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$\mathcal{N} = 2$ higher-derivative couplings from strings

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Abstract: We consider the Calabi-Yau reduction of the Type IIA eight derivative one-loop stringy corrections focusing on the couplings of the four dimensional gravity multiplet with vector multiplets and a tensor multiplet containing the NS two-form. We obtain a variety of higher derivative invariants generalising the one-loop topological string coupling, F_1, controlled by the lowest order Kähler potential and two new non-topological quantities built out of the Calabi-Yau Riemann curvature.

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Contents

1 Introduction and summary

The quantum corrections in $\mathcal{N}=2$ theories have received a great deal of attention. These are of two types: corrections proportional to the inverse tension of the string and corrections proportional to the string coupling constant. The former arise from perturbative and instantonic world-sheet corrections and are encoded in higher derivative terms in the ten-dimensional supergravity action, while the latter come from string loops and brane instantons. Perturbative low energy effective actions are expanded in a double perturbation series in the inverse tension and the coupling constant.

These corrections not only affect the moduli spaces of $\mathcal{N}=2$ theories, but are manifested in the higher-derivative couplings. A better control of these couplings is hence essential, as is demonstrated by the study of terms involving the Weyl chiral (supegravity) super field W. However most of the other (higher-derivative) couplings in $\mathcal{N}=2$ theories and their relation to string theory remain largely unexplored. We make some steps in this directions. Our study is mostly restricted to string one loop results and Calabi-Yau compactifications, and will not cover gauged $\mathcal{N}=2$ theories.

The better-understood structures, involving W holomorphically, are captured by the topological string theory. In particular, the F-term in the low energy effective action in four dimensions is related to the scattering amplitude of 2 selfdual gravitons and $(2g-2)$ selfdual graviphotons in the zero-momentum limit and is computed by the genus-g contribution F_g to the topological string partition function [1, 2]. Crucially, the genus-g contribution F_g also determines the partition functions of $\mathcal{N}=2$ global gauge theories.

There exists, however, a continuous deformation of the gauge theory which uses non-trivially the manifest SU(2) R-symmetry of theory. This is what happens in the so-called Omega background [3, 4]. The two-parameter gauge theory partition function in the Omega background has been computed recently and reduces to the standard gauge theory partition function only when the two parameters are set equal. It is an outstanding open problem to find string theory realisation of these backgrounds and understand the extension of the genus-g function F_g which determines the general $\mathcal{N}=2$ partition function, and which should involve scattering amplitude among 2 gravitons, $(2g-2)$ graviphotons, and $2n$ gauge fields in vector multiplets. Theses considerations have lead to a recent interest in explicit realisation of couplings $F_{g,n}W^{2g}V^{2n}$ [5–10].

Let us recall that the genus one partition function F_1 is special due to the fact that it is the only perturbative four-dimensional contribution, which survives the five-dimensional decompactification limit. The ten/eleven dimensional origin of these couplings is related to M5 brane anomalies and they lift to certain eight-derivative terms in the effective action [11, 12]. Until very recently only the gravitational part of these couplings was known (and it was checked that their reduction on CY manifolds does correctly reproduce F_1). At present, we have a much better control of the more general form of the couplings in general string backgrounds with fluxes turned on, so that an explicit calculation of the one-loop four-, six- and eight-derivative couplings in $\mathcal{N}=2$ theories, which should lead to the generalisation of F_1, is now within the reach.

In the four dimensional setting, recent developments in going beyond chiral couplings described by integrals over half of superpace [13], allow us to extend the list of higher derivative terms in several ways. The new couplings are constrained by $\mathcal{N}=2$ supersymmetry to be governed by real functions of the four dimensional chiral fields. The latter naturally include vector multiplets and two types of chiral backgrounds, one of which is the Weyl background, W^2, introduced above. The second chiral background we consider is constructed out of the components of a tensor multiplet containing the NS two-form, the so called universal tensor multiplet[1] and contains four derivative terms on its components, such as $(\nabla H)^2$, where $H = dB$ and B is the NS two-form. These ingredients then allow us to describe couplings which are characterised by polynomials of the type $[F^2+R^2+(\nabla H)^2]^n$, generalising the purely gravitational R^2 couplings discussed above. The function of the vec-

[1]While there is no obstacle in considering a background of an arbitrary number of tensor multiplets in principle, we restrict our considerations to the universal tensor multiplet. We therefore ignore here all the complex deformations of the internal Calabi-Yau; including these in the reduction should yield couplings for generic hyper- matter.

tor multiplet scalars and Weyl background controlling these couplings directly corresponds to the extended couplings $F_{g,n}W^{2g}V^{2n}$, when the tensor multiplet is ignored. Inclusion of the latter results to more general couplings that have not yet been discussed in the literature.

From a quantum gravity point of view, higher-derivative corrections serve as a means of probing string theory at a fundamental level. Even though the complete expansion involves all fields of the theory, so far the attention has been mostly concentrated on the gravitational action. In particular, the one-loop eight derivative R^4 ($\mathcal{O}(\alpha'^3)$) terms stand out among the stringy quantum corrections. Due to being connected to anomaly cancellation, they are not renormalised at higher loops and survive the eleven-dimensional strong coupling limit. These couplings also play a special role in Calabi-Yau reductions to four-dimensional $\mathcal{N}=2$ theories. Firstly, they have been instrumental in understanding the perturbative corrections to the metrics on moduli spaces. In addition, they give rise to the four-derivative R^2 couplings, and as mentioned above agree with F_1W^2.

In order to understand the stringy origin of more general higher derivative couplings in $\mathcal{N}=2$ theories, one needs to go beyond the purely gravitational couplings in ten dimensions. In the NS-NS sector of string theory, H^2R^3 couplings are specified by a five-point function [14]. Direct amplitude calculations beyond this order are exceedingly difficult, but recent progress in classification of string backgrounds using the generalised complex geometry and T-duality covariance provide rather powerful constrains on the structure of the quantum corrections in the effective actions. A partial result for the six-point function, obtained recently, together with T-duality constraints and the heterotic/type II duality beyond leading order, allows to recover the ten-dimensional perturbative action almost entirely [15] (the few yet unfixed terms mostly vanish in CY backgrounds and hence are not relevant for the current project). The eleven-dimensional lift of the modified coupling leads to the inclusion of the M-theory four-form field strength; the subsequent reduction on a non- trivial KK monopole background allows to incorporate the full set of RR fields in the one-loop eight-derivative couplings. This knowledge will be crucially used for obtaining the relevant four-dimensional $\mathcal{N}=2$ couplings.

The goal of this paper is to bring together some of these recent developments. In the process, we shall:

Confirm and specify some of the predictions of general $\mathcal{N}=2$ considerations and fix the a priori arbitrary quantities constrained solely by supersymmetry in terms of Calabi-Yau data

Discover new terms and couplings that have not been previously considered

Provide some tests and justification for the proposed lift of type IIA R^4 terms to eleven dimensions

A brief comment on the last point. Since the lift from ten to eleven dimensions involves a strong coupling limit, ones is normally suspicious of simple-minded arguments associated with just replacing the string theory NS three-form H by a four-form G. In $\mathcal{N}=2$ theories however the three- and four-form give rise to fields in the same super multiplet, namely

JHEP02(2014)093

the (real part of the) scalars u^I and the vectors A^I in the vector matter respectively (here the index I spans the vector multiplets). Hence verifying that the respective couplings involving u^I ad A^I are supersymmetric completions of each other provided a test of the lifting procedure.

We conclude this section by a summary of our results. A variety of four dimensional higher derivative terms of the type $[F^2 + R^2 + (\nabla H)^2]^n$ are characterised by giving the relevant functions of four dimensional chiral superfields that control them. From the point of view of the CY reduction, the order of derivatives of all terms in four dimensions is controlled solely by the power of the CY Riemann tensor appearing in the internal integrals. We therefore find that the eight, six and four derivative terms are controlled by the possible integrals involving none, one or two powers of the internal Riemann tensor, respectively.

We find, in particular, that only the lowest order Kähler potential is relevant for the eight derivative terms, since this is the natural real function of vector multiplet moduli arising in Calabi-Yau compactifications, describing the total volume of the internal manifold. At lower orders in derivatives, the Kähler potential still appears as part of the functions describing the various invariants, combined with the Riemann tensor on the CY manifold X, denoted by R_{mnpq}. Given that all traces of the latter vanish, the relevant internal integrals must necessarily contain the harmonic forms on the CY manifold. In the case at hand, the relevant forms are the $h^{(1,1)}$ two-forms $\omega_I \in H^2(X, \mathbb{Z})$, where $I,\ J = 1, \dots, h^{(1,1)}$, since we ignore the hypermultiplets arising from the $(2,1)$ cohomology. We then obtain the following tensorial objects

$$
\begin{aligned}
\mathcal{R}_{IJ} &= \int_X R_{mnpq}\, \omega_I{}^{mn} \omega_J{}^{pq}\,, \\
X_{IJ} &= \int_X \epsilon_{mnm_1\dots m_4} \epsilon_{pqn_1\dots n_4} R^{m_1m_2n_1n_2} R^{m_3m_4n_3n_4}\ \omega_I{}^{mn}\, \omega_J{}^{pq}\,,
\end{aligned} \tag{1.1}
$$

which control all the couplings that we were able to describe within $\mathcal{N}=2$ supergravity at six- and four-derivative order respectively. Similar to the standard derivation of the lowest order Kähler potential, one can deduce the existence of corresponding real functions whose derivatives lead to the the couplings (1.1). Finally, the inclusion of the Weyl and tensor superfields through additional multiplicative factors leads to the functions that characterise the corresponding couplings involving R^2 and $(\nabla H)^2$ respectively. For example, the R^2F^4 and R^2F^2 couplings lead to the functions

$$
\begin{aligned}
R^2(\nabla F)^2 \quad &\Rightarrow \quad 9\, A_{\mathsf{w}}\, \mathrm{e}^{2\,\mathcal{K}(Y,\bar{Y})}\,, \\
R^2F^2 \quad &\Rightarrow \quad -\frac{3\,\mathrm{i}}{8}\, \frac{\bar{A}_{\mathsf{w}}}{(\bar{Y}^0)^2}\, \mathcal{R}_{IJ} \left(\frac{Y^I}{Y^0} - \frac{\bar{Y}^I}{\bar{Y}^0} \right) \left(\frac{Y^J}{Y^0} + \frac{\bar{Y}^J}{\bar{Y}^0} \right)\,,
\end{aligned} \tag{1.2}
$$

where A_{w} is the scalar in the Weyl multiplet and the Y^I, Y^0 are standard vector multiplet projective coordinates, so that $z^I = Y^I/Y^0$, and $\mathcal{K}(Y, \bar{Y})$ is the lowest order Kähler potential.

There are further invariants arising from the reduction, that cannot be currently described in components[2] within supergravity, and are associated to terms involving H^{2n}

[2]Note, however the final comments in section 6, which may lead to more general couplings.

derivatives / invariants	4	6	8
$F(X, A_{\mathsf{w}}, A_{\mathsf{t}})$	$\underline{R^2}$, $(\nabla H)^2$	–	–
$\big[F^2 + R^2 + (\nabla H)^2\big]^2$	$\boxed{(\nabla F)^2}$	$\underline{R^2 F^2}$, $(\nabla H)^2\, F^2$	R^4, $R^2(\nabla H)^2$, $(\nabla H)^4$
$\big[F^2 + R^2 + (\nabla H)^2\big]^3$	–	$\boxed{F^2(\nabla F)^2}$	$\underline{R^2(\nabla F)^2}$, $(\nabla H)^2(\nabla F)^2$
$\big[F^2 + R^2 + (\nabla H)^2\big]^4$	–	–	$\boxed{(\nabla F)^4}$
Unknown	$H^2 F^2$	$H^2 F^4$	R^4, $H^6 F^2$, $H^2 F^6$

Table 1. A summary of higher-derivative couplings discussed here. The first row corresponds to chiral couplings involving the Weyl and tensor multiplets. The next three rows display the known non-chiral $\mathcal{N}=2$ invariants at each order of derivatives, while the last row summarises the currently unknown invariants that can arise. The double appearance of R^4 at the eight-derivative level corresponds to two different invariants (see (4.2) below).

with n odd, such as H^2F^2, H^2R^2 etc. We comment on some of these terms, either giving the leading terms that characterise them, or pointing out their apparent absence.

An inventory of the four- and six-derivative couplings studied in this paper is given in table 1. The first line in this table describes the terms based only on holomorphic functions of vector moduli and the two chiral backgrounds. These are the only couplings that are controlled by a topological quantity, namely the vector of second Chern classes of the Calabi-Yau four-cycles. The gravitational R^2 coupling is the first nontrivial coupling $F_g W^{2g}$ above, related to the topological string partition function [1, 2]. The second, third and fourth lines correspond to the non-holomorphic couplings of [13], where the tensor multiplet background is included. Note the diagonal of underlined invariants of the type R^2F^{2n}, which correspond to the first nontrivial couplings $F_{g,n}W^{2g}V^{2n}$, for $g = 1$, recently discussed in [6, 8, 9]. The diagonal of the blue boxed invariants gives the one-loop copings of vector multiplets only, controlled by the tensors (1.1) and are the ones defining the structure of all other invariants in each line. To the best of our knowledge, the string original of such couplings have not been discussed in the literature. The remaining invariants in the last line can arise a priori and their description remains unknown within supergravity. We comment on the expected structure of some of these below.[3]

The structure of the paper is as follows: in the next section we shall review briefly the one-loop R^4 couplings as well as some of our conventions and the reduction ansatze. The structure of known higher derivative couplings in four-dimensional $\mathcal{N}=2$ theories is presented in section 3. We then proceed to consider the various higher derivative terms arising from the Calabi-Yau compactification of the one-loop term, organised by the order of derivatives. Hence, in section 4 we consider the eight derivative terms, while in sections 5

[3] In fact, we find that some of these couplings involving the tensor multiplet seem to be missing in the specific compactification we consider, but cannot be excluded if more tensor/hyper multiplets are considered.

JHEP02(2014)093

and 6 we discuss the six and four derivative invariants respectively. Some open questions are listed in section 7. The extended appendices contain further technical details of the structures appearing in the main text. In particular, appendix A contains the fully explicit expressions for the quartic one-loop terms in 10D. Appendices B and C deal with chiral couplings of general chiral multiplets and the composite chiral background of the tensor multiplet respectively. Finally, appendix D reviews the structure of the kinetic chiral multiplet and the various invariants that can be constructed based on it, up to the eight derivative level.

2 Higher derivative terms in Type II theories

The starting point for our considerations is the ten-dimensional eight-derivative terms that arise in Type II string theories. The structure of the gravitational part of these couplings has been known for a long time, but the coupling to the remaining Type II massless fields was not explicitly known. Recently, a more concrete understanding of the terms involving the NS three-form field strength, H, has been achieved [15]. The structure of the corresponding terms involving RR gauge fields is constrained to a large extend, using arguments based on the eleven dimensional uplift to M-theory.

Upon reduction on a Calabi-Yau manifold without turning on any internal fluxes, the NS three form leads to two types of objects in the four dimensional effective theory, namely a lower dimensional three-form field strength and $h^{1,1}$ scalars. The former is naturally part of a tensor multiplet,[4] while the latter are part of vector multiplets in Type IIA and tensor/hyper multiplets in Type IIB.

In this section, we start by giving an overview of the ten-dimensional eight-derivative terms in section 2.1, from which all the lower dimensional higher-derivative terms arise. In section 2.2 we then turn to a discussion of the reduction procedure on Calabi-Yau three-folds, which is central to the derivation of four dimensional couplings.

2.1 The eight-derivative terms in ten dimensions

In summarising the structure of R^4 with the NS three-form H included, it is most convenient to start by introducing the connection with torsion which reads in components

$$\Omega_{\pm\,\mu_1}{}^{\nu_1\nu_2} = \Omega_{\mu_1}{}^{\nu_1\nu_2} \pm \frac{1}{2} H_{\mu_1}{}^{\nu_1\nu_2}. \tag{2.1}$$

The curvature computed out of $\Omega_\pm$ is then

$$R(\Omega_\pm) = R \pm \frac{1}{2} d\mathcal{H} + \frac{1}{4}\mathcal{H}\wedge\mathcal{H}, \qquad \mathcal{H}^{\nu_1\nu_2} = H_{\mu_1}{}^{\nu_1\nu_2} dx^\mu. \tag{2.2}$$

Denoting the Riemann tensor by $R_{\mu\nu}{}^{\nu_1\nu_2}$, we may write in components

$$R(\Omega_\pm)_{\mu_1\mu_2}{}^{\nu_1\nu_2} = R_{\mu_1\mu_2}{}^{\nu_1\nu_2} \pm \nabla_{[\mu_1} H_{\mu_2]}{}^{\nu_1\nu_2} + \frac{1}{2} H_{[\mu_1}{}^{\nu_1\nu_3} H_{\mu_2]\nu_3}{}^{\beta}. \tag{2.3}$$

[4] Upon of the two-form gauge field to a scalar, this leads to the so called universal hypermultiplet, but we will not consider this operation here.

Note that the first and last term in this expression satisfy the pair exchange property, while the second term is antisymmetric under pair exchange due to the Bianchi identity on the three-form.

The Type II eight-derivative terms can be written in terms of two standard "$\mathcal{N}=1$ superinvariants", defined as

$$\begin{aligned}
J_0(\Omega) &= \left(t_8t_8 + \frac{1}{8}\,\epsilon_{10}\epsilon_{10}\right) R^4 \equiv \left(t_8t_8 + \frac{1}{8}\epsilon_{10}\epsilon_{10}\right)^{\nu_1\ldots\nu_8}_{\mu_1\ldots\mu_8} R^{\mu_1\mu_2}{}_{\nu_1\nu_2}\ldots R^{\mu_7\mu_8}{}_{\nu_7\nu_8}\,,\\
J_1(\Omega) &= t_8t_8R^4 - \frac{1}{4}\,\epsilon_{10}t_8BR^4 \equiv t_8t_8R^4 - \frac{1}{4}\,t_{8\,\mu_1\ldots\mu_8}B\wedge R^{\mu_1\mu_2}\wedge\cdots\wedge R^{\mu_7\mu_8}\,,
\end{aligned} \tag{2.4}$$

which provide a convenient way of encoding the kinematic structure of R^4 terms. The tensor t_8 and the associated tensorial structures appearing here are spelled out in appendix A. Note that at this stage the terms (2.4) are build from Levi-Civita connections only, and the three-from H is not included.

It has been argued in [15] that these will be completed with the B-field as follows:

$$\begin{aligned}
J_0(\Omega) \longrightarrow\; & J_0(\Omega_+) + \Delta J_0(\Omega_+, H) \qquad\qquad (2.5a)\\
&= \left(t_8t_8 + \frac{1}{8}\epsilon_{10}\epsilon_{10}\right) R^4(\Omega_+) + \frac{1}{3}\epsilon_{10}\epsilon_{10}H^2R^3(\Omega_+) + \ldots\\
J_1(\Omega) \longrightarrow\; & J_1(\Omega_+) = t_8t_8R^4(\Omega_+) - \frac{1}{8}\epsilon_{10}t_8B\left(R^4(\Omega_+) + R^4(\Omega_-)\right). \qquad (2.5b)
\end{aligned}$$

Note that $J_0(\Omega_+) + \Delta J_0(\Omega_+, H)$ appears at tree level both in IIA and IIB and at one loop in IIB, while $J_0(\Omega_+) - 2J_1(\Omega_+) + \Delta J_0(\Omega_+, H)$ appears at one loop in IIA. The structure of $\Delta J_0(\Omega_+, H)$ is more elaborate and kinematically different form the standard $\frac{1}{8}\epsilon_{10}\epsilon_{10}R^4(\Omega)$ terms, and in fact it is the only part of the eight-derivative action that is not written purely in terms of $R(\Omega_\pm)$.[5] Here we should also use the full six-index un-contracted combination of H^2. These structures receive contributions starting form five-point odd-odd amplitudes:

$$\begin{aligned}
\Delta J_0(\Omega_+, H) = -\frac{1}{3}\,&\epsilon_{\alpha\mu_0\mu_1\cdots\mu_8}\epsilon^{\alpha\nu_0\nu_1\cdots\nu_8}\,R^{\mu_7\mu_8}{}_{\nu_7\nu_8}(\Omega_+)\\
&\times\Big[H^{\mu_1\mu_2}{}_{\nu_0}H_{\nu_1\nu_2}{}^{\mu_0}\,R^{\mu_3\mu_4}{}_{\nu_3\nu_4}(\Omega_+)R^{\mu_5\mu_6}{}_{\nu_5\nu_6}(\Omega_+)\\
&\quad - \frac{3}{16}\left(9\,H^{\mu_1\mu_2}{}_{\nu_0}H_{\nu_1\nu_2}{}^{\mu_0} + \frac{1}{9}\,H^{\mu_1\mu_2\mu_0}H_{\nu_1\nu_2\nu_0}\right)\nabla^{\mu_3}H^{\mu_4}{}_{\nu_3\nu_4}\nabla^{\mu_5}H^{\mu_6}{}_{\nu_5\nu_6}\Big]\\
&+\ldots.
\end{aligned} \tag{2.6}$$

The order H^4R^2 contribution is known up to some ambiguities, while the terms with higher powers of H remain a conjecture. Luckily these terms play little role in $\mathcal{N}=2$ reductions and we comment on the cases where they are relevant below.

The last term in (2.5b), coming form the worldsheet odd-even and even-odd structures corresponds to the gravitational anomaly-canceling term. The relative sign between the

[5] Incidentally, using the connection with torsion $\Omega_\pm$ and $R(\Omega_\pm)$ is not sufficient for writing the two-derivative effective action. For this one also needs the Dirac operator that appears in supersymmetry variations. Note that the Dirac operator, the covariant derivative with respect to $\Omega_\pm$ and the effective action are related via generalisation of the Lichnerowicz formula.

two terms is fixed by the IIA GSO projection, so that the coupling contains only odd powers of B-field. The explicit contribution to the effective action is

$$-(2\pi)^6\alpha'^3 B\wedge \overline{X}_8 = -\frac{(2\pi)^2}{192}\alpha'^3 B\wedge\left(\mathrm{tr}\, R^4-\frac{1}{4}(\mathrm{tr}\, R^2)^2+\text{exact}\right)\,, \qquad (2.7)$$
$$\overline{X}_8=\frac{1}{2}\,\left[t_8R^4(\Omega_+)+t_8R^4(\Omega_-)\right]\,.$$

Since $t_8R^4\sim\frac{1}{4}p_1^2-p_2$ is made of characteristic classes and H enters in (2.7) like a torsion in the connection, its contribution amounts to a shift by exact terms. For completeness, we record the complete expression,

$$\begin{aligned}\overline{X}_8=\frac{1}{192(2\pi)^4}\Bigg[&\left(\mathrm{tr}\, R^4-\frac{1}{4}(\mathrm{tr}\, R^2)^2\right)\\ &+d\,\Bigg(\frac{1}{2}\mathrm{tr}\,\left(\mathcal{H}\nabla\mathcal{H}R^2+\mathcal{H}R\nabla\mathcal{H}R+\mathcal{H}R^2\nabla\mathcal{H}\right)\\ &\quad-\frac{1}{8}\left(\mathrm{tr}\, R^2\,\mathrm{tr}\,\mathcal{H}\nabla\mathcal{H}+2\,\mathrm{tr}\,\mathcal{H}R\,\mathrm{tr}\, R\nabla\mathcal{H}\right)\\ &\quad+\frac{1}{16}\,\mathrm{tr}\,\left(2\mathcal{H}^3(\nabla\mathcal{H}R+R\nabla\mathcal{H})+\mathcal{H}R\mathcal{H}^2\nabla\mathcal{H}+\mathcal{H}\nabla\mathcal{H}\mathcal{H}^2R\right)\\ &\quad-\frac{1}{16}\,\left(\mathrm{tr}\,\mathcal{H}\nabla\mathcal{H}\,\mathrm{tr}\, R\mathcal{H}^2+\mathrm{tr}\, R\nabla\mathcal{H}\,\mathrm{tr}\,\mathcal{H}^3-\mathrm{tr}\,\nabla\mathcal{H}\mathcal{H}^2\,\mathrm{tr}\,\mathcal{H}R\right)\\ &\quad+\frac{1}{32}\mathrm{tr}\,\nabla\mathcal{H}\mathcal{H}^5+\frac{1}{16}\mathrm{tr}\,\mathcal{H}(\nabla\mathcal{H})^3\\ &\quad+\frac{1}{192}\mathrm{tr}\,\nabla\mathcal{H}\mathcal{H}^2\,\mathrm{tr}\,\mathcal{H}^3-\frac{1}{64}\mathrm{tr}\,\mathcal{H}\nabla\mathcal{H}\,\mathrm{tr}\,(\nabla\mathcal{H})^2\Bigg)\Bigg]. \end{aligned}\qquad (2.8)$$

since its reduction will be useful in the following.

The eight-derivative (tree level and one-loop) terms are the origin of the only perturbative corrections to the metrics on the $\mathcal{N}=2$ moduli spaces. The corrections respect the factorisation of the moduli spaces, and the classical metrics on moduli space of vectors and hypers receive respectively tree-level and one-loop corrections, both of which are proportional of the Euler number of the internal Calabi-Yau manifold [12, 16]. Needless to say, our discussion is consistent with these corrections, and from now on we shall concentrate only on the higher-derivatives terms. Recent progress in understanding the hyper-multiplet quantum corrections is reviewed in [17].

As already mentioned, the reduction of type IIA super invariant $J_0(\Omega)-2J_1(\Omega)$ on Calabi-Yau manifolds yields the one loop R^2 terms in $\mathcal{N}=2$ four-dimensional theory, and this is the only known product of the reduction so far that leads to higher derivative terms in 4D. We shall return to the four-dimensional R^2 terms in section 6. Clearly, the inclusion of the B field leads to further couplings to matter upon dimensional reduction, to which we now turn.

2.2 Reduction on Calabi-Yau manifolds

We now consider the reduction of the ten-dimensional eight-derivative action on a Calabi-Yau threefold X, and its relation to the $\mathcal{N}=2$ action. The metric can be reduced in the

JHEP02(2014)093

standard way, as[6]

$$g_{\mu_1\mu_2} = \begin{pmatrix} g_{\mu\nu} & 0 \\ 0 & g_{mn} \end{pmatrix}, \tag{2.9}$$

where g_{mn} is the metric on the Calabi-Yau manifold, X, which we will not need explicitly. The three-form H reduces as

$$H_3 = H + f^I \wedge \omega_I\,, \tag{2.10}$$

where the four-dimensional H is part of the tensor multiplet, and the one-forms f^I can be locally written as $f^I = du^I$, with u^I being a part of the vector multiplet scalars. The index I spans over $h^{1,1}(X)$, and $\omega_I \in H^2(X,\mathbb{Z})$.

Hence reducing the terms built out of $R(\Omega_\pm)$ and H_3 (where $\Omega_\pm = \Omega \pm \frac{1}{2}\mathcal{H}$), one expects at a given order of derivatives various couplings involving the Riemann tensor, R, as well as the tensor multiplet and vector multiplets. For example, at the four-derivative level one recovers the four-dimensional R^2 couplings and expects to obtain further couplings quartic in tensor multiplet and vector multiplets, as well as mixed terms. We use the symbolic computer algebra system Cadabra [18, 19] to systematically derive the structure of these terms.

The vector moduli shall be denoted $z^I = u^I + it^I$, where t^I are the Kähler moduli, defined through the decomposition of the Calabi-Yau Kähler form, J, as $J = t^I\omega_I$. The total volume, $\mathcal{V}$, of the CY manifold is given by the standard volume form, cubic in J as

$$\mathcal{V} = \frac{1}{3!}\int_X J\wedge J\wedge J = \exp[-\mathcal{K}]\,, \tag{2.11}$$

where we defined the 4D Kähler potential. We shall not need the vector fields themselves, but only their field strengths, denoted as F^A, where the index $A = 0, I$ runs over the $h^{1,1}(X)+1$ vector fields (in places where the shorthand notation is used, F will stand for the entire multiplet).

Reducing the NS eight-derivative couplings we obtain couplings that contain u^I and t^I. The couplings to F^I can be recovered by thinking of the (one-loop) couplings as being reduced from five (or eleven) dimensions. In practical terms, one has to add an extra index on $f^I_\mu \mapsto F^I_{\mu\nu}$. Since (the affected parts of) the expressions are even in powers of F^I, the extra index will always be contracted with a similar counterpart. Moreover most of the expressions are only quadratic in F, hence the lifting is unique. A little combinatorial imagination is needed for F^4 terms. This procedure follows the lifting of one-loop NS couplings to eleven dimensions, outlined in [15] and is analogous to the way one can recover graviphoton couplings from the R^2 term - one just has to think of the lifting of the couplings to five dimensions and their consequent reduction. As already mentioned, here we can benefit from the explicit $\mathcal{N}=2$ formalism in verifying that the

[6]We use numbered Greek letters for 10D curved indices, while ordinary Greek letters denote 4D curved indices. We use Latin letters from the beginning of the alphabet for 4D flat indices, and Latin letters from the middle to the end of the alphabet, $m, n, \ldots$ are reserved for CY indices. We reserve the letters i, j, k, l for SU(2) R-symmetry indices. Capital Latin indices $I, J = 1, \ldots, h^{1,1}(X)$ span the matter vector multiplets.

couplings involving t^I and F^I complete each other sypersymmetrically and hence provide a verification of the lifting of the complete one-loop eight-derivative terms from type IIA strings to M-theory.

Since we are focusing on Calabi-Yau compactifications without flux, different pieces in the reduction will involve integrating over X expressions containing some power of the internal curvature and $\omega_I \in H^2(X,\mathbb{Z})$.[7] We shall start with the familiar integrals.

At the four derivative level, one needs to consider terms with exactly two powers of the Riemann tensor in the internal Calabi-Yau manifold. In the purely gravitational sector, one then finds an R^2 term in four dimensions, originating from the R^4 couplings in ten dimensions. In this case, one obtains

$$t_8 t_8 R^4 = -\frac{1}{8}\epsilon_{10}\epsilon_{10}R^4 = 12\, F_1\, R^{\mu\nu\rho\lambda}R_{\mu\nu\rho\lambda}\,, \tag{2.12}$$

where we note that only terms completely factorised in internal and external objects contribute. The function F_1 is an integral over the internal directions that takes the form

$$F_1 = \int_X R^{mnpq}R_{mnpq} = \frac{1}{8}\int_X \epsilon_{mnm_1\dots m_4}\epsilon^{mnn_1\dots n_4}R^{m_1m_2}{}_{n_1n_2}R^{m_3m_4}{}_{n_3n_4} = \alpha_I t^I\,, \tag{2.13}$$

where the first equality holds up to Ricci terms and in the second equality we evaluated the integral.

The fine balance between the two a priori different terms in (2.13) can be extended to more complicated integrals, that are relevant in the reduction of the non-purely gravitational terms. In this case, we have checked explicitly the identity

$$\begin{aligned} &t_{8\mu m\nu n m_1\dots m_4}t_8{}^{\rho p\sigma q n_1\dots n_4}\omega_I{}^m{}_p\omega_J{}^n{}_q\, R^{m_1m_2}{}_{n_1n_2}R^{m_3m_4}{}_{n_3n_4} = \\ &\qquad -\frac{1}{8}\,\delta_\mu{}^\sigma\,\delta_\nu{}^\rho\epsilon_{mnm_1\dots m_4}\epsilon^{pqn_1\dots n_4}\omega_I{}^m{}_p\omega_J{}^n{}_q\, R^{m_1m_2}{}_{n_1n_2}R^{m_3m_4}{}_{n_3n_4}\,, \end{aligned} \tag{2.14}$$

up to Ricci terms. Note that the structure of spacetime indices is different in the two sides, while the remaining terms are purely internal. Upon contraction with the Kähler moduli, each side of (2.14) reduces to the expression in (2.13).

Even further, the identity in (2.15) can be generalised to an identity involving eight indices, as follows.

$$\begin{aligned} &t_{8m_1\dots m_4p_1\dots p_4}t_8{}^{n_1\dots n_4q_1\dots q_4}\omega_I{}^{m_1}{}_{n_1}\omega_J{}^{m_2}{}_{n_2}\,\omega_K{}^{m_3}{}_{n_3}\omega_L{}^{m_4}{}_{n_4}\,R^{p_1p_2}{}_{q_1q_2}R^{p_3p_4}{}_{q_3q_4} = \\ &-\frac{1}{8}\,\epsilon_{p_0q_0m_1\dots m_4p_1\dots p_4}\epsilon^{p_0q_0n_1\dots n_4q_1\dots q_4}\omega_I{}^{m_1}{}_{n_1}\omega_J{}^{m_2}{}_{n_2}\,\omega_K{}^{m_3}{}_{n_3}\omega_L{}^{m_4}{}_{n_4}\,R^{p_1p_2}{}_{q_1q_2}R^{p_3p_4}{}_{q_3q_4}\,, \end{aligned} \tag{2.15}$$

again up to Ricci-like terms. These two expressions are relevant for the terms in $R(\Omega_+)$ that are odd or even under pair exchange, respectively.

[7] Since the four-dimensional three-form H in (2.10) is in the hyper matter, some of the couplings involving hyper multiplets will be discussed here. However we mostly concentrate on the vector multiplets here, and do not consider any internal expressions involving forms in $H^{2,1}(X)$.

The reduction to six- and eight-derivative couplings will require integration over expressions linear or zeroth order in the Riemann tensor of the internal Calabi-Yau manifold. In view of the vanishing of the Calabi-Yau Ricci tensor, these are essentially unique, and given by

$$\begin{aligned} G_{IJ} &= \frac{1}{2}\int_X \omega_I^{mn}\omega_{Jmn}\,, \\ \mathcal{R}_{IJ} &= \int_X R_{mnpq}\,\omega_I{}^{mn}\omega_J{}^{pq}\,, \end{aligned} \tag{2.16}$$

where G_{IJ} ultimately leads to the vector multiplet Kähler metric and $\mathcal{R}_{IJ}$ is a new coupling to be discussed in due time.

3 The four dimensional action

We now describe the structure of the effective $\mathcal{N}=2$ supergravity action in four dimensions, that arises from the reduction of the one-loop Type IIA Lagrangian. Given that the original ten dimensional action contains eight derivatives, one obtains a variety of higher derivative couplings, next to the lowest order two derivative action. In order to describe these in a systematic way, we will consider the off-shell formulation of the theory, which allows to construct infinite classes of higher derivative invariants without modifying the supersymmetry transformation rules. However, since the higher dimensional one-loop action and the reduction scheme are on-shell, one has to deduce the off-shell invariants from the desired terms that result upon gauge fixing to the on-shell theory. In the following, we take the pragmatic approach of matching the leading, characteristic terms in each invariant and promoting to off-shell variables by standard formulae for special coordinates for the vector multiplet scalars. In practice, these choices are essentially unique, and below we comment on this issue in the examples where this is relevant.

The defining multiplet of off-shell $\mathcal{N}=2$ supergravity is the Weyl multiplet, which contains the graviton, e_μ^a, the gravitini, gauge fields for local scale and R-symmetries and various auxiliary fields. Of the latter, only the auxiliary tensor $T_{ab}{}^{ij}$ is directly relevant, since it is identified with the graviphoton in the on-shell formulation of the theory, at the two-derivative level. The reader can find a short account of the Weyl multiplet in appendix B. In what follows, we will mostly deal with the covariant fields of the Weyl multiplet, which can be arranged in a so-called chiral multiplet (see appendix B for more details), which contains the auxiliary tensor $T_{ab}{}^{ij}$ and the curvature $R(M)_{\mu\nu}{}^{ab}$. The latter is identified with the Weyl tensor, up to additional modifications. These observations will be very useful in the identification of the various higher derivative couplings.

There are various matter multiplets that can be defined on a general supergravity background. Here, the fundamental matter multiplets we consider are vector multiplets and a single tensor multiplet, corresponding to the universal tensor multiplet of Type II theories. Both these multiplets comprise $8+8$ degrees of freedom and are defined in appendices B and C respectively, to which we refer for further details. Moreover, they can be naturally viewed as two mutually non-compatible projections of a chiral multiplet, which is central to our considerations.

All Lagrangians considered in this paper are based on couplings of chiral multiplets, which contain $16+16$ degrees of freedom and can be defined on an arbitrary $\mathcal{N}=2$ superconformal background. We refer to appendix B for more details on chiral multiplets. Here, we simply state that these multiplets are labeled by the scaling weight, w, of their lowest component, A, and that products of chiral multiplets are chiral multiplets themselves, obtained by simply considering functions $F(A)$, which must be homogeneous, so that a weight can be assigned to them. As mentioned above, the matter multiplets we consider are also chiral multiplets of $w=1$, on which a constraint projecting out half of the degrees of freedom is imposed and the same property holds for the covariant components of the Weyl multiplet. This implies that actions for all the above multiplets can be generated by considering expressions constructed out of chiral multiplets, which are invariant under supersymmetry.

3.1 Two derivatives

The prime example is given by the invariant based on a $w=2$ chiral multiplet, implying that its highest component, C, has Weyl weight 4, and chiral weight 0, as is appropriate for a conformally invariant Lagrangian in four dimensions. It can be shown that the expression

$$e^{-1}\mathcal{L}=C-\frac{1}{16}A(T_{ab\,ij}\varepsilon^{ij})^2+\text{fermions}\,, \tag{3.1}$$

is the bosonic part of the invariant, including a conformal supergravity background described by the auxiliary tensor $T_{ab\,ij}$ of the gravity multiplet. The two derivative action for vector multiplets is now easily constructed, by setting the chiral multiplet in this formula to be composite, expressed in terms of vector multiplets labeled by indices $I,J,\dots=0,1,\dots,n_{\rm v}$. It is possible to show (cf. (B.6)) that the relevant terms of such a composite multiplet are given by[8]

$$\begin{aligned}
A&=-\frac{\mathrm{i}}{2}F(X)\,,\\
C&=\mathrm{i}\,F(X)_I\,\square_{\rm c}\bar{X}^I+\frac{\mathrm{i}}{8}\,F(X)_{IJ}\big[B_{ij}{}^I B_{kl}{}^J\,\varepsilon^{ik}\varepsilon^{jl}+X^I\,G_{ab}^{+\,J}T^{ab}{}_{ij}\varepsilon^{ij}-2\,G_{ab}^{-\,I}G^{-abJ}\big]\,,
\end{aligned} \tag{3.2}$$

where F_I and F_{IJ} are the first and second derivative of the function F, known as the prepotential and $B_{ij}{}^I$, $G_{ab}^{-\,I}$ are the remaining bosonic components of the chiral multiplets (which in this case are constrained by (B.7) for vector multiplets). As the bottom composite component, A, has $w=2$, the function $F(X)$ must be homogeneous of degree two in the vector multiplet scalars X^I. Taking into account the constraints in (B.7), the bosonic terms of the Lagrangian following from (3.1) read

$$\begin{aligned}
8\pi\,e^{-1}\,\mathcal{L}_v=&\;\mathrm{i}\mathcal{D}^\mu F_I\,\mathcal{D}_\mu\bar{X}^I-\mathrm{i}F_I\,\bar{X}^I\left(\frac{1}{6}R-D\right)-\frac{1}{8}\mathrm{i}F_{IJ}\,Y_{ij}^I Y^{Jij}\\
&+\frac{1}{4}\mathrm{i}F_{IJ}\left(F_{ab}^{-I}-\frac{1}{4}\bar{X}^I T_{ab}^{ij}\varepsilon_{ij}\right)\left(F^{-Jab}-\frac{1}{4}\bar{X}^J T^{ijab}\varepsilon_{ij}\right)\\
&-\frac{1}{8}\mathrm{i}F_I\left(F_{ab}^{+I}-\frac{1}{4}X^I T_{abij}\varepsilon^{ij}\right)T_{ij}^{ab}\varepsilon^{ij}-\frac{1}{32}\mathrm{i}F(T_{abij}\varepsilon^{ij})^2+\text{h.c.}\,,
\end{aligned} \tag{3.3}$$

[8] The function G in (B.6) is conventionally chosen as $G(X^I)=-\frac{\mathrm{i}}{2}F(X)$ in this context.

JHEP02(2014)093

where in the last line we added the hermitian conjugate to obtain a real Lagrangian. Here, F_{ab}^I are the vector multiplet gauge field strengths, R is the Ricci scalar and D is the auxiliary real scalar in the gravity multiplet. This Lagrangian is invariant under scale transformations and can be related to an on-shell Poincaré Lagrangian by using a scale transformation to set the coefficient of the Einstein-Hilbert term, $\mathrm{Im}(F_I \bar{X}^I)$, to a constant. For standard Calabi-Yau compactifications of Type II theories, one obtains a cubic prepotential, as

$$F = -\frac{1}{6}\frac{C_{IJK}Y^I Y^J Y^K}{Y^0}\,, \tag{3.4}$$

where the constant tensor C_{IJK} stands for the intersection numbers of the manifold.

As it turns out, the Lagrangian (3.3) is inconsistent as it stands, so that one needs to add at least one auxiliary hypermultiplet, which is to be gauged away by superconformal symmetries, similar to the scalar $\mathrm{Im}(F_I \bar{X}^I)$ above. In addition, in this paper we consider a single tensor multiplet, corresponding to the universal hypermultiplet upon dualisation of the tensor field. We refer to appendix C for more details on this multiplet. For later reference, we display the bosonic action for the auxiliary hypermultiplet and the physical tensor multiplet that needs to be added to (3.3) to obtain a consistent on-shell theory with a physical tensor multiplet, as

$$\begin{aligned} 8\pi\, e^{-1}\,\mathcal{L}_t = &-\frac{1}{2}\varepsilon^{ij}\,\Omega_{\alpha\beta}\left[\mathcal{D}_\mu A_i{}^\alpha\,\mathcal{D}^\mu A_j{}^\beta - A_i{}^\alpha A_j{}^\beta\left(\frac{1}{6}R + \frac{1}{2}\,D\right)\right] \\ &- \frac{1}{2}\,F^{(2)}\,\mathcal{D}_\mu L_{ij}\,\mathcal{D}^\mu L^{ij} + F^{(2)}\,L_{ij}\,L^{ij}\left(\frac{1}{3}R + D\right) + F^{(2)}\left[E_\mu\,E^\mu + G\bar{G}\right] \\ &+ \frac{1}{2}\mathrm{i}e^{-1}\varepsilon^{\mu\nu\rho\sigma}\,\frac{\partial F^{(2)}}{\partial L_{ij}}\,E_{\mu\nu}\,\partial_\rho L_{ik}\,\partial_\sigma L_{jl}\,\varepsilon^{kl}\,. \end{aligned} \tag{3.5}$$

Here, $A_i{}^\alpha$ is the hypermultiplet scalar, described as a local section of $\mathrm{SU}(2)\times\mathrm{SU}(2)$, and $\Omega_{\alpha\beta}$ is the invariant antisymmetric tensor in the second $\mathrm{SU}(2)$. The on-shell fields of the tensor multiplet are the triplet of scalars L_{ij} and the two-form gauge field, $B_{\mu\nu}$, while

$$E^\mu = \frac{1}{2}\mathrm{i}\,e^{-1}\,\varepsilon^{\mu\nu\rho\sigma}\partial_\nu B_{\rho\sigma}\,, \tag{3.6}$$

is the dual of its field strength and G is a complex auxiliary scalar. The functions $F^{(2)}$ and $F^{(3)}$ can be viewed as the second and third derivative of a function of the L_{ij}, that can easily be generalised to an arbitrary number of tensor multiplets [20] (see (C.4)–(C.5)). For a single tensor multiplet, there is a unique choice, as

$$F^{(2)} = \frac{1}{\sqrt{L_{ij}L^{ij}}}\,, \tag{3.7}$$

which we will assume throughout. However, as we ignore all tensor multiplet scalars in our reduction scheme, all scalars and $F^{(2)}$ are kept constant and only appear as overall factors.

3.2 Higher derivatives

In this paper we construct higher derivative actions based on the properties of chiral multiplets, as discussed above. One way of doing this is to consider the function F in (3.2) to depend not only on vector multiplets, but also on other chiral multiplets, which are treated as background fields. Alternatively, one may consider invariants more general than (3.1), containing explicit derivatives on the chiral multiplet fields. Here we use both structures, which we discuss in turn, emphasising the methods and the structure of invariants rather than details, which can be found in [13, 20, 21].

We consider two chiral background multiplets, one constructed out of the Weyl multiplet and one constructed out of the tensor multiplet, whose lowest components we denote as A_{w} and A_{t} respectively. These are proportional to the auxiliary fields $(T_{ab}{}^{ij}\varepsilon_{ij})^2$ of the Weyl and and G of the tensor multiplet and we refer to appendices B and C for more details on their precise definition. Considering a function $F(X^I, A_{\mathsf{w}}, A_{\mathsf{t}})$ leads to a Lagrangian of the form (3.3), where the set of vector field strengths is extended to include the Weyl tensor $R(M)_{ab}{}^{cd}$ in (B.3) and the combination $\nabla_{[a}E_{b]}$, so that four derivative interactions of the type

$$\mathcal{L} = \int F(X^I, A_{\mathsf{w}}, A_{\mathsf{t}}) \propto \int \left(\frac{\partial F}{\partial A_{\mathsf{w}}} \, R(M)^{-\,2} + \frac{\partial F}{\partial A_{\mathsf{t}}} \, (\nabla_{[a}E_{b]^-})^2 + \dots \right) , \tag{3.8}$$

are generated. The explicit expressions for the relevant chiral multiplets can be found in (B.11) and (C.11) respectively.

These couplings are distinguished, in the sense that they are described by a holomorphic function and correspond to integrals over half of superspace. The R^2 term has been studied in detail, especially in connection to BPS black holes, see e.g. [12, 22–25]. The full function $F(X^I, A_{\mathsf{w}})$ is in this case related to the topological string partition function [1, 2]. We will only be concerned with the linear part of this function, originating in the one-loop term in section 2.1, which is controlling the R^2 coupling through (3.8). The $(\nabla E)^2$ term has appeared more recently [20], without any coupling to vector multiplets.

More general higher derivative couplings can be constructed by looking for invariants of chiral multiplets that contain explicit derivatives, unlike (3.1). Indeed, such invariants can be derived by considering a chiral multiplet whose components are propagating fields, i.e. described by a Lagrangian containing derivatives. This can be done in the standard way, by writing a Kähler sigma model, which in the simple case of two multiplets reads

$$\int \mathrm{d}^4\theta \, \mathrm{d}^4\bar\theta \; \Phi \, \bar\Phi' \approx \int \mathrm{d}^4\theta \, \Phi \, \mathbb{T}(\bar\Phi') \, , \tag{3.9}$$

where both Φ and Φ' must have $w = 0$ for the integral to be well defined. In the second form of the integral we defined a new chiral multiplet, $\mathbb{T}(\bar\Phi')$, the so called kinetic multiplet, since it contains the kinetic terms for the various fields. This multiplet was constructed explicitly in [13] and is summarized in appendix D below (see also [26] for a recent generalisation). In practice, one can think of the operator $\mathbb{T}$ as an operator similar to the Laplacian, acting on the components of the multiplet, as we find

$$\int \mathrm{d}^4\theta \, \Phi \, \mathbb{T}(\bar\Phi') = C \, \bar C' + 8 \, \mathcal{D}_a F^{-ab} \, \mathcal{D}^c F'^{+}{}_{cb} + 4 \, \mathcal{D}^2 A \, \mathcal{D}^2 \bar A' + \cdots \, , \tag{3.10}$$

where we only display the leading terms.

One can now simply declare the chiral multiplets Φ, Φ' to be composite by imposing (3.2), where the corresponding functions F, F' can depend on vector multiplet scalars, as well as the Weyl and tensor multiplet backgrounds, exactly as described above. As described in section D and in [13], this leads to a real function $\mathcal{H} = F\bar{F}' + \text{c.c.}$, homogeneous of degree zero, which naturally describes a variety of higher derivative couplings, corresponding to the combinations generated by

$$\left[F^{-\,2} + R^{-\,2} + (\nabla E)^{-\,2}\right] \otimes \left[F^{+\,2} + R^{+\,2} + (\nabla E)^{+\,2}\right], \tag{3.11}$$

where the $\pm$ stand for selfdual and anti-selfdual parts. Each of these is controlled by a function of the vector multiplet moduli as

$$\begin{aligned}
\mathcal{H}(X^I, A_{\mathsf{w}}, A_{\mathsf{t}}, \bar{X}^I, \bar{A}_{\mathsf{w}}, \bar{A}_{\mathsf{t}}) = \sum_{i+j\leq 2} \mathcal{H}^{(i,j)}&(X^I, \bar{X}^I)(A_{\mathsf{w}})^i\,(\bar{A}_{\mathsf{t}})^j + \text{c.c.} \Rightarrow \\
\mathcal{H}^{(0,0)}(X^I, \bar{X}^I) \qquad &\Rightarrow \qquad (\nabla F)^2, F^4 \\
\left[\mathcal{H}^{(1,0)} A_{\mathsf{w}} + \text{c.c.}\right] \qquad &\Rightarrow \qquad R^2 F^2 \\
\left[\mathcal{H}^{(0,1)}\, A_{\mathsf{t}} + \text{c.c.}\right] \qquad &\Rightarrow \qquad (\nabla E)^2 F^2 \\
\mathcal{H}^{(2,0)} A_{\mathsf{w}}\, \bar{A}_{\mathsf{w}} \qquad &\Rightarrow \qquad R^4 \\
\mathcal{H}^{(0,2)}\, A_{\mathsf{t}}\, \bar{A}_{\mathsf{t}} \qquad &\Rightarrow \qquad (\nabla E)^4 \\
\left[\mathcal{H}^{(1,1)}\, A_{\mathsf{w}}\, \bar{A}_{\mathsf{t}} + \text{c.c.}\right] \qquad &\Rightarrow \qquad R^2 (\nabla E)^2\,,
\end{aligned} \tag{3.12}$$

where we display the characteristic terms at each order. Note that we consider a function at most quadratic in A_{w}, A_{t}, since a higher polynomial would lead to the same terms, multiplied by additional powers of these auxiliary scalars. These are analogous to the non-linear parts of the chiral coupling in (3.8) and go beyond one-loop terms, so we do not consider them in the following. Finally, note that due to the expansion (3.12), the functions $\mathcal{H}^{(i,j)}$ are not homogeneous of degree zero for $i, j \neq 0$, but we we will always refer to the corresponding degree zero monomial in (3.12), for clarity.

The invariants based on (3.10) are the simplest ones containing the kinetic multiplet. It is straightforward to construct more general integrals, for example

$$\int \mathrm{d}^4\theta\, \Phi\, \mathbb{T}(\bar{\Phi})\, \mathbb{T}(\bar{\Phi})\,, \qquad \int \mathrm{d}^4\theta\, \Phi\, \mathbb{T}(\bar{\Phi})\, \mathbb{T}(\bar{\Phi})\, \mathbb{T}(\bar{\Phi})\,, \qquad \int \mathrm{d}^4\theta\, \Phi_0 \mathbb{T}(\bar{\Phi})\, \mathbb{T}(\bar{\Phi}_0 \mathbb{T}(\Phi))\,, \tag{3.13}$$

which are the cubic and quartic invariants discussed in section D. In exactly the same way as above, the first of these integrals leads to a homogeneous function of degree -2, describing couplings cubic in F^2, R^2 and $(\nabla E)^2$. Only some of these are relevant in the following, in particular the $(R^2 + (\nabla E)^2)F^4$ and F^6, since the rest contain more than eight derivatives. Finally, the last two integrals describe couplings with at least eight derivatives and lead to homogeneous functions of degree -4. Only the last integral is relevant for us, namely for the F^8 term.

4 Eight derivative couplings

We start by considering terms containing the maximum number of derivatives appearing in the one-loop correction, i.e. we consider the possible eight derivative invariants in four dimensions. This may seem counterintuitive at first and in fact some of these invariants have not been described explicitly. However, the terms that are known in four dimensions are the simplest to describe, setting the stage for the more complicated structures to follow.

Applying the rules and assumptions spelled out in section 2.2, one can characterise the various terms appearing in the reduction by the order of Riemann tensors, tensor multiplet fields strengths and vector multiplet fields strengths arising in four dimensions. Schematically, we then find a decomposition of the type

$$\begin{aligned}\mathcal{L}^{1-\mathrm{loop}} \Rightarrow & R^4 + R^2(\nabla H)^2 + (\nabla H)^4 + \underline{R^4} + \underline{H^6 F^2} + H^4 F^4 \\ & + \underline{H^2 F^6} + R^2\,(\nabla F)^2 + (\nabla F)^4\,, \end{aligned} \tag{4.1}$$

where we write in blue the terms which correspond to the known four-dimensional invariants. The supersymmetric invariants for the underlined (red) terms are not known.

Gravity and tensor couplings. The most obvious and simplest term is the R^4 term, which arises by trivial reduction of the corresponding ten dimensional term. Note that only the even-even contribution survives the reduction and leads to a four dimensional R^4 term as

$$\begin{aligned} t_8 t_8 R^4 \to & \, 192\,(R_{\mu\nu\rho\lambda}R^{\mu\nu\rho\lambda})^2 + 144\,\mathrm{tr}\,[R_{\mu\nu}R_{\rho\lambda}]\mathrm{tr}\,[R^{\mu\rho}R^{\nu\lambda}] + \ldots \\ & = 768\,(R^+)^2(R^-)^2 + 48\,\left((R^+)^2 - (R^-)^2\right)^2 + \ldots\,. \end{aligned} \tag{4.2}$$

The second line corresponds to two different invariants in four dimensions, each with its own supersymmetric completion, corresponding to the double appearance of R^4 in (4.1). The supersymmetrisation of the second term is not known in $\mathcal{N}=2$ supergravity (see however [27] for a discussion in the $\mathcal{N}=1$ setting). The supersymmetric completion of the first term was found in [13, 28], where it was shown that it is governed by a homogeneous degree zero real function of the vector multiplet moduli and the Weyl multiplet scalar A_{w}. In the present case however, (4.2) does not depend on moduli other than the total volume of the CY manifold, so that we can immediately identify the relevant function as depending only on the Kähler potential as

$$\mathcal{H}_{R^4} = \frac{3}{16}\,\mathrm{e}^{2\,\mathcal{K}(Y,\bar{Y})}\,A_{\mathsf{w}}\bar{A}_{\mathsf{w}}\,. \tag{4.3}$$

Here, the function of the off-shell scalars $\mathcal{K}(Y,\bar{Y})$ is very closely related to the lowest order Kähler potential, as

$$\mathrm{e}^{-\mathcal{K}} = 2\,\mathrm{Im}F_{AB}Y^A\bar{Y}^B\,, \tag{4.4}$$

with the prepotential (3.4), and is equal to it once special coordinates are chosen (for $Y^0=1$). Note however, that this is only the most natural choice that results in the first coupling in (4.2) upon taking the on-shell limit and one might consider more elaborate off-shell functions leading to the same result. Upon taking derivatives of this function with respect to the vector multiplet moduli, various couplings involving vector multiplet field strengths and auxiliary fields arise at the off-shell level, resulting to further eight derivative terms in the on-shell theory.

The corresponding purely tensor coupling is the eight derivative term of the tensor multiplet, which takes the form

$$t_8 t_8 R^4 \to 96\,\Big((\nabla_{[\mu}E_{\nu]}\nabla^{[\mu}E^{\nu]})^2 - 4\,\nabla_{[\mu}E_{\nu]}\nabla^{[\nu}E^{\rho]}\nabla_{[\rho}E_{\sigma]}\nabla^{[\sigma}E^{\mu]}\Big)\,. \tag{4.5}$$

These couplings can be described in a way completely analogous to the R^4 term, through a homogeneous real function corresponding to (4.3), as

$$\mathcal{H}_{H^4} = \frac{3}{16}\,\mathrm{e}^{2\,\mathcal{K}(Y,\bar{Y})}\,A_{\mathsf{t}}\bar{A}_{\mathsf{t}}\,. \tag{4.6}$$

The final possible combination at the eight derivative level for NS fields is the R^2H^4 coupling, which in 4D is characterised by the term

$$t_8 t_8 R^4 \to -96\,(\nabla_{[\mu}E_{\nu]_-}\nabla^{[\mu}E^{\nu]})^2 R^-_{\kappa\lambda}{}^{\rho\sigma}R^{-\kappa\lambda}{}_{\rho\sigma}\,. \tag{4.7}$$

These couplings can be described by the obvious mixed combination of the two functions (4.3) and (4.6) above, as

$$\mathcal{H}_{R^2H^4} = \frac{3}{16}\,\mathrm{e}^{2\,\mathcal{K}(Y,\bar{Y})}\,A_{\mathsf{w}}\bar{A}_{\mathsf{t}} + \mathrm{c.c.}\,. \tag{4.8}$$

The last function can be straightforwardly added to the functions above, to define a total function of the vector multiplet moduli and Weyl and tensor multiplet backgrounds, defined as

$$\mathcal{H}^8_{NS} = \frac{3}{16}\,\mathrm{e}^{2\,\mathcal{K}(Y,\bar{Y})}\,|A_{\mathsf{w}} + A_{\mathsf{t}}|^2\,, \tag{4.9}$$

describing the eight derivative couplings of NS sector fields. At this point it is worth pausing, to note that the form of these equations exhibits a correspondence between the graviton and the B-field, since the complete eight derivative action for the gravity and tensor multiplet is controlled by the combination $A_{\mathsf{w}} + A_{\mathsf{t}}$. This will appear in several instances below, at all orders of derivatives, and reflects the structure of the 10D Lagrangian, which is controlled by the combination $R(\Omega_+)$.

Couplings involving vector multiplets. We now turn to some of the eight derivative terms involving derivatives on vector multiplet fields. We start with mixed terms between NS and RR fields, namely the ones where the order of derivatives is balanced between the two sectors. Indeed, it is straightforward to obtain the function characterising the R^2F^4 coupling, which in 4D is described by the cubic invariant in appendix D, where one considers one of the chiral multiplets to be the Weyl multiplet. The R^2F^4 coupling is then characterised by the terms

$$\begin{aligned}&\mathcal{H}^{(8)}_{A_{\mathsf{w}}A\,\bar{B}}R^-_{\mu\nu\rho\lambda}R^{-\,\mu\nu\rho\lambda}\,\Big(\nabla F^{-\,A}\nabla F^{+\,\bar{B}} + \Box X^A\Box X^{\bar{B}}\Big)\\ &+\mathcal{H}^{(8)}_{A_{\mathsf{w}}AB\,\bar{C}\bar{D}}R^-_{\mu\nu\rho\lambda}R^{-\,\mu\nu\rho\lambda}\,F^{-\,A}F^{-\,B}F^{+\,\bar{C}}F^{+\,\bar{D}} + \dots\,.\end{aligned} \tag{4.10}$$

In this case, only the Ricci-like terms contribute to the reduction altogether, so that we obtain for the relevant coupling

$$\mathcal{H}^{(8)}_{A_{\mathsf{w}}I\,J} = -576\int_X \omega_I{}^{mn}\omega_{Jmn} \equiv -576\,G_{IJ}\,, \tag{4.11}$$

i.e. proportional to the lowest order Kähler metric G_{IJ}.

This result determines the coupling of the vector multiplet scalars and the corresponding vector fields, but we still need to fix the couplings to the Type IIA RR gauge field, labeled by 0 in four dimensions. These can be derived by the observation that all field strengths can be introduced by lifting the three-form $H_{\mu_1\mu_2\mu_3}$ to the eleven dimensional four-form field strength $G_{\mu_1\mu_2\mu_3\mu_4}$ and reducing back on a circle, keeping all components. The result of the reduction of the four-form to 4D gauge fields, $F^I_{\mu\nu}$, naturally leads to the combination $F^I_{\mu\nu} + u^I F^0_{\mu\nu}$, which should replace the field strengths in the couplings shown above, so that the full coupling becomes

$$\mathcal{H}^{(8)}_{A_{\mathsf{w}}I\,J} \to \mathcal{H}^{(8)}_{A_{\mathsf{w}}A\,\bar{B}} = -576 \begin{pmatrix} G_{IJ} & G_{IJ}u^J \\ u^I G_{IJ} & G_{IJ}u^I u^J \end{pmatrix} . \tag{4.12}$$

Combined with the fact that the relevant function depends on the Weyl multiplet background only linearly, as implied by (4.10), one can now integrate to obtain

$$\mathcal{H}_{R^2F^4} = 9\, A_{\mathsf{w}}\, \mathrm{e}^{2\,\mathcal{K}(Y,\bar{Y})} . \tag{4.13}$$

This form is in line with the observation that the R^2F^4 coupling can be roughly seen as the product of the chiral R^2 term with the real F^4 term. Note that, unlike for the lowest order Kähler potential, the $0I$ and 00-components of the second derivative $\mathcal{H}^{(8)}_{A_{\mathsf{w}}A\,\bar{B}}$ in (4.10) are physical in this case, since they describe the couplings of the RR one-form gauge field. In fact, the coupling $\mathcal{H}^{(8)}_{A_{\mathsf{w}}A\,\bar{B}}$ is proportional to the real part of the period matrix, which describes the the theta angles in the two derivative theory.

The natural extension of (4.13) to a function where A_{w} is replaced by A_{t} and thus describes couplings of the type $(\nabla E)^2(\nabla F)^2$ is straightforward. However, in the compactification we consider all such terms cancel identically, in a nontrivial way. Similarly, there are no parity odd terms of this type either, so that this particular coupling seems to be absent in four dimensions.

The same conclusion seems to hold for terms of the type $(\nabla H)^2H^2F^2$, which would in principle be characteristic of the H^6F^2 term in (4.2), even though this coupling is not known in four dimensions. Terms of this order in fields do not appear in the odd sector as well.

Finally, we consider the purely vector multiplet eight derivative couplings, corresponding to an F^8 term. This can be obtained by a trivial dimensional reduction, leading to the four dimensional coupling

$$t_8 t_8 R(\Omega_+)^4 \to 72 \left(\int_X \omega_I{}^{mn} \omega_{Jmn} \omega_K{}^{pq} \omega_{Lpq} \right) \partial^{\mu\nu} u^{(I} \partial_{\mu\nu} u^J \partial^{\rho\sigma} u^K \partial_{\rho\sigma} u^{L)} , \tag{4.14}$$

which can be described by the second quartic invariant in (D.11). Since the coupling above is given purely in terms of the product of the $(1,1)$ forms, $\omega_I \cdot \omega_J$, the relevant real function is related to the Kähler potential and is given by

$$\mathcal{H}_{F^8} = 6\, \mathrm{e}^{4\,\mathcal{K}(Y,\bar{Y})} . \tag{4.15}$$

This function is consistent with (4.14) for the I, J, indices and naturally extends to the 0-th gauge field in four dimensions as seen above, but we have not checked those couplings explicitly.

5 Six derivative couplings

At the six derivative level, we need to saturate two of the derivatives in the internal directions, so that exactly one Riemann tensor will appear in the relevant integrals on the Calabi-Yau manifold. This requirement turns out to be quite restrictive, since all traces of the Calabi-Yau curvature vanish. It follows that the internal integrals must also involve harmonic forms on which the indices of the Riemann tensor are contracted. Given that we do not consider any complex structure deformations, this observation directly implies that no invariants involving only NS-NS fields, such as R^2H^2 or H^6, can arise in four dimensions.[9]

However, mixed couplings involving fields from both the NS-NS and the R-R sector are nontrivial and a priori include three types of couplings, namely R^2F^2, $(\nabla E)^2F^2$ and H^2F^4. The latter has not been described in the context of $\mathcal{N}=2$ supergravity, while the former two can be constructed using the techniques in [13]. In addition, a purely vector multiplet coupling including six derivatives on the component fields, i.e. an F^6 invariant arises.

In particular, the R^2F^2 term was already constructed explicitly in [13], and is governed by a function, $\mathcal{H}(X, A_{\mathsf{w}}; \bar{X})$, that is linear in the Weyl multiplet, while the vector multiplet scalars appear through a holomorphic function of degree -2 and an anti-holomorphic function of degree 0. The relevant R^2F^2 coupling is

$$\mathcal{H}_{A_{\mathsf{w}}\,\bar{A}\bar{B}} R^{-}_{\mu\nu\rho\lambda} R^{-\mu\nu\rho\lambda}\, F^{+\bar{A}}{}_{\kappa\sigma} F^{+\bar{B}\kappa\sigma} + \dots\,, \tag{5.1}$$

where the part of the coupling coming from the R-R fields, as derived from the reduction is

$$\mathcal{H}_{A_{\mathsf{w}}\,\bar{I}\bar{J}} = -48 \int_X R_{mnpq}\, \omega_I{}^{mn} \omega_J{}^{pq} \equiv -48\, \mathcal{R}_{IJ}\,, \tag{5.2}$$

where in the second equality we defined the tensor $\mathcal{R}_{IJ}$ for later convenience. This tensor clearly describes a non-topological coupling, since it depends on the curvature of the Calabi-Yau manifold explicitly. In fact, the definition (5.2) is invertible, as one can reconstruct the Riemann tensor R_{mnpq} from $\mathcal{R}_{IJ}$ by contracting with the harmonic two-forms. We record the following properties of $\mathcal{R}_{IJ}$, which will be useful in the discussion below,

$$\mathcal{R}_{IJ} = \mathcal{R}_{JI}\,, \qquad t^I \mathcal{R}_{IJ} = 0\,, \qquad G^{IJ}\mathcal{R}_{IJ} = 0\,, \tag{5.3}$$

where t^I and G^{IJ} are the Kähler moduli and G^{IJ} is the inverse of the Kähler metric.

In order to extend (5.2) to include the 0-th gauge field, we follow the same procedure as in (4.12), to obtain the additional couplings

$$\mathcal{H}^{(6)}_{\bar{A}_{\mathsf{w}}\,AB} = \begin{pmatrix} \mathcal{R}_{IJ} & \mathcal{R}_{IJ}u^J \\ u^I\mathcal{R}_{IJ} & \mathcal{R}_{IJ}u^I u^J \end{pmatrix} \to \begin{pmatrix} \mathcal{R}_{IJ} & \mathcal{R}_{IJ}\mathrm{Re}(z^J) \\ \mathrm{Re}(z^I)\mathcal{R}_{IJ} & \mathcal{R}_{IJ}\mathrm{Re}(z^I)\mathrm{Re}(z^J) \end{pmatrix}. \tag{5.4}$$

We then obtain for the function describing the R^2F^2 invariant

$$\mathcal{H}_{R^2F^2} = -\frac{3\mathrm{i}}{8}\frac{\bar{A}_{\mathsf{w}}}{(\bar{Y}^0)^2}\hat{\mathcal{R}}_{IJ}\left(\frac{Y^I}{Y^0} - \frac{\bar{Y}^I}{\bar{Y}^0}\right)\left(\frac{Y^J}{Y^0} + \frac{\bar{Y}^J}{\bar{Y}^0}\right), \tag{5.5}$$

[9] Note that these will become nontrivial if more hyper/tensor multiplets are included in the reduction.

where $\hat{\mathcal{R}}_{IJ}(Y,\bar{Y}) = \mathcal{R}_{IJ}(t)$ is viewed as a function of the $t^I = \mathrm{Im}(\frac{Y^I}{Y^0})$, as obtained in the standard special coordinates. Note that (5.5) is manifestly homogeneous in the holomorphic scalars Y^A, but non-homogeneous in the anti-holomorphic scalars $\bar{Y}^A$, as expected.

It is straightforward to obtain a term of the type $(\nabla E)^2 F^2$ by simply replacing $A_{\mathsf{w}} \to A_{\mathsf{t}}$ in (5.5), in line with previous observations. It turns out that this invariant is also generated by the reduction, as

$$\mathcal{H}_{A_{\mathsf{t}}\,\bar{A}\bar{B}} \nabla^{[a} E^{b]^-} \nabla_{[a} E_{b]^-} F^{+\bar{A}}{}_{\kappa\sigma} F^{+\bar{B}\kappa\sigma} + \dots\,, \tag{5.6}$$

where the two couplings $\mathcal{H}^{(6)}_{A_{\mathsf{t}}\,\bar{I}\bar{J}} = \mathcal{H}^{(6)}_{A_{\mathsf{w}}\,\bar{I}\bar{J}}$, are equal. By the same argument as above, the function (5.5) can be extended to include the tensor multiplet coupling as

$$\mathcal{H}^{(6)}(\bar{A}_{\mathsf{w}}, \bar{A}_{\mathsf{t}}, Y, \bar{Y}) = -\mathrm{i}\,\frac{\bar{A}_{\mathsf{w}} + \bar{A}_{\mathsf{t}}}{(\bar{Y}^0)^2} \hat{\mathcal{R}}_{IJ} \left(\frac{Y^I}{Y^0} - \frac{\bar{Y}^I}{\bar{Y}^0}\right)\left(\frac{Y^J}{Y^0} + \frac{\bar{Y}^J}{\bar{Y}^0}\right)\,, \tag{5.7}$$

which describes the first row in the six-derivative part of table 1.

We now turn to the F^6 term, which is computationally more challenging than the couplings described above. This is due to the fact that there are no terms cubic in the two-form field strength H in ten dimensions, so that $(\nabla F)^3$ terms do not arise in four dimensions. This is consistent with the fact that similar terms cancel in the F^6 coupling that follows from the cubic invariant described in appendix D. One therefore is forced to consider terms of the type $(\nabla F)^2 F^2$, which are quartic in the $(1,1)$ forms ω_I, from the point of view of the Calabi-Yau reduction.

The result is a coupling containing all possible combinations of an internal Riemann tensor and four ω_I, as in

$$\omega_I{}^{mn}\omega_{Jmn} R^{pqrs}\omega_{Kpq}\omega_{Lrs}\,, \quad \omega_I{}^{mn}\omega_{Jnp} R^{pqrs}\omega_{Kqm}\omega_{Lrs}\,, \quad \dots\,, \tag{5.8}$$

which in principle determine the function controlling the F^6 coupling. However, we also find nontrivial odd terms for the scalars resulting from (2.7), in contrast to the known coupling in section D. These terms include

$$\begin{aligned}\mathcal{L}_{\text{odd}} &\sim Y_{IJKMN}\, du^I \wedge du^J \wedge du^K \wedge (\partial^\mu u^M d\,\partial_\mu u^N)\,,\\ Y_{IJKMN} &= \int \omega_I \wedge \omega^m{}_M \wedge \omega^n{}_N \wedge (2\,R_{np}\omega^{pq}{}_J \omega_{qmK} + \omega_{npJ} R^{pq}\omega_{qmK}) + \dots\,,\end{aligned} \tag{5.9}$$

where the dots stand for terms containing the same objects in double traces rather than s single one. We observe that a term completely antisymmetric in three indices $I\,,J\,,K$ arises and conclude that the known coupling is not sufficient to describe these terms. We leave it to future work to determine the possible new coupling(s) that can complete the structure.

Finally, it is worth discussing in brief the invariant of the type H^2F^4, which is not known explicitly in supergravity. Such terms do appear and seem to be controlled by the same tensor $\mathcal{R}_{IJ}$ in (5.2) above, since we find the characteristic couplings $\mathcal{R}_{IJ}E^2\nabla F^I \nabla F^I$ for all possible contractions of indices between the vector and tensor multiplet field strengths.

JHEP02(2014)093

Similarly, we find the parity odd terms

$$W_{IJKL}\, H\wedge(\partial^\mu u^I d\partial_\mu u^J)(\partial^\nu u^K\partial_\nu u^L)\,,$$
$$W_{IJKL}=\int_X \omega^m{}_I\wedge\omega^n{}_J\wedge\omega_{mK}\wedge\omega^p{}_L\wedge R_{np}+\dots\,, \tag{5.10}$$

where in the last integral we used similar conventions as in (5.9) above. Indeed, the two integrals appear to be closely related, so that the two couplings may have a similar origin in terms of superspace invariants.

6 Four derivative couplings

In order to obtain four derivative couplings in four dimensions from the 10D R^4 invariant, one needs to consider terms that include exactly two Riemann tensors in the internal directions. It follows that the integrals controlling the 4D couplings are quadratic in the Calabi-Yau curvature, in the same way as the six derivative couplings of the previous section are controlled by the Calabi-Yau Riemann tensor through (5.2) above.

At this level in derivatives, four structures can appear, namely R^2, F^4, H^2F^2, and H^4. Given our assumption of no hyper/tensor multiplets other than the universal tensor multiplet, all of these structures will be described by functions involving vector multiplet scalars, but only the latter two involve the tensor multiplet explicitly. All couplings except the H^2F^2 terms can be described straightforwardly in $\mathcal{N}=2$ supergravity, and we now discuss each in turn.

The $\boldsymbol{R^2}$ term. The R^2 term has been known for quite some time [12, 29], and arises from terms that can be completely factorised in internal and external indices, as

$$\left(t_8t_8-\frac{1}{8}\epsilon_{10}\epsilon_{10}\right)R(\Omega_+)^4\to\alpha_I t^I\,\left(R_{\mu\nu\rho\lambda}(\Omega_+)R^{\mu\nu\rho\lambda}(\Omega_+)+\epsilon_{abcd}R_2^{ab}(\Omega_+)\wedge R_2^{cd}(\Omega_+)\right)$$
$$B\wedge t_8\left[R^4(\Omega_+)+R^4(\Omega_-)\right]\to\alpha_I u^I\left(\operatorname{tr}R(\Omega_+)\wedge R(\Omega_+)+\operatorname{tr}R(\Omega_-)\wedge R(\Omega_-)\right) \tag{6.1}$$

where we used (2.13) and

$$\alpha_I=\int_X\omega_I\wedge\operatorname{tr}R^2\,, \tag{6.2}$$

are the the second Chern classes of the Calabi-Yau four-cycles. Note that this is a topological quantity, unlike the objects controlling higher derivative couplings described above, as e.g. in (5.2).

The supergravity description requires to allow the lowest order prepotential to depend on the Weyl multiplet through A_{w} [21], so that the explicit prepotential arising from (6.1) is given by

$$F=-\frac{1}{6}\frac{C_{IJK}Y^IY^JY^K}{Y^0}+\frac{1}{24\cdot 64}\frac{\alpha_I Y^I}{Y^0}A_{\mathsf{w}}\,, \tag{6.3}$$

where we remind the reader that the physical moduli are given by $z^I=\frac{Y^I}{Y^0}$ in terms of the scalars Y^A above.

The H^4 term. Turning to the tensor multiplet sector, an explicit computation using (2.14) leads to the following terms in four dimensions

$$\left(t_8t_8 - \frac{1}{8}\,\varepsilon_{10}\varepsilon_{10}\right) R^4 = 48\, R^{mnpq} R_{mnpq}\, \left(2\,\partial^{[\mu} E^{\nu]}\,\partial_{[\mu} E_{\nu]} + \frac{3}{4}\,(E^\mu E_\mu)^2\right)\,, \tag{6.4}$$

where, in complete analogy with the R^2 terms, only factorised traces contribute. It follows that the four dimensional Lagrangian contains the four derivative tensor multiplet invariant arising from (C.11), controlled by exactly the same prepotential in (6.3), upon extending the term containing the Weyl background to include the tensor background, as

$$F = -\frac{1}{6}\,\frac{C_{IJK}Y^IY^JY^K}{Y^0} + \frac{1}{24\cdot 64}\,\frac{\alpha_I Y^I}{Y^0}\,(A_{\mathsf{w}} + 8\,A_{\mathsf{t}})\,, \tag{6.5}$$

with A_{t} as in (C.7). This function describes the couplings in the first line of table 1.

Once again we observe the close relation between the R^2 and tensor multiplet couplings, which are characterised by exactly the same functional form in terms of the corresponding chiral backgrounds. This structure arises despite the fact that in $\mathcal{N}=2$ supergravity in four dimensions the tensor H and gravity are not in the same multiplet anymore, so that, a priori, more flexibility, parametrized by two functions is allowed. However, we find that the Calabi-Yau reduction leads to a single function of vector multiplet scalars for both couplings.

The F^4 term. In the purely RR sector, an invariant quartic in derivatives on the vector multiplet components exists in 4D, which is characterised by an $(\nabla F)^2$ coupling. As above, we analyse the terms arising from the odd term under pair exchange in $R(\Omega_+)$ in order to obtain these explicitly. Using (2.14), the terms coming from non-Ricci combinations cancel, and the remaining ones are Laplacians of four dimensional fields. Explicitly, we obtain that the total $(\nabla F)^2$ term reads

$$\left(t_8t_8 - \frac{1}{8}\,\epsilon_{10}\epsilon_{10}\right) R^4 \to 3\, X_{IJ} \nabla^2 u^I \nabla^2 u^J\,, \tag{6.6}$$

where X_{IJ} is the tensor

$$X_{IJ} = \int_X \epsilon_{mnm_1\dots m_4}\epsilon_{pqn_1\dots n_4} R^{m_1m_2n_1n_2} R^{m_3m_4n_3n_4}\ \omega_I{}^{mn}\,\omega_J{}^{pq}\,, \tag{6.7}$$

and is explicitly given by

$$\begin{aligned} X_{IJ} = 8\ \big[&(R_{mnpq})^2\omega_I{}^{rs}\omega_{Jrs} - 8\, R^{mnpq}R_{mnp}{}^r\omega_{Iq}{}^s\omega_{Jrs} + 4\, R^{mnpq}R_{mn}{}^{rs}\omega_{Ipr}\,\omega_{Jqs} \\ &+ 2\, R^{mnpq}R_{mn}{}^{rs}\omega_{Ipq}\,\omega_{Jrs} - 8\, R^{mnpq}R_m{}^r{}_p{}^s\omega_{Inr}\,\omega_{Jqs}\big] \end{aligned} \tag{6.8}$$

The gauge field partner of these scalar couplings is obtained by lifting to eleven dimensions the original expression and reducing back on a circle times a Calabi-Yau. It then follows that the result for gauge fields takes the form

$$\left(t_8t_8 - \frac{1}{8}\,\epsilon_{10}\epsilon_{10}\right) R^4 \to 3\, X_{IJ} \nabla^a F^I_{ac} \nabla^b F_b{}^{cJ}\,. \tag{6.9}$$

Comparing (6.6) and (6.9) to the known F^4 term in supergravity, given in (D.7), we find that the interactions of the vector multiplets arising from expansion along the second cohomology are governed by the tensor X_{IJ} above.

We now turn to the three-index structure, and compute the parity odd term quadratic in 4D field strengths. Following the same lifting and reducing procedure, we find that the only even-odd term quadratic in 3-from field strengths is

$$t_{8\mu_1\dots\mu_8} B \wedge \nabla G^{\mu_1\mu_2\mu_9} \wedge \nabla G^{\mu_3\mu_4}{}_{\mu_9} \wedge R^{\mu_5\mu_6} \wedge R^{\mu_7\mu_8}\,, \tag{6.10}$$

which upon reduction to 4D gives rise to a term of the type

$$6\,u^K\,H_{IJ,K}\,\epsilon^{\mu\nu\rho\lambda}\nabla_\mu F^I{}_\nu{}^\kappa \nabla_\rho F^I{}_{\lambda\kappa} \simeq -3\,H_{IJ,K}\,\epsilon^{\mu\nu\rho\lambda}\nabla_\mu u^K\,F^I{}_{\nu\rho}\nabla_\kappa F^I{}_{\lambda\kappa} + \dots\,, \tag{6.11}$$

where in the second step we partially integrated and the dots denote terms involving the derivative of the coupling $H_{IJ,K}$. The explicit expression for this three index coupling follows from the relation

$$\begin{aligned} u^K\,H_{IJ,K} = -16\int_X B\wedge\big[&R^{mn}\wedge R_{mn}\omega_I{}^{rs}\omega_{Jrs} - 8\,R^{pq}\wedge R_p{}^r\omega_{Iq}{}^s\omega_{Jrs} \\ &- 4\,R^{pq}\wedge R^{rs}\omega_{Ipr}\,\omega_{Jqs} + 2\,R^{pq}\wedge R^{rs}\omega_{Ipq}\,\omega_{Jrs}\big]\,, \end{aligned} \tag{6.12}$$

where we note the important identities

$$t^K\,H_{IJ,K} = -X_{IJ}\,, \qquad t^I X_{IJ} = 32\,\alpha_I\,. \tag{6.13}$$

The remaining interactions with the ten dimensional RR gauge field, F^0, described by $\mathcal{H}_{0I}$ and $\mathcal{H}_{00}$, are obtained by viewing F^0 as a Kaluza-Klein gauge field, coming from the reduction from 11D. As these must necessarily be quadratic in the Kaluza-Klein gauge fields, only the factorised term in the 10D invariant contributes. It then follows that, as far as terms quadratic in Riemann tensors are concerned, the lifting and reducing procedure is identical to the 4D/5D connection studied in [25]. Therefore, we can simply add the couplings $\mathcal{H}_{0I}$ and $\mathcal{H}_{00}$ found in that work, given by

$$\begin{aligned} \mathcal{H}_{0\bar I}\big|_{\rm KK} &= -\,12\,\mathrm{i}\alpha_I = -\frac{3}{8}\,\mathrm{i}\,X_{IJ}t^J\,, \\ \mathcal{H}_{0\bar 0}\big|_{\rm KK} &= 24\,\alpha_I t^I = \frac{3}{4}\,X_{IJ}t^I t^J\,, \end{aligned} \tag{6.14}$$

to the ones in (6.6) and (6.9) above.

After adding the extra contribution in (6.14) to (6.6), and performing the by now standard shift in (4.12) to account for the axionic coupling to the 0-th gauge field strength, we obtain the final form of the coupling $\mathcal{H}_{AB}$, as

$$\mathcal{H}_{A\bar B} = \begin{pmatrix} X_{IJ} & -X_{IJ}z^J \\ -\bar z^I X_{IJ} & X_{IJ}z^I\bar z^J \end{pmatrix} \to |Y^0|^{-4}\begin{pmatrix} |Y^0|^2\,X_{IJ} & -\bar Y^0\,X_{IJ}Y^J \\ -Y^0\,\bar Y^I X_{IJ} & X_{IJ}Y^I\bar Y^J \end{pmatrix}\,, \tag{6.15}$$

where in the second step we passed from the special coordinates z^I to the projective coordinates Y^A. Note that the coupling X_{IJ} is real and depends only on the Kähler moduli t^I, similar to the lowest order Kähler potential.

The couplings (6.15) satisfy the condition $Y^A \mathcal{H}_{A\bar{B}} = 0$, so that they belong to the class of [13]. Recently, a more general class of F^4 invariants appeared in [26], which allows for $Y^A \mathcal{H}_{A\bar{B}} \neq 0$ and contains additional terms quadratic in the Ricci tensor. However, we find that no such extra terms appear in the reduction of the 10D action, beyond the one in the familiar (Weyl)^2 term, consistent with the properties of $\mathcal{H}_{A\bar{B}}$ above.

On H^2F^2 terms. Finally, we comment on the possible four derivative terms which mix tensor and vector multiplets. Such terms have not been explicitly constructed in the literature and it is a interesting open problem to tackle, even for rigidly supersymmetric theories. Indeed, a construction of such an invariant is likely to lead to insight into more general mixed terms of the type $H^{2n}F^{2m}$, where n is odd, examples of which have been mentioned above (e.g. the H^2F^4 term).

An explicit computation of the terms arising from reduction of the parity even terms at this order reveals that terms involving derivatives of H and F do not arise. However, we do find nontrivial terms involving field strengths only, e.g.

$$\mathcal{L} \propto X_{IJ} E^\mu E^\nu \partial_\mu u^I \partial_\nu u^J \,, \tag{6.16}$$

and terms related to this by introducing the gauge field strengths, i.e. $X_{IJ} E^\mu E^\nu F^I{}_\mu{}^\rho F^J{}_{\nu\rho}$, where X_{IJ} is the integral defined in (6.7) above. In order to obtain this result, we used (2.14) and we note that the additional terms $\Delta J_0(\Omega_+, H)$ in (2.6) are nontrivial in this case.

In addition, the parity odd terms are also nontrivial for these couplings, since one can easily verify that the parity odd term (2.7) leads to couplings of the type

$$Y_{IJ}\, H \wedge \partial^\mu u^{[I} d\partial_\mu u^{J]} \,, \tag{6.17}$$

where Y_{IJ} is the integral

$$Y_{IJ} = \int \left(R^{mn} \wedge R_{np} \wedge \omega_I{}^p \wedge \omega_{Jm} - \frac{1}{8} R^{mn} \wedge R_{mn} \wedge \omega_I{}^p \wedge \omega_{Jp} \right) . \tag{6.18}$$

In the last relation, the two-forms ω_I are viewed as vector valued one-forms, for convenience. Note that the term (6.17) is linear in the tensor field strength, unlike the parity even coupling (6.16). This may seem counterintuitive, but we stress that our simplifying choice of ignoring the scalars in the tensor multiplet may obscure the connection between tensor multiplet couplings that are expected to be controlled by appropriate functions of these scalars. Finally, we point out that Y_{IJ} is by definition antisymmetric in its indices, which is similar to the corresponding six derivative terms in (5.9)–(5.10) above. This type of odd terms is somewhat unconventional in the $\mathcal{N}=2$ setting and may point to a common origin of all these unknown invariants.

One possible way to construct couplings of this type is to make use of the results of [30], on arbitrary couplings of vector and tensor multiplet superfields. In terms of the superfields G^2 and W^A describing the tensor and vector multiplets respectively, one may

JHEP02(2014)093

consider an integral of the type[10]

$$\int d^4\theta d^4\bar{\theta}\,\mathcal{H}(W,\bar{W})\,G^2\,, \tag{6.19}$$

in order to describe couplings such as above, where the function $\mathcal{H}$ must be such that the couplings (6.16)–(6.17) are reproduced. It is worth mentioning that including kinetic multiplets in (6.19) may lead to even higher derivative couplings that can account for some of the unknown couplings pointed out above, i.e. of the type $H^{2n}F^{2m}$, where n is odd. The explicit realisation of the possible Lagrangians following from the integral (6.19) in components would require the construction of a density formula for a general real multiplet of $\mathcal{N}=2$ supergravity and falls outside the scope of the present work.

7 Some open questions

We shall conclude with a list of some open questions.

One immediate consequence of this work is the prediction of new four-dimensional higher-derivative $\mathcal{N}=2$ invariants. It would be nice to be able to verify this prediction by explicitly constructing some of these terms, either using the structure in (6.19), or new techniques. It is interesting to point out that the new invariants involve terms, descending from the eleven-dimensional anomalous terms $C_3\wedge X_8$, which are top-form Chern-Simons-like couplings. Examples of these at the six-and four-derivative are discussed in sections 5 and 6 respectively. It would also be very interesting to verify whether the terms that we find to be vanishing but could in principle be nontrivial, such as the H^6F^2 and $(\nabla H)^2(\nabla F)^2$ terms, do exist or not. Moreover, we stress that we have been focusing on the leading terms, matching to the invariants constructed in [13] and disregarding the possibility of more detailed structures that might appear. While we have not found any inconsistencies, we cannot exclude the existence of subleading terms that are not captured here. For example, the types of invariants recently constructed in [26] allow for additional couplings proportional to the square of the Ricci tensor, rather than the Weyl tensor alone.

There is a number of important omissions here. We have worked exclusively with one-loop terms, and avoided the discussion of the dilaton. Our excuse can be that the tree-level terms neither survive the eleven-dimensional limit, nor contribute to the well studied R^2 terms in four dimensions. Yet they are important for understanding the corrections to the moduli spaces. In addition the dilaton is subtle and important enough to merit a discussion.

As already mentioned, we have largely ignored the complex deformations of the internal CY. It might be of some interest to extend our results to generic hyper-matter, since that would most likely turn on the couplings that we find to be vanishing.

We have concentrated only on CY compactification and hence ungauged $\mathcal{N}=2$ theories. Quantum corrections to the super potential have been much studied and are of obvious interest. It would be very interesting to extend the discussion of (at least some of) the

[10] We thank Daniel Butter for pointing out this possibility.

JHEP02(2014)093

higher derivative couplings to the gauged theories. The fact that the couplings described here have an off-shell formulation is helpful in that respect.

The relation of our calculation to the topological string calculations needs further elucidation. Most of our CY integrals are not topological and one may ask if there is an extension or refinement of topological strings that may capture the physical string theory couplings described here. Our calculations are exclusively one-loop, but one might hope that the structure of the terms discussed here, and the relations between different supersymmetric invariants are sufficiently restricted by supersymmetry to extend to all genus calculations.

The structure of the various functions describing the coupling of the gravity and tensor multiplets seem to treat the two backgrounds on the same footing, somehow reflecting the structure of the ten dimensional action built out of the torsionfull curvature tensor $R(\Omega_+)$. Given that this structure was instrumental in checking T-duality in [15], it would be interesting to consider the properties of our couplings under the c-map, which is the lower dimensional analogous operation. Note that this would explicitly relate the vector and tensor multiplets, especially in view of the fact that the various couplings mix the two kinds of multiplets.

The new terms discussed here are not relevant for BPS black hole physics, at least at the attractor [13, 31, 32], as they vanish by construction on fully BPS backgrounds and do not affect the entropy and charges. However, our results are relevant for non-BPS black holes and may be related to the one-loop modifications to the entropy of such objects, as in [33].

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A Tensor structures in ten dimensions

We define the tensor, t_8, as having four antisymmetric pairs of indices and given in terms of its contraction with an antisymmetric tensor $F^{\mu\nu}$ by

$$t_8 F^4 = 24\,\mathrm{tr}\,F^4 - 6\,(\mathrm{tr}\,F^2)^2\,. \tag{A.1}$$

Taking derivatives of this identity with respect to F one can obtain the explicit tensor t_8. The Type IIA one-loop correction in ten dimensions contains terms quadratic in t_8 and quartic in the modified curvature $R(\Omega_+)_{\mu_1\mu_2}{}^{\mu_3\mu_4}$. The latter is antisymmetric in each pair of indices, but does not satisfy the Bianchi and pair exchange identities. Considering a general tensor, $\mathcal{R}$, with these symmetries, the relevant expression reads

$$t_8 t_8 \mathcal{R}^4 = 192\,\mathcal{R}_1 + 384\,\mathcal{R}_2 + 24\,\mathcal{R}_3 + 12\,\mathcal{R}_4 - 96\,(\mathcal{R}_{5a} + \mathcal{R}_{5b}) - 48\,(\mathcal{R}_{6a} + \mathcal{R}_{6b})\,, \tag{A.2}$$

where the $\mathcal{R}_i$ are defined in (A.4) below. Similarly, we display for completeness the full expression for the odd-odd term quartic in $\mathcal{R}$ as

$$\begin{aligned}
-\frac{1}{8}\,\varepsilon_{10}\varepsilon_{10}\mathcal{R}^4 = {} & 192\,\tilde{\mathcal{R}}_1 + 24\,\tilde{\mathcal{R}}_3 + 12\,\tilde{\mathcal{R}}_4 - 192\,\tilde{\mathcal{R}}_5 - 384\,\tilde{\mathcal{R}}_6 - 384\,\tilde{A}_7 \\
& + 4\,\mathcal{R}\,\mathcal{R}\,\left(\mathcal{R}\,\mathcal{R} + 6\,\mathcal{R}^{\mu_1\mu_2\mu_3\mu_4}\mathcal{R}_{\mu_3\mu_4\mu_1\mu_2} - 24\,\mathcal{R}\,\mathcal{R}\,\mathcal{R}^{\mu_1\mu_2}\mathcal{R}_{\mu_2\mu_1}\right) \\
& + 384\,\mathcal{R}\,\mathcal{R}^{\mu_1\mu_2}\left(\mathcal{R}_{\mu_2}{}^{\mu_3}{}_{\mu_1}{}^{\mu_4}\mathcal{R}_{\mu_4\mu_3} - \mathcal{R}_{\mu_2}{}^{\mu_3\mu_4\mu_5}\mathcal{R}_{\mu_4\mu_5\mu_1\mu_3} + \frac{2}{3}\,\mathcal{R}_{\mu_2}{}^{\mu_3}\mathcal{R}_{\mu_3\mu_1}\right) \\
& + 32\,\mathcal{R}\,\mathcal{R}^{\mu_1\mu_2\mu_3\mu_4}\,\left(\mathcal{R}_{\mu_3\mu_4}{}^{\mu_5\mu_6}\mathcal{R}_{\mu_5\mu_6\mu_1\mu_2} - 4\,\mathcal{R}_{\mu_3}{}^{\mu_5}{}_{\mu_1}{}^{\mu_6}\mathcal{R}_{\mu_4\mu_6\mu_2\mu_5}\right) \\
& + 96\,\mathcal{R}^{\mu_1\mu_2}\mathcal{R}_{\mu_2\mu_1}\,\left(2\,\mathcal{R}^{\mu_3\mu_4}\mathcal{R}_{\mu_4\mu_3} - \mathcal{R}^{\mu_3\mu_4\mu_5\mu_6}\mathcal{R}_{\mu_5\mu_6\mu_3\mu_4}\right) \\
& + 768\,\mathcal{R}^{\mu_1\mu_2}\mathcal{R}_{\mu_2}{}^{\mu_3}{}_{\mu_1}{}^{\mu_4}\big(\mathcal{R}_{\mu_4}{}^{\mu_5\mu_6\mu_7}\mathcal{R}_{\mu_6\mu_7\mu_3\mu_5} \\
& \qquad\qquad - \mathcal{R}_{\mu_4}{}^{\mu_5}{}_{\mu_3}{}^{\mu_6}\mathcal{R}_{\mu_6\mu_5} - 2\,\mathcal{R}_{\mu_4}{}^{\mu_5}\mathcal{R}_{\mu_5\mu_3}\big) \\
& + 384\,\mathcal{R}^{\mu_1\mu_2}\mathcal{R}_{\mu_2}{}^{\mu_3}\,\left(2\,\mathcal{R}_{\mu_3}{}^{\mu_4\mu_5\mu_6}\mathcal{R}_{\mu_5\mu_6\mu_1\mu_4} - \mathcal{R}_{\mu_3}{}^{\mu_4}\mathcal{R}_{\mu_4\mu_1}\right) \\
& + 384\,\mathcal{R}^{\mu_1\mu_2}\mathcal{R}_{\mu_2}{}^{\mu_3\mu_4\mu_5}\big(\mathcal{R}_{\mu_4\mu_5\mu_1}{}^{\mu_6}\mathcal{R}_{\mu_6\mu_3} - \mathcal{R}_{\mu_4\mu_5}{}^{\mu_6\mu_7}\mathcal{R}_{\mu_6\mu_7\mu_1\mu_3} \\
& \qquad\qquad + 2\,\mathcal{R}_{\mu_4}{}^{\mu_6}{}_{\mu_1\mu_3}\mathcal{R}_{\mu_5\mu_6} + 4\,\mathcal{R}_{\mu_4}{}^{\mu_6}{}_{\mu_1}{}^{\mu_7}\mathcal{R}_{\mu_5\mu_7\mu_3\mu_6}\big)
\end{aligned} \tag{A.3}$$

where $\mathcal{R}_{\mu_1\mu_2} = \mathcal{R}_{\mu_1\mu_3\mu_2}{}^{\mu_3}$ is a non-symmetric tensor corresponding to the Ricci tensor and the scalar $\mathcal{R}$ is its trace. The various non-Ricci combinations appearing in both the even-even and odd-odd structures are defined as

$$\begin{aligned}
\mathcal{R}_1 &= \operatorname{tr}\mathcal{R}^{\mu_1\mu_2}\mathcal{R}_{\mu_2\mu_3}\mathcal{R}^{\mu_3\mu_4}\mathcal{R}_{\mu_4\mu_1}\,, & \tilde{\mathcal{R}}_1 &= \operatorname{tr}\mathcal{R}_{\mu_1\mu_2}\tilde{\mathcal{R}}^{\mu_2\mu_3}\mathcal{R}_{\mu_3\mu_4}\tilde{\mathcal{R}}^{\mu_4\mu_1}, \\
\mathcal{R}_2 &= \operatorname{tr}\mathcal{R}^{\mu_1\mu_2}\mathcal{R}_{\mu_2\mu_3}\mathcal{R}_{\mu_1\mu_4}\mathcal{R}^{\mu_4\mu_3}, & & \\
\mathcal{R}_3 &= \operatorname{tr}\mathcal{R}_{\mu_1\mu_2}\mathcal{R}^{\mu_3\mu_4}\operatorname{tr}\mathcal{R}^{\mu_1\mu_2}\mathcal{R}_{\mu_3\mu_4}, & \tilde{\mathcal{R}}_3 &= \operatorname{tr}\mathcal{R}^{\mu_1\mu_2}\tilde{\mathcal{R}}^{\mu_3\mu_4}\operatorname{tr}\mathcal{R}_{\mu_3\mu_4}\tilde{\mathcal{R}}_{\mu_1\mu_2}, \\
\mathcal{R}_4 &= \operatorname{tr}\mathcal{R}^{\mu_1\mu_2}\mathcal{R}_{\mu_1\mu_2}\operatorname{tr}\mathcal{R}^{\mu_5\mu_6}\mathcal{R}_{\mu_5\mu_6}, & \tilde{\mathcal{R}}_4 &= \operatorname{tr}\mathcal{R}^{\mu_1\mu_2}\tilde{\mathcal{R}}_{\mu_1\mu_2}\mathcal{R}^{\mu_5\mu_6}\tilde{\mathcal{R}}_{\mu_5\mu_6}, \\
\mathcal{R}_{5a} &= \operatorname{tr}\mathcal{R}_{\mu_1\mu_2}\mathcal{R}^{\mu_2\mu_5}\operatorname{tr}\mathcal{R}_{\mu_5\mu_6}\mathcal{R}^{\mu_6\mu_1}, & \mathcal{R}_{5b} &= \operatorname{tr}\tilde{\mathcal{R}}^{\mu_3\mu_4}\tilde{\mathcal{R}}_{\mu_3\mu_5}\operatorname{tr}\tilde{\mathcal{R}}^{\mu_5\mu_8}\tilde{\mathcal{R}}_{\mu_4\mu_8}, \\
\tilde{\mathcal{R}}_5 &= \operatorname{tr}\mathcal{R}^{\mu_1\mu_2}\tilde{\mathcal{R}}_{\mu_1\mu_5}\operatorname{tr}\mathcal{R}^{\mu_5\mu_6}\tilde{\mathcal{R}}_{\mu_2\mu_6}, & & \\
\mathcal{R}_{6a} &= \operatorname{tr}\mathcal{R}^{\mu_1\mu_2}\mathcal{R}^{\mu_5\mu_6}\operatorname{tr}\mathcal{R}_{\mu_1\mu_5}\mathcal{R}_{\mu_2\mu_6}, & \mathcal{R}_{6b} &= \operatorname{tr}\tilde{\mathcal{R}}^{\mu_3\mu_4}\tilde{\mathcal{R}}_{\mu_5\mu_6}\operatorname{tr}\tilde{\mathcal{R}}_{\mu_3\mu_5}\tilde{\mathcal{R}}_{\mu_4\mu_6}, \\
\tilde{\mathcal{R}}_6 &= \operatorname{tr}\mathcal{R}_{\mu_1\mu_2}\tilde{\mathcal{R}}^{\mu_5\mu_6}\mathcal{R}_{\mu_8\mu_5}{}^{\mu_7\mu_1}\mathcal{R}_{\mu_7\mu_6}{}^{\mu_8\mu_2}, & & \\
\tilde{A}_7 &= \mathcal{R}_{\mu_1\mu_2}{}^{\mu_3\mu_4}\mathcal{R}_{\mu_3\mu_5}{}^{\mu_1\mu_6}\mathcal{R}_{\mu_4\mu_7}{}^{\mu_5\mu_8}\mathcal{R}_{\mu_6\mu_8}{}^{\mu_2\mu_7}\,, & &
\end{aligned} \tag{A.4}$$

for any tensor $\mathcal{R}_{\mu_1\mu_2}{}^{\mu_3\mu_4}$ that is antisymmetric in each pair of indices, but does not satisfy the Bianchi identity and we use the shorthand notation $\tilde{\mathcal{R}}_{\mu_1\mu_2}{}^{\mu_3\mu_4} = \mathcal{R}^{\mu_3\mu_4}{}_{\mu_1\mu_2}$ in order to keep expressions compact. Note that if $\mathcal{R}$ is identified with a Riemann tensor, all tilded quantities become equal to their untilded counterparts.

B Off-shell $\mathcal{N}=2$ supergravity and chiral multiplets

In this appendix we summarise some general formulae on the $\mathcal{N} = 2$ Weyl multiplet in four dimensions and the chiral multiplets in a general superconformal background. Our conventions are as in [13], where the reader can find a more detailed account.

JHEP02(2014)093

	Weyl multiplet											parameter	
field	$e_M{}^A$	$\psi_M{}^i$	b_M	A_M	$\mathcal{V}_M{}^i{}_j$	$T_{AB}{}^{ij}$	χ^i	D	ω_M^{AB}	$f_M{}^A$	$\phi_M{}^i$	ϵ^i	η^i
w	-1	$-\frac{1}{2}$	0	0	0	1	$\frac{3}{2}$	2	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
c	0	$-\frac{1}{2}$	0	0	0	-1	$-\frac{1}{2}$	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
γ_5		+					+				$-$	+	$-$

Table 2. Weyl and chiral weights (w and c) and fermion chirality (γ_5) of the Weyl multiplet component fields and the supersymmetry transformation parameters.

JHEP02(2014)093

$\mathcal{N}=2$ superconformal gravity. The off-shell formulation of four-dimensional $\mathcal{N}=2$ supergravity is based on the Weyl multiplet of conformal supergravity, whose components are given in table 2. This consists of the vierbein $e_\mu{}^a$, the gravitino fields $\psi_\mu{}^i$, the dilatational gauge field b_μ, the R-symmetry gauge fields $\mathcal{V}_{\mu i}{}^j$ (which is an anti-hermitian, traceless matrix in the SU(2) indices i,j) and A_μ, an anti-selfdual tensor field $T_{ab}{}^{ij}$, a scalar field D and a spinor field χ^i. All spinor fields are Majorana spinors which have been decomposed into chiral components. The three gauge fields $\omega_\mu{}^{ab}$, $f_\mu{}^a$ and $\phi_\mu{}^i$, associated with local Lorentz transformations, conformal boosts and S-supersymmetry, respectively, are not independent as will be discussed later.

The infinitesimal Q, S and K transformations of the independent fields, parametrized by spinors ϵ^i and η^i and a vector $\Lambda_{\mathrm{K}}{}^A$, respectively, are as follows,

$$\begin{aligned}
\delta e_\mu{}^a =& \,\bar\epsilon^i\,\gamma^a\psi_{\mu i} + \bar\epsilon_i\,\gamma^a\psi_\mu{}^i\,,\\
\delta\psi_\mu{}^i =& \,2\,\mathcal{D}_\mu\epsilon^i - \frac{1}{8}T_{ab}{}^{ij}\gamma^{ab}\gamma_\mu\epsilon_j - \gamma_\mu\eta^i\\
\delta b_\mu =& \,\frac{1}{2}\bar\epsilon^i\phi_{\mu i} - \frac{3}{4}\bar\epsilon^i\gamma_\mu\chi_i - \frac{1}{2}\bar\eta^i\psi_{\mu i} + \text{h.c.} + \Lambda^a_K e_{\mu a}\,,\\
\delta A_\mu =& \,\frac{1}{2}\mathrm{i}\bar\epsilon^i\phi_{\mu i} + \frac{3}{4}\mathrm{i}\bar\epsilon^i\gamma_\mu\,\chi_i + \frac{1}{2}\mathrm{i}\bar\eta^i\psi_{\mu i} + \text{h.c.}\,,\\
\delta\mathcal{V}_\mu{}^i{}_j =& \,2\,\bar\epsilon_j\phi_\mu{}^i - 3\bar\epsilon_j\gamma_\mu\,\chi^i + 2\bar\eta_j\,\psi_\mu{}^i - (\text{h.c. ; traceless})\,,\\
\delta T_{ab}{}^{ij} =& \,8\,\bar\epsilon^{[i}R(Q)_{ab}{}^{j]}\,,\\
\delta\chi^i =& -\frac{1}{12}\gamma^{ab}\,\not{D}T_{ab}{}^{ij}\,\epsilon_j + \frac{1}{6}R(\mathcal{V})_{\mu\nu}{}^i{}_j\gamma^{\mu\nu}\epsilon^j - \frac{1}{3}\mathrm{i}R_{\mu\nu}(A)\gamma^{\mu\nu}\epsilon^i\\
&+ D\,\epsilon^i + \frac{1}{12}\gamma_{ab}T^{abij}\eta_j\,,\\
\delta D =& \,\bar\epsilon^i\,\not{D}\chi_i + \bar\epsilon_i\,\not{D}\chi^i\,.
\end{aligned} \tag{B.1}$$

Here, D_μ denotes the full superconformally covariant derivative, while $\mathcal{D}_\mu$ denotes a covariant derivative with respect to Lorentz, dilatation, and chiral SU(2) $\times$ U(1) transformations, e.g.

$$\mathcal{D}_\mu\epsilon^i = \left(\partial_\mu - \frac{1}{4}\omega_\mu{}^{cd}\,\gamma_{cd} + \frac{1}{2}\,b_\mu + \frac{1}{2}\mathrm{i}\,A_\mu\right)\epsilon^i + \frac{1}{2}\,\mathcal{V}_\mu{}^i{}_j\,\epsilon^j\,. \tag{B.2}$$

Under local scale and U(1) transformations the various fields and transformation parameters transform as indicated in table 2.

The various quantities denoted by $R(\mathcal{Q})$, and appearing in the supersymmetry variations above denote the supercovariant curvature tensors corresponding to each generator, $\mathcal{Q}$, whose detailed definition can be found in [13]. Here, we only give the following

$$\begin{aligned}
R(P)_{\mu\nu}{}^a &= 2\,\partial_{[\mu}\,e_{\nu]}{}^a + 2\,b_{[\mu}\,e_{\nu]}{}^a - 2\,\omega_{[\mu}{}^{ab}\,e_{\nu]b} - \frac{1}{2}(\bar\psi_{[\mu}{}^i\gamma^a\psi_{\nu]i} + \text{h.c.})\,,\\
R(Q)_{\mu\nu}{}^i &= 2\,\mathcal{D}_{[\mu}\psi_{\nu]}{}^i - \gamma_{[\mu}\phi_{\nu]}{}^i - \frac{1}{8}\,T^{abij}\,\gamma_{ab}\,\gamma_{[\mu}\psi_{\nu]j}\,,\\
R(M)_{\mu\nu}{}^{ab} &= 2\,\partial_{[\mu}\omega_{\nu]}{}^{ab} - 2\,\omega_{[\mu}{}^{ac}\omega_{\nu]c}{}^b - 4f_{[\mu}{}^{[a}e_{\nu]}{}^{b]} + \frac{1}{2}(\bar\psi_{[\mu}{}^i\,\gamma^{ab}\,\phi_{\nu]i} + \text{h.c.})\\
&\quad + \left(\frac{1}{4}\bar\psi_\mu{}^i\,\psi_\nu{}^j\,T^{ab}{}_{ij} - \frac{3}{4}\bar\psi_{[\mu}{}^i\,\gamma_{\nu]}\,\gamma^{ab}\chi_i - \bar\psi_{[\mu}{}^i\,\gamma_{\nu]}\,R(Q)^{ab}{}_i + \text{h.c.}\right)\,,
\end{aligned}\tag{B.3}$$

which are necessary to introduce the conventional constraints

$$\begin{aligned}
R(P)_{\mu\nu}{}^a &= 0\,,\\
\gamma^\mu R(Q)_{\mu\nu}{}^i + \frac{3}{2}\gamma_\nu\chi^i &= 0\,,\\
e^\nu{}_b\,R(M)_{\mu\nu a}{}^b - \mathrm{i}\tilde R(A)_{\mu a} + \frac{1}{8}T_{abij}T_\mu{}^{bij} - \frac{3}{2}D\,e_{\mu a} &= 0\,,
\end{aligned}\tag{B.4}$$

defining the composite gauge fields associated with local Lorentz transformations, S-supersymmetry and special conformal boosts, $\omega_M{}^{AB}$, $\phi_M{}^i$ and $f_M{}^A$, respectively.

Chiral multiplets. Chiral multiplets are the basic building blocks of all supersymmetric invariants in this paper. We therefore give a concise overview of their most basic properties, to be used in the various constructions.

Chiral multiplets are complex, carrying a Weyl weight w and a chiral U(1) weight c, which is opposite to the Weyl weight, i.e. $c = -w$, while anti-chiral multiplets can be obtained from chiral ones by complex conjugation, so that anti-chiral multiplets will have $w = c$. The components of a generic scalar chiral multiplet are a complex scalar A, a Majorana doublet spinor Ψ_i, a complex symmetric scalar B_{ij}, an anti-selfdual tensor G^-_{ab}, a Majorana doublet spinor Λ_i, and a complex scalar C. The assignment of their Weyl and chiral weights is shown in table 3. The Q- and S-supersymmetry transformations for a scalar chiral multiplet of weight w, are as follows

$$\begin{aligned}
\delta A &= \bar\epsilon^i\Psi_i\,,\\
\delta\Psi_i &= 2\,\not{D}A\epsilon_i + B_{ij}\,\epsilon^j + \frac{1}{2}\gamma^{ab}G^-_{ab}\,\varepsilon_{ij}\epsilon^j + 2\,wA\,\eta_i\,,\\
\delta B_{ij} &= 2\,\bar\epsilon_{(i}\not{D}\Psi_{j)} - 2\,\bar\epsilon^k\Lambda_{(i}\,\varepsilon_{j)k} + 2(1-w)\,\bar\eta_{(i}\Psi_{j)}\,,\\
\delta G^-_{ab} &= \frac{1}{2}\varepsilon^{ij}\,\bar\epsilon_i\not{D}\gamma_{ab}\Psi_j + \frac{1}{2}\bar\epsilon^i\gamma_{ab}\Lambda_i - \frac{1}{2}(1+w)\,\varepsilon^{ij}\bar\eta_i\gamma_{ab}\Psi_j\,,\\
\delta\Lambda_i &= -\frac{1}{2}\gamma^{ab}\not{D}G^-_{ab}\epsilon_i - \not{D}B_{ij}\varepsilon^{jk}\epsilon_k + C\varepsilon_{ij}\,\epsilon^j + \frac{1}{4}\big(\not{D}A\,\gamma^{ab}T_{abij} + w\,A\,\not{D}\gamma^{ab}T_{abij}\big)\varepsilon^{jk}\epsilon_k\\
&\quad - 3\,\gamma_a\varepsilon^{jk}\epsilon_k\,\bar\chi_{[i}\gamma^a\Psi_{j]} - (1+w)\,B_{ij}\varepsilon^{jk}\,\eta_k + \frac{1}{2}(1-w)\,\gamma^{ab}\,G^-_{ab}\eta_i\,,
\end{aligned}$$

	Chiral multiplet					
field	A	Ψ_i	B_{ij}	G^-_{ab}	Λ_i	C
w	w	$w+\frac{1}{2}$	$w+1$	$w+1$	$w+\frac{3}{2}$	$w+2$
c	$-w$	$-w+\frac{1}{2}$	$-w+1$	$-w+1$	$-w+\frac{3}{2}$	$-w+2$
γ_5		$+$			$+$	

Table 3. Weyl and chiral weights (w and c) and fermion chirality (γ_5) of the chiral multiplet component fields.

	vector multiplet				tensor multiplet			
field	X	W_μ	Ω_i	Y^{ij}	L^{ij}	$B_{\mu\nu}$	φ_i	G
w	1	0	$\frac{3}{2}$	2	2	0	$\frac{5}{2}$	3
c	-1	0	$-\frac{1}{2}$	0	0	0	$-\frac{1}{2}$	1
γ_5			$+$				$-$	

Table 4. Weyl and chiral weights (w and c) and fermion chirality (γ_5) of the vector multiplet and the tensor multiplet.

$$\begin{aligned}\delta C =& -2\,\varepsilon^{ij}\bar\epsilon_i \not{D}\Lambda_j - 6\,\bar\epsilon_i\chi_j\,\varepsilon^{ik}\varepsilon^{jl}B_{kl}\\ &-\frac{1}{4}\varepsilon^{ij}\varepsilon^{kl}\big((w-1)\,\bar\epsilon_i\gamma^{ab}\not{D}T_{abjk}\Psi_l + \bar\epsilon_i\gamma^{ab}T_{abjk}\not{D}\Psi_l\big) + 2\,w\varepsilon^{ij}\bar\eta_i\Lambda_j\,. \end{aligned} \tag{B.5}$$

Any homogeneous function of chiral superfields constitutes a chiral superfield, whose Weyl weight is determined by the degree of homogeneity of the function at hand. Indeed, one can show that a function $G(\Phi)$ of chiral superfields Φ^I defines a chiral superfield, whose component fields take the following form,

$$\begin{aligned} A|_G =& \,G\,,\\ \{\,\Psi_i\,,B_{ij}\,,G^-_{ab}\,\}|_G =& \,G_I\,\{\,\Psi_i{}^I\,,B_{ij}{}^I\,,G^-_{ab}{}^I\,\}\,,\\ \Lambda_i|_G =& \,G_I\,\Lambda_i{}^I - \frac{1}{2}G_{IJ}\left[B_{ij}{}^I\varepsilon^{jk} + \frac{1}{2}\,G^-_{ab}{}^I\gamma^{ab}\delta_i^k\right]\,\Psi_k{}^J\,,\\ C|_G =& \,G_I\,C^I - \frac{1}{4}G_{IJ}\big[B_{ij}{}^I B_{kl}{}^J\,\varepsilon^{ik}\varepsilon^{jl} - 2\,G^-_{ab}{}^I G^{-abJ}\big]\,, \end{aligned} \tag{B.6}$$

where G_I, G_{IJ} etc. are the derivatives of the function G with respect to the scalars A^I and we omitted all terms nonlinear in fermions for brevity.

Chiral multiplets of $w=1$ are special, because they are reducible upon imposing a reality constraint. The two cases that are relevant are the vector multiplet, which arises upon reduction from a scalar chiral multiplet, and the Weyl multiplet, which is a reduced anti-selfdual chiral tensor multiplet.

JHEP02(2014)093

The constraint for a scalar chiral superfield implies that $C|_{\rm vector}$ and $\Lambda_i|_{\rm vector}$ are expressed in terms of the lower components of the multiplet, and imposes a reality constraint on $B|_{\rm vector}$ and a Bianchi identity on $G^-|_{\rm vector}$ [34–36], as

$$
\begin{aligned}
A|_{\rm vector} =& \, X \, , \\
\Psi_i|_{\rm vector} =& \, \Omega_i \, , \\
B_{ij}|_{\rm vector} =& \, Y_{ij} = \varepsilon_{ik}\varepsilon_{jl} Y^{kl} \, , \\
G^-_{ab}|_{\rm vector} =& F^-_{ab} - \frac{1}{4}\, \bar X \, T_{ab}{}^{ij}\, \varepsilon_{ij} \, , \\
\Lambda_i|_{\rm vector} =& -\varepsilon_{ij} \not{D} \Omega^j \\
C|_{\rm vector} =& -2\,\Box_{\rm c} \bar X - \frac{1}{4} G^+_{ab}\, T^{ab}{}_{ij}\varepsilon^{ij} \, ,
\end{aligned}
\tag{B.7}
$$

where $F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}$ is the field strength of a gauge field, A_μ. The corresponding Bianchi identity on G_{ab} can be written as,

$$
D^b\left(G^+_{ab} - G^-_{ab} + \frac{1}{4} X T_{abij}\varepsilon^{ij} - \frac{1}{4}\bar X T_{ab}{}^{ij}\varepsilon_{ij}\right) = 0 \, , \tag{B.8}
$$

where in both (B.7) and (B.8) we again omitted terms nonlinear in fermions. The reduced scalar chiral multiplet thus describes the covariant fields and field strength of a *vector multiplet*, which encompasses $8+8$ bosonic and fermionic components. Table 4 summarizes the Weyl and chiral weights of the various fields belonging to the vector multiplet: a complex scalar X, a Majorana doublet spinor Ω_i, a vector gauge field A_μ, and a triplet of auxiliary fields Y_{ij}.

The Q- and S-supersymmetry transformations for the vector multiplet take the form,

$$
\begin{aligned}
\delta X =& \, \bar\epsilon^i \Omega_i \, , \\
\delta\Omega_i =& \, 2\not{D} X \epsilon_i + \frac{1}{2}\varepsilon_{ij} G_{\mu\nu}\gamma^{\mu\nu}\epsilon^j + Y_{ij}\epsilon^j + 2X\eta_i \, , \\
\delta A_\mu =& \, \varepsilon^{ij}\bar\epsilon_i(\gamma_\mu\Omega_j + 2\,\psi_{\mu j} X) + \varepsilon_{ij}\bar\epsilon^i(\gamma_\mu \Omega^j + 2\,\psi_\mu{}^j \bar X) \, , \\
\delta Y_{ij} =& \, 2\,\bar\epsilon_{(i}\not{D}\Omega_{j)} + 2\,\varepsilon_{ik}\varepsilon_{jl}\,\bar\epsilon^{(k}\not{D}\Omega^{l)} \, ,
\end{aligned}
\tag{B.9}
$$

and, for $w = 1$, are in clear correspondence with the supersymmetry transformations of generic scalar chiral multiplets given in (B.5).

We now turn to the covariant fields of the Weyl multiplet, which can be arranged in an anti-selfdual tensor chiral multiplet, whose chiral superfield components take the following form,

$$
\begin{aligned}
A_{ab}|_W =& \, T_{ab}{}^{ij}\varepsilon_{ij} \, , \\
\Psi_{abi}|_W =& \, 8\,\varepsilon_{ij} R(Q)^j_{ab} \, , \\
B_{abij}|_W =& -8\,\varepsilon_{k(i} R(\mathcal{V})^-_{ab}{}^k{}_{j)} \, , \\
\left(G^-_{ab}\right)^{cd}|_W =& -8\,\hat R(M)^-_{ab}{}^{cd} \, , \\
\Lambda_{abi}|_W =& \, 8\left(\mathcal{R}(S)^-_{abi} + \frac{3}{4}\gamma_{ab}\not{D}\chi_i\right) \, , \\
C_{ab}|_W =& \, 4D_{[a}\, D^c T_{b]c\,ij}\varepsilon^{ij} - \text{dual} \, .
\end{aligned}
\tag{B.10}
$$

Note that all quantities involved in the components above are either manifestly supercovariant curvatures or (covariant) auxiliary fields of the Weyl multiplet. In particular, $\mathcal{R}(S)_{abi}$ is the curvature of the S-supersymmetry gauge field, which is solved in terms of the derivative of the gravitino curvature, $\mathcal{R}(Q)_{abi}$, due to the conventional constraints.

All higher derivative terms involving powers of the Weyl tensor in this paper are constructed by couplings of the scalar chiral multiplet with $w = 2$ is obtained by squaring the Weyl multiplet above. The various scalar chiral multiplet components of this multiplet are given by,

$$
\begin{aligned}
A_{\mathsf{w}} &= (T_{ab}{}^{ij}\varepsilon_{ij})^2 \,, \\
\Psi_{\mathsf{w}i} &= 16\,\varepsilon_{ij} R(Q)^j_{ab}\, T^{klab}\,\varepsilon_{kl} \,, \\
B_{ij\mathsf{w}} &= -\,16\,\varepsilon_{k(i} R(\mathcal{V})^k{}_{j)ab}\, T^{lmab}\,\varepsilon_{lm} - 64\,\varepsilon_{ik}\varepsilon_{jl}\,\bar{R}(Q)_{ab}{}^k\, R(Q)^{l\,ab} \,, \\
G_{\mathsf{w}}^{-ab} &= -\,16\,\hat{R}(M)_{cd}{}^{ab}\, T^{klcd}\,\varepsilon_{kl} - 16\,\varepsilon_{ij}\,\bar{R}(Q)^i_{cd}\gamma^{ab} R(Q)^{cd\,j} \,, \\
\Lambda_{i\mathsf{w}} &= 32\,\varepsilon_{ij}\,\gamma^{ab} R(Q)^j_{cd}\,\hat{R}(M)^{cd}{}_{ab} + 16\,(\mathcal{R}(S)_{ab\,i} + 3\gamma_{[a}D_{b]}\chi_i)\, T^{klab}\,\varepsilon_{kl} \\
&\quad - 64\, R(\mathcal{V})_{ab}{}^k{}_i\,\varepsilon_{kl}\, R(Q)^{ab\,l} \,, \\
C_{\mathsf{w}} &= 64\,\hat{R}(M)^{-cd}{}_{ab}\,\hat{R}(M)^{-ab}_{cd} + 32\, R(\mathcal{V})^{-ab\,k}{}_l\, R(\mathcal{V})^{-l}_{ab\;k} \\
&\quad - 32\, T^{ab\,ij}\, D_a\, D^c T_{cb\,ij} + 128\,\bar{\mathcal{R}}(S)^{ab}{}_i\, R(Q)_{ab}{}^i + 384\,\bar{R}(Q)^{ab\,i}\gamma_a D_b \chi_i \,.
\end{aligned}
\tag{B.11}
$$

In practice, we will only use the lowest component, A_{w}, to construct functions that define composite chiral multiplets, as in (B.6), which determines completely all instances of the higher components in the relevant couplings. The components (B.11) can then be substituted straightforwardly in the final expressions to obtain the explicit couplings to the fields of the Weyl background.

C Tensor multiplet as a chiral background

We now turn to the tensor multiplet, which is also defined as an off-shell multiplet in an arbitrary superconformal background. The field content of this multiplet includes a pseudoreal triplet of scalars, L_{ij}, a two-form gauge potential, $B_{\mu\nu}$, a Majorana fermion doublet, φ^i, and an auxiliary complex scalar, G, with the Weyl and chiral assignments given in 4. The corresponding supersymmetry transformation rules are as follows

$$
\begin{aligned}
\delta L_{ij} &= 2\,\bar{\epsilon}_{(i}\varphi_{j)} + 2\,\varepsilon_{ik}\varepsilon_{jl}\,\bar{\epsilon}^{(k}\varphi^{l)} \,, \\
\delta\varphi^i &= \not{D} L^{ij}\,\epsilon_j + \varepsilon^{ij}\,\hat{\not{E}}^I\,\epsilon_j - G\,\epsilon^i + 2L^{ij}\,\eta_j \,, \\
\delta G &= -\,2\,\bar{\epsilon}_i \not{D}\,\varphi^i - \bar{\epsilon}_i\left(6\,L^{ij}\,\chi_j + \frac{1}{4}\gamma^{ab}T_{abjk}\,\varphi^l\,\varepsilon^{ij}\varepsilon^{kl}\right) + 2\,\bar{\eta}_i\varphi^i \,, \\
\delta B_{\mu\nu} &= \mathrm{i}\bar{\epsilon}^i\gamma_{\mu\nu}\varphi^j\,\varepsilon_{ij} - \mathrm{i}\bar{\epsilon}_i\gamma_{\mu\nu}\varphi_j\,\varepsilon^{ij} + 2\mathrm{i}\, L_{ij}\,\varepsilon^{jk}\,\bar{\epsilon}^i\gamma_{[\mu}\psi_{\nu]k} - 2\mathrm{i}\, L^{ij}\,\varepsilon_{jk}\,\bar{\epsilon}_i\gamma_{[\mu}\psi_{\nu]}{}^k \,,
\end{aligned}
\tag{C.1}
$$

and we refer to [20] for the precise definitions of the superconformally covariant derivatives on the various fields. The vector $\hat{E}^\mu$ is the superconformal completion of the dual of the three-form field strength, $\hat{E}^\mu = \frac{1}{2}\mathrm{i}\, e^{-1}\,\varepsilon^{\mu\nu\rho\sigma}\partial_\nu B_{\rho\sigma}$.

The couplings of the tensor multiplets are given in terms of composite vector multiplets [20], described by functions of a set of tensor multiplets, labeled by I. To this end, we define the first component, the scalar X_I as

$$X_I = \mathcal{F}_{I,J}\,\bar{G}^J + \mathcal{F}_{I,JK}{}^{ij}\,\bar{\varphi}_i{}^J\varphi_j{}^K\,, \tag{C.2}$$

which, by (C.1), transforms according to the first of (B.5) into the remaining bosonic components of the vector multiplet, as

$$\begin{aligned}
Y_{ij\,I} &= -2\,\mathcal{F}_{I,J}\left[\Box^{\rm c} L_{ij}{}^J + 3\,D L_{ij}{}^J\right] - 2\,\mathcal{F}_{I,JKij}\,(\bar{G}^J\,G^K + \hat{E}_\mu{}^J\,\hat{E}^{\mu K})\,,\\
&\quad - 2\,\mathcal{F}_{I,JK}{}^{kl}\,(D_\mu L_{ik}{}^J\,D^\mu L_{jl}{}^K + 2\,\varepsilon_{k(i}\,D_\mu L_{j)l}{}^J\,\hat{E}^{\mu K})\\
F_{\mu\nu\,I} &= -2\,\mathcal{F}_{I,JK}{}^{mn}\,\partial_{[\mu}L_{mk}{}^J\,\partial_{\nu]}L_{nl}{}^K\,\varepsilon^{kl}\\
&\quad - 4\,\partial_{[\mu}\left(\mathcal{F}_{I,J}\,\hat{E}_{\nu]}{}^J - \frac{1}{2}\,\mathcal{F}_{I,J}\,\mathcal{V}_{\nu]}{}^i{}_j\,L_{ik}{}^J\,\varepsilon^{jk}\right)\,,\\
C_I &= -2\,\Box_{\rm c}(\mathcal{F}_{I,J}\,G^J) - \frac{1}{4}\left(F^+_{ab\,I} - \frac{1}{4}\,\mathcal{F}_{I,J}\,\bar{G}^J T_{ab\,ij}\varepsilon^{ij}\right) T^{ab}{}_{ij}\varepsilon^{ij}\,,
\end{aligned} \tag{C.3}$$

where we suppressed all fermions and the component C_I is consistent with (B.7). In order for this multiplet to be well defined, the first derivative of $\mathcal{F}_{I,J}(L)$ with respect to $L^{K\,ij}$, denoted by $\mathcal{F}_{I,J,Kij}$, must satisfy the constraints

$$\mathcal{F}_{I,J,Kij} = \mathcal{F}_{I,K,Jij}\,, \qquad \varepsilon^{jk}\,\mathcal{F}_{I,J,Kij,Lkl}(L) = 0\,, \tag{C.4}$$

while Weyl covariance requires the condition

$$\mathcal{F}_{I,JKik}\,L^{kjK} = -\frac{1}{2}\delta_i{}^j\,\mathcal{F}_{I,J}\,, \tag{C.5}$$

which implies that the function $\mathcal{F}_{I,J}$ is SU(2) invariant and homogeneous of degree -1, so that it has scaling weight -2.

The expressions for the composite chiral supermultiplet above can be used to construct actions with higher derivative couplings. In general, one can use (C.2)–(C.3) on the same footing as any vector multiplet to obtain actions containing vector-tensor couplings. This is beyond the scope of this paper, where we only consider a background chiral multiplet containing four derivatives on the components of a single tensor multiplet, similar to [20] but allowing for couplings depending on vector multiplet scalars as well.

For a single tensor multiplet, the functions $\mathcal{F}_{I,J}$ in (C.2) reduce to a single function $\mathcal{F}(L)$, while the constraints (C.4)–(C.5) imply the constraint,

$$\frac{\partial^2\mathcal{F}(L)}{\partial L^{ij}\,\partial L_{ij}} = 0\,. \tag{C.6}$$

We then consider the chiral multiplet of $w = 2$ defined by its first component as the square of (C.2), through

$$\hat{A}^{\rm t} = \mathcal{F}^2\bar{G}^2 + 2\,\mathcal{F}\,\mathcal{F}^{ij}\,G\,\bar{\varphi}_i\varphi_j = \mathcal{H}\,G^2 + \mathcal{H}^{ij}\,\bar{G}\,\bar{\varphi}_i\varphi_j\,, \tag{C.7}$$

JHEP02(2014)093

where we defined the function $\mathcal{H}(L) = [\mathcal{F}(L)]^2$, and its derivatives, as

$$\mathcal{H}^{ij} = \frac{\partial \mathcal{H}}{\partial L_{ij}}\,, \qquad \mathcal{H}^{ij,kl} = \frac{\partial^2 \mathcal{H}}{\partial L_{ij}\,\partial L_{kl}}\,. \tag{C.8}$$

As noted in (3.7), for a single tensor multiplet the function $\mathcal{F}$ is essentially unique, so that $\mathcal{H}$ is simply given by its square, as

$$\mathcal{H} = \frac{1}{L_{ij}L^{ij}}\,, \tag{C.9}$$

where in the reduction we consider in the main text, the scalars L_{ij} contain the dilaton and are kept constant throughout.

The remaining components of this composite background multiplet are given by (B.6) for $G(A) = A^2$, as follows from (C.7). For completeness, we display their form for a general function $\mathcal{H}$, as follows

$$\begin{aligned}
B^{\mathrm{t}}{}_{ij} =& \,2\,\bar{G}\,\Big(2\,\mathcal{H}\,\Big[\Box^{\mathrm{c}} L_{ij} + 3\,D L_{ij}\Big] - \mathcal{H}_{ij}\,(|G|^2 + \hat{E}_\mu\,\hat{E}^\mu) \\
& -\mathcal{H}^{kl}\,(D_\mu L_{ik}\,D^\mu L_{jl} + 2\,\varepsilon_{k(i}\,D_\mu L_{j)l}\,\hat{E}^\mu)\Big)\,, \\
G^{\mathrm{t}-}{}_{ab} =& \,-2\,\mathcal{H}^{mn}\,\bar{G}\,\mathcal{D}_{[a}L_{mk}\,\mathcal{D}_{b]}L_{nl}\,\varepsilon^{kl} - 8\,\mathcal{H}\,\bar{G}\,\Big(\mathcal{D}_{[a}\hat{E}_{b]} - \frac{1}{4}\,R_{ab}{}^{i}{}_{j}(\mathcal{V})\,L_{ik}\,\varepsilon^{jk}\Big) \\
& -4\,\bar{G}\mathcal{H}_{mn}\,\mathcal{D}_{[a}L^{mn}\hat{E}_{b]} - \frac{1}{2}\,\mathcal{H}\,|G|^2 T_{ab}{}^{ij}\varepsilon_{ij}\,,
\end{aligned} \tag{C.10}$$

for the lower components and

$$\begin{aligned}
C^{\mathrm{t}} =& \,\mathcal{H}(L)\Big\{ - 4\bar{G}\Box_{\mathrm{c}}G - 2\,(\Box_{\mathrm{c}}L_{ij} + 3\,D\,L_{ij})^2 + 16\,\mathcal{D}_{[a}E_{b]^-}\,\mathcal{D}^{[a}E^{b]^-} \\
& - 8\,\mathcal{D}_a E_b\,\Big(R^{abi-}{}_{j}(\mathcal{V})L_{ik}\varepsilon^{jk} - \frac{1}{4}[T^{abij}\varepsilon_{ij}\,G + \text{h.c.}]\Big) \\
& + \frac{1}{16}\,((T_{ab}{}^{ij}\varepsilon_{ij})^2 G^2 + 2\,(T_{abij}\varepsilon^{ij})^2\bar{G}^2) + 12\,(|G|^2 + E^2)\,D \\
& + R^{abm}{}_{n}(\mathcal{V})L_{ml}\varepsilon^{nl}\,\Big(R_{ab}{}^{i-}{}_{j}(\mathcal{V})L_{ik}\varepsilon^{jk} - \frac{1}{2}[T_{ab}{}^{ij}\varepsilon_{ij}\,G + \text{h.c.}]\Big)\Big\} \\
& + \mathcal{H}^{ij}(L)\Big\{\Big(\Box_{\mathrm{c}}L^{kl} + 3\,D\,L^{kl}\Big)\,(\mathcal{D}_\mu L_{ik}\mathcal{D}^\mu L_{jl} - 4\,\varepsilon_{ik}E^\mu\mathcal{D}_\mu L_{jl}) \\
& - 2\,\Box_{\mathrm{c}}L_{ij}(2\,|G|^2 + E^2) - 4\,\bar{G}\,\mathcal{D}^\mu G\,\mathcal{D}_\mu L_{ij} \\
& - 4\,\Big(E_b\mathcal{D}_a L_{ij} + \frac{1}{2}\mathcal{D}_a L_{ik}\mathcal{D}_b L_{jl}\varepsilon^{kl}\Big)\,\Big(R^{abm-}{}_{n}(\mathcal{V})L_{mo}\varepsilon^{no} - \frac{1}{4}\,T^{abmn}\varepsilon_{mn}G\Big) \\
& + 8(\mathcal{D}_a L_{ik}\mathcal{D}_b L_{jl}\varepsilon^{kl} - 2\,E_a\mathcal{D}_b L_{ij})\mathcal{D}^{[a}E^{b]^-}\Big\}
\end{aligned}$$

JHEP02(2014)093

$$
\begin{aligned}
&+\mathcal{H}^{ij,kl}(L)\Big\{-\varepsilon_{ik}\varepsilon^{pq}\mathcal{D}^\mu L_{mp}\mathcal{D}^\nu L^{mn}\mathcal{D}_\mu L_{jn}\mathcal{D}_\nu L_{ql} \\
&\qquad -8\,\varepsilon_{ik}E^b\mathcal{D}^a L_{jm}\left(\mathcal{D}_a L^{mn}\mathcal{D}_b L_{nl}+\frac{1}{6}\epsilon_{abcd}\mathcal{D}^c L_{mn}\mathcal{D}^d L_{nl}\right) \\
&\qquad +2\,\mathcal{D}_\mu L_{ik}\mathcal{D}^\mu L_{jl}|G|^2-(|G|^2+E^2)\,\left(\varepsilon_{ik}\varepsilon_{jl}(|G|^2+E^2)+4\varepsilon_{ik}E^\mu\mathcal{D}_\mu L_{jl}\right) \\
&\qquad +2\varepsilon_{ik}\varepsilon^{mn}\mathcal{D}_\mu L_{jm}\mathcal{D}^\mu L_{nl}\,E^2+4\varepsilon_{ik}(\mathcal{D}_\mu L_{jm}\mathcal{D}_\nu L_{ln}\varepsilon^{mn})E^\mu E^\nu\Big\}\,, \qquad \text{(C.11)}
\end{aligned}
$$

for the top component.

D The kinetic multiplet and supersymmetric invariants

The central object in constructing the various higher derivative invariants of the type $R^{2n}F^{2m}$ in this paper is the so called kinetic chiral multiplet. The term 'kinetic' multiplet was first used in the context of the $N=1$ tensor calculus [37], because this is the chiral multiplet that enables the construction of the kinetic terms, conventionally described by a real superspace integral, in terms of a chiral superspace integral. In [13, 36] a corresponding kinetic multiplet, $\mathbb{T}(\bar\Phi)$, for a chiral $w=0$ multiplet, Φ, was identified for $N=2$ supersymmetry, which now involves four rather than two covariant $\bar\theta$-derivatives. It follows that $\mathbb{T}(\bar\Phi)$ contains up to four space-time derivatives, so that the expression

$$
\int \mathrm{d}^4\theta\;\mathrm{d}^4\bar\theta\;\Phi\,\bar\Phi' \approx \int \mathrm{d}^4\theta\;\Phi\,\mathbb{T}(\bar\Phi')\,, \qquad \text{(D.1)}
$$

corresponds to a four derivative coupling. Expressing the chiral multiplets in terms of (functions of) reduced chiral multiplets, (D.1) leads to higher-derivative couplings of vector multiplets and/or the Weyl multiplet.

Denote the components of a $w=0$ chiral multiplet by (A,Ψ,B,G^-,Λ,C), out of which we construct the components of $\mathbb{T}(\bar\Phi_{w=0})$, denoted by $(A,\Psi,B,G^-,\Lambda,C)|_{\mathbb{T}(\bar\Phi)}$. In [13] the following relation was established,

$$
\begin{aligned}
A|_{\mathbb{T}(\bar\Phi)} &= \bar C\,, \\
\Psi_i|_{\mathbb{T}(\bar\Phi)} &= -\,2\,\varepsilon_{ij}\,\not{D}\Lambda^j-6\;\varepsilon_{ik}\varepsilon_{jl}\chi^j B^{kl}-\frac{1}{4}\varepsilon_{ij}\varepsilon_{kl}\,\gamma^{ab}T_{ab}{}^{jk}\overleftrightarrow{\not{D}}\,\Psi^l\,, \\
B_{ij}|_{\mathbb{T}(\bar\Phi)} &= -\,2\,\varepsilon_{ik}\varepsilon_{jl}\big(\Box_{\mathrm{c}}+3\,D\big)B^{kl}-2\,G^+_{ab}\,R(\mathcal{V})^{ab\,k}{}_i\,\varepsilon_{jk}\,, \\
G^-_{ab}|_{\mathbb{T}(\bar\Phi)} &= -\left(\delta_a{}^{[c}\delta_b{}^{d]}-\frac{1}{2}\varepsilon_{ab}{}^{cd}\right)\left[4\,D_cD^eG^+_{ed}+(D^e\bar A\,D_cT_{de}{}^{ij}+D_c\bar A\,D^eT_{ed}{}^{ij})\varepsilon_{ij}\right] \\
&\qquad +\Box_{\mathrm{c}}\bar A\,T_{ab}{}^{ij}\varepsilon_{ij}-R(\mathcal{V})^-{}_{ab}{}^i{}_k\,B^{jk}\,\varepsilon_{ij}+\frac{1}{8}T_{ab}{}^{ij}\,T_{cdij}G^{+cd}\,, \\
\Lambda_i|_{\mathbb{T}(\bar\Phi)} &= 2\,\Box_{\mathrm{c}}\not{D}\Psi^j\varepsilon_{ij}+\frac{1}{4}\gamma^c\gamma_{ab}(2\,D_cT^{ab}{}_{ij}\,\Lambda^j+T^{ab}{}_{ij}\,D_c\Lambda^j) \\
&\qquad -\frac{1}{2}\varepsilon_{ij}\big(R(\mathcal{V})_{ab}{}^j{}_k+2\mathrm{i}\,R(A)_{ab}\delta^j{}_k\big)\,\gamma^c\gamma^{ab}D_c\Psi^k \\
&\qquad +\frac{1}{2}\,\varepsilon_{ij}\left(3\,D_bD-4\mathrm{i}D^aR(A)_{ab}+\frac{1}{4}T_{bc}{}^{ij}\overleftrightarrow{D}_a\,T^{ac}{}_{ij}\right)\gamma^b\Psi^j
\end{aligned}
$$

$$
\begin{aligned}
&- 2\,G^{+ab}\,\not{D} R(Q)_{abi} + 6\,\varepsilon_{ij} D\,\not{D}\Psi^j \\
&+ 3\,\varepsilon_{ij}\left(\not{D}\chi_k\,B^{kj} + \not{D}\bar{A}\,\not{D}\chi^j\right) \\
&+ \frac{3}{2}\left(2\,\not{D}B^{kj}\varepsilon_{ij} + \not{D}G^{+}_{ab}\gamma^{ab}\,\delta_i^k + \frac{1}{4}\varepsilon_{mn}T_{ab}{}^{mn}\,\gamma^{ab}\,\not{D}\bar{A}\,\delta_i{}^k\right)\chi_k\,,
\end{aligned}
$$

$$
\begin{aligned}
C|_{\mathbb{T}(\bar{\Phi})} = {}& 4(\Box_{\rm c} + 3\,D)\Box_{\rm c}\bar{A} - \frac{1}{2}D_a\left(T^{ab}{}_{ij}\,T_{cb}{}^{ij}\right)D^c\bar{A} + \frac{1}{16}(T_{abij}\varepsilon^{ij})^2\bar{C} \\
&+ D_a\left(\varepsilon^{ij}D^aT_{bcij}\,G^{+bc} + 4\,\varepsilon^{ij}T^{ab}{}_{ij}\,D^cG^{+}_{cb} - T_{bc}{}^{ij}\,T^{ac}{}_{ij}\,D^b\bar{A}\right) \\
&+ \left(6\,D_bD - 8\mathrm{i}D^aR(A)_{ab}\right)D^b\bar{A}+\,,
\end{aligned}
\tag{D.2}
$$

where we suppressed terms nonlinear in the covariant fermion fields. Observe that the right-hand side of these expressions is always linear in the conjugate components of the $w = 0$ chiral multiplet, i.e. in $(\bar{A}, \Psi^i, B^{ij}, G^{+}_{ab}, \Lambda^i, \bar{C})$.

Using the result (D.2) one can construct a large variety of superconformal invariants with higher-derivative couplings involving vector multiplets, as well as the tensor and Weyl chiral backgrounds. The construction of the higher-order Lagrangians therefore proceeds in two steps. First one constructs the Lagrangian in terms of unrestricted chiral multiplets of appropriate Weyl weights, in the form

$$
\int \mathrm{d}^4\theta\,\Phi_0\,\mathbb{T}^{(n_1)}\,\mathbb{T}^{(n_2)}\cdots\,\mathbb{T}^{(n_k)}\,. \tag{D.3}
$$

Here, the n-th power of the kinetic multiplet is defined recursively as $\mathbb{T}^{(n)} = \mathbb{T}(\bar{\Phi}_n\,\mathbb{T}^{(n-1)})$ for Φ_n of appropriate weight. Subsequently, one expresses the unrestricted supermultiplets in terms of the reduced supermultiplets in section B. In these expressions it is natural to introduce a variety of arbitrary homogeneous functions, so that resulting final Lagrangian is controlled by a function of given homogeneity and holomorphicity in the various fields, corresponding to the original structure in (D.3).

In this work, we will make use of invariants of the type (D.3), where one, two or three kinetic multiplets appear, and are naturally quadratic, cubic and quartic in chiral multiplet components, respectively. While the first of these was described in detail in [13], the other two have not appeared in the literature. These are straightforward to write, using the formulae above and in [13] but are rather unilluminating, so that we prefer to emphasise the structure of the corresponding Lagrangians, restricting ourselves to the leading terms.

The quadratic invariant. The simplest case of a Lagrangian involving a kinetic multiplet is the one in (D.1), where a $w = 0$ chiral multiplet is multiplied with the kinetic of an antichiral one. In components, the leading bosonic terms in the resulting Lagrangian read

$$
\begin{aligned}
e^{-1}\mathcal{L} = {}& C\,\bar{C} + 8\,\mathcal{D}_aF^{-ab}\,\mathcal{D}^cF^{+}{}_{cb} + 4\,F^{-ac}\,F^{+}{}_{bc}\,R(\omega,e)_a{}^b \\
&+ 4\,\mathcal{D}^2A\,\mathcal{D}^2\bar{A} + 8\,\mathcal{D}^\mu A\left[R_\mu{}^a(\omega,e) - \frac{1}{3}R(\omega,e)\,e_\mu{}^a\right]\mathcal{D}_a\bar{A} \\
&- \mathcal{D}^\mu B_{ij}\,\mathcal{D}_\mu B^{ij} + \left(\frac{1}{6}R(\omega,e) + 2\,D\right)B_{ij}B^{ij} + \cdots\,,
\end{aligned}
\tag{D.4}
$$

where we suppressed the prime on the second chiral multiplet indicated in (D.1) for brevity. The next step is to consider the components of the chiral and anti-chiral multiplet in (D.4) to be composite, given as holomorphic and anti-holomorphic functions, F, $\bar{F}$ of the fundamental vector, tensor and Weyl multiplet respectively. The result is a Lagrangian that is controlled by a homogeneous function of degree zero,

$$F(X^A, A_{\mathsf{w}}, A_{\mathsf{t}})\, \bar{F}(\bar{X}^A, \bar{A}_{\mathsf{w}}, \bar{A}_{\mathsf{t}}) \sim \mathcal{H}(X^A, A_{\mathsf{w}}, A_{\mathsf{t}}, \bar{X}^A, \bar{A}_{\mathsf{w}}, \bar{A}_{\mathsf{t}})\,, \tag{D.5}$$

which depends on the vector multiplets scalars, X^A, and the Weyl and tensor multiplet composites, A_{w} and A_{t}. This invariant corresponds to higher derivative couplings that are quadratic in the leading terms, F^2, R^2 and $(\nabla E)^2$ respectively. The arbitrariness of the function in A_{w} is analogous to the similar dependence of the chiral couplings, $F(X^A, A_{\mathsf{w}})$ which describes the full topological string partition function. Note that the various combinations have different order of derivatives, as e.g. F^4 comprises only four derivatives, while R^2F^2, $(\nabla E)^2F^2$ contain six derivatives and R^4, $R^2(\nabla E)^2$, $(\nabla E)^4$ contain eight derivatives. However, all these invariants have a common structure, found by substituting the definitions of the chiral multiplets in terms of F, $\bar{F}$ and $\mathcal{H}$ in (D.4).

This was done in [13], where the F^4 coupling was constructed, based on a real function $\mathcal{H}(X, \bar{X})$, which plays the role of a Kähler potential, as it is defined up to a real function, as

$$\mathcal{H}(X, \bar{X}) \to \mathcal{H}(X, \bar{X}) + \Lambda(X) + \bar{\Lambda}(\bar{X})\,. \tag{D.6}$$

The explicit form of the Lagrangian is

$$\begin{aligned}
e^{-1}\mathcal{L} = {} & \mathcal{H}_{IJ\bar{K}\bar{L}}\Big[\frac{1}{4}\Big(G^{-}_{ab}{}^{I}\, G^{-ab\,J} - \frac{1}{2}Y_{ij}{}^{I}\, Y^{ijJ}\Big)\Big(G^{+}_{ab}{}^{K}\, G^{+ab\,L} - \frac{1}{2}Y^{ijK}\, Y_{ij}{}^{L}\Big) \\
& \qquad + 4\,\mathcal{D}_a X^I\, \mathcal{D}_b \bar{X}^K \Big(\mathcal{D}^a X^J\, \mathcal{D}^b \bar{X}^L + 2\, G^{-\,ac\,J}\, G^{+\,b}{}_{c}{}^{L} - \frac{1}{4}\eta^{ab}\, Y_{ij}^{J}\, Y^{L\,ij}\Big)\Big] \\
& + \Big\{\mathcal{H}_{IJ\bar{K}}\Big[4\,\mathcal{D}_a X^I\, \mathcal{D}^a X^J\, \mathcal{D}^2 \bar{X}^K - \mathcal{D}_a X^I\, Y_{ij}^{J}\, \mathcal{D}^a Y^{K\,ij} \\
& \qquad - \Big(G^{-ab\,I}\, G^{-}_{ab}{}^{J} - \frac{1}{2}Y_{ij}^{I}\, Y^{Jij}\Big)\Big(\Box_{\mathrm{c}} X^K + \frac{1}{8}G^{-}_{ab}{}^{K}\, T^{abij}\varepsilon_{ij}\Big) \\
& \qquad + 8\,\mathcal{D}^a X^I G^{-}_{ab}{}^{J}\Big(\mathcal{D}_c G^{+\,cb\,K} - \frac{1}{2}\mathcal{D}_c \bar{X}^K T^{ij\,cb}\varepsilon_{ij}\Big)\Big] + \text{h.c.}\Big\} \\
& + \mathcal{H}_{I\bar{J}}\Big[4\Big(\Box_{\mathrm{c}} \bar{X}^I + \frac{1}{8}G^{+}_{ab}{}^{I}\, T^{ab}{}_{ij}\varepsilon^{ij}\Big)\Big(\Box_{\mathrm{c}} X^J + \frac{1}{8}G^{-}_{ab}{}^{J}\, T^{abij}\varepsilon_{ij}\Big) + 4\,\mathcal{D}^2 X^I\, \mathcal{D}^2 \bar{X}^J \\
& \qquad + 8\,\mathcal{D}_a G^{-\,abI}\, \mathcal{D}_c G^{+c}{}_{b}{}^{J} - \mathcal{D}_a Y_{ij}{}^{I}\, \mathcal{D}^a Y^{ij\,J} + \frac{1}{4}T_{ab}{}^{ij}\, T_{cdij}\, G^{-ab\,I} G^{+cd\,J} \\
& \qquad + \Big(\frac{1}{6}\mathcal{R} + 2\,D\Big)\, Y_{ij}{}^{I}\, Y^{ij\,J} + 4\, G^{-ac\,I}\, G^{+}{}_{bc}{}^{J}\, \mathcal{R}_a{}^{b} \\
& \qquad + 8\Big(\mathcal{R}^{\mu\nu} - \frac{1}{3}g^{\mu\nu}\mathcal{R} + \frac{1}{4}T^{\mu}{}_{b}{}^{ij}\, T^{\nu b}{}_{ij} + \mathrm{i}R(A)^{\mu\nu} - g^{\mu\nu}D\Big)\, \mathcal{D}_\mu X^I\, \mathcal{D}_\nu \bar{X}^J \\
& \qquad - \Big[\mathcal{D}_c \bar{X}^J\, \Big(\mathcal{D}^c T_{ab}{}^{ij}\, G^{-\,I\,ab} + 4\, T^{ij\,cb}\, \mathcal{D}^a G^{-}_{ab}{}^{I}\Big)\, \varepsilon_{ij} + [\text{h.c.}; I \leftrightarrow J]\Big] \\
& \qquad - \Big[\varepsilon^{ik}\, Y_{ij}{}^{I}\, G^{+ab\,J}\, R(\mathcal{V})_{ab}{}^{j}{}_{k} + [\text{h.c.}; I \leftrightarrow J]\Big]\Big]\,,
\end{aligned} \tag{D.7}$$

JHEP02(2014)093

where (we suppress fermionic contributions),

$$
\begin{aligned}
G_{ab}^{-}{}^{I} &= F_{ab}^{-}{}^{I} - \frac{1}{4}\,\bar{X}^{I}\,T_{ab}{}^{ij}\varepsilon_{ij}\,,\\
\Box_{\rm c} X^{I} &= \mathcal{D}^{2}X^{I} + \left(\frac{1}{6}\mathcal{R} + D\right)\,X^{I}\,.
\end{aligned}
\tag{D.8}
$$

One can obtain the more general couplings as discussed above, resulting in similar expressions. For example, the R^2F^2- and R^4-type couplings feature terms found by substituting $F^2 \to R^2$ and similarly for the other components in (D.7) and are discussed in [13].

The cubic invariant. The next more complicated example of Lagrangians containing kinetic multiplets is to consider an integral quadratic in kinetic multiplets, as

$$
\int \mathrm{d}^4\bar{\theta}\,\bar{\Phi}_0\mathbb{T}(\Phi_1)\,\mathbb{T}(\Phi_2)\,,
\tag{D.9}
$$

where Φ_0 is a $w = -2$ chiral, while Φ_1 and Φ_2 are $w = 0$ anti-chirals, as above. It is straightforward to apply the multiplication rule for chiral multiplets, to obtain the analogous master formula of the type (D.4), in this case. The result takes the form

$$
\begin{aligned}
e^{-1}\mathcal{L} = {}& C^2\,\bar{C} - \frac{1}{16}\,\bar{A}\,C^2(T^+)^2\\
&+ 2\,\bar{A}\,C\left[4\,(\Box + 3\,D)\Box A + \frac{1}{16}\,C\,(T^-)^2 + 4\,D_a\big(T^{+\,ab}\,D^cG^+_{cb}\big) + \ldots\right]\\
&- \bar{A}\,C\left[2\,\varepsilon^{ik}\varepsilon^{jl}(\Box + 3\,D)B_{ij}\,(\Box + 3\,D)B_{kl} - D_{[a}\big(D^cG^-_{cb]_+}\big)D^{[a}\big(D_cG^{-\,cb]_+}\big) + \ldots\right]\\
&+ 2\,C\left[B^{ij}\,(\Box + 3\,D)B_{ij} - G^{+\,ab}\left(D_{[a}\big(D^cG^-_{cb]_+}\big) - \Box A\,T^+_{ab}\right) + \cdots\right],
\end{aligned}
\tag{D.10}
$$

which is manifestly quadratic in holomorphic and linear in anti-holomorphic components. Note that we again use a simplified notation that naively identifies the three a priori independent multiplets, despite the fact that the anti-chiral multiplet is of weight -2, while the chiral ones are of $w = 0$. The most general invariant follows by completing the combinations given above with the components of the kinetic multiplet given in (D.2) and viewing the holomorphic components as quadratic forms in the components of the two chiral multiplets in (D.9), as done in (D.4).

It is now straightforward, if cumbersome, to consider the three multiplets in (D.9) as functions of the vector multiplets, the tensor multiplet and the Weyl multiplet, as done in (D.5), leading to a Lagrangian described by a function, $\mathcal{H}(X^A, A_{\mathsf{w}}, A_{\mathsf{t}}, \bar{X}^A, \bar{A}_{\mathsf{w}}, \bar{A}_{\mathsf{t}})$, which is homogeneous of degree zero in the holomorphic components and homogeneous of degree -2 in the anti-holomorphic components. We refrain from giving the corresponding expression (D.7) in this case, since we will only be dealing with the leading terms and the properties of the corresponding function $\mathcal{H}$.

Once again, the generic function of all available multiplets leads to various invariants, which contain different orders of derivatives but share the same structure, as in (D.10). The prototype of these terms is the F^6 invariant arising by taking $\mathcal{H}(X^A, \bar{X}^A)$, i.e. a function of vector multiplet scalars only. Allowing for holomorphic/anti-holomorphic dependence on the scalars A_{w} and A_{t} leads to terms of the type R^2F^4, R^4F^2, $(\nabla E)^2F^4$ and so on for all possible combinations. Note that many of these contain more than eight derivatives and therefore fall outside the scope of this work.

The quartic invariants. We finally consider integrals of the type (D.3) which are cubic in the kinetic multiplet operator, $\mathbb{T}$, in which case we find two possibilities. Indeed, this is the first case where one needs to consider nested kinetic multiplets, since the two possible integrals,

$$\int \mathrm{d}^4\bar{\theta}\,\bar{\Phi}_0\mathbb{T}(\Phi_1)\,\mathbb{T}(\Phi_2)\,\mathbb{T}(\Phi_3)\,, \qquad \int \mathrm{d}^4\bar{\theta}\,\bar{\Phi}_0\mathbb{T}(\Phi_1)\,\mathbb{T}(\Phi_0'\mathbb{T}(\bar{\Phi}_2))\,, \tag{D.11}$$

are not equivalent upon partial integration. Here, the first integral is the straightforward extension of (D.1) and (D.9), while in the second integral Φ_0 and Φ_0' are $w=-2$ chirals, while Φ_1 and Φ_2 are $w=0$ chirals, as above.

Once again, one can apply the multiplication rule for chiral multiplets, to obtain the analogous master formula of the type (D.4), in these cases. The expression for the first integral is similar to (D.10), where three chiral multiplets appear and is not used in this paper. The second integral is more cumbersome, but can be easily computed by an iterative procedure, by noting that $\bar{\Phi}_0\mathbb{T}(\Phi_1)$ and $\Phi_0'\mathbb{T}(\bar{\Phi}_2)$ are $w=0$ multiplets, so that (D.4) applies for their components. One can then obtain the result to the integral by making the following substitutions

$$\begin{aligned}
A &\to A_0\, A|_{\mathbb{T}(\bar{\Phi})}\,,\\
B_{ij} &\to B_{0\,ij}\, A|_{\mathbb{T}(\bar{\Phi})} + A_0\, B_{ij}|_{\mathbb{T}(\bar{\Phi})}\,,\\
G^{-\,ab} &\to G_0^{-\,ab}\, A|_{\mathbb{T}(\bar{\Phi})} + A_0\, G^{-\,ab}|_{\mathbb{T}(\bar{\Phi})}\,,\\
C &\to C_0\, A|_{\mathbb{T}(\bar{\Phi})} + A_0\, C|_{\mathbb{T}(\bar{\Phi})} - \frac{1}{4}\left(\varepsilon^{ik}\varepsilon^{jl}B_{0\,ij}\, B_{kl}|_{\mathbb{T}(\bar{\Phi})} - 2\, G^{-}_{0\,ab}\, G^{-\,ab}|_{\mathbb{T}(\bar{\Phi})}\right),
\end{aligned} \tag{D.12}$$

in (D.4), where the components labeled with $|_{\mathbb{T}(\bar{\Phi})}$ are as in (D.2).

As above, allowing for the four chiral multiplets involved to depend on the vector, tensor and/or the Weyl multiplet, exactly as in (D.5), one obtains various higher derivative invariants, sharing the same structure. However, all but one of the invariants described by each of the two integrals in (D.11) necessarily contain more than eight spacetime derivatives if the Weyl and tensor multiplet backgrounds are allowed, so that they are not relevant for our consideration. The exception is the case where all the composite chiral multiplets only depend on the vector multiplets, in which case we obtain two F^8 invariants from (D.11).

References

[1] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, *Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes*, Commun. Math. Phys. **165** (1994) 311 [hep-th/9309140] [INSPIRE].

[2] I. Antoniadis, E. Gava, K. Narain and T. Taylor, *Topological amplitudes in string theory*, Nucl. Phys. **B 413** (1994) 162 [hep-th/9307158] [INSPIRE].

[3] N.A. Nekrasov, *Seiberg-Witten prepotential from instanton counting*, Adv. Theor. Math. Phys. **7** (2004) 831 [hep-th/0206161] [INSPIRE].

[4] N. Nekrasov and A. Okounkov, *Seiberg-Witten theory and random partitions*, hep-th/0306238 [INSPIRE].

[5] J.F. Morales and M. Serone, *Higher derivative F terms in $N = 2$ strings*, Nucl. Phys. **B 481** (1996) 389 [hep-th/9607193] [INSPIRE].

[6] I. Antoniadis, S. Hohenegger, K. Narain and T. Taylor, *Deformed Topological Partition Function and Nekrasov Backgrounds*, Nucl. Phys. **B 838** (2010) 253 [arXiv:1003.2832] [INSPIRE].

[7] Y. Nakayama and H. Ooguri, *Comments on Worldsheet Description of the Omega Background*, Nucl. Phys. **B 856** (2012) 342 [arXiv:1106.5503] [INSPIRE].

[8] I. Antoniadis, I. Florakis, S. Hohenegger, K. Narain and A. Zein Assi, *Worldsheet Realization of the Refined Topological String*, Nucl. Phys. **B 875** (2013) 101 [arXiv:1302.6993] [INSPIRE].

[9] I. Antoniadis, I. Florakis, S. Hohenegger, K. Narain and A. Zein Assi, *Non-Perturbative Nekrasov Partition Function from String Theory*, Nucl. Phys. **B 880** (2014) 87 [arXiv:1309.6688] [INSPIRE].

[10] M. Billó, M. Frau, F. Fucito and A. Lerda, *Instanton calculus in RR background and the topological string*, JHEP **11** (2006) 012 [hep-th/0606013] [INSPIRE].

[11] S. Ferrara, R.R. Khuri and R. Minasian, *M theory on a Calabi-Yau manifold*, Phys. Lett. **B 375** (1996) 81 [hep-th/9602102] [INSPIRE].

[12] I. Antoniadis, S. Ferrara, R. Minasian and K. Narain, *R^4 couplings in M and type-II theories on Calabi-Yau spaces*, Nucl. Phys. **B 507** (1997) 571 [hep-th/9707013] [INSPIRE].

[13] B. de Wit, S. Katmadas and M. van Zalk, *New supersymmetric higher-derivative couplings: Full $N = 2$ superspace does not count!*, JHEP **01** (2011) 007 [arXiv:1010.2150] [INSPIRE].

[14] K. Peeters, P. Vanhove and A. Westerberg, *Chiral splitting and world sheet gravitinos in higher derivative string amplitudes*, Class. Quant. Grav. **19** (2002) 2699 [hep-th/0112157] [INSPIRE].

[15] J.T. Liu and R. Minasian, *Higher-derivative couplings in string theory: dualities and the B-field*, arXiv:1304.3137 [INSPIRE].

[16] I. Antoniadis, R. Minasian, S. Theisen and P. Vanhove, *String loop corrections to the universal hypermultiplet*, Class. Quant. Grav. **20** (2003) 5079 [hep-th/0307268] [INSPIRE].

[17] S. Alexandrov, J. Manschot, D. Persson and B. Pioline, *Quantum hypermultiplet moduli spaces in $N = 2$ string vacua: a review*, arXiv:1304.0766 [INSPIRE].

JHEP02(2014)093

[18] K. Peeters, *A field-theory motivated approach to symbolic computer algebra*, Comput. Phys. Commun. **176** (2007) 550 [arXiv:cs/0608005].

[19] K. Peeters, *Introducing Cadabra: A Symbolic computer algebra system for field theory problems*, hep-th/0701238 [INSPIRE].

[20] B. de Wit and F. Saueressig, *Off-shell $N = 2$ tensor supermultiplets*, JHEP **09** (2006) 062 [hep-th/0606148] [INSPIRE].

[21] B. de Wit, *N=2 electric-magnetic duality in a chiral background*, Nucl. Phys. Proc. Suppl. **49** (1996) 191 [hep-th/9602060] [INSPIRE].

[22] J.M. Maldacena, A. Strominger and E. Witten, *Black hole entropy in M-theory*, JHEP **12** (1997) 002 [hep-th/9711053] [INSPIRE].

[23] T. Mohaupt, *Black hole entropy, special geometry and strings*, *Fortsch. Phys.* **49** (2001) 3 [hep-th/0007195] [INSPIRE].

[24] G. Lopes Cardoso, B. de Wit, J. Kappeli and T. Mohaupt, *Stationary BPS solutions in $N = 2$ supergravity with R^2 interactions*, JHEP **12** (2000) 019 [hep-th/0009234] [INSPIRE].

[25] N. Banerjee, B. de Wit and S. Katmadas, *The Off-Shell 4D/5D Connection*, JHEP **03** (2012) 061 [arXiv:1112.5371] [INSPIRE].

[26] D. Butter, B. de Wit, S.M. Kuzenko and I. Lodato, *New higher-derivative invariants in $N = 2$ supergravity and the Gauss-Bonnet term*, JHEP **12** (2013) 062 [arXiv:1307.6546] [INSPIRE].

[27] F. Moura, *Type II and heterotic one loop string effective actions in four dimensions*, JHEP **06** (2007) 052 [hep-th/0703026] [INSPIRE].

[28] F. Moura, *Four-dimensional 'old minimal' $N = 2$ supersymmetrization of R^4*, JHEP **07** (2003) 057 [hep-th/0212271] [INSPIRE].

[29] E. Bergshoeff, M. de Roo and B. de Wit, *Extended Conformal Supergravity*, Nucl. Phys. **B 182** (1981) 173 [INSPIRE].

[30] D. Butter and S.M. Kuzenko, *New higher-derivative couplings in 4D $N = 2$ supergravity*, JHEP **03** (2011) 047 [arXiv:1012.5153] [INSPIRE].

[31] A. Dabholkar, J. Gomes and S. Murthy, *Quantum black holes, localization and the topological string*, JHEP **06** (2011) 019 [arXiv:1012.0265] [INSPIRE].

[32] J. Gomes, *Quantum entropy and exact 4d/ 5d connection*, arXiv:1305.2849 [INSPIRE].

[33] B. Sahoo and A. Sen, *Higher derivative corrections to non-supersymmetric extremal black holes in $N = 2$ supergravity*, JHEP **09** (2006) 029 [hep-th/0603149] [INSPIRE].

[34] R. Firth and J. Jenkins, *Super-Symmetry with Isospin*, Nucl. Phys. **B 85** (1975) 525 [INSPIRE].

[35] M. de Roo, J. van Holten, B. de Wit and A. Van Proeyen, *Chiral Superfields in $N = 2$ Supergravity*, Nucl. Phys. **B 173** (1980) 175 [INSPIRE].

[36] B. de Wit, J. van Holten and A. Van Proeyen, *Structure of $N = 2$ Supergravity*, Nucl. Phys. **B 184** (1981) 77 [*Erratum ibid.* **B 222** (1983) 516] [INSPIRE].

[37] S. Ferrara and P. van Nieuwenhuizen, *Tensor Calculus for Supergravity*, Phys. Lett. **B 76** (1978) 404 [INSPIRE].