Anomalies in Field and String Theories, Renormalization Group, 1/N Expansion

ARXIV EPRINT: [1207.5806](https://arxiv.org/abs/1207.5806)

---

## Contents

|          |  |           |
|----------|--|-----------|
| <b>1</b> | <b>Introduction</b>                                  | <b>1</b>  |
| <b>2</b> | <b>Calculation of the CVE coefficient</b>            | <b>2</b>  |
| 2.1      | Coupling to scalars                                  | 2         |
| 2.2      | Coupling to gauge fields                             | 5         |
| <b>3</b> | <b>Connection to 3 dimensional Chern-Simons term</b> | <b>7</b>  |
| 3.1      | Quantization of the level                            | 7         |
| 3.2      | Zero coupling example                                | 9         |
| <b>4</b> | <b>Conclusion</b>                                    | <b>10</b> |

---

## 1 Introduction

Recently, the manifestation of chiral anomaly in hydrodynamics has been under considerable attention. Since the original observations in gauge/gravity duality [1–3], it has been shown that the hydrodynamic equations of systems with anomalies necessarily contain additional terms if both the equations of anomalies and the second law of thermodynamics are to be preserved [4]. These new terms lead to two new effects: the chiral magnetic effect (CME) and the chiral vortical effect (CVE), i.e. the appearance of a current in the presence of a magnetic field or a vorticity of a fluid flow. These effects are dissipationless and hence are proportional to equilibrium thermodynamic quantities rather than kinetic coefficients [5, 6].

It has been found that anomaly matching within hydrodynamics fixes the coefficients of the CME and CVE up to a single numerical coefficient [4, 7]. This coefficient determines the magnitude of the CVE at zero chemical potential. Consider a fluid flow with a velocity profile  $u^\mu(x)$ . The coefficient under consideration is  $C_1$  in the following relationship

$$j^{5\mu} = C_1 T^2 \frac{1}{2} \epsilon^{\mu\nu\lambda\rho} u_\nu \partial_\lambda u_\rho. \tag{1.1}$$

Landsteiner et al. [8, 9] show that in theories with gravity dual, the value of this coefficient in holography (i.e. at strong coupling) coincides with its value at zero coupling, suggesting coupling-constant independence (For a different approach based on kinetic theory see [10]). In a theory with a single fermion of right-handed chirality,  $C_1 = 1/12$ . Landsteiner et al. suggested that this coefficient is related to a gravitational anomaly (more precisely, the gravitational contribution to the divergence of an axial current). Although the connection between the zero-chemical-potential CVE and the gravitational anomaly is direct in holography, it has not been proven outside of holography. In fact, naive power counting indicates that the effects of gravitational anomaly can appear only in higher-order (namely, third-order) hydrodynamics.

In this paper, we demonstrate that in a system of fermions coupled through Yukawa interactions, the CVE coefficient at zero chemical potential is independent of the coupling-constant. We first provide a perturbative analysis of diagrams to show that the one-loop fermion graph is the only graph contributing to the relevant Green function appearing in the Kubo's formula for zero- $\mu$  CVE coefficient. We then show a connection between the CVE and the 3D Chern-Simons term appearing in the dimensionally reduced effective field theory of the thermal system. Using this connection, we show that we can understand the numerical value of  $C$  from the summation formula  $\sum_{n=1}^{\infty} n = -1/12$ . (The non-renormalization in this case can also be inferred from the results in [11], where a different approach based on fluid dynamics in terms of group-valued variables was taken.)

However, it turns out that for the case of coupling to gauge fields, the CVE coefficient is not in fact protected against renormalization. We demonstrate that even in the large  $N$  limit, there is a single class of diagrams that contribute to the CVE coefficient and explicitly evaluate this at the two loop order.

The paper is organized as follows. In section 2 we present a perturbative proof of the non-renormalization of the chiral vortical coefficient  $C_1$ . Our proof is based on similar proof, by Coleman and Hill, of the non-normalization of the fermion-induced Chern-Simons terms in (2+1) dimensions. We elaborate more on the connection to the connection to the 3D Chern-Simons theory in section 3. We conclude in section 4.

**Note:** as this work was nearing completion, we were informed of [12] which deals with issues similar to the ones considered in this paper. The first version of this paper included an error which was kindly pointed out by H. Ren et al. who also have a forthcoming paper discussing its implications [13].

## 2 Calculation of the CVE coefficient

In this section, we look at two cases. First we consider a Dirac fermion coupled to scalars through a chiral Yukawa term and show that the CVE is independent of this interaction. In the second part of this section, we show that this is also the case for fermions coupled to a dynamical gauge field in leading order in the large  $N$  expansion, where  $N$  is the number of colors associated with the gauge field.

### 2.1 Coupling to scalars

To be concrete, we first concentrate on a simplest interacting quantum field theory with a conserved axial current; namely, a linear sigma model which contains a Dirac fermion interacting with a scalar field.

$$S = \int dt d^3x \left[ i\bar{\psi}\gamma^\mu D_\mu\psi - D_\mu\phi^* D^\mu\phi - m^2\phi^*\phi - \lambda(\phi^*\phi)^2 - g \left( \bar{\psi}\frac{1+\gamma^5}{2}\psi\phi^* + \bar{\psi}\frac{1-\gamma^5}{2}\psi\phi \right) - \frac{1}{2}h_{\mu\nu}T^{\mu\nu} + \mathcal{O}(h^2) \right], \quad (2.1)$$

with

$$\begin{aligned}
 D_\mu \psi &= (\partial_\mu - i\gamma^5 A_\mu)\psi, \\
 D_\mu \phi &= (\partial_\mu - 2iA_\mu)\phi, \\
 T^{\mu\nu} &= \frac{i}{4}\bar{\psi}\gamma^\mu(\overrightarrow{D}^\nu - \overleftarrow{D}^\nu)\psi + \mu \leftrightarrow \nu + \text{scalar contributions}.
 \end{aligned}
 \tag{2.2}$$

We have include in the action the coupling to external axial U(1) gauge field  $A_\mu$  and small metric perturbation  $h_{\mu\nu}$ .

The symmetries associated with the background fields are anomalous, and the anomalies are captured by the Ward identities:

$$\partial_\mu J_5^\mu = -\frac{1}{48\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma},
 \tag{2.3}$$

$$\partial_\mu T^{\mu\nu} = F^{\nu\rho}\left(J_\rho - \frac{1}{12\pi^2}\epsilon_\rho^{\sigma\alpha\beta}A_\sigma F_{\alpha\beta}\right),
 \tag{2.4}$$

where  $J_5^\mu = \bar{\psi}\gamma^\mu\gamma^5\psi$ ,  $F_{\mu\nu}$  is the field strength associated with  $A_\mu$ . We note that the axial current is conserved in the absence of sources.

The goal here is to calculate the chiral vortical coefficient at zero chemical potential ( $A_0 = 0$ ). As explained in refs. [9], this coefficient is not a proper kinetic coefficient, but is basically an equilibrium quantity given by the behavior of the retarded two-point Green's function between the current  $J_A^i$  and the momentum density  $T^{0j}$  at zero frequency and small momenta,

$$G_R^{i,0j}(\omega, k)\Big|_{\omega=0} = i\epsilon_{ijn}k_n\sigma_A^{\mathcal{V}} + O(k^2)
 \tag{2.5}$$

The fact that we are interested in the zero-frequency limit allows the chiral vortical coefficient to be related to an Euclidean Green's function

$$iG_R^{i,0j} = \mathcal{G}^{i,0j} = \frac{\delta}{\delta A_i}\frac{\delta}{\delta g_{oj}}Z = \langle J_A^i T^{0j} \rangle + \text{contact terms},
 \tag{2.6}$$

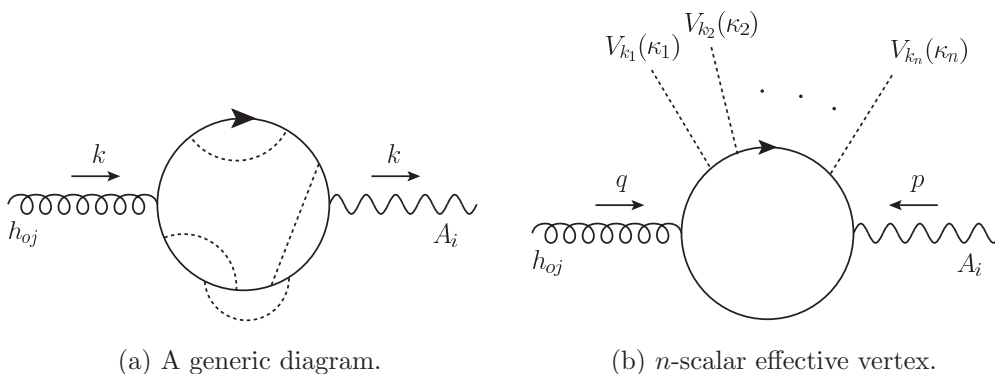
where  $\mathcal{G}^{i,0j}$  is the Euclidean (Matsubara) Green's function. At zero frequency we can take our source fields  $A_\mu$  and  $h_{\mu\nu}$  to be time independent from the start. With this configuration, it is easy to see that the right hand side of the anomalous Ward identities in equations (2.4) vanish and the symmetry is effectively restored.

We will demonstrate diagrammatically, using an argument similar to that of Coleman and Hill who showed the non-normalization of the Chern-Simons term obtained by integrating out massive fermions in 3D,<sup>1</sup> that only diagrams that are present in the zero-coupling limit give non-zero contributions to the CVE coefficient [15]. The calculation at zero coupling is given in [8], and our argument would show that the calculation is exact at any value of the coupling constant.

Let us now look at a generic diagram at non-zero coupling (see figure 1a). All such diagrams can be obtained from the  $n$ -scalar effective vertices with exactly one insertion

---

<sup>1</sup>It is known that there are caveats to the Coleman-Hill theorem (see for example [14]). Namely when there are parity-violating interactions in the initial Lagrangian. However, as there are no such terms, they do not concern the treatment in this paper.



**Figure 1.** (a) A generic diagram at finite coupling. The internal lines are the scalar field and the external lines are the graviton and the axial U(1) field. (b) An  $n$  scalar effective vertex. We can get all diagrams of type (a) by contracting the external scalar lines on type (b) diagrams.

of  $J_5^\mu$  and  $T_{\mu\nu}$  by contracting external scalar potentials and integrating over the scalar momenta. Let us look at the central block of the construction:

$$\Gamma_{ij}^{(n)}(p, q, \kappa_1, \dots, \kappa_n). \tag{2.7}$$

Graphically, this is a one-loop graph with  $n$  external dynamical scalars, 1 external graviton  $b_i$  and 1 external source gauge boson  $a_i$  (figure 1b).

The gauge invariance of the time independent source fields implies:

$$\begin{aligned} p^i \Gamma_{ij}^{(n)}(p, q, \kappa_1, \dots, \kappa_n) &= 0, \\ q^j \Gamma_{ij}^{(n)}(p, q, \kappa_1, \dots, \kappa_n) &= 0. \end{aligned} \tag{2.8}$$

Differentiating these equations with respect to  $p^r$  and  $q^r$  and then letting  $p$  and  $q$  be zero respectively we get:

$$\begin{aligned} \Gamma_{ijk_1 \dots k_n}^{(n)}(0, q, \kappa_1, \dots, \kappa_n) = 0 &\Rightarrow \Gamma_{ijk_1 \dots k_n}^{(n)}(p, q, \kappa_1, \dots, \kappa_n) = \mathcal{O}(p) \\ \Gamma_{ijk_1 \dots k_n}^{(n)}(p, 0, \kappa_1, \dots, \kappa_n) = 0 &\Rightarrow \Gamma_{ijk_1 \dots k_n}^{(n)}(p, k, \kappa_1, \dots, \kappa_n) = \mathcal{O}(k), \end{aligned} \tag{2.9}$$

where we have used the fact that these functions are analytic in  $k$ . For  $n = 0$  the two momenta  $p$  and  $q$  are not independent and hence we derive:

$$\Gamma_{ijk_1 \dots k_n}^{(0)}(p, -p) = \mathcal{O}(p). \tag{2.10}$$

However, for  $n > 0$  these two momenta are independent and (2.9) implies:

$$\Gamma_{ijk_1 \dots k_n}^{(n)}(p, q, \kappa_1, \dots, \kappa_n) = \mathcal{O}(pq). \tag{2.11}$$

This is all we need for our proof. The full contribution to the CVE coefficient is the sum of the zero coupling diagram calculated above and diagrams with internal scalars. The latter can be constructed from our effective  $n$  point vertices by contracting the external scalar lines.

Two cases arise. If the source fields  $a_i$  and  $b_i$  are attached to the same fermion loop, the total diagram is  $\mathcal{O}(k^2)$  by equation (2.11). Using the same argument, if they are attached to two different fermion loops, we again get that the total diagram is  $\mathcal{O}(k^2)$  by applying equation (2.9) for each loop. The same reasoning can be applied to the case with the scalar stress tensor insertion and the case with scalar internal loops, implying that all such diagrams are  $\mathcal{O}(k^2)$ . On the other hand the chiral vortical coefficient is the term linear in the external momentum of the two-point Green's function. Thus, only the one-loop free diagram can contribute to the CVE. This completes the proof.

It is important to note that the analyticity arguments given above require the mass of the scalars to be non-zero. Given the fact that scalars generically acquire mass at finite temperature, this requirement is equivalent to the statement that we are not sitting at a second order phase transition.

## 2.2 Coupling to gauge fields

We now turn to the case with fermions coupled to dynamical gauge fields. As claimed in the introduction, in this case, the CVE in fact does receive corrections. We show that for the non-abelian theory at leading order in  $1/N$  there is a single class of diagrams that give non-zero contributions to the CVE coefficient. To demonstrate this, we point out where the arguments of the previous section are affected. The action is:

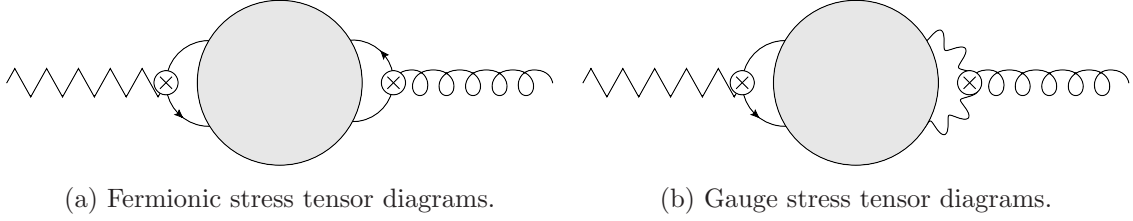
$$S = \int dt d^3x \left( i\bar{\psi}\not{D}\psi - \frac{1}{4g^2}V_{\mu\nu}V^{\mu\nu} + eA_\mu J_5^\mu - \frac{1}{2}h_{\mu\nu}T^{\mu\nu} + \mathcal{O}(h^2) \right), \quad (2.12)$$

where  $D$  and  $V_{\mu\nu}$  are respectively the covariant (vector-like) coupling and curvature tensor associated with the dynamical gauge field  $V_\mu$ . We assume one flavor of fermions and we suppress the Lie group indices.

Following the steps of the scalar-coupling case discussed above, we come across two issues and we tackle them one by one, one of which ultimately leads to radiative corrections. The first difference is that there are now potentially gapless excitations associated with the gluons. The infrared singularities associated with such massless modes threaten to ruin the analyticity arguments that were crucial in the proof. This is the same caveat that also came up in the original Coleman and Hill paper [15]. However, since we are working with non-abelian theories, the solution provided there does not apply.

Fortunately, non-abelian theories at finite temperature do produce a mass gap non-perturbatively [16]. To take advantage of this fact, we divide the arguments of the previous section into two steps. First, we perturbatively integrate out the fermionic fields to give effective vertices with gauge fields, the axial source and the graviton (respectively  $V_\mu$ ,  $A_i$  and  $h_{0i}$ ) as external lines. According to the previous section, the diagrams with more than two external legs are  $\mathcal{O}(k^2)$ .

Next, we evaluate the expectation value of these effective vertices non-perturbatively. This method allows us to use both our perturbative diagrammatic arguments of the previous section while steering clear of the IR singularities that plague perturbative analysis of gauge fields.



**Figure 2.** The two diagram classes.

The second difference is that the anomaly (equation (2.4)) no longer vanishes in the absence of external sources and is now given by:

$$\partial_\mu J_5^\mu = -\frac{g^2 C(r)}{16\pi^2 \sqrt{-g}} \epsilon^{\mu\nu\rho\sigma} V_{\mu\nu} V_{\rho\sigma} + \dots, \quad (2.13)$$

where we have omitted the terms higher order in the external sources. This is a big problem as the arguments of the previous section all depended on the conservation of the axial current.

At this point, we split the diagrams into two groups depending on the form of the Energy-Momentum tensor (see figure 2). The first group has a fermionic Energy Momentum insertion (figure 2a) and the second has the gauge part (figure 2b). For the first group, at leading order in  $1/N$ , we have a single fermion loop to which both the axial current  $J_5^i$  and the energy-momentum tensor  $T^{0i}$  are attached. The anomalous contribution of such one loop diagrams is exactly captured in equation (2.13). Expanding the right hand side in metric perturbations, we see that to linear order, the anomaly only depends on the trace of the perturbation and thus does not contribute to correlators involving a single insertion of  $T^{0i}$ .

However, it turns out that there are certain diagrams in the second class that do in fact contribute to the CVE. Since in the absence of the anomaly in equation (2.13), there would be no radiative corrections, it is clear that the only contributing diagram are those that include a triangle subdiagram as in figure 3a. Figure 3b shows the leading order diagram in this class which we now proceed to calculate.

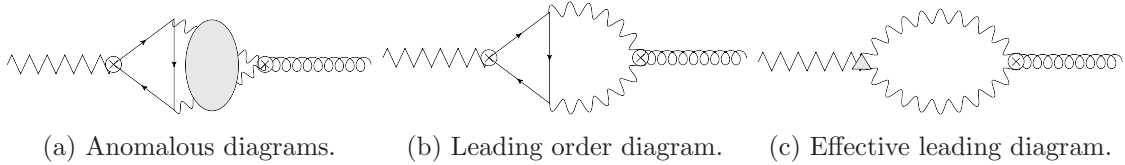
Since we already know that the only contribution comes from the anomaly in (2.13), we can replace the triangle part of the two loop diagram with the effective vertex:

$$J_{\text{Anom}}^\mu = -\frac{g^2 C(r)}{4\pi^2 \sqrt{-g}} \epsilon^{\mu\nu\rho\sigma} V_\nu \partial_\rho V_\sigma \quad (2.14)$$

which captures the divergence of the axial current, and we are effectively left with only a one loop diagram (figure 3c). The calculation of the diagram is slightly complicated by the fact that different components of the gauge field receive different effective masses which are in general very important for our arguments. However, in this case it turns out that there are no infrared divergences even at zero effective mass and we present the calculation with this simplification.

The anomalous contribution to the Euclidean Green's function is:

$$\mathcal{G}_{\text{Anom}}^{i,0j} = \langle J_{\text{Anom}}^i T^{0j} \rangle = \frac{-g^2 T C(r) d(G)}{4\pi^2} \epsilon^{ijm} k_m \int \frac{d^3 p}{(2\pi)^3} \frac{\omega^2 - \frac{p^2}{3}}{(\omega^2 + p^2)^2} + \dots \quad (2.15)$$



**Figure 3.** (a) The only class of diagrams with non-vanishing corrections to the CVE in the large  $N$  limit. (b) The leading order diagram. (c) After replacing the leading order diagram with the effective vertex  $J_{\text{Anom}}^\mu$ . The triangle vertex represents the anomalous effect of the triangle diagram.

where  $d(G)$  is the dimension of the adjoint representation and we have expanded up to first order in  $k$ . It is now a simple matter to carry out the sum and the integral:

$$\mathcal{G}_{\text{Anom}}^{i,0j} = \frac{-g^2 C(r) d(G)}{48\pi^4} \epsilon^{ijm} k_m \int dp p \frac{1 - e^{-2\frac{p}{T}} - 4e^{-\frac{p}{T}} \frac{p}{T}}{\left(1 - e^{-\frac{p}{T}}\right)^2} \rightarrow \frac{g^2 C(r) d(G)}{48\pi^4} \epsilon^{ijm} k_m, \quad (2.16)$$

where in the last step we have removed the divergent zero temperature contribution before doing the integral. In fact, at zero temperature, this contribution is not allowed by full Lorentz symmetry and vanishes (This is also verified by direct calculation). Therefore, any regulator that respects full Lorentz symmetry in the zero temperature limit would reproduce the same result.

We now read off the leading order correction to the CVE coefficient in the large  $N$  limit:

$$\sigma_{\text{Correction}}^\nu = \frac{g^2 C(r) d(G)}{48\pi^2} T^2. \quad (2.17)$$

### 3 Connection to 3 dimensional Chern-Simons term

The diagrammatic arguments put forward in section 2 were originally presented as proof of non-renormalization of the  $U(1)$  Chern-Simons term in 3 dimensions. It is well known that the topological nature of the non-abelian Chern-Simons term leads to its quantization and therefore, does not receive radiative corrections. Similar arguments for the Abelian case can be made if one assumes a compact spatial manifold.

In this section, we will show that under similar assumptions, one can show that the quantization of the Abelian Chern-Simons term in 3-dimensions leads to the non-renormalization of the CVE coefficient. First, we show that gauge invariance leads to the quantization of the level and then, as an example, we verify that the free theory satisfies these quantization conditions.

#### 3.1 Quantization of the level

In order to see the connection between the 3-dimensional Chern-Simons term and the CVE, we dimensionally reduce the theory on the compactified time direction [5]. To this end, we rename the relevant sources:

$$a_i = A_i(x), \quad (3.1)$$

$$b_i = T g_{0i}, \quad (3.2)$$



which transform under gauge and coordinate transformations with respective parameters  $\alpha$  and  $\epsilon_\mu$  as follows:

$$a_i \rightarrow a_i + \partial_i \alpha, \tag{3.3}$$

$$b_i \rightarrow b_i + T (\nabla_i \epsilon_0 + \nabla_0 \epsilon_i), \tag{3.4}$$

where  $\epsilon_\mu$  is the diffeomorphism parameter ( $x^\mu \rightarrow x^\mu + \epsilon^\mu$ ). As we argued in section 2, these 3-dimensional gauge transformations are not anomalous, even in the presence of anomalies in the underlying 4-dimensional theory.<sup>2</sup> The term of concern in the effective action is:

$$S_{\text{eff}} = i \kappa \int d^3x \epsilon_{ijk} a_i \partial_j b_k = i \kappa T \int d^4x \epsilon_{ijk} a_i \partial_j b_k, \tag{3.5}$$

where in the final equality we have inserted a spurious factor of  $T$  along with a trivial time integral in order to make contact with the 4-dimensional 2 point function at hand. With this definition, the Green's function becomes:

$$\mathcal{G}^{i,0j} = T^2 \frac{\delta}{\delta a_i} \frac{\delta}{\delta b_j} Z_{\text{eff}} = -\kappa T^2 \epsilon_{ijk} p_k, \tag{3.6}$$

which leads to a contribution of the CVE coefficient of

$$\sigma_A^{\mathcal{V}} = \kappa T^2. \tag{3.7}$$

To see the quantization of the coefficient  $\kappa$ , we put the system on a 3 torus of sides  $L_1$ ,  $L_2$  and  $L_3$ . The periodicity of the manifold dictates the periodicity of the compact gauge fields  $a_i$  and  $b_i$ . Using large gauge transformations  $\alpha = \frac{2\pi x_i}{e L_i}$  and  $\epsilon_0 = \frac{2 x_i}{T L_i}$ , we see:

$$a_i \equiv a_i + \frac{2\pi}{e L_i}, \tag{3.8}$$

$$b_i \equiv b_i + \frac{2}{L_i}, \tag{3.9}$$

where  $e$  is the electric charge and the factor 2 in the time translation parameter  $\epsilon_0$  is compensating for the anti-periodicity of fermions in the time direction.

We then perform a large gauge transformation on the  $a_i$  fields with  $\alpha = 2\pi n \frac{x_3}{e L_3}$ , with  $n \in \mathbb{Z}$ . The change in the action is:

$$\delta S_{\text{eff}} = 2\pi n i \frac{\kappa}{e} \int_{x_3=0} \epsilon_{3jk} \partial_j b_k. \tag{3.10}$$

Choosing a field configuration with non-trivial winding around the compact directions such as  $b_2 = \frac{2x_1}{L_1 L_2}$ , the variation becomes:

$$\delta S_{\text{eff}} = 2\pi n i \frac{\kappa}{e} \int_{x_3=0} \epsilon_{312} \partial_1 b_2 = 4\pi n i \frac{\kappa}{e}, \tag{3.11}$$

which gives us the desired quantization result of  $\kappa \in \frac{e}{2} \mathbb{Z}$ .<sup>3</sup>

<sup>2</sup>For coupling to gauge fields, this is only true at leading order in  $1/N$ .

<sup>3</sup>This is half the expected result of integer  $\kappa$  values and is solely due to the fact that the fermions are anti-periodic in the thermal direction as stated under equation (3.9).

### 3.2 Zero coupling example

We now look at the Kaluza-Klein reduction of the action (2.1) at zero coupling. The fermions decompose on the thermal circle with periodicity  $\beta = \frac{1}{T}$  as:

$$\psi(t, x) = e^{2\pi i n \frac{t}{T}} \psi_n(x), \quad n \in \mathbb{Z} + \frac{1}{2}, \quad (3.12)$$

and we assume the background sources have no time dependence. At zero coupling the different Matsubara frequencies do not mix and we can analyze the partition function mode by mode. For the  $n$ 'th mode we have:

$$S = \frac{1}{T} \int d^3x \left( i\bar{\psi}_{-n} \not{\partial} \psi_n + A_\mu J_{5,n}^\mu - h_{0i} T_n^{0i} \right), \quad (3.13)$$

with

$$\begin{aligned} J_{5,n}^\mu &= \bar{\psi}_{-n} \gamma^\mu \gamma^5 \psi_n, \\ T_n^{0i} &= \frac{i}{4} \bar{\psi}_{-n} \gamma^0 (\overrightarrow{\partial}^i - \overleftarrow{\partial}^i) \psi_n - n\pi \bar{\psi}_{-n} \gamma^i \psi_n. \end{aligned} \quad (3.14)$$

The diagram to be evaluated is given in figure 1a with no internal vertices. We have:

$$\mathcal{G}^{i,0j} = -T \int \frac{d^3p}{(2\pi)^3} \text{Tr} \frac{(\not{p} + \not{k})_3 - 2\pi m T \gamma^0}{(p+k)^2 + 4\pi^2 m^2 T^2} \left( \frac{\gamma^0}{4T} (2p+k)^j - m\pi \gamma^j \right) \frac{\not{p}_3 - 2\pi m T \gamma^0}{p^2 + 4\pi^2 m^2 T^2} \gamma^i \gamma^5, \quad (3.15)$$

where  $\not{p}_3$  denotes contraction with  $\gamma^i$ , the spatial gamma matrices. We have:

$$\epsilon_{ijn} \mathcal{G}^{i,0j} = -T \int \frac{d^3p}{(2\pi)^3} \frac{\epsilon_{ijn} \text{Tr}(\gamma^0 \gamma^r \gamma^s \gamma^i \gamma^5) N_{rs}^j}{((p+k)^2 + 4\pi^2 m^2 T^2)(p^2 + 4\pi^2 m^2 T^2)}, \quad (3.16)$$

where

$$\begin{aligned} N_{rs}^j &= (-2\pi m T)(-\pi m \delta_r^j) p_s - (p+k)_r \frac{1}{4T} (2p+k)^j p_s + (p+k)_r (-\pi m \delta_s^j) (-2\pi m T) \\ &= -2\pi^2 m^2 T \delta_r^j k_s - \frac{1}{4T} k_r p_s (2p+k)^j + \text{Sym in } r \leftrightarrow s. \end{aligned} \quad (3.17)$$

Using the above and taking the limit of  $k \rightarrow 0$ , we have:

$$\lim_{k \rightarrow 0} \epsilon_{ijn} \mathcal{G}^{i,0j} = T \int \frac{d^3p}{(2\pi)^3} \frac{\epsilon_{ijn} 4\epsilon^{rsi} (2\pi^2 m^2 T \delta_r^j k_s + \frac{1}{2T} k_r p_s p^j)}{(p^2 + 4\pi^2 m^2 T^2)^2}. \quad (3.18)$$

This integral is of course divergent and needs to be regularized. Here we simply use the relation between different integrals in dimensional regularization.<sup>4</sup> Pauli-Villars regularization gives the same result:

$$\begin{aligned} \lim_{k \rightarrow 0} \epsilon_{ijn} \mathcal{G}^{i,0j} &= 16m^2 T^2 k_n \int_0^\infty dp \frac{p^2}{(p^2 + 4\pi^2 m^2 T^2)^2}, \\ &= 2|m| T k_n. \end{aligned} \quad (3.19)$$

<sup>4</sup>Here we just need the simplest of these relations:

$$\int \frac{d^d l}{(2\pi)^d} \frac{l^\mu l^\nu}{(l^2 + \Delta)^2} = \frac{-\Delta}{d-2} \int \frac{d^d l}{(2\pi)^d} \frac{1}{(l^2 + \Delta)^2}.$$

And finally we write down the contribution to the CVE coefficient for each mode:

$$\sigma_{A,m}^{\mathcal{V}} = |m|T. \tag{3.20}$$

Which satisfies the quantization condition in the previous section. We evaluate the full CVE coefficient:

$$\sigma_A^{\mathcal{V}} = T \sum_{m=-\infty}^{\infty} \sigma_{A,m}^{\mathcal{V}} = 2T^2 \sum_{m=\frac{1}{2}}^{\infty} m \rightarrow \frac{1}{12}T^2. \tag{3.21}$$

Where we have used the  $\zeta$ -function regularization. This is the same result as [8]. In essence, this calculation is the same with the order of the 3-dimensional momentum integral and the Matsubara sum.

## 4 Conclusion

We have shown in detail that, in so far as the perturbative loop expansion is valid, the zero chemical potential, temperature dependent part of the chiral vortical effect coefficient does not receive any radiative corrections from Yukawa type couplings. We have also shown that this non-renormalization is related to the quantized nature of the 3-dimensional effective action.

None of this holds, however, when there are couplings to dynamical gauge fields present in the theory. In this case, even at large  $N$ , radiative corrections are still present and we have calculated the leading correction to the CVE coefficient in this limit.

It is curious that the actual value of the CVE coefficient does not satisfy the quantization condition, despite the fact that the contribution from each Matsubara mode does. Considering the fact that the proof of the quantization requires invariance under *both* large axial gauge and large diffeomorphism transformations, it seems very probable that this value would be fixed by considerations of global anomalies [17, 18]. This would also explain why arguments based on anomalous Ward identities such as in [6], which are consequences of anomalies only in the infinitesimal transformations, would not be able to determine this coefficient.

## Acknowledgments

This work is supported, in part, by DOE grant DE-FG02-00ER41132. The authors thank David B. Kaplan and Larry Yaffe for discussion. We are also grateful to Hai-cang Ren for pointing out an error in the first version of this paper.

**Open Access.** This article is distributed under the terms of the Creative Commons Attribution License ([CC-BY 4.0](https://creativecommons.org/licenses/by/4.0/)), which permits any use, distribution and reproduction in any medium, provided the original author(s) and source are credited.

## References

- [1] S. Bhattacharyya, S. Lahiri, R. Loganayagam and S. Minwalla, *Large rotating AdS black holes from fluid mechanics*, *JHEP* **09** (2008) 054 [[arXiv:0708.1770](#)] [[INSPIRE](#)].
- [2] J. Erdmenger, M. Haack, M. Kaminski and A. Yarom, *Fluid dynamics of R-charged black holes*, *JHEP* **01** (2009) 055 [[arXiv:0809.2488](#)] [[INSPIRE](#)].
- [3] N. Banerjee et al., *Hydrodynamics from charged black branes*, *JHEP* **01** (2011) 094 [[arXiv:0809.2596](#)] [[INSPIRE](#)].
- [4] D.T. Son and P. Surowka, *Hydrodynamics with triangle anomalies*, *Phys. Rev. Lett.* **103** (2009) 191601 [[arXiv:0906.5044](#)] [[INSPIRE](#)].
- [5] N. Banerjee et al., *Constraints on fluid dynamics from equilibrium partition functions*, *JHEP* **09** (2012) 046 [[arXiv:1203.3544](#)] [[INSPIRE](#)].
- [6] K. Jensen, *Triangle anomalies, thermodynamics and hydrodynamics*, *Phys. Rev. D* **85** (2012) 125017 [[arXiv:1203.3599](#)] [[INSPIRE](#)].
- [7] Y. Neiman and Y. Oz, *Relativistic hydrodynamics with general anomalous charges*, *JHEP* **03** (2011) 023 [[arXiv:1011.5107](#)] [[INSPIRE](#)].
- [8] K. Landsteiner, E. Megias and F. Pena-Benitez, *Gravitational anomaly and transport*, *Phys. Rev. Lett.* **107** (2011) 021601 [[arXiv:1103.5006](#)] [[INSPIRE](#)].
- [9] K. Landsteiner, E. Megias and F. Pena-Benitez, *Anomalies and transport coefficients: the chiral gravito-magnetic effect*, [arXiv:1110.3615](#) [[INSPIRE](#)].
- [10] J.-H. Gao, Z.-T. Liang, S. Pu, Q. Wang and X.-N. Wang, *Chiral anomaly and local polarization effect from quantum kinetic approach*, *Phys. Rev. Lett.* **109** (2012) 232301 [[arXiv:1203.0725](#)] [[INSPIRE](#)].
- [11] V.P. Nair, R. Ray and S. Roy, *Fluids, anomalies and the chiral magnetic effect: a group-theoretic formulation*, *Phys. Rev. D* **86** (2012) 025012 [[arXiv:1112.4022](#)] [[INSPIRE](#)].
- [12] R.L.K. Jensen and A. Yarom, *Hydrodynamics with gravitational anomalies from thermodynamics on cones*, to appear.
- [13] H.L.Y. De-Fu Houya and H. Cang Ren, *A possible higher order correction to the vortical conductivity in a QED plasma*, to appear.
- [14] A. Khare, R. MacKenzie and M.B. Paranjape, *On the Coleman-Hill theorem*, *Phys. Lett. B* **343** (1995) 239 [[hep-th/9408091](#)] [[INSPIRE](#)].
- [15] S.R. Coleman and B.R. Hill, *No more corrections to the topological mass term in QED in three-dimensions*, *Phys. Lett. B* **159** (1985) 184 [[INSPIRE](#)].
- [16] D.J. Gross, R.D. Pisarski and L.G. Yaffe, *QCD and instantons at finite temperature*, *Rev. Mod. Phys.* **53** (1981) 43 [[INSPIRE](#)].
- [17] E. Witten, *Global aspects of current algebra*, *Nucl. Phys. B* **223** (1983) 422 [[INSPIRE](#)].
- [18] E. Witten, *Global gravitational anomalies*, *Commun. Math. Phys.* **100** (1985) 197 [[INSPIRE](#)].